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Mathematical objects and mathematical structure.

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MATHEMATICAL OBJECTS

AND

MATHEMATICAL STRUCTURE

A Dissertation Presented

By

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PREFACE

The primary purpose of this work is to enunciate and defend the view that abstract structure is the proper subject matter of mathematics. As I describe it, this view is Platonistic. Yet it is incompatible with traditional mathematical Platonism, since it denies the existence of special mathematical objects.

Certain branches of mathematics do seem to require special objects. Arithmetic seems to require natural numbers; analysis, real numbers; set theory, pure sets.* But in my first chapter, I argue in favor of a reinterpretation of the special branches, arithmetic and analysis. Arithmetic is not concerned with special objects, natural numbers, but rather with all number-like sequences. Analysis is not concerned with special objects, real numbers, but with all real-number-like systems. I also argue for an expansion of the background theory, set theory, so that it is not concerned specially with pure sets, but with <u>all</u> sets, pure and impure. From my arguments I conclude that the apparent need for special mathematical objects is not genuine.

If there are no mathematical objects, what then is mathematics about? My answer is that mathematics is a very general study of more or less ordinary things. Among "more or less ordinary things", I include functions, relations, and systems of functions and relations. These entities may be quite theoretical, such as the continuum of spatio-

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*A <u>pure</u> set is one founded solely on the empty set. All its members, all members of its members, all members of members of its members, etc., are sets.

temporal distance relations (a real-number-like system), or mundane, such as the ways of rotating an automobile tire (a group of rotations). They are not "ordinary" in the sense of being concrete things, but primarily in the sense that they are not specially mathematical entities--they are objects for empirical sciences.

My second chapter begins with a certain picture of mathematics: Mathematics divides into a number of mathematical theories. Each mathematical theory is concerned with all systems of a certain sort. Thus group theory is concerned with all groups, and arithmetic is concerned with all number-like sequences. Particular systems are provided by the empirical sciences, and they may be construed as ordered n-tuples of objects, functions, relations, and sets. Each theory has its own postulates, postulates telling us which systems are the concern of the theory. These postulates contain only notions of higher-order logic and set theory. Finally, the special notions of a mathematical theory are all the notions which are preserved under isomorphisms of the relevant systems. --This picture incorporates some of the ideas of category theory, and it may serve as a bridge to a more sophisticated view.

From this picture we can see that mathematics is not concerned with all aspects of ordinary things, but only with their abstract structural aspects. Abstract structure is structure generated by the notions of higher-order logic and set theory: the notion of an object possessing a property, the notion of two objects bearing a relation, the notion of one object being the result of applying a function to another object, and the notion of an object being a member of a set. In my second chapter, I give a precise formulation of the view that mathematics is about abstract structure. I also point out strong affinities between this view, _structuralism_, and an older view, _logicism_.

In my final chapter, I discuss set theoretic foundations for structuralist mathematics. Highlights include a rejection of the Axiom of Infinity, defense of a Platonistic conception of sets, and a proposal that sets are identical with certain properties.

In developing structuralism, I have attempted to keep ontology and ideology at a minimum. I state principles in modal form only where the corresponding nonmodal principles are too weak to be interesting. The main structuralist doctrines are stated without modalities. Objects which would outstrip the ranks of set theory, the types of type theory, have been studiously avoided. For example, I postulate no membership relation, since it would be borne by sets of arbitrarily high rank, and no exemplification relation, since it would be borne by objects and properties of arbitrarily high type. Similarly, I have not postulated "abstract structural properties", such as the property an operation has just in case it is a group, and the property a relation has just in case it is a number-like sequence, an _ω-sequence_. These too would be instanced by objects of arbitrarily high type.

Although I postulate no relations of membership and exemplification and no structural properties, I do speak of the _notions_ of membership and exemplification and of structural _notions_. These notions are relation-_like_, or property-_like_ things. For example, just as we may say that two objects bear a relation, we may say that an object and a set "bear" or "satisfy" the membership notion. And just as we may say that an object exemplifies a certain property, so we may say that a function "has" or "satisfies" the group notion.

I shall have much to say about notions, but I do not wish to commit myself on the question of what exactly they are. There are at least four interesting theories:

N1 Notions are Platonic universals which outstrip all types or ranks.

N2 Notions are ways of thinking about things. (For instance, the notion of an ω-sequence is the relation of believing a thing to be an ω-sequence.)

N3 Notions are symbols. (For instance, the notion of an ω-sequence is just the predicate "is an ω-sequence".)

N4 Notions are symbolic constructions. (That is, they are nothing at all: Talk of notions is disguised talk about symbols.)

None of these theories strikes me as especially unreasonable. Each theory explains how notions can be relation-like. According to N1, notions can just be "super-relations". According to N2, a notion can "relate" objects in the sense of being a true way of thinking about them. According to N3, a notion can "relate" objects in the sense of being a true predicate of them. N4 has not been spelled out, but I think that it too would yield a sense in which notions can "relate" things.

Besides relation-like notions, there are also function-like notions, notions which, like functions, "map" objects to other objects. The above theories can easily be extended to cover these notions as well.

In view of my acceptance of the classical, Platonistic conception of set (indeed, of sets, properties, relations, and functions), it may seem strange that I shy away from the super-Platonism of N1. My attitude here is the result of considerable reflection. Within the ranks and types, I have found no acceptable alternative to the Platonistic conception. On the other hand, there is no mathematical need for a rich Platonistic totality of notions. What I call notions might be confined to inscribed symbols or acts of human thought.

I intend that my writing will serve as an introduction to the modern problems of mathematical ontology, problems which can be grasped only by those who see that structuralism is at least a correct view of large portions of mathematics. In order to achieve this purpose, I stick close by all orthodox postions, except where I find them indefensible, and I avoid discussing some of the complexities of opposing views. I think the result is accessible to anyone acquainted with the essentials of higher-order logic and set theory.

The problems of mathematical ontology are of very general philosophical significance and must prove fascinating to anyone possessed of genuine curiosity. On this I base my hope that the rewards of reading this piece outweigh the difficulties.

I am very much indebted to Michael Jubien, both for his extensive criticisms and for many of the ideas that inspired this work. I also thank Gary Hardegree and Ed Zalta for their helpful suggestions.

Alan F. McMichael

ABSTRACT

Mathematical Objects

and

Mathematical Structure

September, 1979

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The primary purpose of this work is to enunciate and defend the view that abstract structure is the proper subject matter of mathematics. As I describe it, this view is Platonistic. Yet it is incompatible with traditional mathematical Platonism, since it denies the existence of special mathematical objects.

Certain branches of mathematics do seem to require special objects. Arithmetic seems to require natural numbers; analysis, real numbers; set theory, pure sets.* But in my first chapter, I argue in favor of a reinterpretation of the special branches, arithmetic and analysis. Arithmetic is not concerned with special objects, natural numbers, but rather with all number-like-sequences. Analysis is not concerned with special objects, real numbers, but with all real-number-like systems. I also argue for an expansion of the background theory, set theory, so

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that it is not concerned specially with pure sets, but with _all_ sets, pure and impure. From my arguments I conclude that the apparent need for special mathematical objects is not genuine.

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In my final chapter, I discuss set theoretic foundations for structuralist mathematics. Highlights include a rejection of the Axiom of Infinity, defense of a Platonistic conception of sets, and a proposal that sets are identical with certain properties.

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CHAPTER I

MATHEMATICS WITHOUT MATHEMATICAL OBJECTS

1. Arithmetic Without Numbers

Statements of arithmetic contain constants, "0", "1", "2", "3", and so on, and quantifiers restricted by the word "number". It is easy to suppose that the constants denote definite objects--particular numbers--and that the quantifiers are restricted to a definite kind--the numbers. However, there is a serious objection to this supposition: Although it is generally agreed that numbers must be abstract objects, there has been no fully successful attempt to identify which abstract objects the numbers are.

The problem is not that we lack theories of number. We have the theory that numbers are pure sets, $\emptyset$, $\{\emptyset\}$, $\{\emptyset, \{\emptyset\}\}$, and so on, the theory that numbers are properties of sets, the property of having no members, the property of having one member, the property of having two members, and so on, the theory that numbers are just numerical symbol-types, "0", "1", "2", and so on, as well as many others. The problem is that we have no good reason to think that any particular <u>one</u> of these theories is correct.*

Arithmetic itself provides incomplete clues. To be sure, numbers are supposed to form an infinite sequence, and there are supposed to be certain operations among numbers, operations having certain formal pro-

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*Paul Benacerraf pointed out the difficulty in his article "What Numbers Could Not Be".

perties. However, these characteristics are insufficient to single the numbers out. If there are any number-like sequences, there are many such sequences. On each such sequence, there are operations with the formal properties arithmetic operations are supposed to have.

It is the function of number terms in nonmathematical discourse that provides the basis for many theories of number.* Yet this basis is a very shaky one. On the one hand, number terms appear to function ambiguously in ordinary discourse. The most pronounced ambiguity is that between _cardinalities_ and _integral proportions_. (See section 2 of this chapter.) On the other hand, it is far from clear that number terms must denote the very same things in both ordinary and mathematical contexts. So even if we arrive at a theory of what "numbers", in ordinary discourse, are, we may not have a good theory of the objects of arithmetic.

At this point, a defender of numbers might object: Why suppose that there is a problem of "identifying" numbers? Numbers might be abstract objects _of a unique sort_, and so not admit of any independent specification.

The trouble with this proposal is that it leaves us with no idea of how we are _acquainted_ with abstract objects of the unique sort.** For typical abstract objects, acquaintance is relatively straightforward. It proceeds by means of _abstraction_, the recognition of common features and

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*The most notable example is the account of number contained in Gottlob Frege's _Foundations of Arithmetic_.

**In "Ontology and Mathematical Truth", Michael Jubien points out the related problem of _referring_ to numbers. However, Jubien thinks that there is a general problem of reference to abstract entities, whereas my arguments concerning acquaintance are directed solely at special mathematical entities.

relations, and is founded ultimately on the experience of particulars. By the usual means of acquaintance, we come to know such things as properties, relations, functions, and sets. Unfortunately, we have found ourselves unable to identify numbers with any of these. If we now erect a separate ontological category for numbers, we are left with the very serious difficulty of saying how we are acquainted with them.

I conclude that since we lack sufficient means for identifying numbers as particular abstract objects, the hypothesis that arithmetic constants denote definite objects, numbers, is very doubtful. Nevertheless, arithmetic statements are meaningful. Some are true; others, false. If they are not statements about numbers, how are they to be interpreted?

My answer is that they may be interpreted as disguised generalizations concerning all number-like sequences. I claim two advantages for this interpretation. First, due to the facts just mentioned, it is far more natural than any interpretation according to which arithmetic is concerned with a particular infinite sequence. Second, it shows how we can pursue arithmetic without committing ourselves to the existence of particular objects, numbers. Thus we may avoid one great stumbling block in the foundations of mathematics, the apparent need to postulate the existence of infinite totalities.

Such an interpretation has been outlined before in Nicholas White's article, "What Numbers Are". However, an exact assessment is impossible unless we work out the rather intricate details. As a reward for our labors, we shall uncover features of arithmetic which can be generalized to all typical branches of mathematics.

A number-like sequence, or _ω-sequence_, is a relation R satisfying the following postulates:

A1ω Nothing bears the relation R to more than one thing.

A2ω No two things bear the relation R to the same thing.

A3ω Everything in the field of R bears R to something.*

A4ω There is a unique object 0 in the field of R such that nothing bears the relation R to it.

A5ω For any set x of objects from the field of R, if 0 is in x, and for any objects a and b, if a is in x and a bears R to b, then b is in x, then every object in the field of R is in x.

These are Dedekind-Peano postulates for "the relation of arithmetic succession", "the relation a number n bears to the number n + 1", except that I am not giving a definite interpretation for the variable "R".

In the interpretation of arithmetic I am proposing, each arithmetic constant denotes a function-like entity which maps any ω-sequence to an object in its field. For example, "0" denotes that function-like entity which maps each ω-sequence R to its first member, that is, to that object in the field of R such that nothing bears R to it.

I say "function-like entities" because these things are not true functions. They transcend all the types of type theory, all the ranks of set theory. This is so because they can map ω-sequences to objects at arbitrarily high types or ranks.** I take it that all true functions belong to some type or rank. Hence these entities are not true functions.

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*The field of a relation R is the set of all objects a such that either a bears R to some object b, or some object b bears R to a.

**If there are any ω-sequences, then there are ω-sequences of arbitrarily high type or rank: Suppose there is an ω-sequence R with members a, a', a'',... Let X be an object of arbitrarily high type or rank. Then clearly there is an ω-sequence R' with members X, a, a', a'',... The type or rank of R' is yet higher than that of X.

I shall call such function-like nonfunctions <u>functional notions</u>. Similarly, there are relation-like nonrelations which I shall call <u>relational notions</u>. These include, for example, the notion of set membership. I shall call unary relational notions <u>property-notions</u>. The notion of an ω-sequence is a property-notion.

Each n-ary arithmetic operation symbol denotes a functional notion which maps each ω-sequence to an n-ary operation on the field of that sequence. Thus "+" denotes a functional notion which yields a binary operation whenever it is applied to an ω-sequence.

Each n-ary arithmetic predicate denotes a functional notion which maps each ω-sequence to an n-ary relation on the field of that sequence. Thus "is divisible by" denotes a functional notion which yields a binary relation whenever it is applied to an ω-sequence.

The statement "2 + 3 = 5" is a truth of arithmetic because for any ω-sequence R, the value at R of the functional notion denoted by "+" is a binary operation which when applied to the value at R of the functional notion denoted by "2" and the value at R of the functional notion denoted by "3" yields a result which is identical to the value at R of the functional notion denoted by "5".--This example illustrates the proposed interpretation. In the truth conditions for a statement of arithmetic, reference is made to all ω-sequences.

We can give a general semantics along these lines. Let us define an <u>assignment</u> to be a function f which maps arithmetic constants, variables, primitives, and complex terms to functional notions, and which meets the following conditions:

(1) For any arithmetic constant K, f_K (meaning: $f(K)$) is the functional

notion k denotes. f_k maps each ω-sequence to an item in its field.

(2) For any primitive n-ary operation symbol γ^n, f_{γ^n} is the functional notion which γ^n denotes. f_{γ^n} maps each ω-sequence to an n-ary operation on the field of R.

(3) For any primitive n-ary predicate π^n, f_{π^n} is the functional notion which π^n denotes. f_{π^n} maps each ω-sequence R to an n-ary relation on the field of R.

(4) For any arithmetic variable η, f_η is <u>some</u> functional notion which maps each ω-sequence to an item in its field.

(5) For any n-ary operation symbol γ^n and any terms $\tau_1, \tau_2, \ldots, \tau_n$, the complex term $\ulcorner\gamma^n(\tau_1, \ldots, \tau_n)\urcorner$ is assigned a functional notion $f_{\ulcorner\gamma^n(\tau_1,\ldots,\tau_n)\urcorner}$ such that for every ω-sequence R,

$$f_{\ulcorner\gamma^n(\tau_1,\ldots,\tau_n)\urcorner}.(R) = f_{\gamma^n}.(R)(f_{\tau_1}.(R), \ldots, f_{\tau_n}.(R)) \; *$$

Notice that by (1), (2), and (3), each assignment is an extension of the natural denotation function for arithmetic primitives. <u>Satisfaction</u> is defined <u>relative</u> to a given ω-sequence R:

(6) An assignment f satisfies an identity formula $\tau = \sigma$ in R if and only if $f_\tau.(R) = f_\sigma.(R)$

(7) An assignment f satisfies an atomic formula $\pi^n(\tau_1, \ldots, \tau_n)$ in R if and only if $f_{\tau_1}.(R), \ldots, f_{\tau_n}.(R)$ bear the relation $f_{\pi^n}.(R)$

(8) An assignment f satisfies a wff $\ulcorner\sim\varphi\urcorner$ in R if and only if it does not satisfy φ in R.

(9) An assignment f satisfies a wff $\ulcorner\varphi \;\&\; \psi\urcorner$ in R if and only if it satisfies both φ and ψ in R.

(10) An assignment f satisfies a wff $\ulcorner\exists\eta(\varphi)\urcorner$ in R if and only if there is an assignment which differs from f, in regard to variables, at most in the case of η, and which satisfies φ in R.

Defining truth is a two-stage process:

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*I use the dot "." to distinguish applications of functional and relational notions from applications of functions and relations. Thus "n.(R)(a, b)" denotes the result of applying the function n.(R) to a and b, where n.(R) is the result of applying the functional notion n to R.

(TO) A wff φ is <u>true of</u> an ω-sequence R if and only if every assignment satisfies φ in R.

(T) A wff φ is <u>true</u> if and only if it is true of every ω-sequence.

The most natural definition of falsehood is:

(F) A wff φ is <u>false</u> if and only if its negation is true.

Nothing in these rules precludes the possibility of truth-value gaps. There are notions which are true of some, but not all, ω-sequences. By (T) and (F), if any such notion corresponds to an arithmetic statement, then that statement is neither true nor false. But it seems clear that every statement of arithmetic <u>is</u> either true or false. Hence there should be some restriction on the interpretation which insures that this is so. What sort of restriction will do?

The constant "0" is most naturally assigned a functional notion which picks out the first member of any ω-sequence. Suppose we introduce a constant "0*". Suppose we let it denote a functional notion which maps an ω-sequence R to its first member if R is not identical to a certain ω-sequence <u>R</u>*, and which maps <u>R</u>* itself to its second member. If this is permissible, then the formula "0 = 0*" is neither true nor false. I am assuming, on the contrary, that there are no truth-value gaps. There is something wrong with the interpretation of "0*". No constant so interpreted can belong to the language of arithmetic.

Similarly, unacceptable interpretations can be concocted for predicates and operation symbols.

Note that the functional notion assigned to "0" maps an ω-sequence R to something definable in terms of R, namely that item in the field of R such that nothing bears R to it. Note also that for every ω-sequence

R, the definition retains its form. This is not the case with the functional notion denoted by "0*". Thus we might look for a definability restriction on our arithmetic interpretation. Unfortunately, there may be no general notion of definability which would solve our problems with predicates and operation symbols.

Roughly speaking, a notion is an acceptable interpretation of an arithmetic constant or primitive just in case it maps the various ω-sequences to things which exhibit a constant structural role in their respective ω-sequences. So, for instance, constants denote functional notions which map ω-sequences to n-th members of those sequences, for some fixed n. Maintaining a fixed structural role from ω-sequence to ω-sequence is more complicated in the case of the notions denoted by predicates and operation symbols. To circumscribe acceptable interpretations in a general way, we need to make use of the notion of an <u>isomorphism</u>, a structure-preserving mapping or function.

Isomorphisms of relations may be defined as follows:

D1 i is an isomorphism of relations from R to R' =df (1) i is a one-to-one function, (2) the domain of i is the field of R, (3) the range of i is the field of R', and (4) for any objects a and b in the domain of i,

$$R[a, b] \text{ iff } R'[i(a), i(b)] \; *$$

Two relations are said to be <u>isomorphic</u> if there is an isomorphism from one to the other. It is a fact that all ω-sequences are isomorphic to one another, and that any relation which is isomorphic to an ω-sequence is itself an ω-sequence.

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*The basic terms for describing functions, "domain", "range", "one-to-one", and so on, are precisely defined in section 4 of chapter II.

Arithmetic notions may be defined as notions which are preserved, in a certain sense, under all ω-sequence isomorphisms. They come in various kinds, corresponding to the different types of arithmetic symbols. Constants of arithmetic denote <u>object notions of arithmetic</u>:

D2ω f is an object notion of arithmetic =df f is a functional notion which maps each ω-sequence to an item in its field, and for any isomorphism i from an ω-sequence R to an ω-sequence R',

$$i(f.(R)) = f.(R')$$

Primitive arithmetic operation symbols denote <u>operation notions of arithmetic</u>:

D3ω f is an n-ary operation notion of arithmetic =df f is a functional notion which maps each ω-sequence to an n-ary operation on the field of that sequence, and for any isomorphism i from an ω-sequence R to an ω-sequence R', and any objects $a_1, \ldots, a_n$ a_{n+1} in the domain of i,

$$f.(R)(a_1, \ldots, a_n) = a_{n+1} \quad \text{iff} \quad f.(R')(i(a_1), \ldots, i(a_n)) = i(a_{n+1})$$

Primitive arithmetic predicates denote <u>relation notions of arithmetic</u>:

D4ω f is an n-ary relation notion of arithmetic =df f is a functional notion which maps each ω-sequence to an n-ary relation on the field of that sequence, and for any isomorphism i from an ω-sequence R to an ω-sequence R', and any objects $a_1, \ldots, a_n$ in the domain of i,

$$f.(R)[a_1, \ldots, a_n] \quad \text{iff} \quad f.(R')[i(a_1), \ldots, i(a_n)]$$

Only arithmetic notions should be assigned to arithmetic constants and primitives. This constitutes our final rule of interpretation:

(11) An assignment to an arithmetic constant or primitive must be an arithmetic notion.

It can be shown that this restriction yields the desired result. It follows from the complete semantics that every statement of arithmetic is either true or false. It follows because of a fact about ω-sequences, namely that they are all isomorphic to one another.

Under the proposed semantics, all the laws of arithmetic are necessarily true. It is easy to convince ourselves of this: (1) Suppose there is an ω-sequence of objects, putative numbers. Then a theorem of arithmetic φ is true when interpreted in terms of those objects. But there is obviously a corresponding general truth about all ω-sequences. According to the proposed semantics, φ expresses this general truth. (2) Suppose, on the other hand, that there is no ω-sequence of objects, nothing to serve as the numbers. Then the proposed semantics tells us that the theorems of arithmetic are true anyway. They are vacuously true of all ω-sequences.*

This way of interpreting arithmetic has some significant advantages. First, it is an especially natural interpretation. On the one hand, arithmetic has all the appearance of an uninterpreted system, a system advitting various applications. On the other hand, arithmetic statements have definite truth-values, so cannot be entirely uninterpreted. The proposed semantics explains these facts. Arithmetic appears to be uninterpreted because while seeming to be a theory about particular objects, it is not really one. Instead it is a theory we can apply to any ω-sequence we encounter. Nevertheless, arithmetic statements have definite truth-values. Each corresponds to a generalization concerning all ω-sequences, a generalization which is definitely true or false. Application of an arithmetic statement is straightforward. We simply consider that instance of the corresponding generalization which involves

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*So too are their negations, in apparent violation of the law of noncontradiction. Later in this section, I shall explain why I find this result harmless.

the ω-sequence with which we are concerned.

Standard interpretations assign objects to arithmetic constants, operations on those objects to operation symbols, and relations on those objects to arithmetic predicates. There are many standard interpretations which promise to yield the correct distribution of truth-values over arithmetic statements. However, each suffers from unnaturalness.

