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Prediction and unsaturation : an essay on Frege's philosophy of logic.

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PREDICATION AND UNSATURATION:

An Essay on Frege's Philosophy of Logic

A Dissertation Presented

By

William Harvey Walters

Submitted to the Graduate School of the
University of Massachusetts in partial
fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

April 1975

Philosophy

PREDICATION AND UNSATURATION:

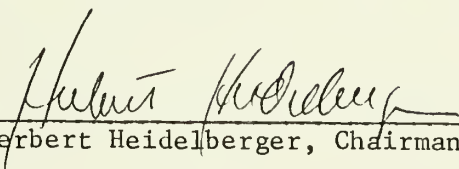
An Essay on Frege's Philosophy of Logic

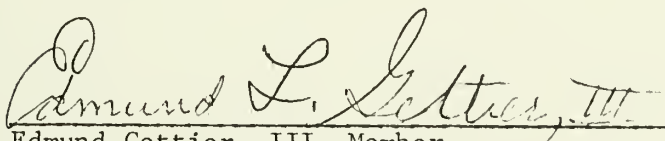
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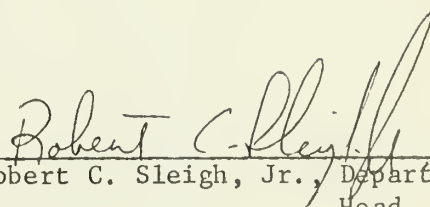
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Of those who have aided me, often in diverse ways, in coming to write this work I should like to mention Daniel Bennett, Herbert Heidelberger, Anne Amster Walters, and especially Philip Nugly.

UNSATURATION AND PREDICATION:

An Essay on Frege's Philosophy of Logic

(April 1975)

William H. Walters

Directed by: Herbert Heidelberger

In the opening chapter I offer an interpretation of Frege's philosophy of mathematics, one which is in keeping with his claim to have argued for the epistemological thesis that arithmetic is comprised of analytic judgments. Central to this interpretation is the suggestion that Frege took a notion of analysis which Kant had directed upon concepts and redirected it upon whole propositional contents. By this means Frege attempts to justify his use, in Die Grundlagen der Arithmetik, of "definitions" which are ontologically augmentative. In particular, Frege holds that the pair

The number of Fs is the same as the number of Gs
There are just as many Fs and Gs

have the same content, though the former makes reference to numbers whereas the latter does not, while the latter is epistemologically prior to the former. In this way Frege gains numbers without needing to call upon non-empirical intuition to account for our arithmetic knowledge. I further argue that Grundgesetze der Arithmetik does not bring any basic change on these matters, that it

could not, if Frege is to continue to hold that arithmetic judgments are analytic and not synthetic a priori.

Chapter II represents an attempt at clarification of Frege's conception of logic. I develop his idea of logic as lingua characterica, where such a lingua would be a canonical notation whose grammar is that of logic and which (potentially) is a universal language, a language in which what can be said can be said. Frege's thinking about logic deepened with his conception of the laws of logic as laws of truth. Frege's thoughts on truth, developed in providing a semantics for his canonical notation, place him squarely in the "correspondence theory of truth" tradition. I suggest that we may look upon Frege's theory of reference -- his semantics of Begriffsschrift -- as an attempt at an exhaustive answer to such a question as how language is "about the world" so that what we say can be true (or false.)

In Chapter III, having expounded Frege's semantics, I examine the origins of this semantics through attempting to display a path which is inviting at its beginnings, and which follows individually appealing steps right up to such "paradoxical" conclusions as that the concept horse is not a concept. And I argue that the unsaturation of concepts which gives rise to such "paradoxes" is more deep-seated in Frege's thought than has been generally acknowledged. Tinkering will not eliminate the "problem"

since unsaturation is an essential element of Frege's semantics of predication -- to reject it is to reject his account of predication, if not his conception of logic. The "problem", most generally stated, is the ineffability of Frege's semantics; the semantics of the lingua cannot be stated within the lingua itself.

Chapter IV opens with a defense, of sorts, of Frege's largely ignored horizontal-stroke. In particular, I argue that Frege was not inconsistent, as is sometimes alleged, in holding both that the denotations of predicates are incomplete and that the denotations of sentences are not. I then turn to inquiring into whether the ineffability which attends Frege's semantics is avoidable through some alternative account of predication. An examination of a representative selection of commonly held views leads me to conclude that Frege is correct in his judgment that a "necessity of language" precludes the statement of an otherwise adequate account of predication in a manner which is consistent with the import of that account. At least this cannot be done while remaining true to Frege's philosophy of logic.

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. . . distinctions between a priori and a posteriori, synthetic and analytic, concern, as I see it, not the content of the judgment but the justification for making the judgment . . . if the truth concerned is a mathematical one . . . [t]he problem becomes . . . that of finding the proof of the proposition, and of following it up right back to the primitive truths. If in carrying out the process, we come only on general logical laws and on definitions, then the truth is an analytic one, . . .

GOTTLOB FREGE

Frege's definition of 'analytic' explicates well what is meant by saying that the logicist (Russell's) philosophy of mathematics has refuted Kant and established the analytic nature of arithmetic truths. There is, however, a subtle difficulty here . . . which is generally overlooked.

ARTHUR PAP

C H A P T E R I

LOGICISM

§1. Frege's Program and a Problem of Definition

The central concern of this essay is, as the title suggests, with Frege's account of predication. On this view, predicative expressions are said to denote concepts. Such denotata are said to be unsaturated; they effectively deflect all attempts at singular reference to themselves. I shall, in Chapter III, be extensively concerned with the whys and therefore of this apparently bizarre doctrine.

"Why bother?" it might be asked, since this view of Frege's would seem to be universally rejected. A desire to straighten the historical record perhaps provides

sufficient justification. Beyond this, however, I have come to believe that many, perhaps most, of those who have dismissed this curious feature of Frege's philosophy have done so too quickly. While some of the consequences of holding that concepts are unsaturated are notorious, others have not been terribly well appreciated. Thus they are not investigated, or not in the light of Frege's general philosophical outlook. Yet upon investigation it will be seen, I believe, that some of these consequences, ones which strike us initially as most distasteful, are nonetheless unavoidable. Or so I wish to argue. I have in mind in particular the ineffability bred of unsaturation. In Chapter IV, which concludes this essay, I shall argue for this, arguing that it is not possible to provide an account of predication the statement of which is consistent with the account provided. At least this cannot be done, so I shall argue, while being true to a certain conception of logic, one on which the laws of logic are laws of truth.

This picture of logic, leaned upon in the later argument, is developed and attributed to Frege in Chapter II. There I explore Frege's idea of a lingua characterica, relate this to his thesis that the laws of logic are laws of truth, and locate Frege squarely in the Aristotelian tradition of viewing truth as correspondence of language with reality.

I begin this essay with some remarks on Frege's philosophy of mathematics. And let us recall that it was Frege's investigations into the foundations of mathematics that moved him to develop his logical theory. Here too my purpose, partially, is to clarify an important episode of our history through procuring a better appreciation of a central aspect of Frege's logicism, one which at first comes into focus over the justification of certain definitions. Additionally, the perspective provided on Frege's logicism should put us in a receptive state of mind to take up next Frege's conception of logic.

In 1879 Frege published a booklet of some 88 pages bearing the title Begriffsschrift, eine der arithmetischen nachgebildete Formelsprache des reinen Denkens.¹ In this he set down modern first-order predicate logic in a complete and consistent form, though he did not attempt to prove either of these results. He used negation and conditionality to give truth-functional logic; universal quantification, for predicate logic. Also included were identity theory, and an essential, though inexplicit, second-order logic. In the "Preface" Frege announced that

1. Begriffsschrift, a Formula Language, Modeled upon That of Arithmetic, for Pure Thought. Translated in From Frege to Gödel, van Heijenoort. Hereafter, Begriffsschrift.

. . . arithmetic was the point of departure for the train of thought that led me to my Begriffsschrift. And . . . I intend to apply it first of all to that science, attempting to provide a more detailed analysis of the concepts of arithmetic and a deeper foundation for its theorems.²

Towards this end he included in Begriffsschrift a definition of the notion of the ancestral of a relation in terms of the logic of his system; this was to figure essentially in the definition of number Frege would later give.

Die Grundlagen der Arithmetik followed five years hence.³ This work divides rather evenly into two parts. It begins with a brilliantly critical discussion of a wide array of views on topics in the philosophy of mathematics, focusing especially upon opinions as to the nature of arithmetic truths and on the concept of number. The latter half of the book is an extended argument for the view, which Frege claims only to have made probable, that

the laws of arithmetic are analytic judgments and consequently a priori. Arithmetic thus becomes simply a development of logic, and every proposition of arithmetic a law of logic, albeit a derivative one.

2. Begriffsschrift, p. 8.

3. The Foundations of Arithmetic. Translated by J.L. Austin.
Hereafter, Grundlagen.

4. Grundlagen, p. 99.

This is the thesis of logicism.

Frege had informed his readers at the outset of Grundlagen that, with regard to the distinctions between the a priori and a posteriori, and the analytic and the synthetic,

. . . I do not . . . mean to assign a new sense to these terms, but only to state accurately what earlier writers, Kant in particular, have meant by them.⁵

Thus it was Frege's aim in Grundlagen to make probable the idea that arithmetic was analytic in Kant's sense. And so to make probable the conclusion that Kant had been mistaken in holding that arithmetic truths were synthetic a priori. We would then be freed from having to evoke and make sense of such ideas as that of Kant's pure intuition, at least as regards our philosophy of number.⁶ That Grundlagen has a general epistemological character was emphasized by Frege in his "Introduction"; while "reception by philosophers will be varied, depending on each philosopher's own position", he hopes that "[s]ome one or another perhaps will take this opportunity to examine afresh the principles of his theory of knowledge."⁷

5. Ibid., p. 3, n. 1.

6. Cf. Parsons, "Mathematics, Foundations of" in Edwards' Encyclopedia, p. 193.

7. Grundlagen, pp. x-xi.

How then did Frege intend to establish his view that arithmetic truths were analytic in nature? Roughly speaking, by convincing us that arithmetic was derivable from logic by purely logical means. Somewhat more exactly, if we suppose we possess an axiomatization of arithmetic, say with Peano's axioms, then, if within an axiomatized system of logic we could derive each of those arithmetic axioms, Frege would regard his thesis as established. Of course such a derivation is impossible. For arithmetic has its distinctive vocabulary with which we speak of numbers and operations upon numbers, and in logic no mention is made of numbers and arithmetic operations, at least not in so many words. Definitions must supplement derivations if the latter are to make the intended point. But then, as Arthur Pap reminds us, "any statement could be made to express an analytic truth in [the] sense [of being derivable from logic alone] if any definition whatever were admissible."⁸ This focuses our attention upon the question of justifying such definitions as we employ in shifting from a purely logical to an arithmetical vocabulary, justifying them in such a way as to sustain the claim that arithmetic is analytic in a non-Pickwickian sense. Actually there are several problems in this area. One very general one

8. Semantics and Necessary Truth, p. 45. .

we might put this way. How are we to get numbers out of logic? Additionally there may arise specific questions over particular definitions, over Frege's definition of the one:one relationship, for instance.⁹ Somewhere in between the very general and the quite specific the question may arise as to the justification, if numbers are to be identified with certain logical objects (sets, perhaps), of identifying the progression of numbers with one rather than another progression of these logical objects. Paul Benacerraf, for one, has argued that since there is no principled way of picking one sequence of sets with which to identify the numbers over other equally suitable candidates, we should reject the claim that individual numbers are particular sets.¹⁰ But I shall not be concerned with this issue. More generally, it will be no part of my aim to resuscitate logicism. My concern will be to show how Frege was attuned to the general problem, which I shall refer to as Pap's problem, of gaining numerical discourse adequate to the workings of arithmetic out of non-numerical discourse, and this in a fashion which opens the way to arguing for the view

9. See Dummett, "Frege, Gottlob" in Edwards' Encyclopedia, p. 234.

10. "What Numbers Could Not Be", Philosophical Review, LXXIV (1965), 47-73.

that arithmetic is analytic in Kant's sense. (The actual argument will proceed over a pavement of major technical accomplishments, but these will not come in for discussion.)

In Grundlagen Frege gains numbers by definition, or what he calls definition. We may distinguish between definitions which allow the elimination of the defined expression and those which do not. Eliminative definition may be accomplished either directly, by providing an alternative expression with which to replace that defined, or paraphrastically, through showing how to avoid use of the defined expression by paraphrasing sentences in which it occurs into others without it. Russell's theory of descriptions popularized this procedure, known as contextual definition, in at least certain segments of the philosophical community.

Michael Dummett believes that contextual definition plays a significant role in the program of Grundlagen.

When Grundlagen is read in its natural sense, without the importation of views stated only in Frege's subsequent writings, it is plain that he regards his principle that words have meaning only in the context of sentences as justifying contextual definitions and took this to be one of its most important consequences.¹¹

And that the use of this type of definition is justified, in Frege's eyes, by what I shall call Frege's

11. "Frege", p. 228.

is better attuned to the function which the Dictum does play for Frege in Grundlagen. Indeed my major aim in this chapter is to bring out how it does perform, which I think has not been well appreciated.

A non-eliminative definition, while not providing means for dispensing with the defined expressions, settles the use of that expression, at least for certain key contexts. Prominent here are recursive, or inductive, definitions. We shall encounter one such, Tarski's for truth-in-L, at the tail end of this essay. For now I simply pass this by, noting only that the "definitions" of Grundlagen which will hold our attention are not of this sort.

The Grundlagen type of definition whereby numbers are gained is non-eliminative however. We may initiate discussion of this by quoting at length a passage of Grundlagen where Frege is discussing Kant on definition.

He seems to think of concepts as defined by giving a simple list of characteristics in no special order; but of all ways of forming concepts, that is one of the least fruitful. . . . A geometrical illustration will make the distinction clear . . . If we represent the concepts . . . by figures or areas in a plane, then the concept defined by a simple list of characteristics corresponds to the area common to all the areas representing the defining characteristics; it is enclosed by segments of other boundary lines. With a definition like this, therefore, what we do -- in terms of our illustration -- is to use the lines already given in a new way for the purpose of demarcating an area. Nothing

essentially new, however, emerges in the process. But the more fruitful type of definition is a matter of drawing boundary lines that were not previously given at all. What we shall be able to infer from it, cannot be inspected in advance; here, we are not simply taking out of the box again what we have just put into it.¹⁴

In part Frege is recommending that definitions need not be limited to the conjunctive; other logical operations may just as legitimately be called upon in providing definitions. But this is not the whole force of the passage. Other logical operations will also simply trace old lines in different patterns, whereas, "the more fruitful type of definition is a matter of drawing boundary lines that were not previously given at all." We must postpone discussion of just what these "definitions" will look like. But Frege implies that they may yield up something "essentially new", and we may anticipate that, in the case of our concern, what is "new" will be numbers. It is through the "more fruitful type of definition" that Frege attempts to solve Pap's problem in a fashion compatible with his program of establishing that the truths of arithmetic are analytic in the sense of Kant. Preparatory to taking this up, let us reminisce a little on the sense of 'analytic' for Kant.

14. Grundlagen, pp. 100-101.

\$2. Analyticity and Analysis

In his Prolegomena Kant states that

. . . there is a distinction in judgments . . . according to which they are merely explicative, adding nothing to the content of knowledge, or expansive, increasing the given knowledge. The former may be called analytical, the latter synthetical judgments.¹⁵

He goes on immediately to say,

Analytical judgments express nothing in the predicate but what has been already thought in the concept of the subject, though not so distinctly or with the same (full) consciousness. When I say: "All bodies are extended," I have not amplified in the least my concept of body, but have only analyzed it, as extension was really thought to belong to that concept before the judgment was made, though it was not expressed. This judgment is therefore analytical.¹⁶

Analytical judgments add "nothing to the content of knowledge." The point is put in the Critique this way: "through analytic judgments our knowledge is not in any way extended, . . ."¹⁷

Companion to this view is the idea that analytic judgments are arrived at through conceptual analysis, "by dissecting given concepts"¹⁸ as Kant says in Prolegomena. In the Critique he speaks of arriving at

15. Prolegomena to Any Future Metaphysics, p. 14.

16. Ibid., p. 14.

17. Immanuel Kant's Critique of Pure Reason, p. 49.

18. Prolegomena, p. 24.

analytical judgments by "merely breaking [a concept] up into those constituent concepts that have all along been thought in it . . ."19 In a similar vein his Logic notes say

Analytic propositions one calls those propositions whose certainty rests on identity of concepts (of the predicate with the notion of the subject.) Propositions whose truth is not grounded on identity of concepts must be called synthetic.20

Here Kant runs together the idea of something being true in virtue of its constituent concepts, and its being known to be true through attention to these concepts. And he seems to assume, here and elsewhere, that something which can be thus known to be true cannot be otherwise known. This is probably false. But these matters do not concern us. The point is the priority of concepts; what is analytic can be known through conceptual analysis.

A third point is that Kant's remarks upon analytic judgments tend to presuppose that all judgments, or propositions, are of what he calls subject-predicate form, where 'All bodies are extended' provides an example.

I shall assume, however, that the analytic-synthetic distinction is intended to be both exclusive and exhaustive. (In these respects this distinction would be like

19. Critique, p. 48.

20. Logic, p. 117.

the a priori-a posteriori distinction.) Exclusive, in that nothing is both analytic and synthetic; nor, I shall assume, analytic at one time (or for one person) and synthetic at another (or for another). This condition requires that what we speak of as analytic, or, alternatively, synthetic, must be thought of as having, or bearing, fixed truth-values. Let us not here worry about what such must, or may, be. The distinction is exhaustive over the domain of truths, or bearers of truth, in that each truth is either an analytic truth or a synthetic truth. (We may conveniently ignore falsehoods.) The analytic-synthetic distinction will thus cover more ground than has been thus far suggested. One trivial addition will be those truths, such as 'Bodies are bodies', where the "predicate" does not explicate (even partially) the "subject", but simply is the "subject". In his Logic Kant says that such truths as these are tautological.²¹

More importantly, the distinction must have application to propositions of any form. As was implicit in the statement of the second point above, I would suggest that there is nothing inherent in Kant's idea of truths knowable through conceptual analysis which restricts its application to propositions of any particular form. 'All

21. See Logic, p. 118.

brothers of sisters are siblings of sisters' would seem to count as analytic as readily as 'Brothers are siblings', or 'Bodies are extended'.

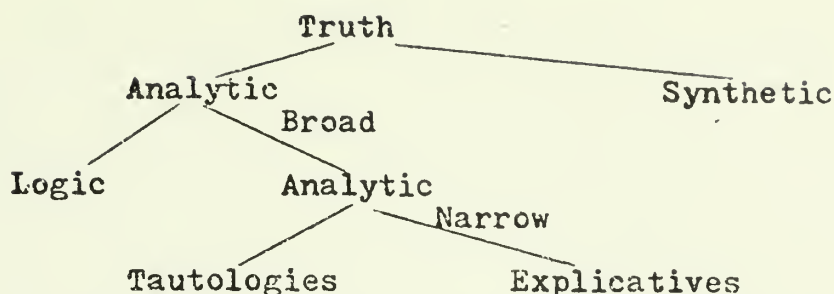
What then of (e.g.) 'If John is a brother of Mary, then John is a sibling of Mary'? Whether this counts as analytic will turn on whether sentences of the form

If S_1 , then S_1

are analytic. And generally, whether the truths of logic are analytic truths. I assume that Kant would (or did) classify logical truths as analytic.²² Certainly this is how Frege took him, for the Grundlagen program presupposes it. And surely if Kant treated the analytic-synthetic distinction as exhaustive he must have considered logic to be analytic. He could hardly have failed to mention that the truths of logic were synthetic a priori if that was his view. But the intuitive justification for classing the truths of logic as analytic seems to be missing.

Before continuing with this, we may summarize the foregoing with the following schematization:

22. "The judgments in the science of logic itself are... entirely a priori. I have not found any statement in Kant that they are analytic and not synthetic, but...I am prepared to assume that they are all properly regarded as analytic." Paton, Kant's Metaphysic of Experience, vol. 1, p. 214.



Let us now consider the question: What, according to Kant, do explicatives and logical truths have in common which warrants according them the same epistemological status? Kant in several places writes as if there was a single source of analyticity and that this was the law of contradiction. Thus in the Prolegomena he states that "[t]he common principle of all analytical judgments is the law of contradiction".²³ The implied claim, that a proposition is analytic if and only if its denial is self-contradictory, may be true, but it hardly seems explanatory. On the one hand, how would we determine that 'Some vixen is not female' is self-contradictory? Presumably we would reason that since a vixen is a female fox the claim in question in effect amounts to stating that something both is and is not female, which is self-contradictory. But clearly the law of contradiction plays a subsidiary role in our being certain that all vixens are female, one subsidiary to the idea of conceptual analysis.

23. Prolegomena, p. 14.

On the other hand, how would we determine that the following is self-contradictory: Though all men are mortal and Socrates is a man, nonetheless Socrates is not mortal. We can derive an explicit self-contradiction from this. But in so doing we will employ various principles of inference. So we shall be justified in calling the claim denied by our initial statement analytic only if we have independent justification for supposing that the principles of inference used in deriving the contradiction take one only from the denial of analytic truths to self-contradictions. (And that the denial of the denial of an analytic truth is itself an analytic truth.) And no such justification is forthcoming. Thus neither the explicatives, nor the logical truths, have their epistemological status explained, at least in any direct way, in terms of the law of contradiction.

I shall not pursue further this matter of a common ground for logical truths and explicatives. It would not seem that a unified account of Kant's analytic judgments would be required for the purpose of assessing Frege's claim to show that arithmetic is analytic in Kant's sense. To this end we need only ask whether Frege's initial resources are analytic in Kant's sense, and whether each step he takes therefrom results, one way or another, in a proposition which should be considered analytic in Kant's sense, and whether in this way we

arrive at arithmetic. Though if we thought that no general account of analyticity could be given, this might diminish in our eyes the epistemological importance of such success as Frege may have achieved with his program.

Before leaving Kant for Frege let us look a bit more closely at those analytic propositions which are, intuitively, most deserving of the label, the explicatives. Let us first note, in passing, what I shall call Kant's conceptual atomism. This is the thought, at least implied in Kant's discussions of analytic judgments, that any given concept of any given proposition has a unique (though perhaps null) analysis into component concepts. Or at least that it is not possible to have differing complete analyses of a given concept, where a complete analysis issues in ultimate constituent concepts, and an ultimate concept is one itself without constituents (i.e., susceptible only of the null analysis.) To provide an analysis of a concept is the same as to define that concept. And we might call a complete analysis a complete definition.²⁴ Kant seems to have this latter in mind when he speaks of "analytic definitions" in Logic as follows.

24. Kant was not as clear as Frege would be that a list of concepts does not constitute a definition. Failure of appreciation on this point forecloses the use of logical operations other than conjunction in one's definitions, which was a complaint Frege lodged against Kant.

All given concepts, be they given a priori or a posteriori, can only be defined through analysis. For given concepts can only be made distinct by making their characteristics successively clear. If all characteristics of a given concept are made clear, the concept becomes completely distinct; and if it does not contain too many characteristics, it is at the same time precise, and from this springs a definition of the concept.²⁵

For he notes,

Since one cannot become certain by any proof whether all characteristics of a given concept have been exhausted by complete analysis, all analytic definitions must be held to be uncertain.²⁶

Analytic truths, at least of the explicative variety, are, we might say, true by definition. But to so remark is in no way to hint of conventionalism.

Explicatives are truths which we may come to know are true through appropriately attending to their component concepts. We can learn, for instance, that vixens are female foxes through learning that the concept of vixen contains both the concept of female and the concept of fox. This is the view. Yet a difficulty lurks. For the description of the case presupposes the learner to possess the concept of vixen, and so, presumably, to already know such trivialities as that vixens are vixens. But if this is so, how would one learn that vixens are

25. Logic, p. 143.

26. Ibid., p. 143.

female foxes, given that the concept of female fox is the concept of vixen? Yet it seems indisputable that this can be done. A person can be in a position of comprehending what to analyze, and yet not possess the analysis. Still we may be puzzled by this combination of knowledge and ignorance.

Concepts, for Kant, have fixed constituents. When we learn that, for instance, vixens are female foxes it seems that, for Kant, what we do is scrutinize the concept vixen so as to somehow apprehend its parts. Yet it is puzzling how we can comprehend the concept and yet fail to apprehend its parts, as if comprehension were like stuffing something in one's pocket paying it no heed. Complementarily, it is unclear on what model we are to imagine the process of attempted analysis. That Kant was alive to such difficulties seems evident from the second passage quoted in this section.

Similar puzzles turn up, though somewhat altered in form, at a crucial juncture of Frege's philosophy of mathematics. In Grundlagen they will arise in connection with definitions of the more fruitful type. Later much the same issues are focused upon a crucial axiom of his system of logic, or so I shall argue further on. The Kantian puzzles of analysis are relocated as a result of Frege's shift from concepts to

propositions as the target of analysis. His success in resolving them will determine, in part, the adequacy of his solution of Pap's problem.

§3. Frege's Solution

Let us begin by recounting and recasting the earlier observation on what Frege need do to establish that arithmetic is analytic in Kant's sense. It is enough if he were to (1) begin with resources which Kant might well accept as purely logical resources, and so also as ones which issue in analytic propositions, and (2) while making no move not analytically implied by propositions already certified as analytic, arrive at arithmetic, by deriving Peano's axioms for instance. To accomplish (1) about all Frege would need to do would be to defend his analysis of statements of generality, such as 'All bodies are extended', in terms of predication, truth-functional conditionality, and universal quantification -- the analysis which is called to mind by the notation: $(x)(Bx \supset Ex)$. To these ideas we need add only those of negation, identity and second-order generality to have the basic ideas of the system of logic presented in Begriffsschrift. It seems unlikely that Kant would dispute that any of these was purely logical. And so the logic of Begriffsschrift, which Frege would later call the fundamental part of logic, would seem to count

as analytic for a Kantian. Anyway I shall assume that this is so.

Grundgesetze der Arithmetik,²⁷ the first volume of which was published in 1893, shows some important changes in Frege's logical system. For the time being we may think of this system as the fundamental logic of Begriffsschrift enriched by set theory. The enrichment occurs via the ill-fated Basic Law V (BLV), the axiom alluded to earlier. Granting that arithmetic can be couched in the notation of set theory and that its axioms so expressed are provable in that theory, to argue that we thereby show arithmetic to be analytic would require a defense of a claim that BLV itself is, in some appropriate sense, analytically true. To this we shall return (§4). First we shall take a closer look at what goes on in Grundlagen.

But let us observe that to show that Frege's logic is essentially Kantian would not itself be to show that Frege and Kant were in agreement on all matters relating to the nature or epistemology of logic. It is possible for two people to agree on the logical truths and yet disagree on how we know them. We shall take up this

27. The Basic Laws of Arithmetic. Translated (in part), with an introductory essay, by Montgomery Furth. Hereafter, Grundgesetze.

topic in Chapter II. Here, in passing, we may acknowledge the presumption that Frege saw himself in substantial agreement with Kant over the epistemology of logic. For he tells us that he is using what he takes to be Kant's notion of analyticity, and he never -- at least during the period of our concern in this essay, roughly up through the early years of this century -- suggests that he thinks of logic, at least in its fundamental part, as other than analytic.

We now turn to the issue of how Frege, in Grundlagen, sought to show that arithmetic was analytic. Philosophically (as opposed to technically) the central passage starts at section 62, which begins with the question: "How . . . are numbers given to us . . .?"²⁸ It may not be immediately clear just what Frege is asking. Our approach will be to let his response instruct us as to his intent. But at the outset we can at least confirm the importance of this question for Frege. In the second volume of Grundgesetze, published in 1902, Frege wrote that "if there are logical objects at all -- and the objects of arithmetic are such objects -- then there must also be

28. Grundlagen, p. 73.
Conversations with Philip Hugly impressed upon me the importance of this section.

a means of apprehending them, of recognizing them."²⁹
 Then, while this volume was awaiting publication, Frege learned from Russell of the inconsistency in the Grundgesetze system of logic. He responded with an appendix to that volume which concludes with this remark.

The prime problem of arithmetic is the question, In what way are we to conceive logical objects, in particular, numbers? By what means are we justified in recognizing numbers as objects? Even if this problem is not solved to the degree I thought it was when I wrote this volume, still I do not doubt that the way to the solution has been found.³⁰

Though the form of words changes, the question of how we are to apprehend numbers, how they are to be given to us, remains in the forefront of Frege's thinking.

Before moving directly to Frege's Grundlagen attempt to answer his question, let us briefly recap certain "results" established by this juncture of the essay. Here we may lean on a summary Frege himself provides.

Let us cast a final brief glance back over the course of our enquiry. After establishing that number is neither a collection of things nor a property of such, yet at the same time is not a subjective product of mental processes either, we

29. Philosophical Writings of Gottlob Frege. Edited and translated by Peter Geach and Max Black, p. 181. The assumption that numbers must be logical objects marks something of a shift in view from that to be found in Grundlagen, and also a hardening of position.

30. Grundgesetze, p. 143.

concluded that a statement of number asserts something objective of a concept.³¹

By "statement of number" Frege intends such statements as 'Jupiter has four moons' and 'Venus has zero moons'. He regarded such use of number words as their "basic use," or their basic use "in the context of a judgment."³² And, whereas he does not say so explicitly, his thought would seem to be this. In the first place we use number words to count. In counting things we are always counting things of some sort. And so when we come to express the result of our countings we offer a statement of how many things there are of the given sort. The statement 'Jupiter has four moons' is such a statement. ("How many moons does Jupiter have?" -- 1, 2, 3, 4 . . . Jupiter has four moons.) Such a statement of number "asserts something objective of a concept" since, in our example, what is being said is that exactly four objects, no more no less, fall under the concept of moon of Jupiter, or more simply that four are the moons of Jupiter.

Continuing with Frege's recapitulation,

We attempted next to define the individual numbers 0, 1, etc., and the step from one number to the next in the number series. Our first attempt broke down, because we had defined only the predicate which we

31. Grundlagen, p. 115.

32. Ibid., p. 59.

said was asserted of the concept, but had not given separate definitions of 0 or 1, which are only elements in such predicates. This resulted in our being unable to prove the identity of numbers. It became clear that the number studied by arithmetic must be conceived not as a dependent attribute, but substantively. Number thus emerged as an object that can be recognized again, although not as a physical or even merely spatial object, nor yet as one of which we can form a picture by means of our imagination.³³

Although the basic use of number words is in attributive constructions such use is not adequate to arithmetic. Briefly, we may think of number words as employed in the attributive construction of "statements of number" as functioning as indices upon numerical quantifiers. Thus, 'Jupiter has four moons' might be symbolized in standard notation as: $(\text{Ex}_4)(x \text{ is a moon of Jupiter})$. Then Frege's point can, I believe, be put as follows. We cannot express such essential facts of arithmetic as that every number has a successor if number words are limited to indexical position of numerical quantifiers. For what would be needed would be to somehow treat the position of (e.g.) '4' in the above as itself open to quantification. But this would be to no longer regard '4' as an indissoluble part of ' Ex_4 '. When we come to state arithmetic

33. Ibid., p. 59.

laws we need numbers;³⁴ numerical quantifiers are inadequate. Thus, though in their basic use number words do not refer to objects, since they function not as singular terms, but rather as attributives modifying sortal terms, or as we have just put it, as indices on numerical quantifiers, numbers nonetheless are objects, Frege concludes. But then how, and with what justification, do we make our acquaintance with numbers, the objects of arithmetic? How, and with what justification, do we make the change from using number words to count with, to using them to refer to numbers?

Continuing once more with Frege's summary remarks,

We next laid down the fundamental principle that we must never try to define the meaning of a word in isolation, but only as it is used in the context of a proposition: only by adhering to this can we, as I believe, avoid a physical view of number without slipping into a psychological view of it.³⁵

This brings us back up to section 62, whose first two sentences run in full,

How, then, are numbers to be given to us, if we cannot have any ideas or intuitions of them? Since it is only in the context of

34. Cf. Frege, "Function and Concept" in Philosophical Writings; for example such remarks as: "The first place where a scientific expression appears with a clear cut reference is where it is required for the statement of a law." (p. 21) Also, the closing remarks of this address.

35. Grundlagen, p. 116.

a proposition that words have any meaning, our problem becomes this: To define the sense of a proposition in which a number word occurs.³⁶

Here, as in the summary, we find Frege referring to what we have called his Dictum. The next stretch of text indicates in an initial way, the role Frege's Dictum is to play in his philosophy of arithmetic, at least in Grundlagen. Frege takes the task at hand to be

to define the sense of the proposition
 "the number which belongs to the
 concept F is the same as that which
 belongs to the concept G";
 that is to say, we must reproduce the content
 of this proposition in other terms, avoiding
 the use of the expression
 "the Number which belongs to
 the concept F".

In doing this, we shall be given a general criterion for the identity of numbers. When we have thus acquired a means of arriving at a determinate number and of recognizing it again as the same, we can assign it a number word as its proper name.³⁷

Abbreviating the sentence 'the Number which belongs to the concept F is the same as . . .' to,

$$NxFx = NxGx,$$

Frege says, in effect, that "we must reproduce the content of this" in non-numerical terms; to accomplish this would be to provide a means of recognizing numbers.

How is this to be done? Frege, citing Hume, puts forward the idea of defining numerical identity in terms

36. Ibid., p. 73.

37. Ibid.: p. 73.

of one:one correlation, the just-as-many-as relationship. This is an idea he will endorse. But, having cautioned that "it raises certain logical doubts and difficulties, which ought not to be passed over without examination",³⁸ he here emphasizes that he is not proposing "to define identity specially for this case."³⁹ Rather the "aim is to construct the content of a judgment which can be taken as an identity such that each side of it is a number",⁴⁰ and so "to use the concept of identity, taken as already known, as a means for arriving at that which is to be regarded as being identical."⁴¹ "Admittedly", he adds, "this seems to be a very odd kind of definition to which logicians have not yet paid enough attention . . ."⁴²

Interrupting Frege, how might we state the definition he has in mind? The thought is that we are to use the idea of one:one correlation to express the content of the previously set-off sentence. When there are just as many Fs as there are Gs, so that the Fs are correlated one:one with the Gs, let us say, simply, that the Fs equal the Gs, underlining 'equal' to signal its somewhat special use.

38. Ibid., p. 74.

39. Ibid., p. 74

40. Ibid., p. 74.

41. Ibid., p. 74.

42. Ibid., p. 74.

Then Frege's definition can be stated this way.

(A) $NxFx = NxGx$ iff the Fs equal the Gs.

It is worth emphasizing that Frege has, in effect, told us that the left and right side of this bi-conditional are the same in content. Otherwise put, an instance of the left side, (e.g.) 'The number of cows is identical with the number of horses', will express the same sense as a "like" instance of the right side, 'The cows equal the horses', which is to say that there are just as many of one as the other.

Continuing with Frege, he says that this "very odd kind of definition", which in fact is one of the "more fruitful type" earlier mentioned, "is not altogether unheard of", as "may be shown by a few examples."⁴³ At this point he launches into a discussion of an example involving parallel lines, which discussion is critical for our understanding of what Frege is trying to show. However I wish to interrupt Frege once more, this time to take cognizance of a remark by Peter Geach.

With our (A) in mind Geach has written that

Given this sharp criterion for identifying numbers Frege thought that only prejudice stood in the way of our regarding numbers

⁴³. Ibid., p. 74

as objects. I am strongly inclined to think
he is right.⁴⁴

With what I take to be the thrust of this remark I am in agreement; on this I shall be more explicit further down. But it is deserving of notice that, as it stands, the remark is both inappropriate and inaccurate. Inappropriate, since, as we have observed, Frege claims to have already established that numbers are "self-subsistent objects" before taking up the problem of how numbers are given to us, which leads him to put forth (A). Inaccurate as regards Frege's views, since, as also noted, Frege in effect tells us that the definition (A) raises "certain logical doubts and difficulties, which ought not to be passed over without examination." Such examination eventually leads Frege to revise, or reformulate, the definition in terms of extensions of concepts. On this Geach writes that "the importance of Frege's doctrine concerning extensions has been grossly exaggerated because it has been thought an essential part of his doctrine concerning numbers".⁴⁵ Geach says that he is going to ignore it, and does. Now, as regards Frege's thinking at the time of Grundlagen I am, on this point, in sympathy with Geach. I shall bring this out through

44. G.E.M. Anscombe and P.T. Geach, Three Philosophers, p. 161.

45. Ibid., p. 158

examining Frege's examination of one of those "doubts and difficulties"; all in due time.

Let us now get back to Frege's discussion of his example involving parallel lines, to which I attach much importance. He writes,

The judgment "line a is parallel to line b", or, using symbols,
 $\underline{a} // \underline{b}$,
 can be taken as an identity. If we do this, we obtain the concept of direction, and say: "the direction of line a is identical with the direction of line b". Thus we replace the symbol // by the more generic =, through removing what is specific in the content of the former and dividing it between a and b. We carve up the content in a way different from the original way, and this yields us a new concept.⁴⁶

Frege seems to have just told us that the pair of sentences

- (a) Line a is parallel to line b.
- (b) The direction of a is identical with the direction of b.

have the same content, but that they "carve up the content" in somewhat different ways. How do these sentences differ? They differ, or seem to, in logical structure. This difference we might show schematically this way:

xRy

$$Tx(Fx) = Ty(Gy).$$

They differ, or seem to, in their logical consequences.

- (a), but not (b), implies something of the form

46. Grundlagen, pp. 74-75.

$(\text{Ex})(\text{xRy})$.

(b), but not (a), implies something of the form

$(\text{Ex})(\text{x} = \text{Ty}(\text{Gy}))$.

