Optical Physics of Rifle Scopes

A Senior Project

By

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1 Introduction

Optical systems are typically consist of multiple lenses and mirrors and are used in a wide variety of fields. While there exist a variety of setups, there are a few key concepts that are at the core of all of them. These could be used from the most advanced scientific research to a simple magnifying glass. They are also used in industry and consumer products such as glasses and cameras. We are going to go over these key concepts and show how they apply to telescopes, or more specifically rifle scopes. First, we will go over Snell's Law of refraction [1] as this is central idea governing almost all of the light interactions within the scope. Then we will apply this to simple one and two lens systems [2]. Larger systems are more tedious to calculate, so we will then go over how to make calculations using matrices [2]. The last optical theories that we will discuss are aberration [2] and reflections [3] along a few experiments to demonstrate their effects. We will also go over bullet trajectory [4] as one needs to be able to know the path of the projectile in order to know where do aim using the scope.

2 Theory

2.1 Snell's Law

We start off by observing the path that light travels through two different media. The Light will travel the path with the shortest travel time given the velocity of the light in each of the given media, which we can measure the values of and know the constant velocity at which light travels for many media such as air or glass.

To show this we start with our two points, one in each of the given media. We then label the distance the the light travels in the x direction, from left to right, X. We the say that the light starts a distance y_1 above the boundary between the two media and ends a distance y_2 below. Using the Pythagorean theorem we determine that

$$d_1 = \sqrt{x^2 + y_1^2},$$

and

$$d_2 = \sqrt{(X - x)^2 + y_2^2}.$$

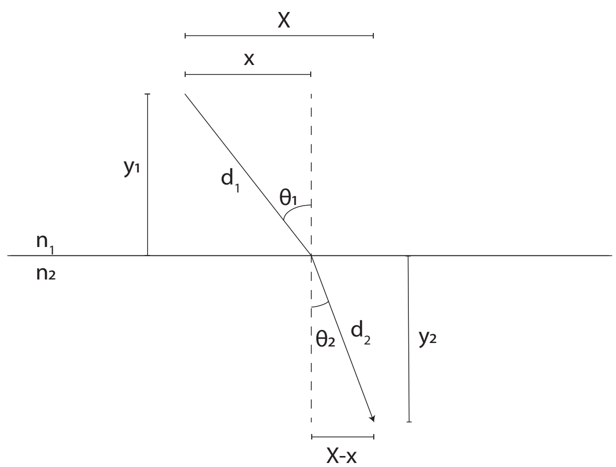

Figure 1: Snell's Law

Because the light travels at a constant speed we also know that

$$d_1 = v_1 t_1,$$

and

$$d_2 = v_2 t_2.$$

This gives us the time it takes the light to travel in each of the media as

$$t_1 = \frac{\sqrt{x^2 + y_1^2}}{v_1}$$
$$t_2 = \frac{\sqrt{(X - x)^2 + y_2^2}}{v_2}$$

To find the value of x that gives the least amount of time we add the two travel times to get the total time and set the derivative with respect to x to zero, or

$$\frac{dt}{dx} = (2x)\frac{\frac{1}{2}(x^2 + y_1^2)^{-1/2}}{v_1} + \frac{\frac{1}{2}\left((X - x)^2 + y_2^2\right)^{-1/2}(-2)(X - x)}{v_2} = 0.$$

We can then simplify this equation as

$$\frac{x}{v_1\sqrt{x^2 + y_1^2}} - \frac{X - x}{v_2\sqrt{(X - x)^2 + y_2^2}} = 0,$$

and looking at the figure we see that

$$\sin\theta_1 = \frac{x}{d_1} = \frac{x}{\sqrt{x^2+y_1^2}},$$

and

$$\sin\theta_2 = \frac{X-x}{d_2} = \frac{X-x}{\sqrt{(X-x)^2+y_2^2}},$$

If we plug this in to the previous equation we find

$$\frac{1}{v_1}\sin\theta_1 - \frac{1}{v_2}\sin\theta_2 = 0.$$

Last we can set the two terms equal to each other and multiply both by c to use the relation $n = \frac{c}{v}$, where n in the index of refraction for a given medium, v is the velocity of light in the medium and c is the speed of light in a vacuum. We then get the equation

$$n_1 \sin\theta_1 = n_2 \sin\theta_2, \tag{1}$$

which is the equation known as Snell's Law. This is important because this tells us how the light travels when it hits boundaries between two media and is the basis for how lenses work.

2.2 Lenses

Now that we have shown how light is transmitted when it refracts at the boundary of two media with respect to their refractive indices, we can begin to see how light goes through a lens. To do this we must first show how the light will refract on a curved surface. For this we will calculate the refraction of light through a spherical surface with radius R.

