Type theoretic weak factorization systems

Paige Randall North

King's College

This dissertation is submitted for the degree of Doctor of Philosophy.

June 2017

Summary

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This thesis presents a characterization of those categories with weak factorization systems that can interpret the theory of intensional dependent type theory with Σ, Π, and identity types.

We use display map categories to serve as models of intensional dependent type theory. If a display map category $(\mathcal{C}, \mathcal{D})$ models Σ and identity types, then this structure generates a weak factorization system $(\mathcal{L}, \mathcal{R})$. Moreover, we show that if the underlying category $\mathcal{C}$ is Cauchy complete, then $(\mathcal{C}, \mathcal{R})$ is also a display map category modeling Σ and identity types (as well as Π types if $(\mathcal{C}, \mathcal{D})$ models Π types). Thus, our main result is to characterize display map categories $(\mathcal{C}, \mathcal{R})$ which model Σ and identity types and where $\mathcal{R}$ is part of a weak factorization system $(\mathcal{L}, \mathcal{R})$ on the category $\mathcal{C}$. We offer three such characterizations and show that they are all equivalent when $\mathcal{C}$ has all finite limits. The first is that the weak factorization system $(\mathcal{L}, \mathcal{R})$ has the properties that $\mathcal{L}$ is stable under pullback along $\mathcal{R}$ and all maps to a terminal object are in $\mathcal{R}$. We call such weak factorization systems *type theoretic*. The second is that the weak factorization system has what we call an `Id`-*presentation*: it can be built from certain categorical structure in the same way that a model of Σ and identity types generates a weak factorization system. The third is that the weak factorization system $(\mathcal{L}, \mathcal{R})$ is generated by a *Moore relation system*. This is a technical tool used to establish the equivalence between the first and second characterizations described.

To conclude the thesis, we describe a certain class of convenient categories of topological spaces (a generalization of compactly generated weak Hausdorff

spaces). We then construct a Moore relation system within these categories (and also within the topological topos) and thus show that these form display map categories with Σ and identity types (as well as Π types in the topological topos).

Declaration

This dissertation is the result of my own work and includes nothing which is the outcome of work done in collaboration except as declared in the Preface and specified in the text. It is not substantially the same as any that I have submitted, or, is being concurrently submitted for a degree or diploma or other qualification at the University of Cambridge or any other University or similar institution except as declared in the Preface and specified in the text. I further state that no substantial part of my dissertation has already been submitted, or, is being concurrently submitted for any such degree, diploma or other qualification at the University of Cambridge or any other University or similar institution except as declared in the Preface and specified in the text.

Paige Randall North
June 2017

Acknowledgements

I would like to thank my advisor, Martin Hyland, whose guidance I did not always follow but whose influence on this thesis is likely stronger than I understand. Thanks also go to the category theory group at Cambridge whose fellowship sustained me. I would also like to thank the group at Chicago, especially Peter May and Emily Riehl, who inspired me as an undergraduate.

To my friends in Cambridge and around the world: thank you for encouraging me. To my parents and brothers: thank you for supporting me. To Benedikt: thank you.

Special thanks go to Martin Hyland, Benedikt Ahrens, and Peter Lumsdaine for reading and critiquing early drafts, and to Nicola Gambino and Andrew Pitts for examining and correcting the final draft.

Contents

Introduction.

This thesis is an attempt to better understand the significance of the word *homotopy* in the phrase *homotopy type theory*.

It has long been observed that `Id` types in type theory resemble the path spaces of traditional homotopy theory (first recorded in [HS98] and [AW09]). In [GG08], it was shown that models of type theory with `Id` types generate weak factorization systems in which the `Id` types appear as path objects. Conversely, many particular weak factorization systems have been studied (e.g., by [War08] and [BG12]) which can interpret type theory with `Id` types. In this thesis, we aim to solidify this connection between weak factorization systems and models of type theory with `Id` types.

To do this, we use *display map categories* as our medium of interpretation, as this structure is closest to that of weak factorization systems on categories. We also consider very weak specifications of Σ types, `Id` types, and Π types (weaker than those considered in [GG08], [War08], or [BG12], for example) which are more readily captured by the language of Quillen model category theory. In return for these concessions, we obtain a complete characterization of weak factorization systems (on finitely complete categories) that can interpret Σ types and `Id` types (Theorem 3.5.2). Furthermore, in the case that the ambient category is locally cartesian closed, such weak factorization systems will interpret not only Σ types and `Id` types but also Π types (Theorem 3.5.3).

Outline

In Chapter 1, we review the basic theory of weak factorization systems. We reformulate the traditional presentation to better suit our purposes. Firstly, we make a point of distinguishing between those concepts which only require the mere existence of some structure and those concepts which explicitly include a choice of the structure in question (Definitions 1.3.2 and 1.6.1). For example, we find it useful to consider both weak factorization *systems* and weak factorization *structures*: the former entail only the existence of factorizations while the latter include particular factorizations as part of their data. Secondly, we will mostly use one characterization of weak factorization structures in later chapters: a special kind of factorization which we call *weakly algebraic* (Definition 1.4.6).

In Chapter 2, we review the theory of display map categories. We consider them as a special kind of comprehension category, and, following the literature on comprehension categories, we define what it means for display map categories to model $\Sigma, \mathtt{Id},$ and Π types. We then consider the case when the ambient category is Cauchy complete. This gives us the first substantial result of this thesis: when $\mathcal{C}$ is a Cauchy complete category and $(\mathcal{C}, \mathcal{D})$ is a display map category modelling Σ and $\mathtt{Id}$ types (respectively, Π types), then $(\mathcal{C}, \overline{\mathcal{D}})$ is a display map category modelling Σ and $\mathtt{Id}$ types (respectively, Π types) (Theorem 2.5.12). Here $\mathcal{D}$ is a special class of morphisms of $\mathcal{C}$, and $\overline{\mathcal{D}}$ is the smallest class of maps which contains $\mathcal{D}$ and is the right class of a weak factorization system on $\mathcal{C}$. Thus, given a Cauchy complete category $\mathcal{C}$, if there is a display map category $(\mathcal{C}, \mathcal{D})$ modelling Σ and $\mathtt{Id}$ types, then not only does this appear within a weak factorization system $({}^{\boxslash}\mathcal{D}, \overline{\mathcal{D}})$ (as [GG08] showed), but the weak factorization system itself forms a display map category $(\mathcal{C}, \overline{\mathcal{D}})$ modelling Σ and $\mathtt{Id}$ types. Thus, in the rest of the thesis, we focus on characterizing those weak factorization systems $(\mathcal{L}, \mathcal{R})$ for which $(\mathcal{C}, \mathcal{R})$ is a display map category modeling Σ and $\mathtt{Id}$ types. In these cases, we say that the data for the model of $\mathtt{Id}$ types is a $\mathtt{Id}$-*presentation* of the weak factorization system $(\mathcal{L}, \mathcal{R})$. We also call *type theoretic* those weak factorization systems $(\mathcal{L}, \mathcal{R})$ in which all objects are fibrant and in which $\mathcal{L}$ is stable under pullback along $\mathcal{R}$. A weak factorization system $(\mathcal{L}, \mathcal{R})$ must be type theoretic if $(\mathcal{C}, \mathcal{R})$ were to model Π types.

In Chapter 3, we focus on characterizing such weak factorization systems. In the first section, we describe diagrams of factorizations, relations (the shape of the data underlying a model of $\mathtt{Id}$ types), and a hybrid which we call *relational factorizations*. Then using this categorical apparatus, we define categories of type theoretic weak factorization structures and categories of those relations which produce $\mathtt{Id}$-presentations of weak factorization systems (Definitions 3.1.48 and 3.1.49). The rest of the chapter is devoted to producing a certain kind of equivalence between these two categories. To establish this equivalence, we introduce a theory of *Moore relation systems* in the second section. This is a certain kind of structure on a relation which ensures that the relation generates a type theoretic weak factorization system. Moreover, this structure is minimal in the sense that it is always entailed by a type theoretic weak factorization system. In the main theorem of this chapter, we show that all of these properties of weak factorization system are equivalent.

Theorem 3.5.2. *Consider a category $\mathcal{C}$ with finite limits. The following properties of any weak factorization system $(\mathcal{L}, \mathcal{R})$ on $\mathcal{C}$ are equivalent:*

1. *it has an* `Id`*-presentation;*
2. *it is type theoretic;*
3. *it is generated by a Moore relation system;*
4. $(\mathcal{C}, \mathcal{R})$ *is a display map category modeling* Σ *and* `Id` *types.*

In Chapter 4, we describe and generalize the construction of certain convenient categories of topological spaces in order to find models of Σ and `Id` types within them. We begin by reviewing the categorical theory of coreflective hulls of subcategories. We apply this to subcategories of the category of topological spaces, generalizing the construction of the category of compactly generated spaces and the category of compactly generated weak Hausdorff spaces. In the last section, we construct Moore relation structures in many of these convenient categories of topological spaces and in the topological topos (introduced in [Joh79]). Thus, we find models of Σ and `Id` types within these categories (Theorem 4.5.25).

Chapter 1

Weak factorization systems and structures.

This chapter is an overview of the theory of factorizations, weak factorization structures, and weak factorization systems which will be used in later chapters. It is intended to fix notation and to record standard results of model category theory for later use. The definitions and results here, while well-known, have been reformulated and optimized for use in the following chapters. Good references for a more standard presentation of this material include [Hov99] and [MP12]. The perspective taken here was introduced by Rosický and Tholen [RT02] and Grandis and Tholen [GT06] and was later promoted in work of Garner (e.g. [Gar09]) and Riehl (e.g. [Rie11]).

1.1 Lifting properties.

Fix a category $\mathcal{M}$. A weak factorization system on $\mathcal{M}$ first of all consists of two classes $\mathcal{L}$ and $\mathcal{R}$ of maps of $\mathcal{M}$ with a certain relationship.

First we need to fix some notation.

Definition 1.1.1. A *lifting problem* in $\mathcal{M}$ is a commutative square in $\mathcal{M}$ as shown on the left-hand side below.

$$\begin{tikzcd} A \arrow[r, "x"] \arrow[d, "\ell"'] & B \arrow[d, "r"] \\ C \arrow[r, "y"'] & D \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "x"] \arrow[d, "\ell"'] & B \arrow[d, "r"] \\ C \arrow[r, "y"'] \arrow[ur, "s"] & D \end{tikzcd}$$

A *solution* to such a lifting problem is a morphism s making the right-hand diagram above commute.

Lifting problems will often be denoted by a diagram of the following form

$$\begin{tikzcd} A \arrow[r, "x"] \arrow[d, "\ell"'] & B \arrow[d, "r"] \\ C \arrow[r, "y"'] \arrow[ur, dashed] & D \end{tikzcd}$$

to indicate that the solid arrows are known and that the dashed arrow is sought.

Definition 1.1.2. Fix two classes $\mathcal{L}$ and $\mathcal{R}$ of morphisms of $\mathcal{M}$. Suppose that every lifting problem as shown below has a solution if $\ell \in \mathcal{L}$ and $r \in \mathcal{R}$.

$$\begin{tikzcd} A \arrow[r, "x"] \arrow[d, "\ell"'] & B \arrow[d, "r"] \\ C \arrow[r, "y"'] \arrow[ur, dashed] & D \end{tikzcd}$$

Then we say that $\mathcal{L}$ has the *left lifting property* against $\mathcal{R}$ or that $\mathcal{R}$ has the *right lifting property* against $\mathcal{L}$, and we write $\mathcal{L} \boxslash \mathcal{R}$. We will often also say that $\mathcal{L}$ *lifts against* $\mathcal{R}$.

The class of all morphisms with the left lifting property against $\mathcal{R}$ is denoted ${}^{\boxslash}\mathcal{R}$, and the class of all morphisms with the right lifting property against $\mathcal{L}$ is denoted $\mathcal{L}^{\boxslash}$.

When any of the classes in the above terminology or notation is a singleton, we will drop the braces: e.g., $\ell \boxslash r$ means $\{\ell\} \boxslash \{r\}$.

Definition 1.1.3. Say that two classes $\mathcal{L}$, $\mathcal{R}$ of maps of $\mathcal{M}$ are a *lifting pair* if $\mathcal{L} = {}^{\boxslash}\mathcal{R}$ and $\mathcal{R} = \mathcal{L}^{\boxslash}$. We call $\mathcal{L}$ the *left class* of this lifting pair and $\mathcal{R}$ the *right class*.

If $\mathcal{L}$ and $\mathcal{R}$ form a lifting pair then not only does every $\ell \in \mathcal{L}$ lift against every $r \in \mathcal{R}$, but we can also determine whether a morphism is in $\mathcal{R}$ (or, respectively, $\mathcal{L}$) by checking whether it lifts against every $\ell \in \mathcal{L}$ (or, respectively, every $r \in \mathcal{R}$). We use this in the following example and propositions.

Example 1.1.4. Consider the category of sets. Let $\mathcal{I}$ denote the class of injections, and $\mathcal{S}$ denote the class of surjections. Then $(\mathcal{I}, \mathcal{S})$ is a lifting pair.

Consider any injection i and surjection s in a lifting problem as below.

$$\begin{tikzcd} A \arrow[r, "x"] \arrow[d, "i"'] & B \arrow[d, "s"] \\ C \arrow[r, "y"'] \arrow[ur, dashed] & D \end{tikzcd}$$

Using the axiom of choice, choose a splitting t of s. Consider C as the union of A and its complement A^c. Then $x \cup ty : A \cup A^c \to B$ is the lift we seek. We have shown that $\mathcal{I} \boxslash \mathcal{S}$, or, in other words, that $\mathcal{I} \subseteq {}^{\boxslash}\mathcal{S}$ and $\mathcal{S} \subseteq \mathcal{I}^{\boxslash}$.

To see that ${}^{\boxslash}\mathcal{S} \subseteq \mathcal{I}$, consider any $i : A \to C$ in ${}^{\boxslash}\mathcal{S}$. If A is empty, then i is an injection. Otherwise, $! : A \to *$ is a surjection, and there is a solution σ to the following lifting problem.

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, "i"'] & A \arrow[d, "!"] \\ C \arrow[r, "!"'] \arrow[ur, dashed, "\sigma"] & * \end{tikzcd}$$

But since $\sigma i = 1$, we can conclude that i is in $\mathcal{I}$.

