MATHEMATICS: WHAT'S SPATIAL, WHAT'S NOT

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Abstract: Probably, almost everyone has some idea of what is meant by the words 'spatial mathematics.' The problem is that 100 people have 100 different ideas, because the concept is not easy to codify. In this paper we suggest a few ways to illustrate differences between 'spatial' and 'non-spatial' concepts, and ways to introduce spatial approaches where none was present before.

Key words: spatial, non-spatial, mathematics, Latin square

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Natural spatial concepts

Almost everyone's first acquaintance with mathematics is a spatial one. While the concept of 'two' is an abstract one, we learn that abstraction from viewing sets of two objects. So by seeing two boxes, two triangles, two dogs, etc., we come to understand the concept of 'two.' To some, that remains a spatial concept. To others, the abstract idea comes to dominate. Certainly, very few have a good spatial concept of 976,827,253,188. So eventually the abstraction gains weight.

Similarly, most of us were exposed to analytic geometry due to the perspicacity of Rene Descartes associating the points in a coordinate plane with an ordered pair of numbers related to a pair of perpendicular coordinate axes. For instance, the ordered pair (4,5) is associated with a number 4 units to the right of a vertical axis and 5 units above a horizontal axis. If the horizontal axis is called the x-axis and the verticalaxis the y-axis, 4 is called the x-coordinate and 5 the y-coordinate (since (4,5) is 4 units horizontally from the y-axis and 5 units vertically from the x-axis). This can be extended relatively easily to 3 dimensions, even though most pictures of 3-dimensional objects are drawn in 2 dimensions.

All of this is useful for describing and picturing solutions of equations. For instance, the solutions of the equation $x + 3y = 15$ are represented by a straight line in 2-space, and the equation $y = x^2$ represents a parabola in 2-space. But $x + y + z = 6$ is a plane in 3-space, and if we add variables, we use analogies. Thus, $x + y + z + w =6$ is called a hyperplane in 4-space, and very few can visualize this spatially.

Since the concept of ordered pair generalizes to ordered triple or ordered n-tuple, we can still talk about solutions of equations in any number of variables, and such problems occur regularly in the real world, even though the spatial part of the solutions becomes less and less visible.

Spatial desirability

Even when concepts become less spatial, people like to regain some spatial ideas. For instance, when we learn addition and multiplication tables, the essential problem is memorization. But we may use flash cards to give the answers some physical representation.

Similarly, it seems that people's names are non-spatial. But when we meet someone, it helps to associate some physical characteristic with the name. So if Sam is six feet six inches tall, it's easier to remember him. If Sandy's hair is sandy, that's easier, too.

Even more unlikely, people choose to adapt complicated mathematical structures for their own spatial use, even when it comes to amusements.

Definition. A Latin square of order n is an $n \times n$ matrix whose entries are the integers 1, 2, ... , n arranged so that each integer appears exactly once in each row and exactly once in each column.

Latin squares have application in design of experiments (Arlinghaus 1991), in finite geometries (Dembowski 1968), and in combinatorial theory (Ryser 1963). But, in the last 10 years, mathematicians have adapted them to their own purposes.

Sudoku

Sudoku (Hayes 2006) is a number puzzle whose object is to fill in a 9×9 Latin square, each of whose 3×3 subgrids contains the integers 1 through 9, given an initial placement of some numbers. It appears to have been invented by an architect, Howard Garns, in 1979. It was popularized in Japan in 1986 and became an international hit in 2009. Fig. 1 shows a Sudoku puzzle layout on the left and its solution on the right.

Today, Sudoku puzzles appear in booklets, magazines, and newspapers. In addition to solving the puzzles, some want to know how few numbers must be supplied for the solution to be unique. Clearly, there is an appeal here to a spatial representation of numbers.

KenKen

KenKen (Shortz 2009) is an arithmetic and logical puzzle invented by Tetsuya Miyamoto in 2004 to help brain training using arithmetic. The object is to produce a Latin square in which the digits obey simple mathematical rules contained in groups of squares called 'cages.' Fig. 2 shows a 6 by 6 puzzle on the left with its solution on the right.

In this case, one can start with a Latin square and invent rules to create various cages, then remove the numbers to create the puzzle. Once again, we see the desire of humans to create a spatial format even for simple mathematics, and again the result is interesting to the daily puzzle aficionado.

