Extension from memory kits to inductively derived views

J. Jude Kline · Thierry Lavendhomme · Samuel Waltener

August 24, 2017

Abstract Inductive Game Theory (IGT) was developed to study the emergence of the subjective views of individuals in a recurrent social situation. We define an extension process (EP) to go from a memory kit to an inductively derived view (i.d.view) and prove that every i.d.view is the result of some EP. We put a restriction on an EP, which we call a linking EP, to address the problem that there are a countably infinite number of i.d.views from a memory kit whenever there is at least one. We study finite existence of i.d.views obtained by linked EP's.

Keywords Inductive game theory · Inductive derivation · Extension process

JEL Classification C70 · C72 · C79

1 Introduction

Inductive Game Theory (IGT) (see Kaneko and Kline (2008a), Kaneko and Kline (2013)) was developed to study the emergence of the subjective views of individuals in a recurrent social situation. Each individual learns about the social situation by his repeated interactions in that situation. Kaneko and Kline (2013) studied the emergence of a variety of subjective views from memories. These were called *inductively derived views* (i.d.views) since they are based on the inductive learning from repeated plays of the game. In the present paper, we consider a step by step process called an *extension Process* (EP) for constructing i.d.views. It is shown that every i.d.view is obtained as the result of some EP.

When someone is involved in a new social situation occurring several times, he tries to represent it by a subjective view of the situation. This view helps to improve his understanding of the world around him and to optimize his future behavior in the recurrent social situation.

We start by giving some remarks on the use of an *information protocol* and *memory function* introduced in (Kaneko and Kline 2008b) for inductive game theory and explain why the standard extensive form game is insufficient for our purposes.

The authors thank Mamoru Kaneko and participants to the 17th SAET conference for helpful comments on an earlier version of this paper.

J.J. Kline
School of Economics, University of Queensland, Brisbane, QLD4072, Australia
E-mail: j.kline@uq.edu.au

Th. Lavendhomme
Faculty of Economics, Social and Political Sciences and Communication, Université Saint-Louis - Bruxelles, Bruxelles, B-1000, Belgium
E-mail: thierry.lavendhomme@usaintlouis.be

S. Waltener
Faculty of Economics, Social and Political Sciences and Communication, Université Saint-Louis - Bruxelles, Bruxelles, B-1000, Belgium
E-mail: samuel.waltener@usaintlouis.be

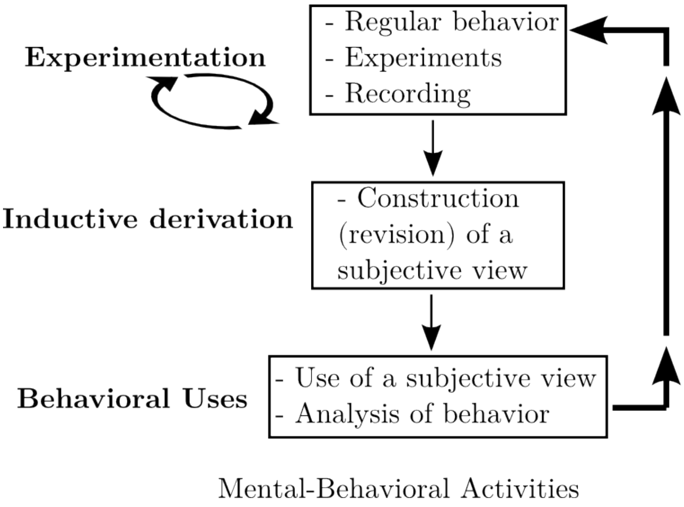

Fig. 1 Stages of IGT (adapted from Fig. 1 in Kaneko and Kline (2013))

First, the i.d.view obtained by an individual depends upon his experiences and memories from repeated interactions in the given social situation. In IGT, we presume that a player does not know the structure of the social situation *ex ante*. Nevertheless, he constructs his view (understanding) of that situation from his memories of his trials and errors over repeated interactions. An information protocol is consistent with this approach since a player receives only pieces of information while interacting in the social situation. These pieces need not contain details about the entire structure of the social situation. The standard theory of extensive games, on the other hand, implicitly presumes knowledge of the underlying game tree describing the social situation. We are interested in precisely how and what a player learns about this structure from his experiences. For this purpose, information protocols are more appropriate than extensive form games.

Second, the approach of IGT treats two types of memories and distinguishes these from the information transmission described by an information partition. The first type of memories are local "short term" memories obtained while interacting in the social situation. These are experienced as the output of his objective memory function. This should be interpreted only as a mechanism describing his local memory at various points in the course of one round of the social situation. The mechanism is not known to him at all. The second type of memory consists of "long term" memories which are the ones that remain in his mind and are described by his *memory kit*. In contrast, the theory of extensive form games mixes information transmission and memories of a player in the information partition. The memory function and the memory kit are additional components to an extensive game. In this paper, we start with the memory kit of a player and discuss how possibly he might construct his i.d.view from that memory kit.

Before summarizing the results of this paper, we give some general comments on the basic scenario of IGT. Kaneko and Kline (2008a) divided the dynamics of a basic scenario for IGT into three main stages (Fig. 1). During the stage of experimentation, the player collects memories from his repeated interactions in the social situation. In the second stage, the player uses his memories to inductively derive his subjective view. This derivation is inductive[1] in the sense that it is based on his collected memories of past experiences. In the third stage, the player uses his subjective view to adjust his behavior in future. Later, he may return to the stage of experimentation, followed by an update of this i.d.view and may repeat the same cycle.

Some parts of the stages of this general scenario have already been explored in more detail. For example, Kaneko et al (2012) analyzed the accumulation of memories from experimentation and the resulting memory kit in the example of Mike's bike commuting. Kaneko and Kline (2013)

[1] In this context, the term "induction" has to be understood as the scientific induction and thus a process of deriving a general statement from a finite number of observations Kaneko and Kline (2008a).

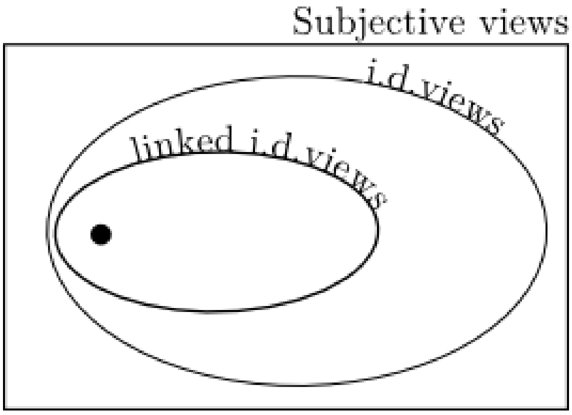

Fig. 2 Inductive processes from Kaneko and Kline (2008b) to this present paper

studied the behavioral use of i.d.views, in particular, the behavioral checking of i.d.views and the effect on decision making.

In the present paper, we focus on the inductive derivation phase from the memory kit to the i.d.view. Kaneko and Kline (2008b) looked first at players having exact and perfect recall memory abilities. In this case, the memory kit of the player already defines a unique subjective i.d.view of the social situation. In Fig. 2, this uniqueness result is symbolised by a point. Kaneko and Kline (2013) next considered i.d.views for the case of partial memories. For this, the notion of an i.d.view had to be extended. The extension guarantees existence, but generates a great multiplicity of views as shown in Fig. 2. The authors interpreted this variety of views as having the potential to describe the diversity of beliefs found in society. As IGT is concerned with bounded rationality, the authors focussed on small i.d.views and introduced the notion of minimal i.d.views. While this approach leads to a finite set of i.d.views, the authors noted (on page 31 of Kaneko and Kline (2013)) that, in the presence of weak memory, minimal i.d.views may not capture essential structures for decision making since those views are typically too small.

We take an alternative approach to dealing with the multiplicity of i.d.views. We first develop an explicit process, called an EP, for extending the memory kit of a player to an i.d.view. We prove that every i.d.view can be obtained from some EP. Next, we consider a restriction on an EP that the memories be linked in a certain sense before they can be used in the EP. This leads to the notion of a *linking* EP which reduces the set of i.d.views to a set of i.d.views called *linked i.d.views* (see Fig. 2). We prove that acyclicity of the memory kit is necessary and sufficient for the set of linked i.d.views to be finite, when at least one exists. We also provide a necessary and sufficient condition on the memory kit for existence of a linked i.d.view. Finally, we study conditions directly on the objective view to ensure the set of linked i.d.views from the memory kit is finite and non-empty.

The remainder of the paper is organized as follows. Sect. 2 explains basic notions of IGT and presents basic results. Sect. 3 presents EP's and shows that they can be used to obtain the full set of i.d.views. Sect. 4 explores linked EP's and studies when an i.d.view can be obtained from a linked EP and when the set of linked i.d.views is finite. Sect. 5 explores sufficient conditions directly on the objective view for existence and finiteness the set of i.d.views obtained by linking EP's. Sect. 6 concludes.

