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SLENDER SHAPES OFFERING MINIMUM DRAG IN FREE-MOLECULAR FLOW

by

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with an Appendix on

Similarity Law for Longitudinal Contours of Optimum Bodies in Free-Molecular Flow

by

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Summary

Analytical expressions are obtained for the optimum shapes which minimise the drag of a slender axisymmetric body in free-molecular flow, provided the drag expression is simplified using the slenderness assumption. The problem is formulated as one of Mayer type in the calculus of variations and solved by using the Buler-Lagrange equations together with the transversality condition. The shapes derived are optimum subject to constraints on thickness, length, wetted area and volume. In the particular cases solved any two of these four quantities are fixed while the remaining two are free. The expression for the shape of the body when thickness is free is obtained in closed form.

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Contents

Notation

a_j, b_j, c_j	coefficients in generalised drag integral (2.6)
$B(a, b)$	Beta function
C, C_1, C_2, C_3	constants
$C_D = 4D/\pi d^2 q$	drag coefficient based on frontal area at $x = \ell$
d	maximum diameter of body
D	drag
f	subscript referring to final point
F	augmented function
i	subscript referring either to initial point or free stream conditions
$I_x(a, b)$	incomplete Beta function, defined in section 5.2.2.
j	parameter taking value 0 in hypersonic extreme and value 1 in low subsonic extreme
K_1, K_2, K_3	defined in (3.4)
ℓ	value of x when $y = \frac{1}{2}d$
$q = \frac{1}{2}\rho U^2$	free stream dynamic pressure
$q_m = \frac{1}{2}\rho U_m^2$	dynamic pressure based on U_m
$s = U/U_m$	speed ratio
S	wetted surface area
T_i	temperature of molecules in free stream
T_r	temperature of reemitted molecules
U	free stream velocity
U_m	most probable velocity of random molecular motion at temperature T_i
V	body volume
x, y	axial and radial coordinate of body
$\dot{y} \equiv dy/dx$	
α, β, γ	integrals defined in section 3

$\alpha_{f\varepsilon}, \alpha_{f\nu}, \beta_f, \gamma_f$	integrals defined in section 4
$\Gamma(n)$	Gamma function
ε_j	coefficient defined in (4.1)
$\eta = y/\tfrac{1}{2}d$	
$\dot{\eta} \equiv d\eta/d\xi$	
θ	slope of surface element to free stream
λ	Lagrange multiplier
ν_j	coefficient defined in (4.1)
$\xi = x/\ell$	
$\sigma = s \sin \theta$	
$\tau = d/\ell$	
ϕ	differential constraint

1. Introduction

Tan[1] has determined the body of revolution offering minimum drag in a free-molecular flow for a given length and a given diameter. In this paper the analysis is extended to include constraints on the wetted surface area and volume as well as on the length and diameter. The reflection process is assumed to be totally diffuse with a complete energy exchange. By employing the assumption that the body is slender it is possible to obtain the shape of the optimum body in closed analytical form.

This topic has already formed the subject of a thesis[5] written by the author's student Lt. G.B. Hall, U.S.N., and the author is grateful to him for the Figures included in this paper. While this paper was being prepared Professor Miele sent the author a copy of Part VI of his coming book[3] in which part of the work described here, section 5.1, is to be the subject of Chapter 28 written by Miele and Pritchard.

2. Formulation of the problem

Consider a slender axisymmetric body in a free-molecular flow whose macroscopic mean velocity is U and equilibtium temperature is T_i. The mean free path of the molecules is so much larger than a characteristic body dimension that collisions with the body dominate over intermolecular collisions between the incoming flow and that reflected from the body, and the latter may be ignored. The resulting flow field may be treated as if it were composed of two independent flow fields. In particular the drag experienced by the slender body is obtained as the sum of the impact drag of the incoming molecules and the reactive drag of the reflected molecules.

The speed ratio s between the free stream velocity U and the most probable velocity of the random molecular motion U_m at temperature T_i is defined by

$$s = U/U_m = U/\sqrt{2RT_i} ,$$

where R is the specific gas constant.