Each standard interpretation suggests a resolution of the irresolvable, our basic confusion over what numbers are. Each obscures the most fascinating fact in the whole subject, the fact that despite our confusion, we are able to employ arithmetic language with great precision.

The proposed semantics has another, more palpable advantage. If arithmetic is interpreted in terms of objects, then acceptance of arithmetic statements involves commitments to the existence of those objects. Under the interpretation I have presented, arithmetic can be pursued without such commitments. This is but the first of several results supporting the conclusion thatthe theorems of classical mathematics do not entail the existence of anything.

Let us pause to consider some objectsions to this nonstandard interpretation:

Objection 1: A sentence is true just in case what it says is true. The given semantics violates this tenet. It says that an arithmetic sentence is true just in case what it says is true of every ω-sequence. Hence this theory of arithmetic truth is not a theory of truth at all.

Reply: There is no good reason to think that truth of arithmetic sentences is just a special case of the truth of ordinary sentences. Arithmetic sentences are quite extraordinary. There is no "obvious" way to interpret them. Arithmetic constants do not stand for definite objects, numbers. Operation symbols of arithmetic do not stand for operations on such definite objects. Arithmetic predicates do not stand for relations among numbers. Among the alternative standard interpretations, no one is especially attractive. Too many are equally attractive. Hence any natural interpretation of arithmetic, like that given above, must be of a new sort.

Objection 2: You say that arithmetic constants are not to be interpreted by definite objects which are the numbers. But the above semantics does assign objects to arithmetic constants, namely functional notions of a certain sort. Why aren't they the numbers?

Reply: We should not consider the assignments to constants in isolation. Part of what is meant when it is said that numerals stand for numbers is that they play a certain role in arithmetic discourse. In that role, they are supposed to denote arguments of operations and relations, operations denoted by arithmetic symbols such as "+", relations denoted by predicates such as "is divisible by". In the proposed semantics, operation symbols do not denote operations on the object assigned to arithmetic constants. They do not denote operations at all. They denote functional notions which are not operations. Similarly, arithmetic predicates do not denote relations of the things denoted by arithmetic constants. Hence, on the new semantics, arithmetic constants do not play the fanciful role of "names of numbers".

Objection 3: Doesn't the new semantics commit us to the existence of things stranger than numbers, namely functional and relational notions?

Reply: I have not said what notions are. Perhaps they are very familiar things. The semantics requires a notion corresponding to each inscribed constant or primitive of arithmetic. It is possible to maintain that these notions are constituted by the inscriptions themselves! Surely we believe in the existence of the inscriptions. We have seen them.

It is slightly more difficult to describe things which may be assigned to variables. For any ω-sequence R, we want at least as many assignable objects as there are items in the field of R. Our requirements are satisfied if for every ω-sequence R and object x in the field of R, there is a notion which maps each ω-sequence to its own "x-th" member. Such a notion may be construed as an ordered pair, (R, x). We specify that:

$$(R, x).(R') = i(x), \text{ where } i \text{ is any isomorphism from the } \omega\text{-sequence } R \text{ to the } \omega\text{-sequence } R'$$

In this way, assignments to variables may be reduced to "logical constructions" on sequences and objects.

Objection 4: The proposed semantics gives a preposterous account of simple arithmetic statements, such as "2 + 3 = 5". Surely these statements are not generalizations about infinite sequences, not even disguised ones.

Reply: This objection is a good one. The new semantics is only suited to statements which appear to involve the whole infinity of numbers, only to quantified arithmetic statements. Is there a way to modify it?

Notions of infinite arithmetic have a "partial extension" to finite sequences.* For example, instead of interpreting "5" with the notion of

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*A sequence may be defined as a relation which satisfies postulates A1ω, A2ω, A4ω, and A5ω. A finite sequence is a sequence which does not satisfy A3ω.

a sixth member of an ω-sequence, we may let it denote the notion of the sixth term of any sequence. That latter notion is defined over many finite sequences. Similarly, "+" can be so interpreted that it yields partial operations on finite sequences. Under such an extended interpretation, "2 + 3 = 5" is not only true of all ω-sequences, but also of all finite sequences over which its component notions are defined. Thus we might give different truth-conditions for such finite statements:

(TU) An unquantified arithmetic sentence φ is true if and only if it is true of every sequence over which its component notions are defined.

We avoid the objection by saying that unquantified arithmetic statements are <u>more</u> general than originally supposed. If true, they must be true of all sufficiently long sequences.*

<u>Objection 5</u>: By interpreting arithmetic in the suggested way, we are supposed to avoid commitments to the infinity of numbers. However, notice that very strange things happen if there are no ω-sequences. On the proposed interpretation, if there are no ω-sequences, both the sentence "There are infinitely many primes" and the sentence "There are not infinitely many primes" are true. Thus the new semantics allows violations of the law of noncontradiction. This can be avoided only at the price of tacking the existence of an ω-sequence onto the truth-conditions. But then arithmetic is not ontologically neutral, and its theorems are not obviously true.

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*Even the modified semantics has some unintuitive results. If there are no ω-sequences, then, under the modified semantics, the quantified sentence "Every number is even" is true. Yet it seems to be falsified on finite sequences. A further modification of the semantics is necessary to take such quantified sentences into account.

Reply: This observation, properly understood, is correct. However, other observations suggest that the result is harmless. First, it is clear that this possibility of arithmetic contradiction is not the possibility of a genuine contradiction. We may not infer what we please from such a contradiction. Second, if we follow the suggestion in the preceding reply, the problem will not affect all of finite arithmetic. If there are no ω-sequences, then all quantified arithmetic statements are true (and false). But unquantified arithmetic statements, such as "2 + 3 = 6", can fall short of truth on finite sequences (by (TU)). Thus a portion of finite arithmetic remains usable. Indeed, it is exactly that portion of arithmetic which can be applied in the given circumstances.

Objection 6: The above semantics is formulated in terms of ω-sequences. But it is clear that it could have been formulated differently. For instance, use could have been made of ω-well-orderings instead. (ω-well-orderings, unlike ω-sequences, are transitive relations.) Or a similar semantics could have been formulated in terms of betweenness or adjacency. Thus the semantics actually given is itself somewhat artificial. Other choices could have been made.

Reply: Indeed, the above semantics cannot be said to capture the meaning of arithmetic statements. However, I claim only (1) that it is far less artificial than standard interpretations, and (2) that it illuminates certain aspects of arithmetic truth. The objection does not contradict these claims.

Moreover, some of the artificiality of the new semantics is removed by the very general definition of an arithmetic notion (D2ω-D4ω). For example, there is an arithmetic notion which, when applied to an ω-sequence, yields the corresponding ω-well-ordering. Likewise, there are arithmetic notions which yield the corresponding betweenness and adjacency relations. Although the selection of an ω-sequence interpretation is somewhat arbitrary, the full scope of arithmetic notions is ultimately taken into account.

Possibly we could give an interpretation which is wholly free of arbitrary elements. Consider the alternative interpretation in terms of ω-well-orderings. For every ω-well-ordering, there is a unique corresponding ω-sequence, and vice versa. Moreover, isomorphisms on ω-well-orderings are also isomorphisms on the corresponding ω-sequences. Thus the notions of the theory of ω-well-orderings mirror those of the theory of ω-sequences, the "notions of arithmetic" according to D2ω, D3ω, D4ω. And theorems about ω-well-orderings mirror theorems about ω-sequences. Therefore, we might conjecture that there is a neutral <u>structured object</u> underlying each ω-sequence and its associated ω-well-ordering. These structured objects may be the real objects of arithmetic.

From this point of view, the ω-sequence postulates are an auxiliary apparatus. Arithmetic is concerned with those structured objects on which "preferred" infinite sequential orderings exist. Undoubtedly these objects can be described in other ways.

Only minor alterations are required in the proposed semantics. References to ω-sequences in the semantical rules must be replaced by references to the structured <u>ω-objects</u>. Isomorphisms of ω-objects may be defined as isomorphisms of the associated ω-sequences.

However, major alterations are required in standard metaphysical schemes if structured objects are admitted. According to conventional views, no countably infinite collection has an objectively "preferred" sequential ordering. Structure simply is not intrinsic to things; it is exhibited only by selected orderings. Thus the notion of a structured object, an object which has a peculiar structure independently of any chosen description, has no currency. Nevertheless, the notion may have a brilliant future.

In summary, the artificiality of the ω-sequence interpretation does not seem to be a serious matter, and there is hope that it will be entirely overcome.

Notes

In "Mathematical Truth", Paul Benacerraf predicted that any theory of mathematical truth will have to conform to the standard pattern established by Tarski. His prediction has not been borne out. The semantics given above is a perfectly sound theory of arithmetic truth, yet, in virtue of its peculiar truth-rule, (T), it does not conform to the standard pattern.

Ed Zalta has given an additional objection to the semantics, one similar to Objection 5: If there are no ω-sequences, then the sentence "There is a largest prime number" is assigned the value true by (T). But intuitively, it is not true. Hence the semantics might not yield the intuitive distribution of truth-values. Yet a primary purpose of giving a semantics is to generate the intuitive distribution!

I think the correct response is that we can have a somewhat different aim in giving a semantics for arithmetic. Under the proposed semantics, every arithmetic theorem is necessarily true. Thus the essential core of the intuitive distribution is preserved. Furthermore, the semantics meshes well with applications of arithmetic. In view of these features, the semantics meets the requirements of both pure mathematics and natural science. The clash with common intuitions does not seem to be a very serious matter.

We can preserve the intuitive distribution by interpreting arithmetic statements as generalizations about all possible ω-sequences. The cost is a loss in simplicity because of the use of modal notions. I have chosen a nonmodal interpretation because I think it is tenable. On the other hand, standard interpretations, which depend on axioms of infinity, are in serious conflict with our intuitions: The theorems of arithmetic are intuitively certain, and axioms of infinity are not.

2. Cardinalities

There are nonarithmetical uses of number words. Prominent among these is what may be called the cardinal use of number words. This is the use of number words to speak of the number of things of a certain kind. It is exemplified by statements of the form: 'There are n A's', where 'n' is a number word and 'A' is a suitable general term.

There are various ways to interpret the cardinal use. According to one simple interpretation, number words so used correspond to certain notions, property-notions which apply to sets.* So to assert that there are n A's is to assert that the set of A's possesses the property-notion n. To say that there are three cows in the field is to say that the set

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*They are property-notions because they are possessed by sets of arbitrarily high rank.

of cows in the field satisfies the property-notion three.

These property-notions may be called <u>cardinalities</u>. They obey two characteristic laws:

L1 For any cardinality C, and any sets x and x', if there is a one-to-one correspondence from x to x', then

$$C.[x] \text{ iff } C.[x']$$

L2 For any cardinality C, and any sets x and x', if C.[x] and C.[x'], then there is a one-to-one correspondence from x to x'.

Taken together, L1 and L2 say that the sets satisfying a given cardinality form an equivalence class under one-to-one correspondence.

Finite cardinalities can be defined using quantification and identity. For example:

$$\underline{2}.[x] =df \exists a \exists b(a \neq b \ \& \ \forall c(c \in x \leftrightarrow c = a \lor c = b))$$

Using some more powerful notions, infinite cardinalities can be defined as well:

$$\aleph_0.[x] =df \text{ x is the field of some } \omega\text{-sequence}$$

Exactly what notions are used in this definition will become clear later, in my discussion of logicism.

I have contended that arithmetic is not about particular objects, numbers. Cardinalities present a challenge to my contention. First, observe that we may always pass from an assertion of the form 'There are n A's' to a corresponding one of the form 'The number of A's is n'. This suggests that there is a common conception of numbers as cardinalities.*

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*The suggestion is not a very strong one. Quine remarks, "We can say what it means for a class to have n members no matter how we construe the numbers, as long as we have them in order. For to say that a class has n members is to say that the members of the class can be correlated with the natural numbers up to n, whatever they are."--"Ontological Reduction and the World of Numbers"

Second, if there is a common conception of numbers, then isn't it likely that this conception underlies arithmetic?

Moreover, cardinalities are mathematically significant notions. They crop up in all branches of mathematics. For example, many theorems about groups are concerned with the <u>orders</u> of groups. The order of a group is the cardinality of the set of its objects. Given the tremendous mathematical importance of cardinalities, it is easy to suppose that there must be a mathematical theory which deals with them exclusively, namely arithmetic.

However, the argument that arithmetic is concerned with cardinalities is not a good one. The cardinal use of number words is not their only nonarithmetical use. Sometimes number words are used to deal with integral values of continuous quantities. For example, we make statements of the form: 'a is n miles from b', where 'n' is a number word. Clearly this is not the cardinal use. To say that a is n miles from b is not to say that there are n things, miles, between a and b. There are miles between things that have nothing between them!

How shall we interpret this use of number words? Perhaps number words so used stand for certain relational notions, <u>proportions</u>. To say that a is n miles from b is to say that n is the proportion of the distance from a to b and the distance one mile. I understand distances to be true relations. I shall say a little more about them in section 4 of this chapter.

Observe that we may always pass from 'a is n miles from b' to 'n is the number of miles from a to b'. Since 'n' stands for a proportion in this context, this suggests that numbers are proportions.

Recall the parallel argument for the conclusion that numbers are cardinalities. Since cardinalities are distinct from proportions, one of the arguments must be no good. Since the arguments are of the same form, they are equally no good. Hence no compelling reason has been given for thinking that cardinalities are the numbers.

Because finite cardinalities have a natural sequential ordering, arithmetic has application to them. This application has a very general utility, but there are other such applications. For example, arithmetic can be applied to integral proportions, or to positions in sequences (first, second, third, and so on). Each of these applications yields a standard interpretation for arithmetic. We can construe arithmetic as a study of cardinalities, or of integral proportions, or of positions in sequences. However, none of these construals is entirely natural, none tells us what the numbers really are.

If arithmetic is not about cardinalities, must there be some other mathematical theory about them? The inclination to think that there must be one arises from a false conception of mathematics. Mathematics is not a study of special mathematical objects. It is a very general study of more or less ordinary things. In that study, cardinality notions are employed, but they are not themselves objects of any mathematical theory. Mathematics is concerned with cardinality, but not with cardinalities.

Of course, more must be said. Cardinalities are at least apparent objects of reference in mathematics. For instance, algebraists speak of "the order" of a group. How are these cases of apparent reference to be explained away?

Let us consider a particular example:

> <u>Lagrange's Theorem</u> If g is a finite group and h is a subgroup of g, then the order of h is a divisor of the order of g.

Without even knowing what groups are, we can do away with the apparent references to orders:

> If g is a finite group and h is a subgroup of g, then there exists a set x of sets of objects from g such that (i) every object of g is contained in exactly one member of x, and (ii) for every member s of x, there is a one-to-one correspondence from s to the set of objects of h.

That is, to say that the order of a subgroup h divides the order of a group g is to say that the set of objects of g can be partitioned into h-sized pieces.

The uses of cardinalities in mathematics are many and various. It is practically impossible to show how to dispense with every case of apparent reference. Fortunately, the burden of proof lies on others. They need only show an insuperable difficulty in one case.

Meanwhile, I shall stick by the conclusions of this section. Cardinalities are important mathematical notions, but they need not be the objects of any mathematical theory. Cardinalities have a <u>predicative</u> role in mathematics. There are theorems about things possessing specific cardinalities, but cardinalities are not themselves <u>subjects</u> of mathematical assertions.

<u>3. The Real Numbers</u>

Natural numbers are the most familiar of all putative mathematical objects. Real numbers may be the most important. The mathematical theory of real numbers has very extensive and direct employment in the physical sciences. The point of view I shall take is that the theory of real numbers is best interpreted not as a theory about particular objects, real numbers, but as a theory concerning all real-number-like systems.

A real-number-like system, or <u>r-system</u>, is an ordered triple (1, +, $<$) satisfying the following axioms:*

A1r + is a binary operation on the field of the binary relation $<$. (That is, + is a binary function such that the field of $<$ is identical to both the first and second domains of + and contains the range of +.)

A2r For any a, b, and c in the field of $<$, $a + (b + c) = (a + b) + c$.

A3r For any a and b in the field of $<$, $a + b = b + a$.

A4r There is a unique object 0 such that for any a in the field of $<$, $a + 0 = 0 + a = a$.

A5r For any a in the field of $<$, there is a unique object -a such that $a + -a = -a + a = 0$.

A6r If $a < b$, then $a + c < b + c$.

A7r If $a < b$ and $b < c$, then $a < c$.

A8r For any a and b in the field of $<$, exactly one of the following holds: $a < b$, $b < a$, or $a = b$.

A9r If $a < b$, then there is a c such that $a < c$ and $c < b$.

A10r For any set s of objects from the field of $<$, if there exists an a such that every member of s bears $<$ to a, then there exists a b such that for any c, if every member of s bears $<$ to c, then $b < c$ or $c = b$.
(That is, every bounded set has a least upper bound.)

A11r For any $a > 0$, if s is a set such that 0 is a member of s and for any member b of s, b + a is a member of s, then there is no object c such that every member of s bears $<$ to c.

A12r $0 < 1$.

Isomorphisms of r-systems can be defined:

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*Ordered triples can be defined set-theoretically, at the cost of added artificiality. Indeed, my description of r-systems involves a number of arbitrary choices. Whether this constitutes an objection to my interpretation depends on matters discussed in the end of section 1.

D1r i is an <u>isomorphism of r-systems</u> from (1, +, <) to (1', +', <') just in case i is a one-to-one correspondence from the field of < to the field of <' such that for any objects a, b, and c in the domain of i,

$$a < b \text{ iff } i(a) <' i(b)$$
$$a + b = c \text{ iff } i(a) +' i(b) = i(c)$$
$$i(1) = 1'$$

The theory of real numbers can be given a nonstandard semantics. This semantics resembles that I have given for arithmetic, except that it involves r-systems instead of ω-sequences. Again, there are restrictions on the interpretation of constants and primitives. For example, for any predicate π^n of the theory of real numbers, the functional notion $f_{\pi}n$ assigned to it must obey the condition:

If i is an isomorphism of r-systems from (1, +, <) to (1', +', <'), then for any objects $a_1, \ldots, a_n$ in the domain of i,

$f_{\pi}n.((1, +, <))[a_1, \ldots, a_n]$ iff $f_{\pi}n.((1', +', <'))[i(a_1), \ldots, i(a_n)]$

In this way, the theory of real numbers can be interpreted as a general study of r-systems.

The advantages of such an interpretation are manifest. We possess no well-supported view that identifies real numbers as particular abstract objects. Hence we have no reason to believe that the constants of the theory of real numbers, such symbols as "1", "2/3", " 2", and "π", denote particular things, real numbers. In these circumstances, any standard interpretation of the theory, one which assigns particular objects to the real number constants, is a complete fabrication. The suggested interpretation fits the evidence better. It shows us how the theory can be pursued in the absence of knowledge of the nature of real numbers. Finally, the standard interpretations involve commitments to the existence of objects, putative real numbers. The nonstandard interpretation avoids these.

The real numbers are used to "construct" the spaces of real analysis, for example, the space of all triples of real numbers. On the nonstandard interpretation, theorems which are apparently about a particular space in fact embody general truths about constructions on r-systems. For instance, a theorem about "the space of triples of real numbers" embodies a general truth concerning any space of triples which is formed out of the objects of an r-system. When so interpreted, the theorems of analysis do not involve commitments to the existence of any particular spaces. (For a treatment of metric spaces, see the Notes of this section.)

A portion of the theory of real numbers, that which treats nonnegative reals, is useful in dealing with physical quantities. This portion can be construed as a theory about all <u>nnr-systems</u>. An nnr-system is a triple $(1, +, <)$ satisfying A1r-A4r, A6r-A12r, and the following replacement for A5r:

A5nnr If $a + b = 0$, then $a = 0$ and $b = 0$.

Having isolated the notion of an nnr-system, it is easy to give the corresponding nonstandard semantics for the theory of nonnegative reals.

The first component of an nnr-system is a unit, 1. In the case of many physical quantities, there appears to be no natural unit. The mathematical study of such quantities is the theory of <u>unitless nnr-systems</u>. A unitless nnr-system is a pair $(+, <)$ satisfying A1r-A4r, A5nnr, and A6r-A11r.

In the theory of unitless nnr-systems, there are no individual constants other than the symbol "0". This is so because for any nonzero object c of a unitless nnr-system (+, <), and any distinct unitless nnr-system (+', <'), there is no unique object c' of (+', <') which may be said to play the same role in (+', <') as does c in (+, <).

Let me explain. For any two unitless nnr-systems there are infinitely many isomorphisms between them. Zero objects are mapped to zero objects by all isomorphisms, but nonzero objects have no unique correspondents under all isomorphisms. This can be verified from the definition of isomorphisms for unitless nnr-systems:

D1nnr i is an <u>isomorphism of unitless nnr-systems</u> from (+, <) to (+', <') just in case i is a one-to-one correspondence from the field of < to the field of <' such that for any objects a, b, and c in the domain of i,

$$a + b = c \text{ iff } i(a) +' i(b) = i(c)$$
$$a < b \text{ iff } i(a) <' i(b)$$

For any nonzero object c of a unitless nnr-system (+, <), and any nonzero object d of a unitless nnr-system (+', <'), there is an isomorphism which maps c to d. Hence there is no unique correspondent of c under all isomorphisms.

Recall that there is a restriction on the interpretation of constants in our mathematical theories:

If i is an isomorphism from (+, <) to (+', <'), then

$$i(f_k.((+, <))) = f_k.((+', <'))$$

Because of the fact just mentioned, this condition cannot be satisfied if the constant *k* does not pick out, from unitless nnr-system to unitless nnr-system, the zero object of each system.

Since the theory of unitless nnr-systems has but one constant, it will never be conceived as a theory about particular mathematical objects.

Notes

A metric space is normally taken to be a binary function whose domains are identical, and whose range is a subset of the nonnegative reals. Since I do not admit the existence of nonnegative reals, I must construe metric spaces differently.

A metric space is an ordered pair (d, (1, +, <)) satisfying the postulates:

A1m (1, +, <) is an nnr-system, and d is a function whose domains are identical, and whose range is a subset of the field of <.

A2m For any objects a and b in the domains of d, if a ≠ b, then 0 < d(a, b), where 0 is the identity element of +(as in A4r).

A3m For any object a in the domains of d, 0 = d(a, a).

A4m For any objects a and b in the domains of d, d(a, b) = d(b, a).

A5m For any objects a, b, and c in the domains of d, d(a, b) + d(b, c) ≮ d(a, c).

Isomorphisms are easily defined:

D1m An isomorphism of metric spaces from (d, (1, +, <)) to (d', (1', +', <')) is a pair (i, j) such that j is an isomorphism of nnr-systems from (1, +, <) to (1', +', <'), i is a one-to-one correspondence from the domains of d to the domains of d', and for any objects a and b in the domain of i,

$$j(d(a, b)) = d'(i(a), i(b))$$

A unitless metric space is an ordered pair (d, (+, <)) which satisfies axioms similar to A1m-A5m. The sole difference is that the second coordinate of a unitless metric space, (+, <), is a unitless nnr-system. It is interesting to note that under standard construals of metric spaces, there is no separate class of unitless metric spaces. Unitless metric space theory can be given a standard development only by ignoring the special status of the real number one. My approach is more natural.