To see that $\mathcal{I}^{\boxslash} \subseteq \mathcal{S}$, consider any $s : B \to D$ in $\mathcal{I}^{\boxslash}$. We construct the dual lifting problem.

$$\begin{tikzcd} \varnothing \arrow[r] \arrow[d] & B \arrow[d, "s"] \\ D \arrow[r, equal] \arrow[ur, dashed] & D \end{tikzcd}$$

Then s must be in $\mathcal{S}$.

The preceding example foreshadows the kind of weak factorization systems that will be studied in the following chapters. There, the right classes will always contain every map to the terminal object. Thus, repeating the argument above, every map in the left class will be a split monomorphism. In Example 1.3.4, we will see a lifting pair for which this is not the case.

Now we prove two useful lemmas concerning lifting pairs.

Lemma 1.1.5. Consider a lifting pair $(\mathcal{L}, \mathcal{R})$ in $\mathcal{M}$. The class $\mathcal{R}$ contains all isomorphisms, and is closed under the following operations:

1. pullbacks: if f is in $\mathcal{R}$ and

$$\begin{tikzcd}
\alpha^*X \arrow[r] \arrow[d, "\alpha^*f"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\
\alpha^*Y \arrow[r, "\alpha"] & Y
\end{tikzcd}$$

   is a pullback square, then α^*f is in $\mathcal{R}$;

2. composition: if $f : X \to Y$ and $g : Y \to Z$ are in $\mathcal{R}$, then $g \circ f$ is in $\mathcal{R}$;

3. products: if $f : W \to Y$ and $g : X \to Z$ are in $\mathcal{R}$, then $f \times g : W \times X \to Y \times Z$ is in $\mathcal{R}$;

4. retracts: if f is in $\mathcal{R}$, and there is the commutative diagram of the form below,

$$\begin{tikzcd}
W \arrow[r, "w"'] \arrow[d, "g"] \arrow[rr, equal, bend left] & X \arrow[r, "x"'] \arrow[d, "f"] & W \arrow[d, "g"] \\
Z \arrow[r, "z"] \arrow[rr, equal, bend right] & Y \arrow[r, "y"] & Z
\end{tikzcd}$$

   then g is in $\mathcal{R}$.

Dually, $\mathcal{L}$ contains all isomorphisms and is closed under

1. pushouts,
2. composition,
3. coproducts, and
4. retracts.

Proof. We will only prove the statements for $\mathcal{R}$. The proofs of the statements concerning $\mathcal{L}$ are dual to these.

To see that $\mathcal{R}$ contains all isomorphisms, consider a lifting problem as below with $\ell \in \mathcal{L}$ and i an isomorphism.

$$\begin{tikzcd}
A \arrow[r, "x"] \arrow[d, "\ell"'] & B \arrow[d, "i"] \\
C \arrow[r, "y"] \arrow[ur, dashed] & D
\end{tikzcd}$$

Then $i^{-1}y$ gives a lift of this diagram. Thus, i has the right lifting property against every $\ell \in \mathcal{L}$, and so is in $\mathcal{L}^{\boxslash} = \mathcal{R}$.

1. To see that $\mathcal{R}$ is stable under pullback, we need to find a solution to the lifting problem below on the left with $\ell \in \mathcal{L}$ and $\alpha^* f$ as in the statement.

$$\begin{tikzcd} A \arrow[r, "x"] \arrow[d, "\ell"'] & \alpha^* X \arrow[d, "\alpha^* f"] \\ C \arrow[r, "y"] \arrow[ur, dashed] & \alpha * Y \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "x"] \arrow[d, "\ell"'] & \alpha^* X \arrow[r, "f^*\alpha"] & X \arrow[d, "f"] \\ C \arrow[r, "y"'] \arrow[urr, dashed, "s"] & \alpha^* Y \arrow[r, "\alpha"'] & Y \end{tikzcd}$$

Consider the lifting problem above on the right. Since $\ell \in \mathcal{L}$ and $f \in \mathcal{R}$, there is a lift s. Then the morphisms s and y induce a morphism $C \to \alpha^* X$ (by the universal property of $\alpha^* X$) which is a solution for the original lifting problem. Thus, $\alpha^* f$ lifts against $\mathcal{L}$, and so is in $\mathcal{R}$.

2. To see that $\mathcal{R}$ is closed under composition, consider the lifting problem below with $\ell \in \mathcal{L}$ and composable $f, g \in \mathcal{R}$.

$$\begin{tikzcd} A \arrow[r, "x"] \arrow[d, "\ell"'] & X \arrow[d, "gf"] \\ C \arrow[r, "y"] \arrow[ur, dashed] & Z \end{tikzcd}$$

We construct the lift in two stages. First we find a lift s as in the diagram below on the left.

$$\begin{tikzcd} A \arrow[r, "fx"] \arrow[d, "\ell"'] & Y \arrow[d, "g"] \\ C \arrow[r, "y"'] \arrow[ur, dashed, "s"] & Z \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "x"] \arrow[d, "\ell"'] & X \arrow[d, "f"] \\ C \arrow[r, "s"'] \arrow[ur, dashed, "t"] & Y \end{tikzcd}$$

Then we can find a lift t as above on the right. This morphism t is also a solution to the original lifting problem. Thus, $gf \in \mathcal{R}$.

3. To see that $\mathcal{R}$ is closed under products, consider f, g as in the statement. The morphisms $f \times 1_X : W \times X \to Y \times X$ and $1_Y \times g : Y \times X \to Y \times Z$ are in $\mathcal{R}$ as they can be constructed as pullbacks of f, g respectively. Then $f \times g$ is the composition of $f \times 1_X$ and $1_Y \times g$, so it is also in $\mathcal{R}$.

4. To see that $\mathcal{R}$ is closed under retracts, consider a lifting problem as below with $\ell \in \mathcal{L}$ and g as in the statement.

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "\ell"'] & W \arrow[d, "g"] \\ C \arrow[r, "c"] \arrow[ur, dashed] & Z \end{tikzcd}$$

To find a solution, we first find a solution to the following lifting problem,

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "\ell"'] & W \arrow[r, "w"] & X \arrow[d, "f"] \\ C \arrow[r, "c"'] \arrow[urr, dashed, "s"] & Z \arrow[r, "z"'] & Y \end{tikzcd}$$

and then xs is a solution to the original lifting problem. Thus $g \in \mathcal{R}$. $\square$

Remark 1.1.6. Note that this result will be employed frequently and so it will be used without citation.

Lemma 1.1.7. Consider categories $\mathcal{M}$ and $\mathcal{N}$ with lifting pairs $(\mathcal{L}_{\mathcal{M}}, \mathcal{R}_{\mathcal{M}})$ and $(\mathcal{L}_{\mathcal{N}}, \mathcal{R}_{\mathcal{N}})$, respectively, and an adjunction

$$L \dashv R : \mathcal{M} \rightleftarrows \mathcal{N}.$$

Then L preserves the left class if and only if R preserves the right class of these weak factorization systems. More precisely, $L(\mathcal{L}_{\mathcal{M}}) \subseteq \mathcal{L}_{\mathcal{N}}$ if and only if $R(\mathcal{R}_{\mathcal{N}}) \subseteq \mathcal{R}_{\mathcal{M}}$.

Proof. Suppose that $L(\mathcal{L}_{\mathcal{M}}) \subseteq \mathcal{L}_{\mathcal{N}}$. To show that $R(\mathcal{R}_{\mathcal{N}}) \subseteq \mathcal{R}_{\mathcal{M}}$ it suffices to show that $\mathcal{L}_{\mathcal{M}} \boxslash R(\mathcal{R}_{\mathcal{N}})$.

To that end, consider a morphism $\ell \in \mathcal{L}_{\mathcal{M}}$, a morphism $Rr \in R(\mathcal{R}_{\mathcal{N}})$, and a lifting problem between them.

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "\ell"'] & RB \arrow[d, "Rr"] \\ C \arrow[r, "c"] \arrow[ur, dashed] & RD \end{tikzcd}$$

Let χ denote the bijection $\hom(LX, Y) \to \hom(X, RY)$. Then we can form the following lifting problem, which has a solution s by hypothesis.

$$\begin{tikzcd} LA \arrow[r, "\chi^{-1}a"] \arrow[d, "L\ell"'] & B \arrow[d, "r"] \\ LC \arrow[r, "\chi^{-1}c"'] \arrow[ur, dashed, "s"] & D \end{tikzcd}$$

Then $\chi(s)$ is a solution to the original lifting problem.

Therefore, $L(\mathcal{L}_{\mathcal{M}}) \subseteq \mathcal{L}_{\mathcal{N}}$ implies $R(\mathcal{R}_{\mathcal{N}}) \subseteq \mathcal{R}_{\mathcal{M}}$. The proof that $R(\mathcal{R}_{\mathcal{N}}) \subseteq \mathcal{R}_{\mathcal{M}}$ implies $L(\mathcal{L}_{\mathcal{M}}) \subseteq \mathcal{L}_{\mathcal{N}}$ is dual to the argument just given. □

1.2 Factorizations.

We now introduce factorizations.

Definition 1.2.1. A factorization (λ, ρ) on a category $\mathcal{M}$ consists of:

1. a function $(\lambda, \rho) : \mathrm{mor}\mathcal{M} \to \mathrm{mor}\mathcal{M} \times \mathrm{mor}\mathcal{M}$ which takes a morphism f as shown below to composable morphisms $\lambda f, \rho f$ whose composition is f; and

$$X \xrightarrow{\quad f \quad} Y \qquad\qquad X \xrightarrow{\lambda f} Mf \xrightarrow{\rho f} Y$$

2. a function M which takes any commutative square $\langle \alpha, \beta \rangle$ as shown below on the left to a commutative diagram as shown below on the right.

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[d, "f"'] & W \arrow[d, "g"] \\ Y \arrow[r, "\beta"] & Z \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "\alpha"] \arrow[d, "\lambda(f)"'] & W \arrow[d, "\lambda(g)"] \\ Mf \arrow[r, "M\langle \alpha, \beta \rangle"] \arrow[d, "\rho(f)"'] & Mg \arrow[d, "\rho(g)"] \\ Y \arrow[r, "\beta"] & Z \end{tikzcd}$$

Note that the second function in the definition above is not traditionally part of the definition of 'factorization'. However, such a function is entailed by the usual definition of 'functorial factorization' which we introduce now.

We first need to establish some categorical notation. Let $\mathbb{2}$ denote the poset containing the natural numbers $0, 1$,

$$0 \longrightarrow 1$$

and let $\mathbb{3}$ denote the poset containing the natural numbers $0, 1, 2$.

$$0 \longrightarrow 1 \longrightarrow 2$$

Then the objects of the category $\mathcal{M}^{\mathbb{2}}$ (the internal hom of $\mathbb{2}$ and $\mathcal{M}$ in the cartesian closed structure on $\mathcal{C}at$) are morphisms of $\mathcal{M}$, and its morphisms $f \to g$ are pairs $\langle \alpha, \beta \rangle$ of morphisms of $\mathcal{M}$ which fit into the following diagram and make it commute.

$$\begin{tikzcd} \cdot \arrow[r, "\alpha"] \arrow[d, "f"'] & \cdot \arrow[d, "g"] \\ \cdot \arrow[r, "\beta"] & \cdot \end{tikzcd}$$

Similarly, the objects of the category $\mathcal{M}^{\mathbb{3}}$ are pairs (λ, ρ) of composable morphisms of $\mathcal{M}$, and its morphisms $(\lambda, \rho) \to (\lambda', \rho')$ are triples $\langle \alpha, \beta, \gamma \rangle$ fitting into the following diagram and making it commute.

$$\begin{tikzcd} \cdot \arrow[r, "\alpha"] \arrow[d, "\lambda"'] & \cdot \arrow[d, "\lambda'"] \\ \cdot \arrow[r, "\beta"] \arrow[d, "\rho"'] & \cdot \arrow[d, "\rho'"] \\ \cdot \arrow[r, "\gamma"] & \cdot \end{tikzcd}$$

There are functors $\delta_0, \delta_1, \delta_2 : \mathbb{2} \to \mathbb{3}$ which map the non-trivial morphism $0 \leqslant 1$ of $\mathbb{2}$ to the morphism $1 \leqslant 2$, $0 \leqslant 2$, or $1 \leqslant 2$ of $\mathbb{3}$, respectively (borrowing simplicial notation). These give rise to functors $\mathcal{M}^{\mathbb{3}} \to \mathcal{M}^{\mathbb{2}}$. The functor $\mathcal{M}^{\delta_0}$ projects (λ, ρ) to ρ, and the functor $\mathcal{M}^{\delta_2}$ projects (λ, ρ) to λ. The functor $\mathcal{M}^{\delta_1}$ maps (λ, ρ) to the composition $\rho\lambda$.

Now note that the first function $\mathrm{mor}\mathcal{M} \to \mathrm{mor}\mathcal{M} \times \mathrm{mor}\mathcal{M}$ required by the definition of factorization can be characterized as a section of the function $\mathrm{ob}\mathcal{M}^{\delta_1} : \mathrm{ob}\mathcal{M}^{\mathbb{3}} \to \mathrm{ob}\mathcal{M}^{\mathbb{2}}$. Moreover, the second function required by the definition of factorization can be characterized as a section of the function $\mathrm{mor}\mathcal{M}^{\delta_1} : \mathrm{mor}\mathcal{M}^{\mathbb{3}} \to \mathrm{mor}\mathcal{M}^{\mathbb{2}}$. Thus, we are led to an obvious strengthening of the notion of 'factorization'.

Definition 1.2.2. A *functorial factorization* on $\mathcal{M}$ is a section of the functor $\mathcal{M}^{\delta_1} : \mathcal{M}^{\mathbb{3}} \to \mathcal{M}^{\mathbb{2}}$.

By the preceding discussion we know that a functorial factorization on $\mathcal{M}$ is in particular a factorization on $\mathcal{M}$.

We now define the categories of these factorizations. To do that we introduce a notion of (unnatural) transformation between factorizations.