2 Inductive Game Theory and Views

IGT considers two types of views: objective views and subjective views. An objective view describes the actual social situation. A subjective view describes a player's construction of the social situation based on his limited experiences and memories.

An objective view consists of two parts. The first part is called an information protocol which describes the information transmission and available actions of the players interacting in the social situation. It corresponds to an extensive game of Kuhn (1953) and is required to satisfy two basic axioms and three non-basic axioms. The second part is called a memory function. It describes the "short term" local memories of a player over a particular interaction in the social situation.

A subjective view corresponds to a player's understanding of the social situation. It is also described by an information protocol and a memory function. However, since it is based on limited experiences and memories of a player, it is required to satisfy only the two basic axioms. Consequently, it may not correspond to an extensive game of Kuhn (1953), but rather to a weaker form of extensive game which is explained in Kaneko and Kline (2008a) and Kaneko and Kline (2008b).

The connection between an objective and a subjective view in IGT is described as follows. The basic ingredients for forming a subjective view are the sequences of information pieces and actions collected as memories of the player over his various encounters in the objective view. This collection of memories is described as a memory kit. The subjective view is inductively constructed by a player from his memory kit. The definition of an i.d.view contains coherency restrictions between the subjective view, memory kit, and the objective view.

In Sect. 2.1 we define objective and subjective views. In Sect. 2.2 we define memory kits and i.d.views followed by some basic results.

Before proceeding, we have one remark. IGT is developed to consider interactive situations between multiple player including how the views develop and how they affect the behavior of the participants. The present paper focuses on the inductive derivation of a view of a single player. As such, for simplicity, we do not introduce the player assignment or payoff assignment. These additional elements can be added without affecting any results of the present paper.

2.1 Information Protocols and Memory functions

We start by describing an information protocol from Kaneko and Kline (2013). It is based on a set of information pieces W and a set of available actions A and is formulated as a causality relation $\prec$ describing the flow of information pieces/actions to the participants in a particular realization of the social situation.

Definition 2.1 An *information protocol* is a triple $\Pi = (W, A, \prec)$, where

- **IP1:** W is a finite nonempty set of *information pieces*;
- **IP2:** A is a finite (possibly empty) set of *actions*;
- **IP3:** The set of *feasible sequences* $\prec$ is a finite nonempty subset of $\bigcup_{m=0}^{\infty}((W \times A)^m \times W)$ where[2] every $w \in W$ and $a \in A$ occur in some sequence in $\prec$.

This definition differs from Kaneko and Kline (2013) in two respects. First, we have dropped the player assignment and payoff functions. This allows us to focus on the structure of the information protocol. The player assignment and payoff functions can be added without affecting any of the results of this paper. Second, in the present definition of IP2, we have allowed the set of actions A to be possibly empty. This change is more substantial as it guarantees the existence of an i.d.view for general memory functions correcting a mistake in the existence proof of Kaneko and Kline (2013).

We sometimes write $[(w_1, a_1), \ldots, (w_n, a_n)] \prec w_{n+1}$ for a feasible sequence $\langle (w_1, a_1), \ldots, (w_n, a_n), w_{n+1} \rangle \in \prec$. A generic sequence in $\bigcup_{m=0}^{\infty}((W \times A)^m \times W)$ is denoted by $\langle \xi, w \rangle$. By the stipulation in IP3 that each information piece and each action occurs in some feasible sequence, an information protocol $\Pi = (W, A, \prec)$ is fully determined by the set $\prec$ of its feasible sequences. Consequently, we will sometimes refer to an information protocol as the one uniquely determined by $\prec$.

[2] The set $(W \times A)^0 \times W$ is stipulated to be W.

Definition 2.2 Let $\Pi = (W, A, \prec)$ be an information protocol and let w be an information piece in W:

1. **(Decision/End Pieces):** w is called a *decision piece* iff w occurs as an information piece in $[(w_1, a_1), \ldots, (w_m, a_m)]$ for some feasible sequence $\langle(w_1, a_1), \ldots, (w_m, a_m), w_{m+1}\rangle \in\prec$. Otherwise, w is called an *end piece*. We denote the set of decision pieces by W^D and the set of end pieces by $W^E = W \setminus W^D$;
2. **(Available Actions):** The set of *available actions* at w is the set $A_w := \{a \in A : (w, a)$ appears in some feasible sequence in $\prec\}$.

Definition 2.2 partitions the set of information pieces into decision pieces and end pieces. The set of decision pieces are received by a player before he takes some action. The set of end pieces are where the game ends. The set of available actions is defined at each information piece so that the set will be non-empty if and only if the information piece is a decision piece. We follow the convention of using u, v, w for decision pieces and z for end pieces.

To add more structure to an information protocol, we will make use of the notions of a "subsequence" and "supersequence" of a sequence. For this, let $\langle\xi, w\rangle = \langle(w_1, a_1), \ldots, (w_m, a_m), w_{m+1}\rangle$ and $\langle\eta, u\rangle = \langle(u_1, b_1), \ldots, (u_n, b_n), u_{n+1}\rangle$ be two sequences taken from $\bigcup_{m=0}^{\infty}((W \times A)^m \times W)$.

Definition 2.3 We say that $\langle\xi, w\rangle$ is a *subsequence* of $\langle\eta, u\rangle$ iff $[(w_1, a_1), \ldots, (w_m, a_m), (w_{m+1}, a)]$ is a subsequence (in the standard sense) of $[(u_1, b_1), \ldots, (u_n, b_n), (u_{n+1}, b)]$ for some a and b. In the dual manner, we say that $\langle\xi, w\rangle$ is a *supersequence* of $\langle\eta, u\rangle$ iff $\langle\eta, u\rangle$ is a subsequence of $\langle\xi, w\rangle$.

When S is a set of sequences taken from $\bigcup_{m=0}^{\infty}((W \times A)^m \times W)$, we use $\Delta(S)$ to denote the set of subsequences of S. The next lemma describes a basic property of Δ that will be used from time to time. The proof is straightforward.

Lemma 2.1 *Let S and T be finite non-empty subsets of $\bigcup_{m=0}^{\infty}((W \times A)^m \times W)$. Then, $\Delta(S) \bigcup \Delta(T) = \Delta(S \bigcup T)$.*

The following notions play key roles in our analysis. For most of this paper, we will take S to be the set of feasible sequences $\prec$ from an information protocol $\Pi = (W, A, \prec)$. However, it helps to have the general notions for an arbitrary set of sequences S in some parts of the analysis.

Definition 2.4 (Maximal sequences, Initial segments, and Positions) Let S be a finite set of sequences.

1. A sequence $\langle\xi, w\rangle$ in S is said to be *maximal* in S iff S contains no proper supersequence of $\langle\xi, w\rangle$.
2. An *initial segment* of a sequence $\langle(w_1, a_1), \ldots, (w_m, a_m), w_{m+1}\rangle$ is $\langle(w_1, a_1), \ldots, (w_k, a_k), w_{k+1}\rangle$ for some $k \leq m$.
3. A *position* $\langle\xi, w\rangle$ is an initial segment of a maximal sequence $\langle\eta, v\rangle$ in S.

When S is the set of feasible sequences $\prec$ from an information protocol $\Pi = (W, A, \prec)$, a position corresponds to an exhaustive history of information pieces and actions occurring in some realization of the social situation before reaching an information piece. We denote the set of positions in an information protocol by Ξ. Consider a position $\langle\xi, w\rangle$ in Ξ. This position ends with either a decision piece or an end piece. In the former case, the position $\langle\xi, w\rangle$ is called a *decision position*. In the latter case, it is called an *end position*. We use Ξ^E and Ξ^D to denote the sets of end positions and decision positions in $\Pi = (W, A, \prec)$.

Kaneko and Kline (2008b, 2013) gave two basic axioms and three non-basic axioms for an information protocol $\Pi = (W, A, \prec)$.

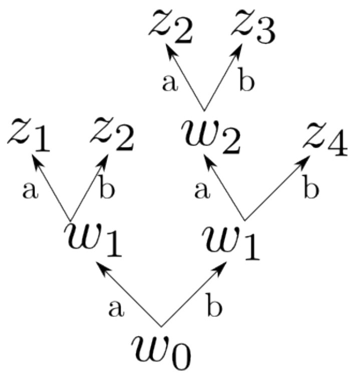

Fig. 3 Objective protocol

Axioms for Information Protocols :

- **B1** (*Subsequence-closedness*) $\prec = \Delta(\prec)$.
- **B2** (*Weak extension*), If $\xi \prec w$ and $w \in W^D$, then there are $a \in A$ and $v \in W$ such that $\langle \xi, (w, a), v \rangle \in \prec$
- **N1** (*Root*) There is a distinguished element $w^0 \in W$ such that $\langle w^0 \rangle$ is an initial segment of every position;
- **N2** (*Determination*) Let $\langle \xi, u \rangle$ and $\langle \eta, v \rangle$ be positions. If $\xi = \eta$ and are nonempty, then $u = v$;
- **N3** (*History-Independent Extension*) If $\langle \xi, w \rangle$ is a position and $[(w, a)] \prec u$, then $\langle \xi, (w, a), v \rangle$ is a position for some $v \in W$.