Here C is the center of the circle, O is the location of the object and I is the point where the image appears. This gives us an exterior angle from triangles CPO $\alpha_1 = \theta_1 + \phi$ and CPI $\alpha_2 = \theta_2 + \phi$. We can then substitute these angles into Snell's Law giving us

$$n_1(\alpha_1 - \phi) = n_2(\alpha_2 - \phi).$$

We can use small angle approximations and plug in the tangents for each angle

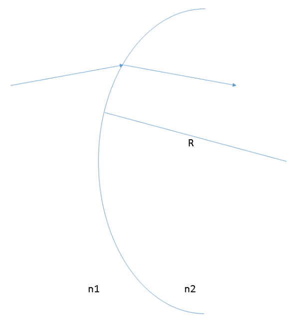

Figure 2: Light refracted off Spherical Surface

$$n_1\left(\frac{h}{s}-\frac{h}{R}\right)=n_2\left(\frac{h}{s'}-\frac{h}{R}\right).$$

Simplifying gives us

$$\frac{n_1}{s}+\frac{n_2}{s'}=\frac{n_2-n_1}{R}. \tag{2}$$

Now we can look at lenses where we have two of these spherical boundaries. Using equation (2) we can find the refraction at each surface to be

$$\frac{n_1}{s_1}+\frac{n_2}{s_1'}=\frac{n_2-n_1}{R_1},$$

and

$$\frac{n_2}{s_2}+\frac{n_1}{s_2'}=\frac{n_1-n_2}{R_2}.$$

We can also note that $s_2 = t - s_1'$ where t is the thickness of the lens. In this case we will be making a thin lens approximation where we set $t = 0$. By doing this and combining the two boundary equations we get

$$\frac{1}{s}+1s'=\frac{n_2-n_1}{n_1}\left(\frac{1}{R_1}-\frac{1}{R_2}\right).$$

Because the definition of the focal length is the image distance for an image at infinity we come to our final lens equation

$$\frac{1}{f} = \frac{n_2 - n_1}{n_1}\left(\frac{1}{R_1} - \frac{1}{R_2}\right), \tag{3}$$

which is also known as the Lens Maker's formula.

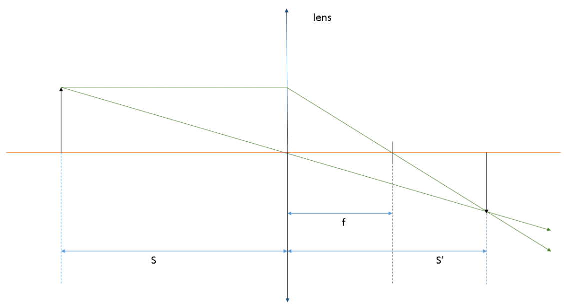

Figure 3: Light through a Lens

2.3 Lens Systems

2.3.1 Simple Magnifier

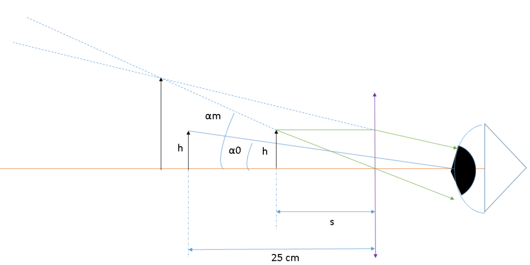

Figure 4: Simple Magnifier

A simple magnifier consists of only a single lens, such as a magnifying glass. We can calculate the angular magnification by using the ratio of the angles shown in the figure.

$$\frac{\alpha_M}{\alpha_0} = \frac{h/s}{h/25} = \frac{25}{h}, \tag{4}$$

where h is the height of the object, α_0 is the angle from the object to the eye at $25cm$, α_M is the angle to the virtual image formed by the magnifier, and s is the distance between the object and the lens.

2.3.2 Microscopes v Telescopes

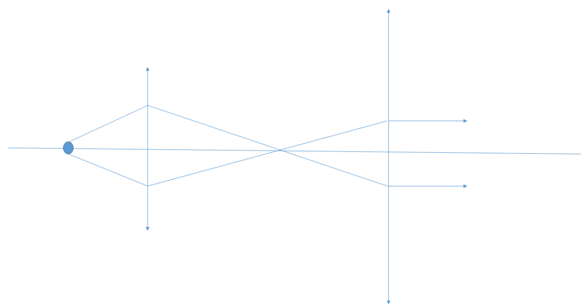

Figure 5: Microscope

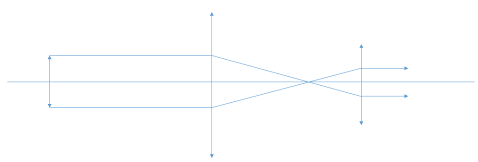

Figure 6: Astronomical Telescope

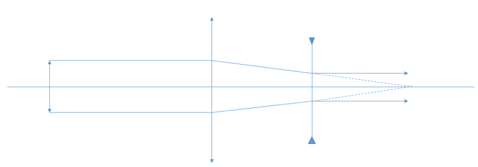

Figure 7: Galilean Telescope

Microscopes are designed to view objects that are very close to the objective lens while telescopes are used to see objects that are far away. Both output parallel rays for our eyes to see. Microscopes input rays have a large divergence and telescopes input rays are parallel. The difference the the way that the two are set up is the ratio of the focal lengths. In a microscope, the eyepiece has a longer focal length than the objective, while in a telescope, the objective lens has the longer focal length. The short focal length of the microscope objective lens is what allows it to focus on small objects that are close to the lens. The telescope objective lens allows it to focus on an object that is farther away.