4. Does Physics Require Real Numbers?

On the face of it, a measurement is an association of a real number with an object or pair of objects. For example, to measure the mass of an object is to assign the object a real number, a real number which represents the mass with respect to a chosen unit. To measure the distance between two objects is to assign them a real number, a real number representing the distance with respect to a chosen unit. Thus the facts of measurement suggest an argument in favor of the real numbers: We have good reason to believe the pronouncements of physical science. But physical science is based on measurement, and measurement requires the existence of real numbers. Hence we should believe in the existence of real numbers.--If this argument is good, then there is reason to believe that the theory of real numbers _is_ about particular objects, real numbers, contrary to what I have said.

Hilary Putnam presents such an argument in _Philosophy of Logic_, ch. V. He concludes:

> "If the numericalization of physical magnitudes is to make sense, we must accept such notions as function and real number; and these are just the notions the nominalist rejects." p. 43

I find the argument for this conclusion to be unsound. Science does not need real numbers.* The facts of measurement can be otherwise explained. Indeed, the alternative explanation is a better one.

Intuitively, the mass of an object is a property it possesses, a property it shares with all objects of equal mass. All the masses that

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*Science does seem to presuppose the literal truth of some parts of classical mathematics, so Putnam's _realism_ may survive my criticism.

objects can have form a continuum. On this continuum, there is a natural ordering relation $<_m$. One mass bears $<_m$ to another just in case it is a mass of lighter objects than the other. There is also a natural additive operation $+_m$. For any masses M_1 and M_2, $M_1 +_m M_2$ is the mass any aggregate would have if it consisted of two discrete objects of masses M_1 and M_2.* The interplay of $<_m$ and $+_m$ can be checked on a balance. We learn that $(+_m, <_m)$ either is or approximates a unitless nnr-system. It is this knowledge which makes mass measurements and their manipulation possible, not knowledge of an association between physical objects and special mathematical objects, real numbers.

By arbitrarily selecting a unit mass 1_m, we obtain an nnr-system $(1_m, +_m, <_m)$. The theory of nonnegative reals has direct application to this system of masses. Since that theory can be interpreted in a nonstandard way, we need not suppose that the application proceeds _via_ an association with real numbers.

Because the system of masses $(1_m, +_m, <_m)$ is just as complex as the system of nonnegative reals is supposed to be, explaining measurement in terms of the first rather than the second is not a straightforward case of economizing. From one point of view, it is the very opposite of economizing. To be consistent, I must explain distance measurements in terms of another nnr-system, $(1_d, +_d, <_d)$. But then we have two complex systems instead of a single system of the nonnegative reals.

In fact, it has been suggested, in the interest of ontological economy, that we dispense with the "impure" systems in favor of the "pure"

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*I am ignoring relativistic effects. They introduce complications which are irrelevant here.

system of nonnegative reals. This is known as the reduction of "impure numbers", masses, distances, and so forth, to "pure numbers", the nonnegative reals.*

Each such reduction must be accompanied by an account of the nonnegative reals. Since evidence on the nature of real numbers is very slim, these accounts are bound to be artificial. However, this is not a serious problem for the proponents of reduction. They care little how we choose to construe the real numbers, so long as we choose one construal over the various impure systems.

The outstanding problem for reductionists is one of establishing the existence of certain objects, namely those required by their accounts of the real numbers. According to the most well-known accounts, real numbers are infinite pure sets, sets founded solely on the empty set. But reductionists have done little to demonstrate the existence of pure sets. They only point out the convenience of pure sets, the fact that pure sets yield simple interpretations for the statements of mathematical physics. However, the need for interpretation does not specifically support the existence of pure sets, since other entities might also provide simple interpretations.

I think that a rational belief in the existence of pure sets must be based on prior acceptance of a full-blown Platonism. That is, only a view that admits properties, relations, and functions can yield up pure sets as well.** If this is correct, we shall not be able to reduce

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*See Quine's "Ontological Reduction and the World of Numbers".

**In ch. III, s. 6, I shall identify sets with "degenerate" properties and identify pure sets with "degenerate" cases of those!

the impure systems of masses, distances, and so forth to pure systems. The objects of impure systems, such as masses and distances, are properties and relations, and they belong to the "main line" of Platonic universals. We cannot dispense with them in favor of the more exotic pure objects, such as pure sets.

Why, indeed, would we want to? The impure systems of masses and distances are intimately connected with empirical reality. It strikes me that they are on a somewhat sounder footing than the pure systems concocted by the reductionists, and so provide a better interpretation of the facts of measurement.

Notes

While we are on the topic of impure systems, it might be useful to look at impure varieties of metric spaces. This will be a natural continuation of the Notes of the preceding section, in which metric spaces were defined.

Of special scientific interest are <u>material metric spaces</u>. These correspond to arrangements of particles in physical space:

D1im (d, $(1_d, +_d, <_d)$) is a material metric space just in case it is a metric space in which $(1_d, +_d, <_d)$ is the nnr-system of physical distances (or, if one prefers, that of spatiotemporal intervals), and only fundamental units of matter are members of the domains of d.

The arrangement corresponding to a material metric space actually obtains just in case the particles in the domain of the d-function do indeed exhibit the distances which the d-function assigns to them. When this happens, we may say that the material metric space is <u>actualized</u>:

D2im A material metric space (d, (1_d, $+_d$, $<_d$)) is actualized just in case for any particles a and b in the domain of d,

$$d(a, b)[a, b]$$

(that is, a bears the relation d(a, b) to b, where d(a, b) is a certain distance relation)

Notice that this treatment of material metric spaces involves no appeal to pure mathematical objects.

5. More Putative Mathematical Objects

Even statements of abstract algebra sometimes contain apparent references to particular objects. For example, a book on group theory might contain the statement:

> The sixth symmetric group is the smallest symmetric group which contains a subgroup of order nine.

Here the phrase "the sixth symmetric group" seems to denote a particular group.

Must we conclude that group theory is concerned with certain mathematical objects, groups? Fortunately, there is a paraphrase which lacks the singular term in question:

> All and only sixth symmetric groups are smallest symmetric groups containing subgroups of order nine.

The original statement was not really about a specific group. It was about a specific sort of group. The notion of such a group is a mathematical notion, but there is no unique corresponding object. Thus the original statement was somewhat misleading. The paraphrase captures most of its content.

Given the availability of such simple paraphrases, we might well wonder why algebraists often prefer the apparent singular terms. Singular terms are used to indicate the fact that the groups in question are "unique up to isomorphism". This is not to say that the groups, which are many, are one. It is simply to say that they are all isomorphic in the group sense. In the above example, sixth symmetric groups are isomorphic. This fact is either stated or presupposed by the original statement, so the paraphrase is not entirely faithful. Nevertheless, having seen why the paraphrase is not completely faithful, we see quite clearly what it was intended to show: The original statement is not concerned with a specific group.

A group is any associative binary operation + with an identity element and inverses:*

A1g + is a binary operation.

A2g For any a, b, and c in the domains of +, a + (b + c) = (a + b) + c.

A3g There is a unique object 0 such that for any a in the domains of +, a + 0 = 0 + a = a.

A4g For any object a in the domains of +, there is a unique object -a such that a + -a = -a + a = 0.

Obviously a group need not be a purely mathematical object. We encounter groups in many concrete situations.

Isomorphisms of groups are easily defined:

D1g i is an <u>isomorphism of groups</u> from + to +' just in case i is a one-to-one correspondence between the objects in the domains of + and the objects in the domains of +' such that for any objects

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*Usually a group is taken to be a pair (x, +), where x is the set of objects on which the operation + operates. But separate mention of x is not necessary, since it is both the first and second domain of +.

a, b, and c in the domain of i,

$$a + b = c \text{ iff } i(a) +' i(b) = i(c)$$

Two groups are <u>isomorphic</u> if there is an isomorphism from one to the other.

A paradigmatic sixth symmetric group is any group of all rearrangements, or <u>permutations</u>, of some given six objects. Obviously there may be many sixth symmetric groups in existence. Nevertheless, all sixth symmetric groups are isomorphic to one another. Algebraists tend to denote them using a singular term, "the sixth symmetric group", because such groups are indistinguishable from the standpoint of group theory.

These remarks can be generalized. Apparent reference to specific objects in abstract algebra results from a convenient "lumping together" of isomorphic objects. Mathematicians are usually quite conscious of this "lumping together" process. They do not confuse "unique up to isomorphism" with "strictly unique".

Nevertheless, this very same "lumping together" process, operating nonconsciously, could be responsible for the illusion that arithmetic is concerned with a specific collection of objects, the natural numbers, and that the theory of real numbers is concerned with a specific continuum, the real numbers. I have proposed that arithmetic is about all sequences of a certain sort, all ω-sequences. Because all ω-sequences are isomorphic, they are easily lumped together under the term "the sequence of natural numbers". After all, there is no arithmetical distinction between diverse ω-sequences. Similarly, all real-number-like systems, or r-systems, are isomorphic. Hence it is easy to denote them with an apparent singular term, "the continuum of real numbers".

The fundamental source of the mistaken belief that there are particular mathematical objects may be this natural tendency to "identify" isomorphic objects. Mathematicians cultivate this tendency, because it is convenient. Philosophers should master it, because it can lead to confusion.

6. Pure Sets

Pure sets are the apparent objects of set theory. They include the empty set, $\emptyset$, and all sets founded upon it, $\{\emptyset\}$, $\{\{\emptyset\}\}$, $\{\emptyset, \{\emptyset\}\}$, and infinitely many others. Typical formulations of set theory assert the existence of the empty set and at least one infinite set.

Should set theory be treated in the way I have treated other mathematical theories? Should it be interpreted not as a theory of particular objects, but as a theory concerning all systems of a certain sort? For several reasons, this may not be a good idea.

First, set theory is normally conceived as a universal "background theory" for mathematics. It is difficult to reconcile this conception with the idea that set theory is a general theory of set-systems. For example, recall that we employed set theory to define ω-sequences and r-systems. This is clear in the Axiom of Induction, A5ω, and the Axiom of Continuity, A10r. If the set theory so used were a general theory of set-systems, then we would only have defined "ω-sequence relative to a set-system" and "r-system relative to a set-system". There is no very appealing way to recover the absolute notions from these.

For instance, suppose we define ω-sequences in the absolute sense as "ω-sequences relative to <u>standard</u> set-systems".* Here the notion of standardness cries out for explanation. Typically standardness is defined <u>via</u> some background theory: A standard set-system is one that contains no infinite descending ϵ-chain, ... $x_3 \in x_2 \in x_1 \in x_0$. However we choose to spell out what a "chain" is, the background theory is bound to bring with it the idea of a definite realm of abstract entities. Thus we might just as well have stuck with a background set theory, one concerned with a definite universe of sets.

Alternatively, we might simply refuse to explain what standard set-systems are. This radical course seems somewhat out of place, since we have decided to define other mathematical systems, such as ω-sequences and r-systems.

Second, set theory is not purely postulational. For every property P we encounter in applications, we may assert outright the existence of a set of all objects exemplifying P. Then the general laws of sets assure us of the existence of an entire system founded on the set of P's. Thus set theory is not the study of set-systems whether they exist or not, but tells us something about the conditions under which systems of sets exist. These conditions are not stringent: Every property P gives rise to an extensive set system.

What justification is there for this conviction, that every property gives rise to a set-system? No answer is provided by the view that set theory is a general theory of set-systems. According to that view, sets

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*ω-sequences of standard set systems are isomorphic to one another and so may qualify as ω-sequences in the absolute sense. But ω-sequences relative to nonstandard set-systems may not belong to the same isomorphism class.

are merely elements of systems obeying certain postulates. No special character is attributed to them which might account for the omnipresence of set-systems. On the other hand, the more usual views of sets promise some relief. According to them, a set is a special sort of abstract entity. Whenever the members of a set exist and are in a certain sense "collectable", the set exists. This accounts for the existence of a set for every property P and of a system founded on that set.

Third, mathematics needs some background theory. The postulates of mathematical theories contain certain universal notions. For instance, the postulates for groups contain the notion of the result of applying a binary operation to a pair of objects. The laws governing these universal notions are the laws of the mathematical background theory. Set theory has been regarded as a specially interesting part of this background. As such, it is a theory of a definite universal notion, the notion of set membership, and of a definite realm of objects, sets.

Therefore, I shall not treat set theory in the way that I have treated other mathematical theories. It is not a theory of all set-systems. It is a theory about a very broad but definite realm of objects, sets. However, I shall argue that we do not need a set theory which is specially concerned with pure sets.

Set theory was developed to serve as a "foundation" for mathematics. Those who developed it thought that a foundation must include natural numbers and real numbers. They chose to construe numbers as certain pure sets. They realized that this construal is entirely artificial. Their project, however, was not one of capturing "the meaning" of arithmetic statements or of ordinary numerical discourse. They wished to show only that standard interpretations are available for arithmetic and the theory of real numbers.

Why did they choose pure sets to be the numbers? Why didn't they choose, for example, sets of concrete objects? It is generally agreed that a set exists only if all its members do. If numbers were construed as sets of concrete objects, their existence would not be certain. That would be contrary to the certitude of arithmetic theorems. Hence numbers should not be so construed.

On the other hand, consider the empty set. It has no members. If it fails to exist, it does not fail on account of its members failing to exist. But how else could it fail to exist? Many would say that there is no other way, and so conclude that the empty set exists of necessity.

Now consider the other pure sets. These are founded on the empty set alone. It is difficult to see how their existence could depend on any contingent circumstance. With these sets, too, it is easy to conclude that they exist of necessity. Thus if numbers are pure sets, their existence appears guaranteed. Such a construal promises to preserve the certitude of arithmetic theorems.

This line of reasoning is very suspicious. I doubt that it can be elaborated in a convincing way. For instance, suppose it is based on the following principle:

The set of F's exists if and only if all the F's exist.

This entails the existence of the empty set, since that set has no members. It also entails the existence of the various pure sets, since their members are ultimately founded on the empty set. Unfortunately, once the existence of the pure sets is secured, the principle entails the existence of a set of them, a universal pure set. Contradictions soon follow.

Even if pure sets are necessary existents, there is no reason to interpret number theory in terms of them. We have seen some nonstandard interpretations that are far more natural. They too preserve the certitude of arithmetic theorems, not by arbitrary selection of necessary existents to be "the numbers", but by presenting arithmetic truths as generalizations concerning all number-like sequences.

Set theory is not needed to provide objects for arithmetic, or for any other branch of mathematics. Mathematics simply is not concerned with specific objects. Thus there is no point in interpreting numbers as pure sets, and no point to a special interest in pure sets.

If set theory is not to be treated as a study of set-like systems nor as a study of pure sets, how should it be pursued? Set theory is indeed very useful in mathematics, not as a study of pure sets, but as a study of the general laws of sets, pure and impure. All familiar principles concerning pure sets can be extended to cover impure sets. And in practice, they certainly are.

A typical mathematical theory is concerned with all systems of a certain sort. Thus group theory is about all groups, and the theory of real numbers is about all r-systems. The objects of these systems need not be pure sets. Hence references to sets founded on these systems extend to impure sets. Since such references are common in mathematical theories, mathematics does deal with impure sets.

For instance, consider the quantification over sets which appears in the axioms of ω-sequences. This quantification is not restricted to pure sets. The axioms are not intended to cover only ω-sequences of pure set theory. Some relation among physical objects might satisfy the axioms. If one does, then axiom A5ω can be applied to sets of physical objects, sets which are manifestly impure. Such an application may involve general set theoretic laws, such as the axioms of Power Set, Union, Replacement, and Extensionality. Their application to impure sets involves no difficulties.

In set theory, as in other branches of mathematics, there is no need for special objects. And in so far as mathematics requires set theoretic foundations, it requires a theory of sets pure and impure.

CHAPTER II

ABSTRACT STRUCTURE AND THE NOTIONS OF MATHEMATICS

1. Typical Branches of Mathematics

By a typical branch of mathematics, I mean a mathematical theory which does not have a special foundational role, one which is not intended to serve as a "background theory" for all of mathematics. Examples of typical branches are numerous: group theory, topology, Euclidean geometry, arithmetic, the theory of real numbers, and many others. Atypical branches, the foundational theories, are few in number. Set theory is the prime example. Category theory, a more recent development, also has a foundational character. Perhaps logic may be counted among these as well.

Typical branches of mathematics share a number of features, which are conveniently exhibited by the three examples of the preceding chapter, arithmetic, the theory of real numbers, andgroup theory. A knowledge of these features will lead to a more comprehensive understanding of mathematics.

A typical branch of mathematics is concerned with all systems of a certain sort. Usually this concern is explicit in the associated theorems. Theorems of group theory are obviously generalizations about groups. Theorems of topology are explicit generalizations about topological spaces.

In the cases of arithmetic and the theory of real numbers, the general concern with systems is disguised. Theorems of these branches appear to be about particular objects, natural or real numbers. As I have argued in the preceding chapter, this appearence is deceptive. There is no natural interpretation of these theorems according to which they are about numbers. However, there are natural nonstandard interpretations. According to the interpretation I proposed for arithmetic, arithmetic theorems are about all ω-sequences. According to the interpretation I proposed for the theory of real numbers, real number theorems are about all r-systems. Thus, under natural interpretations, each of these theories is concerned with all systems of a certain kind.

Of course, the notion of a "system" is very general. A group is simply a binary operation. An ω-sequence is just a binary relation. On the other hand, r-systems are complex. An r-system is a triple of an object, an operation, and a relation. Other mathematical theories deal with still more complex systems. In each case, however, the complex systems can be regarded as n-tuples of objects, functions, relations, and sets.

Each typical branch of mathematics is associated with one or more sets of postulates, postulates describing the corresponding systems. For example, ω-sequences, the systems of arithmetic, are completely described by the Dedekind-Peano axioms A1ω-A5ω. Groups are completely described by the postulates A1g-A4g; r-systems, by the postulates A1r-A12r.

The postulates of mathematical theories are of a very special sort. Some, such as the postulates for groups, can be formulated schematically in elementary logic. Others contain set theoretic notions. This is true of the Axiom of Induction, A5ω, and the Axiom of Continuity, A10r. But in no case do the postulates appeal to notions beyond those of logic and set theory (except perhaps to higher-order logic, which, in the absence of modalities, is no more powerful than set theory).

It is sometimes said that the postulates of mathematical theories are "purely formal". This terminology is somewhat misleading. Mathematical postulates are "formal" in the sense that they employ only universal notions, the notions of logic and set theory. They are also "formal" in the sense that they are not categorically asserted. They are used to characterize the systems of a mathematical theory. They are not asserted of any individual system, except in applications.

However, mathematical postulates need not be "formal" in the strict logician's sense. Some contain notions of set theory. Since those notions do not belong to elementary logic, such postulates are not formal in the strict sense. For example, the Axiom of Induction, A5ω, contains a quantification over sets.* There is no purely formal way to represent the content of that axiom, not even by means of axiom schemata.

We have concluded that the theorems of mathematics are generalizations, or disguised generalizations, 'All Fs are G', where the Fs are systems characterized by set theoretic postulates. We have yet to say what notions may enter into the second parts of these generalizations, the 'G''s.

That there is some restriction on those notions is suggested by our examination of arithmetic. We found that the notions assigned to arithmetic primitives must be "preserved" under ω-sequence isomorphisms. In

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*It can be stated with a quantification over properties instead, but the notion of a property no more belongs to elementary logic than the notion of a set.

precise terms:

For any isomorphism of ω-sequences i from R to R',

(i) if f is an object notion of arithmetic, then

$$i(f.(R)) = f.(R')$$

(ii) if f is an n-ary operation notion of arithmetic, then for any objects $a_1, \ldots, a_n, a_{n+1}$ in the domain of i,

$$f.(R)(a_1, \ldots, a_n) = a_{n+1} \text{ iff } f.(R')(i(a_1), \ldots, i(a_n)) = i(a_{n+1})$$

(iii) if f is an n-ary relation notion of arithmetic, then for any objects $a_1, \ldots, a_n$ in the domain of i,

$$f.(R)[a_1, \ldots, a_n] \text{ iff } f.(R')[i(a_1), \ldots, i(a_n)]$$

This restriction on arithmetic interpretations accomplishes three things. First, it prevents the occurrence of truth-value gaps. Second, it isolates the proper notions of arithmetic. Third, it ameliorates the effects of our somewhat arbitrary choice of arithmetic postulates, A1ω-A5ω (see ch. I, s. 1, Objection 6).

The first gain is peculiar to arithmetic and its disguised generalizations. But the remaining two can be reproduced by isomorphism restrictions on other mathematical theories. The general principle regarding the notions of mathematical theories is:

P1 In the mathematical theory of Fs, for any isomorphism of F-systems i from F to F',

(i) if f is an object notion of F-theory, then

$$i(f.(F)) = f.(F')$$

(ii) if f is an n-ary operation (or, more generally, function) notion of F-theory, then for any objects $a_1, \ldots, a_n, a_{n+1}$ in the domain of i,

$$f.(F)(a_1, \ldots, a_n) = a_{n+1} \text{ iff } f.(F')(i(a_1), \ldots, i(a_n)) = i(a_{n+1})$$

(iii) if f is an n-ary relation notion of F-theory, then for any objects $a_1, \ldots, a_n$ in the domain of i,

$$f.(F)[a_1, \ldots, a_n] \text{ iff } f.(F')[i(a_1), \ldots, i(a_n)] \ *$$

To see how this works, consider the notion of an <u>identity</u> <u>element</u> in group theory. This is an object notion of group theory. It is a functional notion <u>id</u> which maps each group + to its identity element, 0. It is easy to show that <u>id</u> satisfies P1(i):

> For any isomorphism of groups i from + to +', and object a in the domain of i,
>
> $$i(\underline{id}.(+)) = \underline{id}.(+')$$

Another notion of group theory is the <u>inverse</u> notion. It is a unary operation notion of group theory. It maps each group + to a function that maps each object a of the group + to its inverse in +, that is, to that object b such that a + b = b + a = 0, where 0 is the identity element of +. This notion satisfies P1(ii):

> For any isomorphism of groups i from + to +', and objects a and b in the domain of i,
>
> $$\underline{inv}.(+)(a) = b \text{ iff } \underline{inv}.(+')(i(a)) = i(b)$$

In a mathematical generalization 'All Fs are G', the 'G' may contain any of the special notions of F-theory. However, the 'G''s cannot be confined to those notions which satisfy P1. P1 describes only the <u>first-order</u> notions of each mathematical theory. If a notion satisfies P1, then it picks out, from F-system to F-system, some object of that system, or some function or relation on the objects of that system. A <u>higher-order</u> notion of F-theory might pick out, in each F-system, a function or relation on <u>sets</u> of objects from that system. Such notions do not fall

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*P1 does not apply directly to metric spaces. An isomorphism of metric spaces is a pair of functions (i, j), and so has no true domain. However, we might choose to regard the "domain" of the pair to be the union of their domains, and then apply P1 accordingly.

under principle P1. Since higher-order developments are the everyday practice of mathematicians, we clearly need to extend P1.