And they differ in what we might intuitively speak of as their ontologies, the objects (or entities, in general) to which reference is made within the sentences. For (b) speaks of directions, no mention of which is found in (a) -- or so it seems. There can be no question that Frege regards our pair of sentences as sharing their content. But was he prepared to treat these apparent differences in structure, consequences, and ontology as real ones?

In Begriffsschrift Frege had written that

the contents of two judgments may differ in two ways: either the consequences derivable from the first . . . also follow from the second, . . . or this is not the case. . . . [In the first case] I call that part of the content that is the same in both the conceptual content.⁴⁷

This passage has its unclarities. But on a natural way of taking it Frege would be saying that on his view two sentences are the same in conceptual content just in case they have the same logical consequences. This has the consequence that all logical truths come to have the same conceptual content, and it seems to me doubtful that Frege ever held to that. But if Frege did hold a

47. Begriffsschrift, p. 12.

view with this as a consequence at one time, it was no longer his view come Grundlagen.

That we encounter "a new concept" with (b) of the earlier pair of sentences assures us that the sentences differ in logical structure and consequences. And there can be no doubt that Frege means to say that the process of redistributing the content of (a) to get (b) yields us something new. For in his ensuing remarks he tells us that the basic ideas of geometry "must be given originally in intuition," and whereas intuition provides us with an idea of parallel lines no one "has an intuition of the direction of a straight line."⁴⁸ Rather, "the concept of direction is only discovered at all as a result of a process of intellectual activity which takes its start from the intuition . . ."⁴⁹ Thus sentences (a) and (b), which seem to differ in the ways noted, differ in epistemological status, according to Frege. We shall be in a position to affirm the truth of the second, understanding what we affirm, only if we are in a position to similarly affirm the first. But not conversely. We can know two lines to be parallel and yet not know that their directions are identical.

48. Grundlagen, p. 75.

49. Ibid., p. 75.

It is worth noting a similarity between the pair (a) and (b), and the pair (c) Vixens are vixens (d) Vixens are female foxes of our earlier discussion (§2). In each case we have a pair of sentences which agree in content where, so we assume, it is possible for someone to know that the first is true and yet not know that the second is.

As important as the similarity is this difference. In the second case we could argue that the content is the same on the Kantian grounds that the concept of vixen is analyzable into the concepts of female and fox. Whereas in the first case there is no similar concept, or (proper) portion of propositional content, the analysis of which grounds the claim of common content. Rather the focus of the "intellectual activity" must be the propositional content as a whole. Here we see the influence of Frege's Dictum.

Earlier we noted certain puzzles with Kant's idea of analysis. How is a coupling of knowledge and ignorance such as can occur with the pair (c) and (d) possible? How, knowing one, does someone learn the other? Such questions apply also to Frege's idea of alternate carvings of common content. And he, no more than Kant, directly answers our questions. Such hints as he offers turn on the idea of symbolism. Earlier in Grundlagen he suggests

that some concepts can only be attained by means of symbols when, in discussing Mill, he says we must "distinguish between the symbols themselves and their content, even though it may be that the content can only be grasped by their aid."⁵⁰ And in the essay "On the Justification of a Scientific Concept-Script [Begriffsschrift]" of 1882, he had this to say about symbols and symbolism.

Symbols hold the selfsame significance for thinking as did the discovery of using the wind to sail cross-wind for navigation. Let no one be contemptuous of symbols! A good deal depends upon a practical selection of them. Furthermore, their practical value is not diminished by the fact that after much practice we no longer need to speak out loud in order to think. The fact remains that we think in words or, when not in words, then in mathematical or other symbols.

Without symbols we would further hardly raise ourselves to the level of conceptual thought. In giving the same symbol to similar but different things, we no longer symbolize the individual thing but rather that which they have in common -- the concept -- and the concept itself is first gained by our symbolizing it, for, since the concept is of itself imperceptible to the sense, it requires a perceptible representative in order to appear to us. Thus it is that the sensuous opens up for us the world of the non-sensuous.⁵¹

Now if we recall a remark of Frege's mentioned prior to taking up the parallel lines case:

We are therefore proposing not to define identity specially for this case, but to

50. Ibid., p. 22.

51. English translation: Mind, 73 (1964), pp. 155-160.

use the concept of identity . . . as a means for arriving at that which is to be regarded as being identical . . .⁵²

we might try and capture Frege's thinking this way. From the content of 'Line a is parallel to line b' we are to symbolize out the concept of identity, and (somehow) use this as the means by which we redistribute the remainder of the given content, so that now we can represent the content in question this way: The direction of line a is identical with the direction of line b. We do not thereby alter the propositional content in question; rather we come to apprehend it differently, from a new perspective, by means of an alternate symbolism. This picture, as difficult as it is to clarify, was of much importance to Frege's thought. However I do not want to attempt to assess its ultimate usefulness. All that is required for present purposes is that we appreciate that Frege was committed to the view that the likes of (a) and (b) are, for all their differences, the same in content, and, given that, that we may speak of them as analytically equivalent in a natural extension of Kant's usage of 'analytic'. Assuming this, what proves crucial for Frege's purposes is whether by means of the sort of analytic activity he mentions new objects can come into view.

52. Grundlagen, p. 74.

In the parallel lines case it is not just the concept of a line's direction which newly appears; we also come upon directions, or so it seems. Our conceptual activity, if all goes well, enriches our ontology. And it is by analogous procedures that Frege wishes to argue that numbers are given to us.

Frege's next step is to put the (tentative) results of the previous analytic activity into the form of a definition. He opens section 65 with the remarks,

Now in order to get . . . from parallelism to the concept of direction, let us try the following definition:

The proposition

"line a is parallel to line b"

is to mean the same as

"the direction of line a is identical with the direction of line b."⁵³

This is offered as an instance of the "more fruitful type of definition". And, says Frege, offered in place of the numerical example, our (A), "because I can express myself less clumsily and make myself more easily understood. The argument can readily be transferred in essentials to apply to the case of numerical identity."⁵⁴

Frege then takes up some of those "doubts and difficulties" to which we have twice earlier referred. These lead him to reject the above definition and so also that

53. Ibid., p. 76

54. Ibid., p. 76, n. 1.

of (A). But I believe we can best appreciate the thrust of Frege's thought if we temporarily set aside these "doubts and difficulties" and, assuming the legitimacy of our (A), sketch out how Frege's reasoning could proceed from this to his logicist conclusion. This done we shall come back and look at the "doubts and difficulties" and the move Frege makes in response to them. The thought is that it will then be clear why it was appropriate to defer that discussion for the moment.

Let us suppose that (A) legitimates the use of numerical abstraction operators, (e.g.) 'NxFx', that "we have thus acquired a means of arriving at a determinate number and of recognizing it again as the same . . ." Still, since the right side of (A) involves, at least implicitly, the idea of one:one correlation we have not yet, it may be objected, "defines[d] the sense of" the left side in non-numerical terms. Frege removes this objection by providing, in Grundlagen, a definition of one:one correlation in terms of his Begriffsschrift logic. Given then that we are authorized to use numerical abstraction operators we can -- though this was not Frege's actual procedure -- define 'zero', 'successor' and '(natural) number' using just these operators and Begriffsschrift logic, the third of which will make use of the Begriffsschrift definition of the ancestral of a relation. This done, we shall be in a position to

derive Peano's axioms for arithmetic with the Begriffsschrift logic and these definitions, for 'zero', 'successor', and '(natural) number' are the only primitive, non-logical terms occurring in these axioms. The technicalities of these definitions and derivations are now familiar and will not here concern us.⁵⁵ What I wish to emphasize is their dependence upon (A), and the fact that Frege regards (A) as a case where a propositional content, that represented by the right side, has been carved up in an alternative manner so as to give, in this case, the left side with its apparent reference to numbers. We shall return to this, but now let us pick up again with Frege's discussion.

There is one of the "doubts and difficulties" that Frege spends the most time on; it leads him to revise the earlier definitions. We limit ourselves to commenting upon it. Frege writes that

. . . there is [a] doubt which may make us suspicious of our proposed definition. In the proposition
 "the direction of a is identical with the direction of b"
 the direction of a plays the part of an object, and our definition affords us a means of recognizing this object as the same again, in case it should happen to

55. See the discussions of Charles Parsons in "Mathematics, Foundations of", and "Frege's Theory of Number" in Philosophy in America, edited by Max Black, pp. 180-203.

crop up in some other guise, say as the direction of b. But this means does not provide for all cases. It will not, for instance, decide for us whether England is the same as the direction of the Earth's axis. . . Naturally no one is going to confuse England with the direction of the Earth's axis; but that is no thanks to our definition of direction.⁵⁶

It might be thought that we could simply count as false such claims as that England is the direction of the Earth's axis. But how are we to sort out just such "waste cases" so as to then rule them false? Not on the basis of their form; for we can not assume that a direction of a line will only be referred to just that way, i.e., as a direction of a line.

If we possessed the concept of direction, then, says Frege, with regard to

The direction of a is identical with q
 "we could lay it down that, if q is not a direction, our proposition will be denied, while if it is a direction, our original definition will decide whether it is to be denied or affirmed."⁵⁷ But this idea fails, for "we lack . . . the concept of direction."⁵⁸

We have said that it was Frege's view that we attain the concept of direction through redistributing the content

56. Grundlagen, pp. 77-78.

57. Ibid., p. 78.

58. Ibid., p. 78.

of a statement of parallel lines over the symbols of a certain statement of identity. And Frege did say that "if we do this, we obtain the concept of direction." And yet we have just quoted him as saying that, for all that, we do not possess the concept of direction. Have we caught Frege in a contradiction? I think not. But what is the source of this apparent conflict, and how shall it be resolved?

There is a sense in which we do possess the concept and a sense in which we do not. We do not possess the concept in just this sense: we are not able to set out a procedure to deal with the "waste cases" which does not beg the question. For notice that the last mentioned suggestion of Frege's requires us to be able to determine for any object whether or not it is a direction; thus it tells us no better than the previous attempts that England is not a direction.

We do possess the concept in the sense that we know that England is not a direction, know which are "waste cases", and know that a way has not been provided by which to deal with them. This distinction is implicit in Frege's remark "that we cannot by these methods [i.e., those thus far considered] obtain any concept of direction with sharp limits to its application . . ."⁵⁹ Whereas in

59. Ibid., p. 79.

a sense we have a concept of direction, we have not managed a definition of it for all cases. On this ground Frege deems the initially offered definition inadequate. And so he shifts his approach.

"Seeing that we cannot by these methods alone obtain any concept of direction with sharp limits to its application, nor therefore, for the same reasons, any satisfactory concept of Number either, let us try another way."⁶⁰ Here Frege turns to using extensions of concepts. He gives this definition of direction: "the direction of line a is the extension of the concept "parallel to line a"."⁶¹ And then the analogous definition of number: "the Number which belongs to the concept F is the extension of the concept "equal to the concept F"."⁶²

Definitions in this style satisfy Frege on methodological scores in Grundlagen. But the immediate justification of this latter definition is that (A) follows from it.⁶³ It is with (A), the analogue of the extensionless definition of direction, that the philosophical interest lies.

To pull these remarks together, let me recast a point

60. Ibid., p. 79.

61. Ibid., p. 79.

62. Ibid., p. 79-80.

63. See Parsons, "Frege's Theory of Number".

recently made. We may usefully distinguish between epistemological concerns and methodological concerns, and correlatively, between methods of concept attainment and methods of introduction of concept words into scientifically acceptable discourse. "[T]he concept of direction is only discovered at all as a result of a process of intellectual activity which takes its start from intuition . . ." This is reflected in the extensionless definition, since, according to Frege, we proceed from an intuition of parallel lines to the concept of direction. But this definition is not adequate to science; it lacks "sharp boundaries" and so, from the perspective of an adequate methodology, is not a concept at all. Methodological rigor leads Frege to the definition given in terms of extensions.

Analogously, the concept of number is only gained by means of countings, the results of which are expressed in statements of number, wherein number words have their basic use. Through counting too we learn how to establish that there are, or are not, just as many of one sort of thing as there are of another. And then through redistributing the content of the general case of just-as-many-as judgments we arrive at the concept of number. This is reflected in (A). But this, for reasons given in discussing the concept of direction, does not provide a scientifically acceptable concept of number. And so here

too Frege turns to extensions. He adds, in his later summary, that "this way [i.e., the way of extensions] of getting over the difficulty [the one we looked at] cannot be expected to meet with universal approval, and many will prefer other methods . . . I attach no decisive importance even to bringing in the extensions of concepts at all."⁶⁴ Bringing in extensions was simply a device used (rather unreflectively) to meet a self-imposed methodological requirement.

Still we might wonder how the use of extensions helps us at all. Can we not object that Frege's final definition of the concept of direction does not itself lay down that England is or is not an extension and so no more than previous attempts tells us whether England is, or is not, the direction of the Earth's axis? Frege simply passes this over with the remark "I assume that it is known what the extension of a concept is."⁶⁵ But this is an issue that sooner or later must be faced, especially when, as in Grundgesetze, extensions (or sets) come to play a central role. Here we might just lodge a question. The concept of direction was to be made sharp through using extensions, or sets. But if the concept of set is not reducible to, or explainable in, other terms,

64. Grundlagen, p. 117.

65. Ibid., p. 80, n. 1.

how then could we assure ourselves that this concept has sharp boundaries? To this we shall return somewhat later (§13).

In Grundlagen Frege attached no epistemological importance to his use of extensions; this I have been trying to convey. And this is what I take to be the thrust of Geach's remark, earlier quoted. If so, then it should be clear that, and why, I concur with it. Now, how does this bear upon our guiding problem?

Pap's problem, simply put, is how does one get from speaking in purely logical terms to talk of numbers, and arithmetic generally, and what licenses the move? Frege has an answer. It turns on the idea that a given propositional content can be analyzed in different ways, sometimes with the effect that we come to appreciate the presence of new objects. More specifically, with the statements (or statement forms)

There are just as many Fs as Gs

The number of Fs is the same as the number of Gs we have such a situation. The first is epistemologically prior to the second, but shares with it its content. Thus, having acquired the ability to make just-as-many-as judgments we are in a position to resymbolize such and thereby apprehend numbers. Take this together with the technical accomplishments of the definitions of one:one correlation and the ancestral of a relation in (arguably)

purely logical terms and you have the important elements of the Grundlagen picture. And we can appreciate that it was by no means implausible that Frege had shown the way towards demonstrating that arithmetic was analytic in the sense of Kant. As we have seen, this involves not just working with a richer logic, but also extending Kant's conception of a (narrowly) analytic truth so that (A), for instance, may be regarded as an analytical equivalence.

When Frege gets down to the task of working out the details of the Grundlagen picture various alterations occur. Notably, come Grundgesetze, extensions (or sets) no longer have an arguably incidental role. Frege's philosophy of set theory is now pivotal for his philosophy of mathematics. Still, I would, and shall, argue that, epistemologically speaking, there has been no break with the Grundlagen outlook. This will be a subject of comment in the next section and again in (§13), where it can be discussed in terms of the logical theory of Grundgesetze previously set out (§9).

§4. Logicism and Platonism

Frege never sought to deny nor dispense with the assumption that number words, in their arithmetic employment, refer to numbers; in this sense he was a Platonist in his philosophy of mathematics. Let us sharpen this notion of Platonism. We might say that

someone is a Platonist in this sense if that person holds that numbers exist or assigns being to numbers. And we might say that it does not matter whether the person distinguishes various kinds of being, assigning numbers a kind of being not possessed by, say, tables and chairs. To say that numbers are, is enough. But then the question would arise as to how seriously the person takes the kind of being accorded numbers. If this sort of being is also granted, say, Pegasus, or Hamlet, or the present king of France, we may not wish to associate the name of Plato with the view. For our purposes we may cut through such worries over the worthiness of some kind of being by characterizing a Platonist position as one on which the following two claims are adhered to. (1) There are numbers, each of which has a unique successor. And, (2) in science, existence claims, including that of (1), are univocal. (What is, is.) Thus a philosopher who held that the 'there are' of 'There are tables' and 'There are numbers' was ambiguous, perhaps through assigning numbers and tables to different ontological categories, would not be a Platonist in our sense. Nor would someone be a Platonist under our characterization if they, like Carnap, held that the 'there are' of 'There are numbers' and 'There are prime numbers' was ambiguous. However in the sense here given the term Frege clearly was a

Platonist.⁶⁶ Let us refer to such a Platonism as ontological Platonism.

It seems to be generally possible for two philosophers to agree upon the existence of entities of some broad category and yet fail to agree about how it is we know about these things. For instance, many who have agreed that there are physical objects have disagreed amongst themselves as to whether we perceive them, infer them, construct them, or what. There is, however, a certain kind of epistemological attitude prominently associated with ontological Platonism. It is the attitude William Kneale is thinking of when he writes

many . . . accounts of a priori knowledge have been inspired by Plato's notion of contemplation . . . as a kind of intellectual gazing in which the soul may read off facts about super-sensible objects.⁶⁷

This picture is especially associated by philosophers with mathematicians waxing philosophical. Here is one confirming instance from the writings of G.H. Hardy,

I believe that mathematical reality lies outside us, that our function is to discover or observe it, and that the theorems

66. This could be objected to on the grounds that Frege's logical theory is essentially a many-sorted logic. However, there is only one sort of objects. In any case my main concern here is not with ontology, but epistemology.

67. William and Martha Kneale, The Development of Logic, p. 636.

which we prove and which we describe grandiloquently as our 'creations' are simply our notes of our observations.⁶⁸

It is this picture from which I wish to disassociate Frege; he is not, as I shall put it, an epistemological Platonist.

Thinking still of Grundlagen and our (A) I would contrast Frege's thought with the following sort of picture. We can imagine someone viewing (A) as com- pendiously correlating previously unconnected accomplish- ments. On the one hand we have learned to count, express our results in "judgments of number", and so come to comprehend just-as-many-as statements. Independently of this we have acquired some acquaintance with numbers and their properties. And now, putting both hands before us, we realize that an intimate relationship binds the two together, and so come to appreciate the truth of (A). On Frege's picture, having come to comprehend a just-as-many-as statement we are able to recarve the content of it so as to re-express this content by means of a state- ment of numerical identity. Such a transformation is reflected in (A), and it is by such intellectual activity that we apprehend numbers. This picture supports the claim that (A) is an analytical equivalence; the other does not.

68. G.H. Hardy, A Mathematician's Apology, pp. 123-124.

In Grundlagen the likes of our (A) are called definitions; they are those of the "more fruitful type" by means of which we gain something "essentially new". In Grundgesetze however only direct eliminative definitions are counted as definitions.⁶⁹ Thus while in Grundlagen "definitions" were ontologically augmentative, in Grundgesetze all ontology must come through the axioms. Still I regard this more as a shift in the application of 'definition' than an indication of any fundamental divergence from the Grundlagen picture. In particular I would argue that the same type of justification which Frege provides in Grundlagen for (A) lies behind the introduction of Basic Law V of the system of Grundgesetze. BLV is the axiom which enriches Frege's fundamental logic with a theory of the extensions of concepts. We may convey the gist of this axiom, while avoiding for the present Frege's own notation and the intricacies of its interpretation, with the use of contemporary class notation as follows.

$$(B) \quad \hat{x}(Fx) = \hat{x}(Gx) \quad \text{iff} \quad (x)(Fx \text{ iff } Gx).$$

This may be read: The class of Fs is the same as the class of Gs just in case all and only Fs are Gs.

69. "We introduce a new name by means of a definition by stipulating that it is to have the same sense and the same denotation as some name composed of signs that are familiar." Grundgesetze, p. 82.

Dummett, supposing a shift in Frege's practice in the use of contextual definitions from Grundlagen to Grundgesetze and regarding the role of Frege's Dictum in Grundlagen as justifying the use of contextual definition, finds that the Dictum "has no place in Frege's later philosophy . . ." ⁷⁰ I disagree. As has been already implied, I see Frege's Dictum supporting (B) of Grundgesetze in much the fashion as it had (A) of Grundlagen. However Dummett has another reason for supposing Frege's Dictum to be of diminished importance in the later philosophy. This is that "it accords a distinctive position to sentences which he was no longer prepared to recognize." ⁷¹ This, I believe, expresses a genuine insight; to such matters I shall return somewhat later (§15).

I have indicated that with (B), our simplified version of BLV, we have expressed the result of carving up the propositional content represented by the right side so that something "essentially new" comes into view; this is represented on the left side with the statement of set identity. I must defer offering such direct textual support as I can for this interpretation until the logical system of Grundgesetze has been presented. In (§13) the matters under discussion will come up again

70. Dummett, "Frege", p. 233.

71. Ibid., p. 233.

and textual citations made. But in the meantime I can do something better than providing direct textual support; I shall argue that this interpretation must accord with Frege's thinking.

In his "Introduction" to Grundgesetze Frege, having indicated his concern with "the epistemological nature" of the laws of arithmetic, informs us that

with this book I carry out a design announced in my Grundlagen der Arithmetik of 1884. I wish here to substantiate in actual practice the view of Number I expounded in [that] book.⁷²

The "design" of Grundlagen was, of course, to at least make probable that the laws of arithmetic were analytic truths. This, we may suppose, remains the program, though Frege has by and large now dropped the Kantian vocabulary. If we have been on the mark in our portrayal of Frege's reasoning in Grundlagen, we would expect to find in Grundgesetze some juncture at which a transition is made from one mode of discourse to another across a bridge of a common propositional content, a content analyzable first in terms of one mode, subsequently in terms of the other. It is in this way I would have us regard BLV. Let us look at this again, or rather the approximation (B), and set aside our knowledge of its paradoxical implications. How might we picture to ourselves how someone might come

72. Grundgesetze, p. 5.

to grasp this "law"? It might be held that, on the one hand, we may acquire a knowledge of the logical properties of concepts through familiarizing ourselves with fundamental logic, and on the other hand, and independently, we may gain some appreciation of sets and their relations one to another, and then at some point we consider the two together and realize that a systematic connection holds between the domains of concepts and classes which we give expression to with (B), or BLV. As before, this is a picture from which I would disassociate Frege, for if such were his view, the set theory which issues from BLV could not possibly be called analytic in anything like Kant's sense, for BLV itself would surely have to count as synthetic a priori. Hence arithmetic, which is to be made to appear within the resultant theory, would be synthetic a priori also. If arithmetic is to be shown to be analytic, BLV must be thought of in a manner analogous to that discussed with (A). It functions as a general rule for the redistribution of content of certain forms of sentences so that new entities, new objects of logic, come into view. (In this it differs from the other Grundgesetze axioms.) The ultimate basis of our apprehension of numbers is through a generalized analytic equivalence of statements of concept co-extensiveness and statements of set identity. Platonic awareness of sets, or numbers, plays no role.

Logicism is often stated as the thesis that arithmetic is reducible to logic. Then it is observed that for this to be true, or plausible, logic must embrace set theory. So the question arises as to the warrant, or merit, of counting the truths of set theory as truths of logic. For an understanding of Frege's thought this is not the best way of coming at matters. An apter question is whether the truths of set theory are analytic truths. And I portray Frege as arguing that they are; set theory is an analytic extension of logic. Given that, it would matter little what we chose to say on whether set theory is part of logic.

It is also misrepresentative to speak in a single breath of Frege-Russell logicism. For, though I cannot argue it here, I believe it to be clear that Russell never advocated the epistemology of number I am attributing to Frege. Russell from at least the time of his book on Leibniz (1900) until coming under the influence of Wittgenstein around the time of his lectures on Our Knowledge of the External World (1914) was an epistemological Platonist. Under Wittgenstein's influence he moved towards the idea that the truths of mathematics were just so many tautologies. And though he never succeeded in clarifying to his own satisfaction just what this amounted to, it is clear it is a view for which Frege would have had little sympathy.

We have been considering the nature of Frege's response to Pap's problem. We may round off these remarks by considering Frege's position in the light of another complaint lodged against logicism, one Poincaré put this way in 1894.

If . . . all the propositions [which mathematics] enumerates can be deduced one from another by the rules of formal logic, why is not mathematics reduced to an immense tautology? The syllogism can teach us nothing essentially new, and, if everything is to spring from the principle of identity, everything should be capable of being reduced to it. Shall we then admit that the enunciations of all those theorems which fill so many volumes are nothing but devious ways to say A is A?⁷³

Frege had himself raised this issue in Grundlagen a decade earlier.

A very emphatic declaration in favor of the analytic nature of the laws of number is that of W.S. Jevons: "I hold that algebra is a highly developed logic, and number but logical discrimination."

But this view, too, has its difficulties. Can the great tree of the science of number as we know it, towering, spreading, and still continually growing have its roots in bare identities? And how do the empty forms of logic come to disgorge so rich a content?⁷⁴

73. Henri Poincaré, "The Nature of Mathematical Reasoning." Quoted by Tobias Danzig in Number, p. 72. The context of the remark is a discussion of mathematical induction; for a recent discussion of Poincaré's position see Parsons, "Frege's Philosophy of Number."

74. Grundlagen, p. 22.

Given our earlier discussion we might phrase Frege's reply this way. Arithmetic is not just logic, but rather an outgrowth of logic, an analytic extension of logic. Or, to use Poincaré's terms, whereas arithmetic does spring from logic, it is not reducible to logic. However suggestive we find this way of marking a distinction, what is at stake is the conception that given just logic and the idea that a given propositional content may be symbolically re-expressed so that something "essentially new" emerges, arithmetic may be shown to be analytic in, what is arguably, Kant's sense. Thus arithmetic is not synthetic a priori, nor logic empty.

If . . . we set aside all cognitions that we must borrow from objects and reflect solely upon the use of the understanding in itself, we discover those of its rules which are necessary throughout, in every respect and regardless of any special objects, because without them we would not think at all . . . [It] follows that the universal and necessary rules of thought in general can concern solely its form, and not in any way its matter. Accordingly, the science containing these universal and necessary rules is a science of the mere form of our intellectual cognition or of thinking. And we can therefore form for ourselves the idea of the possibility of such a science, just as that of a general grammar which contains nothing beyond the mere form of a language in general, without words, which belong to the matter of language.

Now this science of the necessary laws of the understanding and reason in general, or -- which is the same -- of the mere form of thinking, we call logic.

KANT

The most reliable way of carrying out a proof . . . is to follow pure logic, a way that, disregarding the particular characteristics of objects, depends solely on those laws upon which all knowledge rests.

FREGE

I understand by 'laws of logic' . . . laws of truth.

FREGE

CHAPTER II

LOGIC

\$5. Traits of Logic

Frege, who reflected so deeply on the nature of mathematics, and who so greatly enriched logic, provided us with scant reflection upon the nature of logic itself. What is studied under the heading of 'logic'? What is the logician's subject matter? Let us try and tease out

answers that might well have been Frege's.

It is possible to get rather good agreement upon the extension of 'logic', about what principles are logical principles. As stated previously, logic for Frege in 1879 included predicate logic with identity plus some unheralded second-order logic. With Grundgesetze we find this fundamental logic essentially unchanged (viewed mathematically), except for the addition of notation for definite description and an axiom to govern it. But now the second order logic is explicit, and in its terms is introduced a set theory. And there is more. A many-branching structure of higher-order logic unfolds. If we think in terms of basic ideas -- roughly those of first order logic plus set theory -- many logically minded philosophers would regard all of Frege's logic as logic, and all would regard much of it as logic. But even if agreement were complete on Frege's logic extending to include all, and nothing but, logic, we would not want to identify the two without further ado. For we want to commend Frege on extending logic beyond what it was, and to be able to sensibly inquire into possible additional developments or alternatives. Thus even complete agreement with Frege on the logical truths would still leave the question: What is logic?

The study of logic we might say is the study of the logical truths. Which it is. But the circularity of

this remark is too blatant to serve as an answer to our question.

In logic one studies principles of valid reasoning. True, but here what is intended is not so much pieces of reasonings which people have actually gone through so much as reasonings which are such that if anyone went through them they would be reasoning validly. Or, more simply, we may drop this talk of hypothetical reasonings and speak instead of arguments in the abstract and, in particular, of the relationship of premisses to conclusion in valid arguments, arguments such that were anyone to rehearse them they would be reasoning validly. In this way logic is conceived as a study of the consequence relation, that is, the relation of logical consequence which holds between just the premisses and conclusions of logically valid arguments. And what then is the force of 'logical' in 'logical consequence'? Just this, when a set of premisses bears this relation to a conclusion, conjoining the premisses and taking the result as antecedent of a conditional with the conclusion as consequent yields a truth of a peculiar sort, namely a logical truth. And so it is that we have moved through a rather tight circle back to the idea that logic deals with the truths of logic.

Let us try another tack. Various remarks of Grundlagen suggest that it was Frege's view that the body

of human knowledge divides three ways according to "the ultimate ground upon which [rests] the justification"¹ for particular knowledge claims. Physical science requires the support of sensory evidence, in addition to drawing upon logic and mathematics, and also geometry. Euclidean geometry was for Frege, following Kant, an a priori science; it makes no justificatory use of the senses, but rests upon a special sort of intuition.² The truths of logic -- and arithmetic also, as it was to be shown -- rest upon neither experience nor intuition. But then what do they rest upon? What is the ground of logical truth?

It seems almost to have been Frege's view that it is better not to ask this question, because one is bound to go wrong in trying to respond to it. This sentiment is closely connected with his disgust with what he termed psychologism, the attitude, roughly, that we must look to what goes on in our minds when we speak to understand and clarify the meanings of our words so as to know of what we speak.³ In any case Frege does not directly deal with the question. We are left with the negative characterization of whence the laws of logic gain their authority:

1. Grundlagen, p. 3.

2. See Grundlagen, p. 75 and pp. 101-102.

3. Cf. Dummett, "Frege", p. 225.

neither from the testimony of the senses, nor from intuition.

Asking as to Frege's views on the ground of logical truth has not yielded a positive characterization of logic. Let us look into a closely related matter, that which Frege calls the "domain" of various sciences. He says we may "compare the various kinds of truth in respect of the domains that they govern."⁴ "Empirical propositions hold good of what is physically or psychologically actual, the truths of geometry govern all that is spatially intuitable, whether actual or the product of our fancy."⁵ Whereas "the truths of arithmetic govern all that is numerable. This is the widest domain of all; for to it belongs not only the actual, not only the intuitable, but everything thinkable."⁶ As with arithmetic, so with logic, it too governs "everything thinkable".

Euclidean geometry governs everything that is spatially intuitable, or imaginable. It is not possible to imagine anything non-Euclideanly, though one can reason about non-Euclidean space. Logic, says Frege, governs everything thinkable. But he cannot want to imply that it is not possible for someone to think illogi-

4. Grundlagen, p. 20.

5. Ibid., p. 20.

6. Ibid., p. 21.

cally; he does not wish to deny that people reason fallaciously. It must rather be his view that one cannot think correctly on any topic except in conformity with the laws of logic. But then one cannot (on Frege's view) think correctly about physical objects, the denizens of Euclidean space, except in conformity with the laws of Euclidean geometry. One can however think logically while denying any of these laws. Here then is the difference Frege is after between the domains of geometry and logic. Whereas one can reason in such a manner that one would be reasoning correctly only if some laws of logic were false, it is not possible to reason at all in accordance with a denial of logical or arithmetical laws. Just "try denying any one of them", he says, "and complete confusion sets in. Even to think at all seems no longer possible."⁷

If it is not possible to reason correctly upon any topic, including logic, except in conformity with the laws of logic, then the domain of logic is universal. For, if one could so reason, logic would not apply to that reasoned about, and hence the domain of logic would not be universal. Universality, then, is one mark of logic as Frege conceived it. From this characteristic we can extract a condition which will allow us to compare Frege's logic with its predecessors and possible successors

7. Ibid., p. 21.

or competitors. If we come upon, in any area of thought, principles of inference which are recognizably logical, then these principles would have to turn up in any system of logic which was a complete logic. Thus to know of logical principles not found in someone's logic is to know that logic to be incomplete. Of course it is not always a simple matter either to recognize logical principles or to determine that the effect, at least, of some such principle cannot be gained in a given system. Still this idea does provide us with a hold on comparing one system with another in the absence of anything like a definition of logic.

Of course if there is a principle of inference not reflected in our logic, then there will also be logical truths not be found there either. For, if we take an argument whose conclusion follows validly from its premisses by such an unrecognized logical principle, we may form its corresponding conditional by taking the conjunction of its premisses as the antecedent of a conditional whose consequent is the argument's conclusion. This statement will be a logical truth, but one not recognized by our logic. For if it were, it could be used to justify a rule of inference, since the corresponding conditional of an argument is logically true just in case the argument is logically valid. So our incompleteness criterion, such as it is, can be restated this way.

A complete logic completely covers the truths of logic.

A second characteristic of logic may be mentioned, one which, though largely unremarked upon by Frege, seems clearly to be in keeping with his thinking. This is that logic is evident in the following sense. The fundamental truths of logic are (to be) self-evident, and the basic rules of inference evidently truth-preserving, so that any logical truth which is not itself self-evident can have its truth established either from self-evident truths, or ones which themselves trace back to self-evident truths.⁸

How, we may ask, should we take 'self-evident' in this context? Is it self-evident that two plus two equals four? Is the fact that I now have a hand before my eyes self-evident to me right now? I suppose that for Frege simple arithmetic truths are self-evident in just the sense of that term he required his logical axioms to be self-evident. This would seem to have the consequence that self-evidentness grades off; some arithmetical truths are not quite self-evident, others less so, etc. But I know of no reason to expect that Frege would have balked at this. The same situation holds in logic itself. What Frege required in his formal work was that his axioms be paradigmatically self-evident, so that all theorems of

8. Cf. Quine, Philosophy of Logic, pp. 82-83.

the system would be evident in the sense stated, that is, potentially self-evident. (Setting aside such facts as that some theorems will be too long or too complex to grasp, and others will have proofs too long to take in from beginning to end.)

As to the second case questioned, I would expect that Frege would claim that this fact, that of my hand before my eyes, is not (was not) self-evident, or not self-evident in the same sense of the term as it applies to logic. My reason is that I expect that his insistence on the epistemological importance of the a priori - a posteriori distinction to figure here, though I am not sure just how.⁹

To some extent, these traits of universality and evidentness pull in opposing directions. Universality counsels an expansionist policy; evidentness places constraints. Frege in his attempt to show that logic encompasses arithmetic was forced to strain the evidence condition, as he later admitted. In 1902, speaking in the Appendix to the second volume of Grundgesetze of its

9. This issue was the subject of a debate running through several decades between (prominently) Quine and Carnap, a defender of the Kant-Frege distinction. Quine's Philosophy of Logic contains his reply to Carnap's "W.V. Quine on Logical Truth" in P.A. Schilpp, The Philosophy of Rudolph Carnap (see especially p. 914); see Quine on obviousness, especially pp. 96-97.

Basic Law V, Frege wrote

I have never concealed from myself its lack of the self-evidentness which the others [i.e., the other axioms] possess, and which must properly be demanded of a law of logic, and in fact I pointed out this weakness in the Introduction to the first volume.¹⁰

However what he had actually said in this "Introduction" was,

A dispute can arise, so far as I can see, only with regard to my Basic Law concerning courses-of-values (V), which logicians perhaps have not yet expressly enunciated, and yet is what people have in mind, for example, where they speak of the extensions of concepts. In any event the place is pointed out where the decision must be made.¹¹

I think the tenor of these remarks is the hope, perhaps expectation, that here familiarity will breed self-evidence. And Frege had reason for this hope; his axiom represented a very general example of the sort of case we have discussed where a given propositional content receives alternate symbolizations. What forced the issue was a letter Frege received from Russell in 1902.¹² For, as we have said, the axiom in question was that by which Frege was to derive set theory from his more fundamental logic. It states, in effect, that there is a class

10. Grundgesetze, p. 127.

11. Ibid., pp. 3-4.

12. This letter and Frege's reply are included (in translation) in van Heijenoort, From Frege to Gödel.

answering to every predicative expression; a separate class for non-co-extensive predicates. Russell's letter indicated to Frege that this axiom engenders a contradiction, Russell's paradox.

Logicism requires set theory to be logic or its analytic extension. But what we seemed to have learned since Russell turned up the contradiction named for him is that if a set theory is strong enough to reach arithmetic, its axioms do not have the self-evidentness Frege required, while if one proceeds from only self-evident principles, arithmetic lies beyond reach.¹³ Either way Frege's epistemological program would fail.

\$6. Logic as Lingua Characterica

In a brief note, "Logic as calculus and logic as language",¹⁴ Jean van Heijenoort suggests that "from Frege's writings a certain picture of logic emerges, a conception that is perhaps not discussed explicitly but nevertheless constantly guides Frege", one that finds expression in Frege's conception of his Begriffsschrift as a lingua characterica, and not a mere calculus

13. This judgment is a familiar one; I came to it, I believe, through listening to Richard Cartwright and reading Quine.

14. Jean van Heijenoort, "Logic as Calculus and Logic as Language", Synthese, 17 (1967), pp. 324-330.

ratiocinator. Van Heijenoort in referring to this conception speaks of the universality of logic. We shall explore this idea and try to clarify the picture van Heijenoort attributes to Frege.

Frege typically used the phrase 'lingua characterica' in contexts where he wished to dissociate himself from the Algebra of Logic school of, notably, Boole and Schröder. In particular, he used the phrase to stress what he took to be a difference of purpose between himself and these other logicians.