Definition 1.2.3. Consider two factorizations (λ, ρ) and (λ', ρ') on a category $\mathcal{M}$. A *transformation* $\tau : (\lambda, \rho) \to (\lambda', \rho')$ consists of a commutative diagram in $\mathcal{M}$ of the following shape for each morphism $f : X \to Y$ of $\mathcal{M}$.

$$\begin{tikzcd}
& Mf \arrow[dd, "\tau f"] \arrow[dr, "\rho f"] & \\
X \arrow[ur, "\lambda f"] \arrow[dr, "\lambda' f"'] & & Y \\
& M'Y \arrow[ur, "\rho' f"'] &
\end{tikzcd}$$

This constitutes a category.

Definition 1.2.4. Let $\mathfrak{Fact}^{00}_{\mathcal{M}}$ denote the category of factorizations on a category $\mathcal{M}$ and transformations between them.

Let $\mathfrak{Fact}^{10}_{\mathcal{M}}$ denote the category of functorial factorizations on a category $\mathcal{M}$ and transformations between them.

Let $\mathfrak{Fact}^{11}_{\mathcal{M}}$ denote the category of sections of the functor $\mathcal{M}^{\delta_1} : \mathcal{M}^{\mathbb{3}} \to \mathcal{M}^{\mathbb{2}}$. This is the category of functorial factorizations on a category $\mathcal{M}$ and natural transformations between them.

Notation 1.2.5. The superscript ij of $\mathfrak{Fact}^{ij}_{\mathcal{M}}$ is meant to specify what is functorial or natural. The first bit i specifies whether or not the objects are functorial, and the second bit j specifies whether or not the morphisms are natural.

We can see above that $i = 0$ when the objects are not necessarily functors and that $i = 1$ when the objects are required to be functors. Similarly, $j = 0$ denotes that the morphisms are just transformations whereas $j = 1$ means that the morphisms are natural transformations.

There are natural inclusions between these categories.

$$\mathfrak{Fact}^{11}_{\mathcal{M}} \hookrightarrow \mathfrak{Fact}^{10}_{\mathcal{M}} \hookrightarrow \mathfrak{Fact}^{00}_{\mathcal{M}}$$

The first inclusion is the identity on objects, and the second is the identity on hom-sets.

1.3 Weak factorization structures.

Now we introduce the notion of weak factorization structures. First, we need some terminology.

Definition 1.3.1. Consider a pair $(\mathcal{L}, \mathcal{R})$ of classes of morphisms of a category $\mathcal{M}$ and a factorization (λ, ρ) on $\mathcal{M}$. Say that (λ, ρ) is a factorization *into* $(\mathcal{L}, \mathcal{R})$ if $\lambda(f) \in \mathcal{L}$ and $\rho(f) \in \mathcal{R}$ for every morphism f of $\mathcal{M}$.

Definition 1.3.2. A *weak factorization structure* $(\lambda, \rho, \mathcal{L}, \mathcal{R})$ on $\mathcal{M}$ consists of a factorization (λ, ρ) into a lifting pair $(\mathcal{L}, \mathcal{R})$ of $\mathcal{M}$.

Now we describe a factorization for the lifting pair introduced in the previous section.

Example 1.3.3. Consider the lifting pair $(\mathcal{I}, \mathcal{S})$ in $\mathcal{S}et$ defined in Example 1.1.4.

A factorization into $(\mathcal{I}, \mathcal{S})$ can be given for each $f : X \to Y$ by the coproduct.

$$X \hookrightarrow X \cup Y \xrightarrow{f \cup 1} Y$$

In the following example of a weak factorization structure, the solution to any lifting problem between a morphism of the left class and a morphism of the right is unique. Such a weak factorization structure is called an *orthogonal factorization system*, but these will not be considered further in this work.

However, one aspect of this next example foreshadows the perspective that we will take. To show that a map is in the left class (and respectively, right class), it will suffice here to show that it lifts against its right factor (and respectively, left factor). These particular lifting problems will become of central importance in the next section.

Example 1.3.4. Consider a regular category $\mathcal{C}$, the class $\mathcal{E}$ of regular epimorphisms of $\mathcal{C}$, and the class $\mathcal{M}$ of monomorphisms of $\mathcal{C}$. For example, the category $\mathcal{S}et$ is regular, its regular epimorphisms are the surjections, and its monomorphisms are the injections.

In any such category $\mathcal{C}$, there is a functorial factorization of any map f into

$$X \xrightarrow{e_f} \mathrm{Im}(f) \xrightarrow{m_f} Y$$

where e_f is a regular epimorphism and m_f is a monomorphism. In $\mathcal{S}et$, the object $\mathrm{Im}(f)$ is the usual image of f.

This is a factorization into the pair $(\mathcal{E}, \mathcal{M})$ which we claim is a lifting pair.

To see that $\mathcal{E} \boxslash \mathcal{M}$, consider a lifting problem

$$\begin{tikzcd}
A \arrow[r, "x"] \arrow[d, "e"'] & B \arrow[d, "m"] \\
C \arrow[r, "y"'] \arrow[ur, dashed] & D
\end{tikzcd}$$

where $e \in \mathcal{E}$ and $m \in \mathcal{M}$. Since e is a coequalizer, say of $a, b : Z \rightrightarrows A$, and m is a monomorphism, we have that $xa = xb$.

$$\begin{tikzcd}
Z \arrow[d, shift right, "a"'] \arrow[d, shift left, "b"] & \\
A \arrow[r, "x"] \arrow[d, "e"'] & B \arrow[d, "m"] \\
C \arrow[r, "y"'] \arrow[ur, dashed] & D
\end{tikzcd}$$

This induces a morphism $\sigma : C \to B$ which is a solution to our lifting problem.

Then to see that ${}^{\boxslash}\mathcal{M} \subseteq \mathcal{E}$, we consider for any morphism f in ${}^{\boxslash}\mathcal{M}$, the following lifting problem.

$$\begin{tikzcd}
X \arrow[r, two heads, "e_f"] \arrow[d, "f"'] & \mathrm{Im}(s) \arrow[d, hook, "m_f"] \\
Y \arrow[r, equal] \arrow[ur, dashed, "\sigma"] & Y
\end{tikzcd}$$

There is a solution σ since f is in ${}^{\boxslash}\mathcal{M}$ and m_f is in $\mathcal{M}$. From the diagram, we see immediately that $m_f\sigma = 1$. Since m_f is monic, this gives an isomorphism $\mathrm{Im}(s) \cong Y$ and thus f is a regular epimorphism.

To see that $\mathcal{E}^{\boxslash} \subseteq \mathcal{M}$, consider any $f : X \to Y$ in $\mathcal{E}^{\boxslash}$. Then the following lifting problem has a solution σ.

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, "e_f"'] & X \arrow[d, "f"] \\ \mathrm{Im}(f) \arrow[r, hook, "m_f"'] \arrow[ur, dashed, "\sigma"] & Y \end{tikzcd}$$

Since $\sigma e_f = 1$, we see that e_f is a monomorphism. Thus, $m_f e_f = f$ is also a monomorphism.

Our definition of 'weak factorization structure' deviates from the usual one of 'weak factorization system' in two ways. First of all, the second function required in the definition of factorization is not usually given as part of the definition of weak factorization system. Secondly, we explicitly include the factorization as part of the structure of a weak factorization structure, hence the terminology. In Section 1.6, we will define a 'weak factorization system' where the factorization is decidedly *not* part of the structure. Here, we address the first deviation by showing that this second function mentioned above is actually entailed by the usual definition of 'weak factorization system'.

Proposition 1.3.5. *Consider a function $(\lambda, \rho) : mor\mathcal{M} \to mor\mathcal{M} \times mor\mathcal{M}$ such that $\rho f \circ \lambda f = f$ and $\lambda f \boxslash \rho g$ for any morphisms f, g of $\mathcal{M}$. Then the function (λ, ρ) underlies a factorization on $\mathcal{M}$.*

In particular, if $(\mathcal{L}, \mathcal{R})$ is a lifting pair and the image of (λ, ρ) is in $\mathcal{L} \times \mathcal{R}$, then $(\lambda, \rho, \mathcal{L}, \mathcal{R})$ is a weak factorization structure on $\mathcal{M}$.

Proof. Consider a commutative square $\langle \alpha, \beta \rangle : f \to g$ as below.

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[d, "f"'] & W \arrow[d, "g"] \\ Y \arrow[r, "\beta"] & Z \end{tikzcd}$$

We need to find a morphism $M\langle \alpha, \beta \rangle$ which gives a factorization of the square below on the left. We can find such a morphism by solving the lifting problem on the right below.

$$
\begin{tikzcd}
X \arrow[r, "\alpha"] \arrow[d, "\lambda(f)"'] & W \arrow[d, "\lambda(g)"] \\
Mf \arrow[r, dashed, "M\langle\alpha,\beta\rangle"] \arrow[d, "\rho(f)"'] & Mg \arrow[d, "\rho(g)"] \\
Y \arrow[r, "\beta"] & Z
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "\lambda(g)\alpha"] \arrow[d, "\lambda(f)"'] & Mg \arrow[d, "\rho(g)"] \\
Mf \arrow[r, "\beta\rho(f)"'] \arrow[ur, dashed, "M\langle\alpha,\beta\rangle"] & Z
\end{tikzcd}
$$

□

1.4 An algebraic perspective.

In this section, we show that if a factorization (λ, ρ) on a category $\mathcal{M}$ is part of a weak factorization structure $(\lambda, \rho, \mathcal{L}, \mathcal{R})$, then the lifting pair $(\mathcal{L}, \mathcal{R})$ is uniquely determined by the factorization (λ, ρ). That is, a factorization is part of at most one weak factorization structure.

1.4.1 (Co)algebras.

Notice that in a functorial factorization (λ, ρ) on $\mathcal{M}$, the functor ρ is a pointed endofunctor on $\mathcal{M}^2$. Its point $1 \to \rho$ is given at each morphism f of $\mathcal{M}$ by the following commutative square.

$$
\begin{tikzcd}
X \arrow[r, "\lambda f"] \arrow[d, "f"'] & Mf \arrow[d, "\rho f"] \\
Y \arrow[r, equal] & Y
\end{tikzcd}
$$

(Recall that we always use M to denote $\textsc{cod}\lambda = \textsc{dom}\rho$.)

Dually, λ is a copointed endofunctor $\mathcal{M}^2$ whose copoint $\lambda \to 1$ is given at each f by the following square.

$$
\begin{tikzcd}
X \arrow[r, equal] \arrow[d, "\lambda f"'] & X \arrow[d, "f"] \\
Mf \arrow[r, "\rho f"'] & Y
\end{tikzcd}
$$

Now consider morphisms with algebra structures for the pointed endofunctor ρ. A ρ-algebra structure on a morphism f is a commutative square as on the left below which makes the diagram on the right below commute.

$$
\begin{tikzcd}
Mf \arrow[r, "s"] \arrow[d, "\rho f"'] & X \arrow[d, "f"] \\
Y \arrow[r, "t"'] & Y
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
X \arrow[r, "\lambda f"'] \arrow[d, "f"'] \arrow[rr, bend left, equal] & Mf \arrow[r, "s"'] \arrow[d, "\rho f"] & X \arrow[d, "f"] \\
Y \arrow[r, equal] \arrow[rr, bend right, equal] & Y \arrow[r, "t"] & Y
\end{tikzcd}
$$

We can see immediately that t must be the identity. Then rearranging these diagrams, we see that their commutativity is equivalent to the commutativity of the following diagram.

$$
\begin{tikzcd}
X \arrow[r, equal] \arrow[d, "\lambda f"'] & Y \arrow[d, "f"] \\
Mf \arrow[r, "\rho f"'] \arrow[ur, "s"] & Y
\end{tikzcd}
$$

This proves the following result.

Proposition 1.4.1. *Consider a functorial factorization (λ, ρ) on $\mathcal{M}$.*

An algebra structure for the pointed endofunctor ρ on a morphism $f : X \to Y$ is a solution to the following lifting problem,

$$
\begin{tikzcd}
X \arrow[r, equal] \arrow[d, "\lambda f"'] & X \arrow[d, "f"] \\
Mf \arrow[r, "\rho f"'] \arrow[ur, dashed] & Y
\end{tikzcd}
$$

and a coalgebra structure for the copointed endofunctor λ on a morphism f of $\mathcal{M}$ is a solution to the following lifting problem.

$$
\begin{tikzcd}
X \arrow[r, "\lambda f"] \arrow[d, "f"'] & Mf \arrow[d, "\rho f"] \\
Y \arrow[r, equal] \arrow[ur, dashed] & Y
\end{tikzcd}
$$

We use this to extend the notion of algebra and coalgebra to the non-functorial setting.

Definition 1.4.2. Consider a factorization (λ, ρ) on $\mathcal{M}$.

A λ-*coalgebra structure* on a morphism $f : X \to Y$ of $\mathcal{M}$ is a solution to the following lifting problem,

$$\begin{tikzcd}
X \arrow[r, "\lambda f"] \arrow[d, "f"'] & Mf \arrow[d, "\rho f"] \\
Y \arrow[r, equal] \arrow[ur, dashed] & Y
\end{tikzcd}$$

and a ρ-*algebra structure* on f is a solution to the following lifting problem.

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d, "\lambda f"'] & X \arrow[d, "f"] \\
Mf \arrow[r, "\rho f"'] \arrow[ur, dashed] & Y
\end{tikzcd}$$

Let λ-alg denote the class of morphisms with λ-coalgebra structures, and let ρ-coalg denote the class of morphisms with ρ-algebra structures.