It is assumed that all of the gas molecules are absorbed by the body surface prior to reemission, that the temperature T_r of the reemitted molecules is identical with the surface temperature, and that the velocity distributions of the incident and reflected flows are Maxwellian. Then, as has been shown by Tan[1], the drag may be expressed as

$$(2.1)\qquad D = \tfrac{1}{2}\rho U^2 \int \left\{ \left[\frac{1}{s\sqrt{\pi}} + \frac{\sin\theta}{2s^2}\sqrt{\frac{T_r}{T_i}} \right] e^{-\sigma^2} + \sin\theta \left[1 + \frac{1}{2s^2} + \frac{\sin\theta}{2s}\sqrt{\frac{\pi T_r}{T_i}} \right] \left[1 + \operatorname{erf}\sigma \right] \right\} dS$$

where the surface element dS is at an angle θ to the free stream, $\sigma = s\sin\theta$, and

$$\operatorname{erf}\sigma = \frac{2}{\sqrt{\pi}} \int_o^{\sigma} e^{-t^2} dt \ .$$

Two useful approximations to this drag result are possible. At hypersonic speed ratios, $s >> 1$ and $\sigma^2 >> 1$, (2.1) reduces to

$$(2.2)\qquad D = \rho U^2 \int \left[1 + \frac{\sin\theta}{2s}\sqrt{\frac{\pi T_r}{T_i}} \right] \sin\theta \, dS \ ,$$

while at low subsonic speed ratios, $s << 1$ and $\sigma^2 << 1$, (2.1) reduces to

$$(2.3)\qquad D = \tfrac{1}{2}\rho U^2 \int \left\{ \frac{1}{s} \left[\frac{1}{\sqrt{\pi}} + \frac{\sin^2\theta}{2}\sqrt{\frac{\pi T_r}{T_i}} \right] + \frac{\sin\theta}{2s^2} \left[1 + \sqrt{\frac{T_r}{T_i}} \right] \right\} dS$$

Let the axial coordinate of the axisymmetric body be x and its radius y, then $dS = 2\pi y \sqrt{1 + \mathring{y}^2}\, dx$, and $\sin\theta = \mathring{y}/\sqrt{1 + \mathring{y}^2}$, where $\mathring{y} \equiv dy/dx$. If now the slender body approximation, $\mathring{y}^2 << 1$, is used these may be simplified to $dS = 2\pi y \, dx$ and $\sin\theta = \mathring{y}$. Hence the drag of a slender axisymmetric body, between the stations 0 and x, may be expressed, in the hypersonic extreme, by

$$(2.4)\qquad D(x) = 2\pi\rho U_m^2 \int_o^x y \left[s^2 \mathring{y} + \frac{s}{2}\sqrt{\frac{\pi T_r}{T_i}}\ \mathring{y}^2 \right] dx$$

and, in the low-subsonic extreme, by

$$(2.5)\qquad D(x) = \pi\rho U_m^2 \int_o^x y \left[\frac{s}{\sqrt{\pi}} + \frac{1}{2}\left(1 + \sqrt{\frac{T_r}{T_i}} \right) \mathring{y} + \frac{s}{2}\sqrt{\frac{\pi T_r}{T_i}}\ \mathring{y}^2 \right] dx$$

It is clear from (2.4) and (2.5) that, as the drag expressions are essentially of the same form, the analysis may be made more concise by considering simply

$$D(x) = 2^{(2-j)}\pi q_m \int_0^x y(a_j + b_j\dot{y} + c_j\dot{y}^2)\,dx \tag{2.6}$$

which includes both (2.4) and (2.5) as special cases,

where $a_j = j\left(\frac{s}{\sqrt{\pi}}\right)$, $b_j = \frac{s^{2(1-j)}}{2^j}\left(1 + \sqrt{\frac{T_r}{T_i}}\right)^j$, $c_j = \frac{s}{2}\sqrt{\frac{\pi T_r}{T_i}}$,

and $j = 0$ in the hypersonic extreme, $\sigma^2 \gg 1$,

while $j = 1$ in the low subsonic extreme, $\sigma^2 \ll 1$.

The problem to be considered is to minimise the drag integral, as expressed in (2.6), subject to constraints on the wetted surface area S and volume V. For a slender body

$$S = 2\pi \int_0^x y\,dx \tag{2.7}$$

$$V = \pi \int_0^x y^2\,dx\,. \tag{2.8}$$

In taking the problem in this general form the optimum shapes appropriate for both hypersonic and low-subsonic free molecular flows will be determined.

It is to be noted that in the drag expressions used in this section no allowance has been included for base drag. The base region of any optimum shape which has a blunt base must therefore be assumed to be filled by a cylinder whose axis lies in the stream direction.

