# Shot noise in the edge states of two-dimensional topological insulators

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We calculate the resistance and shot noise in the edge states of a two-dimensional topological insulator that result from the exchange of electrons between these states and conducting puddles in the bulk of the insulator. The two limiting cases where the energy relaxation is either absent or very strong are considered. A finite time of spin relaxation in the puddles is introduced phenomenologically. Depending on this time and on the strength of coupling between the edge states and the puddles, the Fano factor $\mathcal{F} = S_I/2eI$ ranges from 0 to 1/3, which is in an agreement with the available experimental data.

## I. INTRODUCTION

A principal distinctive feature of two-dimensional (2D) topological insulators is the existence of helical edge electronic states in which the electron spin projection is locked to the direction of its momentum. For this reason, the electrons cannot be backscattered unless the time-reversal symmetry is violated. This topological protection of the pair of edge states with opposite spin directions results in the universal value of its conductance $e^2/h$, which should hold in the absence of spin-flip scattering [1]. However, measurements revealed that the conductance appears to be much smaller than this universal value. In most cited papers on HgTe/CdTe quantum wells the suppression of conductance is about 10% as the length of the conducting channel is about 1 $\mu$m [2,3], but in some experiments on these systems, the conductance decreased by two orders of magnitude and the estimates of the coherence-breaking length for the edge states were much lower [4,5]. The common observation for all the experiments was that the suppression of the conductance was weakly temperature dependent. A similar behavior of the conductance was observed in InAs/GaSb/AlSb heterostructures [6,7]. These facts had no satisfactory explanation so far despite a large number of theoretical papers proposing different mechanisms of electron backscattering in the edge states. A number of authors considered spin-flip scattering of electrons on magnetic impurities. A spin relaxation of magnetic impurities due to an interaction with nuclear spins was phenomenologically introduced in Ref. [8], and the authors of Ref. [9] assumed that the impurities lack the axial symmetry, and therefore the corresponding component of the total spin of the electron and the impurity is not conserved. However, this would lead to the Anderson localization of the edge states, which is not experimentally observed. Some authors also considered mechanisms of inelastic scattering on a point defect taking into account a strong electron-electron interaction and a violation of the $S_z$ symmetry in the edge states by spin-orbit coupling [10]. Apart from the scattering by isolated impurities, a capture of electrons from the edge states into the conducting regions in the bulk of the insulator was also considered in Ref. [11], where the formation of these conducting regions was attributed to the potential fluctuations because of impurity doping. But even these processes could not explain the weak temperature dependence of the resistance. In Ref. [12], the authors

suggested that the backscattering could result from a dephasing of electrons captured from the edge state into a quantum puddle with chaotically arranged scatterers, but they could not draw a definite conclusion about the relevance of this mechanism to actual experiments. Two-particle scattering on a defect with localized spin-orbit coupling in a presence of electron-electron interaction was also considered [13]. This process results in a backscattering of electrons in the edge states and a weak temperature dependence of the conductance only if the Luttinger parameter describing the electron interaction in the edge states is very close to the value $K = 1/4$. Nevertheless, there is a general opinion that the most probable reason of the observed breaking of topological protection of the edge states is the presence of structural defects and inelastic-scattering centers. This opinion is supported by the experiment [14] that revealed well-localized scattering centers.

The nonequilibrium electric noise provides important information about the processes of charge transport, which cannot be extracted from measurements of average values. Therefore, a comparison of its theoretical value with the experimental data could allow one to understand the mechanism of conductance suppression in topological insulators. So far, only several theoretical papers on the noise in topological insulators were published, and most of them considered the electron tunneling from one edge of the sample to another [15–18]. These authors neglected the scattering in the edge states themselves and these states were assumed to be noiseless. As far as we know, the noise produced by backscattering in these states was calculated only in Ref. [19], where it resulted from the hyperfine interaction of the electrons with nuclear spins in a presence of nonuniform spin-orbit coupling. The Fano factor of the calculated noise $S_I/2eI$ appeared to be larger than unity in the large-length limit. In recent experiments on the shot noise in the edge states of HgTe based topological insulators, this ratio varied between 0.1 and 0.3 depending on the sample [20]. This suggests that the theoretical model [19] is inapplicable to such systems.

In this paper, we calculate the resistance of the edge states and the nonequilibrium noise in them that result from the tunnel coupling between the edge states and charge puddles in the bulk of the insulators, which is suggested by recent experimental results [3,5,14]. These puddles are believed to form because of inhomogeneous distribution of doping impurities in the adjacent layers of material [11]. We assume that they have a continuous energy spectrum and the

motion of electrons in them is two dimensional, so that the impurity scattering combined with spin-orbit coupling may result in their temperature-independent spin relaxation via the Elliott-Yafet [21] or Overhauser mechanism [22]; see Ref. [23] for a review [24]. Hence the presence of the puddles enables backscattering of the electrons in the edge states and results in their increased resistance along with a finite shot noise in them. We calculate the noise for the two limiting cases where the energy relaxation of electrons in the puddles is either absent or very strong. By comparing the magnitude of the shot noise with the increase in the resistance, one can judge upon the relevance of this model to real topological insulators.

The paper is organized as follows. In Sec. II, we present our model and the kinetic equations for the average current and its fluctuations. In Sec. III, we consider the contribution to the resistance and noise in the absence of energy relaxation in the puddles. In Sec. IV, the opposite limit of a strong energy relaxation is considered, and Sec. V summarizes the results.