The following result from Kaneko and Kline (2013) (Lemma 3.1) states that every information protocol satisfying axioms B1 and B2 can be expressed in terms of its end positions.

Lemma 2.2 *If $\Pi = (W, A, \prec)$ is an information protocol satisfying Axiom B1, then $\prec = \Delta(\Xi)$. Moreover, $\prec = \Delta(\Xi^E)$ whenever Axiom B2 holds as well.*

The information protocol describes the first component of a view. We start with one example that will be used throughout the paper.

Example 2.1 (Information Protocol): Let $\Pi = (W, A, \prec)$ be an information protocol defined by: $W = \{w_0, w_1, w_2, z_1, z_2, z_3, z_4\}$, $A = \{a, b\}$, $\prec = \Delta(\Xi^E)$ where

$$\begin{aligned} \Xi^E = \{ & \langle (w_0, a), (w_1, a), z_1 \rangle, \langle (w_0, a), (w_1, b), z_2 \rangle, \langle (w_0, b), (w_1, a), (w_2, a), z_2 \rangle, \\ & \langle (w_0, b), (w_1, a), (w_2, b), z_3 \rangle, \langle (w_0, b), (w_1, b), z_4 \rangle \} \end{aligned} \tag{1}$$

Here, we use the properties of Lemma 2.2 to simplify the description. A depiction of the information protocol is illustrated in Fig. 3.

The reader can verify that this information protocol satisfies all the axioms. If we delete the end position $\langle (w_0, b), (w_1, b), z_4 \rangle$, then we obtain an information protocol that satisfies all the axioms except N3. If, alternatively, if we modify this end position to be $\langle (w_0, b), (w_1, a), z_4 \rangle$, we obtain an information protocol that violates both axioms N2 and N3.

For the objective view, the second component in Kaneko and Kline (2013, 2008b) consists of a set of memory functions, one for each player in the game. In the present paper, since we focus on the inductive derivations of a single player, we only include a single memory function for the objective view.

Definition 2.5 Let $\Pi = (W, A, \prec)$ be a basic information protocol. A *memory function* m assigns a finite sequence $\langle \eta, v \rangle = \langle (v_1, b_1), \ldots, (v_m, b_m), v \rangle$ to each position $\langle \xi, u \rangle$ in the domain Y where:

1. $\Xi^E \subseteq Y \subseteq \Xi$;

2. $v = u$;
3. $m \geq 0$ and $v_t \in W$, $b_t \in A_{v_t}$ for all $t = 1, \ldots, m$

Item 1 in Definition 2.5 requires the domain Y of the memory function to be some subset of the positions that includes the end positions.[3]. When the player arrives at some position $\langle \xi, u \rangle$ in the domain Y of his memory function, he has the local memory[4] $m\langle \xi, u \rangle = \langle \eta, v \rangle$. We will refer to the value of the memory function at a position as a *memory thread* and the position where the memory occurs $\langle \xi, u \rangle$ as the *current position*. Items 2 and 3 are coherency conditions. Item 2 requires the player to have a correct memory of at least the last information piece of the current position. Item 3 requires the information pieces to occur in the information protocol and the actions in his memory to be available at the corresponding information pieces in his memory thread.

This definition of a memory function allows for partiality of memories even including falsities. Recall that IGT is concerned with bounded rationality which allows for forgetfulness including false memories.

We use two examples of memory functions as illustrations. The first memory function is a benchmark case. It describes perfect memories. The second memory function contains false memories.

Definition 2.6 The *perfect information memory function* m^{PI} is defined over the full domain of positions $Y = \Xi$ by

$$m^{PI}\langle \xi, w \rangle = \langle \xi, w \rangle.$$

The perfect information memory function describes the complete and perfect memory of the history at each position. If a player experiences the full domain of positions by playing enough, and keeps all those memories, then he may be expected to inductively derive the objective information protocol for his subjective view.

In general, however, the memory function will be partial and his subjective view may contain partial elements and falsities. We provide an example now for the information protocol from Fig. 3.

Example 2.2 (Incorrect Recall of Actions) Consider the information protocol of Fig. 3.

Let the domain of the memory function be $Y = \{\langle \xi, w \rangle \in \Xi : w \neq w_2\}$, and let the memory function be defined by $m\langle \xi, w_1 \rangle = \langle (w_0, a), w_1 \rangle$ and $m\langle \xi, w \rangle = \langle \xi, w \rangle$ for all $w \neq w_1$.

With this memory function, the player has perfect memory except at the position ending in w_1 that occurs after action b is taken at w_0. At this position, he recalls, incorrectly, that action a was taken. Here we see how a memory function can introduce some falsity.

Definition 2.7 (Objective and Subjective Views).

1. An *objective view* is a pair (Π^o, m^o) where $\Pi^o = (W^o, A^o, \prec^o)$ is an information protocol satisfying all the basic and non-basic axioms B1, B2, N1, N2 and N3, and m^o is a memory function.
2. A *subjective view* is a pair (Π, m) where $\Pi = (W, A, \prec)$ is an information protocol satisfying the basic axioms B1 and B2 and m is a memory function.

Both views consist of an information protocol and a memory function. The main difference is in terms of the axioms required of the view. In the next section we will describe how the two views are connected.

[3] This differs superficially from Kaneko and Kline (2013) where the domain was required to also include the decision positions of the player. Since we have suppressed the player assignment in this paper, we do not include this additional restriction. If we have the player assignment, the additional restriction can be applied and the results of this paper are unaffected.

[4] When there is no risk of confusion we will omit the parentheses and simply write $m\langle \xi, u \rangle$ for $m(\langle \xi, u \rangle)$.

2.2 Memory Kits and inductively derived views

In IGT, the subjective view is obtained after the accumulation of memories of a player in the objective view. In the present paper, we do not model this accumulation process explicitly as was done in Kaneko et al (2012). Rather, we focus on the resulting set of memory threads from this accumulation process.

Definition 2.8 Let (Π^o, m^o) be an objective view. $D \subseteq Y$ is called a *domain of accumulation* iff D contains at least one end position.

The domain of accumulation is the set of positions in the objective protocol that the player observes during the phase of experimentation described in Fig. 1. For some results of this paper, we focus on domains of accumulation that are closed in a certain sense to be defined presently.

Definition 2.9 A domain of accumulation D is a *closed domain* iff for some non-empty subset $S \subseteq \Xi^E$

$$D = \bigcup_{\langle \xi, w\rangle \in S} \{\langle \eta, v\rangle : \langle \eta, v\rangle \text{ is an initial segment of } \langle \xi, w\rangle\}. \tag{2}$$

When the set S is a singleton, we refer to the domain of accumulation as a *cane domain*. If we take the union of the local memories over the set of positions in a domain of accumulation D, we obtain a memory kit. This set of memories describes the player's long term memory after the stage of experimentation described in Fig. 1.

Definition 2.10 The *memory kit* obtained from the domain of accumulation D is the set $T_D = \{m^o\langle \xi, w\rangle : \langle \xi, w\rangle \in D\}$.

It is important to emphasize that the local memories are accumulated without the order in which they appeared. This assumption is made by Kaneko and Kline (2008b, 2013). It entails some implicit assumption of bounded rationality on the part of the individual. He simply cannot include all the details of the timing of local memories. Rather he focuses only on the timing of information received within one play of the social situation. This approach denies the type of memory used in repeated game theory which allows the entire sequence of rounds of memories to be recorded with a time parameter.

We take the set of accumulated long term memories in the memory kit as the basic ingredients for an i.d.view. Following Kaneko and Kline (2008b, 2013) we presume that:

M1: the player can read all objectively available actions at any information piece in his memory kit.

M1 is a working assumption that allows us to avoid explicitly writing out the set of available actions in a memory thread. It is used to control the set of subjective views that are allowed from a memory kit. The set of allowed subjective views are those that satisfy some consistency requirements involving the memory kit and the objective view. These subjective views, which are called i.d.views will be defined presently.