3. General analysis of hypersonic and low subsonic problems

Define

$$\left.\begin{aligned} \alpha(x) &= \frac{D(x)}{2^{(2-j)}\pi q_m} = \int_0^x y(a_j + b_j\dot{y} + c_j\dot{y}^2)\,dx\,, \\ \beta(x) &= \frac{S(x)}{2} = \int_0^x y\,dx\,, \\ \gamma(x) &= \frac{V(x)}{\pi} = \int_0^x y^2\,dx \end{aligned}\right\} \tag{3.1}$$

Then differentiating with respect to the independent variable yields the following differential constraints for the problem

$$\left.\begin{aligned}\phi_1 &\equiv \dot{\alpha} - y\,(a_j + b_j\dot{y} + c_j\dot{y}^2) = 0\\ \phi_2 &\equiv \dot{\beta} - y = 0\\ \phi_3 &\equiv \dot{\gamma} - y^2 = 0\end{aligned}\right\}$$

The appropriate boundary conditions are

$x_i = y_i = \alpha_i = \beta_i = \gamma_i = 0$ at the initial point and at the final point (subscript f) some but not all of the coordinates will be given.

The problem to be solved may be stated as the following Mayer problem[2]. In the class of functions $y(x)$, $\alpha(x)$, $\beta(x)$, $\gamma(x)$ which are consistent with the differential constraints and the prescribed end conditions, find the set which minimise the difference $\Delta\alpha = \alpha_f - \alpha_i$.

Standard methods of the calculus of variations[2] may be applied, following the introduction of Lagrange multipliers $\lambda(x)$ and the augmented function

$$F = \lambda_1 \left[\dot{\alpha} - y(a_j + b_j\dot{y} + c_j\dot{y}^2)\right] + \lambda_2(\dot{\beta}-y) + \lambda_3(\dot{\gamma}-y^2).$$

The following Euler-Lagrange equations must be satisfied by the extremal curve

$$\frac{d}{dx}\left[-\lambda_1 y(b_j + 2c_j\dot{y})\right] = -\lambda_1(a_j + b_j\dot{y} + c_j\dot{y}^2) - \lambda_2 - 2\lambda_3 y\ .$$

$$\dot{\lambda}_1 = \dot{\lambda}_2 = \dot{\lambda}_3 = 0\ .$$

The last of these equations show that the Lagrange multipliers are constants.

$$\lambda_1 = C_1\ ,\quad \lambda_2 = C_2\ ,\quad \lambda_3 = C_3\ .$$

Because F is formally independent of x the remaining Euler-Lagrange equation has the first integral

(3.2) $$C_1 y(a_j - c_j\dot{y}^2) + C_2 y + C_3 y^2 = C$$

where C is a constant.

The transversality condition, which must be satisfied by all systems of differentials consistent with the prescribed end conditions, takes the form

$$(3.3) \qquad \Big[(C_1+1)d\alpha - Cdx + C_2 d\beta + C_3 d\gamma - C_1 y(b_j + 2c_j\dot{y})dy\Big]_i^f = 0 .$$

Since α_f is free it follows that $C_1 = -1$.

The Legendre-Clebsch condition, which provides a necessary condition for a minimum for all weak variations consistent with the constraints, is in this case

$$-2C_1 \; c_j \; y \; (\delta\dot{y})^2 \geq 0$$

As this is satisfied by $y \geq 0$ it follows that all the extremal arcs which satisfy the conditions of this section provide minimum drag configurations.

The shape of the extremal arc follows from integrating equation (3.2), which gives an expression for the slope of the curve, with $C_1 = -1$. On a forward-facing surface the slope is positive so that

$$x = \int_0^y \left(c_j y \;/\; [C - (C_2-a_j)y - C_3 y^2]\right)^{\frac{1}{2}} dy .$$

Introduce now the dimensionless variables

$$\xi = {}^x/_\ell , \qquad \eta = {}^y/_{\frac{1}{2}d} ,$$

where the maximum ordinate $y = \frac{1}{2}d$ occurs at the station $x = \ell$. The thickness ratio of the body is $\tau = {}^d/_\ell$. In terms of these variables the shape of the extremal on the forward-facing surface is described by

$$(3.4) \qquad \xi = \int_0^\eta (\eta/K_1-K_2\eta-K_3\eta^2)^{\frac{1}{2}} d\eta \Big/\!\!\Big/ \int_0^1 (\eta/K_1-K_2\eta-K_3\eta^2)^{\frac{1}{2}} d\eta$$

in the range $0 \leq \xi \leq 1$, where $K_1 = C(\frac{1}{2}d)^{-1}$, $K_2 = C_2-a_j$, and $K_3 = C_3(\frac{1}{2}d)$.

It will be shown that some of the optimum curves have also a rearward-facing surface following the position of maximum thickness. On such a rearward-facing surface the slope is negative and it is easy to see that the appropriate form for the extremal arc there is

$$\xi = 1 - \left\{ \int_1^{\eta} (\eta/K_1 - K_2\eta - K_3\eta^2)^{\frac{1}{2}} \, d\eta \Big/ \int_0^1 (\eta/K_1 - K_2\eta - K_3\eta^2)^{\frac{1}{2}} \, d\eta \right\}$$

in the range $1 \leqslant \xi \leqslant \xi_f$.