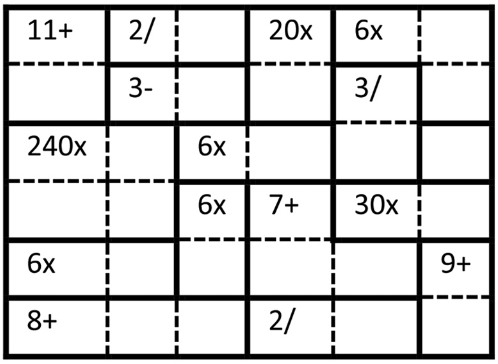
More or Less

These puzzles are more recent additions for the amusement of readers. The object is to fill in a 7×7 Latin square, given some entries and some indications that numbers in a given square

6	4			8				
7			2	1	6			
	1	9					7	
9				7				4
5			9		4			2
8				3				7
	7					3	9	
			5	2	1			6
				9			8	1

6	4	**5**	**7**	8	**9**	**1**	**2**	**3**
7	**8**	**3**	2	1	6	**4**	**5**	**9**
2	1	9	**4**	**5**	**3**	**6**	7	**8**
9	**6**	**1**	**8**	7	**2**	**5**	**3**	4
5	**3**	**7**	9	**6**	4	**8**	**1**	2
8	**2**	**4**	**1**	3	**5**	**9**	**6**	7
1	7	**2**	**6**	**4**	**8**	3	9	**5**
3	**9**	**8**	5	2	1	**7**	**4**	6
4	**5**	**6**	**3**	9	**7**	**2**	8	1

Fig. 1. A Sudoku puzzle layout on the left with its solution on the right



5	6	3	4	1	2
6	1	4	5	2	3
4	5	2	3	6	1
3	4	1	2	5	6
2	3	6	1	4	5
1	2	5	6	3	4

Fig. 2. A 6 by 6 KenKen puzzle layout on the left with its solution on the right

are smaller than or greater than the numbers in a neighboring square (Figure 3).

In this arrangement, one can see that 3, 4, 5, 7 must fill in the squares to satisfy the indicated inequalities. Enough inequalities and numbers are included to force the answers needed to complete the Latin square.

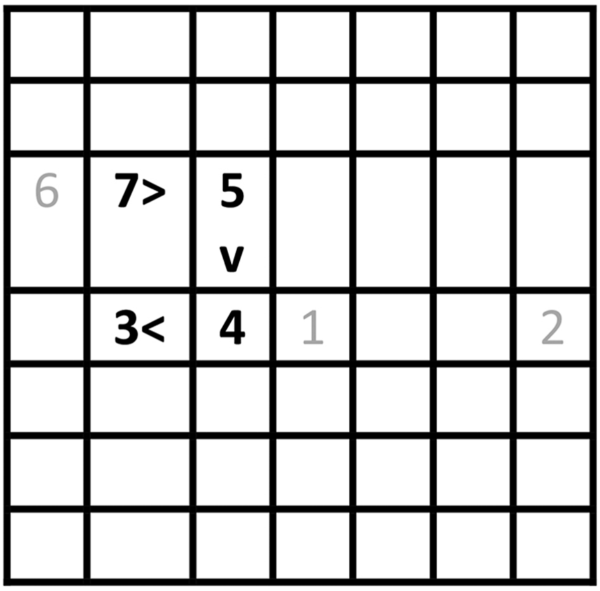


Fig. 3. A 7 by 7 Latin square with superimposed greater-than or smaller-than conditions

Conclusion

It is not easy to separate what is spatial from what is not. Even the most abstract concepts often have spatial components. What is clear is that we have a desire to see things spatially. We imbue abstract numbers with spatial components to clarify concepts, to make the abstract more visible, and even to create amusements for ourselves.

References

Arlinghaus W., 1991. Chapter 10 in Michaels, J. and Rosen, K. *Applications of discrete mathematics*. McGraw-Hill, New York.

Dembowski P., 1968. *Finite geometries*. Springer-Verlag, Berlin.

Hayes B., 2006. Unwed numbers. *American Scientist* 94(1): 12–15.

Ryser H., 1963. *Combinatorial mathematics*. Carus Monograph #14, Mathematical Association of America.

Shortz W., 2009. A new puzzle challenges math skills. *The New York Times*, February 8.