There are two types of simple telescopes, the astronomical and the Galilean. Both have a positive, converging objective lens, but the Galilean telescope has a negative, diverging eyepiece rather than using a positive lens like the astronomical. This allows the Galilean telescope to be more compact and gives an upright image; the astronomical telescope gives an inverted image. For these simple two lens scopes, the length of the scope, or the distance between the lenses, is given by

$$L = f_o + f_e,$$

where f_o is the focal length of the objective lens and f_e is the focal length of the eyepiece. The magnification of the image is given by

$$M = -\frac{f_o}{f_e}. \tag{5}$$

2.4 Matrix Optics

When we have multiple lenses in our optical system, it can become difficult and tedious to solve the thin lens equations to find the images for each of the lenses that are in the system. Instead we can represent each element as a matrix and use matrix multiplication to convert our entire system into a single 2×2 matrix. This is known as the matrix method of geometric optics [2]. For this we solve the matrix equation

$$\begin{bmatrix} y_1 \\ \alpha_1 \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} y_0 \\ \alpha_0 \end{bmatrix},$$

where α_0 is the initial angle, α_1 is the final angle, y_0 is the incoming position above the optical axis, and y_1 is the output position.

2.4.1 Translation

For the case of the Translation Matrix we have

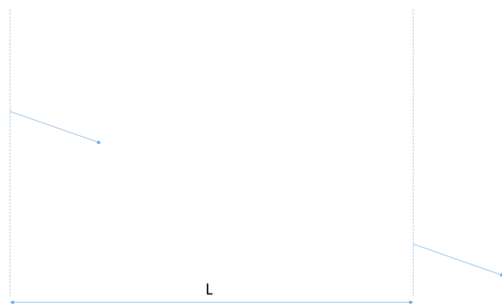

Figure 8: Light Translation

$$\alpha_1 = \alpha_0,$$

and

$$y_1 = y_0 + L \tan \alpha_0,$$

where L is the distance that the light has traveled. We can rewrite these equations as

$$y_1 = (1)y_0 + (L)\alpha_0,$$

and

$$\alpha_1 = (0)y_0 + (1)\alpha_0$$

Plugging these values into the matrix equation we get the equation

$$\begin{bmatrix} y_1 \\ \alpha_1 \end{bmatrix} = \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y_0 \\ \alpha_0 \end{bmatrix},$$

which gives us the translation matrix to be

$$\begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix}. \tag{6}$$

2.4.2 Refraction

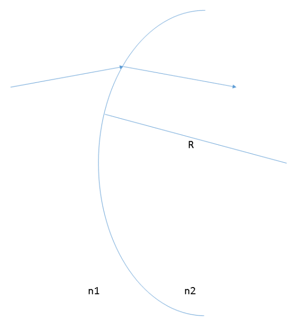

Figure 9: Light Refraction

Now we can look at the case where light is being refracted from a surface such as a lens. For this we will start with the equations:

$$\alpha_1 = \theta_2 - \phi = \theta_2 - \frac{y}{R},$$
$$\alpha_0 = \theta_1 - \phi = \theta_1 - \frac{y}{R},$$

and

$$y_1 = y_0.$$

By using these equations for the incident and refracted angle into Snell's Law we find the new equations to be

$$\alpha_1 = \left(\frac{1}{R}\right)\left(\frac{n_1}{n_2} - 1\right) y_0 + \left(\frac{n_1}{n_2}\right)\alpha_0,$$

and

$$y_1 = (1)y_0 + (0)\alpha_0.$$

This gives us the Refraction Matrix:

$$\begin{bmatrix} 1 & 0 \\ \frac{1}{R}\left(\frac{n_1}{n_2} - 1\right) & \frac{n_1}{n_2} \end{bmatrix}. \tag{7}$$

2.4.3 Lens Matrix

Using the two matrices for translation and refraction, we can find the system matrix for a lens. We will use M_1 as the first surface, M_2 as that translations through the lens, and M_3 for the second surface, or

$$\begin{bmatrix} y_1 \\ \alpha_1 \end{bmatrix} = M_1 \begin{bmatrix} y_0 \\ \alpha_0 \end{bmatrix},$$

$$\begin{bmatrix} y_2 \\ \alpha_2 \end{bmatrix} = M_2 \begin{bmatrix} y_1 \\ \alpha_1 \end{bmatrix},$$

and

$$\begin{bmatrix} y_3 \\ \alpha_3 \end{bmatrix} = M_3 \begin{bmatrix} y_2 \\ \alpha_2 \end{bmatrix}.$$

Using matrix multiplication we combine these expressions to get the system matrix, M, as

$$\begin{bmatrix} y_3 \\ \alpha_3 \end{bmatrix} = M_3 M_2 M_1 \begin{bmatrix} y_0 \\ \alpha_0 \end{bmatrix} = M \begin{bmatrix} y_0 \\ \alpha_0 \end{bmatrix}.$$

For a lens the system matrix is

$$M = \begin{bmatrix} 1 & 0 \\ \frac{n_L - n_2}{n_2 R_2} & \frac{n_L}{n_2} \end{bmatrix} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{n_1 - n_L}{n_L R_1} & \frac{n_1}{n_L} \end{bmatrix},$$

where n_1 is the refractive index of the medium before the lens, n_L is the refractive index of the lens material, n_2 is the refractive index after the lens, t is the thickness of the lens, and R_1 and R_2 are the respective radii of the two spherical surfaces. We can approximate $t = 0$ for a thin lens and simplify the matrix to get our thin lens matrix:

$$M = \begin{bmatrix} 1 & 0 \\ \frac{n_L - n_1}{n_1}\left(\frac{1}{R_2} - \frac{1}{R_1}\right) & 1 \end{bmatrix}. \tag{8}$$