This may be expressed more conveniently in the form

$$\xi = 2 - \left\{ \int_0^{\eta} (\eta/K_1 - K_2\eta - K_3\eta^2)^{\frac{1}{2}} \, d\eta \Big/ \int_0^1 (\eta/K_1 - K_2\eta - K_3\eta^2)^{\frac{1}{2}} \, d\eta \right\} \tag{3.5}$$

$$1 \leqslant \xi \leqslant \xi_f.$$

Particular cases of equations (3.4) and (3.5) will be considered in section 5.

4. Characteristics of the optimum shapes

From equation (2.6) the total drag of an optimum profile for either hypersonic or low-subsonic extremes may be written

$$D = 2^{(2-j)} \pi q_m \int_0^{x_f} y(a_j + b_j\dot{y} + c_j\dot{y}^2) \, dx$$

which in terms of the dimensionless variables becomes

$$D = 2^{(1-j)} \pi q_m \, d\ell \int_0^{\xi_f} \eta(a_j + \tfrac{1}{2}\tau b_j \mathring{\eta} + \tfrac{1}{4}\tau^2 c_j \mathring{\eta}^2) \, d\xi$$

$$= 2^{(1-j)} \pi q_m \, d\ell \left\{ a_j \int_0^{\xi_f} \eta d\xi + \tfrac{1}{4}\tau b_j \eta_f^2 + \tfrac{1}{4}\tau^2 c_j \int_0^{\xi_f} \eta\mathring{\eta}^2 d\xi \right\}$$

Thus the drag coefficient based on the maximum frontal area and the stream velocity is given by

$$C_D = \frac{D}{\frac{1}{4}\pi d^2 q} = \frac{2^{(1-j)} b_j}{s^2} \left\{ \eta_f^2 + \frac{\nu_j}{\tau} \alpha_{f\nu} + \varepsilon_j \tau \, \alpha_{f\varepsilon} \right\} \tag{4.1}$$

where $$b_j = \frac{s^{2(1-j)}}{2^j} \left(1 + \sqrt{\frac{T_r}{T_i}}\right)^j$$

$$\nu_j = \frac{4a_j}{b_j} = \frac{j \; 2^{(j+2)} \; s^{(2j-1)}}{\sqrt{\pi} \left(1 + \sqrt{\frac{T_r}{T_i}}\right)^j}$$

$$\alpha_{f\nu} = \int_0^{\xi_f} \eta d\xi$$

$$\varepsilon_j = \frac{c_j}{b_j} = \frac{2^{(j-1)}\, s^{(2j-1)} \sqrt{\frac{\pi T_r}{T_i}}}{\left(1 + \sqrt{\frac{T_r}{T_i}}\right)^j}$$

$$\alpha_{f\varepsilon} = \int_0^{\xi_f} \eta\dot{\eta}^2 d\xi$$

and $j = 0$ in the hypersonic extreme

$j = 1$ in the low-subsonic extreme.

The wetted area of the optimum body is obtained from

(4.2) $$S = \pi d\, \ell\, \beta_f$$

where $$\beta_f = \int_0^{\xi_f} \eta d\xi$$

and its volume from

(4.3) $$V = \tfrac{1}{4}\, \pi d^2\, \ell\, \gamma_f$$

where $$\gamma_f = \int_0^{\xi_f} \eta^2 d\xi \; .$$

5. Particular results for the hypersonic case

5.1 Thickness fixed

Consider now particular cases of the extremal arc, which has been derived in section 3, in the hypersonic extreme. Thus $j = 0$, $a_0 = 0$, $b_0 = s^2$, $c_0 = \frac{s}{2}\sqrt{\frac{\pi T_r}{T_i}}$, $\nu_0 = 0$, $\varepsilon_0 = \frac{1}{2s}\sqrt{\frac{\pi T_r}{T_i}}$, and

(5.1) $$C_D = 2\left\{\eta_f^2 + \varepsilon_0\, \tau\, \alpha_{f\varepsilon}\right\}.$$

Of the four constraints, length ℓ, thickness d, wetted surface area S, and volume V, two are fixed in each case while the remaining two are free.

When the body thickness, d, and one other from ℓ, S, and V, are fixed the resulting optimum shape is given by a simple power law.