Remember that a factorization (λ, ρ) on $\mathcal{M}$ gives not only a factorization of any morphism of $\mathcal{M}$ but also a factorization of any commutative square in $\mathcal{M}$.

$$\begin{tikzcd}
X \arrow[r, "\alpha"] \arrow[d, "f"'] & W \arrow[d, "g"] \\
Y \arrow[r, "\beta"] & Z
\end{tikzcd}
\quad \mapsto \quad
\begin{tikzcd}
X \arrow[r, "\alpha"] \arrow[d, "\lambda(f)"'] & W \arrow[d, "\lambda(g)"] \\
Mf \arrow[r, "M\langle\alpha,\beta\rangle"] \arrow[d, "\rho(f)"'] & Mg \arrow[d, "\rho(g)"] \\
Y \arrow[r, "\beta"] & Z
\end{tikzcd}$$

Consider a morphism ℓ with a λ-coalgebra structure s and a morphism r with a ρ-algebra structure t. Then we can construct a lift in the square below on the left. First, factor it to get the diagram in the middle.

$$\begin{tikzcd}
X \arrow[r, "\alpha"] \arrow[d, "\ell"'] & W \arrow[d, "r"] \\
Y \arrow[r, "\beta"] & Z
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "\alpha"] \arrow[d, "\lambda(\ell)"'] & W \arrow[d, "\lambda(r)"] \\
M\ell \arrow[r, "M\langle\alpha,\beta\rangle"] \arrow[d, "\rho(\ell)"'] & Mr \arrow[d, "\rho(r)"] \\
Y \arrow[r, "\beta"] & Z
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[rr, "\alpha"] \arrow[d, "\lambda(\ell)"'] & & W \arrow[d, "\lambda(r)"] \\
M\ell \arrow[rr, "M\langle\alpha,\beta\rangle"] \arrow[d, "\rho(\ell)"'] & & Mr \arrow[d, "\rho(r)"] \arrow[u, bend left, "t"] \\
Y \arrow[rr, "\beta"] \arrow[u, bend right, "s"'] & & Z
\end{tikzcd}$$

Then the composite $t \circ M\langle\alpha, \beta\rangle \circ s$ gives a solution to the original lifting problem. Therefore, λ-alg $\boxslash$ ρ-coalg.

Now we have proven the following proposition.

Proposition 1.4.3. *Consider a factorization (λ, ρ) on a category $\mathcal{M}$. Then λ-alg $\boxslash$ ρ-coalg.*

Now we show that if (λ, ρ) is part of a weak factorization system then the left class of morphisms must be λ-alg and the right class must be ρ-coalg. We prove a slightly more general statement since it will be of use in later chapters.

Proposition 1.4.4. *Consider a factorization (λ, ρ) on a category $\mathcal{M}$. Consider also a class of morphisms $\mathcal{R}$ such that $\lambda(f) \boxslash \mathcal{R}$ and $\rho(f) \in \mathcal{R}$ for every morphism f of $\mathcal{M}$. Then λ-alg $= {}^{\boxslash}\mathcal{R}$.*

Proof. To see that ${}^{\boxslash}\mathcal{R} \subseteq \lambda$-alg, consider a morphism $\ell \in {}^{\boxslash}\mathcal{R}$. Since $\rho(\ell) \in \mathcal{R}$, we see that $\ell \boxslash \rho(\ell)$, and thus ℓ has a λ-coalgebra structure.

To see that λ-alg $\subseteq {}^{\boxslash}\mathcal{R}$, first note that Proposition 1.4.3 implies that λ-alg $\subseteq {}^{\boxslash}\rho$-coalg. Since every $r \in \mathcal{R}$ has the right lifting property against $\lambda(r)$, it has a ρ-algebra structure. Thus, $\mathcal{R} \subseteq \rho$-coalg so ${}^{\boxslash}\rho$-coalg $\subseteq {}^{\boxslash}\mathcal{R}$ (since for any classes of maps, $\mathcal{A} \subseteq \mathcal{B}$ implies ${}^{\boxslash}\mathcal{B} \subseteq {}^{\boxslash}\mathcal{A}$). Then we have that λ-alg $\subseteq {}^{\boxslash}\rho$-coalg $\subseteq {}^{\boxslash}\mathcal{R}$. $\square$

Corollary 1.4.5. *Consider a weak factorization structure $(\lambda, \rho, \mathcal{L}, \mathcal{R})$ on a category $\mathcal{M}$. Then we have the following equality.*

$$(\lambda, \rho, \mathcal{L}, \mathcal{R}) = (\lambda, \rho, \lambda\text{-alg}, \rho\text{-coalg})$$

Proof. Applying the preceding proposition to the right class $\mathcal{R}$, we see that λ-alg $= {}^{\boxslash}\mathcal{R} = \mathcal{L}$. Applying the dual of the preceding proposition to $\mathcal{L}$, we see that ρ-coalg $= \mathcal{L}^{\boxslash} = \mathcal{R}$. $\square$

Now we have shown that λ-alg$\boxslash\rho$-coalg for any factorization (λ, ρ), and if this is already part of a weak factorization structure $(\lambda, \rho, \mathcal{L}, \mathcal{R})$, then $(\lambda, \rho, \mathcal{L}, \mathcal{R}) = (\lambda, \rho, \lambda\text{-alg}, \rho\text{-coalg})$.

1.4.2 Weakly algebraic factorizations.

Now we investigate what properties of a factorization (λ, ρ) will ensure that it is part of a weak factorization structure $(\lambda, \rho, \lambda\text{-alg}, \rho\text{-coalg})$.

Reading Definition 1.1.3, we see that λ-alg, ρ-coalg form a lifting pair if and only if

1. λ-alg$^{\boxslash} \subseteq \rho$-coalg , and

2. ${}^{\boxslash}\rho$-**coalg** $\subseteq \lambda$-**alg**

since we already know that λ-**alg** $\boxslash$ ρ-**coalg** by Proposition 1.4.3. Reading Definition 1.3.1, we see that this is a factorization into λ-**alg**, ρ-**coalg** if and only if

3. $\lambda(f) \in \lambda$-**alg** for every morphism f of $\mathcal{M}$,

4. $\rho(f) \in \rho$-**coalg** for every morphism f of $\mathcal{M}$,

But property (3) implies (1), and property (4) implies (2).

To see that (3) implies (1), consider a morphism $r \in \lambda$-**alg**$^{\boxslash}$. Property (3) implies that $\lambda(r) \in \lambda$-**alg**, so there is a solution to the following lifting problem.

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, "\lambda(r)"'] & X \arrow[d, "r"] \\ Mf \arrow[r, "\rho(r)"'] \arrow[ur, dashed] & Y \end{tikzcd}$$

Thus, r is in ρ-**coalg** so property (1) holds. The argument that property (4) implies (2) is dual to this one.

Therefore, a factorization (λ, ρ) underlies a weak factorization structure if and only if $\lambda(f)$ has a λ-coalgebra structure and $\rho(f)$ has a ρ-algebra structure for every morphism f of $\mathcal{M}$. This leads us to the following definition and result.

Definition 1.4.6. A factorization (λ, ρ) on $\mathcal{M}$ is *weakly algebraic* if $\lambda(f)$ has a λ-coalgebra structure and $\rho(f)$ has a ρ-algebra structure for every morphism f of $\mathcal{M}$.

Proposition 1.4.7. *A factorization (λ, ρ) on $\mathcal{M}$ underlies a weak factorization structure if and only if it is weakly algebraic, and this weak factorization structure is $(\lambda, \rho, \lambda$-**alg**$, \rho$-**coalg**$)$.*

Proof. In the above discussion, we saw that a weakly algebraic factorization (λ, ρ) is part of a weak factorization structure (λ, ρ).

Conversely, by Corollary 1.4.5, in a weak factorization structure $(\lambda, \rho, \mathcal{L}, \mathcal{R})$, we have λ-**alg** $= \mathcal{L}$ and ρ-**coalg** $= \mathcal{R}$. Since (λ, ρ) is a factorization into $(\mathcal{L}, \mathcal{R}) = (\lambda$-**alg**$, \rho$-**coalg**$)$, every $\lambda(f)$ has a λ-coalgebra structure, and every $\rho(f)$ has a ρ-algebra structure. $\square$

Remark 1.4.8. In this proposition, 1.4.7, we see that if a factorization (λ, ρ) underlies a weak factorization structure, then it completely determines that weak factorization structure. Thus, in this case, we will call the factorization (λ, ρ) itself a weak factorization structure.

Proposition 1.4.9. *Consider a factorization (λ, ρ) on $\mathcal{M}$. If a morphism $f : X \to Y$ has a λ-coalgebra structure, then it is a retract of $\lambda(f)$ in the slice $X/\mathcal{M}$. If it has a ρ-algebra structure, then it is a retract of $\rho(f)$ in the slice $\mathcal{M}/Y$.*

Proof. We saw in the preceding discussion that if a morphism $f : X \to Y$ has a ρ-algebra structure, then there is a morphism s making the following diagram commute.

$$\begin{tikzcd} X \arrow[r, "\lambda(f)"'] \arrow[rr, bend left, equal] \arrow[d, "f"'] & Mf \arrow[r, "s"'] \arrow[d, "\rho(f)"] & X \arrow[d, "f"] \\ Y \arrow[r, equal] \arrow[rr, bend right, equal] & Y \arrow[r, equal] & Y \end{tikzcd}$$

Thus, f is a retract of $\rho(f)$ in the slice $\mathcal{M}/Y$. □

Corollary 1.4.10. *If λ-**alg** is part of a lifting pair and contains the image of λ, then it is the retract closure of the image of λ. Dually, if ρ-**coalg** is part of a lifting pair and contains the image of ρ, then it is also the retract closure of the image of ρ.*

Proof. Suppose that ρ-**coalg** is part of a lifting pair and contains the image of ρ. Then any ρ-algebra f is a retract of $\rho(f)$ in $\mathcal{M}^2$ by the preceding proposition, 1.4.9. By Proposition 1.1.5, ρ-**coalg** is closed under such retracts. Thus, it is the retract closure of the image of ρ.

The statement concerning λ follows from the dual argument. □

Corollary 1.4.11. *A factorization (λ, ρ) on $\mathcal{M}$ is a weak factorization structure if and only if λ-**alg** is the retract closure of the image of λ and ρ-**coalg** is the retract closure of the image of ρ.*

Proof. Suppose that (λ, ρ) is a weak factorization structure. Then the preceding corollary says that λ-**alg** is the retract closure of the image of λ and ρ-**coalg** is the retract closure of the image of ρ.

Conversely, suppose that (λ, ρ) is a factorization on $\mathcal{M}$ such that λ-**alg** is the retract closure of the image of λ and ρ-**coalg** is the retract closure of the image of ρ. Then λ-**alg** contains the image of λ and ρ-**coalg** contains the image of ρ. Thus, $\lambda(f)$ has a λ-coalgebra structure, and $\rho(f)$ has a ρ-algebra structure for each morphism f of $\mathcal{M}$. Then we can conclude that (λ, ρ) is weakly algebraic and is therefore a weak factorization structure. $\square$

1.4.3 Categories of weak factorization structures.

Since we have shown that any factorization (λ, ρ) is part of at most one weak factorization structure, we can easily define the categories of weak factorization structures.

Definition 1.4.12. Let $\mathfrak{WFS}^{ij}_{\mathcal{M}}$ denote the full subcategory of $\mathfrak{Fact}^{ij}_{\mathcal{M}}$ spanned by those factorizations (λ, ρ) which form a weak factorization structure (λ, ρ) for $ij = 00, 10, 11$.

Now we have defined the following six categories.

$$\begin{tikzcd}
\mathfrak{WFS}^{11}_{\mathcal{M}} \arrow[r, hook] \arrow[d, hook] & \mathfrak{Fact}^{11}_{\mathcal{M}} \arrow[d, hook] \\
\mathfrak{WFS}^{10}_{\mathcal{M}} \arrow[r, hook] \arrow[d, hook] & \mathfrak{Fact}^{10}_{\mathcal{M}} \arrow[d, hook] \\
\mathfrak{WFS}^{00}_{\mathcal{M}} \arrow[r, hook] & \mathfrak{Fact}^{00}_{\mathcal{M}}
\end{tikzcd}$$

The horizontal morphisms are inclusions of full subcategories by definition. As before, the top two vertical inclusions are the identity on objects while the bottom two are the identity on morphisms.

1.5 New weak factorization structures from old.

In this section, we use our characterization of weak factorization structures to describe several situations in which one weak factorization structure induces another. We will need these results in subsequent chapters.

Proposition 1.5.1. *Consider a weak factorization structure $(\lambda, \rho, \mathcal{L}, \mathcal{R})$ on a category $\mathcal{M}$. For any object X, the slice category $\mathcal{M}/X$ inherits a weak factorization structure $(\lambda_X, \rho_X, \mathcal{L}_X, \mathcal{R}_X)$. The factorization λ_X, ρ_X takes any morphism*

$\alpha : f \to g$ *in* $\mathcal{M}/X$ *to*

$$\begin{tikzcd}
\bullet \arrow[r, "\lambda\alpha"] \arrow[dr, "f"'] & M\alpha \arrow[r, "\rho\alpha"] \arrow[d, "g\circ\rho\alpha"] & \bullet \arrow[dl, "g"] \\
& X &
\end{tikzcd}$$

and any commuting square $\langle \gamma, \delta \rangle : \alpha \to \beta$ *in* $\mathcal{M}/X$ *to*

$$\begin{tikzcd}
& \bullet \arrow[rr, "\lambda\beta"] \arrow[ddr, bend right] & & M\beta \arrow[rr, "\rho\beta"] \arrow[ddll] & & \bullet \arrow[ddllll, bend left] \\
\bullet \arrow[ur, "\gamma"] \arrow[rr, "\lambda\alpha"] \arrow[dr] & & M\alpha \arrow[ur, "M\langle\gamma,\delta\rangle"] \arrow[rr, "\rho\alpha"] \arrow[dl] & & \bullet \arrow[ur, "\delta"] \arrow[dlll] & \\
& X & & & &
\end{tikzcd}$$

The lifting pair is given by $\mathcal{L}_X = \textsc{dom}^{-1}\mathcal{L}$*,* $\mathcal{R}_X = \textsc{dom}^{-1}\mathcal{R}$ *(where* $\textsc{dom} : \mathcal{M}/X \to \mathcal{M}$ *maps* $f : D \to X$ *to* D*).*

Proof. First of all, note that a lifting problem in $\mathcal{M}/X$ has a solution if and only if its underlying lifting problem in $\mathcal{M}$ does. That is, if the following two left-most diagrams in $\mathcal{M}$ commute, then we can paste them together to obtain a commuting diagram in $\mathcal{M}$ as shown on the right below.

$$\begin{tikzcd}
\bullet \arrow[r, "\alpha"] \arrow[d, "f"'] \arrow[dd, bend right] & \bullet \arrow[d, "g"] \arrow[ddl, bend left] \\
\bullet \arrow[r, "\beta"] \arrow[d] & \bullet \arrow[dl] \\
X &
\end{tikzcd}
\qquad
\begin{tikzcd}
\bullet \arrow[r, "\alpha"] \arrow[d, "f"'] & \bullet \arrow[d, "g"] \\
\bullet \arrow[r, "\beta"'] \arrow[ur, "\sigma"] & \bullet
\end{tikzcd}
\qquad
\begin{tikzcd}
\bullet \arrow[r, "\alpha"] \arrow[d, "f"'] \arrow[dd, bend right] & \bullet \arrow[d, "g"] \arrow[ddl, bend left] \\
\bullet \arrow[r, "\beta"] \arrow[ur, "\sigma"] \arrow[d] & \bullet \arrow[dl] \\
X &
\end{tikzcd}$$

Therefore, the diagrams of the statement do commute.

