Multivariate Trace Inequalities

Mario Berta

arXiv:1604.03023 with Sutter and Tomamichel (to appear in CMP)
arXiv:1512.02615 with Fawzi and Tomamichel

QMath13 - October 8, 2016

Motivation: Quantum Entropy

- Entropy of quantum states ρ_A on Hilbert spaces $\mathcal{H}_A$ [von Neumann 1927]:

$$H(A)_\rho := -\mathrm{tr}\left[\rho_A \log \rho_A\right] . \tag{1}$$

- Strong subadditivity (SSA) of tripartite quantum states on $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ from matrix trace inequalities [Lieb & Ruskai 1973]:

$$H(AB)_\rho + H(BC)_\rho \geq H(ABC)_\rho + H(B)_\rho . \tag{2}$$

- Generates all known mathematical properties of quantum entropy, manifold applications in quantum physics, quantum information theory, theoretical computer science etc.
- This talk: entropy for quantum systems, strengthening of SSA from multivariate trace inequalities.

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- Mark Wilde at 4pm: *Universal Recoverability in Quantum Information*.

Overview

1. Entropy for quantum systems
2. Multivariate trace inequalities
3. Proof of entropy inequalities
4. Conclusion

Entropy for classical systems

- Entropy of probability distribution P of random variable X over finite alphabet [Shannon 1948, Rényi 1961]:

$$H(X)_P := -\sum_x P(x) \log P(x), \quad \text{with } P(x) \log P(x) = 0 \text{ for } P(x) = 0. \tag{3}$$

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- Extension to relative entropy of P with respect to distribution Q over finite alphabet,

$$D(P\|Q) := \sum_x P(x)\log\frac{P(x)}{Q(x)} \quad \text{[Kullback \& Leibler 1951]}, \tag{4}$$

where $P(x)\log\frac{P(x)}{Q(x)} = 0$ for $P(x) = 0$ and by continuity $+\infty$ if $P \not\ll Q$.

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- Multipartite entropy measures are generated through relative entropy, e.g., SSA:

$$H(XY)_P + H(YZ)_P \geq H(XYZ)_P + H(Y)_P \quad \text{equivalent to} \tag{5}$$

$$D(P_{XYZ}\|U_X \times P_{YZ}) \geq D(P_{XY}\|U_X \times P_Y) \quad \text{with } U_X \text{ uniform distribution.} \tag{6}$$

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- Monotonicity of relative entropy (MONO) under stochastic matrices N:

$$D(P\|Q) \geq D(N(P)\|N(Q))\,. \tag{7}$$

Entropy for quantum systems

- The entropy of $\rho_A \in \mathcal{S}(\mathcal{H}_A)$ is defined as:

$$H(A)_\rho := -\mathrm{tr}\left[\rho_A \log \rho_A\right] = -\sum_x \lambda_x \log \lambda_x \quad \text{[von Neumann 1927]}\,. \tag{8}$$

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$$D_K(\rho\|\sigma) := \sup_{\mathcal{M}} D(\mathcal{M}(\rho)\|\mathcal{M}(\sigma)) \quad \text{[Donald 1986, Petz \& Hiai 1991]}, \tag{9}$$

where $\mathcal{M} \in \mathrm{CPTP}(\mathcal{H} \to \mathcal{H}')$ und $\mathrm{Bild}(\mathcal{M}) \subseteq M \subseteq \mathrm{Lin}(\mathcal{H}')$, M commutative subalgebra.

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- The quantum relative entropy is defined as

$$D(\rho\|\sigma) := \mathrm{tr}\left[\rho\left(\log \rho - \log \sigma\right)\right] \quad \text{[Umegaki 1962]}\,. \tag{10}$$

- Monotonicity (MONO) for $\rho, \sigma \in \mathcal{S}(\mathcal{H})$ and $\mathcal{N} \in \mathrm{CPTP}(\mathcal{H} \to \mathcal{H}')$:

$$D(\rho\|\sigma) \geq D(\mathcal{N}(\rho)\|\mathcal{N}(\sigma)) \quad \text{[Lindblad 1975]}\,. \tag{11}$$

Entropy for quantum systems II

Theorem (Achievability of relative entropy, B. *et al.* 2015)