5.1.1 Given diameter and length

Because the wetted area and volume are free it follows from the transversality condition (3.3) that $C_2 = C_3 = 0$ so that $K_2 = -a_0 = 0 = K_3$. The equation (3.4) for the extremal arc may be integrated immediately to yield

(5.2) $$\eta = \xi^{2/3}$$

which is shown in Fig. 1. The optimum body of revolution is a $2/3$-power law body, for which $\eta_f = 1$, $\alpha_{f\varepsilon} = 4/9$, so that from (5.1) its drag coefficient is given by

(5.3) $$C_D = 2 + \frac{8}{9}\varepsilon_0\tau.$$

The variation of C_D with s for a $2/3$-power law body of thickness ratio $\tau = 1/3$ is illustrated in Fig.3 for various values of the temperature ratio T_r/T_i.

5.1.2 Given diameter and wetted area

As the length and volume are free it follows from (3.3) that $C = C_3 = 0$ so that $K_1 = K_3 = 0$. It is easy to show that the optimum shape is a cone (Fig.1).

(5.4) $$\eta = \xi.$$

Note that this result also applies in the low subsonic extreme. The drag coefficient of the cone is given by

(5.5) $$C_D = 2 + \varepsilon_0\tau$$

5.1.3 Given diameter and volume

As the length and wetted area are free, $C = C_2 = 0$, and $K_2 = -a_0 = 0 = K_1$. The optimum body is here a parabola

(5.6) $$\eta = \xi^2.$$

It will be noticed in Fig.1 that this profile is concave to the flow. Now the drag formula used, (2.1), is based on the assumption that the body is convex to the flow. At first sight it would seem necessary to correct the analysis by including in (2.1) a contribution to the momentum transfer due to those molecules reflected from the surface which hit the surface again. However it has already been shown[4] that for bodies of simple shape and large concavity the effect of these interreflections on the drag is small. For this reason this effect will be neglected here. It is to be expected that the parabolic shape will offer a good approximation to the exact result.

The associated drag coefficient is given by

(5.7) $$C_D = 2 + \frac{8}{5}\,\varepsilon_0\tau .$$

In conclusion it is noted that the slenderness assumption is valid if $\dot{\eta} \leqslant 2$. This condition is violated only by the $^2/3$-power law body, and then only in a small region close to the nose defined by $0 \leqslant \xi \leqslant 0{\cdot}037$. In fact this shape is a close approximation to that obtained by Tan[1] who did not employ the slenderness assumption.

5.2 Thickness free

5.2.1 Given wetted area and volume

Since the length and thickness are free it follows from (3.3) that $C = 0$ so that $K_1 = 0$, and $y_f(b_j + 2c_j\dot{y}_f) = 0$. The last condition requires that either $\eta_f = 0$ or $\dot{\eta}_f = -^1/\tau\varepsilon_j$.

The transversality condition (3.3) in this case reduces to the form

$$\Big[(C_2y + C_3y^2)\,dx\Big]_i^f = 0$$

which yields the condition

(5.8) $$(K_2 + a_j)\,\eta_f + K_3\eta_f^2 = 0\ .$$

On the other hand the first integral reduces to

$$-y(a_j - c_j\dot{y}^2) + C_2y + C_3y^2 = 0$$

giving the condition

(5.9) $$(K_2 + a_j)\,\eta_f + K_3\eta_f^2 = \eta_f(a_j - \tfrac{1}{4}\,c_j\tau^2\dot{\eta}_f^2).$$

Clearly the end condition which will make (5.8) and (5.9) consistent is $\eta_f = 0$.

The maximum ordinate, $\eta_m = 1$, will obviously occur before the end point and the slope will vanish there, $\dot{\eta}_m = 0$. Thus at the maximum ordinate the first integral yields the condition

$$K_2 + K_3 = a_j .$$

Using this relationship together with $K_1 = 0$, (3.4) can be integrated to find the shape of the forward portion of the extremal arc. In the hypersonic extreme, $j = 0$ and $a_0 = 0$, the shape is described by

(5.10) $$\eta = 1 - (1 - \xi)^2, \qquad 0 \leqslant \xi \leqslant 1.$$

Similarly (3.5) gives the shape of the rear portion of the extremal arc as

(5.11) $$\eta = 1 - (\xi - 1)^2, \qquad 1 \leqslant \xi \leqslant 2.$$

This shape is illustrated by the full line in Fig.2.

Evaluating (5.1) with $\eta_f = 0$ yields for the drag coefficient of this optimum body the result

(5.12) $$C_D = \frac{32}{15}\,\varepsilon_0 \tau.$$

5.2.2 Given length and wetted area

The thickness and volume being free requires that $K_3 = 0$ and that either $\eta_f = 0$ or $\dot{\eta}_f = -\,^1/\tau\varepsilon_j$.