Using the equation for the focal length of a thin lens we can simplify this to

$$M = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix}. \tag{9}$$

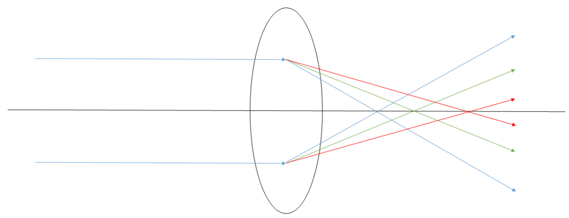

Figure 10: Chromatic Aberration from a single lens

2.5 Chromatic Aberration

What is chromatic aberration? Chromatic Aberration is distortion in the image created by a lens due to the varying electromagnetic wavelengths within the visible light spectrum. A lens actually has a slightly different focal length depending on the wavelength of the light passing through it due to dispersion. This can cause a blurry image over the approximate $400 - 700nm$ range in visible light. Multi-lens optical systems can have a significant problem because there will be chromatic aberration for each of the lenses having a compounded effect on the final image that someone is trying to create. To correct for this, High end optical systems such as rifle scopes will use compound lenses called Achromatic Doublets or Apochromatic Lenses.

Fraunhofer Lines				
λ(nm)	Symbol	Color	n_c	n_f
486.1	F	blue	1.5286	1.7328
589.2	D	yellow	1.5230	1.7205
656.3	C	red	1.5205	1.7076

Table 1: Fraunhofer Lines

To correct for the aberration, it is common to use the Fraunhofer lines. Here λ is the wavelength of the light, n_c is the refractive index of crown glass, and $n_f$is the index of flint glass.

Figure 11: Focal lengths of an Achromatic Doublet

The Achromatic Doublet uses two lenses, rather than just a single lens, to correct for chromatic aberration by converging two wavelengths of light to the same focal point. For rifle scopes, we would create a doublet that converges the ends of the visible light spectrum, represents by the F and C Fraunhofer Lines.

Figure 12: Focal Length of a Apochromatic Lens

The Apochromatic Lens system uses three lenses, rather than just a single lens, in order to further correct for chromatic aberration by converging three wavelengths of light to the same focal point. With this system, we are able to have the F, D, and C Fraunhofer lines converge together.

2.6 Antireflective Coatings

When light hits a surface, not all of the light is transmitted through and a portion of the light is reflected. Once again, when we add more lenses to our system, the issue is compounded. When creating a rifle scope with multiple lenses, we want as much of the incident light as possible to be refracted rather than reflected. To do this lenses are coated with Anti-Reflective(AR) coatings.

2.6.1 Reflections off surfaces

We will start by examining what happens when light is passing through an uncoated lens. We will set up our coordinate system such that the light is propagating in the z direction and is polarized in the x direction. This will give us the following equations for the electric and magnetic fields for the light:

$$\vec{E}_I(\vec{r},t) = E_{0I}e^{k_1 z-\omega t}\hat{x},$$
$$\vec{B}_I(\vec{r},t) = \frac{E_{0I}}{v_1}e^{k_1 z-\omega t}\hat{y},$$
$$\vec{E}_R(\vec{r},t) = E_{0R}e^{-k_1 z-\omega t}\hat{x},$$
$$\vec{B}_R(\vec{r},t) = \frac{E_{0R}}{v_1}e^{k_1 z-\omega t}\hat{-}y,$$
$$\vec{E}_T(\vec{r},t) = E_{0T}e^{k_2 z-\omega t}\hat{x},$$

and

$$\vec{B}_T(\vec{r},t) = \frac{E_{0T}}{v_2}e^{k_2 z-\omega t}\hat{y}.$$

We then apply the following boundary conditions:

$$\epsilon_1 E_{1\perp} = \epsilon_2 E_{2\perp},$$

$$E_{1\parallel} = E_{2\parallel},$$

$$B_{1\perp} = B_{2\perp},$$

and

$$\frac{1}{\mu_1} B_{1\parallel} = \frac{1}{\mu_2} B_{2\parallel}.$$

These boundary conditions give us two equations to solve

$$E_{0I} + E_{0R} = E_{0T},$$

and

$$E_{0I} - E_{0R} = \frac{\mu_1 v_1}{\mu_2 v_2} E_{0T}.$$

If we define $\beta = \frac{\mu_1 v_1}{\mu_2 v_2}$ and solve for these two equations relating the electric fields of the reflected and transmitted light to the incident light we get the following values for E_{0R} and E_{0T} :

$$E_{0R} = \frac{1-\beta}{1+\beta} E_{0I},$$

and

$$E_{0T} = \frac{2}{1+\beta} E_{0I}.$$

Simplifying β gives us that $\beta = \frac{n_2}{n_1}$ where n represents the refractive index for the material. To find out how much light is going through the boundary and how much is reflected back we must look at the intensity of the light,

$$I = \frac{1}{2} \epsilon v E_0^2.$$

To find the amount of light that is reflected or transmitted, we must compare the ratio of the respective light to the intensity of the incident wave.

$$R = \frac{I_R}{I_I} = \frac{\frac{1}{2}\epsilon_1 v_1 E_{0R}^2}{\frac{1}{2}\epsilon_1 v_1 E_{0I}^2}$$

$$T = \frac{I_T}{I_I} = \frac{\frac{1}{2}\epsilon_2 v_2 E_{0T}^2}{\frac{1}{2}\epsilon_1 v_1 E_{0I}^2}$$

This gives us our values of R and T given the refractive indices of the two media.

$$R = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2, \tag{10}$$

and

$$T = \frac{4n_1 n_2}{(n_1 + n_2)^2}. \tag{11}$$

2.6.2 Single Coating

In order to change the values of R and T to get better performance out of the system we would use coatings on the surfaces. We will start by looking at the simple case of only having one coating.