Frege begins the paper "On the scientific justification of a concept-script [Begriffsschrift]", which we have quoted from earlier (§3), with this remark:

The need for some method of avoiding errors in one's own thought as well as misunderstanding on the part of others has time and again made itself felt in the more abstract scientific disciplines. Both these shortcomings are rooted in the imperfection of language, for the fact is that in order to think we must use sense symbols.¹⁵

He goes on, somewhat further down, to state that "language is not in such a way dominated by logical laws that compliance with grammar would of itself guarantee the correctness of thought processes."¹⁶ In addition, "logical rules . . . [have] proved little protection

15. English translation: Mind, 73 (1964), pp. 155-160.

16. Ibid.

[from error] because they have . . . remained external to content."¹⁷ He concludes that "we have need of a system of symbols . . . whose logical form cannot be escaped by the content."¹⁸ This he says, we do not get

with the logical symbolisms stemming from Leibniz which have recently been revived by Boole, R. Grassman, St. Jevons, E. Schroder and others. Here one has the logical forms, if not entirely complete, but the content is lacking. Any attempt to replace the letters in these symbolisms with expressions of content such as analytic equations would with the resulting complexity and ponderousness -- why, even ambiguity -- of the formulas obtained point out how little suited these symbolisms are for the construction of a true Begriffsschrift. For such I would like to demand the following: it must have simple modes of expression for the logical relations which, being limited to the very necessary, can be mastered with ease and sureness. These forms must suited to combine with a content most intimately.¹⁹

Meanwhile Schroder had written a review of Begriffsschrift in which he claimed to find little, other than cumbersome-ness of notation, not already present in the works of Boole.²⁰ To this Frege replied that

17. Ibid.

18. Ibid.

19. Ibid.

20. Zeitschrift fur Mathematik und Physik, 25, Historische literarische Abteilung, 81-94. Referred to by Jean van Heijenoort in his introductory to note to Begriffsschrift in From Frege to Godel, p. 2.

what is primarily overlooked in this reproach . . . is [that] my purpose was other than Boole's. My intention was not to represent an abstract logic in formulas, but to express a content through written signs in a more precise and clear way than is possible through words. In fact I wanted to create not a mere calculus ratiocinator but a lingua characterica in Leibniz's sense.²¹

To these remarks we can add the following of a somewhat later vintage.

In comparison with Boole's Symbolic Logic my Begriffsschrift appears cumbrous; if one considers it merely as Symbolic Logic -- as a calculus ratiocinator and not as a lingua characterica. But this disadvantage becomes an advantage if one uses it for its proper purpose. It is precisely this cumbrousness that makes it possible for the eye to take in in one glance -- at least as regards the principal features -- a complex logical structure. By means of this cumbrousness the more complicated formulae gain a perspicuity that would not be reached without it, and then often would a chaos present itself to the eye which could hardly be extricated from confusion.²²

Frege's Begriffsschrift is intended as a calculus ratiocinator. As Frege puts it, this time in Grundlagen,

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21. "On the Scientific Justification of a Begriffsschrift".
22. Frege's notes (p. 251) to P.E.B. Jourdain's "Development of Theories of Mathematical Logic and the Principles of Mathematics: Gottlob Frege", Quarterly Journal of Pure and Applied Mathematics, 43 (1912), pp. 237-269. Jourdain included a note (p. 237) saying that "Professor Frege has most kindly read this paper in manuscript, and added...notes"; they were apparently written in 1910.

"it is designed to produce expressions which are short and easier to take in, and to be operated like a calculus by means of a small number of standard moves."²³ But as well the Begriffsschrift is to "be suited to combine with a content most intimately." From Grundlagen again, "it is designed . . . to be capable of expressing not only the logical form like Boole's notation, but also the content of a sentence."²⁴ How shall we understand this idea of Frege's that the forms of a lingua characterica shall be suited to the expression of content?

We get some help through considering what Frege found inadequate in, for example, Boole's symbolism. For example, this notation provides no way to symbolize such a simple sentence as 'If something is red and round, then it is red.' For one thing, there are certain technical difficulties with Boole's notation for generality.²⁵ But more importantly, Boole intended that his symbolism was to be adequate to two distinct sorts of interpretation, a class interpretation and a propositional interpretation; indeed, he argued that it was. We were free to interpret the symbols of Boolean formulas as representing either

23. Grundlagen, p. 103.

24. Ibid., p. 103, n. 1.

25. See C.I. Lewis, A Survey of Symbolic Logic, pp. 56-57, and Kneale and Kneale, Development of Logic, pp. 411-412.

classes or propositions.²⁶ Either one or the other, but not both. It is not possible to call upon the resources of each simultaneously. But this we would need to do to symbolize our example, which is why we cannot. With Boole's symbolism we do not have a notation in which we can simultaneously express compounds of sentences while articulating the content of the component sentences in a way which reveals logical relationships among parts of the sentences. There is no way, for instance, to show with the notation that 'Something is red' follows logically from 'Something is red and round'. Frege's Begriffsschrift was intended to allow us to articulate the content of complex thoughts so as to bring out logical relations between these thoughts and others composed in part of their parts, and so forth.

Van Heijenoort remarks as follows on Frege's aim of having a logical notation "suited to combine with a content most intimately". (He passes over the "class interpretation" of Boole's symbolism.)

Frege frequently calls Boole's logic an "abstract logic", and what he means by that

26. Actually the "propositional interpretation" was itself a class interpretation of sorts; in it the various "elective symbols" stand for, not classes of objects, but instead classes of times (or durations, the two not being distinguished), intuitively the times at which the "symbolized" proposition is true. A result was that truth tended to slide out of the focus of concern.

is that in this logic the proposition remains unanalyzed. The proposition is reduced to a mere truth-value. With the introduction of predicate letters, variables, and quantifiers, the proposition becomes articulated and can express a meaning. The new notation allows the symbolic rewriting of whole tracts of scientific knowledge, perhaps all of it, a task that is altogether beyond the reach of the propositional calculus. We now have a lingua, not simply a calculus.²⁷

Adopting van Heijenoort's idea that Frege's notation is intended to allow the "rewriting" of "whole tracts of scientific knowledge" in its terms, and remembering that, if this is so, then much non-knowledge would be similarly expressible, and, further, keeping in mind that the purpose of a Begriffsschrift is to make logical relations manifest, we might be led to the following characterization of a lingua characterica:

A lingua characterica (1) is (or includes) a grammar, a language-frame, which provides a set of linguistic structures adequate to articulate the logical structure of any suitable scientific propositional content so that, when supplemented with appropriate non-logical vocabulary (as allowed for by the open categories of the grammar) any suitably scientific propositional content can be expressed in the terms provided and in such a way as to make manifest the logical relations among it and other similarly expressed propositional contents.

Further, (2) it must always be possible to (finitely) determine logical relationships among expressions of the lingua (at least in principle.)

27. "Logic as Calculus and Logic as Language", p. 325.

So characterized, a lingua characterica must be, in currently accepted usage, a formal system, a system whereby in its specification we are provided with an effective characterization of what is a sentence (formula) of the system, and an effective characterization of what is a proof in the system. That is, we must be provided with a means of determining, for any sequence of expressions of the system, whether or not it counts as a sentence, and for any n -long sequence of sentences of the system, whether or not the sequence of the first $n-1$ sentences counts as a proof in the system of the n th member of the original sequence.

(1) states a strong condition of universality, or potential universality; (2) enforces maximum logical rigor. Did Frege actually regard his Begriffsschrift as meeting these two conditions? On (2) first, it seems clear that he did, and that indeed at least with the presentation of the system in Grundgesetze Frege does attain this level of rigor. It should be noted that Frege does not explicitly set out an effective characterization of what shall count as a Begriffsschrift-sentence, and it is by no means easy to see just how this is to be done. Still it would widely agreed that, with Grundgesetze and perhaps even with Begriffsschrift, we are offered a formal system. Thus, for instance, the judgment of van Heijenoort: "Frege was the first to present, with

all the necessary accuracy, a cardinal notion of modern thought, that of formal system."²⁸

On the first condition (1), whether or not Frege actually regarded his *Begriffsschrift* as such a lingua, I would, echoing van Heijenoort's remark, say that a conception of logic as, ideally, providing the grammar for such a lingua is a conception which accords well with much of what Frege says about logic and can plausibly and instructively be thought of as guiding his work. Except that there is a difficulty in our statement of this idea, at least as it could be held by Frege; this requires a major alteration of (1). Before taking this up let us expend a few words on the working terminology of this essay.

Thus far we have spoken rather haphazardly of the content of sentences, or statements, or judgments, and so forth. This practice is not inappropriate in discussing Frege's works prior to the 1890's, but needs improvement when later writings become the focus of attention, as shall be increasingly the case. For about this time Frege introduced what is most commonly referred to as his distinction of the sense and reference of expressions.

In his "Introduction" to Grundgesetze, in the process

28. Ibid., p. 324.

of indicating changes which he has introduced into his Begriffsschrift since Begriffsschrift, Frege tells us his previous conceptual content "has now split . . . into what I call 'thought' and 'truth-value', as a consequence of distinguishing between sense and denotation of a sign. In this case the sense of a sentence is a thought, and its denotation a truth-value."²⁹ Where Frege here used 'thought' ('Gedanken') Church and others have used 'proposition' to much the same purpose. And both intend much the same in speaking of sentences expressing [ausdrücken] thoughts or propositions. Both would agree that more than one sentence (expression) may express the same thought (sense). And that it is possible for there to be thoughts that no sentence expresses. I shall capitalize 'thought' henceforth when used in this way.

Still talking philosophical lexicon, I shall offer a few words on sentences. Natural language sentences, some of them, clearly enough express Thoughts. Equally clearly not all of what we would ordinarily call sentences do, for a variety of reasons. For instance some are in the wrong mood. It seems to have been Frege's view at the time of Grundgesetze that only declarative sentences have (conceptual) content, that is, express Thoughts, and it will suit the purposes of this essay to concur. Among declar-

29. Grundgesetze, pp. 6-7.

atives most at best only incompletely express Thoughts. Indexical elements of some variety (tense, demonstratives, etc.) are present in most ordinary sentences. Such sentences Frege on occasion speaks of as incomplete sentences. These are noted for their systematic shifting in truth-value with the context of their employment. Even among sentences that do completely express Thoughts problems arise. Some fail of truth-value altogether. This is the case with 'Pegasus flies', for instance. Others, through ambiguity of one sort or another, apparently express more than one Thought and so, in particular cases, may be read with either truth-value. Perhaps we should withhold from the latter the designation 'expresses a Thought', reserving this for expressions whose content is uniquely a Thought. Adopting this suggestion, I propose for the purpose of this essay to so restrict our use of 'sentence' that it apply to only those expressions that express Thoughts and do not fail of truth-value, i.e., to sentences which are either fixedly true or fixedly false. More accurately, I shall have this class in mind in making various "theoretical" remarks, while continuing to use as examples sentences with tensed verbs and other indexical elements. Given this restricted usage of 'sentence' perhaps we may say that all sentences have a suitably scientific sense, as we were, on van Heijenoort's prompting, earlier employing this phrase, and so it may be

dispensed with.

Let us now take up the difficulty alluded to a bit back with condition (1). This condition was stated in terms of the logical structure of Thoughts, as we now say. But such talk of unique logical structure of Thoughts is out of place. This is apparent in a mild way from the Kantian case where the sentences 'Vixens are vixens' and 'Vixens are female foxes' would be held to be expressions of a common Thought. Since the sentences differ in structure, which, if either, will we say shows the logical structure of the Thought in question? Here one might try saying that, if either does, it is the latter, by holding that in analyzing the concept vixen one was also coming to more finely apprehend the logical structure of that concept and so also of Thoughts that contain it. I shall not pursue the plausibility of this approach to the particular example, for it will not work in general. It is not applicable, for instance, to Frege's example where the sentences 'Line a is parallel to line b' and 'The direction of line a is the same as the direction of line b' would be held to express the same Thought. Here the difference in structure is sentence-wide, and so the alternative structures are competing in a way the previous pair were not. And neither can, consistent with Frege's epistemological views, be taken as showing the logical structure of the Thought in question. But the

attempts to speak coherently of Thoughts having determinate logical structure runs afoul of deeper problems than this suggests.

Setting aside for the moment the idea of lingua characterica, we may think of Frege's Begriffsschrift as an attempted axiomatization of logic. Speaking somewhat loosely we may suppose that, in intention, any logical truth would find expression in that system as a theorem of the system. Then consider such a disjunction as 'Snow is white or snow is not white'. How will such a logical truth be expressed in a system such as Begriffsschrift which lacks a symbol for disjunction? We might suppose that our logical truth was inexpressible in such a system, that such a sentence form as

$$\neg S_1 \supset S_2$$

would not be adequate for the expression of disjunctions. And this on the ground that only a disjunctive sentence adequately expresses a disjunctive Thought.

To this it might be replied that we could simply alter the notation to include a symbol for disjunction. This is so. But if we thought it necessary, then we should be thinking that, as it stands, Begriffsschrift, for the lack of a symbol, does not axiomatize truth-functional logic. Further, consider such a sentence as 'Snow is white or snow is white or snow is white' which someone might argue has its (sentential) logical form

best portrayed this way,

$$\vee_3(S_1, S_2, S_3).$$

Where previously we considered a two-place disjunction which might be symbolized with ' $\vee_2(S_1, S_2)$ ', here we have a three-place disjunction. Once this is admitted, it becomes clear that if Begriffsschrift needs a symbol for two-place disjunction if it is to be able to express certain logical truths, then it will also require a symbol for three-place disjunction. And so on. From which it would follow that no system could ever axiomatize sentential logic, at least not if we require a finite vocabulary for our axiomatization.

If we shall not succeed in our purpose through adding additional primitive notation, what about introducing a symbol for disjunction into the notation by definition, perhaps this way:

$$S_1 \vee S_2 =_{df} \neg S_1 \supset S_2.$$

But how are we to take this "definition"? If it simply provides us with short-hand notation, then we shall not have gained the means to express disjunctions; either we already had this, or we still do not. Suppose the "definition" is not just an abbreviation, then what warrants its introduction?

We might try claiming that sentences of one form express Thoughts distinct from, though logically equivalent to, Thoughts expressed by sentences of the other

form. However any such view would seem committed to claiming that the Thought that snow is white is distinct from the Thought that either snow is white or snow is white, that the sentences,

Snow is white

Snow is white or snow is white,

express different Thoughts. But then it seems such a view would also be committed to claiming that each of those sentences expresses a Thought distinct from that expressed by

Snow is white or snow is white or snow is white.

And so forth. Yet it cannot be the case that we encounter a new connective at each iteration. For this would imply an infinite number of such connectives and render our mastery of truth-functional logic inexplicable. So it seems we must be meeting up with the old connectives in new contexts. Now consider the Thought expressed by the last set off sentence and the following two pictures of its logical structure.

$$v_2(S_1, v_2(S_2, S_3))$$

$$v_3(S_1, S_1, S_1)$$

With the first we represent it as built up with two occurrences of a two-place connective. With the second we portray the Thought as containing a single occurrence of a three-place connective. What is to be our ground for preferring the first portrayal to the second? We

might say that only this preference is consistent with our known mastery of truth-functional logic. But this would be wrong. Our knowledge requires a finitely surveyable group of connectives, so I shall assume. But this by itself does not force a choice of the first representation over the second. Nor will it dictate the preference of some combination of one-, two-, and/or three-place connectives over a single four-place connective when we take up the next member of our extendable series of sentences, 'Snow is white or snow is white or snow is white or snow is white'. More importantly, the proffered reason for preferring the first to the second portrayal does not give us grounds for preferring either to the following,

$$\neg S_1 \supset (\neg S_2 \supset S_3).$$

To agree to consider only connectives of at most (say) two places would not be to agree on what connectives occur in any given Thought.

Our recent reflections argue, I believe, for the conclusion that we have no principled way of picking one logical structure from a number of possible alternatives for attribution to a given Thought, and that, consequently, we should reject the idea that we make good sense in speaking, as before, of the logical structure of Thoughts. This conclusion was already explicit in the Grundlagen application of Frege's Dictum which we have discussed. However present reflections go beyond that in the following

respect. When Frege envisaged carving up the content of a Thought in alternative ways both logical and non-logical content were affected. What has been argued just now is that even if we hold non-logical content constant, we still cannot make sense of a Thought having a unique logical structure. Otherwise put, we cannot distinguish among so-called logically equivalent Thoughts where we have agreement in non-logical content.

Our earlier attempt at clarifying the idea of a lingua characterica is in need of revision; it will not do to speak, as we did, of articulating "the logical structure" of any Thought. Nor would it be adequate to require that a lingua be able to express, one way or another, any Thought. For it is crucial to Frege's enterprise that a given Thought can be analyzed in terms of logical structures which are not logically equivalent. So we must require that a lingua have the resources for representing all such alternative analyses in suitably distinct ways. This idea of a lingua being adequate to represent all possible analyses of any Thought is a somewhat elusive one. We might hope to pin it down somewhat by reverting to talk of language as follows: A lingua characterica has the grammatical resources adequate to express any sentence, that is, express the Thought expressed by any sentence, and this by means of a sentence alike in logical structure to that discernible in, or

suggested by, the original. However this will not do, for reasons previously gone over in response to noting that Frege's Begriffsschrift, for instance, does not make use of a symbol for disjunction. The most we can require along the lines of preservation of logical structure is that a lingua provide a structure logically equivalent to that intuitively divined in, or attributed to, a given natural language sentence which enables it to be "rewritten" preserving content or sense. When this condition is met let us speak of the lingua sentence as translating the natural language sentence.³⁰ This gives us a characterization of a lingua characterica, a universal language, more in keeping with Frege's program and his practice.

Earlier (§5) in discussing Frege's idea that the domain of logic is universal, we were led to a view which may be recast, in light of recent remarks, this way: a lingua translates all logically true sentences (given suitable non-logical vocabulary.) Hence Begriffsschrift, for instance, would be incomplete if some logically true sentence could not be translated into it. Now this notion of logical completeness might seem weaker than that inherent in the idea that a lingua can translate any sentence. However a simple argument seems to show that

30. We might wish to additionally require a reasonable sameness of length, but I shall ignore this.

the two ideas are equivalent.

Clearly if a logic is complete in the sense of providing a grammar for a language into which any sentence can be translated, then it is adequate to all logical truths, or all which can be expressed at all. So only the converse can be doubtful. But suppose we had some sentence not translatable into a particular lingua. This sentence will have logical consequences. Picking one we form a corresponding conditional as before; this will be a logical truth. Then if the lingua is adequate to logic, this conditional can be translated into the lingua. But with it (we may suppose) will be translated the antecedent of the conditional, which was the sentence supposedly without translation in the lingua. In terms of an example, consider

(1) Tom is a big butterfly,
which has as a logical consequence (so it seems)

(2) Tom is a butterfly.

It seems we cannot maintain both that (1) has no translation into, say, Begriffsschrift and that Begriffsschrift is adequate to logic. Since if the latter were true, we could translate

(3) (1) only if (2)

into the notation. But to do so would, it seems, provide a translation of (1). I conclude that the two completeness conditions are equivalent.

\$7. Epistemology of Logic

We have previously remarked that much as philosophers can agree, for instance, that there is a tree in the quad while disagreeing over how this is known, Kant and Frege could agree on logic in that they agree on what are the truths of logic and yet not be in agreement in their responses to such questions as:

How do we know logical truths are such?

Why should certain principles, termed logical principles, be accepted by us?

What is the source, or ground, of logical truth?

I do however find it instructive to view Frege's philosophy of logic in a Kantian light. And remarks so made will serve to complement earlier urgings against treating Frege as an epistemological Platonist in his philosophy of number and set. For if, when we turn to Frege's views on fundamental logic we were forced to employ the Platonic picture of mind coming into epistemic contact with something apart from itself, those earlier remarks would be diminished in interest.

Both Kant and Frege would agree, I take it, that there is some sharp and principled way of demarcating the logical truths from other truths. In this sense at least, for both logic is an autonomous domain. So our above questions, which imply this, are not phrased inappro-

priately, at least not on this score.

For Kant, logic abstracts from the content, or matter, of particular cognitions; it is the science of the mere forms of thinking. So conceived, logic encompasses the forms of all possible thinking. To think at all is to think in the terms that logic provides; to reason correctly is to think in conformity with the laws of the science of those forms.

These ideas get sharpened and made concrete with Frege's conception of a lingua characterica, a language receptive to anything which can be said. Instead of speaking of forms of thought we speak of grammar, or syntax. And we are able to so frame our syntax that the logical form of sentences formed from the syntax is manifest, and the logical relationships among such sentences are effectively characterized. In this situation we might find an intuitive justification for thinking of logic as analytic. The logic of a lingua analyzes thought into an exhaustive array of possibilities of its expression.

Indisputably, passages with a Platonic flavor crop up throughout Frege's writings, perhaps increasingly so in the period under consideration. In some ways the most Platonic passage in Grundlagen occurs in the "Introduction" where Frege says

If everything were in continual flux, and nothing maintained itself fixed for all time, there would no longer be any possibility of getting to know anything about the world and everything would be plunged into confusion . . . What is known as the history of concepts is really a history either of our knowledge of concepts or of the meanings of words. Often it is only after immense intellectual effort, which may have continued over centuries, that humanity at last succeeds in achieving knowledge of a concept in its pure form, in stripping off the irrelevant accretions which veil it from the eyes of the mind.³¹

Here it can seem as if Frege is thinking of concepts much as Plato did the Forms. However, in quoting from Grundlagen we left out the middle of the passage. It reads

We suppose, it would seem, that concepts sprout in the individual mind like leaves on a tree, and we think to discover their nature by studying their birth: we seek to define them psychologically, in terms of the nature of the human mind. But this account makes everything subjective, and if we follow through to the end, does away with truth.³²

And replacing it serves to remind us of the context of Frege's remarks, his continuing struggle to free logic and mathematics from the clutches of psychologism. Remarks such as the following dot the pages of Grundlagen.

31. Grundlagen, p. vii.

32. Ibid., p. vii.

. . . sensations are absolutely no concern of arithmetic.³³

. . . psychology should not imagine that it can contribute anything whatever to the foundation of arithmetic.³⁴

. . . the answer to any question whatsoever in psychology must be for mathematics a matter of complete indifference.³⁵

In the closing paragraph of his "Introduction" to Grundgesetze, half of which is taken up with a diatribe against the psychologistic views of Benno Erdmann, Frege gasps,

The distance between my view and the psychological logicians' seems to me so enormous that there is no prospect of my book's having any effect on them at present. It seems to me as if the tree that I have planted would have to lift a colossal weight of stone in order to gain space and light.³⁶

By remembering who is the enemy we shall be prepared for some countering hyperbole, and better placed to maintain perspective.

The earlier quoted Grundlagen passage may be counterbalanced somewhat by this note included in Grundlagen.

An idea in the subjective sense is what is governed by the psychological laws of

33. Ibid., p. v.

34. Ibid., p. vi.

35. Ibid., p. 105.

36. Grundgesetze, p. 25.

association; it is of a sensible, pictorial character. An idea in the objective sense belongs to logic and is in principle non-sensible . . . Subjective ideas are often demonstrably different in different men, objective ideas are the same for all. Objective ideas can be divided into objects and concepts. I shall myself . . . use "idea" only in its subjective sense. It is because Kant associated both meanings with the word that his doctrine assumed such a very subjective, idealist complexion, and his true view was made so difficult to discover. The distinction here drawn stands or falls with that between psychology and logic.³⁷

The counter-weight lies in Frege seeming to portray himself as advocating and developing the "true view" of Kant. Since it is unlikely that anyone would regard Kant as an epistemological Platonist on any topic, we may suppose that Frege did not so regard himself in the sphere of logic.

What is crucial, Frege seems to be saying in speaking as he does of the objectivity of concepts and so of arithmetic, is a sharp distinction between psychology and logic. This Kant also emphasized.

Some logicians presuppose psychological principles in logic. But to bring such principles in logic is as absurd as to bring morality from life.³⁸

Kant's comparison of logic and morality is instructive. It is at least suggestive to think of Frege on

37. Grundlagen, p. 37, n. 1.

38. Logic, p. 16.

logic in parallel with Kant on morality. And this idea is suggested, to some extent, by such remarks of Frege as these.

In one sense a law asserts what is; in the other it prescribes what ought to be. Only in the latter sense can the laws of logic be called 'laws of thought': so far as they stipulate the way in which one ought to think. Any law asserting what is, can be conceived as prescribing that one ought to think in conformity with it, and is thus in that sense a law of thought. This holds for laws of geometry and physics no less than for laws of logic. The latter have a special title to the name "laws of thought" only if we mean to assert that they are the most general laws which prescribe universally the way in which one ought to think if one is to think at all.³⁹

To develop the suggestion a bit, let us recall a portion of Kant's discussion in his Fundamental Principles of the Metaphysics of Morals. In the section headed "Heteronomy of the Will as the Source of All Spurious Principles of Morality" Kant writes that

If the will seeks the law which is to determine it anywhere else than in the fitness of its maxims to be universal laws of its own dictation, consequently if it goes out of itself and seeks this law in the character of any of its objects, there always results heteronomy. The will in that case does not give itself the law, but it is given by the object through its relation to the will.⁴⁰

39. Grundgesetze, p. 12.

40. Immanuel Kant, Fundamental Principles of the Metaphysics of Morals, p. 57.

Subsequently, having divided the possible foundations of morality into those which depend upon empirical principles and those which depend upon rational principles, he tells us that "[e]mpirical principles are wholly incapable of serving as a foundation for moral laws",⁴¹ essentially because this would be to attempt to found the necessary upon the contingent. Much the same reason lies behind the demand of Frege and Kant to keep logic distinct from psychology. Whereas the empirical could conceivably suggest principles, logical or moral, in neither case could it authorize our acceptance of either sort of principle.

In discussing rational principles as a possible foundation of morality Kant maintains that if we are not going to simply build our conception of morality into our conception of the source of morality

the only notion of the Divine will remaining to us is a conception made up of the attributes of desire of glory and domination, combined with the awful conceptions of might and vengeance, and any systems of morals erected on this foundation would be directly opposed to morality.⁴²

Briefly put, might does not make right. Summing up the general outlook, there can be no sort of apprehension whereby mind supposedly gains moral principles, or

41. Ibid., p. 58.

42. Ibid., p. 60.

directives from something independent of it -- be it God, or Nature, or what have you -- which, at the time, authorizes for us the acceptance of these principles. We must ask: Are they right? Are they truly moral principles? And no source other than morality itself can answer these questions. Such, roughly, is Kant's view.

It is the rejection by Frege of a "rational source", just as much as an "empirical source", of logic that I am tentatively, and somewhat controversially, suggesting. Here is textual support from Grundgesetze.

. . . what if beings were found whose laws of thought flatly contradicted ours . . . Anyone who understands laws of logic to be laws that prescribe the way in which one ought to think -- to be laws of truth, and not natural laws of human beings' taking a thing to be true -- will ask, who is right? Whose laws of taking-to-be-true are in accord with the laws of truth?⁴³

It would not matter, I submit, whether the imagined beings were (held to be) divine. Nor would it matter whether we ourselves were the beings in question in this sense. In "consulting the Forms", or whatever, certain principles occur to us which differ from those with which we have previously operated. In any and all such cases we must ask: Which are the laws of truth? Ultimately there is no source outside of logic itself which can authorize

43. Grundgesetze, p. 14.

for us an answer. This seems to be the spirit of the remark that "I take it as a sure sign of a mistake if logic has need of metaphysics and psychology -- sciences that require their own logical first principles."⁴⁴

Yet even if we work ourselves around so that we are sympathetic to this outlook, questions remain, even if they are not just the same ones we started with. Dummett has instructively observed that

From the time of Descartes until very recently the first question for philosophers was what we can know and how we can justify our claim to this knowledge, and the fundamental philosophical problem was how far skepticism can be refuted and how far it must be admitted. Frege was the first philosopher after Descartes totally to reject this perspective, and in this respect he looked beyond Descartes to Aristotle and the Scholastics. For Frege, as for them, logic was the beginning of philosophy; if we do not get logic right, we shall get nothing else right.⁴⁵

But how will we know when we "get logic right"? What do we have to go on here?

§8. Truth

In his Grundgesetze "Introduction" Frege wrote that

Our conception of the laws of logic is necessarily decisive for our treatment of the science of logic, and that con-

⁴⁴. Ibid., p. 18.

⁴⁵. Dummett, "Frege", pp. 225-226.

ception in turn is connected with our understanding of the word "true".⁴⁶

How then did Frege understand the word 'true'? How would he have us speak of truth?

In making that remark Frege, as so often, had psychologism in mind. It is easy, he held, to slide from a recognition that laws of logic prescribe patterns of correct thought, and are, in this sense, laws of thought, "into supposing that these laws govern thinking in the same way as laws of nature govern events in the external world. In that case they can be nothing but laws of psychology . . ."⁴⁷ In this way something being true gets confused with someone's supposing something to be true, until

in the end truth is reduced to individuals taking something to be true. All I have to say to this is: being true is different from being taken to be true, whether by one or many or everybody, and in no case is to be reduced to it. There is no contradiction in something's being true which everybody takes to be false. I understand by laws of logic not psychological laws of takings-to-be-true, but laws of truth.⁴⁸

But this is not the only route to "psychological logic". In his Grundlagen "Introduction" Frege had warned that if his Dictum is not observed "one is almost forced

46. Grundgesetze, p. 12.

47. Ibid., p. 12.

48. Ibid., p. 13.

to take as the meanings of words mental pictures or acts of the individual mind",⁴⁹ and so violate the principle: "always . . . separate sharply the psychological from the logical, the subjective from the objective . . ." ⁵⁰ And in Grundgesetze's "Introduction", while not mentioning the Dictum, he once again offers a diagnosis of a major cause of psychologism.

Because the psychological logicians fail to recognize the possibility of there being something objective that is not actual [i.e., "capable of acting directly or indirectly on the senses"], they take concepts to be ideas and thereby assign them to psychology.⁵¹

The nature of their error lies in forgetting that thoughts are not subjective. So that, for instance, I may have a thought which I relate to you so that you understand it, and convey it, in turn, to someone else, who voices disagreement with it, and so forth. And so, in looking for the meaning of the words used to convey some thought they are led "into the mind" of the one who has the thought; finding there only various ideas, they assign such the task of, individually, being the meanings of words.

Thoughts then become combinations of ideas. And,

[t]hus everything drifts into idealism and from that point with perfect consistency

49. Grundlagen, p. x.

50. Ibid., p. x.

51. Grundgesetze, p. 16.

into solipsism . . . If we could not grasp anything but what was within our own selves, then a conflict of opinions [based on] a mutual understanding would be impossible, . . . there would be no logic to be appointed arbiter in the conflict of opinions.⁵²

Some pages on Frege continues.

If the idealists were consistent, they would put down the sentence "Charlemagne conquered the Saxons" as neither true nor false, but as fiction, just as we are accustomed to regard, for example, the sentence "Nessus carried Deianeira across the river Evenus"; for even the sentence "Nessus did not carry Deianeira across the river Evenus" could be true only if the name "Nessus" had a bearer.⁵³

And again,

. . . I am not designating any of my ideas with the words "This blade of grass is green", and if I were the sentence would be false. Here there enters a second falsification, namely that my idea of green is asserted of my idea of this blade of grass. I repeat: in this sentence there is no talk whatever of my ideas; it is idealists who foist that sense upon us.⁵⁴

Here is revealed the style in which Frege would have us discourse about truth. The sentence 'Charlemagne conquered the Saxons' is true just in case the names 'Charlemagne' and 'the Saxons' have (unique) bearers such that the one stands in the conquering relation to the other. And a familiar style it is. One, often associated with Aristotle, which embodies what A.N. Prior has called

52. Ibid., p. 17.

53. Ibid., p. 20.

54. Ibid., p. 20.

"the nerve of the correspondence theory" of truth.⁵⁵

In a familiar passage of the Metaphysics Aristotle wrote

. . . there cannot be an intermediate between contradictions, but of one subject we must either affirm or deny any one predicate. This is clear, in the first place, if we define what the true and the false are. To say of what is that it is not, or of what is not that it is, is false, while to say of what is that it is, and of what is not that it is not, is true. . .⁵⁶

Here we have a statement -- though not the first, for the idea is present in the Sophist -- of what, following Tarski (and the Warsaw group of Polish philosopher-logicians generally) is commonly termed the classical conception of truth.⁵⁷ Its hallmark is the central role of a notion of reference. In our talk of truth we aim to provide a systematic account of how sentences, or sentences through their parts, make reference to what there is, which account enables us to explain, or understand, how it is that truths are true.

55. A.N. Prior, "Correspondence Theory of Truth" in Edwards' Encyclopedia, p. 224.

56. Metaphysics, 1011b.

57. See, for instance, Alfred Tarski's "The Concept of Truth in Formalized Languages" (p. 153) in Logic, Semantics, and Metamathematics, and (more recently) "Truth and Proof", Scientific American, 220 (June 1969), pp. 63-77, and Kazimierz Ajdukiewicz, Problems and Theories of Philosophy, pp. 9-18.

A variant idea also looks back to Aristotle.

. . . he who thinks the separated to be separated and the combined to be combined has the truth, while he whose thought is in a state contrary to that of the objects is in error. This being so, when is what is called truth or falsity present, and when is it not? We must consider what we mean by these terms. It is not because we think truly that you are pale, that you are pale, but because you are pale we who say this have the truth.⁵⁸

In this perhaps equally well-known passage we find an idea which turns up this way in Russell's Lectures on Logical Atomism.

The first truism to which I wish to draw your attention -- and I hope you will agree with me that these things that I call truisms are so obvious that it is almost laughable to mention them -- is that the world contains facts, which are what they are whatever we may choose to think about them, . . . when I speak of a fact . . . I mean the kind of thing that makes a proposition true or false. . . [Thus, for instance,] if I say "Gravitation varies inversely as the square of the distance", my statement is rendered true by astronomical fact. If I say 'Two and two are four', it is arithmetical fact that makes my statement true.⁵⁹

Again, Dummett, in his widely read article "Truth" tells us that an "important feature of the concept of truth" which "the correspondence theory expresses" is "that a

58. Metaphysics, 1051b.

59. Bertrand Russell, "The Philosophy of Logical Atomism" in Logic and Knowledge (edited by Robert C. Marsh), p. 182.

statement is true only if there is something in the world in virtue of which it is true."⁶⁰

The additional quasi-causal element which appears in these sets of remarks through the expressions 'because', 'makes . . . true', and 'in virtue of' I wish to hold separate from the classical conception of truth as correspondence with reality. I find no hint of it in Frege's writings.

The classical conception of truth seems to carry with it some rather difficult problems. For it seems to presuppose that it is possible for us to relate to the world in such a fashion that we can say how it is. And this would seem to require that our conceptualizations carve reality at its joints, and that our referential apparatus engage with what there is. Only if this were so could our statements be true, that is, say how reality really is. For some, highlighting such assumptions may induce a scepticism, one avoidable only through rejecting the correspondence theory and holding that reality is what we make of it. For others, the impossibility of comparing conceptualization with reality will dictate the view that the scepticism-inducing problem is a pseudo-problem. But this too may lead to a rejection of the classical concep-

60. Michael Dummett, "Truth" in Philosophical Logic (edited by P.F. Strawson), p. 63.

tion on the ground that it is senseless to speak of an independent reality at which we aim our remarks, and so senseless to speak of correspondence between words and world.

While the issues raised here are important I wish to pass them by, since Frege does not seem to have worried the problems, and any discussion I could offer would be too tortuous to be worth the excursion. However I would like just to register the claim that, whereas deeming the question of the ultimate nature of reality a senseless one may lead to a rejection of the classical conception of truth, it need not. And I would mention J.L. Austin and Quine as recent examples of philosophers who have maintained versions of a correspondence theory while rejecting the question of reality's ultimate nature.⁶¹

One or another reaction to the mentioned problems may be a source of adopting one of a variety of non-classical conceptions of truth. Karl Popper has broadly drawn a contrast between the "idea of objective or absolute truth -- that is truth as correspondence with the

61. On Austin, compare the section on "Reality" in "Other Minds" with the essay "Truth", both reprinted in Philosophical Papers. On Quine compare "The Scope and Language of Science" in The Ways of Paradox, especially p. 126, and Chapter III "Truth" of Philosophy of Logic.

facts" and "subjective (or 'epistemic') theories of truth".⁶² The latter

stem from the fundamental subjectivist position which can conceive of knowledge only as a special kind of mental state, or as a disposition, or as a special kind of belief, characterized, for example, by its history or by its relation to other beliefs. . . . [Non-classical theories] all say more or less, that truth is what we are justified in believing or in accepting, in accordance with certain rules or criteria, of origins or sources of our knowledge, or of reliability, or stability, or biological success, or strength of conviction, or inability to think otherwise.⁶³

In this grouping belong the views of Frege's "psychological logicians" with their idealist, coherentist tendencies which he so stoutly opposed.

Having located Frege on the classical side of the truth divide, I wish to indicate a bit more specifically what his view involves. Here, once again, we shall be involved in bringing to the fore a conception which underlies and guides Frege's work, but which receives no explicit discussion on his part. The theme will be how discoursing upon truth can provide what we shall call a semantics for a logical system such as *Begriffsschrift*. Our approach will be slightly oblique.

62. Karl R. Popper, Conjectures and Refutations, pp. 224-225.

63. Ibid., p. 225.

Consider the sentences

- (4) Jack is tired although Jill is not tired
 (5) Jack is tired because Jill is not tired.

Each appears to be composed out of a pair of sentences, the same pair in each case. Further each implies

- (6) Jack is tired.

Thus both of the following would seem to be logical truths.

- (7) (4) only if (6)
 (8) (5) only if (6).

Now it seems we can say why (7) is logically true in a way in which we are unable to say why (8) is so. Of (4) we know that if it is true then both of its component sentences must be true also; and conversely, if the component sentences are both true, then so too must be (4). That is, we can state what are necessary and sufficient conditions of the truth of (4) in terms of its component sentences. Otherwise put, 'although', in (4) expresses truth-functional conjunction. With regard to (5) we know that if it is true then both of its component sentences must be true also, but the converse does not hold. It is not sufficient that Jack be tired and Jill not for (5) to be true. Since each of the seemingly component sentences of (5) is implied by (5), it seems each must, in some way, be a logical component of (5). On the other hand, they do not compound truth-functionally in (5). Indeed we are in the dark as to the logical structure of (5).

