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Higher Order Dependency of Chaotic Maps

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Abstract—Some higher-order statistical dependency aspects of chaotic maps are presented. The autocorrelation function (ACF) of the mean-adjusted squares, termed the quadratic autocorrelation function, is used to access non-linear dependence of the maps under consideration. A simple analytical expression for the quadratic ACF has been found in the case of fully stretching piece-wise linear maps. A minimum bit energy criterion from chaos communications is used to motivate choosing maps with strong negative quadratic autocorrelation. A particular map in this class, a so-called deformed circular map, is derived which performs better than other well-known chaotic maps when used for spreading sequences in chaotic shift-key communication systems.

Key Words—quadratic autocorrelation, piece-wise linear maps, circular maps, negative statistical dependency, chaos shift-keying communication systems

1. Introduction

The linear statistical dependence of fully stretching piece-wise linear maps has been explored in Baranovski & Daems [2], where an explicit expression for their ACF was found, and in Baranovski & Lawrance [1]. This work is extended here by further analyzing the non-linear dynamics of these maps. More specifically, their mean-adjusted quadratic ACF has been found. The result is interesting in the sense that certain collections of fully stretching piece-wise linear maps share the same quadratic ACF.

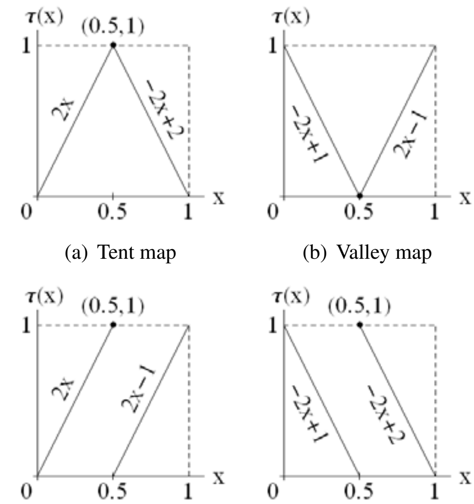
The intention of minimizing the bit energy of a chaotic communication system, eg chaos-shift keying, Lawrance and Ohama [5], led to the formulation of a criterion for assessing the performance of the chaotic map of the system (see Yao [7]). The criterion focuses on the non-linear dependency of the spreading sequence and specifies the form of the quadratic ACF of an effective map.

Finally, the first-order circular map is deformed to decrease its negative quadratic dependency to nearer the absolute lower bound, and so make it nearer to optimal for spreading in chaos-based communication systems.

2. Dependence for Piecewise Linear Maps

2.1. Four Well-Known Maps

We start with the two-branch equi-distributed piece-wise linear maps, namely with the tent and valley map, the positive and negative Bernoulli shift map. They fulfill the equi-distributed property (Khoda & Tsuneda [3], Khoda & Tsuneda [4]) and they are symmetric about the line $y = \frac{1}{2}$, as it can be seen in Figure 1.



(a) Tent map (b) Valley map

(c) Positive sloped Bernoulli shift map (d) Negative sloped Bernoulli shift map

Figure 1: The four two-branch equi-distributed maps defined on [0,1].

A perhaps surprising dependence property of the four maps is that they share the same adjusted quadratic ACF, whereas their linear ACFs differ (Table 1).

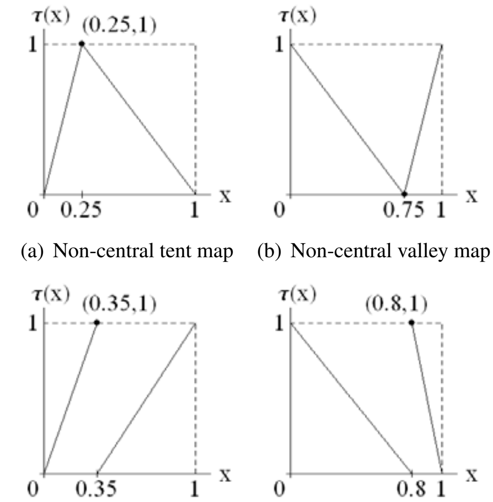
Map	Linear ACF	Quadratic ACF
Tent	0	$\left(\frac{1}{4}\right)^s$
Valley	0	$\left(\frac{1}{4}\right)^s$
Positive Bernoulli	$\left(\frac{1}{2}\right)^s$	$\left(\frac{1}{4}\right)^s$
Negative Bernoulli	$\left(-\frac{1}{2}\right)^s$	$\left(\frac{1}{4}\right)^s$

Table 1: The linear and quadratic ACF of the four maps

2.2. Two-Branch Piecewise Linear Maps

We now consider the more general collection of two-branch piece-wise linear maps. This particular family embraces maps whose monotonicity switches direction at some point $r \in (c, d)$ of their domain $[c, d]$. We introduce the term *non-centrality parameter* to refer to point r. The four equi-distributed maps previously encountered can be seen as special cases arising from the choice $r = 0.5$ of the non-centrality parameter.