We also see that a morphism α of $\mathcal{M}/X$ has a λ_X-coalgebra structure if and only if $\textsc{dom}(\alpha)$ has a λ-coalgebra structure. Since $\lambda\textsc{dom}(\alpha) = \textsc{dom}\lambda_X(\alpha)$ has a λ-coalgebra structure, $\lambda_X(\alpha)$ has a λ_X-coalgebra structure. Then $\mathcal{L}_X = \textsc{dom}^{-1}\mathcal{L}$ and this is the class of morphisms with λ_X-coalgebra structures.

Dually, we find that every $\rho_X(\alpha)$ has a ρ_X-algebra structure and that $\mathcal{R}_X = \textsc{dom}^{-1}\mathcal{R} = \rho_{\mathbf{X}}$**-coalg**.

Thus, (λ_X, ρ_X) is weakly algebraic, so by Proposition 1.4.7, $(\lambda_X, \rho_X, \mathcal{L}_X, \mathcal{R}_X)$ is a weak factorization structure. $\square$

Proposition 1.5.2. *Consider a weak factorization structure* $(\lambda, \rho, \mathcal{L}, \mathcal{R})$ *on a category* $\mathcal{M}$*. Also consider a full subcategory* $\mathcal{N}$ *of* $\mathcal{M}$ *such that for every morphism* f *of* $\mathcal{N}$*, the object* $\text{COD}\lambda(f) = \text{DOM}\rho(f)$ *is in* $\mathcal{N}$ *(i.e., the factorization restricts to* $\mathcal{N}$*). Then the weak factorization structure on* $\mathcal{M}$ *restricts to a weak factorization structure* $(\lambda, \rho, \mathcal{L} \cap \mathcal{N}, \mathcal{R} \cap \mathcal{N})$ *on* $\mathcal{N}$*.*

Proof. Note that a morphism of $\mathcal{N}$ is a λ-coalgebra in $\mathcal{N}$ if and only if it is a λ-coalgebra in $\mathcal{M}$. Thus, every $\lambda(f)$ is a λ-coalgebra in $\mathcal{N}$ since it is one in $\mathcal{M}$. Furthermore, we see that $\mathcal{L} \cap \mathcal{N}$ is the class of morphisms with λ-coalgebra structures in $\mathcal{N}$.

Dually, every $\rho(f)$ is a ρ-algebra in $\mathcal{N}$, and $\mathcal{R} \cap \mathcal{N}$ is the class of ρ-algebras in $\mathcal{N}$.

Therefore (λ, ρ) is weakly algebraic on $\mathcal{N}$, and by Proposition 1.4.7, $(\lambda, \rho, \mathcal{L} \cap \mathcal{N}, \mathcal{R} \cap \mathcal{N})$ is a weak factorization structure on $\mathcal{N}$. $\square$

Now we borrow some terminology from model category theory proper.

Definition 1.5.3. Consider a weak factorization structure $(\lambda, \rho, \mathcal{L}, \mathcal{R})$ on a category $\mathcal{M}$. An object X is *fibrant* if a morphism $X \to *$ to a terminal object is in $\mathcal{R}$.

Corollary 1.5.4. *Consider a weak factorization structure* $(\lambda, \rho, \mathcal{L}, \mathcal{R})$ *on a category* $\mathcal{M}$*. Let* $\mathcal{M}_\mathcal{F}$ *denote the full subcategory of* $\mathcal{M}$ *spanned by its fibrant objects. Then* $\mathcal{M}_\mathcal{F}$ *inherits a weak factorization structure* $(\lambda, \rho, \mathcal{L} \cap \mathcal{M}_\mathcal{F}, \mathcal{R} \cap \mathcal{M}_\mathcal{F})$*.*

Proof. Consider any morphism $f : X \to Y$ in $\mathcal{M}_\mathcal{F}$. Since $\rho(f)$ is in $\mathcal{R}$, and $! : Y \to *$ is in $\mathcal{R}$, the composition $! : Mf \to *$ is also in $\mathcal{R}$. Thus, Mf is in $\mathcal{M}_\mathcal{F}$. Then the preceding proposition, 1.5.2, applies. $\square$

Corollary 1.5.5. *Consider a weak factorization structure* $(\lambda, \rho, \mathcal{L}, \mathcal{R})$ *on a category* $\mathcal{M}$*. Let* $\mathcal{D}$ *be a class of maps of* $\mathcal{M}$ *which contains the image of* ρ *and is closed under composition. For any object* X *of* $\mathcal{M}$*, let* $\{\mathcal{D}, \mathcal{M}\}_X$ *denote the full subcategory of the slice* $\mathcal{M}/X$ *spanned by those objects which are in* $\mathcal{D}$*. Then*

$$(\lambda_X, \rho_X, \mathcal{L}_X \cap \{\mathcal{D}, \mathcal{M}\}_X, \mathcal{R}_X \cap \{\mathcal{D}, \mathcal{M}\}_X)$$

is a weak factorization structure on $\{\mathcal{D}, \mathcal{M}\}_X$*.*

Proof. By Proposition 1.5.1, there is a weak factorization structure

$$(\lambda_X, \rho_X, \mathcal{L}_X, \mathcal{R}_X)$$

in $\mathcal{M}/X$. Consider a morphism $\alpha : f \to g$ in $\{\mathcal{D}, \mathcal{M}\}_X$. The middle object of the factorization of α is the composition $g\rho(\textsc{dom}(\alpha))$ as illustrated below.

$$\begin{tikzcd}
{} \arrow[r, "\lambda(\textsc{dom}\alpha)"] \arrow[dr, "f"'] & {} \arrow[r, "\rho(\textsc{dom}\alpha)"] \arrow[d, dashed, "g\rho(\textsc{dom}\alpha)"] & {} \arrow[dl, "g"] \\
& X &
\end{tikzcd}$$

Since g and $\rho(\textsc{dom}(\alpha))$ are in $\mathcal{D}$, the composition $g\rho(\textsc{dom}(\alpha))$ is also in $\mathcal{D}$. Thus, by Proposition 1.5.2, we see that $\{\mathcal{D}, \mathcal{M}\}_X$ has a weak factorization structure

$$(\lambda_X, \rho_X, \mathcal{L}_X \cap \{\mathcal{D}, \mathcal{M}\}_X, \mathcal{R}_X \cap \{\mathcal{D}, \mathcal{M}\}_X). \qquad \square$$

1.6 Weak factorization systems.

We are also interested in the concept of *weak factorization system*, which for us is similar to a weak factorization structure except that the factorization is not explicitly given as part of the structure, but merely assumed to exist. We take the view that a weak factorization structure is a 'presentation' of a weak factorization system. Indeed the third chapter of this thesis can be understood as an account of finding nice 'presentations' of certain weak factorization systems.

Definition 1.6.1. Say that two weak factorization structures are *equivalent* if they have the same underlying lifting pair. A *weak factorization system* $(\mathcal{L}, \mathcal{R})$ on a category $\mathcal{M}$ is an equivalence class of weak factorization structures with the underlying lifting pair $(\mathcal{L}, \mathcal{R})$.

If W is a weak factorization structure, we will denote the weak factorization system that W represents by $[W]$.

A weak factorization system on a category $\mathcal{M}$ is then a lifting pair $(\mathcal{L}, \mathcal{R})$ for which there exists a factorization (λ, ρ) into $(\mathcal{L}, \mathcal{R})$. This coincides with the usual definition of weak factorization system (cf. §2.1 of [RT02], where the terminology closely matches ours, or Def. 6.2 of [Bou77], where the notion first appeared).

Consider the category $\mathfrak{WFS}^{00}_{\mathcal{M}}$ defined previously. The equivalence relation on weak factorization structures is already encoded by this category.

Proposition 1.6.2. *Consider two weak factorization structures (λ, ρ) and (λ', ρ') on a category $\mathcal{M}$. These are equivalent if and only if there are morphisms $(\lambda, \rho) \leftrightarrows (\lambda', \rho')$ in $\mathfrak{WFS}^{00}_{\mathcal{M}}$.*

Proof. Suppose that there is a morphism $a : (\lambda, \rho) \to (\lambda', \rho')$ in $\mathfrak{WFS}^{00}_{\mathcal{M}}$. We want to show that a morphism has a λ'-coalgebra structure if it has a λ-coalgebra structure. So consider a morphism ℓ with a λ-coalgebra structure σ. The composition $a_\ell \sigma$ as displayed in the following diagram is a λ'-coalgebra structure for ℓ.

$$\begin{tikzcd}
\cdot \arrow[rr, "\lambda'\ell"] \arrow[dd, "\ell"'] \arrow[dr, "\lambda\ell"] & & \cdot \arrow[dd, "\rho'\ell"] \\
& \cdot \arrow[ur, "a_\ell"'] \arrow[dr, "\rho\ell"'] & \\
\cdot \arrow[ur, "\sigma"] \arrow[rr, equal] & & \cdot
\end{tikzcd}$$

Thus, $\lambda\text{-}\mathsf{alg} \subseteq \lambda'\text{-}\mathsf{alg}$, and consequently $\rho'\text{-}\mathsf{coalg} = \lambda'\text{-}\mathsf{alg}^{\boxslash} \subseteq \lambda\text{-}\mathsf{alg}^{\boxslash} = \rho\text{-}\mathsf{coalg}$.

If there is also a morphism $b : (\lambda', \rho') \to (\lambda, \rho)$ in $\mathfrak{WFS}^{00}_{\mathcal{M}}$, then we can conclude dually that $\lambda\text{-}\mathsf{alg} \subseteq \lambda'\text{-}\mathsf{alg}$ and $\rho'\text{-}\mathsf{coalg} \subseteq \rho\text{-}\mathsf{coalg}$.

Therefore, if there are morphisms $(\lambda, \rho) \leftrightarrows (\lambda', \rho')$, then $\lambda\text{-}\mathsf{alg} = \lambda'\text{-}\mathsf{alg}$ and $\rho\text{-}\mathsf{coalg} = \rho'\text{-}\mathsf{coalg}$. Thus, the weak factorization systems (λ, ρ) and (λ', ρ') are equivalent.

Now consider two equivalent weak factorization structures (λ, ρ) and (λ', ρ'). Since every λf lifts against every $\rho' f$ and every $\lambda' f$ lifts against every ρf, we obtain the following morphisms

$$\begin{tikzcd}
\cdot \arrow[r, "\lambda' f"] \arrow[d, "\lambda f"'] & \cdot \arrow[d, "\rho' f"] \arrow[dl, dashed] \\
\cdot \arrow[r, "\rho f"'] \arrow[ur, dashed] & \cdot
\end{tikzcd}
\qquad
\begin{tikzcd}
\cdot \arrow[r, "\lambda f"] \arrow[d, "\lambda' f"'] & \cdot \arrow[d, "\rho f"] \arrow[dl, dashed] \\
\cdot \arrow[r, "\rho' f"'] \arrow[ur, dashed] & \cdot
\end{tikzcd}$$

which assemble into transformations $(\lambda, \rho) \leftrightarrows (\lambda', \rho')$ in $\mathfrak{WFS}^{00}_{\mathcal{M}}$. $\square$

Now we can make this equivalence relation more explicitly categorical by making this category into a proset (pre-ordered set).

Definition 1.6.3. Given a category $\mathcal{C}$, define the proset $|\mathcal{C}|$ to have the same objects as $\mathcal{C}$ and morphisms

$$\hom_{|\mathcal{C}|}(X,Y) = \begin{cases} * & \text{if } \hom_{\mathcal{C}}(X,Y) \text{ is inhabited} \\ \varnothing & \text{otherwise} \end{cases}$$

Note that this constitutes a 2-functor from categories to prosets, which is often called the proset reflection.

Then isomorphism in $|\mathcal{C}|$ defines an equivalence relation on the objects of $\mathcal{C}$. This is the equivalence relation given by $X \simeq Y$ when X and Y are isomorphic in $|\mathcal{C}|$ or, equivalently, when there exist morphisms $X \leftrightarrows Y$ in $\mathcal{C}$. Therefore, we see the following.

Corollary 1.6.4. *Consider two weak factorization structures on a category $\mathcal{M}$. Then these are equivalent if and only if they are isomorphic as objects of $|\mathfrak{WFS}^{00}_{\mathcal{M}}|$.*

In other words, the isomorphism classes of $|\mathfrak{WFS}^{00}_{\mathcal{M}}|$ are the weak factorization systems on $\mathcal{M}$.

Proof. This follows immediately from the preceding proposition, 1.6.2, and the fact that (λ, ρ) and (λ', ρ') are isomorphic in $|\mathfrak{WFS}^{00}_{\mathcal{M}}|$ if and only if there are morphisms $(\lambda, \rho) \leftrightarrows (\lambda', \rho')$ in $\mathfrak{WFS}^{00}_{\mathcal{M}}$. □

1.7 Algebraic weak factorization structures.

Consider a functorial factorization (λ, ρ) on $\mathcal{M}$. We saw in Proposition 1.4.7 that this underlies a functorial weak factorization structure if and only if it is weakly algebraic: that is, if $\lambda(f)$ has a λ-coalgebra structure and $\rho(f)$ has a ρ-algebra structure for every morphism f of $\mathcal{M}$.