For $\rho, \sigma \in \mathcal{S}(\mathcal{H})$ *with* $\rho, \sigma > 0$ *we have*

$$D_K(\rho\|\sigma) \leq D(\rho\|\sigma) \quad \text{with equality if and only if } [\rho, \sigma] \neq 0. \tag{12}$$

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$$D(\rho\|\sigma) = \sup_{\omega>0} \operatorname{tr}[\rho \log \omega] - \log \operatorname{tr}[\exp(\log \sigma + \log \omega)] \quad \textit{[Araki ?, Petz 1988]}. \tag{14}$$

- Golden-Thompson inequality:

$$\operatorname{tr}[\exp(\log M_1 + \log M_2)] \leq \operatorname{tr}[M_1 M_2]\,. \tag{15}$$

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- Proof: new matrix analysis technique asymptotic spectral pinching (see also [Hiai & Petz 1993, Mosonyi & Ogawa 2015]).

Asymptotic spectral pinching [B. *et al.* 2016]

- $A \geq 0$ with spectral decomposition $A = \sum_\lambda \lambda P_\lambda$, where $\lambda \in \mathrm{spec}(A) \subseteq \mathbb{R}$ eigenvalues and P_λ orthogonal projections. Spectral pinching with respect to A defined as

$$\mathcal{P}_A : X \geq 0 \mapsto \sum_{\lambda \in \mathrm{spec}(A)} P_\lambda X P_\lambda \,. \tag{16}$$

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(i) $[\mathcal{P}_A(X), A] = 0$ (ii) $\mathrm{tr}\left[\mathcal{P}_A(X)A\right] = \mathrm{tr}\left[XA\right]$ (iii) $\mathcal{P}_A(X) \geq |\mathrm{spec}(A)|^{-1} \cdot X$

(iv) $|\mathrm{spec}\,(A \otimes \cdots \otimes A)| = \left|\mathrm{spec}\left(A^{\otimes m}\right)\right| \leq \mathcal{O}(\mathrm{poly}(m))\,.$ (17)

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- Golden-Thompson inequality:

$$\log \mathrm{tr}\left[\exp(\log A + \log B)\right] = \frac{1}{m} \log \mathrm{tr}\left[\exp\left(\log A^{\otimes m} + \log B^{\otimes m}\right)\right] \tag{18}$$

$$\leq \frac{1}{m} \log \mathrm{tr}\left[\exp\left(\log A^{\otimes m} + \log \mathcal{P}_{A^{\otimes m}}\left(B^{\otimes m}\right)\right)\right] + \frac{\log \mathrm{poly}(m)}{m} \tag{19}$$

$$= \frac{1}{m} \log \mathrm{tr}\left[A^{\otimes n} \mathcal{P}_{A^{\otimes m}}\left(B^{\otimes m}\right)\right] + \frac{\log \mathrm{poly}(m)}{m} \tag{20}$$

$$= \log \mathrm{tr}[AB] + \frac{\log \mathrm{poly}(m)}{m} \quad \square \tag{21}$$

Entropy for quantum systems III

- The right extension for applications is Umegaki's $D(\rho\|\sigma) = \operatorname{tr}\left[\rho\left(\log\rho - \log\sigma\right)\right]$. Intuition chain rule [Petz 1992] with SSA:

$$H(AB)_\rho + H(BC)_\rho \geq H(ABC)_\rho + H(B)_\rho \quad \text{equivalent to} \tag{22}$$

$$D(\rho_{ABC}\|\tau_A \otimes \rho_{BC}) \geq D(\rho_{AB}\|\tau_A \otimes \rho_B) \quad \text{with } \tau_A = \frac{1_A}{\dim(\mathcal{H}_A)}. \tag{23}$$

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- All known mathematical properties from MONO:

$$D(\rho\|\sigma) \geq D(\mathcal{N}(\rho)\|\mathcal{N}(\sigma)) \quad \Rightarrow \text{ strengthening of MONO/SSA?} \tag{24}$$

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- Equality conditions MONO [Petz 1986]:

Let $\rho, \sigma \in \mathcal{S}(\mathcal{H})$ with $\rho \ll \sigma$ and $\mathcal{N} \in \mathrm{CPTP}(\mathcal{H} \to \mathcal{H}')$. Then, we have

$$D(\rho\|\sigma) - D(\mathcal{N}(\rho)\|\mathcal{N}(\sigma)) = 0 \tag{25}$$

if and only if there exists $\mathcal{R}_{\sigma,\mathcal{N}} \in \mathrm{CPTP}(\mathcal{H}' \to \mathcal{H})$ such that

$$\mathcal{R}_{\sigma,\mathcal{N}} \circ \mathcal{N}(\rho) = \rho \quad \text{und} \quad \mathcal{R}_{\sigma,\mathcal{N}} \circ \mathcal{N}(\sigma) = \sigma\,. \tag{26}$$

The quantum operation $\mathcal{R}_{\sigma,\mathcal{N}}$ is not unique, but can be chosen independent of ρ.