## II. MODEL AND GENERAL EQUATIONS

Consider a pair of helical edge states with linear dispersion $\varepsilon_p = |p| \, v$ connecting electron reservoirs that are kept at constant voltages $\pm V/2$. Each of the two directions of the electron momentum is locked to a definite spin projection, which is labeled by $\sigma = \pm 1$. For simplicity, the interaction between the electrons in these states is neglected. The edge states are tunnel coupled with electron or hole puddles that are formed in the bulk of the insulator because of large-scale potential fluctuations. We also assume that these puddles are sufficiently large to have a continuous spectrum and that the electrons in the puddles are also subject to a spin relaxation because of spin-orbit processes and, in general, to the energy relaxation.

The distribution functions of electrons in the edge states $f_\sigma(x,\varepsilon,t)$ obey the equations

$$\left( \frac{\partial}{\partial t} + \sigma v \frac{\partial}{\partial x} \right) f_\sigma(x,\varepsilon,t)$$
$$= -\sum_i \Gamma_i(x) \left[ f_\sigma(x,\varepsilon,t) - F_{i\sigma}(\varepsilon,t) \right], \quad (1)$$

where $x$ is the coordinate along the edge of the insulator, $\varepsilon$ is the energy, $\Gamma_i(x)$ is the rate of electron tunneling from point $x$ to the puddle $i$, and $F_{i\sigma}(\varepsilon,t)$ is the spin-dependent distribution function of electrons in the puddle $i$. As the conductance of the puddle is much higher than that of the edge states, this distribution function is spatially uniform inside it and obeys the equation

$$\frac{\partial F_{i\sigma}}{\partial t} + \frac{1}{2\pi \hbar v v_i} \int dx \, \Gamma_i(x)[F_{i\sigma}(\varepsilon,t) - f_\sigma(x,\varepsilon,t)]$$
$$+ \frac{1}{2\tau_s}(F_{i\sigma} - F_{i,-\sigma}) = I_\varepsilon(\varepsilon,t), \quad (2)$$

where $v_i$ is the density of states in puddle $i$, $\tau_s$ is the spin-relaxation time, and the collision integral $I_\varepsilon$ accounts for the energy relaxation but conserves the number of electrons with a given spin projection in the puddle. At the zero temperature, the distribution functions of electrons in the right and left reservoirs are Fermi steps, so the boundary conditions for $f_\sigma$

are

$$f_+(0,\varepsilon) = 1 - \Theta(\varepsilon - eV/2),$$
$$f_-(L,\varepsilon) = 1 - \Theta(\varepsilon + eV/2), \quad (3)$$

where $\Theta$ is the Heaviside step function and $V$ is the applied voltage. The current carried by the edge states is given by

$$I = \frac{e}{2\pi \hbar} \int_{-eV/2}^{eV/2} d\varepsilon [f_+(x,\varepsilon) - f_-(x,\varepsilon)]. \quad (4)$$

In a semiclassical system, the dynamics of fluctuations is conveniently described by a set of Langevin equations for the relevant distribution functions. These equations are derived by varying the kinetic equations for the corresponding average quantities and adding Langevin sources to the result of variation [25]. The variation of Eq. (1) with respect to $f_\sigma$ and $F_{i\sigma}$ gives

$$\left( \frac{\partial}{\partial t} + \sigma v \frac{\partial}{\partial x} \right) \delta f_\sigma = -\sum_i \Gamma_i(x)(\delta f_\sigma - \delta F_{i\sigma}) + \sum_i \delta J_{i\sigma},$$
$$(5)$$

where $\delta J_{i\sigma}(x,\varepsilon,t)$ is the Langevin source related to tunneling of electrons from point $x$ of the edge state with spin projection $\sigma$ to puddle $i$ and back. Similarly, the variation of Eq. (2) gives

$$\frac{d}{dt} \delta F_{i\sigma} + \frac{1}{2\pi \hbar v v_i} \int dx \, \Gamma_i(\delta F_{i\sigma} - \delta f_\sigma)$$
$$+ \frac{1}{2\tau_s}(\delta F_{i\sigma} - \delta F_{i,-\sigma})$$
$$= \delta I_\varepsilon - \frac{1}{2\pi \hbar v v_i} \int dx \, \delta J_{i\sigma} + \delta \mathcal{J}_{i\sigma}, \quad (6)$$

where $\delta \mathcal{J}_{i\sigma} = -\delta \mathcal{J}_{i,-\sigma}$ is the Langevin source related to spin-flip scattering. The Langevin source related to energy relaxation is omitted here because it is inessential in the limiting cases considered below. The Langevin sources $\delta J_{i\sigma}$ and $\delta \mathcal{J}_{i\sigma}$ may be treated as independent because they correspond to different scattering processes. As the scattering is assumed to be weak, it may be considered Poissonian, and the correlation functions of the Langevin sources in these equations may be written as the sums of outgoing and incoming scattering fluxes. The spectral density of tunneling-related sources is given by the well-known expression [26]

$$\langle \delta J_{i\sigma}(x,\varepsilon) \delta J_{j\sigma'}(x',\varepsilon') \rangle_\omega$$
$$= 4\pi \hbar v \, \Gamma_i(x) \delta(x - x') \delta(\varepsilon - \varepsilon') \delta_{\sigma\sigma'} \delta_{ij}$$
$$\times [f_\sigma(1 - F_{i\sigma}) + F_{i\sigma}(1 - f_\sigma)], \quad (7)$$

while the spectral density of the sources related to the spin-flip scattering in the puddles equals [27,28]

$$\langle \delta \mathcal{J}_{i\sigma}(x,\varepsilon) \delta j \mathcal{J}_{\sigma'}(x',\varepsilon') \rangle_\omega$$
$$= \frac{1}{\tau_s v} \delta(\varepsilon - \varepsilon') \delta_{ij} (-1)^{(\sigma - \sigma')/2}$$
$$\times [F_{i\sigma}(1 - F_{i,-\sigma}) + F_{i,-\sigma}(1 - F_{i\sigma})]. \quad (8)$$

The boundary conditions for the fluctuations $\delta f_\sigma$ at the ends of the edge states are

$$\delta f_+(0,\varepsilon) = \delta f_-(L,\varepsilon) = 0. \quad (9)$$

Equations (5) and (6) together with the correlation functions (7) and (8) and the boundary conditions (9) allow us to calculate the spectral density of current noise in the edge states.