A ρ-algebra structure for $\rho(f)$ is a morphism $\mu_f : \rho^2(f) \to \rho(f)$ in $\mathcal{M}^2$ making the following diagram commute.

$$\begin{tikzcd} \rho(f) \arrow[r, "\langle \lambda\rho_f, 1\rangle"] \arrow[dr, equal] & \rho^2(f) \arrow[d, "\mu_f"] \\ & \rho(f) \end{tikzcd}$$

If there were a choice of each μ_f, natural in f, and if it satisfied the monad axioms,

$$\begin{tikzcd} \rho \arrow[r, "\langle\lambda,1\rangle"] \arrow[dr, equal] & \rho^2 \arrow[d, "\mu"] & \rho \arrow[l, "\rho\langle\lambda,1\rangle"'] \arrow[dl, equal] \\ & \rho & \end{tikzcd} \qquad \begin{tikzcd} \rho^3 \arrow[r, "\mu"] \arrow[d, "\rho\mu"] & \rho^2 \arrow[d, "\mu"] \\ \rho^2 \arrow[r, "\mu"] & \rho \end{tikzcd}$$

then $(\rho, \langle\lambda, 1\rangle, \mu)$ would be a monad on $\mathcal{M}^2$. Dually, a natural choice of λ-coalgebra structures $\delta_f : \lambda(f) \to \lambda^2(f)$ for each f satisfying the comonad axioms would make λ into a comonad on $\mathcal{M}^2$. Thus, we have the following result and definition.

Definition 1.7.1. An *algebraic weak factorization structure* on a category $\mathcal{M}$ consists of a functorial factorization structure (λ, ρ), a multiplication μ making the pointed endofunctor ρ into a monad, and a comultiplication δ making the copointed endofunctor λ into a comonad.

Note that we do not include a distributivity law in this definition, as is often done.

This defines extra structure on a weakly algebraic factorization (λ, ρ) which takes it from weakly to fully algebraic. Thus, it generates a weak factorization structure (λ, ρ).

This gives us the following result. It is a restatement of Proposition 2.5 of [GT06] in our vocabulary.

Theorem 1.7.2. *Consider an algebraic weak factorization structure on a category $\mathcal{M}$ with underlying factorization (λ, ρ). Then (λ, ρ) is a weak factorization structure on $\mathcal{M}$.*

1.8 Summary.

In this chapter, we defined weak factorization structures and systems on a category $\mathcal{M}$. We showed that a factorization (λ, ρ) is part of a weak factorization structure in at most one way, and this depends on whether or not the factorization is weakly algebraic.

Chapter 2

Models of type theory: display map categories.

In this chapter, we will define our notion of a model of dependent type theory with Σ, Π, and `Id` types in a category. This is the structure which will be studied in the following chapters of this thesis.

There are currently many definitions of a model of dependent type theory in the literature. Ours, which we describe in Sections 2.1 – 2.4, will be very closely related to the notions of a *class of displays* of [Tay99] and *tribe* of [Joy13]. We aim to make our definition a special case of that of *comprehension categories* of [Jac93]. These are well-studied and accepted as a good notion of a model of type theory, and so our models will enjoy results of the literature concerning comprehension categories.

The contents of Section 2.5 aim to simplify the situation when one is considering a Cauchy complete category. The results of this section concerning display map categories are original. In Section 2.6, we use these results to characterize the weak factorization systems that we will consider in the next chapter.

2.1 Display map categories.

Definition 2.1.1 ([Jac93, Def. 4.1]). A *comprehension category* consists of a Grothendieck fibration $G : \mathcal{E} \to \mathcal{C}$ and a cartesian functor $F : \mathcal{E} \to \mathcal{C}^2$ which make the following diagram commute.

$$\begin{tikzcd} \mathcal{E} \arrow[rr, "F"] \arrow[dr, "G"'] & & \mathcal{C}^2 \arrow[dl, "\textsc{cod}"] \\ & \mathcal{C} & \end{tikzcd}$$

Remark 2.1.2. Note that we do not require $\mathcal{C}$ to have all pullbacks, or, in other words, $\textsc{cod} : \mathcal{C}^2 \to \mathcal{C}$ to be a Grothendieck fibration itself.

Remark 2.1.3. In this work, we do not attempt to justify that these structures model type theory. However, in this chapter we will point out which features of the model are meant to represent which features of type theory.

In such a comprehension category, $\mathcal{C}$ is meant to represent a category of *contexts* and *context morphisms*, the fiber $G^{-1}(\Gamma)$ to represent *types in context* Γ, the lifting property of G to represent *context substitution*, and the functor F to represent *context extension* in a type theory.

We consider only certain comprehension categories because we want to find models in structures on categories which are already of interest in category theory and categorical homotopy theory. In particular, we want our notion of model to be characterizable with only concepts from categorical homotopy theory. In doing so, this work is an attempt to understand the relationship between *type theory* and *homotopy theory*.

There are thus two principles that will guide our choice of a definition of model:

1. that the structure to be defined is invariant under the isomorphisms of the category (sometimes called the *equivalence principle*), and

2. that the structure can be found within the category.

The first of these ideologies is well-defined; the second, less so. Many definitions of 'model' take multiple categories and functors between them as input. In particular, the data for a comprehension category include two categories and a Grothendieck fibration between them. Here, we restrict ourselves to considering only one category and a specified subcategory as input.

Admittedly, these principles will make our notion of 'model' incongruent with the strict syntax of type theory. In particular, many constructions of dependent type theory are defined up to equality, not merely up to isomorphism. However, there is a strictification operation [LW15] which replaces a comprehension category with one equivalent to it (but violating ideologies (1) *and* (2)) that better emulates the syntax of type theory. Our display map categories modeling Σ, Π, and `Id` types will be full comprehension categories modeling weakly stable Σ, Π, and `Id` types, in the language of [LW15]. Thus by Theorem 3.4.1 of [LW15], if the display maps are exponentiable, there is an equivalent comprehension category modeling these types strictly. (Note that below, we will only require that display maps satisfy Definition 2.4.1, a strictly weaker property than that of being exponentiable.) We take this as a justification of our consideration of more 'categorical' models of type theory.

Definition 2.1.4. A *category of display maps* consists of a category $\mathcal{M}$ with a terminal object and a class $\mathcal{D}$ of morphisms of $\mathcal{M}$ such that:

1. $\mathcal{D}$ contains every isomorphism;
2. $\mathcal{D}$ contains every morphism whose codomain is the terminal object;
3. every pullback of every morphism of $\mathcal{D}$ exists; and
4. $\mathcal{D}$ is stable under pullback.

Call the elements of $\mathcal{D}$ *display maps*.

This coincides with Joyal's notion of *tribe* [Joy13]. Without conditions (1) and (2), this is the definition of a *class $\mathcal{D}$ of displays* in $\mathcal{M}$ [Tay99, Def. 8.3.2].

In such a category of display maps, the objects of $\mathcal{M}$ are meant to represent contexts and the morphisms of $\mathcal{M}$ represent context morphisms. A morphism $p : E \to B$ of $\mathcal{D}$ represents a type family E dependent on B. The empty context is represented by the terminal object of $\mathcal{M}$, so condition (2) says that every object of $\mathcal{M}$ may be viewed as a type dependent on the empty context. The pullback of a morphism of $\mathcal{D}$ along a morphism of $\mathcal{M}$ represents context substitution.

Consider a category of display maps $(\mathcal{M}, \mathcal{D})$. Let $\{\mathcal{D}, \mathcal{M}\}$ denote the full subcategory of $\mathcal{M}^2$ which is spanned by $\mathcal{D}$, and let $I : \{\mathcal{D}, \mathcal{M}\} \hookrightarrow \mathcal{M}^2$ denote the inclusion. Then, from the category of display maps $(\mathcal{M}, \mathcal{D})$, there arises a comprehension category.

Lemma 2.1.5 ([Jac93, Ex. 4.5])**.** A category of display maps $(\mathcal{M}, \mathcal{D})$ gives rise to the following comprehension category

$$
\begin{tikzcd}
\{\mathcal{D}, \mathcal{M}\} \arrow[rr, hook, "I"] \arrow[dr, "\textsc{cod}"'] & & \mathcal{M}^2 \arrow[dl, "\textsc{cod}"] \\
& \mathcal{M} &
\end{tikzcd}
$$

Proof. First we claim that any morphism $\langle \alpha, \beta \rangle : c \to d$ of $\{\mathcal{D}, \mathcal{M}\}$ is a cartesian morphism if $I\langle \alpha, \beta \rangle$ is a pullback square in $\mathcal{M}$. Consider such a morphism $\langle \alpha, \beta \rangle$. Then for any morphism $\langle \alpha', \beta' \rangle : c' \to d$ in $\{\mathcal{D}, \mathcal{M}\}$ and any morphism $\delta : \textsc{cod}(c') \to \textsc{cod}(c)$ in $\mathcal{M}$ such that $\beta\delta = \beta'$, we can find a unique $\langle \gamma, \delta \rangle : c' \to c$ such that $\textsc{cod}\langle \gamma, \delta \rangle = \delta$ and $\langle \alpha, \beta \rangle\langle \gamma, \delta \rangle = \langle \alpha', \beta' \rangle$ by the universal property of the pullback. Thus, $\langle \alpha, \beta \rangle$ is cartesian.

$$
\begin{tikzcd}
\bullet \arrow[r, dashed, "\gamma"'] \arrow[d, "c'"] \arrow[rr, bend left, "\alpha'"] & \bullet \arrow[r, "\alpha"] \arrow[d, "c"] \arrow[dr, phantom, "\lrcorner", very near start] & \bullet \arrow[d, "d"] \\
\bullet \arrow[r, "\delta"] \arrow[rr, bend right, "\beta'"'] & \bullet \arrow[r, "\beta"] & \bullet
\end{tikzcd}
$$

Now we claim that $\textsc{cod} : \{\mathcal{D}, \mathcal{M}\} \to \mathcal{M}$ is a Grothendieck fibration. Consider an object d of $\{\mathcal{D}, \mathcal{M}\}$ which is a morphism $d : X \to Y$ in $\mathcal{D}$. Also consider a morphism $m : Z \to \textsc{cod}(d) = Y$ in $\mathcal{M}$. Since pullbacks of morphisms in $\mathcal{D}$ exist and are in $\mathcal{D}$, we obtain a morphism $m^*d : m^*X \to Z$ in $\mathcal{D}$ and a morphism $\langle d^*m, m \rangle : m^*d \to d$ in $\{\mathcal{D}, \mathcal{M}\}$ such that $\textsc{cod}\langle d^*m, m \rangle = m$. Since $I\langle d^*m, m \rangle$ is a pullback square in $\mathcal{M}$, it is a cartesian morphism $\{\mathcal{D}, \mathcal{M}\}$.

Lastly, we claim that if $\langle \alpha, \beta \rangle : c \to d$ is a cartesian morphism of $\{\mathcal{D}, \mathcal{M}\}$, then $I\langle \alpha, \beta \rangle$ is a pullback square in $\mathcal{M}$. In other words, we claim that I is a cartesian functor. Consider such a cartesian morphism $\langle \alpha, \beta \rangle : c \to d$. We have just shown that there is another cartesian morphism $\langle d^*\beta, \beta \rangle : \beta^*d \to d$ in $\{\mathcal{D}, \mathcal{M}\}$ such that $\textsc{cod}c = \textsc{cod}\beta^*d$. But since $\langle \alpha, \beta \rangle : c \to d$ and $\langle d^*\beta, \beta \rangle : \beta^*d \to d$ are both cartesian, this induces an isomorphism $\langle \iota, 1 \rangle : c \cong \beta^*d$ making $\langle \alpha, \beta \rangle$ isomorphic to $\langle d^*\beta, \beta \rangle$. Since $I\langle d^*\beta, \beta \rangle$ is a pullback square, $I\langle \alpha, \beta \rangle : c \to d$ is also a pullback square.

$$\begin{tikzcd}
\bullet \arrow[r, dashed, "\iota"'] \arrow[rr, bend left, "\alpha"] \arrow[d, "c"'] & \bullet \arrow[r, "d^*\beta"'] \arrow[dl, "\beta^* d"] \arrow[dr, phantom, "\lrcorner" very near start] & \bullet \arrow[d, "d"] \\
\bullet \arrow[rr, "\beta"'] & & \bullet
\end{tikzcd}$$

□

This lemma defines an injection C from the class of categories of display maps to that of comprehension categories. This gives mathematical content to our attitude that display map categories are a kind of comprehension category.

Remark 2.1.6. Note that since $\mathcal{D}$ has all isomorphisms, our display map category $(\mathcal{M}, \mathcal{D})$ already models unit types ([Jac93, Def. 4.12]) given by the functor $1 : \mathcal{M} \to \{\mathcal{D}, \mathcal{M}\}$ which takes an object M of $\mathcal{M}$ to the display map 1_M.

2.2 Σ types.

Now we consider the representation of Σ types in our category. We use the notion of 'strong sums' in a comprehension category of [Jac93, Def. 5.8]. First, we need some notation.

Notation 2.2.1. For any class $\mathcal{D}$ of morphisms of a category $\mathcal{M}$ and an object X of $\mathcal{M}$, let $\{\mathcal{D}, \mathcal{M}\}_X$ denote the full subcategory of the slice category $\mathcal{M}/X$ which is spanned by those objects which are in $\mathcal{D}$. It is the fiber $\text{COD}^{-1}X$ in $\{\mathcal{D}, \mathcal{M}\}$.

Let $\text{DOM} : \{\mathcal{D}, \mathcal{M}\}_X \to \mathcal{M}$ denote the functor which takes each object $d : Y \to X$ of $\{\mathcal{D}, \mathcal{M}\}_X$ to its domain Y in $\mathcal{M}$.

Consider a category $(\mathcal{M}, \mathcal{D})$ of display maps. The comprehension category obtained from $(\mathcal{M}, \mathcal{D})$, as described in Lemma 2.1.5, has strong sums if and only if properties 1-3, described below, hold.

1. For every $f : X \to Y$ in $\mathcal{D}$, the pullback functor $f^* : \{\mathcal{D}, \mathcal{M}\}_Y \to \{\mathcal{D}, \mathcal{M}\}_X$ has a left adjoint Σ_f.

If property 1 holds, then we can obtain the *Beck-Chevalley natural transformation*

$$\beta : \Sigma_{\alpha^* f}(f^*\alpha)^* \implies \alpha^*\Sigma_f : \{\mathcal{D}, \mathcal{M}\}_X \to \{\mathcal{D}, \mathcal{M}\}_A$$

(for every $f : X \to Y$ in $\mathcal{D}$ and $\alpha : A \to Y$ of $\mathcal{M}$) by taking the unit $1 \to f^*\Sigma_f$, whiskering it with $(f^*\alpha)^*$ to obtain a natural transformation $(f^*\alpha)^* \to (f^*\alpha)^* f^*\Sigma_f \cong (\alpha^* f)\alpha^*\Sigma_f$, and then taking the transpose to obtain β. The next property says that this is an isomorphism.