Let i be an isomorphism of F-theory. Then the <u>ϵ-extension</u> of i, i_ϵ, is a functional notion such that:

(I) For any object a in the domain of i, $i_\epsilon.(a) = i(a)$.

(II) For any set x of objects mapped by i_ϵ,

$$i_\epsilon.(x) = \{i_\epsilon.(y) : y \in x\}*$$

(III) i_ϵ maps no objects other than those forced by (I) and (II).

That is, i_ϵ is the natural extension of i to all sets founded on the objects in its domain.**

I assume that the higher-order notions of a mathematical theory are concerned with <u>sets</u> founded on the associated systems. I call these notions <u>collective</u> <u>notions</u>:

P2 A collective notion x of the mathematical theory of Fs is a functional notion which maps F-systems to sets, and for any F-isomorphism i from F to F', and any y in the domain of i ,

$$y \in x.(F) \text{ iff } i_\epsilon.(y) \in x.(F')$$

An example of a collective notion is the notion of a <u>subgroup</u>. It maps each group + to the set of all @ such that (1) @ is a binary operation in the set theoretic sense, a set of ordered triples, (2) @ is a group, and (3) for all a, b, and c, if a @ b = c, then a + b = c. This notion satisfies P2:

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*If φ is a functional formula and α denotes a set, then "$\{\varphi_y : y \in \alpha\}$" denotes the "range" of φ on the set x.

**The ϵ-extensions of isomorphisms outstrip the ranks of set theory and are rather more numerous than any symbolic constructions. This is a source of some discomfort. Fortunately, the restrictions of ϵ-extensions to a given rank of set theory are genuine functions. Those restrictions may be sufficient for practical purposes.

For any isomorphism of groups i from + to +', and any @ in the domain of i_{ϵ},

$$@ \in \underline{subg}.(+) \text{ iff } i_{\epsilon}.(@) \in \underline{subg}.(+')$$

Because a subgroup @ is an operation in the set theoretic sense, it does fall in the domain of i_{ϵ}. The restriction to set theoretic operations is not harmful. This insures that subgroups obey extensionality, something which mathematicians generally assume. They assume it, for example, when counting the subgroups of a given group.

It is quite essential that we be in possession of such collective notions. The elementary theory of groups is uninteresting. Only the higher-order developments are really fruitful.

Notice that set theory plays a key role in the treatment of higher-order notions. This is a sure sign that set theory is my "background theory". The "background theory" is the theory in which "constructions" are made on the elementary systems.

Our description of typical branches of mathematics is now complete. A typical branch contains several elements:

The mathematical theory of F-systems contains:

(a) Set theoretic postulates which define F-systems.
(b) Isomorphisms of F-systems.
(c) Elementary notions of F-theory, notions which are preserved under F-isomorphisms.
(d) Higher-order notions of F-theory, notions which are preserved under ϵ-extensions of F-isomorphisms.
(e) Theorems of the form 'All Fs are G', where 'G' contains only notions of F-theory.

Mathematical theories can be developed in which (b), (c), and (d) are replaced by set theory itself, so that theorems contain only notions appearing in the basic postulates plus notions of set theory. However, we would find them rather artificial. The actual selection of systems and defining postulates for a mathematical theory involves a number of arbitrary choices. For example, I construed arithmetic as the general study of ω-sequences, when I might just as easily have construed it as the general study of ω-well-orderings. The effects of such choices are ameliorated when we acknowledge the full scope of the special notions of the theory in question. For example, the notion of an ω-well-ordering is no less a part of arithmetic than is the notion of an ω-sequence. But acknowledging the full scope of the special notions involves developing the notion of isomorphism appropriate for that theory. Thus (b), (c), and (d) are natural components of any typical mathematical theory.

Indeed, we might contemplate the opposite sort of economy, doing away with the defining postulates of mathematical theories. We might choose to develop theories directly out of some interesting notions of isomorphism. Sometimes this is quite possible from a purely mathematical standpoint.* Nevertheless, fruitful mathematical theories generally receive some inspiration from the sciences, and the sciences usually provide mathematicians with postulates, not undigested notions of isomorphism. So postulates will remain a part of our mathematical theories, at least for the near future.

Notes

What is the precise relationship of physical and mathematical

*In his Erlanger Program (1872), Felix Klein suggested that every geometrical theory can be developed from its associated <u>automorphisms</u>, isomorphisms of spaces onto themselves. This influential approach does not work well for complex geometries, since automorphisms of the relevant spaces are few and uninteresting. Perhaps category theory, in which isomorphisms are but a special case of the more general <u>morphisms</u>, will succeed where the Erlanger Program failed.

theories? A physical theory says that all physical systems of a certain kind are systems of such-and-such a mathematical theory. For example:

For any time t and set of fundamental particles x, if d is a function which maps any objects a and b in x to their distance at t, and $(1_d, +_d, <_d)$ is the nnr-system of spatial distances, then

$$(d, (1_d, +_d, <_d)) \text{ is a metric space.}$$

The relations and functions of the physical theory are the results of applying relation notions and function notions of the mathematical theory to the relevent physical systems.

2. Mathematics as the Study of Abstract Structure

What constitutes the subject matter of mathematics, if not mathematical objects? According to one promising new view, mathematics is concerned with abstract structure.* But what is abstract structure, and what is it for mathematics to be concerned solely with it?

Let us tackle the second question first. We have found that mathematics is not concerned with special objects. Mathematical laws are generalizations about rather ordinary things. However, mathematics is not concerned with all general aspects of ordinary things. Some laws about ordinary things belong to the special sciences. The hypothesis we are now considering is that mathematics is only concerned with the abstract structural aspects of things. That is, the notions involved in mathematical generalizations pertain only to these structural aspects. What does this mean? I think we may give it the following interpretation:

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*The view can be found in Bourbaki's Elements of Mathematics, especially the volume on sets. I first came across it in Michael Jubien's "Ontology and Mathematical Truth".

No mathematical notion distinguishes objects that have the same abstract structure. Or, so as not to give the impression of hypostatizing structure: No mathematical notion distinguishes objects that are abstractly isomorphic. Thus if N is a mathematical notion, and R and R' are abstractly isomorphic objects, then R satisfies N if and only if R' satisfies N.

The main difficulty with this hypothesis is that of specifying the relevant notion of isomorphism. In our examination of typical branches of mathematics, we found many types of isomorphism. The isomorphisms of a mathematical theory are mappings which "preserve" the special notions of that theory. They are also said to "preserve" the relevant sort of structure. ω-sequence isomorphisms preserve arithmetic notions and arithmetic structure, group isomorphisms preserve group theory notions and group structure, and so on. How can we fit these many notions of isomorphism into one view of abstract isomorphism?

We might say that the isomorphisms of particular mathematical theories are species of a single genus, abstract isomorphisms. However, this proposal does not get us very far. We have not said which sorts of isomorphism are abstract.

Clearly there are nonmathematical sorts of isomorphism. Examples are easy to construct. For example, we can define "friendship isomorphisms":

> i is a friendship isomorphism from a set of people x to a set of people y =df i is a one-to-one correspondence from x to y such that for any objects a and b in its domain,
>
> a is a friend of b iff i(a) is a friend of i(b)

Surely these have no mathematical significance.

More seriously, there are isomorphisms of physical systems, isomorphisms which preserve various physical notions. These, too, are not mathematical. They are not sufficiently abstract.

The mathematical isomorphisms we have seen do have a common character. Arithmetic isomorphisms are just isomorphisms of relations (D1). Group theory isomorphisms are just <u>isomorphisms of functions</u>:

D2 i is an isomorphism of n-ary functions from f to f' =df i is a one-to-one correspondence from the union of the domains and range of f to the union of the domains and range of f' such that for any objects $a_1, \ldots, a_n, a_{n+1}$ in the domain of i,

$$f(a_1, \ldots, a_n) = a_{n+1} \text{ iff } f'(i(a_1), \ldots, i(a_n)) = i(a_{n+1})$$

(So in the case of groups we have:
$$a_1 + a_2 = a_3 \text{ iff } i(a_1) +' i(a_2) = i(a_3) \text{)}$$

R-system isomorphisms are only slightly more complex. An isomorphism i from an r-system (1, +, <) to an r-system (1', +', <') satisfies three conditions: (i) It maps 1 to 1'. (ii) It is an isomorphism of functions from + to +'. (iii) It is an isomorphism of relations from < to <'. All three sorts of isomorphism fit into a general scheme, that which includes isomorphisms of relations, isomorphisms of functions, and compounds thereof (possibly preserving units).

We might therefore conjecture that all mathematical isomorphisms fit into this general scheme:

The systems of the mathematical theory of Fs are n-tuples $(X_1, \ldots, X_n)$, where X_n is some n-tuple of units. An isomorphism of F-systems from $(X_1, \ldots, X_n)$ to $(Y_1, \ldots, Y_n)$ is a function i such that (i) i maps each unit in X_1 to the corresponding unit in Y_1, and (ii) for any m such that $1 < m \leq n$, if X_m is a function, i is an isomorphism of functions from X_n to Y_m, and if X_m is a relation, i is an isomorphism of relations from X_m to Y_m. According to this view, mathematics is concerned with structure generated by arbitrary relations, functions, and units. Since the nature of these relations, functions and units is not specified, this is indeed a view according to which mathematics is concerned with abstract structure.

Unfortunately, this view is not general enough. It fails in the case of point-set topology. Point-set topology deals with certain properties of sets, properties satisfying the following axioms:

A1t If x possesses the property P, then x is a set.

A2t The empty set, ∅, possesses P.

A3t If x possesses P, and y possesses P, then the intersection of x and y possesses P.

A4t If z is a set of sets that have P, then the union of the sets in z has P.

A property which satisfies these axioms may be called a topology.

If we cast topological isomorphisms into the scheme suggested above, we obtain a special case of the isomorphism of relations:

> i is an isomorphism from a topology P to a topology Q just in case i is a one-to-one correspondence from the instances of P to the instances of Q such that for any object a in the domain of i,
>
> #. P(a) iff Q(i(a))
>
> (Of course, # follows from the fact that i is a one-to-one correspondence!)

Using this, it follows that two topologies are isomorphic just in case each has the same number of instances, each has the same cardinality! This result is absurd. For instance, it implies that the topology of a line is the same as that of a plane.

An isomorphism of topologies is not a simple isomorphism of properties. A topology is, after all, a property of sets. It has, from the point of view of topology, additional structure beyond the number of its instances. A stronger notion of isomorphism is needed to preserve that structure:

D1t i is an isomorphism from a topology P to a topology Q =df i is a one-to-one correspondence from the instances of P and the members of instances of P to the instances of Q and the members of instances of Q such that for any items x and a in its domain,

$$P(x) \text{ iff } Q(i(x))$$
$$a \in x \text{ iff } i(a) \in i(x)$$

These isomorphisms are mappings not only on the instances of properties, but also on the members of those instances. They preserve a finer structure. In fact, they preserve enough structure to serve as isomorphisms for topology.

Different sorts of isomorphism preserve different degrees of structure. Many mathematical theories are concerned only with ways in which objects may be ordered by relations and functions. Their isomorphisms are of the sort specified in the general scheme above. Topology does not fit into this scheme. It is concerned not only with the partitions generated by certain properties, but also with membership in the instances of those properties. Its isomorphisms go "one level deeper". Could we say then that all isomorphisms are of the "function-relation" type or the "one level deeper" type? Undoubtedly we can imagine mathematical isomorphisms that violate this new scheme, ones that go "another level deeper". And we can imagine mathematical isomorphisms that "cut across the levels".

Because of these informal considerations, it would be unwise to attempt a general definition of mathematical isomorphism. It simply may not be possible to anticipate all the forms that mathematical isomorphisms will take on.*

What then becomes of the hypothesis that mathematical notions are notions preserved under all abstract isomorphisms? We are not in a position to explain abstract isomorphisms as a genus which includes the isomorphisms of all mathematical theories, for we have not been able to say generally what mathematical isomorphisms are. However, there is another account of abstract isomorphism which may prove more fruitful: Let us suppose that abstract isomorphisms are mappings which preserve not merely the structure which is the concern of this or that mathematical theory, but which preserve <u>all</u> mathematical structure.

Is such a view possible? Recall how we extended the weaker "function-relation isomorphisms to deal with the case of topology. Suppose we iterate this process indefinitely. Perhaps we will obtain isomorphisms which preserve total mathematical structure.

Simple isomorphisms of properties are defined over the instances of those properties. Topological isomorphisms are defined over the instances of properties and the members of those instances. Isomorphisms which preserve total mathematical structure are defined over the <u>closures</u> of objects:

D3 S is the closure of X =df

S is the union of the following inductively defined sets:

S_0 = the set which has X as its sole member

S_{k+1} = the set consisting of

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*Metric space isomorphisms, which are defined in the Notes of section 3 of chapter I, are particularly unusual. They too are not of the "function-relation" sort.

(i) all $a_1, \ldots, a_n$ such that for some property or relation R in S_k, $R[a_1, \ldots, a_n]$

(ii) all $a_1, \ldots, a_n, a_{n+1}$ such that for some function f in S_k, $f(a_1, \ldots, a_n) = a_{n+1}$, and

(iii) all a such that for some set x in S_k, $a \in x$

Notice that if X is a pure set, S is the so-called <u>transitive</u> <u>closure</u> of X.

Let us define abstract isomorphisms as follows:

D4 i is an abstract isomorphism from X to Y =df i is a one-to-one correspondence from the closure of X to the closure of Y such that

(A) for any n-ary relation R in the domain of i, and objects $a_1, \ldots, a_n$ in the domain of i,

$$R[a_1, \ldots, a_n] \text{ iff } i(R)[i(a_1), \ldots, i(a_n)]$$

(B) for any n-ary function f in the domain of i, and objects $a_1, \ldots, a_n, a_{n+1}$ in the domain of i,

$$f(a_1, \ldots, a_n) = a_{n+1} \text{ iff } i(f)(i(a_1), \ldots, i(a_n)) = i(a_{n+1})$$

(C) for any set x in the domain of i, and object a in the domain of i,

$$a \in x \text{ iff } i(a) \in i(x)$$

Clearly these isomorphisms are "abstract". They are defined entirely in terms of logic and set theory. Notice that they preserve all the structure we can possibly describe in logical or set theoretic terms. If they do not preserve total mathematical structure, I cannot imagine what would.

We can define abstractly isomorphic in the obvious way:

D5 X and Y are abstractly isomorphic =df there is an abstract isomorphism from X to Y

Now our hypothesis concerning mathematical notions can be stated quite precisely:

P3 If F is a property-notion of mathematics, then for any abstractly isomorphic objects X and Y, X has F if and only if Y has F.

The notions of ω-sequence, group, r-system, and topology all satisfy this principle. (How these examples may be established is shown in the Notes of this section.)

P3 covers the property notions of mathematics. What about the more complex relational and functional notions of mathematics? For example, is there another principle which is satisfied by the notion of the identity element of a group?

Assuming the usual set theoretic construal of ordered n-tuples, we might try the following:

P4a If R is an n-ary relational notion of mathematics, then for any abstractly isomorphic n-tuples $(X_1, \ldots, X_n)$ and $(Y_1, \ldots, Y_n)$,

$$R.[X_1, \ldots, X_n] \text{ iff } R.[Y_1, \ldots, Y_n]$$

b If f is an n-ary functional notion of mathematics, then for any abstractly isomorphic n+1-tuples $(X_1, \ldots, X_n, X_{n+1})$ and $(Y_1, \ldots, Y_n, Y_{n+1})$,

$$f.(X_1, \ldots, X_n) = X_{n+1} \text{ iff } f.(Y_1, \ldots, Y_n) = Y_{n+1}$$

The notion of the identity element of a group satisfies P4. So too does the topological notion of the boundary of a set. Indeed, I can think of no counterexample to P4. (A method for proving examples is given in the Notes.)

The view that mathematics is concerned with abstract structure reduces to these two principles, P3 and P4. I believe that these two principles are correct, and so believe that the general view is correct.

Moreover, P3 and P4 are very nearly definitive of mathematical notions. Any notion which satisfies the conditions mentioned in those principles is extensionally equivalent to some mathematical notion.

Every mathematical predicate or function-phrase stands for a mathematical notion. Thus the rejection of mathematical objects in chapter I is a natural accompaniment of the present principles. For example, if we had allowed "+" to stand for an operation among numbers, it would stand for something which pertains only to numbers, and not to abstractly isomorphic things--contrary to P4. We may conclude that traditional mathematical Platonism is incompatible with the newer "structuralism".

Notes

I have asserted that mathematical property-notions satisfy P3, and that relational and functional mathematical notions satisfy P4. Direct verification of cases is rather tedious. Fortunately, we shall later have a theorem which renders such verification unnecessary, theorem T3 of ch. II, s. 6. T3 tells us how to recognize abstract structural notions _from their very definitions_. Thus, for example, a mere glance at the group axioms is enough to assure us that the notion of a group satisfies P3. Nevertheless, the reader might appreciate at least one example of direct verification. I shall therefore treat the case of groups in this manner.

Let i be an abstract isomorphism from an object X to an object Y. Suppose X is a group. By theorem t2 (from the Notes of ch. III, s. 2), i(X) = Y. Thus to verify that P3 holds of the group notion, we must show that Y satisfies the group axioms.

Since i is an abstract isomorphism, we know that for any objects a, b, and c in the domains of X,

\# X(a, b) = c iff Y(i(a), i(b)) = i(c)

Suppose the domains of Y are not identical. Suppose, for example, that an object a' is in the first domain of Y but not in its second domain. Then there are objects b' and c' such that Y(a', b') = c'. Since i is a one-to-one correspondence from the closure of X to the closure of Y, there are objects a, b, and c such that a' = i(a), b' = i(b), and c' = i(c). By #, X(a, b) = c. But since X is a binary operation, there are objects d and e such that X(d, a) = e. By #, Y(i(d), a') = i(e). So a' is in the second domain of Y.--a contradiction. A similar contradiction arises if the second domain of Y contains an object not in its first domain. Thus the domains of Y must be identical.

Suppose the range of Y is not contained in its domains. Then there is an object a' in the range which does not belong to the domains. Since a' is in the range, there are objects b' and c' such that Y(b', c') = a'. Since i is a one-to-one correspondence, there are objects a, b, and c such that a' = i(a), b' = i(b), and c' = i(c). By #, X(b, c) = a. But since X is a binary operation, a is in the domains of X. That is, for some d and e, X(a, d) = e. By #, Y(a', i(d)) = i(e), and a' is a member of the domains of Y--a contradiction. Hence the domains of Y contain its range.

Because the domains of Y are identical and contain the range, Y is a binary operation. Y satisfies the first axiom of groups, A1g.

For any objects a, b, and c, X(a, X(b, c)) = X(X(a, b), c), by the associativity of the group X. Using several applications of #, it follows that for any objects a, b, and c in the domain of i, Y(i(a), Y(i(b), i(c))) = Y(Y(i(a), i(b)), i(c)). But since every member of the domains of Y is the image of some object under i, we may conclude that for any objects a', b', and c' in the domains of Y, Y(a', Y(b', c')) = Y(Y(a', b'), c'). That is, Y satisfies the Associative Axiom, A2g.

X has an identity element, 0: For any object a in the domains of X, X(a, 0) = X(0, a) = a. By #, it follows that for any object a in the domains of X, Y(i(a), i(0)) = Y(i(0), i(a)) = i(a). But since every object in the domains of Y is the image under i of some object in the domains of X, this implies that i(0) is an identity element for Y. Since there obviously cannot be two identity elements, Y satisfies A3g.

Let a' be any object in the domains of Y. Since i is a one-to-one correspondence, there is an object a such that i(a) = a', and a is in the domains of X. Because X is a group, a has a unique inverse -a: X(a, -a) = X(a, -a) = 0. By #, Y(a', i(-a)) = Y(i(-a), a') = i(0), so i(-a) is an inverse for a'. Thus every member of the domains of Y has an inverse. Since inverses are necessarily unique, Y satisfies A4g.--This completes the proof that Y is a group.

3. The Logicist Account of Mathematical Notions

The "structuralism" of principles P3 and P4 is not the only existing account of mathematical notions. The logicists have propounded another:*

LOG The notions of mathematics are definable from the basic notions of logic.

This is in fact a very successful account of mathematical notions, and it will be worth our while to compare it with the structuralist view. But

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*The view is stated in the Preface of Bertrand Russell's Principles of Mathematics (1903) and in the opening remarks of Rudolph Carnap's "The Logicist Foundations of Mathematics" (1931).

first we need a more complete understanding of the account itself.

In order to understand LOG, we obviously need to know what the basic logical notions are. The logicists gave roughly the following list:*

BLN (1) Logical Connectives: $\sim$, &, $\vee$, $\rightarrow$, $\leftrightarrow$
(2) Quantifiers: $\exists$, $\forall$
(3) Identity: =
(4) Set Membership: $\in$
(5) Forms of Predication: P[a], R[a, b], S[a, b, c],...
(6) Forms of Functional Application: f(a), g(a, b),... **

The first two items on this list are a bit problematic. Assuming that connectives are notions, they are neither clearly relational nor clearly functional. However, I think that this will not cause us any difficulty. The last two items on the list are most in need of explanation.

A form of predication is what we obtain from a proposition when all its specialized constituents are replaced by variables. For example, 'Socrates is a man' yields the form of simple predication, P[a], the notion of an object possessing a property. This is of course a relational notion, since it corresponds to the relational predicate "object ___ possesses property ...". It is not a true relation, because it transcends all the types of type theory, all the ranks of set theory. 'John loves Mary' has the form of binary relation, R[a, b], the notion of an object a bearing a relation R to an object b. This is a ternary relational notion, since it corresponds to the ternary predicate "object ___ bears relation ... to object ---". For any n, there is a form of n-ary relation, and it is an n+1-ary relational notion. Note also that

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*Compare this with the list of Russell, op. cit., s. 1, and that of Carnap, op. cit., s. I.

**I shall use all simple Roman letters as variables; underlined words and letters, as constants. I understand variables to range over objects of all types, so that there are no explicit type distinctions.

the form of simple predication is for intensions, properties, what set membership is for extensions, sets.*

A form of functional application is what we obtain from a functional complex when all its specialized constituents are replaced by variables.** For example, 'the mother of Meinong' has the form of simple application, f(a), the notion of the result of applying a function f to an object a. This is a functional notion, the one corresponding to the function-phrase "the result of applying ____ to ...". For every n, there is a form of n-ary application, and it is an n+1-ary functional notion.