Figure 2 exemplifies the collection of non-central two-branch maps and Table 2 provides their statistical dependence.



(a) Non-central tent map (b) Non-central valley map (c) Non-central positive sloped Bernoulli shift map (d) Non-central negative sloped Bernoulli shift map

Figure 2: Piece-wise linear examples of non-central two-branch maps. The non-centrality parameter r has been set to $0.25,\ 0.75,\ 0.35$ and 0.8 in Figures (a), (b), (c) and (d), respectively.

Map	Linear ACF	Quadratic ACF
Tent	$(2r-1)^s$	$(3r^2-3r+1)^s$
Valley	$(1-2r)^s$	$(3r^2-3r+1)^s$
Positive Bernoulli	$(2r^2-2r+1)^s$	$(3r^2-3r+1)^s$
Negative Bernoulli	$(-2r^2+2r-1)^s$	$(3r^2-3r+1)^s$

Table 2: The linear and quadratic ACF of the four non-central maps

It is worth mentioning that the four non-central maps of Table 2 are uniformly $\mathbb{U}(0,1)$ distributed regardless of the non-centrality parameter r. But the most intriguing fact is that they share the same adjusted quadratic autocorrelation function, provided that they all have the same non-centrality parameter r; this point is still open to intuitive explanation.

2.3. Fully Stretching Piece-wise Linear Maps

The highest level of our generalization involves fully stretching piece-wise linear maps $\tau : [0,1] \to [0,1]$ with more than two branches. We assume that τ is composed of $k+1$ linear branches $\tau_i(x) = \lambda_i x + \mu_i$, i.e. that τ has k non-centrality parameters $r_i,\ i \in \{1, 2, \ldots, k\}$, such that $0 \equiv r_0 < r_1 < r_2 < \ldots < r_k < r_{k+1} \equiv 1$.

It can be shown that the uniform $\mathbb{U}(0,1)$ satisfies the Perron-Frobenius equation, that is $\mathbb{U}(0,1)$ can be adopted as the invariant density of the chaotic sequence produced by the map τ.

The linear ACF of fully stretching piece-wise linear maps is known (see Baranovski & Daems [2]), and is neatly given by

$$Corr(X_t, X_{t+s}) = \left(\sum_{i=1}^{k+1} \frac{1}{\lambda_i |\lambda_i|} \right)^s, \ s \in \mathbb{N}, \tag{1}$$

where λ_i is the slope of the i-th branch τ_i of τ.

We report that the adjusted quadratic ACF of fully stretching piece-wise linear maps is correspondingly given by

$$Corr\left[\left(X_t - \frac{1}{2} \right)^2, \left(X_{t+s} - \frac{1}{2} \right)^2 \right] = \left(\sum_{i=1}^{k+1} \frac{1}{\lambda_i^2 |\lambda_i|} \right)^s, \ s \in \mathbb{N}. \tag{2}$$

An alternative expression for (2), in terms of the non-centrality parameters $r_i,\ i \in \{1, 2, \ldots, k\}$, of τ is provided as:

$$Corr\left[\left(X_t - \frac{1}{2} \right)^2, \left(X_{t+s} - \frac{1}{2} \right)^2 \right] = \left\{ \sum_{i=1}^{k+1} \left[(r_i - r_{i-1})^3 \right] \right\}^s. \tag{3}$$

We point out the conclusions which can be drawn from our work so far:

- Equation (3) generalizes and clarifies the property we came across in Tables 1 and 2; all fully stretching piece-wise linear maps with the same non-centrality parameters $r_i,\ i \in \{1, 2, \ldots, k\}$, share the same adjusted quadratic ACF. On the other hand, they do not exhibit identical linear dependence, since their linear autocorrelations do not coincide.
- A comparison between equations (1) and (2) justifies the choice of adjusting the quadratic ACF for the mean; the resemblance between the two equations suggests that the adjusted quadratic ACF is a natural measure of non-linear dependence, principally because it is not affected by the mean.
- Finally, it can be deduced from equation (3) that the adjusted quadratic autocorrelations of any lag of any fully stretching piece-wise linear map is positive. However, this is precisely not wanted for chaos-based spreading in communication systems, as will be seen in section 3.