Definition 2.11 An *i.d.view* from the memory kit T_D is a subjective view (Π, m) that satisfies:

- **ID1 :** $W = W^{T_D} \equiv \{w \in W^o : w \text{ occurs in some sequence in } T_D\}$, $W^D \subseteq W^{oD}$ and $W^E \subseteq W^{oE}$;
- **ID2 :** $A_w \subseteq A^o_w$ for each $w \in W$;
- **ID3 :** $\Delta(T_D) \subseteq \prec$.
- **ID4 :** m is the perfect information memory function m^{PI} for Π

Since we require an i.d.view to be a subjective view, it must satisfy Axioms B1 and B2. Conditions ID1, ID2, and ID3 are the consistency conditions mentioned above. The first part of ID1 requires that the set of information pieces W in the i.d.view are exactly those that appear in the memory kit. The second part of ID1 requires that the player does not mix up the decision pieces and end pieces in his i.d.view. ID2 requires that the available actions at an information piece w in his i.d.view are objectively available at w. This requirement makes sense in the presence of M1. ID3 requires that all memory threads in the memory kit appear in the causality relation of the i.d.view. Finally, we require by ID4 that the player has the perfect information memory function. This make sense since the subjective view is in the mind of the player. The same assumption was taken in Kaneko and Kline (2013).

To present some basic results from Kaneko and Kline (2013) we introduce the following notions.

Definition 2.12 (T_D-based Sequences and Conservative Supersets)

1. A sequence $\langle (w_1, a_1), \ldots, (w_m, a_m), w_{m+1} \rangle$ is T_D*-based* iff each of $w_1, \ldots, w_{m+1}$ occur in $\Delta(T_D)$ and $a_t \in A^o_{w_t}$ for $t = 1, \ldots, m$.
2. A superset F of $\Delta(T_D)$ is *conservative* iff every sequence in F is T_D-based.

The notion of a conservative superset of the memory kit allowed Kaneko and Kline (2013) to state a general existence result and to characterize the existence in a simple manner. Unfortunately, the existence proof contains a mistake if we require the set of available actions to be non-empty as the following example shows.

Example 2.3 Let $\Pi^o = (W, A, \prec)$, where $W = \{w, z\}$, $A = \{b\}$ and $\prec = \{\langle w \rangle, \langle (w, b), z \rangle\}$. Let the domain of the memory function Y be the same as the domain of accumulation $D = \{\langle (w, b), z \rangle\}$ which consists of the unique end position in Π^o. Consider the memory function $m^o\langle (w, b), z \rangle = \langle z \rangle$ which generates the memory kit $T_D = \{\langle z \rangle\}$.

If we require the set of actions A in the i.d.view to be non-empty, then there is no i.d.view from the memory kit. To see the problem, observe that by ID1, the set of information pieces is a singleton, that is, $W = \{z\}$. Since z is objectively an end piece, and thus has no available action, it follows by ID2 that the available action set A_z in any i.d.view must be empty. Since the action set A of the i.d.view is required to be non-empty, there must be some action a in A. Then, by the condition IP3, a must occur in some sequence in the information protocol of the i.d.view. But this implies $a \in A_z$ which contradicts our earlier finding that A_z must be empty.

In the present paper, we allow the set of available actions in an information protocol to be empty which fixes the existence problem. The following theorem summarizes the existence Theorem 4.1 and the characterization Theorem 4.2 for i.d.views from Kaneko and Kline (2013).

Theorem 2.1 *Let (Π^o, m^o) be an objective view and let D be a domain of accumulation.*

1. *(Existence) There is an i.d.view from T_D.*
2. *(Conditions on $\prec$) Let $\prec$ be an arbitrarily given set of feasible sequences. There is an i.d.view from T_D with the set of feasible sequences $\prec$ if and only if:*
   (a) $\prec$ is a conservative superset of $\Delta(T_D)$;
   (b) $\prec = \Delta(\prec)$;
   (c) $w \in W^{oE}$ for any maximal sequence $\langle \xi, w \rangle \in \prec$.

Before moving on, we mention two important consequences of these results. First, if there is no decision piece that occurs in the memory kit T_D, then there is a unique i.d.view from T_D. Second, if there is at least one decision piece in T_D, then there are an infinite number of i.d.views from T_D.

The result of uniqueness whenever there is no decision piece in T_D follows from Theorem 2.1 part 2. When there is no decision piece, $\Delta(T_D)$ is the *only* conservative superset of $\Delta(T_D)$ and it defines the unique i.d.view from T_D.

The result that there are an infinite number of i.d.views whenever there is at least one decision piece in the memory kit was explained in Kaneko and Kline (2013) (p. 44). We paraphrase the argument here. Consider any i.d.view from T_D and call it Π. We can create a new i.d.view from Π by adding a decision piece/action pair from T_D to the front of each maximal sequence in Π.

This great multiplicity was partially dealt with in Kaneko and Kline (2013) by introducing the notion of "minimal i.d.views". Minimality, however, had some issues in that the minimal i.d.views may be too small for adequate decision making due to partiality of the memory function. In the present paper, we take an alternative route for dealing with this multiplicity which involves restricting the ways of extending a memory kit to obtain an i.d.view.

3 Extension Process

We start our analysis with an explicit constructive extension process. We will show that this process can be used to obtain all i.d.views.

Definition 3.1 An *extension process* (EP) from a memory kit T_D is a finite sequence $\{\prec^0, \ldots, \prec^\tau\}$ such that :

1. (initial step) $\prec^0 = \Delta(T_D)$,
2. (inductive step) for each $t > 0$, there is a T_D-based sequence α^t not in $\prec^{t-1}$ such that $\prec^t = \prec^{t-1} \bigcup \Delta(\{\alpha^t\})$,

An extension process takes the subsequence closure of a memory kit and extends it by adding the subsequence closure of new T_D sequences, one by one.

Definition 3.2 An *extended memory kit* (EMK) from a memory kit T_D is the last step of an EP from T_D.

Recall the remark after Definition 2.1 that an information protocol is completely determined by its set of feasible sequences $\prec$. Since an EP starts with a subsequence closed set and unites it with a set of subsequence closed sequences, it follows by Lemma 2.1 that each step in the process is an information protocol satisfying axiom B1. However, it may not satisfy axiom B2.

Definition 3.3 An EMK $\prec$ is said to be *satiated* iff $w \in W^{oE}$ for any maximal sequence $\langle \xi, w \rangle \in \prec$.

Satiation of an EMK is precisely condition (c) of part 2 of Theorem 2.1 which characterizes when a conservative superset of T_D is an i.d.view. Conditions (a) and (b) of that theorem are already guaranteed by requirements 1 and 2 of Definition 3.1 of an i.d.view. We thus have the following result that the set of i.d.views from T_D is exactly the set of satiated EMK's from T_D.

Theorem 3.1 *(Equivalence) Let (Π^o, m^o) be an objective view and let T_D be a memory kit obtained on a domain of accumulation D. The set of i.d.views from T_D is equivalent to the set of satiated EMK's from T_D.*

Proof (Only-if part) Let $\prec$ be an i.d.view from T_D. Since every i.d.view is a subjective view, $\prec$ must satisfy Axioms B1 and B2 and Ξ^E must be a finite non-empty set. By Lemma 2.2, $\prec = \Delta(\Xi^E)$.

Consider the set of end positions in $\prec$ that are not in $\Delta(T_D)$, which is denoted by $\Xi^E \setminus \Delta(T_D)$. If this set is empty, then $\prec = \Delta(T_D)$ and the process $\{\prec^0\}$ is an EP that generates the EMK $\prec$. Hence, by Theorem 2.1 (only-if part of part 2), $\prec$ is a satiated EMK.

If $\Xi^E \setminus \Delta(T_D)$ is non-empty, then we can enumerate it's elements as $\langle \xi^1, w^1 \rangle, \ldots, \langle \xi^\tau, w^\tau \rangle$ for some $\tau > 0$. Let $\prec^0 = \Delta(T_D)$, and for each t such that $0 < t \leq \tau$ inductively define $\prec^t = \prec^{(t-1)} \bigcup \Delta(\{\langle \xi^t, w^t \rangle\})$. By construction, $\{\prec^0, \ldots, \prec^\tau\}$ is an EP. It follows from Lemma 2.1 that this EP generates $\prec$. Finally, by Theorem 2.1 (only-if part of part 2), $\prec$ is a satiated EMK.

(If part) Let $\prec$ be a satiated EMK. By Definition 3.3 and repeated applications of Lemma 2.1, $\prec$ is seen to be a conservative superset of T_D satisfying Axiom B1. By the fact that the EMK obtained is satiated, it follows from Theorem 2.1(if-part of part 2) that there is an i.d.view from T_D with $\prec$ as the set of feasible sequences. $\square$

We end this section with an example based on the objective protocol of Fig. 3. It shows how a diversity of views can arise even with a simple cane domain.

Example 3.1 We consider the cane domain defined by the end position $\langle (w_0, a), (w_1, a), z_1 \rangle$ on the left most side of Fig. 3. We presume the player recalls correctly at each position only the previous information piece and action if they exist. The resulting memory kit is expressed by $T_D = \{\langle w_0 \rangle, \langle (w_0, a), w_1 \rangle, \langle (w_1, a), z_1 \rangle\}$.