Only the first three items on the list, connectives, quantifiers, and identity, appear in <u>elementary</u> logic. Set membership, forms of predication, and forms of functional application are <u>nonelementary</u> logical notions. They belong to higher-order logic and set theory. The logicists evidently believed that logic includes set theory and higher-order logic.

Some philosophers will say that only elementary logic is <u>really</u> logic. As an objection to LOG, this misses the point. The boundaries of logic are unclear. The important fact is that BLN happens to be a pretty good basis on which to define mathematical notions.

LOG is a generalization: Every mathematical notion is definable from BLN. This generalization has a large number of confirming instances, many of which are exhibited in <u>Principia Mathematica</u>. From BLN,

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*For a discussion of these forms, see Russell's <u>Introduction to Mathematical Philosophy</u>, ch. XVIII, p. 198-9.

**In speaking of propositions and functional complexes, I do not wish to commit myself to the view that these are something beyond sentences and function-phrases.

we may define each mathematical notion we have encountered in our discussion, for each was introduced in purely logical or set theoretic terms. Thus we may define such simple notions as the domain of a function and the transitivity of a relation, and we may define the more interestion mathematical notions of ω-sequence, group, identity element, r-system, and topology.

But a single disconfirming instance topples LOG, and I believe that there is one.

4. Objection to the Logicist Account

The simplest objection to LOG begins with the observation that the basic logical notions, the members of the list BLN, are countable. If we understand definition in the ordinary way, the notions definable from the basic logical notions are also countable. Thus LOG entails that there are only countably many mathematical notions. This restriction leads to difficulty.

To see the difficulty clearly, we need some definitions. Some of these can be framed in terms of the basic logical notions:*

d1 $\forall a(a \in \underline{\text{field}}.(R) \leftrightarrow \exists b(R[a, b]) \vee \exists b(R[b, a])$

d2 $\underline{\text{well-ordering}}.[R] \leftrightarrow \forall a \forall b(R[a, b] \rightarrow \sim R[b, a])\ \&$
$\forall a \forall b(R[a, b] \vee a = b \vee R[b, a])\ \&$
$\forall a \forall b \forall c(R[a, b]\ \&\ R[b, c] \rightarrow R[a, c])\ \&$
$\forall x(\forall a(a \in x \rightarrow a \in \underline{\text{field}}.(R))\ \&\ \exists a(a \in x) \rightarrow$
$\exists a(a \in x\ \&\ \sim\exists b(b \in x\ \&\ R[b, a])))$

d3 $\underline{\text{initial segment}}.[I, R]$
$\exists p(p \in \underline{\text{field}}.(R)\ \&\ \forall a \forall b(I[a, b] \leftrightarrow R[a, p]\ \&\ R[b, p]\ \&\ R[a, b]))$

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*These are definitions of notional constants in the object language.

d4 $\forall a(a \in$ domain.$(f) \leftrightarrow \exists b(f(a) = b))$ *

d5 $\forall a(a \in$ range.$(f) \leftrightarrow \exists b(f(b) = a))$

d6 one-to-one.$[f] \leftrightarrow \forall a \forall b(f(a) = f(b) \rightarrow a = b)$

d7 one-to-one correspondence.$[f, x, y] \leftrightarrow$ domain.$(f) = x$ & range.$(f) = y$ & one-to-one.$[f]$

These notions are fairly familiar, but their definitions do serve to illustrate what can be constructed from BLN.

However, the most forceful statement of my objection requires notions beyond the power of the logicist systems. These invoke the theory of possible worlds and possible objects. This is a departure from the extensionalist methods of previous sections, but perhaps in the end it will prove dispensible.

d8 A function f is a transworld function just in case for any possible objects x and y, if it is possible that f(x) = y, then necessarily f(x) = y.

I call such functions "transworld" because they do exactly the same thing from world to world. Relying heavily on the theory of possible objects, I shall suppose that contingently existing objects can appear in the domains and ranges of such functions. One contingent object can be the result of applying a transworld function to another even in worlds where neither exists!

d9 A function f is a transworld isomorphism from a relation R in a possible world w to a relation R' in a possible world w' just in case (1) f is a transworld function, (2) f is a one-to-one correspondence from the field of R in w to the field of R' in w', and (3) for any possible objects x and y,

$$R[x, y] \text{ in } w \quad \text{iff} \quad R'[f(x), f(y)] \text{ in } w'$$

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*The k-th domain of an n-ary function f is the set of all objects a such that for some $b_1, \ldots, b_n$, $f(b_1, \ldots, b_{k-1}, a, b_k, \ldots, b_{n-1}) = b_n$.

For any such function f, the way in which objects are arranged by R in w is exactly duplicated by the way in which their corresponding objects, their images under f, are arranged by R' in w'.

d10 A relation R in a world w is isomorphic to a relation R' in a world w' just in case there is an isomorphism from R in w to R' in w'.

Isomorphic relations are simply relations which arrange their fields in exactly the same way.*

d11 A well-ordering R in w is shorter than a well-ordering R' in w' just in case R in w is isomorphic to some initial segment of R' in w'.

Shortness in well-orderings is just what one would expect it to be. Indeed, we can prove the following theorem about it: For any well-orderings R in w and R' in w', exactly one of the following holds: (i) R in w is shorter than R' in w', (ii) R' in w' is shorter than R in w, or (iii) R in w is isomorphic to R' in w'.

d12 A notion T is a relation type just in case there is a relation R' and a possible world w' such that for every relation R and world w, R satisfies T in w if and only if R in w is isomorphic to R' in w'.

This is similar to Bertrand Russell's definition of a relation number.** But a relation number is a class of relations which are isomorphic in the actual world. I am concerned with the corresponding notions, whatever they may be.

d13 A notion T is a well-order type just in case T is a relation type such that for any relation R and world w, if R satisfies T in w, then R is a well-ordering in w.

Since every relation isomorphic to a well-ordering is itself a well-

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*They are also relations which share the same arrow diagrams. See Carnap, The Logical Structure of the World, s. 11.

**Principles of Mathematics, s. 253.

ordering, any relation type which is possessed by a well-ordering in some world is a well-order type.

Certain well-order types can be defined from the basic logical notions. A trivial example is the well-order type of two-element well-orderings:

$$\underline{\text{2-well-ordering}}.[R] \leftrightarrow \exists a \exists b(a \neq b \ \& \ \forall c \forall d(R[c, d] \leftrightarrow c = a \ \& \ d = b))$$

A non-trivial example is the type of ω-well-orderings:

$$\underline{\omega\text{-well-ordering}}.[R] \leftrightarrow \underline{\text{well-ordering}}.[R] \ \& \ \forall a(a \in \underline{\text{field}}.(R) \rightarrow \exists b(R[a, b])) \ \& \ \exists a \forall b(\forall c(R[c, b] \rightarrow \exists d(R[c, d] \ \& \ R[d, b])) \leftrightarrow b = a)$$

(that is, R is an infinite well-ordering with a unique element that is not the immediate successor of any other)

The relation $<$ among natural numbers is supposed to be an ω-well-ordering. So is the relation ∈ among the members of the Von Neumann set theoretic ordinal ω.

Since these well-order types can be defined from the basic logical notions, they are the sort of notions LOG counts as mathematical. In so far as it does this, LOG is correct. These well-order types are mathematical notions. But I think we can assert a much stronger claim:

p1 All well-order types, whether definable from the basic logical notions or not, are mathematical notions.

This principle can be derived from the view that mathematics is concerned with abstract structure. A well-order type is simply a notion which is necessarily satisfied only by relations exhibiting a peculiar abstract relational structure. Thus it is an abstract structural notion. As an independent basis for p1, I can only cite mathematical practice: Single out a type of well-orderings, and it is bound to become a subject of mathematical study.*

The problem for LOG is that there is a well-order type which is not definable from the basic notions of logic, namely:

UWOT The well-order type corresponding to the predicate "is a shortest well-ordering not of a type definable from BLN".

To complete the objection, all we need show is that the notion UWOT is indeed a well-order type.

The major premise is: Uncountable well-orderings are possible.-- This premise would be granted by most working mathematicians. Moreover, anyone who believes the axioms of set theory are possibly true is committed to it. This follows from the fact that the existence of an uncountable ordinal, $\aleph_1$, is provable from the axioms of set theory.**

Let us therefore suppose that there is an uncountable well-ordering R in a world w. Each object x in the field of R in w corresponds to a unique initial segment of R in w, the restriction of R to the objects which bear R to x. Since R is uncountable, it has uncountably many initial segments. No two of these are of the same well-order type (as can be seen from the fact mentioned after d11). Since only countably many notions can be defined from BLN, not every initial segment has a definable well-order type. Since R is a well-ordering, there is a least x in the field of R such that the corresponding initial segment is not of a type definable from BLN. Let I be this initial segment. It is the shortest initial segment of R which is not of a type definable from BLN.

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*Probably it would be studied in set theory, studied by postulating the existence of a pure set, an ordinal, which represents the type.

**$\aleph_1$ is the set of all countable ordinals. Its members are well-ordered by the relation $\in$ restricted to that set.

Let I' be any relation which satisfies UWOT in a world w'. That is, it is in w' a shortest well-ordering not of a type definable from BLN. Then I in w is not shorter than I' in w'--otherwise it would be isomorphic to a definable initial segment of I' in w' and so be of a definable type. Suppose on the other hand that I' in w' is shorter than I in w. By the definition of shortness, I' in w' is isomorphic to an initial segment of I in w. That initial segment is shorter than the whole, I in w. By the selection of I, it must be of a definable type. But since I' in w' is isomorphic to it, I' in w' is of the very same type, a type definable from BLN. But this contradicts our original hypothesis. Hence I' in w' is not shorter than I in w. Since both relations are well-orderings in the respective worlds, and since neither is shorter than the other, they are of equal lengths and isomorphic.--We have shown that if a relation satisfies UWOT in a world, then it is isomorphic to I in w.

Let I' in w' be isomorphic to I in w. Then I' in w' is not of a type definable from BLN, since if it were, so would I in w be. Suppose I'' in w' is a well-ordering shorter than I' in w'. Since I' in w' is isomorphic to I in w, I'' in w' is shorter than I in w. That is, it is isomorphic to an initial segment of I in w. But since that initial segment must be of a type definable from BLN, so too must I'' in w'. Hence any well-ordering which is shorter than I' in w' is of a type definable from BLN. That is, I' satisfies UWOT in w'.--Moreover, we have shown that for any I' in w' which is isomorphic to I in w, I' satisfies UWOT in w'.

Putting these results together, we have shown that UWOT is a well-order type, the type satisfied by all relations I' and worlds w' which are isomorphic to I in w. By p1, UWOT is a mathematical notion. However, it is obviously not definable from BLN. So LOG is false. There are mathematical notions which cannot be defined from the basic notions of logic.

It might be objected that I have construed "definition" too narrowly. I have said that only countably many notions can be defined from BLN. This is so if definitions are restricted, in the usual way, to finite combinations of defining notions. But mightn't the logicists have construed definition more broadly? Mightn't they have allowed definitions by some form of infinite combination?

Carnap rules out the possibility by calling for "explicit definitions". Russell is not quite so clear, although the Preface of <u>Principles of Mathematics</u> describes that work as an attempt to establish the logicist position by "strict symbolic reasoning". The definitions that can be given explicitly in symbols are all of the finite sort.

If the logicists did not mean to confine themselves to definitions in the ordinary sense, they certainly omitted any explanation of a broader sense of definition. They gave no rules for definition by infinite combination. Rather than attribute such an omission to them, it is better to suppose that they gave LOG a quite clear meaning. In that case, however, the argument against LOG holds.

LOG has some attractive results. Many notions of mathematics are definable from BLN. But others are not. What exactly went wrong?

LOG places a <u>formalistic</u> restriction on mathematical notions, the finitude of definition. It is because of this restriction that LOG falls prey to the argument above. Maybe <u>one</u> of the ideas behind LOG is right, namely that mathematical notions are in some way dependent on the basic notions of logic. Maybe LOG goes wrong only becuase it identifies the dependence as explicit definition.

Notes

A similar objection to logicism appears in John L. Pollock's article "On Logicism". Pollock believes his objection is entirely decisive. In what follows, I hope to show that a fairly simple modification of logicism survives the attack.

The argument of this section does not undermine variants of logicism which include all ordinals, or all pure sets, among the basic logical notions. My argument is aimed only at "reductive" identifications of mathematics and logic. These extravagant variants are evidently not reductive.

5. Overcoming the Limitations of Definition

Mathematical notions are not just those which may be defined from the basic logical notions. However, there is another way in which they might depend on them.

Before proceeding, it is convenient to recall a distinction: Among the basic notions of logic, some are _elementary_. These include the connectives, the quantifiers, and identity. Set membership, forms of predication, and forms of functional application are _nonelementary_ logical notions. They appear only in the higher forms of logic and in set theory.

With this distinction, we may restate logicist thesis LOG:

LOG' The notions of mathematics are definable from the nonelementary logical notions using the elementary logical notions.

That is, let us now imagine that nonelementary logical notions are the "material" from which logicist definitions are made, and that definition proceeds within the "framework" of elementary logical notions. Concentrating on the "material" will simplify matters for us. If we were to attend to the "framework", we would have to ascertain the "semantic categories" of the elementary logical notions. In the case of quantifiers and connectives, that would not be easy.

The nonelementary logical notions impose an order on the objects of any possible world. For certain pairs of possible worlds, these orders are exactly resembling, or _isomorphic_. This happens just when there is a one-to-one correspondence between objects in the first world and objects in the second world such that objects in the first world are related in a certain way by the nonelementary logical notions if and only if their correspondents in the second world are also so related by those notions. These correspondences are _isomorphisms with respect to_ the nonelementary logical notions. They are functions which may be said to _preserve_ the nonelementary logical notions.

They are also functions which preserve all notions definable from the nonelementary logical notions. That is, for any such isomorphism i between worlds u and v, and any notion L definable from the nonelementary notions of logic, an object x satisfies L in u if and only if i(x) satisfies L in v. Moreover, if we define a _derivative_ of the nonelementary logical notions to be any notion which is preserved under all such isomorphisms, then we obtain a sense of dependence which is broader than definition. That is, everything which can be defined from the nonelementary logical notions is a derivative of them, but there are derivatives of the nonelementary logical notions which cannot be so defined.

The isomorphisms we need are isomorphisms with respect to some given set of objects S. And for our purposes, they must be transworld functions:

D6 i is an S-isomorphism from a world w to a world w' just in case (1) i is a transworld function, (2) i is a one-to-one correspondence from the objects existing in w to the objects existing in w', and

(i) for every n-ary relation R that belongs to S in w, and all $a_1, \ldots, a_n$ in the domain of i,

$$R[a_1, \ldots, a_n] \text{ in } w \text{ iff } R[i(a_1), \ldots, i(a_n)] \text{ in } w'$$

(ii) for every n-ary function g that belongs to S in w, and all $a_1, \ldots, a_n, a_{n+1}$ in the domain of i,

$$g(a_1, \ldots, a_n) = a_{n+1} \text{ in } w \text{ iff } g(i(a_1), \ldots, i(a_n)) = i(a_{n+1}) \text{ in } w'$$

(that is, i is a transworld function which preserves the things in S)

Derivative relations and functions can be defined in accordance with my informal remarks:

D7 An n-ary relation R is a derivative of the things in S just in case for any S-isomorphism i from a world w to a world w', and any possible objects $a_1, \ldots, a_n$ in the domain of i,

$$R[a_1, \ldots, a_n] \text{ in } w \text{ iff } R[i(a_1), \ldots, i(a_n)] \text{ in } w'$$

(that is, R is preserved under any isomorphism which preserves the objects in S)

D8 An n-ary function g is a derivative of the things in S just in case for any S-isomorphism i from a world w to a world w', and any possible objects $a_1, \ldots, a_n, a_{n+1}$ in the domain of i,

$$g(a_1, \ldots, a_n) = a_{n+1} \text{ in } w \text{ iff } g(i(a_1), \ldots, i(a_n)) = i(a_{n+1}) \text{ in } w'$$

(that is, g is preserved under any isomorphism which preserves the notions in S)

It follows trivially from these definitions that a function or relation which is a member of a set of objects is a derivative of those objects.

There is a more interesting consequence:

T1 Anything which is definable from a set of relations and functions, using the elementary notions of logic, is a derivative of them.

This theorem says that derivatives include definables. It is proved in the Notes of this section.

T1 suggests a modification of the logicist account of mathematical notions, LOG. That view was based on the idea of definition and was found to be too narrow. Perhaps an account based on the idea of derivative relations and functions would not have that defect:

LOG+ The notions of mathematics are derivatives of the nonelementary notions of logic.

Notice that the step from LOG to LOG+ can be imitated in other contexts, since the definitions D6, D7, and D8 are completely general.*

Unfortunately, there is some difficulty involved in applying the definitions to the set of nonelementary logical notions. The definitions refer explicitly to <u>functions</u> and <u>relations</u> of the given set S, not to relational and functional notions. The problem is that we probably do not want to count the nonelementary notions of logic among relations and functions. To do so would invite contradiction.

For example, suppose the form of simple predication, P[a], is a

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*Indeed, the idea behind the step from LOG to LOG+ was borrowed from remarks on geometry by Hermann Weyl: "A point relation is said to be objective if it is invariant with respect to every automorphism," and "... an automorphism is a one-to-one mapping p ⟶ p' of the point field onto itself which leaves the basic relations undisturbed..." (<u>Philosophy of Mathematics and Natural Science</u>, p. 72-3). Weyl's remarks belong to a tradition established by Klein's Erlanger Program (see footnote, p. 48 of the present discussion).

relation. We may deduce from this the existence of a property $\exists Q[P[Q]]$, the property of having an instance. Then we may deduce that there is a property $\exists Q[P[Q]]$ & $\sim P[P]$, the property of having an instance but not being an instance of itself. Call this property "H". Then from the fact that there are properties which have instances but are not instances of themselves, we may deduce a contradiction: $\underline{H}[\underline{H}] \leftrightarrow \sim\underline{H}[H]$.

For this reason, it seems best to suppose that the nonelementary logical notions are not relations and functions. They are of course like relations and functions, they are relational functional notions. But speaking strictly, D6 does not apply to them.

Probably this difficulty can be surmounted. Probably definitions can be constructed that apply even in the case of nonelementary logical notions. I shall not try to do so, however, since I shall be defending a principle stronger than LOG+. Nevertheless, it is good to see how LOG+ would avoid the problems of LOG, were better definitions constructed.

The objection to LOG began with the observation that only countably many notions can be defined from the basic logical notions. Hence the objection does not even get off the ground against LOG+. It cannot be shown that only countably many notions are derivatives of the nonelementary logical notions.

Moreover, the well-order type which gave LOG trouble, UWOT, is no problem for LOG+. Although it is not definable from the basic logical notions, it is a derivative of them: For suppose it were not. Then there would be an isomorphism with respect to the nonelementary logical notions, i, between worlds w and w', and relations R and R' such that i maps R to R', but R satisfies UWOT in w, and R' does not satisfy UWOT in w'. However, notice that this isomorphism would contain as a part an isomorphism of relations from R in w to R' in w'. Hence R and R' <u>are</u> of the same well-order type--a contradiction. Therefore, the well-order type UWOT is a derivative of the nonelementary notions of logic, and it may qualify as a mathematical notion according to LOG+.

LOG placed a formalistic restriction on the notions of mathematics. LOG+, assuming that it can be patched up, removes this restriction. Moreover, LOG+ inherits all the virtues of LOG, since, by T1, every notion which qualifies as mathematical according to LOG also qualifies according to LOG+. In particular, every mathematical notion defined in <u>Principia Mathematica</u> is a derivative of the nonelementary notions of logic.

Notes

In proving T1, I shall make use of a well-known theorem concerning isomorphic interpretations of first-order languages.* In order to do this, I need some definitions:

d15 $I_{L,w}$, <u>the w-relativized interpretation of an interpreted language L</u>, is a function defined on the constants and primitives of L such that

(i) for any constant k, $I_{L,w}(k)$ is the object k denotes,

(ii) for any primitive n-ary predicate π, $I_{L,w}(\pi)$ is a relation such that for any possible objects $a_1, \ldots, a_n$,

$$I_{L,w}(\pi)[a_1, \ldots, a_n] \text{ iff } R[a_1, \ldots, a_n] \text{ in } w,$$

where R is the relation π denotes, and

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*I shall use a stylistic variant of Proposition 2.33 in Elliott Mendelson's <u>Introduction to Mathematical Logic</u>.

(iii) for any primitive n-ary function symbol γ, $I_{L,w}(\gamma)$ is a function such that for any possible objects $a_1, \ldots, a_n, a_{n+1}$, $I_{L,w}(\gamma)(a_1, \ldots, a_n) = a_{n+1}$ iff $f(a_1, \ldots, a_n) = a_{n+1}$ in w where f is the function γ denotes.

d16 f is a <u>w-relativized assignment for L</u> =df f is a function defined on the constants, primitives, variables, and complex terms of L such that

(i) for any constant or primitive χ of L, $f(\chi) = I_{L,w}(\chi)$,

(ii) for any variable η of L, $f(\eta)$ is some object existing in w,

(iii) for any complex term $\ulcorner\gamma(\tau_1, \ldots, \tau_n)\urcorner$ of L,

$$f(\ulcorner\gamma(\tau_1, \ldots, \tau_n)\urcorner) = f(\gamma)(f(\tau_1), \ldots, f(\tau_n))$$

The notion of being definable from a set of functions and relations occurs in T1. This must be defined for both functions and relations:

d17 <u>A formula</u> φ <u>of a language</u> L <u>defines an n-ary relation</u> R <u>in terms of</u> S =df φ is a formula of L such that

(i) every nonlogical primitive of φ denotes a member of S,

(ii) φ has n free variables, $\eta_1, \ldots, \eta_n$, and

(iii) for any possible world w and possible objects $a_1, \ldots, a_n$, $R[a_1, \ldots, a_n]$ in w iff for every w-relativized assignment f of L, if $f(\eta_1) = a_1, \ldots, f(\eta_n) = a_n$, then f satisfies φ

d18 <u>A term</u> τ <u>of a language</u> L <u>defines an n-ary function</u> g <u>in terms of</u> S =df τ is a term of L such that

(i) every nonlogical primitive of τ denotes a member of S,

(ii) τ has n free variables $\eta_1, \ldots, \eta_n$, and there is a variable η_{n+1} of L which is distinct from these, and

(iii) for any possible world w and possible objects $a_1, \ldots, a_n$, a_{n+1},

$g(a_1, \ldots, a_n) = a_{n+1}$ in w iff for every w-relativized assignment f of L, if $f(\eta_1) = a_1, \ldots, f(\eta_n) = a_n$, $f(\eta_{n+1}) = a_{n+1}$, then f satisfies $\ulcorner \tau = \eta_{n+1} \urcorner$

Obviously, a function or relation is definable from a set S of functions and relations just in case there is a term or formula in some language which defines it in terms of S.