The noted difference between (4) and (5) can be alternatively put in earlier terms; we know how to translate (4) into a notation such as Begriffsschrift (assuming the "internal structure" of the component sentences is unproblematic), but not how to so translate (5). It seems appropriate to register this difference in our knowledge of (4) and (5) by saying that we know why (4) implies (6), but not why (5) does; equivalently, that we know why (7) is a logical truth, but not why (8) is. The sense of 'explain', or 'knowledge' at work here seems to be this. We know how to determine the truth-value of the complex sentence (4) from its component sentences, and knowing just this enables us to know that (7) is true. We do not know why (8) is logically true, since we do not possess knowledge of (5) analogous to what we have of (4) which would enable us to determine, using just it, that (8) is true.

Now consider the following.

- (6) Jack is tired
 - (9) Jack is very tired
 - (10) Something is tired.
- (10) would seem to be a logical consequence of both (6) and (9), thus each of the following would be a logical truth.
- (11) (6) only if (10)
 - (12) (9) only if (10).

Since (10) is not itself a logical truth nor either (6) or (9) a logical falsehood, it seems that if either (11) or (12) is logically true, its antecedent must in some way guarantee the truth of its consequent.

For the likes of (6) Frege offers the following account, an account taken up in detail in Chapter III. 'Jack' is a proper name, and so purports to refer to something, viz., some object; 'is tired' is a predicate, and also purports to refer to something, viz., a particular concept; and so (6) itself will be true just in case the referential relations hold, and the object in question falls under the concept in question. If we accept this account of (6), then if we knew that (6) was true we would be able to determine, on the basis of that knowledge, that (10) was true. Or rather we would if we accept some account of (10) on which it gets treated as claiming in effect that at least one object falls under the concept of being tired; and let us. Given this account we know why if (6) is true, then so too must be (10); that is, we know why (11) is a logical truth.

We are not so well off as regards (12). Neither Frege, nor anyone else to my knowledge, has an account to offer for (9) such that on that account we would know why if (9) were true, so too must be (10), and so why it is that (12) is logically true. Frege's accounting of (6) does not apply in parallel to (9) since there is no

concept answering to the phrase 'is very tired'. (I assume this will be granted and do not consider how we know that this is so.) Once again, the difference in our knowledge of (11) and (12) is reflected in the fact that we know how to translate (6) into a notation such as Begriffsschrift, but we do not know how to similarly translate (9); whereas we can express (6) and (10) in Begriffsschrift so that (10) is derivable from (6), we are unable to do the same with (9) and (10). Again, the logical structure of (6) is -- as treated here -- that of a simple predication wherein a proper name and a predicative expression concatenate to give a sentence. Frege's account of predication tells us (if correct) the conditions under which predications are true, and in so doing instructs us, by implication, on why the likes of (11) must be true -- or rather, accomplishes this, if at all, in conjunction with his account of existential statements.

We have considered two examples of logical truths with regard to which we are able to explain why it is that they are logically true. I wish to generalize a thought implicit in the examples. In general, we may say, we know why a sentence which is a logical truth is such when we are able to recognize parts and modes of combination of these parts in the particular case so that, in terms of a general account of the referential and (perhaps) non-referential function of sentence parts of

these sorts, and a general account of how such parts form wholes, we can show why -- given that the referential work is performed according to plan -- the sentence must be true. To provide such an account, in general, for the logical truths of some system of logic would be to give a semantics for the system. I believe Frege intended remarks of his in Grundgesetze to constitute a semantics for Begriffsschrift in this sense. At the same time, Begriffsschrift is conceived, in the ideal, as a lingua characterica, a universal language. Indeed I think we can capture much of what Frege had in mind in speaking, in his "Introduction" of Grundgesetze of a "thoroughgoing development of my logical views"⁶⁴ with the idea that Frege now thinks of a lingua as requiring a semantics as an integral part.

Since any sentence figures in logical truths, not just as a whole, but through its various parts, to possess a lingua would be to possess the means for spelling out, with regard to any sentence in the lingua, the conditions under which it would be true, its truth-conditions. And since any sentence translates into a lingua, in effect a lingua would provide the means for stating truth-conditions for any sentence. To possess a lingua then, would be to have an understanding of an exhaustive set of ways

64. Grundgesetze, p. 6.

in which a piece of language could be true. It would be, we might say, to have an answer to the question: How is truth possible? For we could respond to this question ostensively: In one of these ways. That is, by sentence parts of such and such sorts coming together in one of thus and so ways. (Remember, this question "How is truth possible?" is not to be confused with the problems earlier mentioned and set aside.)

It is in this light that I would have us consider the theory of reference Frege provides for his Begriffsschrift in Grundgesetze. Perhaps the most important conclusion that Frege drew from his attempt to say how truth is possible was that, as he might have put it, unsaturation is unavoidable. Understanding what he meant by unsaturation and why he was led to this conclusion will be major concerns in the ensuing chapter.

Consider the sentence 'Two is a prime number'. Linguistically we distinguish two parts: a subject 'two' and a predicative part 'is a prime number'. With the latter we usually associate assertive force. Yet this is not essential. If an actor utters a statement on the stage, it cannot be said that he actually asserts anything, nor is he responsible for the truth of that statement. Let us eliminate the assertive force from the predicative part, as it is inessential. The two parts of the sentence will still remain as distinctly different as they are, and it is important to grasp the point that this difference cuts deep and must not be blurred. The first part, 'two', is a proper name of a certain number, denotes an object; it itself is something complete, which does not require a complement. The predicate part, 'is a prime number', on the other hand, does require a complement, and does not denote an object. I shall also call the first part saturated and the second part unsaturated. To this distinction among the symbols there naturally corresponds an analogous distinction in the realm of denotations: to a proper name corresponds an object, and to the predicative part corresponds what I will call a concept. This is not meant to be a definition. For the decomposition into saturated and unsaturated parts must be regarded as a primitive feature of logical structure, which must simply be recognized and accepted but which cannot be reduced to something more primitive.

FREGE

C H A P T E R I I I

THE SEMANTICS OF FUNCTION AND ARGUMENT

§9. Exposition of Notation

We begin this chapter with a sketch of the basic ideas of Frege's system of logic, as presented in Grundgesetze, by following Frege through his exposition of primitive notation, supplementing this with occasional remarks of historical or clarificatory nature. In this exposition I follow Frege's order of introduction of

primitive names, with one exception. Discussion of the notation for definite description is deferred to the next section (§10) where it serves to set the stage for what comes next. This is an attempt to elucidate Frege's semantics through seeing it as flowing naturally and powerfully from certain assumptions about apparently complex names. After some further discussion of certain aspects of Frege's philosophy of set theory, we take up the so-called paradox that the concept horse is not a concept and attempt to assess the importance of this claim within Frege's philosophy.

In Grundgesetze Frege builds from eight primitive names. Each is a function-name. There are no purely logical singular terms which are primitive. Singular terms, or proper names, do occur in the purely logical notation; they make their first appearance in Frege's exposition when the resources for constructing sentences become present. This since Frege holds that sentences are proper names, each a name of one of two objects, the True and the False. The existence of this pair of logical objects is assumed in the ensuing explanations.

The first introduced name is the symbol

— ().

This little stroke, with its attendant argument place shown by the pair of parentheses, is simply called the horizontal. The horizontal, it is explained, is a

function-name, a one-place function-name. It denotes a one-place function which takes objects as arguments and yields objects as values; in particular, when the True is taken as argument this function yields the True as value, and in all other cases, i.e., for the False, or any other object as argument, it yields the False as value. The horizontal has the effect, then, of double-negation. It might be read 'It is the case that . . .', at least when a sentence fills its blank. Since it denotes a function from objects to objects, only names of objects, proper names, may fill its blank. I shall generally use open parentheses as above to indicate positions for proper names, and use apostrophes to indicate distinct positions as required.

In Begriffsschrift what was outwardly the same sign had been called the content-stroke. Its function had been somewhat vaguely characterized as serving to combine "the signs that follow it into a totality" so that they "have a content that can become a judgment."¹ This sort of talk is absent from Grundgesetze. Semantically speaking, the horizontal is on a par with the sign for negation, next introduced. Still the horizontal is peculiar, and its peculiarity is reflected in the fact that it is a syntactic requirement of Frege's Begriffsschrift that

1. Begriffsschrift, p. 12.

a well formed formula begin with the horizontal stroke.
 In (§15) I shall comment further upon this.

The negation-stroke is introduced as a short vertical line placed mid-way along the horizontal thus

$$\neg ().$$

This expression is also a one-place function-name. It denotes a function which yields the False when the True is taken as argument, and yields the True for any other object as value. So, for instance, if Frege had a proper name for the Sun, say '☉', then

$$\neg \text{☉}$$

would be a name of the True.

The horizontal and the negation-stroke are thus, in effect, the two singular truth-functions. However, (i) they are names of functions, not syncategoremata, and (ii) these functions are explained for all objects, not just truth-values, as arguments. Further, each function takes only objects as arguments; they are first-level functions. As well each yields, for any object as argument, a truth-value as value; Frege terms function-names which denote functions with this property concept-words. Both the horizontal and the negation-stroke are one-place, first-level concept-words. Concept-words, as one might expect, denote concepts; concepts are a special case of functions, viz., those which yield a truth-value for every argument. Predicative expressions are, for Frege,

concept-words, but no primitive logical expressions are predicative expressions, unless the identity-sign is so counted.

Frege has, as primitives, two two-place, first-level function-names. Each of these denotes a function which uniformly yields a truth-value when both places have been filled with arguments. Frege calls such two-place functions relations, and the expressions which denote them, relation-words. However, it will often prove simpler if we allow 'concept' to apply generally to functions which yield only truth-values when fully completed, so that we have one-place, two-place, n-place concepts, and similarly with concept-words.

The symbol for identity, '=', is a primitive two-place concept-word. Frege explains

$$()=(\circ)$$

as denoting a function which yields the True just when the "()-argument" is the same as the "(\circ)-argument"; in all other cases it yields the False. Here we make use of our apostrophe convention for holding open distinct places in the concept-word. Note that identity, like all other first-level functions, takes only objects as

arguments.²

In Begriffsschrift Frege had held that what he there called "identity of content" differs from (e.g.) negation and conditionality in that "it applies to names and not to contents."³ There he used a triple-bar, and regarded names which flank this sign as thereby giving up their ordinary function of referring to objects in favor of referring to themselves in some manner, so that identity statements spoke of the flanking expressions, saying of them that they had the same content. Evidently Frege was moved to this view, which on its face is puzzling and difficult to integrate with the other doctrines of Begriffsschrift,⁴ through noticing that different identity

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2. Knowing that negation and conditionality, which is to be introduced, are an adequate basis for truth-functional logic, and that '=' functions as the bi-conditional when it is flanked by sentences, we may wonder whether Frege needed '=' as a piece of primitive notation. He knew that conjunction and disjunction could be defined in terms of negation and conditionality. So could we not define '=' as we would the bi-conditional, as a conjunction of conditionals in the standard manner? No. For then all identity statements (whose terms do not refer to truth-values) would come out true given the Begriffsschrift interpretation of conditionality. Frege in fact held that identity was an indefinable logical notion.
 3. Begriffsschrift, p. 20.
 4. The axiom Frege gives for identity in Begriffsschrift may be expressed: (i) either not $(\forall x \text{ and } x=y) \text{ or } \forall y$, and (ii) $x=x$. But given Frege's interpretation 'x' must be ambiguous, meaning one thing in its first occurrence in (i) and something else in its second occurrence and in (ii).

statements may differ in epistemic interest, while at the same time holding that one had exhausted the meaning of a name when its denotation had been specified. The fact that the morning star is the evening star requires the confirmation of astronomy, while this is not so with the fact that the morning star is the morning star. Yet if reference is all there is to content (at least in the case of names of objects) how shall we account for the difference in interest that attaches to these two claims? In "On Sense and Reference" Frege rejected the Begriffsschrift reply of ruling names ambiguous in reference between their occurrences on one side of an identity sign and elsewhere. Here and henceforth he distinguishes between the sense and the denotation of proper names, and signs generally. The view which Frege came to hold would seem to be one on which the content of no name is exhausted by its having denotation; there are (e.g.) no Russellian proper names. Any expression of which we could sensibly inquire into its denotation has sense, though some of these same expressions fail of reference. This allows the Grundgesetze explanation of '='. Our differing interest in the members of such pairs of identities as 'The morning star is the morning star' and 'The morning star is the evening star' is seen as arising from difference in sense of the names

which flank the expression of identity.⁵

The next symbol, or rather family of symbols, Frege introduces is designed to capture the idea of generality, or to be quite explicit, generality-with-respect-to-objects. Here is an example,

$$\underbrace{\alpha}_{\text{}} - \phi(\alpha).$$

In this ' ϕ ' is a place-holder for a first-level function-name; it functions analogously to our open parentheses in first-level function-names. And we shall use distinct Greek capitals to mark distinct argument positions. All but ' ϕ ', in the example, is the name of a particular function. It is a second-level function-name; it denotes a second-level function. As first-level functions take objects as arguments, so second-level functions take first-level functions as arguments. However the value of this function for appropriate arguments is an object, and this is true for functions generally. This generality

5. Senses come in only for incidental discussion in this essay. Still I wish to at least register the idea that Frege's talk of senses need not commit him to an epistemological Platonism in his theory of meaning. The thought is that we would construe the move from S and R are the same in sense to the sense of S is the same as the sense of R on the model of the move from line a is parallel to line b to the direction of line a is the same as the direction of line b; thus to avoid epistemological Platonism, unless it can be shown that simply holding that sameness and difference in meaning are objective relations itself involves a commitment to epistemological Platonism in one's theory of meaning.

function takes one-place, first-level functions as arguments and yields the True when the argument function itself yields the True for all objects as arguments, and yields the False in all other cases. Other names of this generality function may be formed by using other letters uniformly in place of '□' of our example. However before this becomes necessary we shall have reverted to more common orthography.

In this notation we have, in effect, the universal quantifier.⁶ But note that (i) like the previous "connectives", this symbol has denotation, and (ii) that the generality function is explained for first-level, one-place functions generally as arguments. Its denotata will, however, in every case yield a truth-value as value; so this function is itself a concept, a second-level

6. It is worth recording Frege's reasons for introducing some such notation as that of his concavity symbolism for the expression of generality. These concern the need to keep track of scope distinctions. Near the end of "On the Purpose of the Begriffsschrift" (1883) Frege mentions the need to distinguish between putting something forward as conditional upon a generality and putting forward the generality of a conditional; schematically, the difference is between $(x)A \supset B$ and $(x)(A \supset B)$. Similar remarks occur in Begriffsschrift, p. 25. In Grundgesetze (p. 41) he mentions the need to distinguish the negation of a generality from the generality of a negation. These, by the way, are just the reasons which Alonzo Church gives to support his use of the quantifier notation, as an examination of the pair of examples on p. 46 of Introduction to Mathematical Logic will show.

concept, and the "quantifier" is a second-level concept-word.⁷

Thus the following

$$\neg \sqcup (\sqcup),$$

where the initial horizontal has been absorbed into the line segment preceding the concavity,⁸ is a name of the False. And so is

$$\neg \sqcup \top (\sqcup).$$

Whereas both

$$\top \sqcup (\sqcup)$$

and

$$\top \sqcup \top (\sqcup)$$

7. We may note that, in more modern notation, '(x)Fx' has just the two significant parts, '(x) . . . x' and 'F()', on Frege's analysis. Hence there can be no "vacuous quantification" nor any question of quantifier variables matching quantified variables in Frege's system. If we wish to follow contemporary practice and speak of the variable 'x' occurring in such a formula as the above it would be more appropriate, as Furth suggests (Editor's Introduction, *Grundgesetze*, p. xxxii), to use Peano's term "apparent variable" instead of Hilbert's "bound variable". For real (free) variables Frege uses Roman letters, letters of a style distinct from the apparent (bound) variables. These he also uses for stating the axioms of his system. For more on this see note 27.
8. "...we regard ' \top ' as composed of the small vertical stroke, the negation-stroke, and the two portions of the horizontal stroke, each of which may be regarded as horizontals in our sense. The transition from ' $\top(\neg \Delta)$ ' or ' $\neg \top \Delta$ ' to ' $\top \Delta$ ', as well as from ' $\neg \neg \Delta$ ' to ' $\neg \Delta$ ', I call amalgamation of horizontals." *Grundgesetze*, pp. 39-40.

are names of the True, since they are negations of the preceding pair. The last example we might, with caution, read as

There is something,
since

$$\exists x \phi(x)$$

is, in effect, the existential quantifier; as Frege says, ". . . this is how 'there is' is to be rendered in the Begriffsschrift."⁹ This view is also stated explicitly by Frege in his essay "Function and Concept",¹⁰ and was present in Begriffsschrift¹¹ and Grundlagen,¹² though without the semantics of Grundgesetze. Also in "Function and Concept" Frege echoes his Grundlagen opinion that "the ontological proof of God's existence suffers from the fallacy of treating existence as a first-level concept."¹³ We learn from Carnap that Frege continued in this belief in 1913.¹⁴

It is instructive to contrast Frege's views on

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9. Grundgesetze, p. 42.
 10. Philosophical Writings, pp. 37-38.
 11. Begriffsschrift, p. 27.
 12. Grundlagen, p. 65.
 13. Philosophical Writings, p. 38. Cf. Grundlagen, p. 65.
 14. Rudolph Carnap, "Intellectual Autobiography" in The Philosophy of Rudolph Carnap, edited by P.A. Schilpp, p. 6.

quantification with those of the Algebra of Logic school (\$6). Here we may quote van Heijenoort again.

The universality of logic expresses itself in an important feature of Frege's system. In that system the quantifiers binding individual variables range over all objects. . . . Boole has his universe class, and De Morgan his universe of discourse, denoted by '1'. But these can be changed at will. The universe of discourse comprehends only what we agree to consider at a certain time, in a certain context. For Frege it cannot be a question of changing universes. One could not even say that he restricts himself to one universe. His universe is the universe, . . .¹⁵ [which] consists of all that there is . . .¹⁵

We have mentioned that Boole thought of his symbolism as admitting of alternative interpretations, and the objection of Frege that since Boole's interpretations are incompatible they cannot be called upon simultaneously so that much we wish to say cannot be articulated in the symbolism. However, from Frege's point of view, there is a more fundamental difference between himself and Boole. Boole regarded his notation as requiring some interpretation or other, whereas Frege's Begriffsschrift was intended as a language whose formulas stood in no more need of interpretation than the sentences of natural languages; indeed less so, since they would be unambiguous. To understand ' $(x)(x=x)$ ', or ' $\neg \exists \neg \forall = \forall$ ', it is no more

15. van Heijenoort, "Logic as Calculus and Logic as Language", p. 325.

necessary to specify a universe of discourse than it is to do so for the sentence 'Everything is identical with itself'. This point must not be confused with the fact that any new notation must be explained to us so that we catch on to how it goes.

This difference in attitude is still very much with us. Quine, for instance, speaks for Frege, as well as himself, in such a passage as this.

The existential quantifier . . . is a logically regimented rendering of the "there is" idiom. The bound variable 'x' ranges over the universe, and the existential quantifier says that at least one of the objects in the universe satisfies the appended condition¹⁶

On the other hand, when Donald Kalish, whom we may take as a representative of the current model theoretic approach to logic, provides what he calls a semantics for a syntactical system, not in terms of the notion of truth, but instead in terms of "the more general notion of truth in a model,"¹⁷ he identifies himself with the Boolean attitude. For one element of a model is a specified set of entities intended to serve as the values of the variables of the syntactical system. There is nothing contra-Fregean about model theory, per se. The point is just

16. Quine, "Existence and Quantification" in Ontological Relativity, p. 94.

17. Donald Kalish, "Semantics" in Edwards' Encyclopedia, p. 351.

that if, with Frege, you regard the laws of logic as laws of truth you will not think of yourself, while doing model theory, as studying a "more general notion" than that of truth.¹⁸ To similar matters we shall return in (§17).

To pull together some of the remarks on Frege's notation made thus far, let us introduce into the notation the expressions 'the morning star' and 'the evening star', abbreviated 'MS' and 'ES' respectively, each with its customary sense and reference. Then each of the following

$$\text{MS} = \text{MS}$$

$$\text{MS} = \text{ES}$$

denotes the True, and so likewise does

$$(\text{MS} = \text{MS}) = (\text{MS} = \text{ES}).^{19}$$

18. Gödel has remarked on much this contrast as follows. "Mathematical Logic . . . has two quite different aspects. On the one hand, it is a section of Mathematics treating of classes, relations, combinations of symbols, etc., instead of numbers, functions, geometric figures, etc. On the other hand, it is a science prior to all others, which contains the ideas and principles underlying all sciences. It was first conceived by Leibniz in his Characteristica universalis, of which it would have formed a central part. But it was almost two centuries after his death before his idea of a logical calculus really sufficient for the kind of reasoning occurring in the exact sciences was put into effect (in some form at least, if not the one Leibniz had in mind) by Frege . . . (Kurt Gödel, "Russell's Mathematical Logic" in The Philosophy of Bertrand Russell, edited by P.A. Schilpp., p. 125.)

19. Note that these three expressions are not formulas of Begriffsschrift since they lack an initial horizontal.

We may, from this denotation of the True, selectively delete two occurrences of the proper name 'MS' so as to obtain the one-place function-name

$$(() = MS) = (() = ES).$$

This in turn may be universally quantified; that is, we may replace the place-holding 'ø' in the second-level function-name previously discussed so that we obtain

$$\forall x ((x = MS) = (x = ES)).$$

This we know also denotes the True. We may express this knowledge otherwise by saying that the concept-words,

$$() = MS$$

$$() = ES,$$

denote concepts which are such that whenever they take the same object as argument, they yield the same object as value.

Identity, we know, holds only of objects. Thus we are precluded from inquiring into the "identity-conditions" of concepts. Nevertheless it is clear that Frege viewed concepts (and functions generally) as indistinguishable when, as with the example pair, for every argument they yield the same value.²⁰

We may also note that the above pair of concept-words must, by reasoning parallel to that implicit earlier on, be distinguishable in sense, since completing each with

20. See note 51, and the passage it notates.

(e.g.) 'MS' gives a pair of sentences distinct in sense.

Before getting back to Frege's exposition of his primitive function-names, here seems to be a good place to mention briefly an expression passed over. I refer to the turnstile, '⊢', which Frege calls the assertion-sign. It is Frege's contention that with such (complete) formulas as we have exhibited thus far we have expressions which "only designate a truth-value, without its being said which of the two it is."²¹ To write, for instance,

$$\text{---} \sqcup \text{---} \alpha = \alpha,$$

is merely to denote a truth-value, and not, as we would wish to assert thereby that everything is the same as itself. "We therefore require another special sign to be able to assert something as true."²² For this purpose Frege lets '⊢' precede the name of a truth-value -- more strictly, the assertion-sign prefixes what we have called well-formed formulas, a requirement upon which is that the left-most symbol be the horizontal -- so that, for example, with

$$\text{⊢} \sqcup \text{---} \alpha = \alpha,$$

it is asserted that everything is the same as itself.

Frege regards the assertion-sign as composed of a vertical component, which he calls the judgment-stroke,

21. Grundgesetze, p. 37.

22. Ibid., p. 37.

and a horizontal, in the explained sense, thereby doubly guaranteeing that what is asserted is either true or false, or at least denotative of either the True or the False.

Frege's remarks on assertion and its sign, those quoted and those companion to them, are puzzling and worth puzzling over. On these matters I shall have something to say later (§15). At this juncture I shall simply record two observations. First, the assertion-sign is neither a name, nor a part of any name; in this respect it is like the sign which Frege uses to introduce definitions, the double-stroke of definition, and the signs which he uses to mark inferences, called transition-signs.²³ Secondly, Frege tells us that he calls "the presentation in Begriffsschrift of a judgment by use of the sign '⊢' . . . a proposition of Begriffsschrift or briefly, a proposition."²⁴ Since Frege's Basic Laws (axioms) are propositions, and the rules of inference are stated to insure that only propositions may be inferred from one or more propositions, it is propositions, and not names of truth-values, which form the substance of Frege's logical theory, or so we might wish to say.

23. See Grundgesetze, pp. 82-83.

24. Grundgesetze, p. 38. Frege's term here is 'Satz'; Furth renders it with 'proposition' for reasons he gives on pp. lv-lvi of his "Editor's Introduction" in Grundgesetze. Later, in (§15), I shall call these expressions assertions.

Returning to Frege's exposition, after the generality notation which we have discussed Frege introduces his notation for the course-of-values of a function,

$$\lambda\phi(x).$$

This, like the generality notation, is a one-place, second-level function-name, ' λ ' serving, as before, to hold open a place to be filled with a one-place, first-level function-name. However the function which it denotes is not a concept; in this respect it differs from functions met with through primitive notation to this point. This function takes its argument, a one-place, first-level function, into what Frege calls the course-of-values of the function; as a special case it takes one-place, first-level concepts into their extensions. What then are courses-of-values, or extensions? Frege stipulates that whenever two one-place, first-level function-names denote functions which yield, in parallel fashion, the same values for the same arguments, the course-of-values-names formed from the two function-names denote one and the same course-of-values. And generally, such indistinguishable functions have identical courses-of-values. Such is the force of Basic Law V. It is this which enriches Frege's fundamental logic by set theory, and in such a fashion that the way is open to Russell's paradox.

If we inquire via current set-theoretic notions as

to what courses-of-values are the most appropriate answer would be that each is an infinite set of ordered pairs $\langle x, y \rangle$ each of whose y-member is the value of the particular function in question for the x-member as argument. Extensions are sets of ordered pairs all of whose y-members are truth-values. If we go through an extension and gather together all the x-members whose co-ordinate y-member is the True we are left with the class of objects which satisfy the concept whose extension we have gone through. In this way we can "recover" classes from extensions. Subsequently I shall, on occasion, speak of such classes and extensions more or less interchangeably.

Next Frege uses the course-of-values notation to introduce a surrogate definite article. I hold off on this notation and its explanation until the next section.

Frege then introduces the second primitive, two-place, first-level function-name,

$$\neg(\ ').$$

With it the truth-functional and first-order part of the notation is complete. This denotes the conditionality-function which is explained as yielding the False when the "lower-argument" is the True and the "upper-argument" is any object other than the True, and yielding the True for all other combinations. Thus, the earlier remarks on "connectives" being not syncategoremata and taking

objects in general as argument apply here as well. So, for instance, if the Sun was the "lower-argument", then the True will be the value for whatever object is the "upper-argument".²⁵

Some pages on Frege completes the primitive vocabulary he will actually use in Grundgesetze derivations with notation for second-order generality; more specifically, generality-with-respect-to-(one-place)-first-level-functions. Here is an example

$$\underbrace{f}_{\mu\beta}(f(\beta)).$$

In this ' $\mu\beta - \beta$ ' is a place-holder for a second-level function-name of one argument -- the concavity notation for "universal quantification", for example -- where ' μ ' stands for what will indicate the particular second-level function, and ' β ' marks the "apparent variable"

25. For notation thus far introduced Frege provides the following axioms.

BLI. $\begin{array}{l} \vdash a \\ \vdash b \\ \hline a \end{array}$ $\vdash a$ BLIIa. $\begin{array}{l} \vdash f(a) \\ \vdash f(\alpha) \\ \hline f(\alpha) \end{array}$

BLIII. $\begin{array}{l} \vdash g \underbrace{\vdash f(a)}_{f(b)} \\ \vdash f(b) \\ \hline g(a=b) \end{array}$ BLIV. $\begin{array}{l} \vdash (\neg a) = (\neg b) \\ \vdash (\neg a) = (\vdash b) \end{array}$

BLV. $(\underbrace{\vdash f(c)}_a = \underbrace{\vdash g(a)}_a) = (\neg \underbrace{\vdash f(a)}_a) = g(a)$

The remaining axioms are given in notes 26. and 29. See notes 7. and 27. on the use of Roman letters (free variables) in the axioms.

which will occur, for example, within and without a particular "universal quantifier". What is named is a function which takes one-place, second-level functions as arguments. So it is a third-level function. In particular, this function yields the True as value when the second-level function which is the argument itself yields the True as value for all its arguments, and yields the False for second-level functions which yield the False for any argument. Here again the function in question is, as well, a concept. A simple example of a Begriffsschrift formula which uses a name of this function is

$$\neg \forall f (f(\perp)).$$

This is a name of the False, since not every object falls under every function.²⁶

Frege's system allows for an ascending hierarchy of levels of functions and function-names. In a way his semantics compels such ascent. Further, functions of mixed levels are possible; these ramify the hierarchy. However Frege makes no use of such mixed functions, nor does he go beyond the third-level of functions in his

26. This notation is governed by the following axiom.

$$\text{BLI Ib. } \begin{array}{l} \vdash m_{\beta} (f(\beta)) \\ \quad \vdash m_{\beta} (f(\beta)) \end{array}$$

Grundgesetze derivations. 27

\$10. Definite Description

Frege's logical theory has struck many as rich in ontology, what with its truth-values, truth-functions,

27. I have observed (note 7.) that Frege states his axioms with Roman letters which in effect are free variables. Thus the axioms as they stand are without truth-value (see Grundgesetze, pp. 67-68), and so we need to think of them as implicitly universally quantified. (See Grundgesetze, p. 90.) But note that the introduction of appropriate "quantifiers" would introduce names of particular functions into the axioms. If instead of an axiom ' $x=x$ ' -- which is not itself a Grundgesetze axiom -- we were to use ' $(x) (x=x)$ ', we would have introduced a second-level function-name. And then, given Frege's semantics, we would be licensed to replace this name with an (appropriate) variable, binding it with an initial "existential quantifier". But then here too we would have, in the new "quantifier", a function-name, specifically a third-level function-name. And, once again, Frege's semantics licenses "existentially generalizing" upon the position it occupies. And so forth. At each step we introduce an expression which the semantics says is a name, and hence one which occupies a position accessible to quantification. Perhaps Frege's reason for formulating his axioms with free variables was, at least partially, the potentiality for such a regress.

What this points up is the semantic incompleteness of Frege's system of logic; the semantics always tells us of logical truths not covered by a particular set of axioms, or does so provided only that Frege's semantical pattern is not arbitrarily distorted at some place. (This point may be clearer after we have spent some time working out the semantical pattern.) And this incompleteness (which is not to be confused with Gödel incompleteness) shows that Frege's semantics is at odds with the idea, discussed earlier (§5), that logic is throughout evident, at least if this is taken to imply that the truths of logic can be encompassed by a system with self-evident axioms.

generality and existence functions, et al. This abundance of entities is closely bound up with that most puzzling of Frege's views, that which received its most well-known expression in the remark that the concept horse is not a concept. Our discussion of definite description, besides being of some independent interest, will serve to set the stage for an extended look at the intertwined origins of Frege's ontology and his views on unsaturation.

Frege desired that his notation have a symbol which would substitute for the ordinary definite article. In accordance with what in Grundgesetze he described as his "basic principle", "that every correctly-formed name is to denote something",²⁸ Frege required a piece of notation which would have denotation whenever completed to a well-formed name. When the definite article aids in definitively describing an object, our surrogate notation should do likewise. But what will be the denotations for those visibly similar expressions which do not manage to single out just one thing? Frege's suggestion amounts to treating expressions of the form 'the F', when there are no, or more than one, Fs, as denoting the class of Fs -- more exactly, they shall denote the course-of-values of the F-function. For the intended surrogate he introduces

28. Grundgesetze, p. 9.

the notation,

$\lambda()$.

This is to denote a one-place, first-level function in accordance with the following stipulation. If the argument is such that there is an object, z , such that the argument is $\overset{?}{x}(z=x)$, then z is the value of this function for that argument; if there is no such object, z , then the value of this function for that argument is the argument itself. Intuitively, if the argument of the function is a class, then if the class is a unit-class, the value of the function will be the sole member of that class, and if the class is not a unit-class, the class itself will be the value of the function. And if some object other than a class is the argument of the function, then that object will be the value of the function.²⁹

A few examples are helpful. In each case let 'a' abbreviate the given name of the argument.

(i) If a is the course-of-values of the function, natural satellite of the earth, $\lambda \underline{a}$ is the moon, since it is an object, z , such that $\underline{a} = \overset{?}{x}(z=x)$, i.e., the identity statement,

29. This notation is governed by the following axiom.

$$\text{BLVI. } \vdash \underline{a} = \lambda \overset{?}{\epsilon}(a=\epsilon)$$

Frege collects his axioms together on p. 105 of Grundgesetze.

$\lambda(x \text{ is a natural satellite of the earth}) = \lambda(x=\text{the moon})$, denotes the True.

(ii) If \underline{a} is the course-of-values of the function, author of Principia Mathematica, $\lambda \underline{a}$ is the course-of-values of this function, on the grounds that since the work was co-authored there is no object, z , such that $\underline{a} = \lambda(z=x)$.

(iii) If \underline{a} is $\lambda(2=x)$, $\lambda \underline{a}$ is $\lambda(2=x)$ -- and similarly for any course-of-values which is not the extension of a concept.

These three cases are the ones Frege mentions. He does not explicitly consider what happens when, say, 'Everything is self-identical' or 'Frege' completes the function-name. But it is easily seen that, just as with (iii), so with these cases the value of the function will just be the argument. Thus the following will be true: $\lambda \text{Frege} = \text{Frege}$.

Frege concludes this section of Grundgesetze by remarking on a certain "logical danger". If, from such an ordinary expression as 'author of Principia Mathematica' (I change Frege's example) we were to form the proper name 'the author of Principia Mathematica',

we should commit a logical error, because this proper name, in the absence of further stipulations would be ambiguous, hence even devoid of denotation . . . and if we were to give this proper name a denotation expressly, the object denoted would have no connection with the formation of the name,

and we should not be entitled to infer it was [an author of Principia] while yet we should be only too inclined to conclude just that. This danger about the definite article is here completely circumvented, since ' $\bar{x}\emptyset(x)$ ' always has a denotation, whether the function $\emptyset(x)$ be not a concept, or a concept under which falls no object or more than one, or a concept under which falls exactly one object.³⁰

Perhaps Frege is here inveighing against the sort of suggestion he once offered for dealing with rotten descriptions, those which fail to describe uniquely. Repair the rot "by means of a special stipulation, e.g., by the convention that 0 shall count as its reference, when the concept applies to no object or to more than one."³¹ Still the intent of the above passage is not clear. What advantage does Frege see gained by the Grundgesetze technique? The inference which we are "only too inclined" to make is no better, since the conclusion on either of the two suggestions will be false. Perhaps an advantage is seen in this. On the Grundgesetze view we shall be less inclined to fallaciously draw the conclusion. Hence the virtue of the apparently elaborate symbolism of Frege's surrogate definite article.

If the goal of weakening an inclination to draw what may be an illogical inference is a worthy one, we might

30. Grundgesetze, pp. 50-51.

31. "On Sense and Reference" in Philosophical Writings, p. 71.

wonder whether it could not be achieved some other way. A suggestion to the point would be to weaken Frege's requirement that the surrogate 'the F' itself denote at every occurrence to the weaker requirement that all expressions (not themselves definite descriptions) which contain definite descriptions denote. Let us hold onto this idea for a paragraph.

A puzzling feature of Frege's theory is that the function-name which goes surrogate for the ordinary descriptor is a first-level function-name. Confronted with the theory we naturally take '\ ' as the canonical substitute for the ordinary 'the' (in at least one prominent use of this expression.) But when thus viewed, the canonical 'the' seems to perform a double role; it takes the function-name which follows it into its course-of-values, and then yields the "sole member" or the course-of-values itself, as the case may be, as value. Would we not gain a closer fit with intuition, and at the same time simplify theory, if we introduced the descriptor function as a second-level function understood as yielding, for F as its argument, the sole F, if such there be, and otherwise yielding, say, the class of Fs. The effect of this would be to locate description at the same level

with quantification.³²

The point of these reflections is this. If we combine the weakened requirement of a paragraph back with the suggestion that we treat definite description and quantification as both second-level functions we shall be led naturally and directly to a version of Russell's theory of descriptions. Setting aside the rotten cases, 'The F is G' may be understood by a Fregean as 'There is just one F and it is G'. It is then not difficult to see how to use quantification and identity to re-express 'The F is G' in a way that rules the rotten cases false, provided we have allowed to lapse the requirement that 'the F' must always have denotation.³³

Why did Frege treat his descriptor as a first-level

32. Geach makes this point in a reverse manner when he observes that

when Russell says that expressions like 'the King of France' are not names but incomplete symbols, he is saying what would be put thus in Frege's terminology: 'In "the King of France is bald", "the King of France" is not a name of an object; what it stands for is something incomplete, ... a second-level concept, within which a concept falls if and only if there falls under it someone who is a King of France and apart from whom nobody is a King of France. (Philosophical Writings, editor's note, p. 51.)

33. The reader unfamiliar with the procedure may consult Quine's Methods of Logic, for one work, where will be found a clarity of exposition surpassing any Russell himself provided.

function-name? I do not know. Perhaps it was through treating such expressions as 'the natural satellite of the earth' on the model of 'the successor of zero'. Whatever may have suggested the view to Frege, we can come to understand why a Russell-like treatment would have remained alien to him even were he to have located description up with the quantifiers. To see why this is so is to be well into an appreciation of the considerations moving Frege towards his ontology of truth-values, truth-functions, et al., with its attendant unsaturation. To anticipate, I shall suggest that Frege would have felt great resistance to our suggestion that expressions which in many ways behave like singular terms can be treated as contributing to the reference of sentences which contain them without themselves being treated as names, expressions with a denotation, that is, he would not like our weakening of his "basic principle". Subsequent remarks on the "origin" of Frege's semantics will bring out why.