The first integral yields the condition

(5.13) $$(K_2 + \tfrac{1}{4}\,c_j \tau^2 \dot{\eta}_f^2)\,\eta_f = K_1.$$

The transversality condition gives no further information. If η_f is zero then so is K_1 from (5.13). With $K_1 = K_3 = 0$ (3.3) gives a cone for the extremal arc, but this clearly contradicts the requirement $\eta_f = 0$. Hence the appropriate end-conditions for this problem are $\dot{\eta}_f = -\,^1/\tau\varepsilon_j$, $\eta_f \neq 0$.

At the maximum ordinate $\eta_m = 1$ the slope $\dot{\eta}_m$ vanishes and the first integral yields $K_2 = K_1$. The forward portion of the optimum body is described by

$$(5.14)\qquad \xi = \frac{2}{\pi}\left[\arcsin\sqrt{\eta} - \sqrt{\eta(1-\eta)}\right], \qquad 0 \leqslant \xi \leqslant 1.$$

and the rear portion by

$$(5.15)\qquad \xi = \frac{2}{\pi}\left[\pi - \arcsin\sqrt{\eta} + \sqrt{\eta(1-\eta)}\right], \qquad 1 \leqslant \xi \leqslant \xi_f.$$

This shape is optimum for both the hypersonic and low subsonic extremes. The coordinates of the end point depend on the thickness ratio τ, the speed ratio s, and the temperature ratio T_r/T_i, being determined by the condition $\dot{\eta}_f = -1/\tau\varepsilon_j$. Typical positions of this end point for the hypersonic case are indicated in Fig.2 on the rear portion of the optimum body which is shown dotted.

To evaluate the drag coefficient it is appropriate to take $\alpha_{f\varepsilon}$ in the form

$$\alpha_{f\varepsilon} = \int_0^{\eta_f} \eta \frac{d\eta}{d\xi}\, d\eta$$

noting that the integration on η extends over the range 0 through 1 (the forward portion) and back to η_f (the rear portion). For convenience this may be used in the alternative form

$$\alpha_{f\varepsilon} = 2\int_0^{1} \eta \frac{d\eta}{d\xi}\, d\eta - \int_0^{\eta_f} \eta \frac{d\eta}{d\xi}\, d\eta$$

where $d\eta/d\xi$ is to be evaluated from the positive slope of the forward portion.

Since

$$\frac{d\xi}{d\eta} = \frac{2}{\pi}\left(\frac{\eta}{1-\eta}\right)^{\frac{1}{2}}, \qquad 0 \leqslant \xi \leqslant 1$$

$$\alpha_{f\varepsilon} = \frac{\pi^2}{16}\left[2 - I_{\eta_f}\left(\frac{3}{2}, \frac{3}{2}\right)\right]$$

where $I_x(a, b)$ is the incomplete Beta function defined by

$$I_x(a,b) = \frac{B_x(a,b)}{B(a,b)} = \frac{\int_0^x u^{a-1}(1-u)^{b-1}\, du}{\int_0^1 u^{a-1}(1-u)^{b-1}\, du}$$

and $B(a,b)$ is the Beta function, $a,b > 0$, $0 \leqslant x \leqslant 1$.

Thus the drag coefficient of the optimum body of given length and wetted area is given by

(5.16) $$C_D = 2\left\{\eta_f^2 + \frac{\pi^2}{16}\varepsilon_0\tau\left[2 - I_{\eta_f}\left(\frac{3}{2}, \frac{3}{2}\right)\right]\right\}$$

5.2.3 Given length and volume

The thickness and wetted area are free so that $K_2 = -a_j$ and either $\eta_f = 0$ or $\dot{\eta}_f = -{}^1/\tau\varepsilon_j$.

The first integral gives the condition

(5.17) $$(K_3\eta_f - a_j + \tfrac{1}{4}c_j\tau^2\dot{\eta}_f^2)\,\eta_f = K_1.$$

If η_f is zero K_1 is also zero. In the hypersonic extreme, $j = 0$, $a_0 = 0$, (3.3) shows that the shape of optimum body, with $K_1 = K_2 = 0$, would be the parabola $\eta = \xi^2$, which clearly does not satisfy the condition $\eta_f = 0$. The appropriate end-conditions are thus $\dot{\eta}_f = -{}^1/\tau\varepsilon_0$, $\eta_f \neq 0$.

Applying the first integral at the maximum point gives the condition $K_3 = K_1$ for the hypersonic case. Thus the fore part of the extremal arc is described by

(5.18) $$I_{\eta^2}(\tfrac{3}{4}, \tfrac{1}{2})$$

and the rear part by

(5.19) $$2 - I_{\eta^2}(\tfrac{3}{4}, \tfrac{1}{2})$$

This shape is shown in Fig.2 with the rear of the body dotted and typical end points indicated.