One can construct cane domain of accumulation as a satiated EMK by the following EP denoted by A. The initial step of the EP is $\prec^0 = \Delta(T_D)$. The first and last inductive step of the EP A is $\alpha_A^1 = \langle (w_0, a), (w_1, a), z_1 \rangle$. Then,

$$\prec_A^1 = \prec^0 \cup \Delta(\{\langle (w_0, a), (w_1, a), z_1 \rangle\}) \tag{3}$$

The EMK $\prec_A^1$ is satiated and thus, by Theorem 3.1, it is an i.d.view (Fig. 4a).

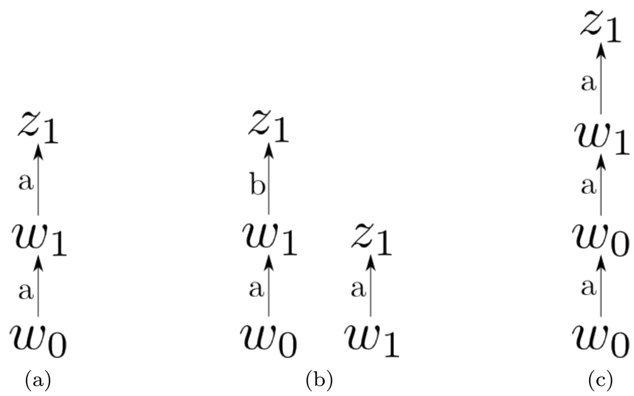

Fig. 4: Examples of i.d.views

Consider the alternative EP denoted by B which is formed by adding the T_D-sequence $\langle (w_0, a), (w_1, b), z_1 \rangle$ to $\Delta(T_D$ as the first and only inductive step. This results in the different satiated EMK:

$$\prec_B^1 = \prec^0 \cup \Delta(\{\langle (w_0, a), (w_1, b), z_1 \rangle\}) \tag{4}$$

Action b was not experienced by the player over this domain. Nevertheless, the sequence $\langle (w_0, a), (w_1, b), z_1 \rangle$ is a T_D-based sequence since $b \in A_{w_1}^o$ and by M1 the player can see b is written on w_1. The EMK $\prec_B^1$ is also satiated and it describes a different i.d.view from $\prec_A^1$.

As noted in Kaneko and Kline (2013), we encounter an infinite number of possible i.d.views whenever there is at least one. This can be seen here by extending $\prec_A^1$, using the T_D-sequence $\langle (w_0, a), (w_0, a), (w_1, a), z_1 \rangle$ to obtain:

$$\prec^2 = \prec_A^1 \cup \Delta(\{\langle (w_0, a), (w_0, a), (w_1, a), z_1 \rangle\}) \tag{5}$$

Even though the player has not memorized that w_0 is a successor of w_0, $\langle (w_0, a), (w_0, a), (w_1, a), z_1 \rangle$ is nevertheless a T_D-based sequence and another i.d.view is reached (Fig. 4c). By extending this memory kit further in the same manner, we can obtain an infinite number of satiated EMK's. As we will see in the next section, avoiding such repetitions is one way to restrict the number of i.d.views.

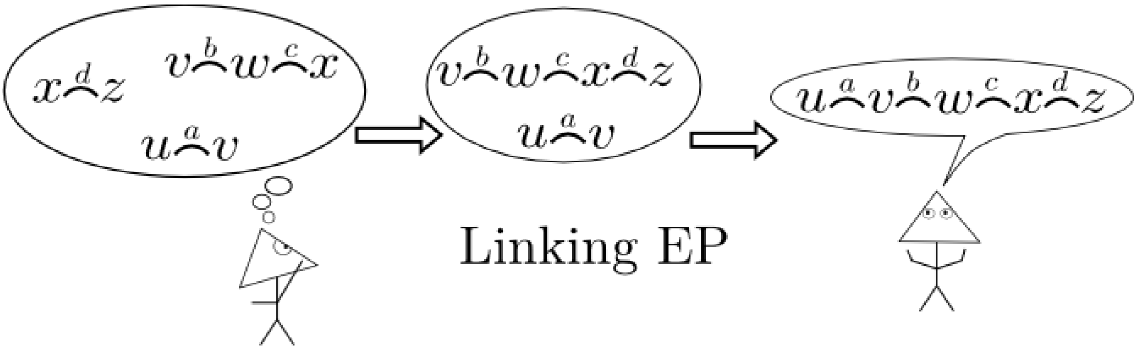

Fig. 5: Jigsaw analogy

We finish this example and section by showing that the perfect information memory function does not always reduce the set of i.d.views to a finite set. With the perfect information memory function on the same cane domain, the memory kit expands to: $T_D^* = \{\langle w_0\rangle, \langle (w_0, a), w_1\rangle, \langle (w_0, a), (w_1, a), z_1\rangle\}$. One effect of this increased memory ability is that $\Delta(T_D^*)$ is itself a satiated EMK equivalent to (3). Nevertheless, an infinite number of i.d.views can be reached by adding repetitions of w_0 in front of the sequence as above.

4 Jigsaw analogy and linking extension processes

An EP can be loosely compared to the manner in which one constructs a jigsaw puzzle. The memory kit T_D serves as a partially constructed puzzle lying on the table. Next to the puzzle is a set of jigsaw pieces which are T_D-based sequences. In each successive step of the EP, the player takes one new jigsaw piece and uses it to solve some part of the puzzle. This continues until he reaches a solution which is an i.d.view.

The person in Fig. 5 constructs a puzzle in this manner. He takes the three threads in his memory kit at the left and links them together to obtain a long memory thread at the right. This linking requirement is an additional restriction to EP's that we will explore in this section. This restriction has the potential to limit the set of i.d.views obtained to be a finite set. On the other hand, the restriction may be too strong and lead to no i.d.view.

In what follows, let S be a subset of $\bigcup_{m=0}^{\infty}((W \times A)^m \times W)$.

Definition 4.1 A triple $(w, a, v) \in W \times A \times W$ is called an *adjacent triple* in S iff there is a sequence $\langle (w_1, a_1), \ldots, (w_m, a_m), w_{m+1}\rangle \in S$ such that $(w, a, v) = (w_k, a_k, w_{k+1})$ for some $k \le m$. We denote by $\mathcal{T}(S)$, the set of adjacent triples in S.

In Fig. 5, the set of adjacent triples in the memory kit of the player is $\mathcal{T}(T_D) = \{(x, d, z), (v, b, w), (w, c, x), (u, a, v)\}$. In contrast, (v, b, z) is not an adjacent triple in $\mathcal{T}(T_D)$ even though $\langle (v, b), z\rangle$ is a T_D-based sequence. We will use the set of adjacent triples $\mathcal{T}(T_D)$ to restrict the set of T_D-based sequences that can be used to extend a memory kit.

Definition 4.2 A T_D-based sequence $\langle \xi, w\rangle$ is called *T_D-linked* iff $\mathcal{T}(\{\langle \xi, w\rangle\}) \subseteq \mathcal{T}(T_D)$.

For instance, $\langle (u, a), (v, b), w\rangle$, $\langle (w, c), (x, d), z\rangle$ and $\langle (v, b), (w, c), (x, d), z\rangle$ are three examples of T_D-linked sequences for the memory kit of Fig. 5. In contrast, the T_D-sequence $\langle (v, b), z\rangle$ mentioned after Definition 4.1 is not T_D-linked.

Definition 4.3 An EP $\{\prec^0, \ldots, \prec^\tau\}$ is called a *linking EP* iff for each $t > 0$, the sequence α^t introduced in step t is T_D-linked.

EMK's and satiated EMK's are called linked when they are generated by linking EP's. By the equivalence Theorem 3.1, every linked satiated EMK will be an i.d.view. Correspondingly, we define linked i.d.views as follows.

Definition 4.4 An i.d.view from T_D with the set of feasible sequences $\prec$ is called a *linked i.d.view* iff $\prec$ is a linked satiated EMK from T_D.

Let's return to Fig. 5. The player starts with a memory kit described by the left bubble. In the first step, he links the sequence $\langle(v,b),(w,c),x\rangle$ with the sequence $\langle(x,d),z\rangle$ using the T_D-linked sequence $\langle(v,b),(w,c),(x,d),z\rangle$. This generates the EMK in the middle bubble. We have excluded the full set of subsequences in the bubbles for simplicity. These can be generated by applying Δ to the set of sequences in each bubble. Finally, the player links the sequence he created in step 1 to the sequence $\langle(u,a),v\rangle$ using the adjacent triple (u,a,v) to obtain the long sequence in the right bubble. Under the presumption that z is the unique end piece, this constitutes the unique linked i.d.view.

We can see that the linking restriction limits the set of i.d.views that can be obtained from a memory kit. In Example 3.1, there are only two adjacent triples, (w_0,a,w_1) and (w_1,a,z_1), in T_D. There is a unique linked i.d.view which is described by (3). Here, the T_D-linked restriction allows us to avoid the repetitions that lead to an infinite number of i.d.views.