Before leaving definite descriptions for the present, a few additional observations are worth registering. One of Russell's motivations in coming up with his theory of descriptions stems from problems incurred in saying what does not exist. One type of negative existential claim is that there are no things of a certain kind, that unicorns do not exist, for instance. This yields unproblematically to Fregean, as to Russellian, resources, going

straightforwardly into a negated existential: $\neg(\exists x)(x \text{ is a unicorn})$. Another type of negative existential involves attaching a denial of existence to a (putative) singular term. Thus we have 'The greatest prime number does not exist', for instance; or again, 'Pegasus does not exist'. The difficulty here is to see how these claims could be true, as we assume they are. Although Frege does not discuss this question, we can see how with the first example, that of 'the greatest prime', we might fashion an answer. Keeping the explanation of his surrogate definite article in mind we give this case the following treatment: $\neg(\exists x)(x = \lambda y(y \text{ is a prime greater than all others}))$ and $x \neq \lambda y(y \text{ is a prime greater than all others})$. Here we incorporate Frege's descriptor and course-of-values abstractor into more common orthography.

What now of the other example, that involving 'Pegasus'? It is widely believed that Frege held what Geach has called the "disguised-description theory of proper names",³⁴ that is, that such unstructured expressions as 'Frege', 'Aristotle', and 'Pegasus' are equivalent in content to some (possible) definite description. For example, 'Frege' might have the sense of 'the greatest logician of the 19th century' for some. This is Frege's view with regard to numerals. But it must be said that

34. Three Philosophers, p. 137.

textual evidence that he held it generally is very slim indeed. There is the footnote in "On Sense and Reference",³⁵ but other than this the only passages which bear directly on the question (in writings Frege published) are in a quite late paper.³⁶ Perhaps the widespread attribution of the "disguised-description" thesis to Frege reflects, in part, a difficulty in imagining what would be the sense of a simple proper name if it were not that of some possible description. Be that as it may, would this thesis open the way to a treatment of (e.g.) 'Pegasus does not exist'? Yes, at least schematically; simply replace, in the earlier formula, 'is a prime greater than all others' uniformly with 'Pegasizes', which itself may be regarded as an abbreviation for (and failing that a place-holder for) an appropriate predicative expression.

Let us be clear that our use of 'Pegasizes' here does not imply any claim about the eliminability of proper names. (We are not using 'Pegasizes' as Quine might.) Nor do we want it to. For we should not confuse the view that any name can be replaced with a definite description, which there is some reason to believe Frege held, with the view that every name can be replaced by a definite description, which there is no reason to suppose Frege

35. Philosophical Writings, p. 58.

36. "The Thought" (1918) in Philosophical Logic.

held. That is, we must not equate the two claims,

A. Any occurrence of a simple name in any sentence can be replaced by some definite description without change of sense.

B. All occurrences of simple names in all sentences are (simultaneously) replaceable by definite descriptions without change of sense.

These are not equivalent since in some cases any definite description capable of replacing a name may itself contain a name replaceable only by a description itself containing such a name, and so forth. So the thesis that every simple name has the sense of some definite description does not imply the eliminability of names.

It has seemed to Leonard Linsky that the way was open for a somewhat different treatment of negative existentials than the one just suggested. He writes that

Frege held that existence is a second-level concept, . . . He also held that the sense of what is ordinarily considered a proper name is the same as that of some definite description. Thus the sense of 'Homer' is perhaps the same as the sense of 'the author of The Illiad and The Odyssey' . . . 'Homer does not exist', correctly understood, must be taken as . . . [asserting] what is asserted by 'It is not the case that one and only one person authored The Illiad and The Odyssey'. Though this may be true, it does not contain any referenceless names, so the problems of negative existentials cannot arise . . . [This] solution . . . is never explicitly formulated by Frege as an answer to the problem . . . rather [it] is a by-product of

Frege's treatment of existence as a second-level concept.³⁷

Linsky seems to suggest that his "solution" simply falls out of the dual thesis that existence is a second-level concept and names have the sense of definite descriptions. But this is not so. A critical step in Linsky's reasoning is the move from (e.g.)

The object which Pegasizes exists,
where 'Pegasizes' is used as before, to

There is exactly one object which Pegasizes.
There is no reason for Frege not to agree that 'there is exactly one object which . . .' denotes a second-level concept; and he might well agree that such pairs of sentences as the above express the same Thought. But this agreement would not commit Frege, in his own eyes, to claiming that the sentences agree in other important respects, such as logical structure and implications. This we know from earlier discussion (§3). So Linsky's elimination requires further argument, at least if it is to be offered on Frege's behalf.

We can see in Linsky's idea a suggestion for the elimination of complex names. But to eliminate complex names would be to eliminate the rationale of Frege's semantics, as we shall see. This is the source of the

37. Leonard Linsky, Referring, p. 29.

resistance I would imagine Frege to have to proceeding along Linsky's lines.

\$11. Complex Names and Predications

In discussing Frege's Dictum earlier on (§1) we mentioned Quine's remark to the effect that it signalled an "important reorientation in semantics away from name, or term, semantics." We also noted Quine's view that this reorientation underlies Russell's theory of descriptions with its use of contextual definitions. I do not wish to take exception to either point. And yet Russell's approach is, I have claimed, alien to Frege. The nub of the reason concerns the eliminability of complex names. For when Frege turned to developing a semantics for his Begriffsschrift he took complex singular terms rather than sentences as his model of significant complexity. In particular, it seems that he was led to his full-blown semantics through generalizing certain impressions gained through pondering the nature of complex arithmetical singular terms.

Central to this semantics is the view that concepts, and functions generally, are unsaturated. If we ask just what Frege means in calling concepts unsaturated, and correlatively, speaking of objects as not so, but rather complete, we get this sort of reply. "'Complete' and 'unsaturated' are of course only figures of speech; but

all that I wish or am able to do here is to give hints."³⁸
 If we ask for Frege's reasons for maintaining this view, it must be admitted that we find very little in the way of direct argument. Still I believe that a more sympathetic understanding of this pivotal idea of Frege's is possible than is commonly evinced. In this section and the next we shall attempt to trace out a plausible path, which Frege may have followed, which leads to the semantics of unsaturated functions. It is hoped that ensuing remarks will constitute something of an explication of such a passage as this one of Frege's.

. . . the words 'the concept square root of 4' have an essentially different behavior, as regards possible substitutions, from the words 'square root of 4' in our original sentence ['there is at least one square root of 4']; i.e. the reference of the two phrases is essentially different.³⁹

Frege's model of significant complexity was that exemplified by such expressions as: $2+3$. We shall focus attention on this for awhile.

Frege did not doubt but what there are numbers, and in Grundlagen he had argued that numbers are objects. Let us assume this is correct. Then the relation between '2', or 'two', as in 'Two is a prime number', and the

38. "On Concept and Object", in Philosophical Writings, p. 55.

39. Ibid., p. 50.

number two may well strike us as transparent and paradigmatic -- or at least as providing one paradigm -- of the way words relate to things. Accordingly let us say that '2' denotes, or names, the number two, and speak of the relation in question is that of denotation. Just as paradigmatically do 'three', '3', '4', '5', 'seventeen', etc. (at least in certain uses) denote numbers; each belongs to a recognizable class of expressions whose members we shall call number-names.

And now consider '2+3'. If we are holding that numbers are nameable, it is not difficult to think of '2+3' itself as naming a number, specifically the number five. Let us then count '2+3' into our class of number-names, and so also 'two plus three', '3-1', '12-5', '(2+3)+7', etc.

We are assuming that '2', at least on occasion, denotes the number two; looking then at '2+3', and assuming it to be a name, it would be difficult to avoid the conclusion that '2' denotes the number two in its occurrence in '2+3'. So we are led to conclude that some number-names have parts which name numbers, that there are complex number-names.

This conclusion, which I have approached as innocently as possible, has an appeal that is easily felt. It is one Frege acceded to unhesitatingly. Still, accepting the idea that there are names with constituent parts

which are also names is a crucial step. Frege took it, so we shall follow.

We assume that '2+3' denotes five and shift our concern to providing an account of how it accomplishes this. That is, taking it to be a fact that '2+3' denotes five we seek an explanation of this fact. But what is the nature of the sought-for explanation? In what terms might we anticipate its framing? Here we should recall the sort of account envisaged in (§8) of why particular logical truths are logically true. And how this led us to the idea that to have a lingua would be to have an answer to the question which we phrased this way: How is truth possible? Though in the case at hand, we are not (directly) concerned with truth, but with the (supposed) fact that '2+3' denotes the number five, we should nonetheless expect what we say here to tie in at some point to our earlier question. Here it seems we wish to require that an essential ingredient of any explanation of the fact that '2+3' denotes five be the fact that '2' is a part of '2+3' and in this occurrence denotes two. It is to be further expected that the explanation will be completed by further specifying "parts" of '2+3' and indicating in some analogous way "relations" these bear to other things. This is rather vague, but that is as it should be at this stage of the game. We may expect that the attempt to provide the explanations we seek will make it clearer to

us what we are willing to count as an explanation. And this process will undoubtedly yield some gain in clarity of the relevant notions of part and relation as invoked above. In the meantime, and again remembering earlier remarks, we shall use 'semantical' to qualify these terms; '2' is to be a semantical part of '2+3', and denotation is a (paradigmatic) semantical relation.

Returning to '2+3', we have recorded the fact that '2', in '2+3', denotes two. A parallel observation may be made regarding '3'. Each of these facts is necessary, we assume, to explain that '2+3' denotes five. But it is clear that they are not, individually or jointly, sufficient. Montgomery Furth has put the point this way.

. . . we cannot regard as a final semantical account of a denoting complex name . . . a simple enumeration of the denotations of all its component parts. For while it is true that a complex complete name is a complete name containing complete parts, and we are taking each complete part to have denotation on its own, an account that stopped here would leave unexplained the most remarkable fact of all, that the complete name is itself a name, denoting a single determinate object, . . . So we know that a further semantical role is being played within the name, over and above the denoting of their respective denotations by the respective complete parts of the name.⁴⁰

These remarks occur in the context of an assumption

40. Furth, "Two Types of Denotation", Studies in Logical Theory (American Philosophical Quarterly Monograph Series, No. 2, (1968), p. 23.

on Furth's part that with (e.g.) '2+3' the only denoting component parts are '2' and '3'. However Furth intends the conclusion more generally. He assumes that the further semantical role played in (e.g.) '2+3' cannot be played by a part which itself denotes an object. Or rather, he assumes that the further semantical role is not played by some (perhaps other) part denoting an object. Though I am inclined to agree with this, I do not see that Furth has set out support for this conclusion. For suppose the reply is made that, appearances to the contrary notwithstanding, the expression which remains when '2' and '3' are deleted from '2+3' itself denotes an object, and that the account we seek of the fact that '2+3' denotes a number, indeed, the number five, is to be found upon examining the nature of this newly introduced object. Furth wants to say that no such reply can be adequate, but I do not find an argument for this rejection. In what follows I shall offer something like an argument for Furth's conclusion, one which plays an essential role in the genesis of unsaturation.

Getting back to the fact that '2+3' denotes five, let us, for simplicity's sake, think of '2+3' as breaking up into just '2' and what remains when '2' is deleted from this. These parts go together, we are assuming, to produce a result which denotes the number five. Our question may then be put as how what remains when '2' is

deleted contributes to this. At this point we may at least infer that the remainder is not itself a name of a number. For if it were, then '2+3' would be a two-item list and not, as it is, a name of a number (assuming it could not be both.) This I think we know. And I think it fair to assume that the difference between a name and a list is reasonably clear in this case. Thus with regard to the role of our "remainder", we are in a position to say this. Either it bears some semantical relation (as we say) other than (what we are calling) denotation to the number two, or it bears some semantical relation (possibly, but not necessarily that of denotation) to something other than two, or any other number. The alternatives, while perhaps not exclusive, are exhaustive.

It is Frege's view that functional expressions, such as are obtained through deleting a number-name from a number-name, have denotation; they denote functions. Thus he takes the second path. Why?

Perhaps he reasoned along these lines. Certainly there are arithmetic functions, or operations, such as those of addition and subtraction. For we know, for instance, that addition is commutative whereas subtraction is not. Since with (e.g.) '2+3' we represent the result of adding together two numbers, surely addition, that is, the arithmetic function of addition, must get mentioned one way or another with the expression '2+3'. And, given

that, it seems clear that '+', or '()+()', must do the mentioning. Or, to revert to our simplified case, it seems that the result of deleting '2' from '2+3', viz. '()+3', must mention an arithmetic function, that of adding three. So we conclude, more generally, that functional expressions mention functions. But why, having come this far, should we conclude that they denote functions? Why, that is, should we liken the semantical relationship holding between functional expressions and functions to that holding between arithmetic names and numbers? A thought here might run this way. Part of the force of saying that number-words denote numbers is that, from their legitimate arithmetic use we can infer that there are numbers. Analogously, it seems we can infer from the legitimate arithmetic use of functional expressions that there are functions and so find appropriate speaking of functional expressions as denoting functions. I touch this thought lightly here; I shall look more closely at a similar idea further down. In any case, let us, for whatever reason, follow Frege in deeming functional expressions denotative of functions.

Before continuing with our semantical journey, let us pause for a few words on functions. Compare '2+3' with the following:

$$1+3 \quad (2+3)+3 \quad (5-2)+3;$$

each has a common part which we can indicate with

()+3.

This incomplete expression (as we may call it, with Frege) along with others which similarly arise, has several notable features. One is that whenever it is completed with a number-name, however complex, a name of a number results. Another is that when two number-names which share their denotations complete such incomplete expressions, the resulting complete expressions share their denotation. So we may think of these incomplete expressions as mapping expressions; they systematically relate the denotations of the completing number-names to the denotations of the completed number-names. In other words, and stated more generally, the reference of a complex number-name is a function of the denotations of its number-naming parts. Functions we may think of as entities through which this mapping is accomplished, and which are such that no two functions perform exactly the same mapping service. More precisely, but less generally, if two function-names are such that whenever each is completed with an arithmetic name (simple or complex, the same or different) of a particular number, the resulting expressions are names of the same number, then the function-names denote (as we are saying, with Frege) the same function. In a word, functions are extensional. This was Frege's view, and it accords with mathematical

practice then and now.⁴¹ What is distinctive of Frege's conception of functions is that they are, as he puts it, unsaturated⁴² (ungesättigt); in contrast to objects, functions are not self-subsistent, but rather stand in need of completion. We are working towards an understanding of this claim and its motivations.

We set ourselves the problem of accounting for the fact that '2+3' denotes five. Frege's parsing of this expression, under our simplifying assumption, yields as significant parts just '2' and '()+3', for each of which denotation is claimed. Accepting the claims, do we have sufficient material with which to construct an account as desired? It would go like this: Since '2' denotes two and '()+3' denotes a function which maps a number into that which is three greater, '2+3' denotes a number that is three greater than two. Is this explanatory in the sense we are after? Let us proceed by granting an

41. But Frege did not simply take over current conceptions of functions; he significantly sharpened them. See Church, Introduction, pp. 22-23.

42. Peirce, independently of Frege, used 'unsaturated' to speak of certain expressions much as Frege used it, or 'incomplete', to speak of certain expressions; however he apparently did not, as Frege did, also use it in speaking of entities referred to by such expressions. Peirce adopted the term from its use in chemistry, which may well have been Frege's source also. I.M. Bochenski's A History of Formal Logic, pp. 323-24, is my source on Peirce.

initial plausibility to this (sort of) account. In this context we want to examine a particular feature of it.

To rehearse earlier observations with a different example, consider the expression $'(2+3)+7'$, which we will, for the present, think of as breaking up into the two parts $'(2+3)'$ and $'()+7'$. If we replace the constituent part $'(2+3)'$, which is evidently a number-name, with another arithmetic name of the same number, say $'5'$, the expression that results not only names a number, but names the same number as the original. Now if $'()+7'$ denoted a number, some numeral would likewise, yet no numeral may replace this incomplete expression with an expression we recognize as a number-name as the result; at best we come up with a listing of numbers. So such incomplete expressions are not names of numbers. This was our earlier conclusion, and not a dramatic one.

Now, considering the language of arithmetic in a broader context, we generalize from our class of number-names, whose members serve as guiding paradigms, to a wider class of expressions all of whose members shall be suitably similar to number-names. Note that here what we "generalize on" is the idea of an expression being a number-name, not on that of an expression being a name of something, or having a denotation. What makes for suitable similarity, or common linguistic role, is difficult to specify, though behavior about signs of

identity (equality) figures importantly. For instance,

9 = (2+3)
 9 is the number of planets
 The third planet from the Sun is Venus
 Venus is the sun,

form a kind of progression which suggests that '9', 'the third planet from the sun' and 'Venus' may replace one another in these contexts (though in some cases with a greater or lesser degree of incongruity) so that the other two will join '9' in our expanded class of expressions. The guiding idea is (or seems to be) that expressions gain entry to our class, and let us call its members singular terms, by intelligibly replacing paradigms (occurring paradigmatically, we should perhaps add.)

Notice, first, that sentences (identities) are used to motivate the expansion; this may not be incidental. Secondly, such motivation as is provided hardly looks like a conclusive argument for placing 'Venus' in the same semantic bag with '9'. Rather, attention is directed to certain features of language which, at least while attended to, pull us in a certain direction. Furthermore, there are other examples which induce an undercurrent. Were we to yield to the initial suggestion, what would we make of the likes of 'Venus + 3' where 'Venus' replaces '2' is a paradigm occurrence? We might decide --

Wittgenstein apparently did⁴³ -- that such examples halt our generalization before it gets going, that they show that the whole endeavor to provide a semantical account of complex names is misguided. In response to this I think all we can do is to (tentatively) treat these examples as misleading, as exceptions to be dealt with later on, once the ball gets rolling. Only by adopting such a high-handed attitude can we proceed, and only through proceeding will we be able to determine whether we have been justified in proceeding, whether in fact we can treat such examples as exceptions to a pattern persuasively portrayed. We shall, thus, with Frege, succumb to the set-out temptation, and explore the consequences.

To continue, whereas we previously thought about replacing a functional expression with a number-name, let us now consider doing likewise with any member of our generalized class of singular terms. The envisioned result is comparable to the earlier one. But now there is a bite. Before we were led only to reject the obviously false suggestion that functional expressions denote numbers. The view we are now led to reject is that it is possible to refer to the denotation of a functional expression with a singular term. Otherwise put, we cannot

43. This, I take it, is a message of the Tractatus. See Tractatus 5.25, and also Anscombe's Introduction to Wittgenstein's Tractatus, p. 120.

generalize the idea of a number-name to sufficiently enrich the class of singular terms to a point where it includes expressions that may denote the denotations we have assumed for such incomplete expressions as ' $()+3$ '. More generally, to assume that every significant part of a complex name itself has denotation seems to require the view that at least one such denotation cannot be denoted by a singular term. Frege assumed that functions were denoted by functional expressions and this, given his assumptions as to the components of complex arithmetic names, seems to require that functions cannot be referred to with singular terms. It is this "result" which stands behind Frege's view that functions are unsaturated. Re-phrasing our conclusion, functions must lie outside the scope of singular terms if the earlier account of ' $2+3$ ' is to be explanatory. Frege sought some such account and so was led to conclude that functions are insusceptible of singular reference. This, I submit, is just what the claim of unsaturation comes to. Ultimately, the suggestion that we can consider functions incomplete in a fashion analogous to the way in which functional expressions themselves are incomplete is unhelpful. And the same is true of other metaphors Frege hoped would be

elucidating.⁴⁴

I have stated, in effect, that the semantical analyses Frege gave to sentences were derived from those originally given to arithmetical singular terms. We have just seen, through a few examples, how Frege's account goes and why it brings in its wake the view that functions are unsaturated. Next we shall look at some simple sentences in much the spirit as we have examined '2+3'. This will, eventually, serve to motivate Frege's wider application and generalization of his semantics of complex arithmetic names.

Sentences are expressions with truth-value. (Here we continue in our earlier conceit (§6).) With or without an emphasis on names, it might well strike us that the simplest sort of sentence is one in which something is named which the remainder of the sentence says something about, as is apparently the case with (e.g.)

Jane is an actress.

This sentence, it happens, we know to be true, at least if we are together on our Jane. We say that 'Jane' names the actress in question, and thus speak of the relation between 'Jane' and Jane in the same terms as that between

⁴⁴. In "Frege on Functions" Black subjects several of Frege's metaphors to scrutiny and finds them wanting. This essay is reprinted in Essays on Frege, edited by E.D. Klemke.

'2' and two. But for the moment we shall set aside those earlier reflections, not drawing upon them yet. As we here shall use 'name', the paradigmatic cases of names are expressions that apply to things to which we might well be able to point, perhaps so to name. And this is certainly not the case with number-names. Previously we took arithmetic as providing us with the clear cases of what we called names; now we think in terms of naming, or pointing, to gain the clear cases of what we call names.

Sentences such as 'Jane is an actress', 'Tom is tall', 'Peter is next to Jane', 'George is between Tom and Sally', etc., we may call simple predications. With sentences of this sort, deleting a (complete) name, as 'Jane' in 'Jane is an actress', leaves an incomplete expression, e.g.,

() is an actress,

which we will call a predicate, or predicative expression. A predication, then, is the completion of a predicative expression by a complete name (or pair of names if the

predicate is doubly-incomplete, etc.) into a sentence.⁴⁵

I now wish to develop a line of thought, rather parallel to that gone through with '2+3', which arises in response to the question: How are we to account for the fact that 'Jane is an actress' has truth-value, indeed, is true, in terms of what is to be said of its parts, 'Jane' and '() is an actress'? (Note here the assumption that such most simple predications, as that of our example, shall have such account as we desire of them framed in terms of just two (proper) parts; what we here assume, with Frege, comes in for mention in (§11) and extended discussion in (§16).) That 'Jane' names, or denotes, Jane is necessary to the desired account, but clearly not

45. We must note that the process of deleting names from sentences to gain predicates is not a wholly transparent one. Witness the fact that if we delete the name 'John' from 'John loves John' we are left with the appearance of a one place predicate '() loves ()', whereas we want to say that loving is a binary relationship, for we recognize the same predicate in 'John loves Mary'. Even Frege slipped on this; see Grundgesetze, p. 81. The moral here would seem to be that we wish to speak of gaining predicates through deleting occurrences of names from sentences. But not any occurrence of a name in any sentence. Dropping 'France' from 'The capital of France is Paris' does not give us a predicate; nor does deleting '2' from '2+3=5'. Nor is it clear that the result of deleting 'Peter' from 'Jane is an actress and Peter is blonde' should be considered a predicate. The moral here is that there is no substitute for the judicious use of examples in introducing the idea of a predicate, or the predicative occurrence of an expression. For a related point see note 72.

sufficient for it. It is also clear that '() is an actress' does not (or not only) function to name Jane, or any other person, since if it did, 'Jane is an actress' would be, in effect, a two-person list, but it is not, since it is a sentence. Here we draw on an assumed distinction between a sentence and a list, in particular, between a sentence about a person and a list of persons, which parallels the earlier distinction between a list of numbers and a name of a number. We then conclude that '() is an actress' either bears other than denotation to Jane, or it bears some semantical relation, perhaps that of denotation, to something other than Jane (or anybody else.)

Frege held that predicative expressions denote what he called concepts. Why was this? In a sense the answer is quite simple. Frege intends to quantify (generalize) at the predicative-position, and, as Quine would say, he knew that to quantify at the place of an expression (i.e., to replace it by a variable bound by a quantifier) was inseparable from the assumption that the expression which occupies that position has denotation. But this then raises questions as to why Frege did so quantify, or what reason can be offered in motivation of this move.

A rather common thought on this topic (similar to that previously offered regarding functions) finds expression in these words of Furth's.

. . . In ordinary speech we move unhesitatingly from (e.g.) "Napoleon was a great general and McClellan was not" to "Napoleon was something that McClellan was not", yet . . . our entitlement to do so presupposes that the predicate generalized upon be thought of as naming in some way comparable to that required of singular terms open to the parallel operation.⁴⁶

Now of course we do commonly enough in ordinary speech move as Furth indicates. And we may agree, for the purposes of argument, that to generalize at the predicate-place is to presuppose that predicates denote. But it may be doubted whether, in the move Furth cites, a predicative expression pure and simple is generalized on. Consider simply,

Napoleon was a great general.

In this we have the predicative expression

() was a great general.

But to replace this, in the above sentence, with a variable which we bind with an existential quantifier gives something which might look this way

(Ex) (Napoleon x).

I have difficulty in understanding this. At the very least it does not admit of a reading parallel to that which we give the result of existentially generalizing

46. "Two Types of Denotation", p. 29. See also Strawson's "Singular Terms and Predication" in Philosophical Logic, pp. 80ff., and Dummett's "Frege on Functions: A Reply" in Essays on Frege, p. 272.

at the position of 'Napoleon'. That gives us

(Ex) (x was a great general),

which unproblematically says that something was a great general. The second existential invites the question: Who (or what) was a great general? Here an appropriate answer is: Napoleon. Now, judging from the quoted passage, Furth would suggest that we understand the first existential as saying that Napoleon was something. Then it invites the question: What was Napoleon? And here an appropriate answer would be: A great general. Or: He was a great general. In answer to the first question we give (or may give) the expression on which we had previously generalized. But this is not so with our answer to the second question. Might we not suppose that this indicates that it was not the position of the predicate that was generalized upon?

These reflections cast some doubt that the move to which Furth directs our attention involves generalization on a predicative expression and hence gives us reason to suppose that predicates denote. And Frege never, so far as I know, offered such an argument for his view that predicates denote. I think that we would do better to seek Frege's reasons for assuming concepts as the denotations of predicates in the context of the demands of logicism as Frege saw them. Reverting to the earlier approximation of Grundgesetze's Basic Law V,

(B) $\hat{x}(Fx) = \hat{x}(Gx)$ iff $(x)(Fx \text{ iff } Gx)$,

if this is going to serve Frege's program of providing us with the means of apprehending sets, and so ultimately numbers, then 'F' and 'G' must be construed as genuine variables, at all occurrences.⁴⁷ This, together with the claim that (e.g.) 'All and only vixens are vixens' has the logical form of the right side of (B), yields the conclusion that predicative expressions have denotation. For the present let us just accept the idea that predicates denote.

Predicates denote concepts. What then are concepts? Well they are the sorts of things which, e.g., '() is an actress' denotes, in its occurrence in 'Jane is an actress', and are such that the fact that '() is an actress' denotes the concept that it does, together with the fact that 'Jane' denotes Jane, is sufficient to account for the fact that the sentence has the truth-value that it does. We proceed in analogy with our discussion of '2+3'. There we assumed a distinction between names and lists; here, between predications (and sentences generally) and lists. There we generalized out from arithmetic singular terms; here we envisage a similar generalization, but one where the paradigmatic singular terms are names of things we could handle or point at --

47. I discuss this more explicitly in (§13).

expressions that Strawson, for instance, would incline to count as singular terms in the first instance. Thus, much as we know that concepts are not persons, or anything else to which we might be able to point, since if they were we would be able to provide the denotation of '() is an actress' with a name, which upon substitution would transmute a predication into a list, so we conclude more generally that concepts are entities which cannot be denoted by singular terms. Concepts, like functions, are unsusceptible of singular reference; they too are unsaturated.

We have sketched independent accounts of complex singular terms and simple predications. That they run as parallel as they do to some extent reflects shared assumptions; still the parallel is striking. The urge toward integration of theory is strong. Do we then have a choice in the direction of integration between assimilating complex names to predications or conversely?

Frege's move was to count predications (and sentences generally) as names, therewith absorbing concepts into functions, as a special case. Being able to thus treat concepts as functions was undoubtedly felt as an evident gain; functions were a mathematical commonplace, whereas concepts were entangled in speculations of uncertain usefulness and some obscurity. Also, where in our development we have made use of two somewhat different

clusters of intuitions as to what a name is, under Frege's integration we have a single category comprising both arithmetical names and "natural" names. And we might feel that the differences in our regard of names serve to reinforce one another with the cumulative effect of a clearer categorization. (Though the resultant inclusion of sentences into this category is a doubtful gain on this score.) Of course, this is dependent upon being able to make out a view on which sentences have denotation, which we have yet to take up.

Whereas this assimilation of sentences to complex names is thus rather straightforward, there is no equally simple way of effecting the reverse assimilation of complex names to sentences. It is, however, possible to go the other way. The possibility is provided by Russell's theory of descriptions. The process proceeds by, first, finding what looks to be a predicative expression in a definite description; thus (e.g.) locating '() is a father of Jane' in 'the father of Jane' so that the latter may be recast as 'that which is a father of Jane'. Then, with any sentence in which the complex phrase occurs, the sentence as a whole is reworked in the familiar fashion with the result that the appearance of a complex name is eliminated. It then remains to show how other apparently complex names, e.g., '2+3', can be treated as definite descriptions, and so on their way to being not

names at all. Such transformations are today a commonplace.⁴⁸

We can now appreciate the resistance Frege might well have felt to a Russellian account of definite description. Carried through thoroughly, this approach threatens to obliterate the very phenomenon reflection upon which led Frege to his semantics, the phenomenon of the semantically complex name. And recognizing this we, in turn, might be led to wonder to what extent, and with what effect, Frege's semantical assimilation of sentences to complex names obliterates the phenomenon of sentences? To this we shall return in (§15).

§12. The Semantics Extended; Problems

If sentences denote, what do they denote? In Begriffsschrift Frege had, at times, spoken somewhat vaguely of sentences and facts, or circumstances, with the implication that (i) sentences stand in some semantical relation to facts, and (ii) the facts are many -- perhaps not a fact for every sentence, but at least different facts for many different sentences. On Frege's later view there are just two facts, if facts are subject to a constraint implicit in Begriffsschrift that the

48. In Quine's Mathematical Logic this, and more, is carried out.

fact (if such there be) to which a sentence is related contains, in some sense, the objects denoted by singular terms occurring in the sentence. These two facts Frege calls the True and the False. Needless to say, facts in such short supply hardly seem facts at all. Why does Frege hold this view?

Suppose that this sentence is true,

George is a Republican.

Assume that just as 'George' denotes George, the containing sentence denotes its associated fact, and that George, however referred to, is a constituent of this fact.

Presumably then any expression that denotes George may be exchanged with 'George', in the example, with the result that the new sentence refers to the same fact as the old. Thus,

Tom's father is a Republican

is co-denotative with the example if George is Tom's father. And if this fellow is the oldest man ever to five-putt the fourteenth at Pebble Beach while standing on his left foot whistling Dixie, then each of the preceding two must be co-referential with

The oldest man ever to five-putt the fourteenth at Pebble Beach while standing on his left foot whistling Dixie is a Republican,

and if all and only men are featherless male bipeds, and if Dixie is the best loved song in Little Rock, then

The oldest featherless male biped to
 five-putt the fourteenth at Pebble
 Beach while standing on his left foot
 whistling the best loved song in Little
 Rock is a Republican

in turn agrees in denotation with each of the other three.
 And so forth.

Having encouraged us in "Sense and Reference" to
 reflect on such examples Frege queries: "what else but
 the truth-value could be found that belongs quite gener-
 ally to every sentence if the denotation of its components
 is relevant, and remains unchanged by substitutions of
 the kind in question?"⁴⁹

If sentences denote, and if the contribution of a
 singular term, occurring as a part of a sentence, to the
 fact that the sentence denotes what it does lies completely
 in the term denoting a particular object, then we may be
 convinced that we have no choice but to treat sentences
 alike in truth-value as co-referential. Then again we
 may not. Can we conjure up an argument that carries us
 directly to this conclusion?

We are assuming that predicates denote, and we may
 suppose that just as the contribution of the subject of
 'George is a Republican' to the fact that the sentence
 denotes what it does is exhausted in its denoting George,
 likewise the contribution of the predicate lies just in

49. Philosophical Writings, p. 64.

its denoting a particular concept. Hence any predicate co-denotative with '() is a Republican' is interchangeable with it, in our example, salva veritate. And if concepts are construed as a special case of functions, two concept-words (predicates) will be co-denotative just in case whenever co-denotative singular terms complete them to form sentences the results have the same denotation. But clearly this information is of no use to us in trying to argue that sentences equal in truth-value are co-denotative.

We try another tack. Sentences denote facts, so we are assuming, but not every sentence denotes a fact distinct from what any other sentence denotes. Some sentence pairs are co-denotative. For instance, a pair of sentences which differ only in that one arises from the other through a substitution of one singular term for another will be co-denotative if the singular terms are. Further, if

S

abbreviates an arbitrary sentence, then its self-conjunction

S and S

will surely be sufficiently equivalent to S to insure that the pair are alike in denotation. And similarly with self-disjunction: surely

S or S

will be co-denotative with the first, and so also with

the second of the above pair.

Since sentences denote we can make use of other of Frege's logical notions so as to form such singular terms as 'the unique thing that is the same as S', briefly,

$$(Tx)(x=S).$$

This we may be sure denotes what S does so that

$$(Tx)(x=S) = S$$

must be true. As, of course, will be

$$(Tx)(x=S) = (Tx)(x=S).$$

Here our interest is less with this being true than with the fact that it is a sentence of Frege's logical theory. Let us look just at the left term. It contains as a part the predicative expression '()=S'; we can thus schematize this term with

$$(Tx)(Fx).$$

Recall that on Frege's theory of descriptions such expressions (or the equivalent ones in his notation) denote the sole F, if such there be, otherwise, (roughly) the class of Fs. Now let us consider the slightly more complex predicate 'S, and ()=S' with which we replace 'F' to obtain

$$(Tx)(S, \text{ and } x=S).$$

What does this denote? A moment's reflection reveals that if S is true, then this expression denotes just what S does, and if S is false it denotes a certain class (roughly, the null class.) Thus the following

$$(Tx)(S, \text{ and } x=S) = (Tx)(x=S)$$

and the original S will be true and false together. Here as before the equivalence is determined through our understanding of the logical ideas drawn upon, and so, as before, we may conclude that the pair are co-denotative.

Now we are in a position to argue straightforwardly for Frege's conclusion that if sentences denote facts, there are at most two facts. We begin with two sentences, S and R, alike in truth-value. Now, (a) and (b)

$$\begin{aligned} (a) & S \\ (b) & (Tx)(S, \text{ and } x=S) = (Tx)(x=S), \end{aligned}$$

we have just shown to share their reference. Then so too must (b) and (c)

$$(c) (Tx)(R, \text{ and } x=S) = (Tx)(x=S),$$

since by our assumption S and R are alike in truth-value, and so (c) arises from (b) through substitution of co-denotative left terms. The expressions '(Tx)(S, and x=S)' and '(Tx)(R, and x=S)' are co-denotative since if both S and R are true each description picks out whatever it is that the sentence S denotes, and if both S and R are false then each description will denote (roughly) the null class as no object will satisfy either of the conditions 'S, and x=S' and 'R, and x=S'. Finally (c) and (d)

$$(d) R$$

will share their reference for the same reason that (a) and (b) do. Since the argument works for any pair of

sentences alike in truth-value, we have reached Frege's conclusion that all the true sentences denote one thing, and all the false ones another.⁵⁰ The common denotation is called the True in the one case, the False in the other.

Just before launching the above argument we reasoned that two concept-words will share denotation just in case sentences that result from them through completion with co-denotative singular terms themselves are co-denotative. Having arrived at the view that sentences denote truth-values we can be more specific. Two concept-words are co-denotative just in case whenever a name completes each to form a sentence the resultant sentences are alike in truth-value. Since concepts are functions, this is no more than what we would expect. But it is worth lingering over this a moment, since there has been a history of misinterpretation of Frege at this point. The source of the trouble lies undoubtedly in the fact that Frege does not come out and say that concepts that uniformly yield the same values for the same arguments are one and the

50. This Fregean argument (called by some Davidson's favorite argument) was first sharpened into something like this form by Church in his review of Carnap's Introduction to Semantics (Philosophical Review, 52 (1943), p. 298-304). It has recently come in for considerable discussion; see, for instance, John Wallace's "Propositional Attitudes and Identity" (Journal of Philosophy, 66 (1969), pp. 145-152) and references therein.

same. And so it has proven easy to assume that he intended some stricter identity-conditions for concepts. However this is mistaken. As emphasized earlier, identity is a first-level function (relation), one which holds only for objects, and so there can be no question of identity conditions for concepts. Still one may feel that even if we are precluded from asking under what conditions are "two" concepts identical, a somewhat analogous question can still be raised. To such the following can be taken as Frege's reply:

...coinciding in extension is a necessary and sufficient criterion for the holding between concepts of the relation corresponding to identity for objects. (Identity does not in fact, properly speaking, hold for concepts.)⁵¹

By 1955 Geach⁵² had directed attention to the fact that identity was inapplicable to concepts and to the above quoted passage; this should have served to clear matters up. And with the publication of Furth's fine introductory essay, excuse for error was removed altogether. That error over the "extensionality" of Frege's concepts has

51. The passage is from Frege's review of Husserl's Philosophie der Arithmetik (Leipzig, 1891), extracts from which will be found in Philosophical Writings, pp. 79-85. However I have quoted from Furth's "Editor's Introduction", p. xliv.

52. "Class and Concept", The Philosophical Review, LXIV (1955), pp. 561-570. Reprinted in Essays on Frege; see p. 284.

been so widespread,⁵³ particularly among writers with a logical bent, is perhaps somewhat accounted for by the fact that Carnap, a known student of Frege's, went awry in The Logical Syntax of Language, where misinterpretation first occurs, to my knowledge.⁵⁴ Nevertheless it is regrettable that, for instance, Quine continues in his wayward way in his recent Philosophy of Logic.⁵⁵

Recapitulating recent thoughts, we have seen what motivates treating predications as complex names. But then if predications have denotation it is natural to assume this is so for sentences generally. Given this, we have constructed an argument of some force that issues in the conclusion that the range of denotations of sentences contains just two members, the True and the False, so called. But if sentences denote truth-values it is appealing to treat truth-functional connectives as a special case of concept-words, that is, as incomplete expressions that denote functions taking objects (in

53. See Furth's "Editor's Introduction", p. xxxviii, for a partial catalogue.

54. The Logical Syntax of Language (first published in 1934; English translation, 1937). See pp. 136-137 and 259-260.

At one place in Grundlagen (p. 80, n. 1) Frege seems to accept the idea that different concepts may have the same extension. But of course this is prior to the development of his semantics on which concepts are functions.