T1 Anything which is definable from a set S of functions and relations is a derivative of them.

I shall prove this for relations only. The case of functions may be proved similarly.

<u>Proof</u>: Suppose K is a k-ary relation definable from S. Let L be a language whose only nonlogical primitives are those used in the definition φ of K. Let i be an arbitrary S-isomorphism from w to w'.

Note that interpretations $I_{L,w}$ and $I_{L,w'}$ are isomorphic. To see this, recall that i is an S-isomorphism:

$$R[a_1, \ldots, a_n] \text{ in w iff } R[i(a_1), \ldots, i(a_n)] \text{ in w'}$$

$$f(a_1, \ldots, a_n) = a_{n+1} \text{ in w iff } f(i(a_1), \ldots, i(a_n)) = i(a_{n+1}) \text{ in w'}$$

for all integers n, and all R and f in S. Therefore,

$$I_{L,w}(\pi)[a_1, \ldots, a_n] \text{ iff } I_{L,w'}(\pi)[i(a_1), \ldots, i(a_n)]$$

$$I_{L,w}(\gamma)(a_1, \ldots, a_n) = a_{n+1} \text{ iff } I_{L,w'}(\gamma)(i(a_1), \ldots, i(a_n)) = i(a_{n+1})$$

for any integer n, and primitives π and γ of L. But given that L has no constants, these latter equivalences just say that $I_{L,w}$ and $I_{L,w'}$ are isomorphic.

For any w-relativized assignment f of L, define a corresponding assignment to be any w'-relativized assignment f' such that for every variable η of L, $f'(\eta) = i(f(\eta))$.

The theorem concerning isomorphic interpretations tells us that if f is a w-relativized assignment for L, and f' is a corresponding assignment, then

! f satisfies φ iff f' satisfies φ

Since φ defines K, we also have:

!! $K[a_1, \ldots, a_k]$ in w iff for every w-relativized assignment f of L, if $f(\eta_1) = a_1, \ldots, f(\eta_k) = a_k$, then f satisfies φ

!!! $K[i(a_1), \ldots, i(a_k)]$ in w' iff for every w'-relativized assignment f' of L, if $f'(\eta_1) = i(a_1), \ldots, f'(\eta_k) = i(a_k)$, then f' satisfies φ

where $\eta_1, \ldots, \eta_k$ are the free variables of φ.

Suppose it is not the case that $K[i(a_1), \ldots, i(a_k)]$ in w'. Then for some w'-relativized assignment f' of L, $f'(\eta_1) = i(a_1), \ldots, f'(\eta_k) = i(a_k)$, but f' does not satisfy φ. Let f be a w-relativized assignment of L such that for every variable η, $f(\eta)$ is the inverse image of $f'(\eta)$ under i (the object a such that $i(a) = f'(\eta)$). Then f' is a corresponding assignment for f. By !, f itself does not satisfy φ. But by definition of f, $f(\eta_1) = a_1, \ldots, f(\eta_k) = a_k$. So by !!, it is not the case that $K[a_1, \ldots, a_k]$ in w.--Thus if $K[a_1, \ldots, a_k]$ in w, then $K[i(a_1), \ldots, i(a_k)]$ in w'.

Suppose it is not the case that $K[a_1, \ldots, a_k]$ in w. Then for some w-relativized assignment f of L, $f(\eta_1) = a_1, \ldots, f(\eta_k) = a_k$, but f does not satisfy φ. Let f' be a w'-relativized assignment for L such that for any variable η, $f'(\eta) = i(f(\eta))$. Then f' is a corresponding assignment for f. By !, f' does not satisfy φ. But $f'(\eta_1) = i(a_1), \ldots, f'(\eta_k) = i(a_k)$. So by !!!, it is not the case that $K[i(a_1), \ldots, i(a_k)]$ in w'.-- Thus if $K[i(a_1), \ldots, i(a_k)]$ in w', then $K[a_1, \ldots, a_k]$ in w.

We have shown that for arbitrary $a_1, \ldots, a_k$ in the domain of i,

$$K[a_1, \ldots, a_k] \text{ in } w \quad \text{iff} \quad K[i(a_1), \ldots, i(a_k)] \text{ in } w'$$

Since we have shown this for an arbitrary S-isomorphism i, K is a derivative of the functions and relations in S. Q.E.D.

Notice that the proof depends on the fact that the members of S are relation-like and function-like, not on the stronger fact that they are true relations and true functions. Hence the proof is easily extended to the case in which S is the set of nonelementary logical notions.

6. A Comparison of Logicism and Structuralism

Although the modified logicist account LOG+ has not been formulated in a completely precise way, we need not shy away from all comparison with the structuralist view underlying P3 and P4. We are in a position to uncover some important relationships.

The major obstacle to a direct comparison is the fact that logicism has been formulated in intensional terms, while structuralism has been formulated in extensional terms. We can remove the obstacle by giving the intensional counterparts of the structuralist principles, P3 and P4.*

*Could we effect a comparison by giving an extensional version of LOG+? This would not be interesting: There are too few isomorphisms-with-respect-to-the-nonelementary-logical-notions from this world to itself.

This involves defining an intensional version of the fundamental structuralist notion, abstract isomorphism:

D4' i is a transworld abstract isomorphism from X in w to Y in v =df (1) i is a transworld function, (2) i is a one-to-one correspondence from the closure of X in w to the closure of Y in v, and (3) the following conditions are satisfied:

(A) for any n-ary relation R in the domain of i, and objects $a_1, \ldots, a_n$ in the domain of i,

$$R[a_1, \ldots, a_n] \text{ in } w \text{ iff } i(R)[i(a_1), \ldots, i(a_n)] \text{ in } v$$

(B) for any n-ary function f in the domain of i, and objects $a_1, \ldots, a_n, a_{n+1}$ in the domain of i,

$$f(a_1, \ldots, a_n) = a_{n+1} \text{ in } w \text{ iff } i(f)(i(a_1), \ldots, i(a_n)) = i(a_{n+1}) \text{ in } v$$

(C) for any set x in the domain of i, and object a in the domain of i,

$$a \in x \text{ in } w \text{ iff } i(a) \in i(x) \text{ in } v$$

Using this notion of abstract isomorphism, it is easy to reformulate P3 and P4:

P3' If F is a property notion of mathematics, and X in w is abstactly isomorphic to Y in v, then X has F in w if and only if Y has F in v.

P4'a If R is an n-ary relational notion of mathematics, and $(X_1, \ldots, X_n)$ in w is abstractly isomorphic to $(Y_1, \ldots, Y_n)$ in v, then

$$R.[X_1, \ldots, X_n] \text{ in } w \text{ iff } R.[Y_1, \ldots, Y_n] \text{ in } v$$

b If f is an n-ary functional notion of mathematics, and $(X_1, \ldots, X_n, X_{n+1})$ in w is abstractly isomorphic to $(Y_1, \ldots, Y_n, Y_{n+1})$ in v, then

$$f.(X_1, \ldots, X_n) = X_{n+1} \text{ in } w \text{ iff } f.(Y_1, \ldots, Y_n) = Y_{n+1} \text{ in } v$$

LOG+ says that the notions of mathematics are derivatives of logical notions, are preserved under isomorphisms with respect to those logical notions. P3'-P4' says that the notions of mathematics are abstract structural notions, are preserved under abstract isomorphism. A close link between these views can be revealed through another definition:

D9 i is an S-isomorphism from a set x in a world w to a set y in a world w' just in case (1) i is a transworld function, (2) i is a one-to-one correspondence from x in w to y in w', and

(i) for every n-ary relation R from S in w, and all $a_1, \ldots, a_n$ in the domain of i,

$$R[a_1, \ldots, a_n] \text{ in } w \text{ iff } R[i(a_1), \ldots, i(a_n)] \text{ in } w'$$

(ii) for every n-ary function g from S in w, and all $a_1, \ldots, a_n, a_{n+1}$ in the domain of i,

$$g(a_1, \ldots, a_n) = a_{n+1} \text{ in } w \text{ iff } g(i(a_1), \ldots, i(a_n)) = i(a_{n+1}) \text{ in } w'$$

(that is, i is a transworld function which preserves the structure imposed on x by the things in S)

This is simply the restriction of S-isomorphism (D6) to a set x. Looking carefully at conditions (A), (B), and (C) of D4', we can see that these state that the nonelementary logical notions are preserved, in a restricted way, by abstract isomorphisms. In fact, we may observe that:

p2 An abstract isomorphism from X to Y is merely an isomorphism-with-respect-to-the-nonelementary-logical-notions from the closure of X to the closure of Y.

Thus the difference between LOG+ and P3'-P4' amounts to this: LOG+ says that mathematical notions are preserved under global isomorphism with respect to the nonelementary logical notions. P3'-P4' says that mathematical notions are preserved under certain restricted isomorphisms with respect to the nonelementary logical notions.

From a superficial philosophical viewpoint, there is little to choose between structuralism and modified logicism. According to both views, mathematics is concerned with structure generated by logical, or set theoretic, notions.

What is the precise relationship between LOG+ and P3'-P4'? Let us observe that:

P3 For any isomorphism-with-respect-to-the-nonelementary-logical-notions i from a world w to a world w', and any object X in the domain of i, i contains as a part an abstract isomorphism from X in w to i(X) in w'.

A fairly simple consequence is:

T2 Every abstract structural notion (every notion which is preserved, in the sense of P3'-P4', under abstract isomorphisms) is a derivative of the nonelementary notions of logic.

A proof is offered in the Notes of this section. T2 tells us that P3' and P4' jointly entail LOG+. That is, structuralism contains the germ of truth in logicism.

However, we cannot prove the converse of T2. Some notions which qualify as mathematical according to LOG+ may not be mathematical according to P3'-P4'. Indeed, we cannot even assert the counterpart of T1. There may be notions definable from the nonelementary notions of logic which are not preserved under abstract isomorphisms (in the sense of P3'-P4').

This seems to put P3'-P4' in a precarious situation. LOG+ is supported by the vast number of mathematical definitions which have been constructed out of the basic notions of logic. P3'-P4' does not directly inherit that support. Indeed, we have not eliminated the possibility that some mathematical definition provides a notion that violates P3'-P4'.

It can be proven that definitions of certain special kinds do define notions that conform to P3'-P4'. One such kind is that of <u>closure-restricted</u> definitions:

D10 A closure restricted definition is a definition expressed by a sentence of one of two forms:

(1) $\pi.[\eta_1, \ldots, \eta_n] \leftrightarrow \varphi'$

(2) $\gamma.(\eta_1, \ldots, \eta_{n-1}) = \eta_n \leftrightarrow \varphi'$

where φ' is obtained from a formula φ by restricting bound variables with the formula '$\eta \in$ closure.$(\eta_1) \vee \ldots \vee \eta \in$ closure.(η_n)', and φ is a formula which (i) is constituted solely of variables and items definable from the list BLM, and (ii) has only variables $\eta_1, \ldots, \eta_n$ free.

Many mathematical definitions are trivially equivalent to closure-restricted definitions. An example is the definition of a semigroup:

d14 semigroup.[o] $\leftrightarrow$ $\forall x(\exists y \exists z(o(x, y) = z) \leftrightarrow \exists y \exists z(o(y, x) = z))$ & $\forall x(\exists y \exists z(o(y, z) = x) \rightarrow \exists y \exists z(o(x, y) = z))$ & $\forall x \forall y \forall z(o(o(x, y), z) = o(x, o(y, z)))$

(that is, a semigroup is an associative binary operation)

This is clearly equivalent to a closure-restricted definition, since every bound variable in the *definiens* is confined to the domains and ranges of o. Similarly, the definitions of groups, Boolean algebras, fields, rings, vector spaces, and partial orderings are all equivalent to closure-restricted definitions.

However, other definitions are not equivalent to closure-restricted definitions. The definition of well-orderings (d2) is an example. It contains the following clause:

$$\forall x(\forall a(a \in x \rightarrow a \in \text{field}.(R)) \ \& \ \exists a(a \in x) \rightarrow \exists a(a \in x \ \& \ {\sim}\exists b(b \in x \ \& \ R(b, a)))$$

Here the variable "x" cannot be confined to the closure of R. "x" must range over *subsets* of the field of R, and the closure of R need not contain those.

The same problem is encountered in definitions of ω-sequences, r-systems, and topologies. In each case, there is a variable over sets that need not appear in the closures of the relations or operations involved.

Nevertheless, these set variables are confined to sets <u>founded on</u> the closures of the objects involved. For example, the set variable in the definition of well-orderings ranges over subsets of the field of the given relation. Since the relation's field is part of its closure, subsets of the field are indeed founded on the closure. The notion of an object being founded on another set of objects can be precisely defined:

D11 $\underline{\text{founded on}}.[X, Y] \longleftrightarrow \forall A(A \in \underline{\text{closure}}.(X)\ \&\ {\sim}\exists B(B \in A) \rightarrow A \in \underline{\text{closure}}.(Y))$

Using this, we can define another kind of definition:

D12 A <u>closure-foundation</u> <u>definition</u> is ...

(Same definition as D9, except that restriction is <u>via</u> formulas of the form '$\underline{\text{founded on}}.(\eta, \underline{\text{closure}}.(\eta_1) \cup \underline{\text{closure}}.(\eta_2) \cup \ldots \cup \underline{\text{closure}}.(\eta_n)$', where "$\cup$" stands for the ordinary union of two sets.)

The definitions of well-orderings, ω-sequences, r-systems, and topologies are equivalent to closure-foundation definitions. Indeed, I can think of no mathematical definition which is not so equivalent. I therefore advance the hypothesis:

H1 Every definition of a mathematical notion from the basic notions of logic is equivalent to some closure-foundation definition.

In view of the vast number of mathematical definitions which can be given in terms of the basic logical notions, H1 supports P3'-P4' if we can prove:

T3 Every closure-foundation definition defines a notion which qualifies as mathematical by P3'-P4'.

I offer a proof of this theorem in the Notes of this section.

T3 is a surprising result. Whether or not a notion meeting the conditions of P3'-P4' is satisfied by objects depends solely on how their closures are arranged by the nonelementary logical notions. But closure-foundation definitions can contain quantifiers ranging over things outside of the closures of the objects to which the defined notion applies. Why can't such definitions be used to define notions which do not conform to P3'-P4', which are not preserved under all abstract isomorphisms?

The answer lies in the "well-roundedness" of the universe of sets: Select any two sets, X and Y, such that (1) they are of the same cardinality, and (2) their members are all non-sets. Then the sets founded on X are exactly like the sets founded on Y.--That is, they are ordered in the same way by the membership relation.

It follows that if two objects are abstractly isomorphic, then the things founded on their closures also exhibit a common structure. The proof of T3 is not far away.

Let me summarize: The intensional version of structuralism is a stronger view than modified logicism, LOG+. But logicism is supported by the fact that many mathematical definitions have been framed in terms of the basic notions of logic. Is there any reason to believe that the notions so defined conform to the structuralist principles? There is no general assurance that definitions in terms of logical notions will satisfy those principles. Yet we have found one kind of logical definition which always yields structural notions, closure-foundation definitions.

If all logical definitions of mathematical notions are equivalent to closure-foundation definitions, as I have hypothesized, then the notions defined do conform to the structuralist principles.

There are marked similarities between structuralism and logicism. Logicism is really concerned with structure imposed by the nonelementary notions of logic. Structuralism is also concerned with such structure, but only within a limited context, the closure of an object.

Notes

Proving T2 is a fairly simple matter. I shall do the case of relational structural notions.

T2 Every abstract structural notion is a derivative of the nonelementary notions of logic.

Proof: Suppose K is a k-ary relational abstract structural notion. Let i be an isomorphism with respect to the nonelementary logical notions from a world w to a world w'. Let $a_1, \ldots, a_k$ be any objects of w. Then the restriction of i to the closure of $(a_1, \ldots, a_k)$ in w is an abstract isomorphism from $(a_1, \ldots, a_k)$ in w to $(i(a_1), \ldots, i(a_k))$ in w'. (Here I assume t2, which is introduced in the Notes of ch. III, s. 2.) Thus $K.[a_1, \ldots, a_k]$ in w if and only if $K.[i(a_1), \ldots, i(a_k)]$ in w'. Since we have shown this for arbitrary i and $a_1, \ldots, a_k$, K is a derivative of the nonelementary logical notions. Q.E.D.

The proof of T3 follows the same pattern as T1, but we need more restricted notions of interpretation and assignment:

d19 $I_{L,w,P}$, the w,P-relativized interpretation of an interpreted language L, is a function defined on the constants and primitives of L such that

(i) for any constant k, $I_{L,w,P}(k)$ is the object k denotes,

(ii) for any primitive n-ary predicate π, $I_{L,w,P}(\pi)$ is a relation such that for any possible objects $a_1, \ldots, a_n$,

$$I_{L,w,P}(\pi)[a_1, \ldots, a_n] \text{ iff } P[a_1] \& \ldots \& P[a_n] \& R[a_1, \ldots, a_n] \text{ in } w$$

where R is the relation π denotes, and

(iii) for any primitive n-ary function symbol γ, $I_{L,w,P}(\gamma)$ is a function such that for any possible objects $a_1, \ldots, a_n, a_{n+1}$,

$$I_{L,w,P}(\gamma)(a_1, \ldots, a_n) = a_{n+1} \text{ iff } P[a_1] \& \ldots \& P[a_n] \& P[a_{n+1}] \& f(a_1, \ldots, a_n) = a_{n+1} \text{ in } w$$

where f is the function γ denotes.

d20 f is a <u>w,P-relativized assignment for L</u> =df f is a function defined on the constants, primitives, and variables of L such that

(i) for any constant or primitive χ of L, $f(\chi) = I_{L,w,P}(\chi)$,

(ii) for any variable η of L, $f(\eta)$ is some object which has P in w, and

(iii) for any complex term $\ulcorner\gamma(\tau_1, \ldots, \tau_n)\urcorner$ of L,

$$f(\ulcorner\gamma(\tau_1, \ldots, \tau_n)\urcorner) = f(\gamma)(f(\tau_1), \ldots, f(\tau_n))$$

I prove T3 for the case of relational notions:

T3 Every closure-foundation definition defines a notion which qualifies as mathematical according to P3'-P4'.

<u>Proof</u>: Suppose K is a k-ary relational notion. Suppose φ is a <u>definiens</u> of a closure-foundation definition for K. Let L be a language whose only nonlogical primitives are those used in φ. Let i be an arbitrary abstract isomorphism from $(a_1, \ldots, a_n)$ in w to $(i(a_1), \ldots, i(a_n))$ in w'. (Again, I assume t2, which appears in the Notes of ch. III, s. 2.)

Since φ defines K, we have:

$K[a_1,\ldots, a_k]$ in w iff for every w-relativized assignment f of L, if $f(\eta_1) = a_1,\ldots, f(\eta_k) = a_k$, then f satisfies φ

where $\eta_1,\ldots, \eta_k$ are the free variables of φ. But since φ is a formula of a closure-foundation definition, the right side of the above equivalence may be exchanged for that of the equivalence below:

$K[a_1,\ldots, a_k]$ in w iff for every w,P-relativized assignment f of L, if $f(\eta_1) = a_1,\ldots, f(\eta_k) = a_k$, then f satisfies φ

where P is the property of being founded on the union of the closures of $a_1,\ldots, a_k$. Similarly,

$K[i(a_1),\ldots, i(a_k)]$ in w' iff for every w',P'-relativized assignment f' of L, if $f'(\eta_1) = i(a_1),\ldots, f'(\eta_k) = i(a_k)$, then f' satisfies φ

where P' is the property of being founded on the union of the closures of $i(a_1),\ldots, i(a_k)$. Furthermore, $I_{L,w,P}$ and $I_{L,w',P'}$ are isomorphic interpretations. The isomorphism is the ϵ-extension of i, i_ϵ. Hence for any w,P-relativized assignment f and corresponding w',P'-relativized assignment f',

f satisfies φ iff f' satisfies φ

By the sort of reasoning contained in the proof of T1, we may deduce:

$K[a_1,\ldots, a_k]$ in w iff $K[i(a_1),\ldots, i(a_k)]$ in w'

Thus K meets the conditions of P3'-P4'. It is preserved under abstract isomorphisms. Q.E.D.

7. The Universality of Mathematics

Mathematics is the study of abstract structure, or so the foregoing discussion indicates. However, this fact does not by itself imply that mathematics has a special place among the sciences. We can just as well say that biology is concerned with biological structure, or that physics is concerned with physical structure. What then is so special about abstract structure?

Structure, as we presently understand it, is relative. Different notions impose different kinds of structure, and these different kinds of structure are preserved under different sorts of isomorphism (*vide* D9). It happens that mathematical structure is structure imposed by the non-elementary logical notions and preserved under isomorphisms with respect to them. Thus if there is something special about abstract structure, this something must be manifested by those very notions.

In what sense are the nonelementary notions of logic special? It will not do to say that they are notions of logic and *ipso facto* special. On the one hand, the boundaries of logic are unclear. Indeed, they are unclear at the very point at which nonelementary notions are admitted. Some would not count these notions as logical ones. On the other hand, it must be explained why logic is so special. This cannot be taken for granted.

Perhaps the distinguishing feature of the nonelementary logical notions is their *universality*. Recall again what they are: set membership, forms of predication, and forms of functional application. These notions are *jointly indispensable*. We cannot make an ordinary statement without employing one of them. If we predicate one thing of others, we employ a form of predication. If we say one thing is the such-and-such of others, we employ a form of functional application. If we say one thing belongs to a certain set of things, we employ the notion of set membership. Thus every statement of every empirical science employs a nonelementary logical notion.*

Of course, a set of jointly indispensable notions can contain irrelevant items. If the nonelementary logical notions are jointly indispensable, so are those notions plus the property of being red--Every statement of every empirical science employs some nonelementary logical notion or the property of being red. Do we have any reason to believe that the list of nonelementary logical notions contains no irrelevant items?

I think we do have a reason: The nonelementary logical notions are <u>intersubstitutable</u> <u>in</u> <u>practice</u>. For all scientific purposes, we could use forms of predication exclusively, ignoring set membership and forms of functional application. Our scientific statements would be expressed in terms of properties and relations, rather than sets or functions. Likewise, we could work solely with functions; or even solely with sets, although the extensionality of sets may make this alternative philosophically unpalatable. The main point is that there is no simple practical consideration indicating that one nonelementary logical notion is somehow less fundamental than the others. The list of nonelementary logical

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*Bare identities, 'a = b', do not employ the nonelementary logical notions. Nor do certain <u>extra</u>ordinary statements, such as philosophical assertions about those notions.

notions contains no irrelevant items.