2. (Beck-Chevalley condition) For every $f : X \to Y$ in $\mathcal{D}$ and $\alpha : A \to Y$ of $\mathcal{M}$, the Beck-Chevalley natural transformation

$$\beta : \Sigma_{\alpha^* f}(f^*\alpha)^* \implies \alpha^*\Sigma_f : \{\mathcal{D}, \mathcal{M}\}_X \to \{\mathcal{D}, \mathcal{M}\}_A.$$

is an isomorphism.

If properties 1 and 2 hold, then since postcomposition with f is left adjoint to pullback along f (as a functor $\mathcal{M}/Y \to \mathcal{M}/X$), we find the following bijection

$$\mathcal{M}/Y(fg, y) \cong \{\mathcal{D}, \mathcal{M}\}_Y(\Sigma_f g, y)$$

(for $f : X \to Y$ in $\mathcal{D}$, $g \in \{\mathcal{D}, \mathcal{M}\}_X$, and $y \in \{\mathcal{D}, \mathcal{M}\}_Y$). Thus, replacing y with $\Sigma_f g$, we find natural transformation $\gamma : f \circ - \implies \Sigma_f : \{\mathcal{D}, \mathcal{M}\}_X \to \{\mathcal{D}, \mathcal{M}\}_Y$. Then whiskering γ with $\textsc{dom}$ we find a natural transformation $\textsc{dom}\gamma : \textsc{dom} \implies \textsc{dom}\Sigma_f : \{\mathcal{D}, \mathcal{M}\}_X \to \mathcal{M}$ (since $\textsc{dom}(f \circ g) = \textsc{dom}g$). The next property says that this is an isomorphism.

3. For every $f : X \to Y$ in $\mathcal{D}$, the induced natural transformation

$$\textsc{dom}\gamma : \textsc{dom} \implies \textsc{dom}\Sigma_f : \{\mathcal{D}, \mathcal{M}\}_X \to \mathcal{M}$$

is an isomorphism.

Now we show that these three properties imply that Σ_f is actually postcomposition with f.

Proposition 2.2.2. *A category $(\mathcal{M}, \mathcal{D})$ of display maps has strong sums if and only if $\mathcal{D}$ is closed under composition. When this is the case, $\Sigma_f g \cong fg$ for composable $f, g \in \mathcal{D}$.*

Proof. Suppose that $\mathcal{D}$ is closed under composition. Then for every $f : X \to Y$ in $\mathcal{D}$, postcomposition with f is a left adjoint to $f^* : \{\mathcal{D}, \mathcal{M}\}_Y \to \{\mathcal{D}, \mathcal{M}\}_X$.

The isomorphism required in (3) above is given by the identity. The Beck-Chevalley isomorphism in property (2) is the isomorphism $\alpha^*(fg) \cong \alpha^* f \alpha^* g$ for composable f, g in $\mathcal{D}$.

Suppose that the category $(\mathcal{M}, \mathcal{D})$ of display maps has strong sums. Then for any $f : X \to Y$ and $g : W \to X$ in $\mathcal{D}$, we obtain the morphism $\gamma_g : fg \to \Sigma_f g$, as described before property 3 above. But γ_g is a morphism in the slice $\mathcal{M}/Y$, and property 3 says that its underlying morphism in $\mathcal{M}$ is an isomorphism. Thus γ_g is an isomorphism itself. We can then conclude that the composition fg is in $\mathcal{D}$ (since $\Sigma_f g$ is) so $\mathcal{D}$ is closed under composition. Furthermore, $\gamma : f \circ - \to \Sigma_f$ is a natural isomorphism. $\square$

Thus, we make the following definition.

Definition 2.2.3. A category of display maps $(\mathcal{M}, \mathcal{D})$ *models* Σ *types* if $\mathcal{D}$ is closed under composition. Call a composition gf of display maps a Σ *type*.

2.3 `Id` types.

Here, we define our notion of `Id` types. We will consider a category $(\mathcal{D}, \mathcal{M})$ of display maps which already models Σ types.

Note that $\{\mathcal{D}, \mathcal{M}\}_Y$ is closed under products in $\mathcal{M}/Y$ since $\mathcal{D}$ is closed under pullback and composition.

Definition 2.3.1. Consider a category of display maps $(\mathcal{M}, \mathcal{D})$ which models Σ types. We say that it *models* `Id` *types* if for every $f : X \to Y$ in $\mathcal{D}$, the diagonal $\Delta_f : f \to f \times f$ in $\{\mathcal{D}, \mathcal{M}\}_Y$ has a factorization $\Delta_f = \epsilon_f r_f$

$$\begin{tikzcd} X \arrow[r, "r_f"] \arrow[dr, "f"'] & \mathtt{Id}(f) \arrow[r, "\epsilon_f"] \arrow[d, "\iota_f"] & X \times_Y X \arrow[dl, "f \times f"] \\ & Y & \end{tikzcd}$$

in $\{\mathcal{D}, \mathcal{M}\}_Y$ where ϵ_f is in $\mathcal{D}$ and for every morphism $\alpha : A \to X$ in $\mathcal{M}$, the pullback $\alpha^* r_f$, as shown below, is in ${}^{\boxslash}\mathcal{D}$ for each $i = 0, 1$.

$$
\begin{tikzcd}
& \alpha^*\mathtt{Id}(f) \arrow[rr] \arrow[dd] & & \mathtt{Id}(f) \arrow[dd, "\pi_i\epsilon_f"] \\
A \arrow[ur, "\alpha^* r_f"] \arrow[rr] \arrow[dr, equal] & & X \arrow[ur, "r_f"] \arrow[dr, equal] & \\
& A \arrow[rr, "\alpha"] & & X
\end{tikzcd}
\tag{$*$}
$$

Call the morphisms $\iota_f : \mathtt{Id}(f) \to Y$ the $\mathtt{Id}$ *types* in $\{\mathcal{D}, \mathcal{M}\}_Y$.

Remark 2.3.2. Note that this definition is slightly stronger than that which is usually given for $\mathtt{Id}$ types. Usually, only pullbacks of r of the following form

$$
\begin{tikzcd}
& \alpha^*\mathtt{Id}(f) \arrow[rr] \arrow[dd] & & \mathtt{Id}(f) \arrow[dd, "\iota_f"] \\
\alpha^* f \arrow[ur, "\alpha^* r_f"] \arrow[rr] \arrow[dr] & & X \arrow[ur, "r_f"] \arrow[dr, "f"] & \\
& A \arrow[rr, "\alpha"] & & Y
\end{tikzcd}
\tag{$**$}
$$

are required to be in ${}^{\boxslash}\mathcal{D}$. Since the map r_f is defined 'in the context Y', requiring that these pullbacks of diagram ($**$) are in ${}^{\boxslash}\mathcal{D}$ can be interpreted as ensuring that this property of r_f (of being in ${}^{\boxslash}\mathcal{D}$) is stable under any substitution $\alpha : A \to Y$.

For the rest of this discussion, we will denote by $P(*)$ (respectively, $P(**)$) the property that the pullbacks of r_f of the form ($*$) (respectively, ($**$)) are in ${}^{\boxslash}\mathcal{D}$.

To see that $P(**)$ is weaker than $P(*)$, consider the situation displayed in diagram ($**$) and assume that $P(*)$ holds. We can obtain the morphism $\alpha^* r_f$ in diagram ($**$) by first pulling back α along f and then pulling back r_f along $f^*\alpha$, as shown below in diagram (†).

$$
\begin{tikzcd}
& \alpha^*\mathtt{Id}(f) \arrow[rr] \arrow[dd] & & \mathtt{Id}(f) \arrow[dd, "\pi_i\epsilon_f"] \\
A \arrow[ur, "\alpha^* r_f"] \arrow[rr] \arrow[dr, equal] & & X \arrow[ur, "r_f"] \arrow[dr, equal] & \\
& \alpha^* A \arrow[rr, "f^*\alpha"] \arrow[d] \arrow[drr, phantom, "\lrcorner", very near start] & & X \arrow[d, "f"] \\
& A \arrow[rr, "\alpha"] & & Y
\end{tikzcd}
\tag{†}
$$

Since the triangular prism in diagram (†) above is of the form of that in diagram $(*)$, $\alpha^* r_f$ is in ${}^{\boxslash}\mathcal{D}$.

We need to make the stronger assumption $P(*)$ if the factorization data $(r, \mathtt{Id}, \epsilon)$ is going to generate a weak factorization system. Let $\alpha_i^* r_f$ denote the pullback in diagram $(*)$ in the above definition for $i = 0, 1$. We will see in Proposition 2.3.4 that requiring that all $\alpha_0^* r_f$ are in ${}^{\boxslash}\mathcal{D}$ is exactly the requirement that the factorization given there is one into $({}^{\boxslash}\mathcal{D}_X, \mathcal{D}_X)$ in each $\{\mathcal{D}, \mathcal{M}\}_X$. Then requiring that each $\alpha_1^* r_f$ is in ${}^{\boxslash}\mathcal{D}$ could be justified by a desire for symmetry, but we will also see in Lemma 3.3.9 that this is actually a consequence of requiring that all $\alpha_0^* r_f$ are in ${}^{\boxslash}\mathcal{D}$.

Furthermore, $P(*)$ entails to a common variant of path induction. When the α of diagram $(*)$ is a point $* \to X$, then this is called *based path induction* or *Paulin-Mohring elimination*, and, in fact, it is equivalent to path induction in the presence of Π types (see [Uni13, §1.12.2] or [Str93], where the crux of the proof first appeared as a theorem due to Martin Hofmann). Then the property $P(*)$ itself can be understood as 'parametrized' version of based path induction.

In [GG08] and [BG12], the authors also work with a stronger variant of `Id` types, called strong `Id` types in [BG12]. Lemma 11 of [GG08] shows that our definition of `Id` types follows from theirs. In Appendix A, we translate that proof into our context, and also show that their definition follows from ours.

The definition we have given here of `Id` types treats each slice $\{\mathcal{D}, \mathcal{M}\}_X$ equally, as the syntax of type theory does. We claim however, that requiring this structure in $\{\mathcal{D}, \mathcal{M}\}_* \cong \mathcal{M}$ is sufficient. We show below that the structure in each $\{\mathcal{D}, \mathcal{M}\}_X$ generates a weak factorization structure (Proposition 2.3.4 and Corollary 2.3.8). We know that a weak factorization structure in $\mathcal{M}$ induces a weak factorization structure in each $\{\mathcal{D}, \mathcal{M}\}_X$ (Corollary 1.5.5). In Corollary 2.3.8, we show that these two weak factorization structures in $\{\mathcal{D}, \mathcal{M}\}_X$ are equivalent. It then remains to be seen that this weak factorization structure in $\mathcal{M}$ also entails the structure of `Id` types in each $\{\mathcal{D}, \mathcal{M}\}_X$. Unfortunately, we will not have the machinery to prove this until the next chapter so this appears in Appendix A as Proposition A.1.5.

Definition 2.3.3. Consider a category $(\mathcal{M}, \mathcal{D})$ of display maps which models Σ types. We say that it *models* `Id` *types of objects* if it has all `Id` types in $\{\mathcal{D}, \mathcal{M}\}_* \cong \mathcal{M}$. That is: if for every X in $\mathcal{M}$, the diagonal $\Delta_X : X \to X \times X$ has a factorization

$$X \xrightarrow{r_X} \mathtt{Id}(X) \xrightarrow{\epsilon_X} X \times X$$

in $\mathcal{M}$ where ϵ_X is in $\mathcal{D}$ and for every morphism $\alpha : A \to X$ in $\mathcal{M}$, the pullback $\alpha^* r_X$, as shown below, is in ${}^{\boxslash}\mathcal{D}$ for each $i = 0, 1$.

$$\begin{tikzcd}
 & \alpha^*\mathtt{Id}(X) \arrow[rr] \arrow[dd] & & \mathtt{Id}(X) \arrow[dd, "\pi_i\epsilon_X"] \\
A \arrow[ur, "\alpha^* r_X"] \arrow[rr] \arrow[dr, equal] & & X \arrow[ur, "r_X"] \arrow[dr, equal] & \\
 & A \arrow[rr, "\alpha"] & & X
\end{tikzcd}$$

Now we show that this structure generates a weak factorization structure on $\mathcal{M}$.

The following theorem uses ideas from the proof of Theorem 10 of [GG08], where a weak factorization structure is constructed in the classifying category of a dependent type theory. A categorical version appears as Theorem 2.8 in [Emm14].

Proposition 2.3.4. *Consider a category of display maps $(\mathcal{M}, \mathcal{D})$ which models Σ types and* Id *types of objects. There exists a weak factorization structure $(\lambda, \rho, {}^{\boxslash}\mathcal{D}, ({}^{\boxslash}\mathcal{D})^{\boxslash})$ in $\mathcal{M}$ where the image of ρ is contained in $\mathcal{D}$.*

Proof. The factorization is defined in the following way for any $f : X \to Y$ in $\mathcal{M}$. We have a factorization

$$Y \xrightarrow{r_Y} \mathtt{Id}(Y) \xrightarrow{\epsilon_Y} Y \times Y$$

of the diagonal $\Delta : Y \to Y \times Y$. Now we define the factorization of f to be

$$X \xrightarrow{1 \times r_Y f} X \times_Y \mathtt{Id}(Y) \xrightarrow{\pi_1 \epsilon_Y} Y$$

where the middle object is obtained in the following pullback.

$$\begin{tikzcd}
X \times_Y \mathtt{Id}(Y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathtt{Id}(Y) \arrow[d, "\pi_0\epsilon_Y"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

The left factor

$$\lambda(f) := 1 \times r_Y f : X \to X \times_Y \mathtt{Id}(Y)$$

is obtained as the following pullback of r_Y.

$$\begin{tikzcd}
& f^*\mathtt{Id}(Y) \arrow[rr] \arrow[dd] & & \mathtt{Id}(Y) \arrow[dd, "\pi_0 \epsilon_Y"] \\
X \arrow[ur, "f^* r_Y"] \arrow[rr] \arrow[dr, equal] & & Y \arrow[ur, "r_Y"] \arrow[dr, equal] & \\
& X \arrow[rr, "f"] & & Y
\end{tikzcd}$$

Thus, it is in ${}^{\boxslash}\mathcal{D}$.