55. See pp. 66-67.

particular truth-values) into truth-values. And given that predicative expressions denote functions an account of universal and existential quantification seems at hand. The universal (existential) closure of a predicative expression denotes the True just in case the completed concept-word denotes a concept that yields the True for all (some) arguments. Seen this way quantifiers too seem something like functional expressions. Treating them as second-level function-names denoting functions in their own right then provides an account of third-level function-names in terms of these second-level functions similar to that of second-level function-names in terms of first-level functions. And so forth.

Our aim has been to lay out, compellingly if possible, natural steps to Frege's semantics, what Frege might have called the semantics of unsaturation. This unsaturation, the unspecifiable ontology, is problematic, and one way or another we shall be concerned with it for most of the remainder of this essay. However there are problems with Frege's semantics which have nothing directly to do with the view that functions are unsaturated, which it behooves us to mention, if not mull over.

There is a problem with multiple quantification. We understand, given the foregoing, the conditions under which something of the form

$$(\forall x)F(x)$$

is true -- just in case F denotes a function which yields the True for all arguments. But what are we to say with regard, for instance, to something of the form

$$(x)(y)(Rxy)?$$

Frege does not tell us how to regard the semantic contribution of an embedded quantifier.

There is an analogous difficulty with connectives. Conditionality is explained for all objects as arguments, but this does not tell us the semantic import of the conditional in sentences of, for instance, the form

$$(x)(Fx \supset Gx).$$

The conditional-sign, the horseshoe or Frege's, has not been explained for contexts in which it is not completed to a proper name by proper names, but rather serves itself to complete a second-level function-name to a proper name. It is through such a use of connectives that we enrich the expressive powers of a finite supply of predicates. So we could just as well say that the problem here mentioned is with the semantics of complex predicates as that it is with connectives in quantificational contexts (and higher-order contexts generally.)

Related to this is a problem with what Frege tells us about predication itself. Something of the form

$$F_a$$

will be true just in case a falls under the concept F . But what are we to say of polyadic predications; of, for

instance, something of the form

$Ra, b?$

In "Function and Concept" Frege writes

In

$x > y$

we have a function with two arguments, one indicated by 'x' and the other by 'y'; and in

$3 > 2$

we have the value of this function for the arguments.⁵⁶

But of course we do not have the value of the function for the arguments, for there is no unique value of the function for the arguments. Rather, what we have is the value of the function for the arguments taken in a certain order, in the order in which we most naturally would take them. I quote Frege's words here not just to catch him up, but to highlight the fact that his practice in this passage is his practice generally. Always in discussing polyadic predications he relies on the natural convention of plugging mentioned arguments into available argument-places of the predicate so that the first mentioned argument goes into the "alphabetically earliest" position, and so on. But this means that Frege in fact has no way of keeping track of the argument places in many-place functions. At least he has given no sense to talk of a particular function having a first, second, . . . ,

56. Philosophical Writings, p. 39.

kth argument place, anymore than he has to a particular function having a top and a bottom, or a left and right side.

This problem Tarski solved with sequences. And it, in turn, led to solution of the problems with complex predicates and multiple quantifications, which, after all, are somewhat artificially parcelled out as separate problems. Sequences are no solution for Frege, however. For sequences are "set-like", indeed, definable set-theoretically. Thus for the semantics to call upon sequences would be for it to undercut the very philosophy of mathematics that was intended to be argued for through the use of the logical notation. It is not clear whether there is any way of providing an adequate semantics for Frege's logic which at least accords with the spirit of his logicism. To Frege's idea of reducing arithmetic to logic we return in the next section, in particular we shall look at his philosophical attitude towards set theory. First, let me repeat that the problems just considered are independent of those caused by Frege's deeming functions insusceptible of singular reference; these matters we return to in (§14).

§13. Classes, Concepts, and Consistency

In developing Frege's semantics we worked on the assumption that such simple subject-predicate sentences

as 'Jane is an actress' and 'George is a Republican' have
just two significant parts. Thus in framing an account
of how such succeed in having truth-value we employed, on
the linguistic side, just (e.g.) 'Jane' and '() is an
actress'. And we have seen how this assumption, coupled
with the view that incomplete expressions are denotative,
can lead to Frege's view that concepts are unsaturated.
But such unsaturated concepts are problematic; apparently
we cannot talk about them as we would wish. Might it be
possible to avoid the introduction of unsaturated entities
by rejecting the assumption of a two-part analysis of
subject-predicate sentences?

Let us consider the possibility of developing a

three-way analysis that treats the traditional copula of
simple subject-predicate sentences as itself a significant
part of the sentence. Here, however, we shall use this
idea as an entry point to further discussion of Frege's
thoughts on concepts and classes. We shall take up the
topic for its own sake in Chapter IV, where our specific
concern will be over whether unwanted consequences of
Fregean unsaturation are altogether avoidable.

Consider the sentence
Peter is next to Mary.

Of this relational statement Frege would say that
() is next to ()

is a doubly-incomplete expression which joins the com-

plete expressions which flank it into a sentence, that each of these three parts denotes something, and that the doubly-incomplete expression has a doubly-unsaturated denotation. The suggested alternative is to try to make something like this analysis work for such as 'Jane is an actress' through treating 'is' as a two-place connecting expression, but in such a way that unsaturated entities are not forced upon us.

Let us assume that, if we treat the copula as denotative, previous reasoning will lead us to the conclusion that its denotation is unsaturated, doubly-so. And it is clear that Frege would have held that if the copula is to be construed as a genuine relational expression, then it must be thought of as denoting something. That is, if the alternate analysis is otherwise acceptable, it requires that we count the copula among the denoting expressions of, for instance, 'Jane is an actress'. That this would be his view is borne out by these remarks.

Somebody may think . . . that there is no need at all to take account of such an unmanageable thing as what I call a concept; that one might . . . regard an object's falling under a concept as a relation, in which the same thing could occur now as object now as concept. The words 'object' and 'concept' would then serve only to indicate the different positions in the relation. This may be done; but anybody who

thinks the difficulty is avoided this way is very much mistaken; it is only shifted.⁵⁷

We have suggested that Frege had sufficient reason to hold that, on his analysis, predicates denote, since he was to quantify at the predicate place. And that such quantification was essential to his logicist program. In effect, Frege developed a set theory through quantifying over concepts; indeed he maintained that this was the only way one could intelligibly ascend into set theory. But this reason for quantification, and so for the assumption that predicative expressions such as '() is an actress' have denotation, does not carry over to support the view that on a three part parsing of subject-predicate sentences the putative relational expression must denote. For it seems that we gain no additional logical power through quantifying at the position of the copula. Either set theory arises independently of such quantification, or not at all. So perhaps the copula need not denote, and hence not denote something unsaturated.

But if we are to make out an alternative three-way analysis, we shall have to explain, or make clear, what it is that is denoted by such general terms as 'an actress' and 'a Republican'. This, Frege would claim, we cannot do, except through also assuming concepts. To understand

57. Ibid., pp. 54-55.

Frege's thought on this will both clarify his thinking on set theory and help us to understand his commitment to concepts.

On the imagined alternative analysis, as applied to 'Jane is an actress', 'Jane' denotes Jane. 'Is' is non-denotative; its semantic function lies in other than having denotation. This aside, what of the remaining piece of the new picture? 'An actress' denotes. But what? (Of course this expression does not look much like a name, but then neither does 'is an actress'.) Clearly 'an actress' does not denote an actress. Perhaps it denotes, collectively, all the actresses. But then presumably, in like fashion, 'a two-headed actress', as in 'Jane is a two-headed actress', denotes the two-headed actresses. But since there are no such creatures, apparently there is nothing for 'a two-headed actress' to denote. But notice that if 'Jane' failed of denotation -- suppose we learn that there were really twins, only one of whom appeared publicly at any time -- we would probably say of 'Jane is an actress' that it failed of truth-value, and so dropped off the stage of our concern. (This feeling would undoubtedly be enhanced if just one of the twins was an actress.) While the fact that there are no two-headed actresses does not incline us in the least to suppose that 'Jane is a two-headed actress' lacks truth-value. On the contrary, it is straightforwardly false.

So, since 'Jane is a two-headed actress' has truth-value if 'Jane is an actress' does, if 'an actress' has denotation, then 'a two-headed actress' must as well. The problem is that there does not seem to be anything for it to denote.

Nowadays an answer is readily forthcoming. Let such general terms as are here required to have denotation denote classes, and so, in particular, 'a two-headed actress' will denote a class, the class that has no members. How adequate a reply is this? Well, what is a class?

Frege found discussions by his contemporaries full of mistake and confusion. There was psychologistic confusion found in the talk of forming sets by abstraction, or putting things together in the mind by a process of attention, etc. There was failure to distinguish the relation of an object falling under a concept (class membership) from that of one concept falling within another (class inclusion), and either of these from the part-whole relation of one object to another. And there were mistakes arising from failure to distinguish adequately concept from object. Thus, for instance, it was not sufficiently recognized that whereas there could be no non-self-consistent objects, there was nothing improper about a self-contradictory concept. Related to this was the attempt, by Schröder for instance, to by-pass concepts

and consider the elements of a class as constitutive of the class. This resulted in the situation that the idea of the null class, the class without members, was unintelligible.⁵⁸

What of our contemporaries? Let us listen to one of the clearest. In his Set Theory and Its Logic Quine says, in the first paragraph of the introduction, that

We can say that a class is any aggregate, any collection, any combination of objects of any sort, if this helps, well and good. But even this will be less help than hindrance unless we keep clearly in mind that the aggregating or collecting here is to connote no actual displacement of the objects, and further that the aggregation or collection or combination of say seven given pairs of shoes is not to be identified with the aggregation, or collection, or combination of those fourteen shoes, nor with that of the twenty-eight soles and uppers. In short, a class may be thought of as an aggregate or collection or combination of objects just so long as 'aggregate' or 'collection' or 'combination' is understood strictly in the sense of 'class'.⁵⁹

It might seem that if Quine is not canceling out with the last sentence all salient suggestions of the first, the problem of the denotation of 'a two-headed actress' re-emerges, for the notion of an aggregate or collection or combination of nothing is unintelligible. And this, again, was Frege's primary objection to set

58. See Grundgesetze §0. and "A Critical Elucidation of Some Points in E. Schröder's Algebra der Logik" in Philosophical Writings.

59. Set Theory and Its Logic, p. 1.

theory as he knew it. Thus Frege directs these words against Schröder.

A class, in the sense in which we have so far used the word, consists of objects; it is an aggregate, a collective unity, of them; if so, it must vanish when these objects vanish. If we burn down all the trees of a wood, we thereby burn down the wood. Thus there can be no empty class.⁶⁰

. . . there cannot be an empty class if we take a class to be a collection or totality of individuals, so that, as the author says, the class consists of individuals or individuals make up the class.⁶¹

But it is doubtful that in this we have an objection to Quine, who did, we now remind ourselves, preface the passage we have quoted with the claim that "[t]he notion of class is so fundamental to thought that we cannot hope to define it in more fundamental terms",⁶² and who continues in his second paragraph to characterize this notion as follows.

We can be more articulate on the function of the notion of class. Imagine a sentence about something. Put a blank or variable where the thing is referred to. You have no longer a sentence about that particular thing, but an open sentence, so called, that may hold true of each of various things and be false of others. Now the notion of class is such that there is supposed to be, in addition to the various things of which that sentence is true, also

60. Philosophical Writings, p. 89.

61. Ibid., p. 102.

62. Set Theory, p. 1.

a further thing which is the class having each of those things and no others as member. It is the class determined by the open sentence.⁶³

Though the null class does not come in for explicit mention in this passage, it does implicitly; 'x is a two-headed actress' like any other open sentence determines a class.

We must, of course, take the talk of an open sentence determining a class in its mathematical sense. Quine does not think that open sentences, or general terms, create classes, since, among other reasons, classes outrun our resources for talking about them.⁶⁴ On this Frege agrees. Since Quine, no more than Frege, construes classes as the denotations of predicative expressions, perhaps it would be best to say simply that for (most) every general term (or open sentence) there is assumed an associated class, though not conversely. (The parenthetical 'most' is a hedge against Russell's paradox, which will remain off-stage for now.) We can then go on, with Quine, and say that if two general terms are true of just the same things, the class associated with the one is just that associated with the other.

Up to this point Quine's classes and Frege's extensions

63. Ibid., p. 102.

64. See Set Theory, p. 2.

seem to run quite parallel. Co-extensive concept-words are associated with the same course-of-values, albeit through the denoted concept. So it would seem that, up to this point, Frege could have no complaint with Quine's class talk. Could we now treat Quine's classes as the denotations of general terms so as to serve the needs of our alternative analysis without incurring Fregean objections?

Frege might well object that we still had not made clear this notion of class for the following reason. A class A and a class B are to be one and the same when each is associated with a general term and the associated terms are true of just the same things. But not only when, since classes outrun general terms. So how are we to express the desired extensionality condition generally? In words, classes are identical when they have just the same members. But here we speak of sameness of class in terms of sameness of members, and now our concern refocuses upon the latter notion, which is as yet unexplained. If class A and class B are the same when they share their members, then if neither has any members they share no members; once again the null class is unintelligible, Frege might have argued.

Since Frege's set theory is developed within a second-order logic, he can state the desired extensionality condition for classes (extensions), and courses-

of-values generally, with his Basic Law V,

$$-(\hat{x}\phi(x) = \hat{x}\psi(x)) = (\exists \alpha \neg \phi(\alpha) = \psi(\alpha)).^{65}$$

Here ' ϕ ' and ' ψ ' are variables ranging over one-place, first-level concepts (and functions generally), including those -- or rather that -- under which no object falls, whence the null class.

Now we might expect a Quine to retort that one could only think that the notion of a concept with an empty extension is clearer than the notion of a class with no members if he were clearer about what is a concept than about what is a class. And just what is a concept? To this Frege may reply that concepts are just a special case of functions, and what a function is is clear enough. Now Quine would agree that functions, as they are called upon in classical mathematics, are unproblematic; they are on a par with, and submit to much the same treatment as, ordered pairs. What is problematic Quine might insist is Frege's insistence that functions, and so concepts, are unsaturated. And if we then turn to ask why Frege maintains this view we are taken back to our beginnings. And led in something of a circle. For what initiated the discussion of this section was the question of whether

65. Here I use the modified Fregean notation of our exposition and have ignored the assertion-sign; this may be compared with the statement of BLV in note 25.

it would be possible to provide a relational analysis of such a sentence as 'Jane is an actress' in a way which would avoid commitment to unsaturated entities. For us here the answer to this question turned on the possibility of construing (e.g.) the denotation of 'an actress', in the example sentence, as a class, and in such a way as to overcome the objection that on such a view 'a unicorn', or 'a two-headed actress' would require denotation, yet fail of it for want of an appropriate class.

What is at issue in our imagined exchange between Frege and Quine is whether, as Quine contends, the notion of class, or class-membership, is fundamental to thought. The reason Frege gives for denying this, which we have developed, is that it would render the notion of a null class unintelligible. But this may be symptomatic. For to hold that the notion of class was fundamental might seem to commit us to viewing BLV as a synthetic a priori truth (if a truth at all), and hence to commit us to the sort of epistemological Platonism which I have urged we

err in attributing to Frege.⁶⁶

Earlier (§4) I claimed that Frege regarded BLV (or its instances) as containing alternative analyses of a given Thought. Let me now offer some textual evidence for this interpretation. In "Function and Concept" Frege writes of the "transformation" involved in BLV this way.

We have generally:

$$x^2 - 4x = x(x-4)$$

whatever number we take for x . . . I express this as follows: the function $x(x-4)$ has the same course-of-values as the function x^2-4x . [In the example] we have not put one function equal to the other, but only the values of one equal to those of the other . . . we have thus expressed that an equality holds generally. But we can also say: 'the course-of-values of the function $x(x-4)$ is equal to that of the function x^2-4x ,' and here we have an equality holding generally between courses-of-values.⁶⁷

Frege puts this equality, or identity, into his notation this way.

$$(a) \quad \overset{3}{x}(x^2-4x) = \overset{3}{x}(x \cdot (x-4)).$$

If we set (a) to the left of the identity sign while

66. However Quine is someone who both regards the notion of class as fundamental and is not an epistemological Platonist. I concur in J.J.C. Smart's assessment. "We need not think that a realist philosophy of mathematics [ontological Platonism] need depend on synthetic a priori intuitions of a special intellectual realm. Such a Platonistic epistemology cannot be squared with a biological view of man. On Quine's account we need no such non-empirical intuitions. ("Quine's Philosophy of Science", Synthese 19 (1968), p. 5.)

67. Philosophical Writings, p. 26. (I have modified the translation to standardize usage with Furth.)

putting on its right side the universal closure of Frege's example,

$$(b) \quad (x)((x^2-4x) = x(x-4)),$$

to its right, we thereby form an instance of BLV.

Now how should we take Frege's remark to the effect that such a pair as (a) and (b) say the same thing? Frege answers when he says, of his previous example, that

If we understand [it] in the same sense as before [i.e., as its universal closure, (b)] this expresses [ausdrücken] the same sense [Sinn], but in a different way. It presents the sense as an equality holding generally; whereas the newly introduced expression [(a)] is simply an equality, and its right side and equally its left side, stands for something complete in itself.⁶⁸

Here we may assume that Frege is using 'sense' and 'expresses' as he does in "Sense and Reference", which had been written, though not published, at the time of this address. Thus he is saying that such pairs as (a) and (b) express the same Thought, though clearly they differ in logical structure and ontology.

In Grundgesetze itself we do not find such an explicit assertion that "instances" of BLV have the same sense. But we do get explicit denial of this in the case of a rather similar pair of sentences. Suppose we have a function F which yields different values for different arguments generally. Then if we apply this function to

68. Ibid., p. 27.

a "pair" of courses-of-values we will get the same object as value just in case the courses-of-values are one, not two; that is, the following sentences are true and false together:

$$(i) \quad F(\tilde{x}(Gx)) = F(\tilde{x}(Hx))$$

$$(ii) \quad \tilde{x}(Gx) = \tilde{x}(Hx),$$

where 'G' and 'H' are any one-place, first-level function-names. Then, given BLV, (i) will also share truth-value with

$$(iii) \quad (x)(Gx = Hx),$$

for any G and H. Having noted all this Frege makes it a point to add, in a note, that "this is not to say that the senses [of (i) and (iii)] are the same."⁶⁹ Under the advocated interpretation Frege held that (ii) and (iii) were the same in sense. That he does not deny it in this context, provides some, indirect, support for this.

On the understanding here advocated, BLV was to provide Frege with the means of apprehending the objects of arithmetic. For numbers are to be gained set-theoretically, and BLV gives us our sets. Sets (or classes, or extensions) are not reducible to some other sort of entity. Nor can discourse about sets be forgone in favor of more basic discourse. Still, for Frege, the notion of class is not an epistemologically fundamental one, at least not as

69. Grundgesetze, p. 46.

compared with that of concept. For we are to gain it through redistributing the content of an expression which makes mention of concepts but not classes. Here we may recall our discussion of Frege's Grundlagen example of judgments concerning parallel lines and directions (§3). Directions are not reducible to other geometric entities, nor talk of directions eliminable in favor of more basic geometric discourse. Still the concept of direction is not, according to Frege, epistemologically on a par with such a concept as that of one line being parallel to another. Parallel lines, we might say, are geometrically prior to their directions. It is an analogous vein that I would take Frege's remark that "the concept is logically prior to its extension."⁷⁰ And similarly, such a remark as this.

In many phrases of ordinary mathematical terminology, the word 'function' certainly corresponds to what I have here called the course-of-values of a function. But function, in the sense of the word employed here, is the logically prior [notion].⁷¹

Now I wish to return to another theme of the earlier discussion of Grundlagen. As we read Frege, such "definitions" as

(A) $NxFx = NxGx$ iff the Fs equal the Gs,

70. Philosophical Writings, p. 106.

71. Ibid., p. 26, note.

were methodologically unsatisfactory in that they did not provide sharp boundaries for the introduced concept. In Grundgesetze, Vol. II, Frege expresses himself this way.

A definition of a concept . . . must be complete; it must unambiguously determine as regards any object, whether or not it falls under the concept . . . though for us men, with our defective knowledge, the question may not always be decidable. We may express this metaphorically as follows: the concept must have a sharp boundary.⁷²

He subsequently adds that

[t]he law of excluded middle is really just another form of the requirement that the concept should have a sharp boundary. Any object Δ that you choose to take either falls under the concept \emptyset or does not fall under it; tertium non datur.⁷³

Further, it is instructive to see that this requirement is equivalent to Frege's "basic principle" of Grundgesetze which we have already mentioned, the "principle that every correctly formed name is to denote something. . ."; a principle which, says Frege, "is essential for full rigor."⁷⁴ For, if some concept-word denoted a concept such that some object neither did nor did not fall under the concept, then there would be sentences which fail of denotation, and so are neither true nor false. If, on the other hand, we had, for instance, a (one-place) concept-

72. Ibid., p. 159.

73. Ibid., p. 159.

74. Grundgesetze, p. 9.

word and a proper name, each of which succeeds in denoting but which are such that the completion of the concept-word by the proper name does not denote, i.e., denotes neither the True nor the False, this would mean that the object denoted by the proper name neither did nor did not fall under the concept denoted by the concept-word.

In Grundgesetze Frege sought to guarantee that all concepts, of whatever level or degree, which can be denoted by primitive notation of Begriffsschrift however complexly put together, have sharp boundaries through insuring that the "basic principle" was in force. His approach was to insure that each of his eight primitive names, each of them function-names, uniquely has denotation and then to guarantee that the rules by which new names may be formed from old ones pass this feature on. This he set out to do in Sections 28-31 of Grundgesetze. This procedure was of considerable importance. For it is reasonably clear that Frege's axioms would be true, if the names of the Begriffsschrift univocally had denotation, and likewise that the rules of inference would be truth-preserving. Thus, to show that the "basic principle" was in force would be to establish that the system of logic of Grundgesetze was consistent. For no falsehood, and hence no contradiction, could be forthcoming.

It seems clear that Frege held that a system of logic must be constructed in such a way that we can be certain

from the outset that it is free from contradiction. For in Grundlagen he had written that

it must still be borne in mind that the rigour of the proof remains an illusion, even though no link be missing in the chain of our deductions, so long as the definitions are justified only as an afterthought, by our failing to come across any contradiction. By these methods we shall, at bottom, never have achieved more than an empirical certainty, and we must face the possibility that we may still in the end encounter a contradiction which brings the whole edifice down in ruins. For this reason I have felt bound to go back rather further into the general logical foundations of our science than perhaps most mathematicians will consider necessary.⁷⁵

Yet I know of no place where he indicates that his "basic principle" was intended for this purpose. (Still I feel that this was the case. I am inclined to think that the fact Russell's fateful letter evoked such a dramatic reaction from Frege was due, in part, to Frege's believing that he had proven that no such contradiction could arise.) Since Frege's logic is susceptible to Russell's paradox we know that Frege failed in his attempt to show that Begriffsschrift names univocally denote. Not surprisingly, the problem centers upon the notation for the course-of-values of a function. In Section 31 "Our simple names denote something" Frege errs in his attempt

75. Grundlagen, p. ix. Remember that with Grundgesetze, axioms take up the role of introducing ontology which Grundlagen definitions had been allowed to perform.

to show that his course-of-values function-name yields a denotation of an object whenever appropriately completed. But I shall not here trouble to detail his procedure or his mistake.

Russell's paradox dealt a severe blow to Frege's logicism. It showed that the two "sides" of BLV did not express the same Thought, and so discredited a central idea of the Grundlagen program, that which we focused upon before (§3). Reportedly Frege later came to regard set theory as an intellectual aberration.⁷⁶

§14. The Concept Horse and Consequences of Unsaturation

Let us review and recast our earlier discussion of Frege's semantics of predication, which we saw to be a major stepping stone to his general semantics.

We seek an account of the fact that such a sentence as

Native Dancer is a horse

has truth-value. We assume that, in this case, the answer is to be framed, on the linguistic side, just in terms of the two (proper) parts, 'Native Dancer' and '() is a horse'. Suppose then that we are led to regard both positions in our example sentence as open to quantification, that each of the sentence parts denotes something. We

76. See Dummett, "Frege", p. 227.

then encounter a conundrum. If the entire contribution of each of the two expressions to the fact that the sentence has truth-value lies in each having denotation, then the positions of our predication would be semantically indistinguishable with the result that replacing '() is a horse' with 'Native Dancer' should be a sentence, a bearer of truth-value. But this is not so.

Unsaturation is a solution. Frege ruled that whereas both singular terms and predicative expressions have denotation, the entities they denote are significantly different, different enough that the denotation of one type of expression cannot be denoted by expressions of the other type. The denotations of predicates are insusceptible of singular reference. Correlatively, the denotations of singular terms are insusceptible of predicative reference. The difference in denoted entities justifies the assumption of positions of two distinct types in such simple predications as 'Native Dancer is a horse'.

Thinking of predications as resulting from names completing predicative expressions, which are otherwise incomplete, the denotations of predicates are spoken of by Frege as analogously incomplete, or unsaturated.

Once sentences are construed as names of truth-values, the denotations of predicates then are treated as functions which always yield a truth-value as value,

i.e., as concepts. Then predicative expressions, such as '() is a horse', are said to be concept-words, that is, expressions which denote concepts.⁷⁷ Similarly, singular terms, such as 'Native Dancer', are said to be proper names, that is, expressions which denote objects. Given this we may say that 'Native Dancer is a horse' is truth-valued since 'Native Dancer' is a proper name and '() is a horse' is a concept-word. And so, the sentence will actually be true if Native Dancer falls under the concept horse; false, if it does not fall under this concept. However, this last statement cannot be taken literally by the true Fregean, since it attempts, with the phrase, 'the concept horse', to refer by means of a singular term to a particular concept.

Discussions of Fregean unsaturation have tended to

77. There is a point worth commenting upon here. Furth says that 'occurs predicatively' and 'denotes a concept' "are intended to be related by that strictest of equivalence of meaning which connects explicandum and explicans." ("Two Types of Denotation", p. 22.) But these phrases do not cover the same ground. In 'The class each of whose members is a prime is a prime', or ' $\hat{x}(x \text{ is a prime}) \text{ is a prime}$ ', the first occurrence of '() is a prime' is not predicative, the second is; yet at both places the expression denotes a concept. And we have a similar situation with (e.g.) 'That which is a father of Tom is a father of Tom', or ' $(Tx)(x \text{ is a father of Tom}) \text{ is a father of Tom}$ '. In other words, predicates do not always occur predicatively, at least not in Frege's grammar. And there is no reason to suppose that Frege thought otherwise.

focus upon what has been called the paradox of the concept horse. In "On Concept and Object" Frege responds to remarks of Benno Kerry this way.

Kerry . . . gives the following example: 'the concept horse is a concept easily attained', and thinks that the concept horse is an object, in fact, one of the objects that fall under the concept concept easily attained. Quite so; and the three words 'the concept horse' do designate an object, but on that very account they do not designate a concept, as I am using the word.⁷⁸

Apparently it is Frege's view that

(1) The concept horse is a concept is false. False, since the object denoted by the first three words is mapped by the denotation of '() is a concept' into the False.⁷⁹ This is regarded as paradoxical since the expression 'the concept horse' has been designed explicitly for the use which, according to Frege, it cannot perform.

Frege's reasons for denying that (1) is true come, it should be clear, straight from the heart of his semantics. It is perhaps less clear why Frege must claim

78. Philosophical Writings, p. 45.

79. We might go so far as to say that (1) is logically false on the grounds that any sentence which results from filling out the context 'the concept—is a concept' will be false. This would lead us to construe 'the concept horse' as a complex singular term formed through completing a one-place, second-level function-name by a one-place, first-level concept word.

that (1) is false. Both Dummett and Geach, in rather different ways, have suggested that Frege need not say this. Exploring these suggestions will serve to deepen and round out our understanding of Frege's semantics, in particular his semantics of predication.

Here is what Dummett has to say in his paper "Frege on Functions: A Reply".

The difficulties which Frege unsuccessfully tried to overcome . . . and which threaten to destroy his whole theory, could have been avoided simply by adopting (to use well-worn jargon) the formal instead of the material mode of speech. Most people would admit that Frege made clearer than anyone had done before him the radical difference in logical role of what he called proper names, concept-words, second-level concept-words, and so on . . . Now if Frege had confined himself to talking about these various types of expression, instead of that for which they stood, the appearance of paradox, the awkwardness of phrasing, the resort to metaphor, which pervade his writing would all have been avoided. . . . In the material mode of speech Frege was forced into such at least superficially contradictory expressions as 'the concept horse is not a concept', 'the function x^2 is not a function'; but when we are talking about expressions, then we have no motive for denying the obvious fact that the predicate 'is a horse' is a predicate . . . ⁸⁰

What is Dummett saying? For one thing, he sees Frege's denial of (1) as a threat to "his whole system". Not that he regards this denial as capricious. If I read him correctly he sees the "paradox" forced upon

80. Dummett, "Frege on Functions" in Essays on Frege, p. 269.

Frege for very much the reason I have suggested.⁸¹ If predicates, like singular terms, are denotative, their denotations must differ significantly, if predication at its simplest is a two-party affair. Since Dummett sees great value in the distinctions of Frege's grammar I take it he accepts Frege's parsing of (e.g.) 'Native Dancer is a horse'. And so Dummett would trace Frege's difficulty to the claim that predicates denote, which he regards as mistaken.

However the quoted passage seems to carry another message. First, Dummett wants to say of, for instance,

- (2) The concept-word '() is a horse' is a concept-word,

that this is unproblematical from Frege's point of view.

In this regard he directs our attention to a note in which Frege says that

[a] similar thing happens when we say as regards the sentence 'this rose is red': The grammatical predicate 'is red' belongs to the subject 'this rose'. Here the words 'The grammatical predicate "is red"' are not a grammatical predicate but a subject. By the very act of explicitly calling it a predicate, we deprive it of this property.⁸²

81. We cannot refer to the relation by using a definite description like 'the relation of being greater', because a definite description must stand for something complete, i.e., an object, and thus cannot stand for anything incomplete like a relation; this is so by the definition of 'object' and 'relation'." ("Frege on Functions", Essays on Frege, p. 281.

82. Philosophical Writings, p. 46, note.

But, claims Dummett, Frege had no reason to deny, for instance, that

- (3) The grammatical predicate 'is a horse' is a grammatical predicate.

And no more does he have reason to deny the truth of (2). Since (2), like (3), is unproblematic we may have recourse to it in place of the "superficially contradictory" (1). And similar shifts will avoid other problems, or apparent problems, arising with "the material mode of speech".

This is Dummett's idea.

It is important for our understanding of Frege to realize how mistaken this is. To this end, consider

- (4) The concept-word '() is a horse' denotes the concept horse.

For Frege, this does not state that a (first-level) relation holds between a concept-word and a concept. It is not, in this respect, similar to 'John loves Mary' which does state that a particular relation holds between John and Mary. It does not, since if it did then a concept would be denoted by a proper name, which is impossible, given Frege's account of predication. Now Dummett recognizes this, and, as well, sees that Frege also recognized this. But it is additionally the case that (4) does not state the "intended" relation for the reason that this would require a concept-word to be denoted by a singular term, viz. 'the concept-word '() is a horse'', and this too cannot be the case, given Frege's account of predi-

cation. This cannot be, since a concept-word, it is explained, is an expression which denotes a concept.

Thus, (4) is equivalent to

(5) '() is a horse', which is an expression which denotes a concept, denotes the concept horse.

And this implies

(6) The expression '() is a horse' denotes a concept.

But if (6) were true a concept would be denoted by a singular term, which we know is not possible. Within the confines of Frege's theory we can no more use a singular term to refer to a concept-word than we can use one to refer to a concept. What is insusceptible of singular reference may be referred to only by what itself is likewise beyond the reach of singular terms.⁸³

We see then why if Frege was to call (1) false we would expect him to say the same of (2). But what of (3)? If the intended reference of 'the grammatical predicate 'is a horse'' is the concept-word of 'Native Dancer is a horse' then since the expression 'the grammatical predicate 'is a horse'' is a singular term, (3) would be false for the same reason that (2) would be; singular terms cannot denote concept-words.

83. A similar situation holds for senses. What denotes something unsaturated has as its sense something unsaturated, which sense can only be expressed, or denoted, by something itself insusceptible of singular reference.

We must be careful to keep in mind that in speaking of concept-words, and of expressions generally, we do not have in mind various "mounds of ink", that is, certain physical objects. Words get spoken and written down with ink, chalk, and so forth, but particular words are not to be identified with particular configurations of sound waves, chalk marks, or ink spots. Any ink spot, however configured, can be pointed at by someone in its vicinity. This Frege does not deny. But we might suppose that we could in pointing at an ink spot point through it, by what Quine calls deferred ostension,⁸⁴ to the word (if any) written down with that ink. This Frege would deny with respect to concept-words, and function-names generally. Though this requires qualification. To take an example of Frege's, consider

(7) Trieste is no Vienna.

Here 'Trieste' is a proper name. 'Vienna' is not; it is either a concept-word, or part of one. Though there are other contexts in which 'Vienna' is a proper name, for instance,

(8) Vienna is no Trieste.

Expressions, at least in natural language, play particular semantical roles in particular contexts, and can

84. Quine, "Ontological Relativity" in Ontological Relativity, p. 40.

play one role in one place and another (or none at all) in another.⁸⁵ This sort of ambiguity would be banished from a lingua characterica.

Returning to Dummett, it should be clear that he has seriously misappreciated the semantics of Frege's notation, at least as Frege understood it; he does not sense how deeply ingrained in Frege's thinking is the phenomenon of unsaturation and the importance of this. In this, however, he is hardly unique.⁸⁶ However, the situation with Dummett is complicated by the fact that he has his own axe to grind on related matters. It is not simply, as earlier suggested, that Dummett would reject Frege's view that predicative expressions have denotation. Rather, it seems, Dummett would have us, while recognizing the merits of Frege's grammar, reject, not in detail, but in conception, Frege's truth-oriented semantics. With this goes a rejection of the classical conception of

85. Another case Frege discusses concerns the sentences (i) 'The morning star is Venus', and (ii) 'The morning star is no other than Venus'. In (i), the expression 'is' denotes the relation of identity; in (ii) it plays no independent semantical role, being a non-denotative part of another expression which denotes the relation of identity. See Philosophical Writings, p. 44.

86. For instance William Kneale says that "[i]f we distinguish carefully between expressions and designations, we can talk about functions without falling into Frege's perplexities..." (Development of Logic, p. 622.)

truth.⁸⁷ But within this essay these must remain passing remarks.

Frege is committed, we have said, to the view that concept-words, and function-names generally, are, like functions, unsaturated. This suggests that function-names are a special case of functions. And likewise that proper names, including sentences, are a special case of objects.⁸⁸ Geach advances this suggestion. "The sign of a function", he says, "is itself a function, and not an actual quotable expression; if so, it is futile to try to make our "function sign" to be a more intelligible term than "function"."⁸⁹ Geach then would find no appeal for Frege in Dummett's way out.

87. See "Truth", especially pp. 55-56.

88. There is, however, the following problem. If all objects fall within the range of any first-level function, then not only can we take 'the moon', for instance, as argument for '() is prime', but we should also be able to take the moon itself as argument for this function. In the first case the value of the function for the argument is the sentence 'The moon is prime'. But what will it be in the second case? And notice that whereas in the first case we may say that 'the moon is prime' denotes the False, there appears to be no analogous way in which we can specify the value yielded in the second case, in a manner consonant with Frege's semantics. We cannot say, for instance, that the value of '() is prime' for the moon as argument is such and such an object, for this would be to treat a function, here a linguistic function, as admitting of singular reference. (And similar puzzles will come up with senses.)

89. Three Philosophers, p. 147.

Nonetheless Geach feels that Frege could have avoided claiming that (1) is false, and so at least this appearance of paradox. He would have Frege reject the idea that such expressions as 'the concept horse' in (1) have denotation. He suggests that just as

sentences with 'some man' as their grammatical subjects are not assertions about something named by 'some man', . . . similarly, 'the concept man stands neither for a concept nor for an object; where it is legitimately used, its apparent unity breaks up under logical analysis.⁹⁰

The idea is that, just as we, with Frege, regard the structure of 'Some man is wise' as made more explicit with 'Something is a man and is wise', so (e.g.) 'The concept man is realized' can be, and should have been by Frege, regarded as a sentence with the same logical properties as 'Something is a man'. And as to the sentence 'The concept horse is not a concept', which Frege was prepared to count true, it and other "sentences not explicable in some such innocent way . . . may be regarded as nonsensical."⁹¹ In another place Geach puts his thought this way: "'The concept horse' would have to stand for a concept if it stood for anything; in fact it does not, and sentences in which it occurs are at best circumlocutory ("falls under the concept horse" = "is a horse") and at

90. "Frege's Grundlagen", in Essays on Frege, p. 477.

91. Ibid., p. 477.

worst philosophers' nonsense."⁹²

It goes with this view that such sentences as

Native Dancer is a horse

and

Native Dancer falls under the concept horse
are just stylistic variants; they do not differ in any way in their semantical properties, which are more perspicuously displayed in the former. A similar attitude would be taken by someone adopting Russell's theory of descriptions with respect to such a pair as

The winged horse flies

There is one and only one winged horse and it flies.

Here the latter would be preferable on the grounds of avoiding an inappropriate appearance of singular reference to a particular object. When the offending appearance of a singular term cannot be removed by a paraphrase Geach would have Frege count the containing sentence as nonsense.