## III. PURELY ELASTIC SCATTERING

First consider the case where there is no inelastic scattering of electrons in the puddles. Then the collision integral in the right-hand sides of Eqs. (2) and (6) may be omitted, and the energy dependences of the distribution functions are determined solely by the boundary conditions (9). Hence at low temperatures the distribution functions $f_\sigma$ and $F_{i\sigma}$ have the characteristic two-step [29] shape: they are equal to 1 at $\varepsilon < -eV/2$, to zero at $\varepsilon > eV/2$, and to some position-dependent but energy-independent intermediate value at $-eV/2 < \varepsilon < eV/2$. These partially occupied states give rise to a finite shot noise that does not vanish even if the conductance is strongly suppressed by the puddles.

### A. Scattering off a single puddle

First consider the case of a single puddle without energy relaxation, which is described by a single spin-dependent electron distribution $F_\sigma(\varepsilon)$ and single tunneling rate $\Gamma(x)$. In what follows, we will be interested only in the range $-eV/2 < \varepsilon < eV/2$, where $f_\sigma$ and $F_\sigma$ are different from zero and 1 and do not depend on $\varepsilon$. Introducing a new coordinate variable

$$\phi(x) = \int_0^x \frac{dx'}{v}\Gamma(x'), \qquad (10)$$

one may write the formal solution of stationary Eq. (1) for $-eV/2 < \varepsilon < eV/2$ in the form

$$\begin{aligned} f_+(\phi) &= e^{-\phi} + (1 - e^{-\phi})F_+, \\ f_-(\phi) &= (1 - e^{\phi-\phi_L})F_-, \end{aligned} \qquad (11)$$

where $\phi_L \equiv \phi(L)$ describes the total strength of the coupling between the puddle and the edge states. In terms of the new variable $\phi$, Eq. (2) may be recast in the form

$$\left(\phi_L + \frac{\pi\hbar v}{\tau_s}\right)F_\sigma - \frac{\pi\hbar v}{\tau_s}F_{-\sigma} = \int_0^{\phi_L} d\phi\, f_\sigma(\phi). \qquad (12)$$

A substitution of Eq. (11) into Eq. (12) results in a closed system of equation for $F_\sigma$ with the solution

$$F_+ = \frac{1}{2}\frac{1 + 2\eta_1}{1 + \eta_1}, \quad F_- = \frac{1}{2}\frac{1}{1 + \eta_1}, \qquad (13)$$

where $\eta_1 = (1 - e^{-\phi_L})\tau_s/2\pi\hbar v$ is the ratio of the spin-flip time and the dwell time of an electron in the puddle. Substituting these values into Eqs. (11) and making use of the expression for the current (4), one easily obtains that

$$I = \frac{e^2 V}{4\pi\hbar}\frac{1 + e^{-\phi_L} + 2\eta_1}{1 + \eta_1}. \qquad (14)$$

Hence the conductance of the system varies from $e^2/2\pi\hbar$ for a weak coupling to the puddle or slow spin relaxation in it to the minimal value of $e^2/4\pi\hbar$ for a strong coupling and fast spin relaxation (see Fig. 1).
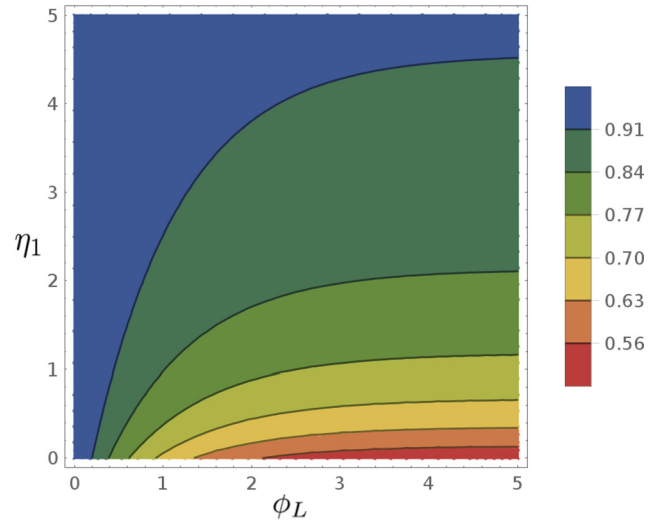


FIG. 1. Contour plot of the conductance for one puddle given by Eq. (14) in coordinates effective coupling strength $\phi_L$—normalized spin-flip time $\eta_1$. As the coupling and the spin-flip rate increase, it decreases from $e^2/2\pi\hbar$ to $e^2/4\pi\hbar$.