After some reduction the drag coefficient of this body can be shown to be

(5.20) $$C_D = 2\left\{\eta_f^2 + \frac{2\pi^{3/2}}{5}\frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})}\left[2 - I_{\eta_f^2}\left(\frac{3}{4}, \frac{3}{2}\right)\right]\right\}$$

where $\Gamma(n)$ is the gamma function.

6. Conclusions

The shape of the slender body of revolution, which minimises the drag subject to constraints on the length, thickness, wetted area and volume, has been investigated. The analysis has been presented in a general form which provides expressions, in both the hypersonic and low subsonic extremes, for the shape of the extremal arc and its associated drag coefficient. Detailed results have been presented for the optimum hypersonic bodies only. By using the slenderness assumption to simplify the expression for the drag it has been possible to find closed analytical forms for these optimum shapes. The problem is solved as one of Mayer type in the calculus of variations, two of the constraints on thickness, length, wetted area and volume being fixed while the other two remain free.

When the thickness of the body is one of the quantities fixed the optimum shapes are expressed by simple power laws. They are illustrated in Fig.1. Thus for given thickness and length the optimum contour is a $^2/_3$-power law, while for given thickness and wetted area it is a cone, and for given thickness and volume a parabola.

When the thickness of the body is free the maximum diameter occurs ahead of the base of the body. The optimum contours are illustrated in Fig.2. For given wetted area and volume the body closes. For both given length and wetted area and given length and volume the body has a blunt base. The coordinates of the end points depend on the thickness ratio τ, speed ratio s, and temperature ratio $^{T_r}/_{T_i}$, being determined by the value of the final slope. For values of the temperature ratio likely to be experienced the magnitude of the blunt base will be very small.

For thickness fixed bodies the drag coefficient has a constant value 2, due to the incident molecules, to which is added a small contribution due to the reemitted molecules, which depends on the thickness ratio, speed ratio, and temperature ratio. Typical values of C_D for the $^2/_3$-power law body are shown in Fig.3.

References

[1] H.S. Tan, Nose drag in free-molecule flow and its minimisation, Journal of the Aero/Space Sciences, 6, 26 (1959), 360-366.

[2] G. Leitmann (Editor), Optimisation techniques with applications to aerospace systems, Academic Press 1962. Chapter 4.

[3] A. Miele (Editor), Theory of optimum aerodynamic shapes, Academic Press 1965. Chapter 28.

[4] M.J. Pratt, The free-molecule flow characteristics of concave surfaces, College of Aeronautics Note No.155, June 1963.

[5] G.B. Hall, Slender shapes of minimum drag in free-molecule flow, College of Aeronautics Thesis, Cranfield, 1965. Unpublished.

APPENDIX

SIMILARITY LAW FOR LONGITUDINAL CONTOURS OF OPTIMUM BODIES IN FREE-MOLECULAR FLOW

The particular longitudinal contours derived in section 5 of this paper minimise the drag, in free-molecular flow, of a body of revolution. The author has since been able to show that these solutions are not restricted to this one case but can be used without further analysis to find the corresponding longitudinal contours which minimise the drag of bodies of arbitrary cross section. This is possible provided the bodies are slender and each cross section is geometrically similar to the base cross section. For these bodies the following similarity law exists :

> The shape of the optimum longitudinal contour of a body of arbitrary, but prescribed, cross section is identical with the optimum longitudinal contour of an axisymmetric body, provided the drag, wetted area and volume of the axisymmetric body, and two parameters depending on the speed and temperature ratios of the flow, are replaced by proportional quantities appropriate to the body of arbitrary cross section, the factors of proportionality depending only on the shape of this prescribed cross section.

The detailed form and the derivation of this law will be given in the paper referenced. Clearly this similarity law offers an important generalisation of the results obtained in the main paper.

Reference

E.A. Boyd. Similarity law for optimum bodies in free-molecular flow. Astronautica Acta. To be published.

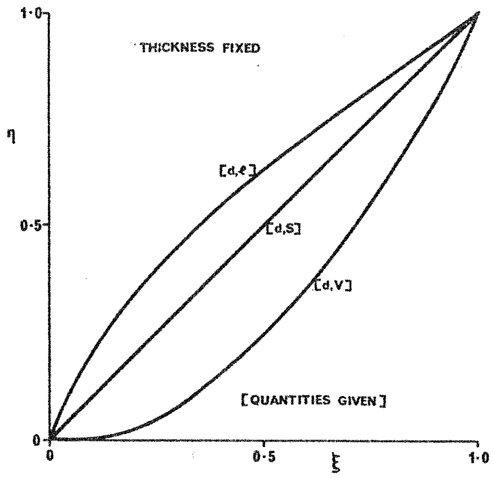

FIG.1. OPTIMUM BODIES OF REVOLUTION IN HYPERSONIC FREE MOLECULAR FLOW. THICKNESS FIXED.