It must be granted, I believe, that Frege was committed to viewing certain occurrences of such expressions as 'the concept horse' as non-denotative. These are those like the occurrence of 'the concept horse' in the phrase 'the extension of the concept horse'. My reasons for

92. Three Philosophers, p. 156.

claiming this is that such phrases are represented in the Begriffsschrift with course-of-values notation by such expressions as: $\hat{x}(x \text{ is a horse})$. (See §9.) And this contains no proper part which itself is a proper name. Still Frege would, I think, have been disinclined to take Geach's advice. For one thing, Geach is assuming of such sentence pairs as those lately discussed that if they express the same Thought, then they are to be accorded the same semantical accounting. But this is not a view Frege accepts; his logicism requires its denial.⁹³ For another, the appeal of Geach's suggestion for paraphrasing out offending expressions will depend significantly on being able, in this or some other way, to handle all the problematic cases. But to deal with (1), and others, Geach appeals to a doctrine of philosophers' nonsense. It may be that Frege, even while recognizing that much of his expositional discourse went irretrievably wide of the mark, would have balked at the idea that it was "philosophers' nonsense".⁹⁴

93. See also Philosophical Writings, pp. 47-49.

94. In Three Philosophers Geach offers Frege another solution; this is to use such sentences as 'Horses are a kind of thing' in place of the likes of (1). But it is difficult to see how Geach will put this in terms of Frege's grammar in a way compatible with the truth of, for instance, 'Unicorns are a kind of thing'. We might try ' $(x)(x \text{ is a horse} \Rightarrow x \text{ is a horse})$ ', but I doubt whether Geach has this in mind, for, intuitively speaking, there seems little sameness in meaning between it and (1).

Suppose Frege were not to adopt Geach's advice and continued to maintain that (1) was false. The question then arises as to what object 'the concept horse' denotes. Frege, so far as I know, never discussed this question in his published writings. This is particularly strange since there is a natural candidate for the job. I suggested earlier (in note 78 of this chapter) that if such an expression as

the concept horse

is to be a proper name its logical structure is that of a second-level function-name completed by a first-level concept-word. And this is just the situation with Begriffsschrift expressions of the form

$\overset{2}{\lambda}(x \text{ is a horse}).$

So the natural thought is to treat the former, problematic proper name as denoting extensions of concepts; on this suggestion the above two names would agree semantically part for part.

Why did Frege not endorse, or even discuss, this

suggestion?⁹⁵ There have been those who thought that Frege's concepts were, like the attributes of more recent literature, in greater supply than the classes associated with them. If this were the case then expressions like 'the concept horse' could not (at least generally) denote classes. Two decades ago Rulon Wells was moved by this consideration to coin the term 'concept-correlate' to

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95. But perhaps in the juxtaposition of the following passages we do find what amounts to an endorsement.

In logical discussions one quite often needs to assert something about a concept, and to express this in the form usual for such assertions -- viz., to make what is asserted of the concept into the content of the grammatical predicate. Consequently, one would expect that the reference of the grammatical subject would be the concept; but the concept as such cannot play this part in view of its predicative nature; it must first be converted into an object, or speaking more precisely, represented by an object. We designate this object by prefixing the words 'the concept'; e.g.: 'The concept man is not empty.' (Philosophical Writings, pp. 46-47.)

It has been suggested . . . that in further developments, instead of second-level functions, we may employ first-level functions . . . this is made possible through the functions that appear as arguments of second-level functions being represented by their courses-of-values -- though of course not in such a way that they simply give their places to them, for that is impossible. (Grundgesetze, p. 92.)

speak of the denotations of the problem phrases.⁹⁶ And this term has enjoyed some currency, particularly with Gustav Bergmann and his students.⁹⁷ However the term was introduced under false pretences. As we have said earlier, co-extensive predicates are co-denotative; thus no class can have more than one concept associated with it.

If our suggestion were adopted, tagging along would be certain bizarre consequences such as that the concept horse is the class of horses. But I doubt whether here we find a reason for reticence on Frege's part. His account of definite description has consequences such as that the winged horse is the class of horses. Though undoubtedly aware of this, he advocated the view nonetheless.

As far as I can see the suggested identification is perfectly coherent with Frege's other views. But let us not forget that all he gains in claiming that (e.g.) 'the concept horse' denotes an object is an avoidance of

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96. R.S. Wells, "Frege's Ontology" in Essays on Frege.
97. Gustav Bergmann, "Frege's Hidden Nominalism" in Essays on Frege. The idea, however, is more widespread than the usage of this term. Thus Newton Garver says that "even Frege wished to talk about concepts, and hence he had to suppose that each concept has a special object associated with it which serves only as an object to talk about when we mean to discuss the concept." ("Subject and Predicate" in Edwards' Encyclopedia, Vol. 8, pp. 36-37.) And Wilfrid Sellars speaks of "the peculiar objects which, according to Frege, one talks about when one attempts to talk about concepts." (Science, Perception, and Reality, p. 228, n. 1.)

meaninglessness for the likes of (1). For Frege is precluded by what he speaks of as "a kind of necessity of language"⁹⁸ from stating literally what we all can feel he wished to state. So our proposed identification, while frictionless, is useless. Perhaps we may reflect Frege's attitude this way. Given that, for all his philosophy, the concept horse could be a class it did not matter what it was.

A couple of observations will serve the purposes of summary. In all but his earliest writings Frege rigorously distinguished between the use and mention of expressions, to use Quine's terminology. He was, perhaps, the first philosopher both to see the need for this and to minister to it. It is thus somewhat ironical that, excepting proper names, all expressions which play a semantical role for Frege are, in isolation, unmentionable. These include, besides predicative expressions, connectives, quantifiers, indeed, all the primitive symbols of his Begriffsschrift.⁹⁹ This is the first observation.

The second builds on the first. A legacy of Frege's work is the notion of a formal system, with its methodological distinction between constructed system, object

98. Philosophical Writings, p. 54.

99. The assertion-sign (and kindred expressions) may be an exception to this.

language so-called, and meta-language. In the latter proceeds discourse about the system, including, of course, directions for the construction of the system. One commentator has gone so far as to call "the distinction contained in his work between formal language and meta-language by means of which is accomplished the construction of a given formal language," "the greatest merit of Frege".¹⁰⁰ But from the point of view of Frege's semantics, since the primitive symbols are unmentionable, formation rules are unstatable. Thus in place of the distinction of language and meta-language we are left with one between literal language and metaphor. "I must confine myself to hinting at what I have in mind by means of a metaphorical expression . . . I rely on my readers agreeing to meet me half-way."¹⁰¹

In Chapter II a conception of logic was set out under which logic is to provide the grammar of a (potentially) universal language. Attributing this conception to Frege leads us to consider his *Begriffsschrift* as providing, in intention, such a universal grammar, or at least a start of one. But logic involves more than grammar. In Frege's mature thought laws of logic are called laws of truth. Also in II, discussion of this idea led us to attribute

100. B.V. Birjukov, Two Soviet Studies on Frege, p. 44.

101. Philosophical Writings, p. 115.

to Frege a conception of logic under which logic provides, schematically, an answer to the question: How is truth possible? It does this through our providing a referential, truth-oriented semantics for the grammar of logic. Frege's semantics, in particular that of predication, has been our concern in this chapter. What has emerged is that the semantics Frege provides for his logic seems to place a limit upon the universality of his lingua. For the semantics of the lingua cannot be stated within the lingua. Though we should not put the point in a way which suggests that the semantics can be stated, just not in the preferred notation of Begriffsschrift. For a "necessity of language" blocks the expression of Frege's semantics; we are confronted with an essential inadequacy of language. We might sum up Frege's view here this way. It is not possible to say how what we can say can be true.

The fact that Frege's semantics cannot be (literally) stated consistent with its own import, what has been called its self-referential inconsistency,¹⁰² has been

102. By M.D. Resnick in "Frege's Theory of Incomplete Entities", Philosophy of Science, 32 (1965), pp. 329-341; he terms this "fatal" to Frege's view. Similarly Sellars claims "it is reasonable to demand of a philosophy that it be self-referentially consistent; i.e., that its claims be consistent with its own meaningfulness, let alone truth." (Science, Perception, and Reality, p. 208.) Here Sellars is speaking of the Tractatus, but would undoubtedly apply such remarks to Frege's philosophy as well.

widely taken to establish that something is wrong, that at least some portion of Frege's views must be given up or excised. Thus Dummett, having discussed (e.g.) Frege's claim that (1) is false, concludes that

[t]his (at first sight trivial) difficulty shows conclusively that the two parts of Frege's theory -- the method of classifying expressions into "proper names," first- and second-level concept-words, etc., and the doctrine that each of these kinds of expression stands for something -- will not hang together; some modification is called for.¹⁰³

Max Black speaks of "the disastrous consequences . . . of the view that it is logically impossible to refer to a function"¹⁰⁴ and adds,

if Frege's view implies that the very formulation of the view is nonsensical, no further refutation is needed.¹⁰⁵

William Kneale, having all so diplomatically mused that

[a] philosopher is in a very awkward position when he finds himself driven to say that his thought cannot be expressed adequately,¹⁰⁶

goes on to claim that Frege's situation "is due to a serious defect in his theory of language."¹⁰⁷

103. Dummett, "Frege on Functions", pp. 282-283.

104. Black, "Frege on Functions", p. 242.

105. Ibid., p. 242.

106. Development of Logic, pp. 500-501.

107. Ibid., p. 501.

So much for sampling of negative reaction. Subsequently I shall want to argue that the fact that Frege's semantics is subject to "self-referential inconsistency" is not sufficient reason for rejecting his views, in whole or in part. For, I shall argue, this situation is unavoidable.

In the meantime there is a somewhat more internal, and specific, objection that can be made against Frege's semantics. The unmentionability and inexpressibility problems trace to the presence of unsaturation in the Fregean scheme of things. Unsaturation in turn, is induced through the assumption that predicative expressions relate to items of what there is in a fashion analogous to that of singular terms. But, we may object, the effect produced by unsaturation strains to the point of breakage the analogy which leads to its appearance. The denotation relation between a proper name and its denotation is a two-place, first-level (semantical) relation taking as arguments a pair of objects, typically a linguistic object and a non-linguistic object. Whereas the denotation relation of a (one-place) concept-word and its denotation is a two-place, second-level (semantical) relation taking as arguments a pair of (one-place) first-level functions, typically a linguistic function and a non-linguistic function. For want of common terms of comparison the

analogy between these modes of reference breaks.¹⁰⁸ We might conclude that Frege's account of predication cannot withstand the breakdown of this analogy.

108. Much this point was made to me by Hugly.

. . . on thorough investigation it will be found that the obstacle is essential, and founded on the nature of our language;

FREGE

C H A P T E R I V

UNSATURATION: AN ASSESSMENT

\$15. Names, Sentences, Horizontals, and Assertions

We shall begin here by reviewing once more the reasoning which leads, with some plausibility, to the conclusions that arithmetic functions are unsaturated and in parallel manner to the conclusion that concepts are unsaturated.

Taking the example '2+3' and assuming that, semantically speaking, its (proper) parts are '2' and '()+3' -- this involves the previous simplifying assumption -- we begin by stating that '2+3' is a name of an object, viz., the number five. '2' also names an object, and we assume that it is through this that it makes its contribution to the fact that '2+3' has denotation. What of '()+3'? If it denotes an object, then that object has, or can be provided with, a proper name, so that, if the contribution of '()+3' to the fact that '2+3' has denotation lies in this fact, i.e., that '()+3' denotes an object, then replacing '()+3', in '2+3', with a proper name of the denoted object would not affect the semantic

character of the larger expression, i.e., the result of replacement would, like '2+3', be a (complex) name, indeed it would, in this case, denote the number five. But this is not so; replacing '()+3' with a proper name does not leave a (complex) proper name. Therefore, we conclude that '()+3' does not denote an object. (At least that that is not the (sole) source of its contribution to the fact that '2+3' has denotation.)

In much the same way, taking the sentence 'Jane is an actress' and assuming that its (proper) semantical parts are 'Jane' and '() is an actress', we say that 'Jane' is a name. In this, we assume, lies its contribution to the fact that 'Jane is an actress' has truth-value. What then of '() is an actress'? If it, like 'Jane', denotes an object, then that object has, or can be provided with, a proper name, so that, if the contribution of '() is an actress' to the fact that 'Jane is an actress' has truth-value lies in this fact, i.e., that '() is an actress' denotes an object, then replacing '() is an actress', in 'Jane is an actress', with a proper name of the denotation of '() is an actress' would not affect the semantic character of the containing expression, i.e., the result of replacement would, like 'Jane is an actress', be a sentence, indeed one which, like 'Jane is an actress', would be true. But this is not so; replacing '() is an actress' with a proper name

does not leave a sentence. Therefore, we conclude that '() is an actress' does not denote an object, or at least that that is not the (sole) source of its contribution to the fact that 'Jane is an actress' has truth-value.

In these reasonings we employ the ideas of object and name of object, i.e., proper name, and it might be objected that we have nowhere explained them. But this is not quite correct. What we did, previously (§11), was to begin with certain words and things, and speak of such words naming, or denoting, such things. We subsequently roamed over portions of our language -- or imagined such roamings -- collecting together other words which struck us as suitably similar to those begun with. And then thought of these on the model of the first as denoting objects. Thus we increased the extension of the denotation relation at both ends; we gained more objects and more names of such. By such linguistic bootstrapping we moved towards the general idea of an expression denoting an object, which, once in hand, we then used to explain proper names as just such expressions. To concede a point, explain the idea of (e.g.) proper name, for instance, we perhaps have not done. But if not, then explanation is not possible here. This situation should be likened to that of our inability to provide an explanation of negation, for instance, on account of the problem

of circularity.¹

Returning to our examples, what shall we say is the semantic function of '()+3', on the one hand, and '() is an actress' on the other? We may regard the earlier reasoning as establishing this much. If such expressions are to have denotations accorded them, their denotations must differ essentially from objects, the (possible) denotations of proper names. This was Frege's position. For reasons stemming from his logicism Frege counted predicative expressions as denotative, and drew our conclusion that concepts, the denotations of predicates, are unsaturated, this being their essential difference from objects. Similarly, for functional expressions and functions. A result of this is that sentences get classed as a species of complex proper name.

This conclusion seems to present a problem, one we have thus far passed by. If (e.g.) 'Jane is an actress' itself denotes an object, then should it not be the case that it and (e.g.) 'Jane' are intersubstitutable? And if they are not, then should we not conclude, by parity of reasoning, that sentences do not denote objects? Yet sentences must denote objects if predicates denote functions. There is, I believe, a wide-spread suspicion that Frege's semantics harbors a contradiction in this area.

1. See Quine, Philosophy of Logic, p. 40..

The principles which (in the context of certain assumptions) lead to the conclusion that the denotations of predicates are unsusceptible of singular reference also (in the context of similar assumptions) lead -- so runs the suspicion -- to the conclusion that the denotations of sentences are likewise beyond the reach of singular terms. Yet Frege maintained that sentences are proper names, a thesis which cannot be given up without unraveling the whole semantics.

Black has voiced this thought this way.

The following argument seems to me to be a sufficient refutation of Frege's view that sentences are designations of truth-values. We may assume that if A and B are designations of the same thing the substitution of one for the other in any declarative sentence will never result in nonsense. This assumption would not have been questioned by Frege. Let A be the sentence "Three is prime" and B the expression "the True". Now "if three is prime then three has no factors" is a sensible declarative sentence; substitute B for A and we get the nonsense "If the True then three has no factors" . . . Hence, according to our assumption A and B are not designations of the same thing -- which is what we set out to prove.²

Black here argues that if 'Three is prime' is true it would, on Frege's view, be co-denotative with 'the True'. But since two expressions which have the same denotation are intersubstitutable, such expressions as 'If the True then three has no factors' should make sense. Since they

2. Black, "Frege on Functions", pp. 229-230.

do not, Frege is wrong in his view that sentences denote.

Black assumes that Frege would not question the principle that co-denotative expressions are intersubstitutable without going from sense to nonsense. I agree, but wish to go further and attribute a somewhat broader principle to Frege. The intuitive idea is that expressions which make the same semantic contribution to containing expressions are so intersubstitutable; classes of such expressions comprise, we may say, semantic categories. I can provide no general characterization of the idea of same semantic contribution, or semantic category. However proper names should comprise one such category, on Frege's semantics. Thus the broader principle requires that any name of an object can replace any other name of an object, so occurring, preserving nonsenselessness, these exchanges not being limited to the co-denotative. So also first-level function-names of like polyadicity should comprise semantic categories. Similarly with second-level function-names; and so forth. We may state this general semantical principle (GSP) this way.

If A and B are expressions of the same semantical category, then the result C of replacing A with B in any expression E will be of the same semantical category as E .

The idea this principle aims to capture is an old one, one which guides Plato in the Sophist I should say. More to the point of this essay's interests, adherence to this

principle is required by the philosophy of logic we have attributed to Frege (§8). If we see ourselves as attempting to say how truth is possible through spelling out how words relate to the world, then we must suppose that expressions which bear the same semantical relations to the same (ontological) sorts of worldly items are making the same type of semantical contribution to wholes which have them as (semantical) parts and so can replace one another preserving the semantical character of the wholes. That Frege came to hold that functions are unsaturated I take to confirm that he was guided by our GSP. But what then of Black's charge that Frege was wrong on his own principles to maintain that sentences denote objects? I want to show that Frege has a reply to the reasoning Black employs, one which leans heavily on the peculiar function-name, the horizontal. Still in all, Frege's semantics does violate our GSP, indeed violate it wholesale, and the source of the problems does lie with the view that sentences are denotative. But when we get a better overall view of the lay of the land I think we shall not be inclined to see in such violations a rejection by Frege of the general semantical principle. Nor shall we find reason to retract our attribution to him of the philosophy of logic of (§8).

First let us see how Frege can reply to Black's objection; here we shall use the example

(1) Theaetetus sits only if Plato flies.

In this we cannot, we might suppose, replace 'Theaetetus sits' with 'Theaetetus' while saving the semantical character of the containing expression. Certainly the result

(2) Theaetetus only if Plato flies

is not grammatical. However I shall argue that, in this case, the failure of grammaticality is rather incidental, at least for thoroughgoing Fregeans.

Let us remind ourselves that (1) would go into Begriffsschrift notation this way.

(3) $\begin{array}{l} \text{Flies}(\text{Plato}) \\ \text{Sits}(\text{Theaetetus}). \end{array}$

The vertical is Frege's condition-stroke to which the horizontal is attached at three places. And let us remark again on certain differences between the Fregean (3) and, for instance,

$\text{Sits}(\text{Theaetetus}) \supset \text{Flies}(\text{Plato})$

as standardly understood. First, whereas both ' \vdash ' and ' \supset ' express truth-function conditionality, Frege's sign is a two-place, first-level function-name denoting a function which takes any object as an argument, not just truth-values. The horseshoe is most typically considered as non-denotative, and, whether sentences are treated as names or not, only sentences are allowed to complete it to a sentence.

More importantly, Frege's (3) differs from the above in that the horizontal is counted as a piece of notation on a par with the condition-stroke. It too is a function-name; it denotes a function which yields the True when its argument is the True, and yields the False when any complete denotation other than the True is its argument. Thus ' $\neg \alpha$ ' will denote a truth-value provided only that α denotes an object, any object.

Holding onto these reminders, we may formulate our understanding of (3) by saying that it is true (denotes the True) just in case if ' $\neg \text{Sits}_{(\text{Theaetetus})}$ ' denotes the True, which it will just if ' $\text{Sits}_{(\text{Theaetetus})}$ ' denotes the True, then ' $\neg \text{Flies}_{(\text{Plato})}$ ' denotes the True, which it will just if ' $\text{Flies}_{(\text{Plato})}$ ' does also. Or somewhat more simply, (3) is true under just these conditions: Either ' $\text{Flies}_{(\text{Plato})}$ ' denotes the True, or ' $\text{Sits}_{(\text{Theaetetus})}$ ' does not (i.e., denotes something other than the True.) For Frege such remarks render our understanding of (3). For, "by our stipulations it is determined under what conditions the name denotes the True. [And t]he sense of this name -- the Thought -- is the thought that these conditions are fulfilled."³ To understand under what conditions a sentence would be true, its truth-conditions, is to grasp its sense, the Thought which it expresses.

3. Grundgesetze, p. 90.

Now suppose we replace, in (3), 'Sits_(Theaetetus)' by 'Theaetetus' to get

(4) $\begin{array}{l} \text{---} \text{Flies}_{(\text{Plato})} \\ \text{---} \text{Theaetetus.} \end{array}$

Do we understand the truth-conditions of this? Yes.

(4) is true just in case if '---Theaetetus' denotes the True then '---Flies_(Plato)' does also. Alternatively put, (4) is true just when either 'Flies_(Plato)' denotes the True or 'Theaetetus' does not. And so we know that (4) is true, since we know 'Theaetetus' to denote other than the True. But this is by the way. Of importance is the fact that, as Fregeans, since we understand the truth-conditions of (4) we grasp the Thought which it expresses, for that is the thought that these conditions are fulfilled.

Now we wish to use these observations to answer Black's objection. I take our task to be to show how to render the likes of (4) into recognizable English. We know (2) will not do, since it is ungrammatical and hence not fit to express the Thought of (4). But a suggestion of Dummett's, made in the course of brushing aside the argument of Black's under consideration, is relevant here.

If sentences stand for truth-values, but there are also expressions standing for truth-values which are not sentences, then the objection to allowing expressions of the latter kind to stand wherever sentences can stand and vice versa is grammatical, not logical. We often use the word 'thing' to provide a noun where grammar demands one and we have only an adjective, e.g.,

'That was a disgraceful thing to do'; and we could introduce a verb, say 'trues', to fulfill the purely grammatical function of converting a noun standing for a truth-value into a sentence standing for the same truth-value.⁴

Applying Dummett's suggestion to an example of his, we would turn the ungrammatical construction 'If oysters are inedible, then the False' into something we can, with good grammatical conscience, count among our English grammatical constructions, namely, 'If oysters are inedible, then the False trues'.

On first reflection it might seem that Dummett's suggestion would not be applicable to our problem with (2) since with 'Theaetetus only if Plato flies' we have no "noun standing for a truth-value" which we could convert "into a sentence standing for the same truth-value." We do have 'Theaetetus', but this, we assume, denotes Theaetetus, and not some truth-value. And it would defeat our purpose to think that this expression had somehow shifted its denotation. For the point of this exercise is, as we may put it, to see how to defend the claim that when (e.g.) 'Theaetetus', with its sense and reference, replaces a sentence the result expresses a Thought.

Upon second reflection, however, it should occur to us that there is no reason not to extend (if that is what

4. Dummett, "Truth", pp. 49-50.

we do) Dummett's suggestion to apply to the general case. Our situation is that of having a Thought, that expressed by (4), to render in a recognizably English sentence. Why not employ Dummett's device to tailor the nongrammatical (2) to our needs thus: Theaetetus trues only if Plato flies. It must be admitted that this is somewhat altered English. But then we might expect some grammatical adjustments to follow in the wake of looking upon sentences as denoting expressions. The suggested adjustment is to allow (or require) that a singular term "grow a verb" when replacing a sentence. Such an adjustment is merely grammatical. We must keep in mind that the expression which occupies the sentential position, the singular term with its attendant "verb", has the same sense and denotation as that we have (or may have) substituted, (e.g.) 'Theaetetus'; "growing a verb" in no way affects the sense or denotation of the substituted singular term. Such, then, is the defense I would have Frege offer against the criticism of Black.

We have just seen how such a problematic case as that of (2) can be made sense of. But remarks made thus far may not make apparent why Frege's horizontal is essential to the line of reply I am offering to the charge of inconsistency. This since it is possible to treat the condition-stroke in such a way as to avoid the need for its pair of trailing horizontals. On Frege's

understanding ' $\overline{\text{L}}_{\alpha}^{\beta}$ ': denotes the True just in case either α does not denote the True or β does, and if we interpret the condition-stroke just this way we need not attach any notational significance to either of the trailing horizontal strokes; they become just pieces of punctuation. (And similar observations can be made of the negation-stroke.)

But there will remain the "limiting case" where, to move back into English, a singular term is substituted for a sentence which is not itself a truth-functional component of a larger sentence. What do we say of the result of substituting for (1), in entirety, 'Theaetetus'? Adopting Dummett's grammatical convention the result will read: Theaetetus trues. But what Thought is thereby expressed? If we think of such substitutions as proceeding within a suppressed initial horizontal, then we may say, generally, that ' α trues' will be true when α denotes the True, false when α denotes other than the True. The Thought expressed is thus that this condition is fulfilled where α is 'Theaetetus', that, in other words, 'Theaetetus' denotes the True. And recalling a remark of (§9), a Fregean can make explicit the presumed implicit "English horizontal" with the phrase 'it is the case that'; thus we may render (4) as: It is the case that Theaetetus trues.

We have considered interpreting the results of a

sentence being replaced by a proper name. For completeness we need to consider what we shall say of the results of a reverse substitution, for instance,

(5) Theaetetus sits sits,

in which the sentence 'Theaetetus sits' replaces 'Theaetetus' in 'Theaetetus sits'? What sense can we make of this? As Frege understands the construction of predication, a most simple predication is true just when the function denoted by the predicate yields the True as value for the denotation of the proper name as argument. When we think of 'Theaetetus sits' as itself an expression which denotes an object then the Thought in question is that the function denoted by '() sits' takes the denotation of 'Theaetetus sits' into the True. This is the Thought which (5) would express were it grammatical, and so eligible to express a Thought. Is it grammatical? If the foregoing has had its intended effect this question will not strike us as terribly important from the point of view of present concerns. If (5) is not grammatical, let us "punctuate" it, perhaps with a sprinkling of hyphens or parentheses, so as to render it grammatically acceptable to us. Once again our position, if we are Fregeans in our semantics, is one of grasping a certain Thought which is (perhaps) not readily expressible in the confines of received English. (This would not be the only place where Frege's theory had an impact on English

constructions; think of the phraseology we accept in rendering quantifications, or negations, for that matter, back into our mother tongue.) But again, such alterations as attend the expression of these thoughts are "meaning neutral". We might compare them in this respect with the grammatical adjustment of the ungrammatical 'Some cow are brown' to the grammatically acceptable 'Some cows are brown'.

Summing up, the point of the last several pages is that Frege's semantics is not shown to be incompatible with the general semantical principle by the line of reasoning Black sets out. We mobilized Frege's horizontal to this purpose. Let us look further at this piece of notation.

The presence of the horizontal in the notation as laid out in Grundgesetze is puzzling. What work does it do? Why does Frege include it? Is this simply a case of a piece of machinery held over from Begriffsschrift which, having been stripped of its earlier function, had to be supplied with some explanation? Our response to Black perhaps provides something of an answer. For there is a natural inclination to treat sentences as comprising a semantic category.⁵ Indeed this would seem to be required

5. And in effect we have earlier. On this see note 13, below.

by Frege's Dictum. This category disappears once sentences are construed semantically as a case of complex proper names. The inclusion, or retention, of the horizontal, with its Grundgesetze explanation, provides at least the semblance of a class of sentences. For we may think of all ordinary sentences as simply having 'it is the case that' as a suppressed prefix so that the results of substituting ordinary names for ordinary sentences get accommodated into an expanded class of "sentences" along the lines sketched out in reply to Black.⁶ Thus we may see Begriffsschrift formulas and their natural language translations as forming a kind of surrogate-sentence class, the class of horizontals.

However our maneuver does not serve to square Frege's semantics with our general semantical principle, for the class of horizontals does not itself comprise a semantic

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6. If we do not think of English sentences as having a suppressed initial horizontal, then we shall encounter the following situation. There will be expressions which we shall not be able to determine whether or not they are names of truth-values, even knowing of their (proper) parts whether these are truth-value names. Consider $(\text{Tx})(x=S \text{ or } x=R)$. If 'S' and 'R' are co-denotative, then the whole expression denotes either the True or the False. But if 'S' and 'R' are not co-denotative, then (on Frege's convention) the whole denotes the class whose members are the True and the False. So we cannot determine whether the whole denotes a truth-value, or as we might also say, whether it expresses a Thought, except by determining if 'S' and 'R' are alike in truth-value.

category, since for every member of this class there are other, co-denotative, expressions themselves not horizontals. Thus, let S be any surrogate-sentence; then ' $(\text{Tx})(x=S)$ ' denotes what S does but, lacking as it does an initial horizontal, it is not itself a surrogate-sentence. The same is true of ' $(\text{Tx})((x=S) \text{ and } S)$ '; and so forth. Any such expression may be replaced by a horizontal, saving semantical character, but not conversely. And generally, a horizontal may replace a proper name, saving semantical character, but not conversely. The horizontal-stroke performs the suggested role of forming a surrogate-sentence class through giving sense to the results of replacing sentences, naturally taken, by names, naturally taken; the semantics of the horizontal provides Thoughts for such replacements to express. But it can do this only at the cost of its own immovability from the position of initial prefix to Begriffsschrift formulas.⁷ So it is that the class of horizontals does not comprise a semantic category.

So it is also that the class of proper names, which includes truth-value names, is not a semantic category.

7. It could, with justice, be pointed out that in Frege's grammar the signs for both negation and conditionality suffer a similar syntactic immovability. The difference is that there is no external requirement for this in their case as there is with the horizontal.

And for similar reasons one-place, first-level function-names do not form a semantic category. The horizontal is such a function-name, and while it may replace any similar function-name, saving semantical character, the converse does not hold.

Indeed, for similar reasons, it follows that Frege's semantics of Begriffsschrift flouts the general semantical principle at every turn; his semantics admits of no semantic categories. By picking an appropriate Begriffsschrift formula and making an appropriate deletion of some name or names, we can display a function-name of any particular level and degree; this name may then replace others of like level and degree, but not conversely. As an example, take the Begriffsschrift translation of 'Everything sits',

$$\text{---}\overset{\text{H}}{\text{---}}\text{--- Sits}_{(\text{H})},$$

and delete the one-place, first-level function-name 'Sits()'. This leaves the one-place, second-level function-name,

$$\text{---}\overset{\text{H}}{\text{---}}\text{--- } \emptyset_{(\text{H})},$$

where ' \emptyset ', as before, is a place-holder. This may replace any other similar function-name, but other such function-names may not, in general, replace it in the above. For instance it cannot be replaced, saving semantical character (or in conformity with Frege's syntax), by the similar course-of-values notation.

Frege's semantics is in complete violation of our

GSP. Still I would say that Frege was guided by some such principle in his semantics. For we can look at this situation this way. Deeming sentences denotative seemingly induces failures of substitution. The horizontal serves to localize these violations so that failure of substitution occurs only at the horizontal. Unfortunately such failure is infectious, and the horizontal touches expressions of all would-be semantic categories. So the whole system suffers the disease. Looked at this way the wholesale violation of the GSP may be seen as resulting from an attempt to handle the more blatant violations of this principle brought on by counting sentences as names.

Still we should emphasize that since in the end Frege's semantics is devoid of semantic categories, there is in particular no such category whose members are distinctively concerned with truth. Frege then has not given us a lingua with respect to which we might entertain the idea of answering our question 'How is truth possible?' in the ostensive fashion previously considered. For such an answer -- "These ways" -- would seem to presuppose that forms therewith indicated, or all lexical fleshings out of them, possess a distinctive semantical character, the character possessed by those linguistic items at which we may sensibly direct the query: Is it true? Again we observe that this situation results from holding that sentences are proper names. I would have us see

how this, in the end, is incompatible with a conception of logic, that of (§8), which, in Frege's case, was in part responsible for including sentences with names in the first place. Here we may find a reason for thinking that move to be mistaken.

There is another, somewhat subtle, difficulty which results from the inclusion of the horizontal in the system of Grundgesetze. Take, for instance, the truism that the disjunction of a truth with itself is a truth. We would expect a system of logic such as Frege's to encompass this, perhaps through showing by some means or other that a self-disjunction

$$S_1 \vee S_1$$

is true just in case S_1 itself is true. However there is a clear sense in which Begriffsschrift is incompetent to this task. And the inadequacy in no way turns upon the lack of a (primitive) symbol for disjunction, which we may suppose is present. The problem is that a self-disjunction, like all disjunctions, has a symbol for disjunction as its main (truth-functional) connective, and no Begriffsschrift formula can be so structured. Since all Begriffsschrift formulas lead off with the horizontal, it will always be the main connective.

We may put the same point in terms of a mode of inference. To move to a conclusion by modus ponens we require two premisses; one is a conditional which has as

its antecedent the sentence of the second premiss and as its consequent that of the conclusion. But since no Begriffsschrift formula will be a conditional, that is, a formula whose main connective is a sign for conditionality, no Begriffsschrift moves can be made in accordance with modus ponens, at least as this is intuitively understood. Similar problems affect (first-order) quantificational rules, and so forth. Since, as suggested, the presence of the horizontal in Grundgesetze reflects the decision to count sentences as names, in the inadequacy of Begriffsschrift to intuitive logic we may find a second reason for rejecting that decision.

Curiously, this problem of capturing intuitive logic turns up in Grundgesetze not just once, but twice over. Recall that the Basic Laws (axioms), and so too the theorems, of Grundgesetze are not themselves names of anything. They are expressions wherein the assertion-sign prefixes a proper name. Equivalently described, they are expressions wherein the judgment-stroke prefixes a horizontal. Such expressions we might naturally call assertions.⁸ (But remember we speak of strings of symbols of Frege's system, not speech acts or the like.) Since the assertion-sign can only occur in a left-most position,

8. In (§9) we followed Furth in calling them propositions.

we cannot, for instance, truth-functionally compound assertions. Hence the problems just noticed with horizontals attend assertions also.

Assertions are problematic in other ways. Since an assertion of Begriffsschrift is a law of logic it must be true; laws of logic are laws of truth. But now consider the fact that of some linguistic items it is appropriate to say that the truth-value of the whole is a function of the truth-values of the parts. This requires that we be able to speak in one breath of wholes and their parts as being true or false. But this is not so with assertions. The sense in which an assertion is true cannot be that in which any of its parts may be. For an assertion cannot contain an assertion as a (proper) part. And, whereas truth-valued parts of assertions are names of truth-values, an assertion is not a name.⁹

It seems that Frege has, or owes us, two theories of truth, both at the level of linguistic artifact. One in terms of semantical relations between words and things; this is well articulated. And another which discusses how truth-value accrues to such non-names as assertions; this we do not get. Further we would like to know, for instance,

9. And we have similar problems with derivations. Rules of inference are officially stated for assertions, but, for instance, when Frege derives Russell's paradox (Grundgesetze, pp. 130-132) he avoids using formulas with a prefixed assertion-sign.

why it is appropriate to use the word 'true' both to say that assertions are true and that (some) sentences denote the True.

Further, we may wish to ask of some theorem of Begriffsschrift: Is it true? And do so even knowing that it is a theorem. But it seems that Frege precludes this, for to do so we would have to be able to consider an assertion non-assertively.

I do not wish to pursue further the topic of Frege on assertion except to remark that it is not implausible that the assertion-sign, like the horizontal, found its way into the Grundgesetze system in part at least under the felt pressure to recoup what gets lost when we lose sentences as a semantical category.¹⁰ Here we may find further reason to judge mistaken the treating of sentences as names.

We shall now return to Frege's philosophy of logic as earlier discussed and inquire as to whether, consistent with this and so with the enunciated semantical principle required by it, it is possible to frame a semantics for predications which avoids the unwanted consequences of Fregean unsaturation.

10. It might be argued that if you could assert the True, then you could assert the number two -- which is impossible; hence truth-value names are not assertible, and a new type of expression is needed.

\$16. Alternative Semantics of Predication (1)

On Frege's view predicative expressions have denotation; they denote functions. These denotations, he says, are unsaturated; they lie beyond the reach of singular terms. As a consequence Frege's semantics of predication cannot be stated consistent with its own import.

Thus for the Fregean there is something, indeed much, which cannot be said. For instance, that '() is an actress' denotes a particular concept in its occurrence in 'Jane is an actress'. Here we can, in a certain fashion, show that the expression is, or is taken to be, denotative; we do this by existentially generalizing upon the predicate position. And in this way generally we may show that a certain expression is denotative, while working within the system; for Frege any denotative expression (excepting a lead horizontal) may yield up to an existential generalization. Sometimes what can be so shown can also be stated. Thus we may, in good conscience, state that 'Jane' denotes Jane. But this is the exception. Only objects admit of such specification of denotation.

The non-believer finds Frege guilty of saying what he says cannot be said; he accuses Frege of "self-referential inconsistency". This "inconsistency" and the above ineffability are two descriptions, or accounts, of a

common phenomenon; we may call it the Frege phenomenon.

The question which shall now exercise us is whether, and if so how, this phenomenon can be avoided or done away with. More exactly, given the Fregean conception of logic keyed to the question "How is truth possible?", can we provide a semantics of predication which avoids the ineffability, or inconsistency, attendant to Frege's account? As to what could count as a "semantics of predication", I shall rely on the content this idea has acquired thus far in the course of the essay, though this will be extended and modified in various ways in ensuing investigations. But a cautionary note is in order.

Whereas we shall be focusing our attention upon predication we will do well to keep in mind that this construction interlocks with others, notably quantification, in notations such as Begriffsschrift.

We shall survey a number of alternatives to Frege's account of predication, investigating something more than a random selection, but perhaps something less than all possible alternatives, even within our task-set boundaries. The hope is that techniques displayed in discussing these examples will be applicable to variations which may be unmentioned so that generality is achieved. In this section we shall look at a number of alternatives which we may see as agreeing with Frege in counting predicative expressions as denotative. In the subsequent, and final,

section alternatives which arise through denying this feature of Frege's semantics will come in for consideration. Each of the various alternatives will resemble the views of certain well-known figures, some of whom, at least, will be indicated. However in critical discussion I shall stick with the basic idea of each alternative and not specifically take up the positions of others, who, in any case, may not intend, or wish, their views to be taken as attempts to deal with our question. An exception to this policy will be the position of Quine, a closer look at which will close the essay.