The nonelementary logical notions are universal in the sense that they are both jointly indispensable and intersubstitutable in practice. Because they are indispensable, they span the sciences. Because they are intersubstitutable, each can contain nothing that is special to any one science. Mathematics is the study of structure imposed by such universal notions.

Mathematical notions, which are derivatives of the nonelementary logical notions, inherit some of their generality. They too crop up in quite diverse sciences. In virtue of this fact, mathematical insights can be transferred freely from one realm of study to another.

Mathematics anticipates the sciences. We can devise mathematical theories that have no known applications. All we need do is produce set theoretic postulates describing all systems of a certain sort. We can work out the consequences of such postulates even if no such systems are known to us. And because of the abstract nature of the postulates, there may be no good way to tell where, among all the sciences, such a system might finally appear.

In this spirit, Hermann Weyl has described mathematical postulate sets as "logical molds for possible sciences".* A mathematical postulate set is an abstract structural description of systems. Such systems may or may not be given to us by empirical science.

Is universality an essential feature of the nonelementary logical notions? Certainly it is difficult to imagine a science devoid of these

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*Philosophy of Mathematics and Natural Science, p. 25.

notions. How could we make statements without employing predication, functional application, or set membership? What forms of language would we use? There are no immediate answers.

From another point of view, it seems quite possible. The nonelementary logical notions are subject to laws, the laws of higher-order logic and set theory. I think we can imagine tossing all the laws aside in favor of new ones. But that would be tantamount to the acceptance of new universal notions.

For example, predication is not mere concatenation of symbols. It is a notion we take to satisfy certain laws. We can imagine a language in which concatenation corresponds to a notion satisfying entirely different laws. Once we accept a scientific theory expressed in terms of such a language, predication is no longer a universal notion.

It is perhaps an accident of history that the logicist's nonelementary logical notions are indeed universal. Likewise, it may be accidental that mathematics is the study of abstract structure, in the sense of P3-P4, and that mathematical notions are derivatives of the nonelementary logical notions, LOG+. The true definition of mathematics seems to be: Mathematics is the study of structure imposed by universal notions, whatever they happen to be.

This definition allows for the possibility that set theory and higher-order logic be replaced by the newer category theory. I am not suggesting that this replacement will or should come about. I do not think the relative merits of these background theories are well understood. Nevertheless, it is good to have a characterization of mathematics that can survive changes in the background theory.

CHAPTER III

CLASSICAL FOUNDATIONS

1. The General Laws of Sets

Set theory has been used to serve three foundational purposes. First, it has been used to specify the systems of mathematical theories. The axiomatic descriptions of well-orderings, ω-sequences, and r-systems contain quantifiers ranging over sets. These systems cannot be specified without using something very much like set theory. Second, set theory has been used to pursue higher-order development of mathematical theories. Mathematicians are not interested merely in the roles of individuals in the systems described by their mathematical theories. They are interested in subsystems and in functions from one system to another. For example, the elementary consequences of the group theory axioms are fairly trivial. Only in theorems about subgroups and group homomorphisms does group theory assume its characteristic richness. Set theory is the measure by which we determine what subsystems and transformations there are. Third, it has been used to provide mathematical objects, objects for such theories as arithmetic and the theory of real numbers.

Of these purposes, the third is superfluous. As I have pointed out in my first chapter, there is no need for mathematical objects. Arithmetic is better interpreted as a theory about all ω-sequences than as a theory about special objects, numbers. Similarly, the theory of real numbers is better interpreted as a theory about all r-systems than as a theory of a particular r-system of real numbers. And in general, a mathematical theory is about all systems of a certain sort, not about some peculiar system of pure mathematical objects.

The two legitimate purposes of set theory can be fulfilled without categorical assertions of existence. In particular, there need not be an axiom asserting the existence of an infinite set.

The sets used to specify the systems of a mathematical theory are sets founded on those systems. Hence they are sets which exist if the systems exist. Their existence need not be asserted categorically.

Subsystems and transformations of systems are also objects founded on the systems themselves. They too exist if the systems exist, and their existence need not be asserted categorically.

Thus the two legitimate purposes of set theory are served by those general laws of sets which tell us what sets exist given the existence of certain other things. These include the Power Set Axiom:

A1s $\forall x(\underline{set}.[x] \rightarrow \exists y(\underline{set}.[y] \;\&\; \forall z(z \in y \leftrightarrow \underline{set}.[z] \;\&\; \forall w(w \in z \rightarrow w \in x))))$

This says that for any set, there is a set of all its subsets. Obviously some such axiom is needed in the higher-order development of mathematical theories. Another important basic law is the Union Axiom:

A2s $\forall x(\underline{set}.[x] \rightarrow \exists y(\underline{set}.[y] \;\&\; \forall z(z \in y \leftrightarrow \exists w(w \in x \;\&\; z \in w))))$

This allows us to combine arbitrarily many sets. Finally, we also need the Replacement Schema:

A3s $\forall u(\underline{set}.[u] \;\&\; \forall s(s \in u \rightarrow \exists r \forall t(\varphi_{s,t} \leftrightarrow t = r)) \rightarrow$
$\exists v(\underline{set}.[v] \;\&\; \forall t(t \in v \leftrightarrow \exists s(s \in u \;\&\; \varphi_{s,t}))))$

This says that for any set u and functional formula φ, there is a set v which is the "range" of φ on u. The Replacement Schema "rounds out" the universe of sets. Nearly as useful, however, is the weaker Separation Schema:

A3s' $\forall x(\underline{set}.[x] \rightarrow \exists y(\underline{set}.[y] \;\&\; \forall z(z \in y \leftrightarrow z \in x \;\&\; \varphi_z)))$

This says that for any set x and formula φ, there is a subset of x consisting of just those things in x which satisfy φ. It tells us something about the "richness" of sets. The subsets of a set x are at least as rich as the formulable distinctions among members of x.

Of course, sets also satisfy the Extensionality Axiom:

A4s $\forall x \forall y(\underline{set}.[x] \;\&\; \underline{set}.[y] \;\&\; \forall z(z \in x \leftrightarrow z \in y) \rightarrow x = y)$

And they are generally taken to satisfy the Regularity Axiom:

A5s $\forall x(\underline{set}.[x] \;\&\; \exists y(y \in x) \rightarrow \exists y(y \in x \;\&\; \forall z(z \in x \rightarrow z \notin y)))$

Notice that I have formulated the axioms in such a way that they apply to impure sets, sets founded on nonsets. This is a desirable feature. The systems treated by mathematical theories include systems discovered by empirical science. Mathematical descriptions of those empirical systems contain references to sets founded upon them, sets which are manifestly impure.

Axioms A1s-A5s provide a pretty good basis for all of classical mathematics, once we dispense with the idea that there must be special mathematical objects. We lack only the Axiom of Choice. I shall argue later that there is reason to admit that axiom as well.

There is also some uncertainty about how the schematic axioms, A3s and A3s', are to be understood. I shall argue that they should not be construed in a purely formal manner. That is, the expression "φ" should not be confined to the expressions of any given formal system.

2. Dealing with Functions and Relations

Mathematics is concerned with structure generated by the nonelementary notions of logic. Those notions include not only set membership, but also forms of predication and forms of functional application. Thus it seems that a foundation for mathematics should contain laws pertaining to relations and functions, as well as laws pertaining to sets.

This is true, but the laws required are only a very limited extension of the laws of sets. Some are quite simple:

A1c $\forall R \exists x \forall a_1 \ldots \forall a_n ((a_1, \ldots, a_n) \in x \leftrightarrow R[a_1, \ldots, a_n])$

A2c $\forall f \exists x \forall a_1 \ldots \forall a_n \forall a_{n+1} ((a_1, \ldots, a_n, a_{n+1}) \in x \leftrightarrow f(a_1, \ldots, a_n) = a_{n+1})$

A3c $\forall x \exists R \forall a_1 \ldots \forall a_n (R[a_1, \ldots, a_n] \leftrightarrow (a_1, \ldots, a_n) \in x)$

A4c $\forall x \exists f \forall a_1 \ldots \forall a_n \forall a_{n+1} (f(a_1, \ldots, a_n) = a_{n+1} \leftrightarrow (a_1, \ldots, a_n, a_{n+1}) \in x)$

These rules state correspondences between relations, functions, and sets of ordered n-tuples. Notice that I have not said that relations and functions <u>are</u> sets of ordered n-tuples. On the one hand, such identification may be impossible. Relations and functions seem to violate extensionality. On the other hand, there is no mathematical need for such identification.

These correspondences are not quite sufficient to complete the foundation. It is compatible with A1s-A5s and A1c-A4c that a property be an instance of itself. But I believe that this cannot happen.* Properties exhibit a regularity comparable to that of sets, which, by A5s, cannot be members of themselves. This regularity follows from a more

*That it cannot happen is a consequence of most existing type theories, and so of most systematic attempts to deal with the logical paradoxes.

complex correspondence:

A5c There is a functional notion <u>itext</u> (the "iterated extension notion") from objects to objects such that

(i) <u>itext</u>.(a) = a, if a has no members, maps nothing, and relates nothing

(ii) $\underline{\text{itext}}.(x) = \{\underline{\text{itext}}.(b) : b \in x\}$, for any set x

(iii) $\underline{\text{itext}}.(R) = \{(\underline{\text{itext}}.(b_1), \ldots, \underline{\text{itext}}.(b_n)) : R[b_1, \ldots, b_n]\}$, for any n-ary relation R

(iv) $\underline{\text{itext}}.(f) = \{(\underline{\text{itext}}.(b_1), \ldots, \underline{\text{itext}}.(b_n), \underline{\text{itext}}.(b_{n+1})) : f(b_1, \ldots, b_n) = b_{n+1}\}$, for any n-ary function f

In virtue of A5s, sets come in layers called <u>ranks</u>. In virtue of A5c, relations and functions also come in layers, <u>via</u> the ranks of their iterated extensions. A layer of relations and functions may be called a <u>type</u>. Thus the logic of relations and functions proposed here resembles a cumulative theory of simple types, except that I have dispensed with the usual restrictions on type theoretic language.

The correspondence rules may enable us to specify infinite sets. For example, consider the property of <u>being a distance relation</u>. By A1c, there is a set of all distance relations. But physical science seems to tell us that there is a continuum of distances which objects may bear to one another.* Hence the set of all distance relations has the power of the continuum and so is infinite.

Perhaps this argument for infinite sets is not decisive. Perhaps there is a less Platonistic interpretation of physical science under which the argument does not go through. However, the problem of inter-

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*Indeed, an infinite part of this continuum is actually exemplified when one object moves continuously with respect to another.

preting physical science is much too difficult for any thorough treatment here. The point relevant to the philosophy of mathematics is that there can be empirical arguments for the existence of infinite sets.

Notes

Usually sets are divided into ranks by assigning a pure set, an ordinal, to each. Since I claim there is no special interest in pure sets, I feel obliged to give the rank construction without appealing to them. I shall do this by describing the rank-ordering associated with any transitive set. (A transitive set is one which contains all members of its members.)

Each rank-ordering is a quasi-well-ordering:

d21 A quasi-well-ordering is a pair of relations (R, E) such that
(i) R and E share the same field
(ii) E is an equivalence relation
(iii) R is transitive
(iv) for any a and b in the field of R, it is not the case that both R[a, b] and R[b, a]
(v) for any a and b in the field of R, either R[a, b] or E[a, b] or R[b, a]
(vi) for any nonempty subset x of the field of R, there is a member a of x such that for any member b of x, either R[a, b] or E[a, b]

For each quasi-well-ordering (R, E), there is an associated well-ordering, namely the natural well-ordering of the equivalence classes under E.

d22 A rank-ordering on a transitive set x is a quasi-well-ordering (R, E) such that
(i) the field of R is x
(ii) for any memberless objects a and b in x, E a, b
(iii) for any object y in x, (a) for any member b of y, R[b, y], and (b) for any member z of x, if for any member b of y, R[b, z], then either R[y, z] or E[y, z]
(that is, any set y in x is a supremum of its elements)

We can prove that wherever rank-orderings intersect, they must agree, so are independent of the transitive background. A set is absolutely of lesser rank than another if it is so in the rank-ordering of some transitive set.

Abstract isomorphisms preserve the ranks of iterated extensions:

t1 For any abstract isomorphism i, if i(A) = B, then <u>itext</u>.(A) and <u>itext</u>.(B) are of equal rank.

The proof relies on the inductive character of quasi-well-orderings, d21(vi): If the theorem fails for an abstract isomorphism i, then, in the rank-ordering of iterated extensions, there must be a least rank at which it fails.

One corollary appears in the proof of T3:

t2 If i is an abstract isomorphism from X to Y, then i(X) = i(Y), and if j is an abstract isomorphism from $(X_1, \ldots, X_n)$ to $(Y_1, \ldots, Y_n)$, then $i(X_1) = Y_1, \ldots, i(X_n) = Y_n$.

The first part follows from t1 plus the fact that the iterated extension of an object X is of greater rank than the iterated extension of any other object in X's closure. The second part may be proved analogously.

<u>3. Infinity</u>

Of all the mathematical axioms in <u>Principia Mathematica</u>, two have borne the brunt of criticism, the Axiom of Reducibility and the Axiom of Infinity. The Axiom of Reducibility states that for any propositional function which involves quantification over objects of type greater than or equal to the type of its arguments, there is an equivalent propositional function which involves no such quantification. In the ramified type theory of <u>Principia Mathematica</u>, the type of a propositional function may be determined by the types of variables bound within it. Hence the axiom is a rule for introducing propositional functions of reduced type. Since the theory of ramified types has fallen out of favor, and since those theories which are in favor do not require an Axiom of Reducibility, criticisms of that axiom are not relevant to our concerns.

The Axiom of Infinity, on the other hand, is worthy of some discussion. That axiom asserts the existence of a propositional function of infinitely many arguments, or, for our purposes, the existence of an infinite set.

No one has succeeded in deriving this axiom from any more fundamental principles. If accepted, it must be accepted either on the basis of self-evidence or on the basis of some empirical justification.

Is the Axiom of Infinity self-evident? One line of reasoning suggests that it is: The empty set, which has no members, does not depend for its existence on the existence of anything else. It exists of necessity. Using the general laws of sets, we can prove the existence of sets founded upon it. For instance, we can prove the existence of $\{\emptyset\}$, $\{\{\emptyset\}\}$, $\{\{\{\emptyset\}\}\}$, and so on. But surely this sequence of sets can be collected to form a single set: $\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \ldots\}$. Thus there necessarily exists an infinite set.

About the first step, that the empty set exists of necessity, I shall not quibble. If only that part of the reasoning fails, it would still show that given the existence of any one thing, there exists an infinite set.

I object to the collecting step: "But surely this sequence of sets can be collected to form a single infinite set..." What is the principle of collection operating here? It cannot be the principle that for every way of specifying things, there is a set consisting of just those things specified. That principle leads us straight into Russell's Paradox.

Perhaps the real reasoning is something like this: The existence of the sets $\emptyset$, $\{\emptyset\}$, $\{\{\emptyset\}\}$, and so on is ensured by the set theoretic axioms. But for any bunch of sets whose existence is ensured by the axioms, there is a set consisting of all those sets. Hence there is an infinite set: $\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \ldots\}$.

Here the working assumption is that there is a set of all sets whose existence is ensured by the axioms. That is, there is a set which is a _model_ for the axioms. The trouble with this assumption is that it is no easier to accept than the Axiom of Infinity itself.*

In conclusion, I fail to see the self-evidence of the Axiom of Infinity. Attempts to point it out have amounted to disguised "proofs", proofs from axioms that are more doubtful.

Against the possibility of empirical justification, Russell himself argues:** Suppose the only justification of the Axiom of Infinity is empirical. Such a justification can only show that the Axiom of Infinity is true, not that it is _tautologous_, or _necessary_. But every truly mathematical proposition must be necessary. Hence an empirical justification of the Axiom of Infinity does not justify its inclusion among the prin-

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*Against this, someone might argue, "The statement that there is a model for the axioms is equivalent to the statement that the axioms are consistent. But the statement that the axioms are consistent seems quite safe indeed. Hence we are warranted in asserting the existence of a model." The trouble with this reasoning is that the theorem asserting the equivalence of consistency and the existence of models is based on a certain assumption--the existence of at least one infinite set.

**This argument is implied by remarks in the "Introduction to the Second Edition" of _Principles of Mathematics_.

ciples of mathematics.

This argument is not wholly persuasive. As Saul Kripke has urged, the necessity of some propositions can be discovered empirically.* The Axiom of Infinity is no clear exception. Couldn't we discover that infinity is in the nature of things? Couldn't we end up with a physical theory that requires infinitely many objects? (Haven't we? See the conclusion of the preceding section.)

One might reply that physical science can only tell us what things are physically necessary, that even if the equations of physics require an infinity of things, it remains metaphysically possible that there be no infinite sets.

Those who adopt this reply are obliged to explain the distinction between physical necessity and metaphysical necessity. Moreover, they must establish that the accepted propositions of mathematics are metaphysically necessary in the sense explained. These are no mean tasks. All considered, I think some hope remains that the Axiom of Infinity can be given an empirical justification.

In view of the flurry caused by the Axiom of Infinity, one might suppose that it plays an essential role in the foundations of mathematics. Actually, this is far from clear. Set theory serves two legitimate foundational purposes. It is needed in the specification of the systems of mathematical theories and in the higher-order development of those theories. These purposes are fulfilled by axioms of hypothetical existence, axioms which tell us that if the systems exist, so too do certain sets

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*"Identity and Necessity"

founded on them. Neither purpose requires an axiom categorically asser- the existence of an infinite set.*

Why then does the Axiom of Infinity appear so frequently in foundational studies? Russell provides one answer:

> "In practice, a great deal of mathematics is possible without assuming the existence of anything. All the arithmetic of finite integers and rational fractions can be constructed; but whatever involves infinite classes of integers becomes impossible. This excludes real numbers and the whole of analysis. To include them, we need 'the axiom of infinity'..."
> Principles of Mathematics, "Introduction to the Second Edition", p. viii.

According to Russell, the existence of infinite sets is required in the development of the theory of real numbers. As Russell conceives it, the theory of real numbers is about particular objects, real numbers. Since that theory involves apparent quantifications over infinite sets of real numbers, it appears to involve commitments to the existence of infinite sets.**

I have urged that it is wrong to conceive the theory of real numbers as a theory of particular objects. It is a theory of all real-number-like systems, or r-systems. Thus a statement which appears to assert the existence of a certain infinite set of real numbers really asserts the existence of such a set in every r-system. Such a statement is true even if the Axiom of Infinity is false, for, in that case, the statement is vacuously true of all r-systems.

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*There is one fine point: Higher-order development does not reach infinite ranks. If it did, infinite sets would be needed even in the case of finite systems.

**Russell himself construes real numbers as infinite sets of rationals. Thus even his quantifications over real numbers involve commitment to the existence of infinite sets.

More generally, the Axiom of Infinity has been used to provide mathematical objects for mathematical theories. If I am right, there are no mathematical objects, so the axiom is not needed for this purpose.

Can we dispense with the Axiom of Infinity in mathematical reasoning? Mathematicians appear to use the axiom in presenting counterexamples to putative theorems. For example, suppose someone asserts, "Every Boolean algebra is atomic." A mathematician might reply, "No. Consider an infinite Boolean algebra of the following sort..." If the Axiom of Infinity is false, then the mathematician has no counterexample, and the statement stands. But in actual practice, such replies are immediately devastating, and no one bothers to check whether the Axiom of Infinity is true.

How are we to make sense of this phenomenon? One explanation has it that the Axiom of Infinity is accepted by mathematicians as uncontroversially true. Is there a more economical alternative?

According to one alternative, what is devastating to the putative theorem is not the actual existence of the counterexample described. It is the <u>recognized possibility</u> of such a counterexample. Perhaps all Boolean algebras are atomic, but until all the infinite counterexamples are ruled out, this cannot be accepted as mathematical fact.

I leave open the interpretation of "recognized possibility". It may well be that nonatomic Boolean algebras are possible in some absolute metaphysical sense. But certainly they are "epistemically" possible. Nothing we know rules them out. As long as this remains the case, we should not believe that all Boolean algebras are atomic.

Since this alternative interpretation seems tenable, the Axiom of

Infinity does not appear essential to mathematical reasoning.

In summary, we have found no very good reason to include the Axiom of Infinity in the set theoretical foundations of mathematics. I have rejected the argument that the Axiom of Infinity is contingent and so not mathematical. But the Axiom of Infinity is not required to ensure the existence of mathematical objects, nor is it required to produce infinite counterexamples. If we conceive of mathematics as the general study of abstract structure, then it is fairly easy to see that there is no mathematical need for an absolute assertion of the existence of an infinite set.*

4. The Classical Conception of Set

Mathematics is commonly viewed as a Platonistic enterprise. But this view rests partly on the misconception that mathematics is concerned with special abstract objects, numbers and pure sets. Once this misconception is cast aside, can we still judge that mathematics is Platonistic?

I have described mathematics as the study of abstract structure. One might be tempted to say that therefore mathematics is about certain Platonic entities, abstract structures. However, this conclusion is by no means forced upon us. The doctrine that mathematics is about abstract structure can be interpreted to mean that mathematics makes no distinctions between abstractly isomorphic objects, and abstract isomorphism can

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*My adoption of a nonabsolutist view on the existence of infinite sets is due in large part to a study of the considerations advanced in Michael Jubien's "Formal Semantics and the Existence of Sets".

be defined without reference to something, abstract structure, which is shared (see D4 and D5).* Thus mathematics can be about abstract structure without there being something, abstract structure, which it is about!

Perhaps the systems of mathematical theories are Platonic entities. Although they are not specially mathematical objects, mathematical theorems apply to them. Maybe this fact makes mathematics Platonistic.

That systems of mathematical theories are Platonic entities should not be accepted without question. To be sure, I have described these systems in Platonistic terms: An ω-sequence is a relation, a group is a binary function, an r-system is a triple of a unit, a function, and a relation, and a topology is a property of sets. Yet use of Platonistic terminology does not guarantee that the things described are the universals we would normally expect them to be. We have not yet ruled out the view that sets, relations, and functions are symbolic constructions. That view is consistent with the formal set theory consisting of A1s-A5s and A1c-A5c.

What debars us from regarding sets, relations, and functions as symbolic constructions, and what indeed makes modern mathematics Platonistic, is the fact that the standard, <u>classical</u> conception of set is far richer than the formal theory. Although we can imagine models for the formal theory in which sets, relations, and functions are symbolic constructions, these models are very unlike the classical realm of sets,

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*If there were universals, abstract structures, which abstractly isomorphic objects share, they would outstrip the ranks of set theory, the types of type theory. Abstract isomorphisms, on the other hand, have been defined as functions in the ordinary sense.

relations, and functions.