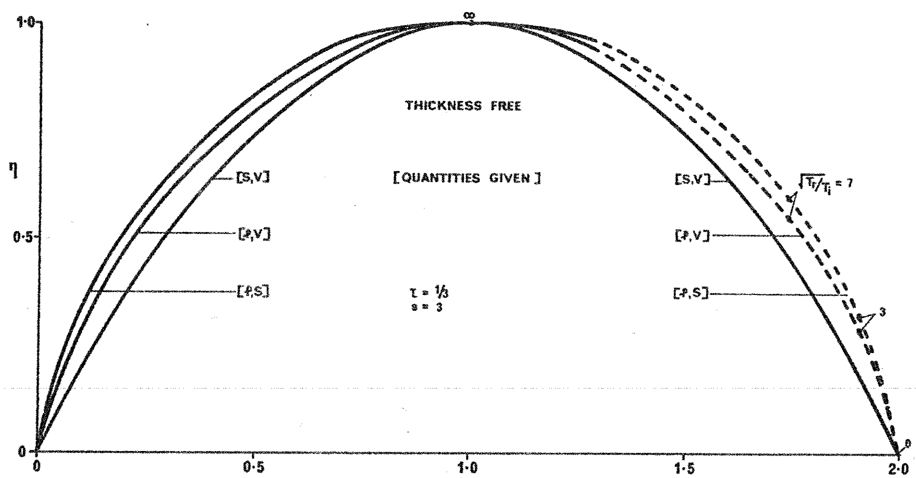

FIG. 2. OPTIMUM BODIES OF REVOLUTION IN HYPERSONIC FREE MOLECULAR FLOW. THICKNESS FREE. FINAL ORDINATES FOR GIVEN [ℓ,V] AND [ℓ,S] DEPEND ON VALUE OF $\sqrt{T_r/T_i}$.

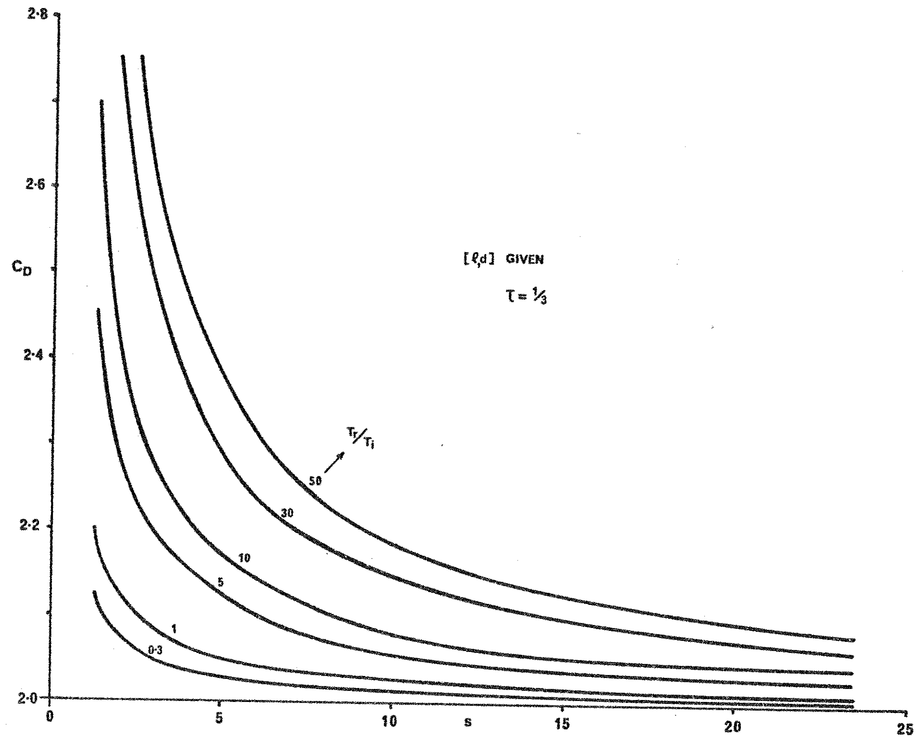

FIG 3. VARIATION OF C_D WITH SPEED RATIO s OF OPTIMUM BODY, FOR GIVEN ℓ AND d, IN HYPERSONIC FREE MOLECULAR FLOW.