At low frequencies, the fluctuations of the distribution functions $f_\sigma$ are easily obtained from Eq. (5) in the form

$$\delta f_+(\phi) = (1 - e^{-\phi})\delta F_+ + \int_0^\phi d\phi' e^{\phi'-\phi}\Gamma^{-1}\delta J_+,$$

$$\delta f_-(\phi) = (1 - e^{\phi-\phi_L})\delta F_- + \int_\phi^{\phi_L} d\phi' e^{\phi-\phi'}\Gamma^{-1}\delta J_+. \qquad (15)$$

Equation (6) may be rewritten in the quasistationary case as

$$\left(\phi_L + \frac{\pi\hbar v}{\tau_s}\right)\delta F_\sigma - \frac{\pi\hbar v}{\tau_s}\delta F_{-\sigma}$$

$$= \int_0^{\phi_L} d\phi(\delta f_\sigma - \Gamma^{-1}\delta J_\sigma) + 2\pi\hbar v\, \delta\mathcal{J}_\sigma, \qquad (16)$$

and a substitution of Eqs. (15) results in a closed system of algebraic equations for $\delta F_\sigma$. Making use again of Eqs. (15) and the linearized Eq. (4), eventually one arrives at the expression for the fluctuation of the current in the form

$$\delta I = \frac{e}{1 + \eta_1}\int d\varepsilon\left[v\eta_1\delta\mathcal{J}_+ + \frac{1}{4\pi\hbar}\int_0^{\phi_L}d\phi\,\Gamma_e^{-1}\right.$$

$$\left. \times (e^{\phi-\phi_L}\delta J_+ - e^{-\phi}\delta J_-)\right]. \qquad (17)$$

Multiplying two instances of Eq. (17) and making use of the spectral densities of Langevin sources Eqs. (7) and (8), one obtains the equation for the spectral density of the noise

$$S_I = \frac{e^2}{4\pi\hbar}\frac{1}{(1 + \eta_1)^2}\int d\varepsilon\left(2\eta_1(1 - e^{-\phi_L})\right.$$

$$\times [F_+(1 - F_-) + F_-(1 - F_+)]$$

$$+ \int_0^{\phi_L}d\phi\{e^{2(\phi-\phi_L)}[f_+(1 - F_+) + F_+(1 - f_+)]$$

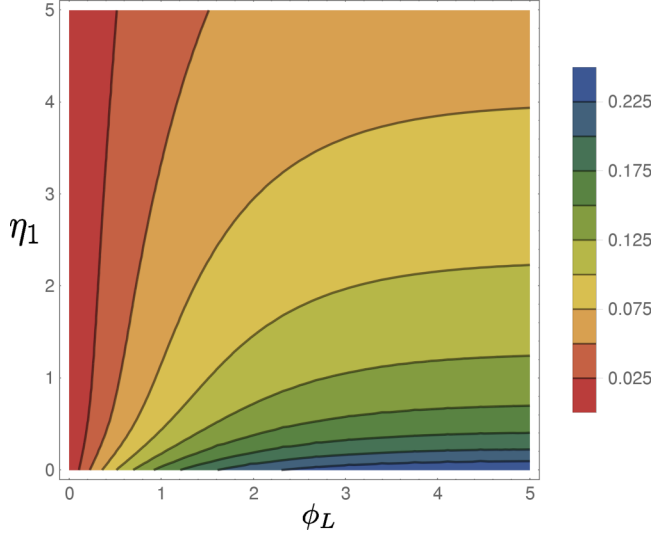$$\left. + e^{-2\phi}[f_-(1 - F_-) + F_-(1 - f_-)]\}\right). \qquad (18)$$

FIG. 2. Contour plot of Fano factor for one puddle in the absence of energy relaxation given by Eq. (19) in coordinates effective coupling strength $\phi_L$—normalized spin-flip time $\eta_1$. The maximum Fano factor corresponds to the maximum resistance of the edge states.

The Fano factor $\mathcal{F} = S_I/2eI$ is given by the expression

$$\mathcal{F} = \frac{1}{4}(1 - e^{-\phi_L})\frac{1 + e^{-\phi_L} + 4\eta_1 + 4\eta_1^2 + 4\eta_1^3}{(1 + e^{-\phi_L} + 2\eta_1)(1 + \eta_1)^3}. \quad (19)$$

The contour plot of $\mathcal{F}$ is shown in Fig. 2. It varies from zero for $\phi_L = 0$ or infinitely large $\eta_1$ to $1/4$ for strong coupling $\phi_L$ and short spin-flip times $\eta_1 = 0$. Hence the maximum Fano factor corresponds to the maximum resistance of the edge states. Note that $\mathcal{F}$ is not a unique function of the conductance.

### B. Multiple puddles in the continuous limit

Consider now the case where the edge states are weakly tunnel coupled to many conducting puddles. As the distribution functions only slightly change from one puddle to another, it is possible to go to the continuum limit and assume that the number $n$ of the puddles per unit length of the insulator edge, the density of states in the puddles $\nu$, the coupling constant

$$\Gamma(x) = \frac{1}{\Delta x}\sum_{i \in [x, x+\Delta x]}\int dy\, \Gamma_i(y),$$

and the electron distributions in the puddles are smooth functions of the coordinate $x$. Hence one may just omit the summation over the puddle number $i$ and replace $F_{i\sigma}(\varepsilon, t)$ by $F_\sigma(x, \varepsilon, t)$ in the right-hand side of Eq. (1). Along with this, one may factor out $f_\sigma$ from the integral in the left-hand side of Eq. (2) so that it becomes local in space and assumes the form

$$\frac{\partial F_\sigma}{\partial t} + \frac{1}{\tau_d}(F_\sigma - f_\sigma) + \frac{1}{2\tau_s}(F_\sigma - F_{-\sigma}) = 0, \quad (20)$$