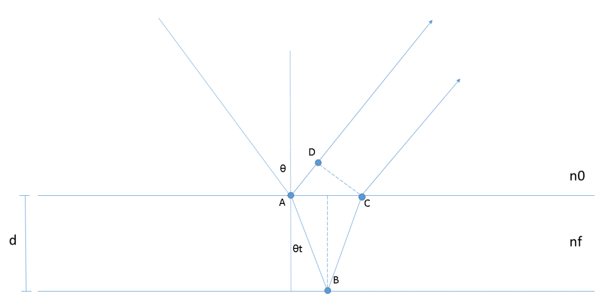

Figure 13: Reflection from a Thin Film

When we have a coating on our surface some of the light will be reflected of of the coating and some will be reflected off of the surface itself as shown by the two paths in the figure. If we look at what the difference is between the two paths of light we see that

$$\Delta = n_f(AB + BC) - n_0(AD).$$

This gives us the path length difference of

$$\Delta = 2n_f d \cos\theta_T. \tag{12}$$

When the two paths are out of phase with each other, we observe destructive interference resulting in the minimum amount of light reflected of of our surface, or

$$\Delta = 2d\cos\theta = (m + \frac{1}{2})\lambda,$$

where λ is the wavelength of light. Using small angle approximations, we can treat the light as if it is at normal incidence with the surface. For $m = 0$ this simplifies the equation to

$$\Delta = 2d = \frac{1}{2}\lambda$$

showing that the optimum thickness for the single thin film is $\frac{\lambda}{4}$. We can then find the percentage of light that is reflected and transmitted.

2.6.3 Two layer Coating

To minimize the reflection for a range of wavelengths we can use the same concepts while using more than one film. The more coatings we can use, the wider the spectrum of light that we can minimize the reflection for. Lets look at the case where we have two films on our surface. There are three possible ways for the light to be reflected off of one of the boundaries and still be transmitted through. The figure below shows four paths that the light can travel through the films.

There are three boundaries that we must consider for this system: air to film one, film one to film two, and film two to glass. We assign reflective and transmission coefficients as R_1, T_1, R_2, T_2, R_3, and T_3 as shown in the figure. There is also a phase shift when comparing two of the paths given by the following equations:

$$\delta = k_0\Lambda,$$
$$k_0 = \frac{2\pi}{\lambda},$$

and

$$\Lambda = \frac{2nd}{\cos\theta_T}.$$

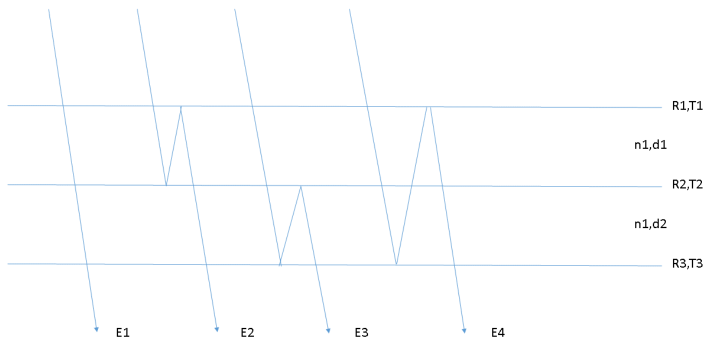

Figure 14: Possible Paths through a Multicoating

For the paths, we have the following Equations:

$$E_1 = T_1 T_2 T_3,$$

$$E_2 = T_1 T_2 T_3 R_1 R_2 e^{-i\delta_1},$$

$$E_3 = T_1 T_2 T_3 R_2 R_3 e^{-i\delta_2},$$

and

$$E_4 = T_1 T_2 T_3 R_3 T_2 R_1 T_2 e^{-i(\delta_1+\delta_2)}.$$

The possible paths that the light can travel and be transmitted through our two film system are combinations of these equations. We can therefore combine the equation for the actual path that the light travels as

$$E = T_1 T_2 T_3 (R_1 R_2 e^{-i\delta_1})^m (R_2 R_3 e^{-i\delta_2})^n (R_1 R_3 T_2^2 e^{-i(\delta_1+\delta_2)})^p. \tag{13}$$

By taking the sum of this equation we get

$$E_{tot} = \sum_{m=0}^{m'} \sum_{n=0}^{n'} \sum_{p=0}^{p'} T_1 T_2 T_3 (R_1 R_2 e^{-i\delta_1})^m (R_2 R_3 e^{-i\delta_2})^n (R_1 R_3 T_2^2 e^{-i(\delta_1+\delta_2)})^p.$$

We can then let $m', n', p' \to \infty$ to get the triple infinite sum

$$E_{tot} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty} T_1T_2T_3(R_1R_2e^{-i\delta_1})^m(R_2R_3e^{-i\delta_2})^n(R_1R_3T_2^2e^{-i(\delta_1+\delta_2)})^p.$$