3. Criterion for Optimality

The bit energy of several binary communication systems, such as chaos shift-keying, involving chaotic spreading sequences, is a function of the mean-adjusted sum of squares of the spreading, say

$$\sum_{i=1}^{N}(X_i-\mu)^2.$$

See Lawrance & Ohama [5], for instance.

In such systems, the bit error rate can be shown to be mimimised when the bit energy is constant at its minimum value. In particular, this implies that the variance of the bit energy should be zero and this is the starting point of the derivation of an optimum map τ, as set out in Yao [7].

First there is the routine general result

$$var\left\{\sum_{i=1}^{N}(X_i-\mu)^2\right\} = N\sigma^2_{(X-\mu)^2}\left\{1+2\sum_{i=1}^{N-1}\left(1-\frac{i}{N}\right)\rho_{(X-\mu)^2}(i)\right\}, \quad (4)$$

where $\sigma^2_{(X-\mu)^2}$ is the variance of the squares and $\rho_{(X-\mu)^2}(i)$ is the lag i autocorrelation of the squares. This variance is clearly zero when

$$2\sum_{i=1}^{N-1}\left(1-\frac{i}{N}\right)\rho_{(X-\mu)^2}(i) = -1. \quad (5)$$

Continuing in general from this result to specify the map τ looks hopeless, and so in the next section it is usefully specialized.

4. Maps Consistent with the Optimality Criterion

4.1. An Optimal Circular Map

The smallest extent of spreading is $N = 2$ which when used in (5) gives the perfect correlation $\rho_{(X-\mu)^2}(1) = -1$ and more clearly $\rho_{(X-\mu)^2,(\tau(X)-\mu)^2} = -1$, showing that $(X-\mu)^2$ and $(\tau(X-\mu))^2$ are negatively linearly related. Assuming $\mu = 0$ without loss, this perfect negative correlation thus requires $\tau(x)^2 = ax^2 + b,\ a < 0,\ -1 \le x \le 1$, and hence if the square-root linear map is to be fully stretching over $(0, 1)$

$$\tau(\sqrt{x})^2 = -x+1,\ -1 \le x \le 1. \quad (6)$$

This straight line cannot be a proper chaotic map and so dynamical straight line approximations are entertained. The simplest seems to be the negative sloped Bernoulli map illustrated in Figure 1, which suggests taking

$$\tau(\sqrt{x})^2 = \begin{cases} -2x+1, & 0 \le x < \frac{1}{2} \\ -2x+2, & \frac{1}{2} \le x \le 1 \end{cases}. \quad (7)$$

Proceeding with this map, possible forms of the map $\tau(x)$ satisfying (7) are included in the formula

$$\tau(\sqrt{x}) = \begin{cases} \pm\sqrt{-2x+1}, & 0 \le x < \frac{1}{2} \\ \pm\sqrt{-2x+2}, & \frac{1}{2} \le x \le 1 \end{cases}. \quad (8)$$

Choosing the most natural possibility for $\tau(x)$ gives the desired *circular map*

$$\tau(x) = \begin{cases} -\sqrt{-2x^2+2}, & -1 \le x < -1/\sqrt{2} \\ \sqrt{-2x^2+1}, & -1/\sqrt{2} \le x < 1/\sqrt{2} \\ -\sqrt{-2x^2+2}, & 1/\sqrt{2} \le x \le 1 \end{cases}, \quad (9)$$

which is illustrated in Figure 3. Yao [7] proves that the invariant distribution of this map has probability density function $f(x) = |x|,\ -1 \le x \le 1$ and that it is linearly uncorrelated. We report that its quadratic autocorrelation function is

$$Corr(X_t^2, X_{t+s}^2) = \left(-\frac{1}{2}\right)^s,\ s \in \mathbb{N}, \quad (10)$$

still room for improvement of the lag 1 autocorrelation towards -1.

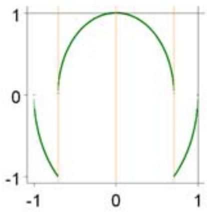


Figure 3: The circular map.