where $\tau_d(x) = 2\pi\hbar\nu\, \nu(x)n(x)/\Gamma(x)$ is the effective dwell time of an electron in the puddle. In the stationary case, Eq. (20)
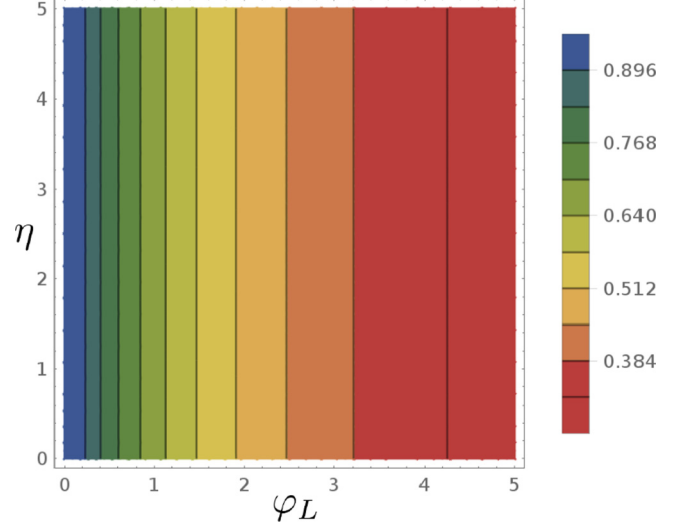


FIG. 3. Contour plot of the conductance for a continuous distribution of puddles given by Eq. (25) in coordinates effective coupling strength $\varphi_L$—normalized spin-flip time $\eta$. As the coupling and the spin-flip rate increase, it decreases from $e^2/2\pi\hbar$ to zero.

is readily solved for $F_\sigma$ giving

$$F_\sigma(x, \varepsilon) = \frac{1}{2}\frac{(1 + 2\eta)f_\sigma + f_{-\sigma}}{1 + \eta}, \quad (21)$$

where $\eta = \tau_s/\tau_d$. A substitution of these values into Eq. (1) results in a closed system of differential equations for $f_\sigma$,

$$\sigma v \frac{df_\sigma}{dx} = -\frac{1}{2}\Gamma_e \frac{f_\sigma - f_{-\sigma}}{1 + \eta}. \quad (22)$$

In terms of a new effective coordinate

$$\varphi(x) = \int_0^x \frac{dx'}{v}\frac{\Gamma(x')}{1 + \eta(x')}, \quad (23)$$

the solutions of this system may be written as

$$f_+(\varphi) = \frac{2 + \varphi_L - \varphi}{2 + \varphi_L}, \quad f_-(\varphi) = \frac{\varphi_L - \varphi}{2 + \varphi_L}, \quad (24)$$

where $\varphi_L \equiv \varphi(L)$. A substitution of these distribution functions into Eq. (4) gives

$$I = \frac{e^2 V}{\pi\hbar}\frac{1}{2 + \varphi_L}, \quad (25)$$

which suggests that the conductance of the edge states tends to zero as the number of puddles increases for any finite spin-flip time (see Fig. 3).

The Langevin equation for the fluctuation $\delta f_\sigma(x, \varepsilon, t)$ is obtained from (5) by omitting the subscript $i$ for all the quantities and replacing $\delta F_{i\sigma}$ by $\delta F_\sigma(x, \varepsilon, t)$. The spectral density of Langevin sources $\delta J_\sigma(x, \varepsilon, t)$ is obtained from Eq. (7) in a similar way.

The Langevin equation for $\delta F_\sigma(x, \varepsilon, t)$ becomes local in space and may be written as

$$\frac{\partial \delta F_\sigma}{\partial t} + \frac{1}{\tau_d}(\delta F_\sigma - \delta f_\sigma) + \frac{1}{2\tau_s}(\delta F_\sigma - \delta F_{-\sigma})$$
$$= -\frac{\delta J_\sigma(x, \varepsilon, t)}{2\pi\hbar\nu n\nu} + \delta \mathcal{J}_\sigma(x, \varepsilon, t), \quad (26)$$

whereas the spectral density of the spin-flip sources takes up the form

$$\langle \delta \mathcal{J}_\sigma(x,\varepsilon) \delta \mathcal{J}_\sigma(x',\varepsilon') \rangle_\omega$$
$$= \frac{1}{\tau_s n v} \delta(x-x')\delta(\varepsilon-\varepsilon')[F_\sigma(1-F_{-\sigma}) + F_{-\sigma}(1-F_\sigma)]. \tag{27}$$

The system of equations for $\delta f_\sigma$ and $\delta F_\sigma$ is solved in a way similar to the system of kinetic equations for $f_\sigma$ and $F_\sigma$, and the fluctuation of current is expressed in terms of the Langevin sources as

$$\delta I = \frac{e}{2\pi\hbar} \int d\varepsilon \int_0^{\varphi_L} \frac{d\varphi}{2+\varphi_L}$$
$$\times [(\delta J_+ - \delta J_-)/\Gamma + 2\tau_s \delta \mathcal{J}_+]. \tag{28}$$

Hence the expression for the spectral density of current fluctuations is of the form

$$S_I = \frac{e^2}{\pi\hbar} \frac{1}{(2+\varphi_L)^2} \int d\varepsilon \int_0^{\varphi_L} \frac{d\varphi}{1+\eta} \{ f_+(1-F_+)$$
$$+ F_+(1-f_+) + f_-(1-F_-) + F_-(1-f_-)$$
$$+ 2\eta[F_-(1-F_+) + F_+(1-F_-)] \}. \tag{29}$$

The calculation can be carried to the end only if the spatial dependence of $\eta$ is specified. For simplicity, assume that it is constant. Together with Eqs. (24) and (21), it results in a Fano factor

$$\mathcal{F} = \frac{\varphi_L}{3} [\varphi_L(\varphi_L+6)(1+\eta)^3$$
$$+ 12\eta(\eta^2+\eta+1) + 6]/[(2+\varphi_L)(1+\eta)]^3. \tag{30}$$