Strong monotonicity (sMONO)

Theorem (Strong monotonicity (sMONO), B. *et al.* 2016)

For the same premises as before, we have

$$D(\rho\|\sigma) - D(\mathcal{N}(\rho)\|\mathcal{N}(\sigma)) \geq D_K(\rho\|\mathcal{R}_{\sigma,\mathcal{N}} \circ \mathcal{N}(\rho)) \,, \tag{27}$$

with

$\mathcal{R}_{\sigma,\mathcal{N}}(\cdot) := \int_{-\infty}^{\infty} \mathrm{d}t \; \beta_0(t)\sigma^{\frac{1+it}{2}}\mathcal{N}^\dagger\left(\mathcal{N}(\sigma)^{-\frac{1+it}{2}}(\cdot)\mathcal{N}(\sigma)^{-\frac{1-it}{2}}\right)\sigma^{\frac{1-it}{2}} \in \mathrm{CPTP}(\mathcal{H}' \to \mathcal{H})$

and $\beta_0(t) := \frac{\pi}{2}\left(\cosh(\pi t) + 1\right)^{-1}$*.*

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- Special case SSA (sSSA), becomes an equality in the commutative case:

$$D(\rho_{ABC}\|\tau_A \otimes \rho_{BC}) - D(\rho_{AB}\|\tau_A \otimes \rho_B) \geq D_K\left(\rho_{ABC}\| \left(\mathcal{I}_A \otimes \mathcal{R}_{B\to BC}\right)(\rho_{AB})\right) , \tag{28}$$

with $\mathcal{R}_{B\to BC}(\,\cdot\,) := \int_{-\infty}^{\infty} \mathrm{d}t \ \beta_0(t) \rho_{BC}^{\frac{1+it}{2}} \left(\left(\rho_B^{-\frac{1+it}{2}} (\,\cdot\,) \rho_B^{-\frac{1-it}{2}} \right) \otimes 1_C \right) \rho_{BC}^{\frac{1-it}{2}}$, where $\mathcal{R}_{B\to BC} \in \mathrm{CPTP}(\mathcal{H}_B \to \mathcal{H}_B \otimes \mathcal{H}_C)$.

Proof SSA

- Following [Lieb & Ruskai 1973] we have with Klein's inequality

$$D(\rho_{ABC}\|\tau_A \otimes \rho_{BC}) - D(\rho_{AB}\|\tau_A \otimes \rho_B) = D\left(\rho_{ABC}\| \exp\left(\log \rho_{AB} - \log \rho_B + \log \rho_{BC}\right)\right) \tag{29}$$

$$\geq \mathrm{tr}\left[\rho_{ABC} - \exp\left(\log \rho_{AB} - \log \rho_B + \log \rho_{BC}\right)\right] \tag{30}$$

We could conclude SSA if $\mathrm{tr}[\exp(\log \rho_{AB} - \log \rho_B + \log \rho_{BC})] \leq 1$.

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- Golden-Thompson $\operatorname{tr}\left[\exp(\log M_1 + \log M_2)\right] \leq \operatorname{tr}[M_1 M_2]$ to Lieb's triple matrix inequality:

$$\operatorname{tr}\left[\exp(\log M_1 - \log M_2 + \log M_3)\right] \leq \int_0^\infty \mathrm{d}\lambda \ \operatorname{tr}\left[M_1\left(M_2+\lambda\right)^{-1} M_3\left(M_2+\lambda\right)^{-1}\right]$$