Using the mathematical identity that $\sum_{k=0}^{\infty}(ae^{-ib})^k$ converges to $e^{-ib}/(e^{-ib}-a)$, we arrive at the following result:

$$E_{tot} = \frac{T_1T_2T_3e^{-2i(\delta_1+\delta_2)}}{(e^{-i\delta_1}-R_1R_2)(e^{-i(\delta_1+\delta_2)}-R_1R_3)(e^{-i\delta_2}-R_2R_3)}. \tag{14}$$

2.6.4 Multilayer Coatings

We can use the same method as used for the two layer case for analyzing three layers. By finding all the the possible reflections while having the light go completely through, we get

$$\begin{aligned} E = T_1T_2T_3T_4(R_1R_2e^{-\delta_1})^a(R_1R_3T_2^2e^{-i(\delta_1+\delta_2)})^b(R_1R_4T_2^2T_3^2e^{-i(\delta_1+\delta_2+\delta_3)})^c \\ *\,(R_2R_3e^{-i\delta_2})^d(R_2R_4T_3^2e^{-i(\delta_2+\delta_3)})^e(R_3R_4e^{-i\delta_3})^f \end{aligned}$$

for the transmitted field. We can then use the same infinite sum identity as before to get

$$\begin{aligned} E_{tot} = \frac{T_1T_2T_3T_4e^{-i(3\delta_1+4\delta_2+3\delta_3)}}{(e^{-i\delta_1}-R_1R_2)(e^{-i(\delta_1+\delta_2)}-R_1R_3T_2^2)(e^{-i(\delta_1+\delta_2+\delta_3)}-R_1R_4T_2^2T_3^2)} \\ * \frac{1}{(e^{-i\delta_2}-R_2R_3)(e^{-i(\delta_2+\delta_3)}-R_2R_4T_3^2)(e^{-i\delta_3}-R_3R_4)}. \end{aligned} \tag{15}$$

2.7 Bullet Trajectory

Newtonian Mechanics is used to calculate the trajectory that a bullet will travel after leaving the barrel. We'll start with the simple case by neglecting air resistance. We will set up our coordinate system such that the bullet is traveling in the x direction with a height of y above the ground. Using the equation $F = ma$ and plugging in g, the acceleration due to Earth's gravity, for a_y we get the following set of differential equations.

$$F_x = m\frac{d^2x}{dt^2} = 0$$

$$F_y = m\frac{d^2y}{dt^2} = -mg$$

We can now solve the differential equations to find an equation for the bullets trajectory in each direction. We will start by dividing by the mass, m and taking the integral giving us the velocity

$$\frac{dx}{dt} = c_1,$$

and

$$\frac{dy}{dt} = -mgt + c_2.$$

Now that we have the velocity, we can take another integral to get the equations for position

$$x = c_1t + c_3,$$

and

$$y = -\frac{1}{2}mgt^2 + c_2t + c_4.$$

By plugging in our initial conditions, $x(0) = x_0, \frac{dx}{dt}(0) = v_{0x}, y(0) = y_0, \frac{dy}{dt}(0) = v_{0y}$ we get the trajectory of the bullet to be

$$x(t) = v_{0x}t + x_0, \tag{16}$$

and

$$y(t) = -\frac{1}{2}mgt^2 + v_{0y} + y_0. \tag{17}$$

To find a more accurate equation for the motion of the projectile, we must include the drag force due to air resistance. The drag force is given by

$$F_d = \frac{1}{2}\rho v^2 C_d A, \tag{18}$$

where F_d is the drag force, ρ is the density of the fluid that the projectile is moving through, v is the velocity, C_d is the drag coefficient, and A is the cross sectional area of the the projectile. To calculate the drag force, we must first define values for all of the variables. Let's start with ρ, fluid density. For this, we will use the density of air at standard temperature and pressure:

$$\rho = 1.2754kg/m^3.$$

Next we will find the drag coefficient, C_d. To find this we will start with the bullet's Ballistic Coefficient, which is usually given by the manufacturer. We can then use the equation for the ballistic coefficient

$$BC = \frac{SD}{i}, \tag{19}$$

where i is called the form factor and SD is the sectional density of the bullet. Our sectional density is then given as

$$SD = \frac{m}{d^2}, \tag{20}$$

where m is the mass of the bullet, and d is the diameter. Both of these are given to us by the manufacturer. The form factor is defined as

$$i = \frac{C_d}{C_g}, \tag{21}$$

where C_g is the drag coefficient of the standard $G1$ projectile which has a flat base, mass of one pound, 1 inch diameter, 3 inches long, and a 2 inch radius of curvature for the point, and determined to be

$$C_g = .0007383m^2/kg.$$

By plugging in equation (21) and (20) into equation (19) we get the following equation for the ballistic coefficient:

$$BC = \frac{mC_g}{d^2C_d}. \tag{22}$$

We can then plug in our our value for C_g and solve for C_d to be

$$C_d = \frac{mC_g}{(BC)d^2}. \tag{23}$$

The cross sectional area of the bullet can be calculated from the caliber, diameter, of the round that is being used by

$$A = \pi \left(\frac{1}{2}d\right)^2. \tag{24}$$

Now that we have values or equation for all of the variables in our drag force equation, we can plug those in to get

$$F_d = \frac{1}{2}\rho v^2 \frac{mC_g}{d^2(BC)} \pi \left(\frac{d}{2}\right)^2,$$