The contour plot of Eq. (30) is shown in Fig. 4. The Fano factor vanishes in the ballistic limit and tends to its maximum value
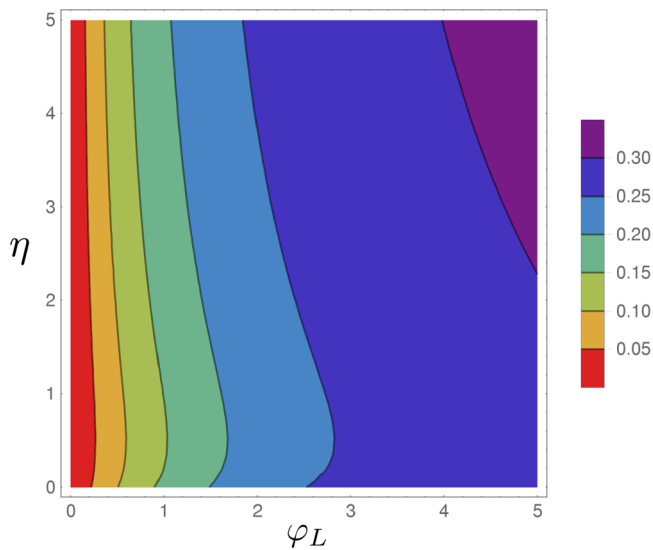


FIG. 4. Contour plot of Fano factor for a continuous distribution of puddles in the absence of energy relaxation in coordinates effective coupling strength $\varphi_L$—normalized spin-flip time $\eta$ according to Eq. (30). The Fano factor vanishes in the ballistic regime and tends to its maximum value 1/3 regardless of $\eta$ if the conductance of the edge states tends to zero.

1/3 regardless of $\eta$ if the conductance of the edge states tends to zero. This behavior is reminiscent of multimode diffusive wires [29,30].

## IV. STRONG ENERGY RELAXATION

Consider now the opposite case of strong energy relaxation in the puddles. At zero temperature, the distribution functions of electrons in the puddles have a steplike shape $F_{i\sigma}(\varepsilon,t) = \Theta(\mu_{i\sigma}-\varepsilon)$, where $\mu_{i\sigma}(t)$ is the spin-dependent chemical potential of electrons in puddle $i$. However, despite the strong energy relaxation, the shot noise in such a system is still possible because in general $\mu_{i+} \neq \mu_{i-}$ and there is a spin imbalance in the puddles.

It is convenient to introduce the excess densities of electrons in the edge states

$$\rho_\sigma(x,t) = \int \frac{d\varepsilon}{2\pi\hbar v}[f_\sigma(x,\varepsilon,t) - \Theta(-\varepsilon)]. \tag{31}$$

Integrating Eq. (1) over the energy results in an equation for $\rho_{i\sigma}(x,t)$ of the form

$$\left( \frac{\partial}{\partial t} + \sigma v \frac{\partial}{\partial x} \right) \rho_\sigma = -\sum_i \Gamma_i \left( \rho_\sigma - \frac{\mu_{i\sigma}}{2\pi\hbar v} \right), \tag{32}$$

which should be supplemented by the boundary conditions

$$\rho_+(0) = \frac{eV}{4\pi\hbar v}, \quad \rho_-(L) = -\frac{eV}{4\pi\hbar v}. \tag{33}$$

Upon integrating Eq. (6) over the energy, the inelastic collision integral drops out because the inelastic scattering conserves the total number of particles, and one arrives at the equation

$$\frac{\partial \mu_{i\sigma}}{\partial t} + \frac{1}{2\pi\hbar v v_i} \int dx \, \Gamma_i(x)(\mu_{i\sigma} - 2\pi\hbar\rho_\sigma)$$
$$+ \frac{1}{2\tau_s}(\mu_{i\sigma} - \mu_{i,-\sigma}) = 0. \tag{34}$$

Equations (32) and (34) together with the boundary conditions (33) form a complete system for determining $\rho_\sigma$ and $\mu_{i\sigma}$, and the current flowing through the edge states equals $I = ev[\rho_+(x) - \rho_-(x)]$.

As the coefficients in Eqs. (1) and (2) are assumed to be energy independent, the energy relaxation in the puddles does not affect the average current. Moreover, $\rho_\sigma(x)$ and $\mu_{i\sigma}$ may be obtained just by integrating $f_\sigma$ and $F_{i\sigma}$ obtained for the elastic case over the energy. Things are different if the spectral density of noise is considered because the correlation functions (7) and (8) are bilinear functions of $f_\sigma$ and $F_{i\sigma}$. Though the expressions for the spectral density of current noise in terms of the average distribution functions remain the same, the resulting values appear to be different.

### A. Single puddle

As in the fully elastic case, we start by considering a system with only one puddle. The average current in it is given by Eq. (14). The spin-dependent chemical potentials of electrons in the puddle may be obtained either by solving Eqs. (32) and (34) or by making use of the elastic distribution function (13) and integrating the difference $F_\sigma(x) - \Theta(-\varepsilon)$ over the energy.