Before explicitly taking up such alternatives we shall spend a little time developing a critical tool, one which derives from, and works well in conjunction with, the general semantical principle which we have seen Frege's conception of logic requires. Consider once more 'Jane is an actress'. On Frege's view this is composed of two semantic parts, the proper name 'Jane' and the function-name '() is an actress'. These expressions are of different semantical type, since, so goes the view, whereas each denotes something and therewith makes its semantic contribution to the fact that the whole expression has truth-value, the denotations involved are essentially different and thus so too are the expressions. But now suppose, for example, we wish a view on which it is possible to refer to the denotations of predicative

expressions with singular terms. Then, barring additional complications, 'Jane' and '() is an actress' will be of the same semantic category. So, with regard to 'Jane is an actress', our general semantic principle will require that some semantical function gets performed other than those accomplished by 'Jane' and '() is an actress' each denoting something. For we must be able, for instance, to replace either one of these names with the other without disturbing the semantical character of the containing expression. But, failing additional interpretation, 'Jane Jane', for instance, lacks truth-value. Hence the need for more in the way of a semantical accounting of 'Jane is an actress' so that such substitutions already licensed may proceed in accordance with our guiding principle. Given that neither 'Jane' nor '() is an actress' be required to perform additional semantical chores, it is most natural to locate the additional semantical feature in the juxtaposition of 'Jane' and '() is an actress'. The new account will in this respect differ from Frege's. On Frege's view each semantical function performed in (e.g.) 'Jane is an actress' receives expressional representation. On the type of alternative to be considered a semantical function is performed by a non-orthographic feature of the sentence, the juxtaposition of subject and predicate.

Now it would seem that we could represent the additional semantic feature in explicit notation. Since 'Jane'

and '() is an actress' are to be counted of a common semantic category we may, without worrying the syntactic details, portray the claim of 'Jane is an actress' more explicitly with a sentence of the form '()R(')'. Here 'R' will stand for the additional notational element; it is to perform as does juxtaposition in 'Jane is an actress'.

Since any sentence, or at least any humanly understandable sentence, must be party to just a finite number of semantic features, it might well seem possible to reflect in explicit notation the semantic features of any sentence. And if this is so, it might seem that we could orthographically represent the semantic features of any sentence with another sentence semantically indistinguishable from the first. (Unless the first already wore its semantic features in its orthography.) To this it could be objected that being able to render any semantical feature orthographically does not imply being able to accomplish this with a semantically indistinguishable sentence. Perhaps rendering explicit previously implicit features induces additional features newly implicit.

We need to look carefully at this idea. Suppose there is a sentence with a non-explicit semantic feature, such as the juxtaposition of 'Jane' and '() is an actress' in 'Jane is an actress' is on a semantics where the two pieces are co-categorematic. What would it mean to say,

for instance, that the semantic function so performed cannot be orthographically represented with a sentence semantically indistinguishable from 'Jane is an actress'? Just this, that the semantic function cannot, literally, be put into words at all. This may not be readily apparent, but reasoning in terms of our example should make it so.

Let ' $\alpha R\beta$ ' be a sentence in which the semantic features of 'Jane is an actress' are rendered explicit through ' α ' functioning as does 'Jane', ' β ', as does '() is an actress', and 'R' as does the juxtaposition of these two elements. Now suppose it is said that ' $\alpha R\beta$ ' is semantically distinguishable from 'Jane is an actress' through containing an additional semantic feature. (Perhaps one shown by the grouping of ' α ' and ' β ' about 'R'.) This would contradict the original supposition, at least if taken in a way intended as supporting the alternative view being considered. If ' α ', ' β ', and 'R' each performs just the semantical roles of 'Jane', '() is an actress', and juxtaposition, respectively, then it will be possible to deploy these elements in such a fashion that a truth-valued expression results. This might require some incidental grammatical adjustments, but cannot require any additional semantical elements. This is so since 'Jane is an actress' itself is a truth-valued expression, and one which gains its semantical character

under our assumption through just the three features in question. Any new semantic feature must be superfluous. Thus if ' $\alpha R\beta$ ' contains an essentially additional feature the original supposition is false. For ease of exposition let us assume that if ' $\alpha R\beta$ ' is essentially richer than the original sentence, then 'R' is our culprit, that 'R' does not function just as does juxtaposition in 'Jane is an actress' contrary to the original supposition. Then we have failed to put into words the semantic function of juxtaposition in our example. Perhaps this failure can be rectified, but only at the cost of rendering implicit something previously explicit. This is the force of the derived contradiction. To maintain focus, let us ignore such complications. So we shall conclude that if ' $\alpha R\beta$ ' is essentially richer than 'Jane is an actress' it will not be possible to use an expression, any expression, to do what is done in 'Jane is an actress' by juxtaposition. If this is the case, then a version of the Frege phenomenon has been forced back upon us.

To facilitate understanding let us approach this through some examples. We may state the semantic function of 'Jane' (in our example) this way,

'Jane' denotes Jane.

Here we use 'Jane' (on the right) in the role we are speaking of. Again, to take a somewhat different case, suppose we were to think of the predicate '() is tall'

in, say, 'Tom is tall', not itself as a denoting expression, but rather as one which performs its semantical function through being true of some, none, or all of various objects. Then we could state its semantic function this way,

'() is tall' is true of what is tall.

And here we use the predicate (on the right) in the role we attribute to it. Now, if we are to be able to state the semantic function of juxtaposition (as in our example) it seems we shall want to say something like this,

Juxtaposition performs thusly.

Here, having indicated the appropriate semantic relation (e.g., denotation, or satisfaction, or what have you) we would use some expression in the role performed by juxtaposition in (e.g.) 'Jane is an actress'. But since we are unable to provide an expression to perform that role, we cannot state what we would wish to say.

Frege was precluded by his semantics from (literally) stating the semantical function of (e.g.) '() is an actress'. What we have just seen is that to hold that some sentence contains a semantic feature which cannot be orthographically represented in another sentence semantically indistinguishable from the first is to be committed to something of a piece with Frege's forced silence. In the example pursued, what we learn, subject to some simplifying assumptions, is that we would be

precluded by our semantics from (literally) stating the semantical function of the juxtaposition of 'Jane' with '() is an actress'. Thus if we are to avoid the ineffability, or inconsistency, attendant to the Frege phenomenon we shall need to require that our semantics, whatever its details, be in accord with the following representation principle:

For any sentence possessing a semantic feature which is not represented in it orthographically, it is possible to construct another, semantically indistinguishable from the first, in which this feature receives orthographic representation.

Since our intention is to attempt a semantics of predication which avoids the consequences of Fregean unsaturation we shall want to adopt this principle.¹¹ But let us keep in mind that to show that a particular semantical interpretation breeds ineffability through violating the representation principle is not, by itself, to provide sufficient grounds for concluding that it is mistaken. For we have not shown that such ineffability is altogether avoidable.

This principle, in tandem with our general semantic

11. Actually this principle is not as strong as we probably would want. It requires the representation of any feature, but not the (simultaneous) representation of all features. This, since it allows the possibility that rendering an implicit feature explicit diminishes one that was explicit so that it is now implicit. However the principle as stated will adequately serve present purposes.

principle, injects a certain rigor into our semantical investigations. When the GSP turns up inexplicit semantical features the representation principle authorizes rephrasal into more explicit forms. Whereupon GSP may again be trained upon our quarry. And so forth. This said, let us get on with the inquiry.

In this section we shall take up four alternatives to Frege's account of predication, each of which agrees with Frege in construing predicates as denoting expressions. The first three will be versions of the idea that in (e.g.) 'Jane is an actress', juxtaposition of the subject and predicate is semantically significant. On one view this has the significance of functional application; on a second, that of a relation; and on the third, that of a non-relational tie. The fourth position to be considered here attempts to avoid the attribution of an additional semantical feature through what we shall call the "meta-language move".

It is convenient to shift to '2 is prime' as our working example of a most simple predication. Then, given that the predicative part '() is prime' has complete denotation as does '2' so that these expressions are semantically co-categorematic and thus interchangeable, we must find additional semantical work done in the sentence. Suppose then that we see in the juxtaposition of subject and predicate notation for the application of a

function to an argument.¹² This is more intuitive if we represent '2 is prime' as

Prime(2).

Here we have two positions; these we may show with

(') ().

The juxtaposition of these positions is to be treated as indicating functional application. This additional semantic role may then be made expressively explicit this way,

A [('), ()],

or, in the particular example, with,

A [(Prime), (2)],

which is intended to be semantically indistinguishable from '2 is prime'. We may read the last set-off expression as 'The application of the prime function to the number two'. And this will be true just in case the function in question yields the True for its argument, which it does, since 2 is prime. However, on the suggested semantics and given our general semantic principle, any name of any object may sit in either of the two available slots. And what, we may wonder, will we say, for instance,

12. This view is similar to those suggested by Alonzo Church in his review of Black's "Frege on Functions", Journal of Symbolic Logic, 21 (1956), pp. 201-202, and Introduction to Mathematical Logic, Chapter 0., and by M.D. Resnick, "Frege's Theory of Incomplete Entities."

when something other than a function is denoted by a name in the left position. We need to provide an interpretation for the notation of functional application which will handle such "waste cases". But note, the need for this does not arise with the introduction of explicit notation. It stems from an even-handedness towards the semantically akin. Perhaps it will do to simply take the idea of "functional application" broadly so that we may say that in all cases where the application of one argument to another does not yield the True, that it yields the False. It is with another aspect of this approach that I wish to be critical.

Taking juxtaposition as having the semantical significance of functional application requires agreement with Frege's view that sentences have denotation. We have already indicated problems with such a view; here we may expand upon these.

If sentences have complete denotation, then our general semantic principle licenses substituting for the sentence '2 is prime' such an expression as, simply, '2'. In this we might suppose we have a violation of our principle, since this substitution takes us from a sentence to a non-sentence. If so, we would have to conclude that '2 is prime' is not semantically explicit. And if it is not, for this reason, then neither would 'A [(Prime), (2)]' be semantically explicit. Indeed, the condition of

explicitness would prove unattainable. But the situation is not as clear as it might be. We violate our guiding principle when replacing one expression by another of its semantic type alters the semantical character of the containing expression. (Where the relevant notion of containment is stretched to the limit to accommodate such substitutions as recently contemplated.) In replacing '2 is prime' with '2' we shift from a truth-valued expression to a non-truth-valued one. But is this a change of semantical character? Constant through such substitutions is the fact that we always have a denoting expression. Is this perhaps not saving semantical character enough? This response is, I believe, sufficient to show that our semantic principle by itself is not able to rule out counting sentences as names.¹³

Let us review the situation with Begriffsschrift. Frege's syntax breaches the general semantical principle through disallowing general replacement of horizontals by proper names. He thereby gains a class of truth-valued expressions. And so can, in effect, provide a semantics for his notation through stating truth-conditions for expressions of this class, the horizontals. And it seems clear that any notation which is intended to be inter-

13. This, and what follows it, is what is referred to in note 5.

preted through the providing of truth-conditions will have to likewise make syntactically impermissible certain semantically authorized substitutions if sentences are names. And so perhaps we should say that whereas replacing '2 is prime' with '2' may not violate the general semantic principle, it does undercut it. For holding sentences to be names, thus licensing such a replacement, is incompatible with the conception of logic that motivates the semantic principle, since, if free replacement is allowed we shall have no means of focusing in upon just those expressions which are truth-valued.

To sum this up, if treating sentences as names does not lead, with Frege, to a flouting of the GSP through judicious syntactical restrictions, it will result in the indiscernibility of bearers of truth-value and so void the possibility of a truth-conditional semantics. Either way Frege's conception of logic suffers. So, I conclude, given that the laws of logic are laws of truth, as previously discussed, sentences are not themselves names. Since treating juxtaposition as functional application requires sentences to be names, this approach must be discarded.

Let us turn to some alternatives which are not faulted in this way. Suppose we treat juxtaposition as

having the semantical import of a relational expression.¹⁴ If predicates, including relational terms, do not denote functions, this move need not force sentences to be names. On this interpretation, '2 is prime' could be rendered semantically more explicit with something of the form ' $\alpha R\beta$ ', where ' α ' does the work of '2', ' β ' that of '() is prime' and 'R' indicates some relation. The particular relation can vary with variants of the basic idea, but it could, for instance, be that of class membership. Then the idea would be that '2 is prime' would be semantically indistinguishable from the more explicit '2 is a member of the class of primes'.¹⁵ However, without fussing over details, a simple argument seems to show the inadequacy of this approach.

If '2 is a member of the class of primes' is a relational statement, then, '() is a member of the class of primes' would be a predicative expression, which, if

14. This approach is quite common with Carnap and students of his. For instance, it is found, attributed to Frege, in Meaning and Necessity. It is often combined with the idea that sentences have denotation, but this is not necessary.

15. Earlier (§13) we discussed one objection Frege had to introducing classes in this way. In passing we may mention another possible objection. If classes are imported with the semantics of predication at the ground level of logic, then there will be hardly anything left to the idea that logic is throughout evident (§5); this thanks to Russell's paradox and the adjustments it requires.

predicates denote objects, would itself have complete denotation. Then, bringing our general semantical principle to bear, we realize that '2 is a member of the class of primes' is not semantically explicit. For '2' and '() is a member of the class of primes' must be substitutable one for another in, for example, '2 is a member of the class of primes', saving semantical character. Nor could '2 is a member of the class of primes' be rendered explicit by interpreting the juxtaposing of '2' and '() is a member of the class of primes' as relational notation. This would be to step off on a vicious regress. So, if the proposed semantics were adequate, '() is a member of the class of primes' could not be a predicative expression. But then neither could '() is a member of ()' be a two-place predicative expression denoting a relation. Since this was to have the same function as juxtaposition in '2 is prime', if the proposed semantics were adequate, juxtaposition could not have the force of a relational expression. But just that was the idea, which must thus be rejected.

Since nothing in this reasoning depends upon the relation in question being that of set-membership, we conclude generally that juxtaposition is not to be accorded the semantic role of a relational expression, not, anyway, if predicates are construed as denoting objects.

Suppose we say that the juxtaposition of '2' and

'() is prime' is not to be considered as an implicit relational expression, but rather what we may call, following Strawson, a non-relational tie.¹⁶ To be as concrete as possible, let us as before take classes to be the denotations of predicates, though nothing shall turn upon the specific nature of these denotations. Then we may offer '2 is tied to the class of primes' as a semantically explicit version of '2 is prime'.

Now we may ask whether '() is tied to the class of primes' itself is a predicative expression. If it were, then '2 is tied to the class of primes' would not be semantically explicit, as we could show through use of the GSP as before. So we conclude that the expression '() is tied to the class of primes' is not a predicate. Likewise, we infer that '() is tied to (')' is not itself a two-place predicate denoting a relation. The reasoning here parallels that gone through with the last proposal. But in this case the conclusion, that since '() is tied to (')' is not a relational expression neither is it the case that juxtaposition is implicitly a relational expression, is not a reductio of the proposal; it is the proposal. It is, anyway, verbally, but the expression 'non-relational tie' remains to be explained.

16. See Strawson, Individuals, p. 169 ff.

The semantical function of juxtaposition may be rendered explicit by '() is tied to ()', or some such expression. (Recall that to deny this, which would be to violate the representation principle, is to introduce the Frege phenomenon we seek to avoid.) But what is this semantic function? More particularly, are we to assume that '() is tied to ()' makes its distinctive semantic contribution to (e.g.) '2 is tied to the class of primes' through itself having denotation? Let us suppose that this is so. And that '() is tied to ()' denotes, not a relation, but what we shall call a tie.¹⁷ Now if '() is tied to ()' has denotation, so too, presumably, does (e.g.) '() is tied to the class of primes'. Then, if this expression were co-categorematic with (e.g.) '2', these would be interchangeable, and so we would learn that (e.g.) the sentence '2 is tied to the class of primes' would not be semantically explicit. Since on the proposal this is explicit, we must conclude that '2' and '() is tied to the class of primes' are not of the same semantic category. This is to conclude that, although '() is tied to the class of primes' is to be accorded denotation, its denotation cannot be denoted by a singular term. And so here, once again, we encounter the Frege

17. For such a view see Bergmann, Logic and Reality, p. 229.

phenomenon. We can encapsulate this reasoning as follows. If relations are denotable by singular terms, it must be that ties are not.¹⁸

At this point we might reject the idea that the semantical function of '()' is tied to (')' lies in its denoting something and explore the possibility of a semantics for predications in other semantical terms. But if we are going to approach the semantics of predication this way, we might be better off to simply treat predicates themselves as non-denotative and inquire directly into the possibility of a semantics for predication on which some semantical idea other than denotation plays the central role. In any case, this is the tack we shall take; we shall pursue this latter idea in the subsequent section. Problems there to be encountered can be applied back to the other idea of rephrasing predications in terms of '()' is tied to (')' and then taking this as non-denotative. With this, we set aside approaching predication through the idea of attributing to juxtaposition the significance of a non-relational tie.

Treating predicates as denotative and then finding a further semantic feature in juxtaposition has not yielded

18. Unless we wish to introduce kinds of objects in the fashion of Russell's theory of types; but this will not enable us to avoid the Frege phenomenon, which now will turn up over such statements as there are objects of various types.

us a semantics of predication which avoids the unwanted consequences of Frege's own account. So let us look at how we might attempt to circumvent the apparent need for such additional semantical accounting through recourse to "metalinguistic discourse". We have been considering Frege's Begriffsschrift, enriched to whatever extent with various proper names and function-names. And we have been attempting to provide it, or substantially it, with (at least the beginnings of) an interpretation, a semantics, other than Frege's own. Our remarks in this vein are not framed with sentences of the notation of which we speak. This talk we may think of as progressing in a metalanguage whose object language is Begriffsschrift, or kindred notations. This distinction of object language and metalanguage is useful, even necessary for the attainment of full methodological rigor. Against this background, to make the metalanguage move is (1) to claim that predicates have complete denotation -- and we may, as before, assume that classes are such denotations, so that (e.g.) '() is prime' denotes the class of primes. (2) deny that, in a given language, say that of Begriffsschrift, the denotations of predicates may also be denoted by singular terms. But (3) deny also that such denotations are altogether unsusceptible of singular reference through allowing the possibility that the (or a) metalanguage may employ

singular terms to just this purpose.¹⁹

But to no avail. This move at best postpones problems. For, if our general semantical principle is in effect, then singular terms occurring (actually and potentially) in the metalanguage limit the predicates which may occur, or be introduced into, the metalanguage. Specifically, the metalanguage may contain no predicate whose extension is that of a class nameable in the metalanguage. Such a metalanguage would pretty clearly seem to be too impoverished to even serve the purpose of laying out the syntax and semantics of its object language. We may bring this point home by observing that if the metalanguage forms class-names by applying an "abstraction operator" to predicates, then the specification of the denotation of '() is prime' will itself draw upon an inadmissible predicate.

It might be rejoindered that this objection presupposes that we will, indeed must, apply the semantics we attribute to the object language in like fashion to the metalanguage, but that this need not be the case. Were we to formalize that metalanguage itself we would be free to offer a different sort of semantical account of its

19. Furth reports that both Richard Montague and David Kaplan made much this suggestion in his discussions with them of Frege. ("Two Types of Denotation", p. 21, n. 27.)

expressions, now in a metametalanguage, than that accorded expressions of the (original) object language.

This type of response underscores a fundamental incompatibility between Frege's picture of logic and one which, when thought in terms of it, encourages such "metalinguistic" scramblings. For to move this way is to reject the idea of a logical notation, such as Begriffsschrift, as (potentially) a universal language.²⁰ To this we shall return later. But for now we have found no way, consonant with Frege's conception of logic, to hold predicates denotative and avoid the Frege phenomenon.

§17. Alternative Semantics of Predication (2)

In this concluding section we shall explore the possibility of diverging more sharply from Frege's semantics of predication through not according predicates denotations at all. For Frege, '2 is prime' has two (proper) semantical parts, '2' and '() is prime'. This can be maintained compatibly with both our general semantical principle and the view that each denotes by ruling the denotations of predicates unsaturated. If however predicate denotations are objects, then our principle requires

20. Recall the earlier remarks on Kalish's idea that truth-in-a-model is a more general idea than that of truth. See also Gödel's remark in note 18, Chapter III. We shall return to this in discussing Quine.

a third semantical role to be played. But not, we have argued, with the effect that the consequences of unsaturation are avoided. If now predicates are to be thought of as non-denotative such an additional semantic feature will be uncalled for. But if they make their semantic contribution otherwise, how so?

Let us first consider a type of view which agrees with Frege's (and other views thus far examined) in this. A predicate stands in some semantical relation to some entity. The divergence is over the relation; it shall not be that of denotation.

We find such an idea in William Kneale's discussion of Frege.²¹ He suggests that we say that, in terms of our example, '() is prime' expresses the attribute of being prime, while maintaining that '2' denotes the number two. The attribute so expressed, and attributes generally, can be referred to with singular terms. 'The attribute of being prime' will do, or, equally well we may use simply 'primeness'. And we can, on this view, state that, for instance, the number two stands in a certain relation to this attribute. For we may say: 2 exemplifies primeness. Further, it seems to be Kneale's view that this sentence shares its content with, expresses

21. Kneale and Kneale, The Development of Philosophy, p. 586.

the same Thought as, '2 is prime'. If so, then this pair would be regarded by Kneale in a way similar to the way in which Frege thinks of '2 is prime' and '2 falls under the concept prime'.²² However, of importance to matters at hand is the fact that on the considered alternative, the pair '2 is prime' and '2 exemplifies primeness' are not semantically indistinguishable; this is indicated by the fact that the latter is genuinely a relational statement, whereas the former is not. Each statement is regarded as semantically explicit. Thus the reasoning, used before, which leads towards a vicious regress does not have a foothold from which to get started.

There are, however, other problems with this type of approach. Consider again the previous statement relating 2 and the attribute of being prime,

2 exemplifies primeness.

Both '2' and 'primeness' are names and so from this we may extract a one-place predicative expression,

() exemplifies (),

of self-exemplification. The negation of this would be,

-[() exemplifies ()],

a predicate of non-self-exemplification. But now if predicates express attributes and this is a predicate, it must express an attribute. But it cannot. For there is

22. Ibid., p. 586.

no attribute of non-self-exemplification for it to express, since if there were it would exemplify itself just in case it did not do just that, which is a contradiction.²³ But if there is no attribute for ' $\neg[(\) \text{exemplifies } (\)]$ ' to express, then, if ' $\neg[(\) \text{expresses } (\)]$ ' is indeed a predicate, it is not the case that predicates, generally, express attributes, and so a semantics of predication cannot be provided in these terms.

On the other hand, if ' $\neg[(\) \text{exemplifies } (\)]$ ' is not a predicate, then neither is ' $(\) \text{exemplifies } (\)$ '.²⁴ Thus, for want of a predicate, we shall not be able to state such relations as hold between objects and attributes as that we seemingly put in words this way: 2 exemplifies primeness. Generally, if there are attributes which things exemplify, we shall not be able to say that this is so, at least not if we are conceiving our semantics of predication as turning upon this relationship. So we see that, once again, ineffability is the cost of gaining an account of predication.

23. Here we encounter a form of Russell's paradox, and we draw Russell's conclusion, the only one compatible with Frege's philosophy of logic, that something we thought we had described does not exist.

24. To deny this would be to deny that every predicate has a negation, and so to reject standard logic. ("[U]nless the negation of a sentence has a sense, the sentence itself is without sense." Frege, Philosophical Writings, p. 104.)

A similar problem occurs over such statements as

'() is prime' expresses primeness,

which purports to state a relation between a predicate and an attribute. From this we may extract a one-place predicative expression,

() expresses (),

of self-expression, which in turn, by negation, gives us one of non-self-expression,

-[() expresses ()].

But, much as before, there is no attribute for this predicate to express, since if there were it would be one which has a particular trait just in case it does not have that trait.²⁵ So if the above are predicates, then it cannot be, generally, that predicates express attributes. In which case we cannot provide an account of predication in these terms. Thus to gain an account of predication along these lines we shall have to deny that '-[() expresses ()]' and '() expresses ()' are predicates. But then we shall not be able to state (e.g.) that '() is prime' expresses primeness, at least not if we are limited in our attempted semantical accounting to

25. What attributes do we suppose we are speaking of? I can say this, but no more. Any expression which expresses an attribute which it itself exemplifies, expresses the attribute of self-expression. And analogously for the attribute of non-self-expression.

drawing upon expressions of semantical categories employed in the account as required by Frege's conception of logic. Thus, much as before, we shall conclude that if there are attributes which predicates express we cannot say that this is the case, or at least not in such a way as to thereby provide an account of predication. Again, if we have an account of predication it is one, as was Frege's, which requires its own ineffability.

It remains to emphasize that these difficulties with the approach just considered are quite general; they attend any approach to predication which proceeds in terms of treating each predicate as relating semantically to some entity.

This may encourage us to try a somewhat different approach, one which does not rely on thinking of each predicate as performing its semantic function through relating to some particular non-linguistic entity. Suppose we assume that, for instance, '2 is prime' has just two (proper) semantical parts, '2' and '() is prime'. And that '2', as all along, denotes the number two. But now, instead of bringing in new entities in terms of which to speak of the semantical contribution of predicates, let us attempt to provide a semantics of predication in terms of just the sort of things which our singular terms are denoting. And so we shall say that a one-place predicate, such as '() is prime', is true of such objects, some,

none, or all of them as the case may be, or that various objects satisfy various predicates. In the case of '2 is prime', we shall say that '() is prime' is true of (or satisfied by) all objects which are prime, and so '2 is prime' will be true just if '2' denotes one of those primes.

Since on this approach we do not proceed in terms of some relation holding of predicates and classes or attributes, or the like, it will be immune to the first of the pair of difficulties with the last considered idea, that is, our account of predication will not uncover a version of Russell's paradox.

However, the second of those difficulties is still with us; we shall, in this semantics, turn up a version of Grelling's paradox. From, for instance,

Two satisfies '() is prime',

we may extract a one-place predicate,

() satisfies (),

of self-satisfaction. Its negation will be,

-[() satisfies ()],

conveying non-self-satisfaction; this will be satisfied by just those things which do not satisfy themselves.

But now what of this predicate itself? It will satisfy itself just in case it is a non-self-satisfier, that is, just if it does not satisfy itself. We come again upon a contradiction. Since we cannot live with this, something

will have to be denied or given up.

In the analogous situation with the previous alternative we denied the existence of a particular attribute, the (supposed) attribute expressed by the predicate of non-self-expression. But since we are no longer viewing predicates as functioning semantically through relating to particular entities our situation is one in which, to put the point in a paradoxical manner, we have no entity around to deny, or none whose denial would be to the point.

In the earlier situation, denying the attribute led to denying that '() expresses ()' was a predicate. There we argued that if this were a predicate and predicates express attributes, then it, and so its negation, would have to express an attribute; but in the latter case at least that there is such an attribute is denied. And here is a parallel with the present situation, for in this case we are constrained to deny that '() satisfies ()' is a predicate. Here we argue that if this were a predicate, so too would be its negation. But then in carrying through with the idea that predicates function semantically through being satisfied by objects -- some, none, or all as the case may be -- we encounter the fact that if '-[() satisfies ()]' is a predicate, then it satisfies itself just in case it does not satisfy itself. Here we are not to deny the existence of the expression '() satisfies ()', which would be absurd. What we deny is the

possibility of this expression, or any other, performing a particular predicative function; we deny that '() satisfies ()' is a predicate. So, once again, we find ourselves in a situation where, if we have an account of predication, it is one which we cannot state, or cannot state compatibly with that account itself. Once again the Frege phenomenon is encountered.

This completes my survey of alternative semantics for predication; I am ready to conclude that the unwanted consequences of unsaturation are unavoidable. However, since the last considered alternative is essentially that of Quine (following Tarski), it will prove instructive to consider what Quine's position is on the Frege phenomenon. This will serve to put the course of our inquiry into perspective. And it may also help to dispel the feeling, which some may have, that the problems we have been repeatedly encountering stem from the sort of failure to keep straight use and mention Tarski wrote of this way.

People have not always been aware that the language about which we speak need by no means coincide with the language in which we speak. They have carried out the semantics of a language in that language itself and, generally speaking, they have proceeded as though there was only one language in the world.²⁶

Our general conclusion does not, I should say, depend

26. Tarski, "The Establishment of a Scientific Semantics" in Logic, Semantics, and Metamathematics, p. 402.

upon inattention to such matters.

I shall briefly review some of Quine's remarks in Chapter 2 "Grammar" and Chapter 3 "Truth" of his recent Philosophy of Logic. In this I will assume rather much as to the reader's familiarity with Quine's use of Tarski's work on defining truth for formalized languages, and also with Tarski's work itself, though more with his "results", how the thing goes, than with the technical details.

For his canonical notation Quine employs a "standard logical grammar" of an "austere" sort, one which draws upon just (nearly enough) four constructions. Predication consists in the adjoining of predicates (of various degree) to (appropriately many) variables. Open sentences, with free variables, result; these are considered sentences for purposes of exposition. (The grammar uses no names and a single style of variables, in Quine's familiar fashion.) Negation consists in prefixing a sign for negation to a sentence. Conjunction consists in joining two sentences with a sign for conjunction. And existential quantification is a construction whereby through prefixing an existential quantifier to an open sentence which has an alphabetically similar free variable a closed sentence may result. (And will result if the open sentence contained just one free variable.)

Subsequently Quine turns to provide a semantics for

this notation, and so for any language with just its grammar. He offers a semantics of predication in terms of a notion of satisfaction which is a generalized version of the idea we have considered. The generalization occurs in speaking of open sentences as satisfied by sequences of objects; this allows its application to such expressions as polyadic predications, which we have conveniently ignored. Indeed the satisfaction concept of Tarski's which Quine employs is adequate to provide a semantics for the entire canonical notation; no other semantical idea need be called upon. This semantics has its penultimate culmination in a recursive, or inductive, definition of satisfaction for a language built upon the grammar of the notation; the definition will have one step for each of the predicates of the particular language, and then one step for each of the constructions of negation, conjunction, and existential quantification. Given this, a definition of truth for a language falls out; a sentence is true if always satisfied, i.e., satisfied by any sequence whatsoever; false, otherwise. Here we shall simply pass by the details. Quine then shows us how to go about transforming such an inductive definition into an explicit, or direct, definition, drawing upon ideas due to Frege. More of this in a moment.

Suppose that the logical grammar has been stocked with some predicates. Then we have not merely a grammar,

or language-frame, but a language in which we are able to say things. Call this the object language. And call then our ordinary, unformalized mother tongue its metalanguage. What can be said in the object language will depend to some extent upon what predicates we have introduced into it. Thus if it lacks the predicate '() is a cow', then we shall not be able to speak in it of cows, unless in other terms. And there will be no point in our taking cows among the values of the variables of the object language. All this, I repeat, from the vantage point of our unreconstructed mother tongue. Similarly, for instance, with the predicate '() is a class' or the relation of set-membership. If the object language lacks the means for speaking of sets, then we have no reason to count sets among the values of its variables. And this even though in providing a semantics for the object language we do, at least in effect, speak of sets through our talk of sequences satisfying expressions.

Let us get back to the matter of turning a recursive definition into a direct one. Set theory is available in our metalanguage. Though it is somewhat indefinite as to just what this set theory is, as to what are its existence assumptions, or requirements. The point at hand is that given sufficient set-theoretic resources in our metalanguage we can turn our recursive definition of satisfaction for the object language into a direct definition

thereby eliminating the talk of truth and satisfaction as directed upon object language sentences in favor of (just) the language of logic and set theory. The trick here is to take the steps, or clauses, of the recursive definition of satisfaction -- there will be $3+n$, where n = the number of predicates -- and replace 'satisfies' by a variable. We can then state that the satisfaction relation, which will be a certain set of ordered pairs, is that relation which answers to the open sentence constructed as just described. This sentence we may abbreviate, with Quine, to ' SR_z '. Again we do not linger over the details, and shall simply assume that we have such a definition of satisfaction for some sample object language. And generally, that we know that no matter how well supplied our object language may be with predicates, we may always directly define satisfaction for it in the metalanguage. (If satisfaction gets defined directly, truth will also, tagging along as before.)

Having sketched this in, Quine proceeds to develop a version of Grelling's paradox. He reasons that if our object language were itself to have the resources of set theory and devices sufficient to enable us to speak with it of expressions of the object language, then it would seem that we could reconstruct in the object language the direct definition of its own satisfaction predicate. Thus we would, in effect, have in the object language a

predicate '() satisfies ()' and so also the predicate '-[() satisfies ()]', from which we move directly to contradiction as before.

The stage is set. What does Quine say now? How does he react to this looming contradiction? This is where our interest lies. Here is the response.

The inductive definition . . . was all right, and can be translated wholly into the object language except of course for the new term that is being inductively defined -- the verb 'satisfies'. Accordingly 'SR_z', which we get from [the steps of our recursive definition of 'satisfies'] by dropping that verb and supplying 'z', is fully translatable into the object language. Moreover 'SR_z' does indeed require of z that its member pairs be precisely the pairs <x,y> such that x satisfies y. So far so good. But is there a set z meeting this requirement? . . . the answer is . . . negative, by reductio ad absurdum; there is no such set z, or we would be back in Grelling's contradiction.²⁷

If '() satisfies ()' were a predicate of the object language and object language predicates are satisfied by sequences of objects, then there would have to be a set of sequences answering to this predicate. And though from the perspective of the metalanguage it seems we can speak of the "satisfaction set" of the object language, it cannot be that this set is among the values of the variables of the object language. It cannot, since otherwise contradiction would ensue. And this is so despite the fact

27. Quine, Philosophy of Logic, pp. 44-45.

that the imagined object language may employ a relation of class-membership.

Thus Quine takes Grelling's paradox as it turns up in the context of a Tarskian semantics in much the spirit of Russell's paradox; each brings home to us that something does not exist. Two paragraphs down he continues on this theme.

. . . we have been forced by Grelling's paradox to repudiate a supposed set z . But it should be noted that this repudiation is a weaker paradox than Russell's in this respect: the z that it repudiates was not a set that was purportedly determined by any open sentence expressible in the object language. It purported indeed to be the set of all pairs $\langle x, y \rangle$ such that x satisfies y , and so purported indeed to be determined by the open sentence ' x satisfies y '. But this is not a sentence of the object language. As Grelling's paradox has taught us, it is untranslatable foreign language.²⁸

Quine's conclusion is an interesting one. Let us hold it in abeyance for a moment while we recall some remarks he has made elsewhere on the theme of the universal adequacy of standard logical notation.

Taking the canonical notation . . . austerely, . . . we have just these basic constructions: predication, universal quantification . . ., and the truth functions (reducible to one) . . . What thus confronts us as a scheme for systems of the world is that structure so well understood by present-day logicians, the logic of quantification or calculus of predicates. Not that the idioms thus renounced are

28. Ibid., p. 45.

supposed to be unneeded in the market place or in the laboratory . . . The doctrine is only that such a canonical idiom can be abstracted and then adhered to in the statement of one's scientific theory. The doctrine is that all traits of reality worthy of the name can be set down in an idiom of this austere form if in any idiom.²⁹

Quine's canonical idiom is much the same as that in which Frege couched fundamental logic in his *Begriffsschrift*. And Quine's doctrine, as here stated, partakes of much the same spirit as the view attributed to Frege in our discussion of the idea of logic as lingua characterica (§6).

Now returning to Quine's conclusion noted before, let us observe that, with regard to our reflections upon the Frege phenomenon, the upshot of Quine's views is much the same as that we have arrived at, with Frege. For when, with Quine, we set out to set down traits of reality, we shall sally forth in the knowledge that we shall be precluded from stating, in a general way, how it is that such truths as we manage to inscribe do succeed in reflecting reality. For to say this we would need, what we cannot have, an expression of the idiom with the meaning of '() satisfies ()'. The semantics of the idiom resists its own statement in the idiom. This we know. Thus we also know that our attempts at its statement are, strictly

29. Quine, Word and Object, p. 228.

speaking, "untranslatable foreign language", that is, meaningless to speakers of the idiom.

Let us cover this ground once more. We begin by carving out a bit of language upon which to reflect: the grammar of logic together with such predicates as we choose to include. This done we speak of it as the object language about which we reason in the remainder of our language, the metalanguage so-called. But then if, with Quine, we think of this "object language" as a canonical notation, as a language in which, given suitable predicates, we can set down any "trait of reality worthy of the name", then these metalinguistic ruminations take on a somewhat different cast. They come to take place in a kind of limbo. For if the import of such remarks should escape expression in the "object language" we shall, ultimately and upon taking up the perspective of the "object language" look back upon this discourse as nonsense. And such, in a curious way, is the situation with Quine's attempt to provide a semantics for the "austere" idiom. From the point of view of this idiom we are able to grasp the point of the direct definition of satisfaction which was offered in the "metalanguage", for we appreciate the point of the recursive steps which lead up to it. This despite the fact that since we cannot pull this talk down into canonical idiom we must, so long as we retain this perspective, regard the earlier discourse

as lacking literal meaning. Ultimately then the meta-language is at best metaphoric. Or, to alter and invert the image, our metalinguistic exercises provide us with a scaffolding with which to construct the object language, and in this they have a pragmatic or instrumental use; still this significance is only instrumental, for, as we have seen, they are like Wittgenstein's ladder, to be discarded once used.³⁰

My conclusion is that the Frege phenomenon is real. Frege's conception of logic is truth-oriented. It encourages us to provide for our preferred notation a clear understanding of how it is that sentences of this notation are true, if true, through supplementing its syntax with a truth-conditional semantics. This, together with a claim to universal adequacy for the notation, focuses attention upon the question: How is truth possible? Frege's answer was unstateable in its own terms. He held, in effect, that a necessity of language precludes the answering of our question. I think he was right. It remains to assess the importance of this.

30. Quine uses Wittgenstein's image to a related purpose in discussing the nature of the completeness proof for quantification theory. ("Reply to Strawson", Synthese, 19 (1968), p. 297.)

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