For some memory kits, however, there may be no linked i.d.view.

Example 4.1 Let a memory kit be $T_D=\{\langle(v,c),u\rangle,\langle(u,d),v\rangle,\langle z_1\rangle\}$ with z_1 the unique end piece. Both (v,c,u) and (u,d,v) are the only adjacent triples in $\mathcal{T}(T_D)$. Since no T_D-linked sequence can reach z_1 from the decision pieces u and v, there is no linked i.d.view from this memory kit T_D.

The following theorem characterizes existence in terms of the set of linked sequences in the memory kit. To state it we use the notion of a path where we will say that a sequence $s=\langle(w_1,a_1),\ldots,(w_m,a_m),w_{m+1}\rangle$ is a *path* from w_1 to w_{m+1}.

Theorem 4.1 *(Existence of a Linked View) Let T_D be a memory kit. The set of linked i.d.views from T_D is non-empty if and only if for each decision piece $w\in W^{T_D}\bigcap W^{oD}$ there is a T_D-linked sequence $\langle\xi,v\rangle$ that is a path from w to an end piece $v\in W^{oE}$.*

Proof Clearly, if the right hand side of the Theorem is not satisfied, then there will be no linked i.d.view because satiation cannot be satisfied. On the other hand, if it is satisfied, then we can construct an EP that generates a linked i.d.view as follows. First, consider the set of maximal sequences in T_D that end with a decision piece $w\in W^{T_D}\bigcap W^{oD}$. If this set is empty, then $\{\Delta(T_D)\}$ is a linking EP generating the linked i.d.view $\Delta(T_D)$.

If the set of maximal sequences in T_D that end with a decision piece $w\in W^{T_D}\bigcap W^{oD}$ is not empty, then list these sequences as $\langle\xi^1,w^1\rangle$,...,$\langle\xi^\tau,w^\tau\rangle$. Let $\langle\xi^t,w^t\rangle$ denote the t'th sequence in the list. By the right hand side of this theorem, there is a T_D-linked sequence s^t forming a path from w^t to some end piece v^t. Since $\langle\xi^t,w^t\rangle$ is a maximal sequence in T_D it must also be T_D-linked. Define α^t be the sequence obtained by concatenating s^t to the end of $\langle\xi^t,w^t\rangle$. Since α^t is the concatenation of two T_D-linked sequences, α^t is T_D-linked.

Let $\prec^0=\Delta(T_D)$, and for each $t=1,...,\tau$, inductively define $\prec^t=\prec^{t-1}\bigcup\Delta(\{\alpha^t\})$. This defines a linking EP that generates a linked i.d.view. □

Let's return to the examples to see how this theorem works. Example 4.1 is a negative result. Due to excessive forgetfulness, there is no T_D-linked sequence that is a path from the decision piece u or v to an end piece. Thus, by Theorem 4.1, there is no linked i.d.view.

As a positive result, consider Example 3.1 where the memory kit is $T_D=\{\langle w_0\rangle,\langle(w_0,a),w_1\rangle,\langle(w_1,a),z_1\rangle\}$. There are two adjacent triples (w_0,a,w_1) and (w_1,a,z_1). The only decision pieces in the memory kit are w_0 and w_1. Since, for each decision piece w, we can find a T_D-linked sequence that is a path to an end piece, it follows, by Theorem 4.1, that there is a linked i.d.view from T_D. For example, the T_D-linked sequence $\langle (w_0, a), (w_1, a), z_1\rangle$ is a path from the decision piece w_0 to the end piece z_1. As mentioned earlier, there is actually a unique linked i.d.view from T_D in the example which is expressed by (3).

In general, however, the set of linked i.d.views will not be unique and it may even be infinite as was the case for i.d.views in general. We are interested in a condition on the memory kit that will ensure the set of linked i.d.views is finite whenever it is non-empty.

Definition 4.5 A set of sequences S is *cyclic* iff there exists ℓ ($\ell \geq 1$) adjacent triples in $\mathcal{T}(S)$ denoted by $tr_{(k)} = (u_k, a_k, v_k)$ (for $k = 1, \ldots, \ell$) such that the ℓ-uple $(tr_{(1)}, \ldots, tr_{(\ell)})$ satisfies

$$v_k = u_{k+1} \text{ for each } k = 1, \ldots, \ell - 1 \text{ and } v_\ell = u_1. \tag{6}$$

We say that S is *acyclic* iff it is not cyclic.

Fig. 6 illustrates two examples of cyclic sets of sequences that represent information protocols. In Fig. 6a the same information piece E appears twice in the position $\langle (E, c), E\rangle$ and the 1-uple $(tr_{(1)})$ with $tr_{(1)} = (E, c, E)$ satisfies (6). The information protocol described in Fig. 6b is also cyclic. For instance the 2-uple $(tr_{(1)}, tr_{(2)})$ with $tr_{(1)} = (u, d, v)$ and $tr_{(2)} = (v, c, u)$ satisfies (6). Numerous examples of acyclic set of sequences has already been encountered. For instance, the memory kit of the player in Fig. 5 or the one of Example 3.1 are acyclic.

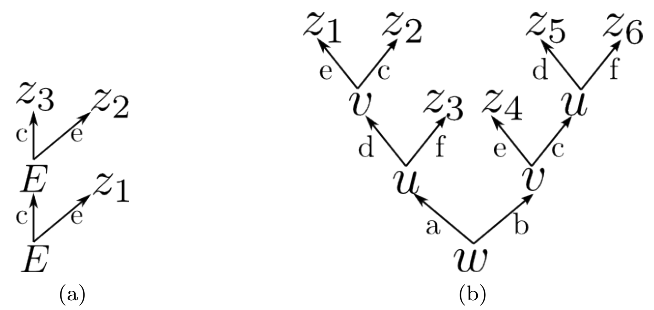

Fig. 6: Examples of cyclic information protocols

We have the following theorem.

Theorem 4.2 *Let T_D be a memory kit that admits at least one i.d.view. The set of linked i.d.views from T_D is finite if and only if T_D is acyclic.*

Proof (if) For proving the finiteness of the set of linked i.d.views, it is sufficient to prove that the set of T_D-linked sequences is finite.

Let's construct the set of T_D-linked sequences. This set is extensively described by $\bigcup_{(u,a,v)\in\mathcal{T}(T_D)} S((u, a, v))$ where $S((u, a, v))$ is defined as the set of sequences of the form $\langle (w_1, a_1), \ldots, (w_n, a_n), w_{n+1}\rangle$ such that $\langle (u, a), v\rangle$ is an initial segment of that sequence and (w_i, a_i, w_{i+1}) for $i = 1, \ldots, n$ is an adjacent triple in $\mathcal{T}(T_D)$.

Since T_D is finite and acyclic, the set of T_D-linked sequences is also finite. Since every linked i.d.view is a subset of T_D-linked sequences, the number of linked i.d.views must also be finite.

(only if) We prove the contrapositive. Let T_D be a cyclic memory kit. There exists an ℓ-tuple $(tr_{(1)}, \ldots, tr_{(\ell)}) = ((u_1, a_1, v_1), \ldots, (u_\ell, a_\ell, v_\ell))$ where $u_1 = v_\ell$ and $v_i = u_{i+1}$ for $i = 1, \ldots \ell - 1$.

By hypothesis, there exists one linked i.d.view which we denote by $\prec$. By ID1, the decision piece u_1 occurs in an end position of $\prec$. Let $\langle (w_1, b_1), \ldots, (w_m, b_m), w_{m+1}\rangle$ be this end position where

$w_t = u_1$ for some $t \in \{1, \ldots, m\}$. Then, $\alpha = \langle (w_1, b_1), \ldots, (w_t, a_1), \ldots, (u_\ell, a_\ell), (w_t, b_t), \ldots, w_{m+1} \rangle$ is a T_D- based sequence that is not in $\prec$ such that $\mathcal{T}(\{\alpha\}) \subseteq \mathcal{T}(T_D)$. As a consequence, $\prec' = \prec \bigcup \Delta(\{\alpha\})$ is a linked i.d.view that is different from $\prec$. The same argument presented above can be conducted on $\prec'$ and so on, as many times as one wishes, leading then to an infinite set of linked i.d.views. □

Let's return to the examples to see how Theorem 4.2 works. First, consider Example 3.1 where the memory kit was $T_D = \{\langle w_0 \rangle, \langle (w_0, a), w_1 \rangle, \langle (w_1, a), z_1 \rangle\}$. There are just two adjacent triples (w_0, a, w_1) and (w_1, a, z_1). This set of adjacent triples does not contain any cycle, and as already noted, there is a linked i.d.view. Hence, by Theorem 4.2, the set of linked i.d.views must be finite. We, in fact, know the set to be a singleton in this example.