The richness of the classical conception stems in part from an informal extension of the Separation Schema:

A3s' $\forall x(\underline{set}.[x] \rightarrow \exists y(\underline{set}.[y] \;\&\; \forall z(z \in y \leftrightarrow z \in x \;\&\; \varphi_z)))$

Construed formally, this tells us that for any set x, and any formula φ of formal set theory, there is a subset of x consisting of just those members of x which satisfy φ. According to the classical conception, the Separation Schema can be extended so that it covers subsets specifiable in ordinary language or in scientific language: For any set x, and any meaningful formula φ, there is a subset of x consisting of just those members of x which satisfy φ.

This extension is not part of the formal theory, since it involves quantification over expressions outside the formalism. Nevertheless, it meets with nearly universal acceptance. Without it, set theory could not be linked with practical applications in a fully satisfactory way. We would have trouble dealing with, for example, sets of symbols or sets of fundamental particles.

Although natural language is certainly less limited than formal set theory, it is still not rich enough to yield the classical conception. Even the extended Separation Schema does not tell us precisely what the subsets of any given set are. Because every language is limited in some respect, there might be subsets that have no linguistic specification. Thus there is more to be said about the classical conception.

The classical conception is an <u>absolute</u> conception.* Language can

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*My use of "absolute" should not be confused with any existing technical uses.

change with time. It can increase in expressive power. But the subsets of a set are fixed for all time:

P5 For any set x and times t and t', the power set of x at t is identical to the power set of x at t'.

Indeed, we might say that subsets are fixed across all possible worlds:

P5' For any set x and worlds w and w', the power set of x in w is identical to the power set of x in w'.

These principles sever any essential connection between sets and symbolic constructions. They are expressions of the Platonistic character of the classical conception.

The classical conception is a <u>maximally</u> <u>rich</u> conception. Indeed, from P5 and Extended Separation, we can show that no matter how language increases in expressive power, we shall never be able to specify a subcollection of a set which is not one of its classical subsets. And that is not the whole of maximal richness: <u>The subsets of a set are as many as they could conceivably be</u>, <u>given the Axiom of Extensionality</u>.*

The significance of this can be illuminated with an example. Suppose we are given an infinite set x of two-membered sets. Suppose each member of x consists of a pair of empirically indistinguishable objects. ** Is there a set consisting of exactly one member from each member of x? It seems that a maximally rich conception must admit the existence of many such sets, for there is nothing to rule them out. (Notice that each such set is simply a subcollection of the union of x.) Yet

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*In "Mathematics Without Foundations", Hilary Putnam gives a novel account of maximal richness, one framed in terms of the possibility of concrete arrow diagrams.

**Such as socks. See Bertrand Russell's <u>Introduction to Mathematical Philosophy</u>, p. 125-7.

these sets involve infinitely many arbitrary "choices", and so cannot be specified in any human language. Thus the classical conception seems to admit subsets that could not be specified in any language whatsoever!

The specific point of this example is that the classical conception supports the Axiom of Choice: For any set x of nonempty sets, there exists a choice set c consisting of exactly one member from each member of x.*

Extended Separation, Absoluteness, and the Axiom of Choice are helpful clarifications of the classical conception. Do they single it out uniquely? If Absoluteness is taken in form P5, but not in form P5', then the classical conception is probably not singled out. If it is taken in form P5' as well, then whether we have succeeded in isolating the classical view depends on how strong the notions of <u>possible world</u> (P5') and <u>language</u> (Extended Separation) are. They might well be strong enough to tell us how sets are arranged in a classical universe.

In any event, these principles distinguish the classical conception from all constructive conceptions. Since the classical view underlies mathematics, mathematics is thereby Platonistic--not of course in the sense that it requires special Platonic objects, but in the sense that it postulates a nonconstructive realm of subsets for any given set.**

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*This seems to follow from P5'. For example, although Russell's socks are indistinguishable in this world, there is a possible world in which exactly one member of each pair has a hole. In that world, the extended version of Separation gives us a choice set, the socks with holes. P5' tells us that the choice set exists in this world as well!

**Notice that the divergence between the classical and constructive conceptions appears only in the case of infinite sets.

Notes

One nick in the armor of Platonism is the fact that so little of the classical conception can actually be manifested in practice. Extended Separation is certainly useful in applications. Absoluteness, on the other hand, is of little practical value. About all that can be said in its favor is that its strong form, P5', does support the mathematically useful Axiom of Choice.

Whether a coherent view can be constructed which accepts Extended Separation but rejects Absoluteness and the Axiom of Choice remains an open question.

5. Attacks on the Classical Conception

In recent years, some philosophers have attacked the classical conception on the ground that it is hopelessly "unclear". In "The Thesis that Mathematics is Logic", Hilary Putnam adopts this position and gives several considerations in support of it.

Suppose we are given a countably infinite set S. Then the classical conception is supposed to give us a definite understanding of what the subsets of S are. Putnam denies that we can acquire such an understanding. He points out that there are uncountably many subsets of S. Hence in any human language, most of these sets are indefinable. Assuming that we have an understanding of definable sets, Putnam does not see how we can ascend to an understanding of sets definable or not. Thus he believes that the general notion of a set remains unclear.

According to the classical conception, every set theoretic question has an answer in the realm of all sets. In particular, the Continuum

Hypothesis has a definite truth-value. Putnam points out that the Continuum Hypothesis is independent of the axioms of each of the usual formal set theories. He says further that both it and its negation are compatible with reasonable extensions of the (unclear) classical notion of set. So Putnam concludes that the classical conception is not clear enough to endow the Continuum Hypothesis with a definite truth-value.

In an attempt to remedy the deficiencies of the classical conception, Putnam suggests a "postulational" approach to set theory. He seems to be reasoning as follows: The classical notion of set is unclear. Various clarifications are possible--we need only add to the postulates sets are supposed to satisfy. But there is no one proper clarification. Therefore, mathematics does not aim to discover truths concerning a definite realm of things, sets. It draws the consequences of various systems of postulates, whether those are postulates concerning sets or not. That is, mathematics does not investigate hypotheses of the form 'H($\in$)', where "$\in$" is the symbol for set membership. Instead, it concerns itself with hypotheses of the form 'If A(R), then H(R)', where A is a conjunction of postulates, and R is a variable over all notions whatsoever.

Putnam's suggestion certainly clashes with mathematical practice. In the pursuit of mathematical theories, a set theoretic background is always presupposed, not postulated. Moreover, the background is the same from theory to theory, at least there is no detectable difference. And this common set theoretic background does not deviate from the classical conception.

Taken as a prescription for future mathematical practice, Putnam's suggestion does not seem to be a very good idea. We can indeed study the consequences of different formal set theories. But Putnam would have us study alternatives to the classical conception of set. Since that conception and its competitors are not purely formal, it is difficult to see how this would be done. We could not merely examine various axiomatic systems.

Nor does Putnam's "undecidability" argument seem strong enough to warrant the change in practice. Putnam says that the models used to show the independence of the Continuum Hypothesis represent reasonable extensions of the standard conception of set. This seems false. It is highly unnatural to think that sets are limited to Godel's "constructable" sets.* Even less natural are the models in which the Continuum Hypothesis has been proved false. No one has really shown that there are ways of filling out the standard conception which yield different truth-values for the Continuum Hypothesis. The independence proof pertains to formal set theory, but the standard conception is not formal.

Of course, "informality" is not a magic word which decides all questions. No one working within the classical conception is quite sure what the truth-value of the Continuum Hypothesis is. Some feel that it is false, but even they are not willing to guess what the power of the continuum really is.** Does this show that the classical conception is unclear?

From the fact that we are uncertain of the truth-value of the Continuum Hypothesis it does not follow that the Continuum Hypothesis lacks

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*As noted by Paul Cohen, Set Theory and the Continuum Hypothesis, p. 150-1.

**See Kurt Godel's article, "What is Cantor's Continuum Hypothesis?", and Paul Cohen, op. cit., p. 150-2.

a truth-value. Hence we have not discovered serious unclarity of the sort Putnam has in mind.

The second problem Putnam raises is based on the observation that not all subsets of a countable infinite set are definable. Thus even if we have an understanding of definable sets, we need not have a clear notion of sets definable or not. Here too a reply is possible. When it is said that not all subsets of a countably infinite set are definable, it is meant that there can be no one human language in which each is defined. It is not meant that there is a subset which is not definable in any possible human language. To make this clearer, suppose we are aware (somehow) of the existence of an ω-sequence. We might, at least in principle, construct a machine generating an infinite random sequence of "1"'s and "0"'s. If we do so, then we can define a predicate φ which is satisfied by the n-th member of the ω-sequence if and only if the machine's n-th output is a "1". The extension of this predicate is a subset of the ω-sequence. But it could be <u>any</u> subset of the ω-sequence, depending on what happens. So there is no subset of the sequence which is indefinable in principle. This suggests, contrary to Putnam, that we do have a clear notion of the subsets of an infinite sequence. Although some subsets must remain undefined, none are indefinable in principle. Thus the gap between the understanding of defined and undefined subsets (none are really indefinable) is not so great as Putnam indicates.

Perhaps an even better course is to reject Putnam's complaint altogether. Why should our inability to specify all the subsets of a given set entail a deficiency in our understanding of what the subsets of that set are? Surely it is possible to understand a kind of thing without being able to specify all the individuals of the kind. Putnam's reasoning makes sense only if sets are things which must admit of specification. But to assume that sets must admit of specification is to opt in favor of a constructive viewpoint, and to beg the question against the classical conception.

Where Putnam points out difficulties for the classical conception, I see little problem at all. Is there no plausibility in the idea that the classical conception is unclear?

The feeling that the classical conception is unclear probably arises from the fact that this conception is not formalizable. In particular, the formal Separation Schema does not quite yield the classical notion of a subset. However, our inability to express the classical conception using formal devices does not necessarily signify any deficiency in that conception.

The Incompleteness Theorem shows that even elementary arithmetic is not formalizable. Yet not even the Intuitionists would maintain that arithmetic is "unclear".

Moreover, the classical conception can be "filled out" informally by saying that it is a maximally rich conception, that the subsets of a set are as many as they possibly could be, given Extensionality. This informal elaboration may not satisfy formal philosophers, but they have not yet found any serious weakness in it.

In "Mathematics Without Foundations", Putnam himself gives a novel expression of the informal content of the classical conception, and he abandons the position I have criticised here.

Notes

The precise result concerning the continuum is this: Assuming the existence of a standard model for Zermelo-Fraenkel set theory, if $\aleph_\tau$, $\tau \geqslant 1$, is any cardinal in the minimal model M, then there is an extension of M in which either (1) there is a one-to-one correspondence between the power set of ω and $\aleph_\tau$(where $\aleph_\tau$ does not have cofinality ω), or (2) there is a one-to-one correspondence between the power set of ω and $\aleph_{\tau+1}$ (where $\aleph_\tau$ does have cofinality ω).*

This result has suggested to some that there is no such thing as "maximal richness", that there is an unending chain of increasingly rich models (or possible models) of set theory, and, in particular, that there is no model in which the continuum is maximally rich, in which ω has all the subsets it possibly could have.

I myself am not inclined to draw this conclusion. Cohen's specific result concerns only <u>cardinals of the minimal model</u>. None of these cardinals is absolutely uncountable. Thus, for example, the result does not assure us of the possibility of a model in which $\overline{\overline{\underline{P}(\omega)}} = \aleph_2$, where $\aleph_2$ is the <u>real</u> second uncountable cardinal.

In all fairness, I should say that the insolubility of the continuum problem has had no great practical effect on contemporary mathematics. The Continuum Hypothesis itself, $\overline{\overline{\underline{P}(\omega)}} = \aleph_1$, has some odd results, but it does not overturn any classical theorems.** Probably other hypotheses concerning the cardinality of the continuum are equally harmless. Thus

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*From Cohen's <u>Set Theory and the Continuum Hypothesis</u>, p. 95-6, 129, 134.

**For some of the odd results, see Godel's "What is Cantor's Continuum Problem?".

there is no pressing need to decide which hypothesis is right.

Maybe some day the continuum problem will be regarded as an avoidable one. However, in the context of present-day foundations, it naturally occurs to us.*

6. What are Sets?: An Identity Theory

The arrangement of the set theoretic universe is not nearly so problematic as the nature of its elements. Sets are commonly described as "collections", but this view is fraught with difficulties. Two of these difficulties are noted by Russell in Introduction to Mathematical Philosophy: If sets are collections, it is hard to see why there is an empty set and why a set of one member differs from that member (ch. XVII, p. 183).

Moreover, the notion of a collection is not antecedently clear. The notion is fairly manageable if we mean by collection a number of things gathered together in one place. But obviously there are sets which are not collections in this sense.

Of course, one might adopt the view that sets are theoretical entities of a unique and fundamental sort. Because they are theoretical entities, they cannot be described in fully ordinary terms. Because they are unique and fundamental, they cannot be described in independent technical terms. Nevertheless, they are not much more suspicious than other theoretical entities. Although we do not observe them directly, they do play a certain role in scientific and mathematical theory.

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*I am indebted to Michael Jubien for raising the issues contained in these Notes.

Against such views, Russell argues that sets cannot be part of the "ultimate furniture" of the universe:

> "If we had a complete symbolic language, with a definition for everything definable, and an undefined symbol for everything indefinable, the undefined symbols in this language would represent what I mean by 'the ultimate furniture of the world.' I am maintaining that no symbols either for 'class' in general or for particular classes would be included in this apparatus of undefined symbols." IMP, ch. XVII, p. 182 *

Russell goes on to describe a clever procedure for eliminating talk of sets in favor of talk of properties.**

Unfortunately, Russell's description of a proper ontology is seriously incomplete. For what does he mean by "definition"? If by a definition he means a rule for replacing one symbol with another which is necessarily equivalent to it, then it does seem that he has "defined away" sets. But under this sense of definition, there are no absolute indefinables. What is undefined in one complete symbolic language is defined in another. For example, one complete symbolic language might contain a symbol for each precise distance relation, x is r units from y. Another might contain no primitive symbols for these relations, but only for relations of the form x is less than r units from y. Thus a strict application of Russell's views leaves us with unpalatable alternatives. Either we conclude that the "ultimate furniture" is language-relative, or we conclude that the world has no "ultimate furniture" whatsoever.

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*Where Russell says "class", we may read "set", since Russell does not include proper classes in the extension of his term. Russell's theory of classes is not quite like modern set theory--it is "stratified" by underlying type distinctions--but the difference is immaterial here.

**I use the term "property" in place of Russell's "propositional function of one variable". The difference in meaning is negligible, except where Russell describes propositional functions as "symbols". The elimination of sets cannot be accomplished via symbols (see section 4).

On the other hand, suppose by "definition" he means analysis. Then he must do more than present a scheme for replacing talk of sets. He must argue that the replacing terms are more basic than the terms replaced. He must argue that properties are more "respectable" than sets.

It does appear that property-terms are essential to any complete symbolic language. We could not get along with merely the corresponding sets. For example, suppose x is a set which is the extension of distinct properties P and Q. Then attributions of membership in x are no replacement for both attributions of P and attributions of Q.

Yet it does not follow that terms for all imaginable properties are essential to a complete symbolic language. In particular, it does not follow that all the properties needed in the elimination of sets are needed in such a language.

Russell himself provides a glaring example. Consider the choice set which picks out one sock from each of infinitely many pairs, the choice set mentioned in Russell's discussion of the Axiom of Choice (IMP, p. 125-7; also, section 4 of this chapter). This choice set involves infinitely many arbitrary selections, selections between indistinguishable objects. So too does any corresponding property. But intuitively no such property belongs to the "ultimate furniture" of the universe.

It is difficult to state this intuition precisely. By hypothesis, such a property belongs to a thing only if it does not belong to some qualitatively identical thing. Thus attributions of such a property do nothing to describe the qualities of objects. Naturally we might conclude that such attributions are inessential to complete symbolic languages. Thus the properties needed to eliminate the choice sets are no more respectable than the sets themselves!

I conclude that Russell has not succeeded in analyzing sets away. And we might have expected this from the beginning. For each set, Russell postulates an equivalent property. Thus little reduction in complexity is possible, and no reduction in number.

Having given up the idea of analyzing sets away, let us return to an idea that Russell rejected, that sets are identical with certain properties. An identity theory affords no genuine ontological gain, but it does allow us to simplify our notation. The notation for membership is adsorbed by the notation for simple predication. Of course, Russell's logical construction theory also simplifies notation, but the notation which replaces membership is far more cumbersome than it would be under an identity theory.

Russell says that for every set, there are many properties which are exemplified by just the members of that set. Russell rejects identity theories because he sees no reason why one property should be the set, and the other equivalent properties not. If we possessed a general means for singling one property out of each class of equivalent properties, then perhaps Russell's objection would be answered.

A finite set $\underline{x}$ can be specified by an enumeration of its elements: $\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_n$. To this way of specifying the set, there corresponds a unique property, the property expressed by the open formula: $a = \underline{a}_1 \vee a = \underline{a}_2 \vee \ldots \vee a = \underline{a}_n$. Such properties may be called enumerative properties. The view I shall propose is that sets are these enumerative properties.

The foremost objection to this view is that infinite sets cannot be enumerative properties, since there can be no enumerating formulae in the case of infinite sets. I shall answer this objection by giving an account of enumerative properties that makes no reference to formulae. Thus the absence of formulae for infinite sets is no bar to the existence of the enumerative properties.

I understand that properties are identical if it is necessarily true that they are exemplified by the same things.* Thus to specify a property, it is sufficient to say what its instances are in every possible world.

Consider the enumerative property corresponding to the set $\{a, b\}$. It is the property of being identical to a or identical to b. In any possible world in which this property exists, it has exactly the instances a and b. That is, it has its instances essentially.

In general, an enumerative property is a property such that anything which could be an instance of it is essentially an instance of it. Obviously such a property might be exemplified by infinitely many things.

A detail remains. Given a set of objects, there might be more than one property such that (1) every one of its possible instances falls in that set, and (2) every one of its possible instances is essentially an instance of it. This is so because two such properties might not exist in the very same worlds. Hence an existence condition must be added if we are to be assured of a unique correspondence between enumerative properties and sets.

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*I do not maintain that predicates which express the same property have the very same "meaning". In order to have the very same meaning, the predicates must be intensionally isomorphic. See Rudolph Carnap's *Meaning and Necessity*, ch. I, s. 13-15.

The condition to require is that an enumerative property exists wherever all its possible instances exist and are "collectable". The instances are collectable in a possible world just in case there is some property which they share in that world:

D13 P is an enumerative property =df P is a property such that (1) if for some possible world w and possible object a, a has P in w, then for any possible world w' in which P exists, a has P in w', and (2) P exists in every world w such that all possible objects which have P in some world both exist in w and share some property Q in w

This definition paves the way for an identity theory:

P6 Sets are enumerative properties. An object is a member of a set just in case it is an instance of it.

The reader should convince himself that enumerative properties satisfy extensionality.

According to this theory, there is an empty set, namely the impossible property, the property of being non-self-identical. Also, a one membered set is distinct from its sole member, since an object a is distinct from the enumerative property of being identical to a.

Why do Russell and others believe that every set has at least one defining property? My explanation is that they take for granted the existence of "merely" enumerative properties. Thus a property exists even for those sets which are not distinguished by any truly general feature. But given the existence of these enumerative properties, there is no reason not to adopt the identity theory.

Notes

The ideas presented in this section can also yield a theory of <u>relations-in-extension</u> and <u>functions-in-extension</u>. For example, the relation expressed by the open formula "a = a & b = b" might be that relation-in-extension which relates just the pair (a, b). This theory of extensional items has fewer arbitrary features than standard set theoretic reductions to sets of ordered n-tuples. Recall that we do need such extensional items. For example, if a group is to have a proper number of subgroups, then subgroups must be functions-in-extension. We noted this fact in ch. II, s. 1.)

BIBLIOGRAPHY

Benacerraf, Paul: "Mathematical Truth", Journal of Philosophy LXX.19 (1973), p. 661-679

__________, Paul: "What Numbers Could Not Be", Philosophical Review 74 (1965), p. 47-73

Bourbaki, Nicholas: Elements of Mathematics, 1: Theory of Sets (Addison-Wesley, Reading, Massachusetts, 1968)

Carnap, Rudolph: The Logical Structure of the World (1928; 2nd ed., 1961), translated by Rolf A. George (U. of California, 1967)

______, Rudolph: "The Logicist Foundations of Mathematics" (1931) in Philosophy of Mathematics (Benacerraf & Putnam, eds.; Prentice-Hall, 1964), translated by Erna Putnam and Gerald Massey

______, Rudolph: Meaning and Necessity (2nd ed., U. of Chicago, 1956)

Cohen, Paul: Set Theory and the Continuum Hypothesis (W. A. Benjamin, Reading, Massachusetts, 1966)

Frege, Gottlob: Foundations of Arithmetic (1884), translated by J. L. Austin (2nd ed., Northwestern U., 1968)

Godel, Kurt: "What is Cantor's Continuum Problem?" (1947) in Philosophy of Mathematics (Benacerraf & Putnam, eds.; Prentice-Hall, 1964)

Jubien, Michael: "Formal Semantics and the Existence of Sets", forthcoming in Nous

______, Michael: "Ontology and Mathematical Truth", Nous 11 (1977), p. 133-150

Kripke, Saul: "Identity and Necessity" in Identity and Individuation (Milton K. Munitz, ed.; New York U., 1971), p. 135-164

Mendelson, Elliot: Introduction to Mathematical Logic (Van Nostrand Reinhold, New York, 1964)

Pollock, John L.: "On Logicism" in Essays on Bertrand Russell (E. D. Klemke, ed.; Illini Books, Illinois, 1970)

Putnam, Hilary: "Mathematics Without Foundations", Journal of Philosophy 64 (1967), p. 5-22

______, Hilary: Philosophy of Logic (Harper & Row, New York, 1971)

Putnam, Hilary: "The Thesis that Mathematics is Logic" in _Bertrand Russell: Philosopher of the Century_ (Ralph Schoenman, ed.; Allen & Unwin, London, 1967)

Quine, W. V.: "Ontological Reduction and the World of Numbers" in _Ways of Paradox_ (2nd ed., Harvard, 1976), p. 212-220

Ramsey, F. P.: _The Foundations of Mathematics_ (1931)

Russell, Bertrand: _Introduction to Mathematical Philosophy_ (Allen & Unwin, London, 1919)

_______, Bertrand and Whitehead, Alfred N.: _Principia Mathematica_ (1910; 2nd ed., 1925)

_______, Bertrand: _Principles of Mathematics_ (1903; 2nd ed., 1938)

Weyl, Hermann: _Philosophy of Mathematics and Natural Science_ (Princeton, 1949)

White, Nicholas P.: "What Numbers Are", _Synthese_ 27 (1974), p. 111-124

Whitehead, Alfred N. and Russell, Bertrand: _Principia Mathematica_ (1910; 2nd ed., 1925)