4.2. A Deformed Circular Map

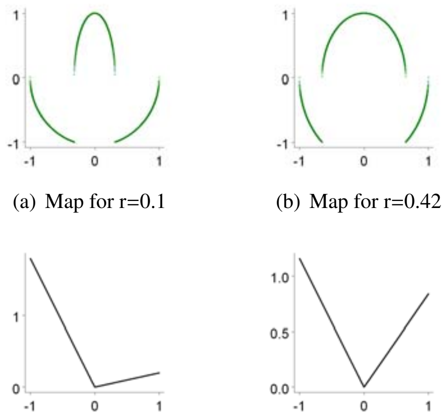
In trying to reduce further the lag 1 quadratic ACF, the circular map is generalized by introducing a *deforming parameter* r, which determines the range of each branch of the map. To be more specific, the *deformed circular map* is defined as:

$$\tau(x) = \begin{cases} -\sqrt{-(1-r)^{-1}x^2+(1-r)^{-1}}, & -1 \le x < -\sqrt{r} \\ \sqrt{-r^{-1}x^2+1}, & -\sqrt{r} \le x < \sqrt{r} \\ -\sqrt{-(1-r)^{-1}x^2+(1-r)^{-1}}, & \sqrt{r} \le x \le 1 \end{cases}. \quad (11)$$

This map has an invariant density $f(x)$ given by

$$f(x) = \begin{cases} -2(1-r)x, & -1 \le x \le 0 \\ 2rx, & 0 < x \le 1 \end{cases}, \quad (12)$$

which satisfies the Perron-Frobenius equation, and is seen to generalize the invariant density $|x|$ of the circular map. Moreover, Figure 4 helps us get a better idea about the shape of deformed circular maps by providing the plots of maps with deforming parameters $r = 0.1$ and $r = 0.42$ as well as their associated invariant densities.



(a) Map for r=0.1 (b) Map for r=0.42

(c) Density for r=0.1 (d) Density for r=0.42

Figure 4: Two examples of deformed circular maps and their invariant densities.

Although the derivation of an analytic expression for the adjusted quadratic ACF appears to be a cumbersome task, at least an explicit formula has been obtained for the first lag. The first lag can be expressed as a function of r only. Figure 5 provides the plot of lag 1 versus r. As can be seen, the function is minimized for $r = 0.42$ and its mimimum equals -0.722. In other words, we have found that the deformed circular map with deforming parameter $r = 0.42$ (Figure 4.(b)) is even better than the circular map ($r = 0.5$, Figure 3) from a communications perspective; it is closer to the requirements imposed by our criterion.

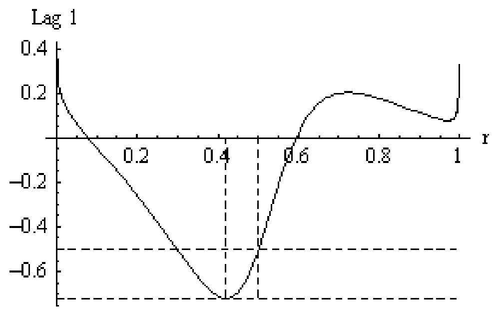


Figure 5: Plot of the first lag of the adjusted quadratic ACF of the deformed circular map versus its deforming parameter r. The first lag takes its minimum value -0.722 for $r = 0.42$.

Since this paper deals with work-in-progress, it is appropriate to point out that there is a further question as to how near -0.722 is to the theoretical lower bound. There is such a correlation lower bound over all bivariate distributions with specified marginal distribution. This is part of an area of distribution theory developed by Frechet (1951), see Ripley [6], for instance. What is needed is the distribution function of $F_{X^2}(\cdot)$ of X^2, from (12) in this case, and the corresponding inverse function $F^{-1}_{X^2}(\cdot)$. The 'degenerate' bivariate distribution of minimum correlation is then that of the variables $F^{-1}_{X^2}(U)$ and $F^{-1}_{X^2}(1-U)$, where U is a $\mathbb{U}(0,1)$ random variable. Since the first of these random variables has the distribution of X^2 by the well known 'inverse transformation' of a uniform random variable U, and because $1-U$ is also a uniform random variable, the second also has the distribution of X^2. In the present context, this degenerate case of minimum correlation can be cast as $X_t = F^{-1}_{X^2}(U)$, $X_{t+1} = F^{-1}_{X^2}(1-U)$. In fact the non-linear relation between the two variables is clearly,

$$X_{t+1} = F^{-1}_{X^2}\Big(1 - F_{X^2}(X_t)\Big) \quad (13)$$

and this would suggest that the map giving minimun correlation should approximate the curve

$$\tau(x) = F^{-1}_{X^2}\Big(1 - F_{X^2}(x)\Big) \quad (14)$$

in a dynamical way.

Acknowledgments

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