In the converse direction, whenever there is a linked i.d.view and the set of adjacent triples has a cycle, we know that the set of linked i.d.views will be infinite. Consider Fig. 6a as an objective protocol Π^o. Let

$$T_D = \{\langle E \rangle, \langle (E, e), z_1 \rangle, \langle (E, c), E \rangle, \langle (E, c), (E, e), z_2 \rangle, \langle (E, c), (E, c), z_3 \rangle\}$$

be the memory kit obtained from the perfect information memory function m^{PI} on $D = \Xi^o$. It is immediately apparent that there is a cycle (E, c, E) in T_D. Hence, if there is a linked i.d.view, then, by Theorem 4.2, the set of linked i.d.views will be infinite. As it happens, ΔT_D is already a linked i.d.view and from it, we can form a infinite number of linked i.d.views by adding (E, c) to the start of each view.

Theorems 4.1 and 4.2 provide us with necessary and sufficient conditions for the set of linked i.d.views to be both non-empty and finite. Since we need acyclicity of T_D for finiteness when there is at least one linked i.d.view, we can simplify the existence condition which we present in the following lemma.

Lemma 4.1 *Let T_D be an acyclic memory kit and let $w \in W^{T_D} \bigcap W^{oD}$. The following two conditions are equivalent:*

1. *there is a T_D-linked sequence $\langle \xi, v \rangle$ that is a path from w to an end piece $v \in W^{oE}$.*
2. *there is an adjacent triple (u_1, a_1, u_2) in $\mathcal{T}(T_D)$ such that $u_1 = w$.*

Proof $(1 \Rightarrow 2)$ Let $w \in W^{T_D} \bigcap W^{oD}$. The first adjacent triple in the T_D-linked sequence $\langle \xi, v \rangle$ guaranteed by 1 fulfils the requirement of 2 for w.

$(2 \Rightarrow 1)$ Let $w \in W^{T_D} \bigcap W^{oD}$. By 2, we can create a T_D-linked sequence $\langle (u_1, a_1), u_2 \rangle$ that is a path from w to some u_2. If $u_2 \in W^{oE}$, we have satisfied 1. If $u_2 \notin W^{oE}$ then we can apply 2 to u_2 to obtain a T_D-linked sequence $\langle (u_1, a_1), (u_2, a_2), u_3 \rangle$ that is a path from w to some u_3. Since T_D is finite and acyclic, after a finite number of applications of 2, we must obtain a path satisfying 1. □

When a memory kit T_D is cyclic, the condition 2 of Lemma 4.1 is not sufficient for satisfying condition 1 which can be seen in Example 4.1.

From Lemma 4.1 and Theorems 4.1 and 4.2, we obtain the following corollary which expresses a joint necessary and sufficient condition for non-emptiness and finiteness of linked i.d.views.

Corollary 4.1 *Let T_D be a memory kit. The set of linked i.d.views from a memory kit T_D is non-empty and finite if and only if T_D is acylic and for each decision piece $w \in W^{T_D} \bigcap W^{oD}$ there is an adjacent triple (u_1, a_1, u_2) in $\mathcal{T}(T_D)$ such that $u_1 = w$.*

Proof (If-part) Since T_D is acylic, non-emptiness of linked i.d.views follows from Lemma 4.1 and the if-part of Theorem 4.1. Having obtained existence, we can apply the if-part of Theorem 4.2 to obtain finiteness.

(Only-if part) By non-emptiness and finiteness, we obtain acyclicity by Theorem 4.2. By non-emptiness and acyclicity, we can apply Theorem 4.1 and Lemma 4.1 to obtain the second condition of the right hand side of this Corollary. □

5 Conditions on the Objective view for Uniqueness and Finiteness of Linked i.d.Views

In the previous section we explored conditions on the memory kit for existence and finiteness of linked i.d.views. It would be beneficial to have conditions directly on the objective view that will generate a memory kit with a finite and non-empty set of linked i.d.views. For this, we need some new definitions.

Definition 5.1 The *information history* of a sequence $\langle \xi, w\rangle = \langle (w_1, a_1), \ldots, (w_m, a_m), w_{m+1}\rangle$ is the sequence $\theta(\langle \xi, w\rangle) = \langle w_1, ..., w_{m+1}\rangle$ of information pieces as they appear in $\langle \xi, w\rangle$.

Definition 5.2 For the information history $\theta(\langle \xi, w\rangle) = \langle w_1, ..., w_{m+1}\rangle$ of a sequence $\langle \xi, w\rangle$ and a non-negative integer k, we define the *information history of length* k by:

$$\theta(\langle \xi, w\rangle)^k = \begin{cases} \langle w_{m-k+1}, \ldots, w_{m+1}\rangle & \text{if } k \leq m \\ \theta(\langle \xi, w\rangle) & \text{if } k > m. \end{cases}$$

In what follows, let (Π^o, m^o) be an objective view and let $\langle \xi, w\rangle$ be a position in the domain Y of the memory function m^o. We will place restrictions directly on the Π^o and m^o to obtain finite existence of linked i.d.views.

We define the Y*-part* of a position in Y. It is the history related to the domain of the player's memory function.

Definition 5.3 The Y*-part* of $\langle \xi, w\rangle = \langle (w_1, a_1), \ldots, (w_m, a_m), w_{m+1}\rangle$, denoted $\langle \xi, w\rangle_Y$, is the maximal subsequence $\langle (w_{j_1}, a_{j_1}), ..., (w_{j_s}, a_{j_s}), w_{j_{s+1}}\rangle$ of $\langle \xi, w\rangle$ with the property that the initial segment of $\langle \xi, w\rangle$ up to each j_t is in Y.

Consider the objective view from Example 2.2 and the end position $\langle (w_0, b), (w_1, a), (w_2, a), z_2\rangle$. The Y-part of this position $\langle (w_0, b), (w_1, a), (w_2, a), z_2\rangle_Y = \langle (w_0, b), (w_1, a), z_2\rangle$ since $\langle (w_0, b), (w_1, a), z_2\rangle$ is not in Y.

Next, consider the following condition an objective view.

Definition 5.4 We say that (Π^o, m^o) satisfies *occurrence distinguishability (OD)* iff for any $\langle \xi, v\rangle$, $\langle \eta, w\rangle \in Y$,

$$\theta(\langle \xi, v\rangle_Y) \neq \theta(\langle \eta, w\rangle_Y) \text{ implies } v \neq w. \tag{7}$$

Occurrence distinguishability is based on the condition of occurrence memory introduced by Okada (1987) to describe a weakening of Kuhn (1953) perfect recall condition in an extensive game. Occurrence distinguishability in IGT, however, is not about memory. Instead, it is a restriction on information transmission in an information protocol since it restricts the information histories that can occur in the objective information protocol. It is weaker than Kuhn's distinguishability condition which was studied in Kaneko and Kline (2013).

The two objective protocols represented in Fig. 6 do not satisfy the OD-condition with $Y = \Xi^o$. It can be seen easily for the left objective protocol by considering the two positions $\langle E\rangle$ and $\langle (E, c), E\rangle$ where the respective information histories are different but the information piece reached is identical. Likewise, $\langle (w, b), (v, c), u\rangle$ and $\langle (w, a), u\rangle$ for the right objective protocol, violates (7).

Return now to the objective view of Example 2.2. It satisfies the OD condition. A justification may be needed for positions $\langle (w_0, a), (w_1, b), z_2\rangle$ and $\langle (w_0, b), (w_1, a), (w_2, a), z_2\rangle$. The Y-part of these positions are respectively $\langle (w_0, a), (w_1, b), z_2\rangle$ and $\langle (w_0, b), (w_1, a), z_2\rangle$ and their information histories are equal. They do, however, differ in terms of actions taken.

Next, we consider a further restriction on the player's objective memory function m^o.

Definition 5.5 The memory function m^o is said to be *informationally* Y*-correct* iff $\theta(m^o\langle \xi, w\rangle)$ is a subsequence of $\theta(\langle \xi, w\rangle_Y)$ for each $\langle \xi, w\rangle \in Y$.

A player's memory function is informationally Y-correct if he only recalls occurrences of past moves correctly, though maybe not the actions taken there. There are many memory functions that satisfy this property, since the player may forget everything, or only some part of his information history. A particular type is called an occurrence recall-k memory function. With this type of memory function, the player correctly recalls the last k information pieces in the Y-part of the information history at each position in Y.

Definition 5.6 The memory function m^o is an *occurrence recall-k* memory function iff for all $\langle \xi, w \rangle$ in Y,

$$\theta(m^o \langle \xi, w \rangle) = \theta(\langle \xi, w \rangle_Y)^k.$$

The notions of informationally Y-correct and occurrence recall-k memory functions are based on the stronger notions of full Y-correctness and recall-k memory functions developed in Kaneko and Kline (2013). The latter notions require the correct memories of both past information pieces and past actions, while the former only requires the correct memories of past information pieces. As such there may be several occurrence recall-k memory functions for each non-negative integer k. It is straightforward to show that every recall-k memory function is an occurrence recall-k memory function, while the converse is not generally true.