This gives us

$$\mu_\sigma = \sigma \frac{eV}{2} \frac{\eta_1}{1 + \eta_1}. \tag{35}$$

The distribution functions are easily obtained by solving Eq. (1) and equal

$$f_+(\phi, \varepsilon) = \begin{cases} 0, & \varepsilon > eV/2, \\ e^{-\phi}, & \mu_+ < \varepsilon < eV/2, \\ 1, & \varepsilon < \mu_+ \end{cases} \tag{36}$$

and

$$f_-(\phi, \varepsilon) = 1 - f_+(\phi_L - \phi, -\varepsilon) \tag{37}$$

because of the electron-hole symmetry. The fluctuation of the current and the spectral density of its noise are given by Eqs. (17) and (18). The substitution of $f_\sigma$ from Eqs. (36) and (37) and $F_\sigma(\varepsilon) = \theta(\mu_\sigma - \varepsilon)$ with $\mu_\sigma$ from Eq. (35) into Eq. (18) results in a Fano factor

$$\mathcal{F} = \frac{1}{2} \frac{1 - e^{-\phi_L}}{1 + e^{-\phi_L} + 2\eta_1} \frac{e^{-\phi_L} + 2\eta_1^2}{(1 + \eta_1)^2}. \tag{38}$$

The contour plot of the Fano factor is shown in Fig. 5. It exhibits a more complicated behavior than in the elastic case and shows two separate maxima. One of them $\mathcal{F} \approx 0.086$ corresponds to the limit of fast spin relaxation $\eta_1 = 0$ and $\phi_L \approx 1$ and results from random tunneling between the puddle and the edge states. The other maximum is nearly of the same magnitude $\mathcal{F} \approx 0.09$ and corresponds to the limit of strong puddle-edge state coupling $\phi_L \to \infty$ and moderate spin relaxation $\eta_1 \approx 1.6$. It stems from random spin-flip scattering in the puddle.
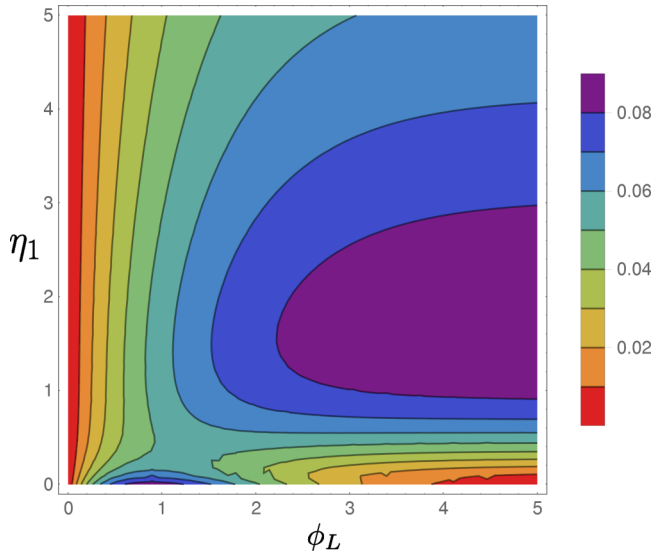


FIG. 5. Contour plot of Fano factor for a single puddle with a strong energy relaxation in coordinates effective coupling strength $\phi_L$—normalized spin-flip time $\eta_1$ according to Eq. (38). The maximum of Fano factor at $\eta_1 = 0$ results from the randomness of tunneling between the puddle and the edge states. The maximum at $\eta_1 \approx 1.6$ stems from the randomness of spin-flip scattering in the puddle.
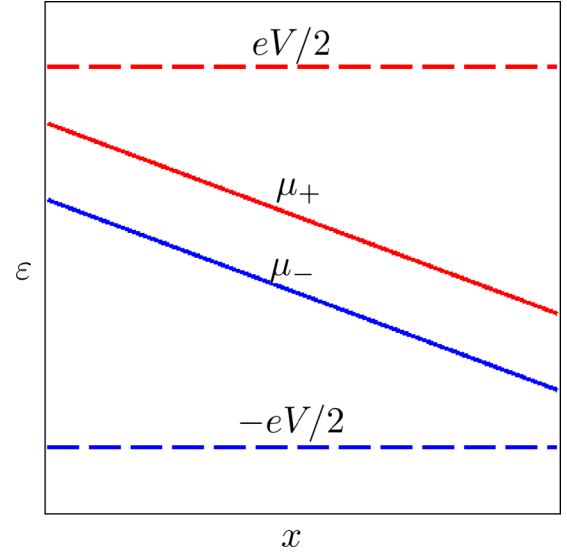


FIG. 6. Approximate coordinate dependence of the chemical potentials for spin-up and spin-down electrons in the puddles for the strong energy relaxation. There are finite jumps between the spin-dependent chemical potentials of the left and right reservoirs and potentials of the puddles.

## B. Continuous limit

If there are many puddles weakly coupled to the edge states and the energy relaxation in the puddles is strong, one may consider the continuous limit much like as in Sec. III B. To this end, we introduce the coordinate-dependent distribution function of electrons in the puddles $F_\sigma(x, \varepsilon, t) = \Theta(\mu_\sigma - \varepsilon)$, where $\mu_\sigma(x, t)$ is the local spin-dependent chemical potential of electrons in the puddle at point $x$. The excess densities of electrons in the edge states (31) obey Eq. (32) with $\mu_{i\sigma}$ replaced by $\mu_\sigma(x)$, and Eq. (34) takes up the form

$$\frac{\partial \mu_\sigma}{\partial t} + \frac{1}{\tau_d}(\mu_\sigma - 2\pi\hbar v\rho_\sigma) + \frac{1}{2\tau_s}(\mu_\sigma - \mu_{-\sigma}) = 0. \tag{39}$$

The solution of these equations is easily obtained, and in terms of the variable $\varphi$ (23), the spin-dependent chemical potentials may be presented in the form

$$\mu_\sigma(\varphi) = \frac{eV}{2} \frac{(1 + \eta)(\varphi_L - 2\varphi) + 2\sigma\eta}{(2 + \varphi_L)(1 + \eta)}. \tag{40}$$

An approximate coordinate dependence of the potentials is shown in Fig. 6. Note that there is a finite jump between the chemical potentials of the reservoir and the puddles at the left end

$$\Delta\mu = \frac{eV}{2} - \mu_+(0) = \frac{eV}{(1 + \eta)(2 + \varphi_L)} \tag{41}$$

and a similar jump at the right end.