$$F_d = \frac{1}{2}(1.2754kg/m^3)\frac{m(.0007383m^2/kg)}{d^2(BC)}\pi\frac{d^2}{4}v^2,$$

and

$$F_d = .00037\frac{v^2 m}{BC}. \tag{25}$$

Adding this to our previous force equation, we get a new pair of differential equations. Unlike the previous set of ordinary differential equation, this set includes the fact that the drag force is nonlinear. Our new force equations are

$$F_x = ma_x = -.00037\frac{1}{BC}v_x^2,$$

and

$$F_y = ma_y = -mg + .00037\frac{1}{BC}v_y^2.$$

The differential equation become:

$$\frac{d^2x}{dt^2} + \frac{.00037}{(BC)m}\left(\frac{dx}{dt}\right)^2 = 0, \tag{26}$$

and

$$\frac{d^2y}{dt^2} + \frac{.00037}{(BC)m}\left(\frac{dy}{dt}\right)^2 = g. \tag{27}$$

We will start by solving the nonlinear differential equation for the projectile motion in the x direction. To do this we will start by rewriting the equation for x as

$$\frac{dv_x}{dt} = -Dv_x^2,$$

where v_x is the velocity of the projectile in the x direction, $\frac{dx}{dt}$, and $D = \frac{.00037}{(BC)m}$. Using this form of the equation we can use separation of variables to get

$$\frac{1}{v_x^2} dv = -D dt.$$

By taking the integral of both sides and solving for v_x we find that the equation for the velocity is

$$v_x = \frac{v_0}{1 + v_0 D t}. \tag{28}$$

Integrating this equation for velocity gives us

$$\frac{v_0}{v_0 D} \ln \left(1 + v_0 D t\right).$$

Simplifying and plugging our value for D gives the equation for the x position as

$$x = \frac{(BC)m}{.00037} \ln \left(1 + \frac{.00037 v_0}{(BC)m} t\right) \tag{29}$$

Now that we have the equation for our x trajectory, we can figure out the motion in the y direction. Unlike the differential equation for x, our equation for y cannot be solved using separation of variables because it is inhomogeneous, there is a non-zero constant term. However, we can simplify the equation by finding the terminal velocity, the velocity at which the gravitational force and drag force are equal and opposite. By setting the acceleration to zero we find the terminal velocity is

$$y_t = \sqrt{\frac{g}{D}}. \tag{30}$$

We can now use this to rewrite the differential equation 27 as

$$\frac{dv}{dt} = g \left(1 - \frac{v}{v_t}\right)$$

Now that our equation is in this form, we can use the separation of variables technique to get

$$\frac{1}{1 - \frac{v^2}{v_t^2}} dv = g dt.$$

Taking the integral of each side gives us

$$v_t \arctan \frac{v}{v_t} = gt.$$

Solving for v tells us that the velocity of the projectile is given by

$$v_y(t) = v_t \tanh \frac{gt}{v_t}, \tag{31}$$

and taking another integral gives a position of

$$y(t) = \frac{v_t^2}{g} \ln \left(\cosh \frac{gt}{v_t} \right). \tag{32}$$

We can now get our final equation for the motion in the y direction by plugging in all of our constants, giving us

$$y(t) = \frac{(BC)}{.00037} \ln \left(\cosh \sqrt{\frac{.00037g}{(BC)m}} t \right). \tag{33}$$

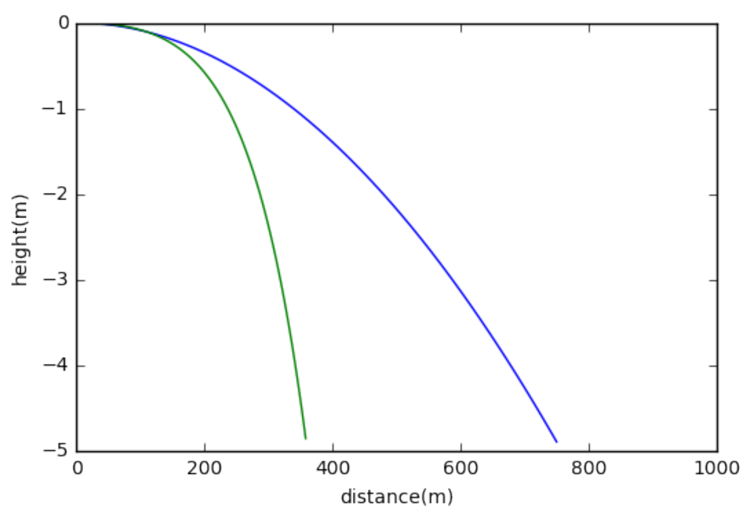

Figure 15: Projectile motion with and without drag

As you can see in figure 15, the same bullet travels a similar path for the first 100 m, but after that the bullet experiencing drag falls significantly earlier in its flight.

3 Experiment

3.1 Chromatic Aberration

Figure 3.1 shows the experimental setup that was used to test for the chromatic aberration of a lens. As mentioned before, the refractive index of a material actually depends on the wavelength of the incident

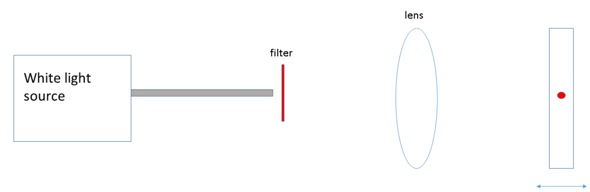

Figure 16: Experimental setup to test for Aberration

light. Since the focal length of a lens is dependent on the index of refraction, it is also effected by the wavelength of the light entering the lens.