We have the following results on finiteness and existence of linked i.d.views which are based on conditions placed directly on the objective view.

Theorem 5.1 *Let (Π^o, m^o) be an objective view where Π^o satisfies the OD-condition and let D be a domain of accumulation.*

1. *If m^o is informationally Y-correct, then the set of linked i.d.views from T_D is finite.*
2. *If m^o is an occurrence recall-k memory function with $k \geq 1$ and D is closed, then the set of linked i.d.views from T_D is finite and non-empty.*

The proof of Theorem 5.1 is based on the following lemma.

Lemma 5.1 *Let (Π^o, m^o) be an objective view where Π^o satisfies the OD-condition. If the memory function m^o is informationally Y-correct then T_D is acyclic.*

Proof By informational correctness and the fact that the definition of acyclicity of a set does not depend on the actions in the adjacent triples, it suffices to show that $\{\langle \xi, w \rangle_Y : \langle \xi, w \rangle \in Y\}$ is acyclic.

Consider an arbitrary sequence $\{(u_t, a_t, v_t)\}_{t=1}^{\ell}$ of adjacent triples in $\{\langle \xi, w \rangle_Y : \langle \xi, w \rangle \in Y\}$ satisfying $v_t = u_{t+1}$ for $t = 1, ..., \ell - 1$. We will show that $v_\ell \neq u_1$. By definition, we can find a sequence $\{\langle \xi_t, v_t \rangle_Y\}_{t=1}^{\ell}$ of Y-parts of positions in Ξ^o such that (u_t, a_t, v_t) is the last adjacent triple in $\langle \xi_t, v_t \rangle_Y$ for each t. Notice also that each $\langle \xi_t, v_t \rangle_Y$ contains a proper initial segment $\langle \xi_t', u_t \rangle_Y$ up to u_t. Thus, for $t = 1, ..., \ell$,

$$\theta(\langle \xi_t', u_t \rangle_Y) \text{ is a proper initial segment of } \theta(\langle \xi_t, v_t \rangle)_Y). \tag{8}$$

Since, $v_t = u_{t+1}$ for all $t < \ell$ it follows from the contrapositive of OD that for each $t < \ell$:

$$\theta(\langle \xi_t, v_t \rangle_Y) = \theta(\langle \xi_{t+1}', u_{t+1} \rangle_Y). \tag{9}$$

Taken together (8) and (9) imply that for each $t < \ell$:

$$\theta(\langle \xi_t', u_t \rangle_Y) \text{ is a proper initial segment of } \theta(\langle \xi_{t+1}, v_{t+1} \rangle_Y). \tag{10}$$

By induction over t:

$$\theta(\langle \xi_1', u_1 \rangle_Y) \text{ is a proper initial segment of } \theta(\langle \xi_\ell, v_\ell \rangle_Y). \tag{11}$$

Since no sequence is a proper initial segment of itself, $\theta(\langle \xi_1', u_t \rangle_Y) \neq \theta(\langle \xi_\ell, v_\ell \rangle_Y)$. Hence, by OD, $u_1 \neq v_\ell$. □

Proof of Theorem 5.1. (1): This follows from Lemma 5.1 and the if-part of Theorem 4.2, since the proof of the latter does not depend on the existence of a linked i.d.view.

(2): Since every occurrence recall-k memory function is informationally Y-correct, it follows by part (1) of this Theorem, that the set of linked i.d.view is finite. By Lemma 5.1, T_D is acyclic. Hence, by Corollary 4.1, it suffices to show that for each decision piece $w \in W^{T_D} \bigcap W^{oD}$ there is an adjacent triple (u_1, a_1, u_2) in $\mathcal{T}(T_D)$ such that $u_1 = w$. Let $w \in W^{T_D} \bigcap W^{oD}$. Since D is closed and the memory function is an occurrence recall $k \geq 1$ memory function, there must be position $\langle \xi, v \rangle \in D$ for which $m^o\langle \xi, v \rangle$ has (w, a, v) as its last adjacent triple for some $a \in A^o_w$. $\square$

We return to Example 2.2. Consider an occurrence recall-1 on $D = Y = \{\langle \xi, w \rangle \in \Xi : w \neq w_2\}$. The resulting memory kit is

$$\begin{aligned} T_D = \{&\langle (w_0, a), w_1 \rangle, \langle (w_1, a), z_1 \rangle, \langle (w_1, b), z_2 \rangle, \langle (w_0, b), w_1 \rangle, \\ &\langle (w_1, a), z_2 \rangle, \langle (w_1, a), z_3 \rangle, \langle (w_1, b), z_4 \rangle\} \end{aligned} \tag{12}$$

By Theorem 5.1, the set of linked i.d.views is non empty and finite. Two examples are illustrated in Fig. 7. Incidentally, none of the possible linked i.d.views satisfies every non-basic axiom. This is the consequence of the fact that the position $\langle (w_0, b), (w_1, a), z_2 \rangle \notin Y$ and the fact that two positions in the objective view reach the same information piece w_1.

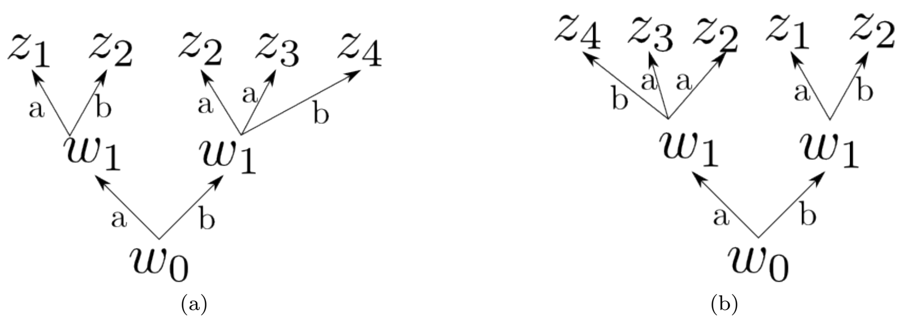

Fig. 7: Examples of linked i.d.views

6 Conclusions

In this paper, we gave an explicit process for going from a memory kit to an i.d.view. The process was called an extension process (EP) and it was shown that every i.d.view in the sense of Kaneko and Kline (2013) can be obtained by an EP. We then restricted EP's by requiring them to be linked in an effort to tackle the multiplicity issue identified by Kaneko and Kline (2013).

The notion of linked i.d.views was explored in the present paper and necessary and sufficient conditions were obtained on the memory kit for existence and finiteness of i.d.views. We also gave sufficient conditions directly on the objective view for finiteness and existence. These conditions are satisfied by occurrence recall-k memory function with $k \geq 1$. These memory functions include the recall-k memory functions defined in Kaneko and Kline (2013). It is shown in Waltener (2017) that the "perfect recall view", that is, the direct i.d.view obtained by a player having a recall-k memory function, is a linked i.d.view in such situations. Thus, a natural candidate for an i.d.view, is obtained by focusing on linked i.d.views. The perfect recall view may fail to be a minimal view which helps justify our concentration on linked i.d.views rather than minimal i.d.views.

By establishing the notion of EP's, this paper paves the way for further natural restrictions on i.d.views, some of which are already explored in Waltener (2017). Those are obtained by considering a wider source of adjacent triples allowed in a linking EP. By this weakening, the linking EP reaches supersets of the set of linked i.d.views studied in this paper and represented in Fig 2.

Since EP's have more structure, it may also be possible to find some measure of the cognitive cost of constructing a particular i.d.view. Since IGT is interested in boundedly rational agents, this may allow us to explore the relationship between the cognitive limitations on a player's memory and limitations on his ability to construct complicated i.d.views. One might expect a trade-off between memory and construction ability; a player with a stronger memory would need a lower ability of construction of i.d.views, and vice versa.

References

Kaneko M, Kline J (2008a) Inductive Game Theory: A Basic Scenario. Journal of Mathematical Economics 44(12):1332–1363

Kaneko M, Kline J (2008b) Information Protocols and Extensive Games in Inductive Game Theory. Game Theory and Applications 13:57–83

Kaneko M, Kline J (2013) Partial Memories, Inductively Derived Views, and their Interactions with Behavior. Economic Theory 53(1):27–59

Kaneko M, Kline J, Akiyama E, Ishikawa R (2012) A Simulation Study of Learning a Structure - Mike's Bike Commuting. In: Pina N, Kacprzyk J, Obaidat M (eds) SIMULTECH, SciTePress, pp 208–217

Kuhn H (1953) Extensive games and the problem of information. In: Kuhn H, Tucker A (eds) Contributions to the Theory of Games II, Princeton University Press, Princeton, pp 193–216

Okada A (1987) Complete inflation and perfect recall in extensive games. International Journal of Game Theory 16(2):85–91

Waltener S (2017) Derivation of personal views in inductive game theory. PhD thesis, UNamur