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- Proof SSA with $M_1:=\rho_{AB}$, $M_2:=\rho_B$, $M_3:=\rho_{BC}$ and $\int_0^\infty d\lambda\, x(x+\lambda)^{-2}=1$. □
- Idea: for sSSA start with the variational formula

$$D(\rho_{ABC}\|\tau_A\otimes\rho_{BC})-D(\rho_{AB}\|\tau_A\otimes\rho_B)$$
$$=\sup_{\omega_{ABC}>0}\operatorname{tr}\left[\rho_{ABC}\log\omega_{ABC}\right]-\log\operatorname{tr}\left[\exp\left(\log\rho_{AB}-\log\rho_B+\log\rho_{BC}+\log\omega_{ABC}\right)\right]. \quad (32)$$

Multivariate trace inequalities

Theorem (Multivariate Golden-Thompson, B. *et al.* 2016)

Let $p \geq 1$*,* $n \in \mathbb{N}$*, and* $\{H_k\}_{k=1}^n$ *be a set of hermitian matrices. Then, we have*

$$\log \left\| \exp \left(\sum_{k=1}^{n} H_k \right) \right\|_p \leq \int_{-\infty}^{\infty} \mathrm{d}t \; \beta_0(t) \log \left\| \prod_{k=1}^{n} \exp\left((1+it)H_k\right) \right\|_p , \tag{33}$$

where $\|M\|_p := \left(\mathrm{tr}\left[\left(M^\dagger M\right)^{p/2} \right] \right)^{1/p}$ *with* $\beta_0(t) := \frac{\pi}{2}\left(\cosh(\pi t) + 1\right)^{-1}$*.*

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- Proof based on Lie-Trotter expansion $\exp\left(\sum_{k=1}^n H_k\right) = \lim_{r\to 0}\left(\prod_{k=1}^n \exp(rH_k)\right)^{1/r}$ extension of [Araki-Lieb-Thirring 1976/1990]:

Lemma (Multivariate Araki-Lieb-Thirring, B. *et al.* 2016)

Let $p \geq 1$, $r \in (0,1]$, $n \in \mathbb{N}$, *and* $\{M_k\}_{k=1}^n$ *be a set of positive matrices. Then, we have*

$$\log \left\| \left| \prod_{k=1}^n M_k^r \right|^{1/r} \right\|_p \leq \int_{-\infty}^{\infty} \mathrm{d}t\ \beta_r(t) \log \left\| \prod_{k=1}^n M_k^{1+it} \right\|_p , \tag{34}$$

with $\beta_r(t) := \frac{\sin(\pi r)}{2r(\cosh(\pi t)+\cos(\pi r))}$.

Complex interpolation theory

- Strengthening of Hadamard's three line theorem [Hirschman 1952]:

Let $S := \{z \in \mathbb{C} : 0 \leq \mathrm{Re}(z) \leq 1\}$, $g : S \to \mathbb{C}$ be uniformly bounded on S, holomorph in the interior of S, and continous on the boundary. Then, we have for $r \in (0,1)$ with $\beta_r(t) := \frac{\sin(\pi r)}{2r(\cosh(\pi t)+\cos(\pi r))}$ that:

$$\log|g(r)| \leq \int_{-\infty}^{\infty} \mathrm{d}t\ \beta_{1-r}(t) \log|g(it)|^{1-r} + \beta_r(t) \log|g(1+it)|^r \tag{35}$$

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Complex interpolation theory

- Strengthening of Hadamard's three line theorem [Hirschman 1952]:

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- Stein interpolation for linear operators [Beigi 2013, Wilde 2015, Junge *et al.* 2015]:

Let $S = \{z \in \mathbb{C} : 0 \leq \text{Re}(z) \leq 1\}$ and $G : S \to \text{Lin}(\mathcal{H})$ be holomorph in the interior of S and continous on the boundary. For $p_0, p_1 \in [1, \infty]$, $r \in (0,1)$, define p_r with $1/p_r = (1-r)/p_0 + r/p_1$. If $z \mapsto \|G(z)\|_{p_{\text{Re}(z)}}$ is uniformly bounded on S, then we have for $\beta_r(t)$ as above:

$$\log\|G(r)\|_{p_r} \leq \int_{-\infty}^{\infty} \mathrm{d}t \, \left(\beta_{1-r}(t) \log\|G(it)\|_{p_0}^{1-r} + \beta_r(t) \log\|G(1+it)\|_{p_1}^{r}\right). \tag{37}$$

Proof of multivariate trace inequalities

Lemma (Multivariate Araki-Lieb-Thirring, B. *et al.* 2016)