To demonstrate this, I measured the focal length of a single lens using different incident light of varying wavelengths. I did so by using a white light source and using color filters to change the light that reaches the lens.

3.2 Antireflective Coatings

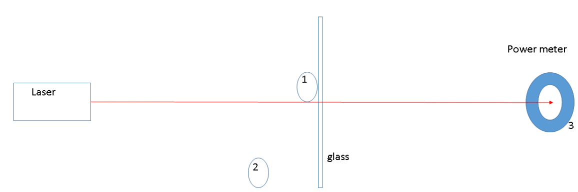

Figure 17: Testing setup for AR coatings

Figure 3.2 shows the setup used to test the effects of having antireflective coatings on the glass surfaces. For this test, a laser was shown into a power meter. First a measurement was taken with no glass present, at position 1 in figure 3.2, and then glass surfaces were placed in the path of the laser and further power readings were recorded for each surface. One measurement was also made at position 2 for the reflection off of the first surface. This was done using uncoated glass and then lenses with coatings designed to let through light with wavelengths between $600 - 900nm$.

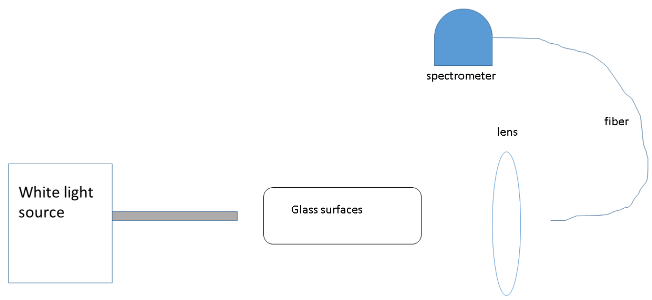

Figure 18: Testing light transmission using a spectrometer

Figure 18 shows the second setup to test the antireflective coatings. In this setup, rather than reading the power from a laser, a white light source was focused onto an optical fiber and into a spectrometer. Once again light was measured without any glass in the path before testing the different surfaces. In this experiment, we will be comparing the transmission of light through bare glass, lenses with coatings designed for red light and a Vortex Diamondback rifle scope with Vortex's proprietary multi-coated lenses. In this setup, all three tests have four glass surfaces that the light is transmitting through before being focused onto the face of the fiber.

4 Data and Analysis

4.1 Aberration

Color filter	Focal length(cm)
None	4.6
Red	4.7
Yellow	4.65
Green	4.5
Blue	4.45

Table 2: Measured focal lengths of a lens from different colored light

Table 2 shows how the focal length of the lens changes when we put different color filters in front. We can see that there is a 1.5mm range in focal lengths; while this does not seem like much, it can make a significant effect in the clarity of the image we would see. This is why most rifle scopes have at least a doublet for the lenses.

4.2 AR Coating

Initial Power Readings	
Control	.02mW
Incident	1.06mW

Table 3: Power readings without glass

In table 3, the control is the power reading when the laser was off. This gives us our background noise so we can subtract that from all further power readings when analyzing the data. The incident reading is the measured power from the laser with no glass surfaces. The next step in the experiment was to measure the power of the reflected and transmitted beams off of a single surface.

Single Surface Power Readings(mW)		
	Reflected	Transmitted
Bare glass	0.12 mW	0.95 mW
Red coating	.04 mW	1.04 mW

Table 4: Power reflected/refracted from single glass surface

As you can see already, the coated glass allows significantly more of the light through. Also note that the reflected and transmitted readings add up to roughly the power of the incident beam. After taking into account for the .02 mW background we have and incident beam of 1.04 mW and the sum of the reflected and transmitted beams are 1.03 mW for the bar glass and 1.04 mW for the coated lens. The uncated glass is off by only 0.01 mW which could be due to light being absorbed by the glass or minor errors in the power meter.

The values for table 4.2 were collected by shining a laser into a power meter and anding glass surfaces in the laser light's path.

Multiple Surfaces		
Number	Bare	Coated
1	0.95	1.04
2	0.92	1.03
3	0.86	1.02
4	0.81	1.01
5	0.75	1.00
6	0.72	0.98
7	0.69	0.97
8	0.63	0.95
9	0.58	0.94
10	0.54	0.93

Table 5: Power Transmisson through multiple surfaces. Note, all power measurements are in mW

5 Discussion

As you can see from the results of the experiments, having AR coatings can make a significant difference in the amount of light going through an optical system, especially when there are several surfaces. We can also see how light separates when it is refracted off a surface which can cause the focal length of lenses to change. This is consistent with what we expected to see. Based on the theory, we knew that the refractive index of a material changed with the frequency of light, and that a thin film would enhance the transfer of light through the material.

6 Conclusions

In order to design a functioning rifle scope, we must first understand Snell's Law. This is the key to explaining how lenses bend light the way that they do so that we can start to build a system of lenses. As we increase the number of lenses, it is useful to understand the matrix method as this allows us to make calculations that would be too difficult or tedious otherwise. In order to increase the quality of the image we see while looking through the scope, we must correct for aberrations and maximize the amount of light that goes through the entire system to improve the image quality.

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