The current is given by the same Eq. (25) as in the purely elastic case. The equation for the fluctuation of current and the expression for its spectral density are the same as Eqs. (28) and (29), but the distribution functions $f_\sigma$ and $F_\sigma$ are now different. To calculate $f_\sigma$ explicitly, one has to specify the coordinate dependence of $\eta$, and we assume it to be constant
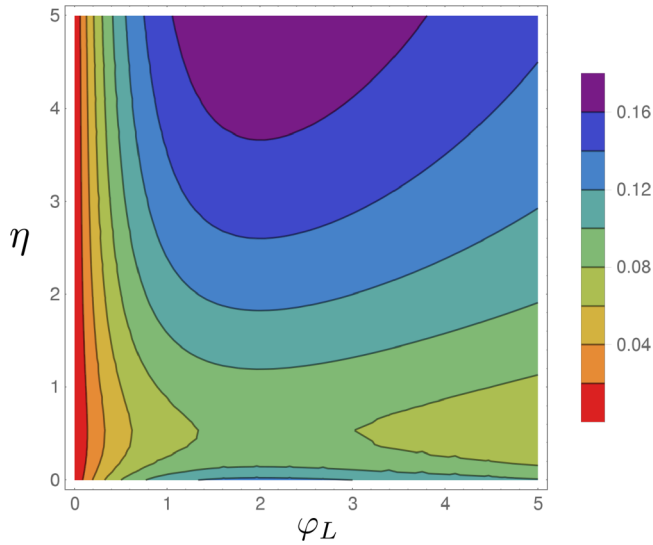
FIG. 7. Contour plot of Fano factor for a continuous distribution of puddles for a strong energy relaxation given by Eq. (43) in coordinates effective coupling strength $\varphi_L$—normalized spin-flip time $\eta$. The smaller maximum of $\mathcal{F}$ is located at $\eta = 0$, and the larger is reached at $\eta \to \infty$. $\mathcal{F}$ varies from zero to 1/4.

like in Sec. III B. Solving Eq. (1) readily gives

$$f_+ = \begin{cases} 1, & \varepsilon < \mu_+(\varphi), \\ \exp\left[\frac{\mu_+(\varphi) - \varepsilon}{\Delta\mu}\right], & \mu_+(\varphi) < \varepsilon < \mu_+(0), \\ \exp\left[\frac{\mu_+(\varphi) - \mu_+(0)}{\Delta\mu}\right], & \mu_+(0) < \varepsilon < eV/2, \\ 0, & \varepsilon > eV/2 \end{cases} \qquad (42)$$

and $f_-$ is related to $f_+$ by the electron-hole symmetry condition Eq. (37).

The resulting Fano factor equals

$$\mathcal{F} = \frac{\varphi_L}{(2 + \varphi_L)^2} \frac{1 + 2\eta^2}{(1 + \eta)^2}. \qquad (43)$$

The contour plot of this equation is shown in Fig. 7. Much like the case of a single puddle, the Fano factor exhibits two isolated maxima. Both of them correspond to $\phi_L = 2$, i.e., to the conductance $e^2/4\pi\hbar$. The smaller maximum $\mathcal{F} = 1/8$ is located at $\eta = 0$, and the larger maximum $\mathcal{F} = 1/4$ is reached at $\eta \to \infty$. Hence $\mathcal{F}$ varies from zero to 1/4, but unlike in

the elastic case, it vanishes in the limit of zero conductance regardless of $\tau_s$.

## V. CONCLUSION

In conclusion, we have calculated the conductance and shot noise of a pair of edge states in a 2D topological insulator using a semiphenomenological model of conducting puddles in the bulk of material that can exchange electrons with the edge states. We have considered two versions of this model. The first version involves one puddle with arbitrary coupling to the edge states, and the second version involved a continuum of puddles weakly coupled by tunneling to these states. The rate of spin relaxation in the puddles was assumed to be finite, and the energy relaxation in them was assumed to be either absent at all or very fast.

In the case of a single puddle without energy relaxation, the conductance decreases with increasing coupling and spin-relaxation rate from $e^2/2\pi\hbar$ to $e^2/4\pi\hbar$. Along with this, the Fano factor increases from zero to 1/4.

In the continuum limit without energy relaxation, the conductance tends to zero as the coupling and spin-relaxation rate increase, while the Fano factor increases from zero to 1/3, as in diffusive metals. One may think that in the most realistic case of several puddles strongly coupled to the edge states, $\mathcal{F}$ lies somewhere between 1/4 and 1/3.

The presence of a strong energy relaxation does not change the conductance but significantly changes the noise. The maximum values of the Fano factor are lower than in the elastic case and are now reached at intermediate values of conductance. Moreover, $\mathcal{F}$ is a nonmonotonic function of both coupling and spin-flip rate and vanishes in the limit of zero conductance.

The experimental values of the Fano factor for the edge states in HgTe topological insulators [20] vary between 0.1 and 0.3, which roughly agrees with the above model of the noise. However, to reliably distinguish between different versions of this model, one has to carefully correlate the Fano factor of the sample with its conductance, which has yet to be done.

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