Let $p \geq 1$*,* $r \in (0,1]$*,* $n \in \mathbb{N}$*, and* $\{M_k\}_{k=1}^n$ *be a set of positive matrices. Then, we have*

$$\log \left\| \left| \prod_{k=1}^{n} M_k^r \right|^{1/r} \right\|_p \leq \int_{-\infty}^{\infty} \mathrm{d}t \; \beta_r(t) \log \left\| \prod_{k=1}^{n} M_k^{1+it} \right\|_p . \tag{38}$$

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and choose

$$G(z) := \prod_{k=1}^{n} M_k^z = \prod_{k=1}^{n} \exp(z \log M_k) \quad \text{sowie} \quad p_0 := \infty, \; p_1 := p, \; p_r = \frac{p}{r} . \tag{40}$$

For positive matrices M_k, M_k^{it} becomes unitary, $\log \|\cdot\|_{p_0}^{1-r}$ in (39) becomes zero, and (38) follows. □

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- Multivariate Golden-Thompson from Lie-Trotter expansion.

Proof of sSSA/sMONO

- The proof of sSSA follows from multivariate Golden-Thompson for $p = 2$ and $n = 4$:

$$\operatorname{tr}\left[\exp(\log M_1 - \log M_2 + \log M_3 + \log M_4)\right] \leq \int \mathrm{d}t\ \beta_0(t) \operatorname{tr}\left[M_1 M_2^{-(1+it)/2} M_3^{(1+it)/2} M_4 M_3^{(1-it)/2} M_2^{-(1-it)/2}\right]. \quad (41)$$

Remark: Lieb's triple matrix inequality is a relaxation of the case $p = 2$ and $n = 3$!

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Remark: Lieb's triple matrix inequality is a relaxation of the case $p = 2$ and $n = 3$!

- Proof: Choose $M_1 := \rho_{AB}$, $M_2 := \rho_B$, $M_3 := \rho_{BC}$, $M_4 := \omega_{ABC}$, and thus

$$D(\rho_{ABC}\|\tau_A \otimes \rho_{BC}) - D(\rho_{AB}\|\tau_A \otimes \rho_B) = D\left(\rho_{ABC}\| \exp\left(\log\rho_{AB} - \log\rho_B + \log\rho_{BC}\right)\right) \quad (42)$$

$$= \sup_{\omega_{ABC}>0} \mathrm{tr}\left[\rho_{ABC}\log\omega_{ABC}\right] - \log\mathrm{tr}\left[\exp\left(\log\rho_{AB} - \log\rho_B + \log\rho_{BC} + \log\omega_{ABC}\right)\right] \quad (43)$$

$$\ge \sup_{\omega_{ABC}>0} \mathrm{tr}\left[\rho_{ABC}\log\omega_{ABC}\right] - \int \mathrm{d}t\ \beta_0(t)\log\mathrm{tr}\left[\omega_{ABC}\rho_{BC}^{\frac{1+it}{2}}\rho_B^{-\frac{1+it}{2}}\rho_{AB}\rho_B^{-\frac{1-it}{2}}\rho_{BC}^{\frac{1+it}{2}}\right] \quad (44)$$

$$\ge D_K\left(\rho_{ABC}\|\int \mathrm{d}t\ \beta_0(t)\rho_{BC}^{\frac{1+it}{2}}\rho_B^{-\frac{1+it}{2}}\rho_{AB}\rho_B^{-\frac{1-it}{2}}\rho_{BC}^{\frac{1+it}{2}}\right) \quad (45)$$

$$= D_K\left(\rho_{ABC}\|\mathcal{R}_{B\to BC}(\rho_{AB})\right) \quad \square \quad (46)$$

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$$\mathrm{tr}[M_1 \# M_2] \leq \mathrm{tr}\left[\exp(\log M_1 + \log M_2)\right] \leq \mathrm{tr}[M_1 M_2] \quad \text{[Hiai \& Petz 1993]}. \tag{47}$$

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$$D_K\left(\rho_{ABC} \| \sigma_{ABC}\right) \leq D(\rho_{ABC} \| \rho_{BC}) - D(\rho_{AB} \| \rho_B) \leq D_B\left(\rho_{ABC} \| \sigma_{ABC}\right), \qquad (48)$$

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- Mark Wilde at 4pm: *Universal Recoverability in Quantum Information*.