The right factor

$$\rho(f) := \pi_1 \epsilon_Y : X \times_Y \mathtt{Id}(Y) \to Y$$

is in $\mathcal{D}$ because it is the composition of a pullback of ϵ_Y with a pullback of $X \to *$.

$$\begin{tikzcd}
X \times_Y \mathtt{Id}(Y) \arrow[r] \arrow[dd] \arrow[dddr, phantom, "\lrcorner", very near start] \arrow[ddd, bend right, "\rho(f)"'] & \mathtt{Id}(Y) \arrow[d, "\epsilon_Y"] \\
& Y \times Y \\
X \times Y \arrow[ur, "f \times 1"] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d] \\
Y \arrow[r] & *
\end{tikzcd}$$

Since $\mathcal{D} \subseteq ({}^{\boxslash}\mathcal{D})^{\boxslash}$, each $\rho(f)$ is also in $({}^{\boxslash}\mathcal{D})^{\boxslash}$. Thus (λ, ρ) gives a factorization into $({}^{\boxslash}\mathcal{D},\ ({}^{\boxslash}\mathcal{D})^{\boxslash})$.

For any morphisms f, g, we have $\lambda(f) \boxslash \rho(g)$ since $\lambda(f) \in {}^{\boxslash}\mathcal{D}$ and $\rho(g) \in \mathcal{D}$. Thus by Proposition 1.3.5, λ and ρ underlie a factorization. Since $\lambda(f) \boxslash \rho\lambda(f)$ and $\lambda\rho(f) \boxslash \rho(f)$, we can find a λ-coalgebra structure for every $\lambda(f)$ and a ρ-algebra structure for every $\rho(f)$. Thus, this factorization is weakly algebraic, and it generates a weak factorization structure (λ, ρ) by Proposition 1.4.7.

By Proposition 1.4.4, we see that $\lambda\textsf{-alg} = {}^{\boxslash}\mathcal{D}$, and thus also that $\rho\textsf{-coalg} = ({}^{\boxslash}\mathcal{D})^{\boxslash}$. Therefore, we have a weak factorization structure $(\lambda, \rho, {}^{\boxslash}\mathcal{D}, ({}^{\boxslash}\mathcal{D})^{\boxslash})$. $\square$

Notation 2.3.5. Let $\overline{\mathcal{D}}$ denote $({}^{\boxslash}\mathcal{D})^{\boxslash}$.

Definition 2.3.6. Consider a category $(\mathcal{M}, \mathcal{D})$ of display maps which models Σ types.

By the preceding proposition, 2.3.4, a model $(\mathtt{Id}, r, \epsilon)$ of `Id` types on objects in $(\mathcal{M}, \mathcal{D})$ generates a weak factorization structure $(\lambda, \rho, {}^{\boxslash}\mathcal{D}, \overline{\mathcal{D}})$.

In this case, we will say that the data $(\mathtt{Id}, r, \epsilon)$ *presents* or *is a presentation of* the weak factorization system $({}^{\boxslash}\mathcal{D}, \overline{\mathcal{D}})$.

Corollary 2.3.7. *Every morphism $f : X \to Y$ of $\overline{\mathcal{D}}$ is a retract in $\mathcal{M}/Y$ of some morphism of $\mathcal{D}$. Thus, the class $\overline{\mathcal{D}}$ is the retract-closure of $\mathcal{D}$.*

Proof. Every morphism $f : X \to Y$ of $\overline{\mathcal{D}}$ is a ρ-algebra, so it is a retract in $\mathcal{M}/Y$ of $\rho(f) \in \mathcal{D}$. Conversely, as the right class of a lifting pair, $\overline{\mathcal{D}}$ is closed under retracts (Lemma 1.1.5). □

Now we show that the structure of `Id` types also generates a weak factorization structure in each $\{\mathcal{D}, \mathcal{M}\}_X$ and that these are all compatible with that in $\{\mathcal{D}, \mathcal{M}\}_* \cong \mathcal{M}$ and thus also with each other.

Corollary 2.3.8. *Consider a category of display maps $(\mathcal{M}, \mathcal{D})$ which models Σ types and* `Id` *types.*

Then for each object X in $\mathcal{M}$, the structure given by the `Id` *types in $\{\mathcal{D}, \mathcal{M}\}_X$ generates a weak factorization structure $(\lambda_X, \rho_X, {}^{\boxslash}\mathcal{D}_X, \overline{\mathcal{D}}_X)$ on $\{\mathcal{D}, \mathcal{M}\}_X$.*

Furthermore, this weak factorization system $({}^{\boxslash}\mathcal{D}_X, \overline{\mathcal{D}}_X)$ on $\{\mathcal{D}, \mathcal{M}\}_X$ coincides with the one $({}^{\boxslash}\mathcal{D}, \overline{\mathcal{D}})$ on $\mathcal{M}$ in the sense that $\textsc{dom}_X^{-1}({}^{\boxslash}\mathcal{D}) = {}^{\boxslash}\mathcal{D}_X$ and $\textsc{dom}_X^{-1}(\overline{\mathcal{D}}) = \overline{\mathcal{D}}_X$ where $\textsc{dom}_X : \{\mathcal{D}, \mathcal{M}\}_X \to \mathcal{M}$ is the domain functor.

Proof. Let $\mathcal{D}_X$ denote the class of morphisms in $\{\mathcal{D}, \mathcal{M}\}_X$ whose underlying morphism in $\mathcal{M}$ belongs to $\mathcal{D}$. Then note that $(\{\mathcal{D}, \mathcal{M}\}_X, \mathcal{D}_X)$ is a category of display maps which models Σ types and `Id` types on objects. By the preceding proposition, this generates a weak factorization structure which we will denote here by $(\lambda_X, \rho_X, {}^{\boxslash}\mathcal{D}_X, \overline{\mathcal{D}}_X)$.

Then from the weak factorization system $({}^{\boxslash}\mathcal{D}, \overline{\mathcal{D}})$ on $\{\mathcal{D}, \mathcal{M}\}_* \cong \mathcal{M}$, we get by Corollary 1.5.5 another weak factorization system $(\textsc{dom}_X^{-1}{}^{\boxslash}\mathcal{D}, \textsc{dom}_X^{-1}\overline{\mathcal{D}})$ on $\{\mathcal{D}, \mathcal{M}\}_X$.

To show that ${}^{\boxslash}\mathcal{D}_X = \textsc{dom}_X^{-1}{}^{\boxslash}\mathcal{D}$ and $\overline{\mathcal{D}}_X = \textsc{dom}_X^{-1}\overline{\mathcal{D}}$, note that $\overline{\mathcal{D}}_X$ is the retract closure of $\mathcal{D}_X = \textsc{dom}_X^{-1}\mathcal{D}$ and that $\overline{\mathcal{D}}$ is the retract closure of $\mathcal{D}$. But a morphism f in $\{\mathcal{D}, \mathcal{M}\}_X$ is a retract of something in $\text{DOM}_X^{-1}\mathcal{D}$ if and only if $\text{DOM}f$ is a retract of something in $\mathcal{D}$ so the retract closure of $\text{DOM}_X^{-1}\mathcal{D}$ is $\text{DOM}_X^{-1}\overline{\mathcal{D}}$. Thus, $\overline{\mathcal{D}}_X = \text{DOM}_X^{-1}\overline{\mathcal{D}}$ and consequently ${}^{\boxslash}\mathcal{D}_X = \text{DOM}_X^{-1 \boxslash}\mathcal{D}$. □

2.4 Π types.

Now we discuss our notion of Π types in a category of display maps. This coincides with the usual definition [Jac93, p. 196].

Definition 2.4.1. A category of display maps $(\mathcal{M}, \mathcal{D})$ *models* Π *types* if

1. for every $f : X \to Y$ in $\mathcal{D}$, the pullback functor $f^* : \{\mathcal{D}, \mathcal{M}\}_Y \to \{\mathcal{D}, \mathcal{M}\}_X$ has a right adjoint Π_f; and
2. (Beck-Chevalley condition) for every morphism $\alpha : A \to Y$ of $\mathcal{M}$, the induced natural transformation

$$\beta : \alpha^*\Pi_f \implies \Pi_{\alpha^* f}(f^*\alpha)^* : \{\mathcal{D}, \mathcal{M}\}_X \to \{\mathcal{D}, \mathcal{M}\}_A$$

   is an isomorphism.

Call such a morphism $\Pi_f g$ a Π *type*.

Remark 2.4.2. The natural transformation β is obtained from the counit of the adjunction $\epsilon : f^*\Pi_f \to 1$. First, pullback ϵ along $f^*\alpha$ to get the following morphism,

$$(f^*\alpha)^*\epsilon : (\alpha^* f)\alpha^*\Pi_f \cong (f^*\alpha)^* f^*\Pi_f \to (f^*\alpha)^*$$

and take the transpose of this to get β.

$$\beta := \overline{(f^*\alpha)^*\epsilon} : \alpha^*\Pi_f \to \Pi_{\alpha^* f}(f^*\alpha)^*$$

Proposition 2.4.3. *A display map category* $(\mathcal{M}, \mathcal{D})$ *models* Π *types if and only if for every* $f : X \to Y$ *in* $\mathcal{D}$ *and* g *in* $\{\mathcal{D}, \mathcal{M}\}_X$*, there exists an object* $\Pi_f g$ *in* $\{\mathcal{D}, \mathcal{M}\}_Y$ *with the universal property*

$$\mathcal{M}/Y(y, \Pi_f g) \cong \mathcal{M}/X(f^*y, g)$$

natural in every y *in* $\mathcal{M}/Y$ *(and in this case the* Π *types are the morphisms* $\Pi_f g$*).*

Proof. First suppose that $(\mathcal{M}, \mathcal{D})$ models Π types. Consider f, y, g as in the statement. For any $\alpha : A \to Y$, the Beck-Chevalley condition says that $\alpha^* \Pi_f g \cong \Pi_{\alpha^* f}(f^*\alpha)^* g$. Therefore, we have the second isomorphism in the following chain of isomorphisms.

$$\begin{aligned}
\mathcal{M}/Y(\alpha, \Pi_f g) &\cong \{\mathcal{D}, \mathcal{M}\}_A(1_A, \alpha^* \Pi_f g) \\
&\cong \{\mathcal{D}, \mathcal{M}\}_A(1_A, \Pi_{\alpha^* f}(f^*\alpha)^* g) \\
&\cong \{\mathcal{D}, \mathcal{M}\}_{A \times_Y X}(1_{A \times_Y X}, (f^*\alpha)^* g) \\
&\cong \mathcal{M}/X(f^*\alpha, g)
\end{aligned}$$

The first and fourth isomorphisms above are applications of the (Σ type) adjunctions $x \circ - \dashv x^*$ and the third is an application of a Π type adjunction.

Now we show the converse. Consider f and g as in the statement. Restricting the universal property of $\Pi_f g$ described in the statement to any y in $\{\mathcal{D}, \mathcal{M}\}_Y$ gives us the universal property

$$\{\mathcal{D}, \mathcal{M}\}_Y(y, \Pi_f g) \cong \{\mathcal{D}, \mathcal{M}\}_X(f^* y, g)$$

required of Π types.

It remains to prove the Beck-Chevalley condition. To that end, consider any $\alpha : A \to Y$ in $\mathcal{M}$. We want to show that $\alpha^* \Pi_f g \cong \Pi_{\alpha^* f}(f^*\alpha)^* g$. Consider the following chain of isomorphisms for any $z : Z \to A$ in $\mathcal{D}$.

$$\begin{aligned}
\{\mathcal{D}, \mathcal{M}\}_A(z, \Pi_{\alpha^* f}(f^*\alpha)^* g) &\cong \{\mathcal{D}, \mathcal{M}\}_{A \times_Y X}((\alpha^* f)^* z, (f^*\alpha)^* g) \\
&\cong \mathcal{M}/X(f^*\alpha \circ (\alpha^* f)^* z, g) \\
&\cong \mathcal{M}/X(f^*(\alpha \circ z), g) \\
&\cong \mathcal{M}/Y(\alpha \circ z, \Pi_f g) \\
&\cong \{\mathcal{D}, \mathcal{M}\}_A(z, \alpha^* \Pi_f g)
\end{aligned}$$

The first and fourth isomorphisms are applications of the universal property of Π, the second and last are applications of the (Σ type) adjunctions $x \circ - \dashv x^*$, and the third comes from the (Σ type Beck-Chevalley) isomorphism $f^*\alpha \circ (\alpha^* f)^* - \cong f^*(\alpha \circ -)$. Then applying the Yoneda lemma, we find that $\Pi_{\alpha^* f}(f^*\alpha)^* g \cong \alpha^* \Pi_f g$. $\square$

2.5 Cauchy complete categories.

We would like to study categories of display maps with the tools of categorical homotopy theory. Thus we would rather study $\overline{\mathcal{D}}$ than $\mathcal{D}$ itself since $\overline{\mathcal{D}}$ is the right class of a weak factorization system and as such can be described by the language of categorical homotopy theory.

We show in this section that if a category $\mathcal{M}$ is Cauchy complete and $(\mathcal{M}, \mathcal{D})$ is a category of display maps which models Σ types and `Id` types, then $(\mathcal{M}, \overline{\mathcal{D}})$ is also a category of display maps which models Σ types and `Id` types. Moreover, if $(\mathcal{M}, \mathcal{D})$ also models Π types, then $(\mathcal{M}, \overline{\mathcal{D}})$ models Π types as well.

Note that in these results, the hypothesis that $\mathcal{M}$ is Cauchy complete will only used to establish that $(\mathcal{M}, \overline{\mathcal{D}})$ is a category of display maps and that $(\mathcal{M}, \overline{\mathcal{D}})$ models Π types when the same are true of $(\mathcal{M}, \mathcal{D})$. The other results, concerning Σ and `Id` types, are proven using more general, homotopical methods.

For those results which do utilize Cauchy completeness, the proofs use the following idea. In both cases, we need to prove that a certain functor, built out of elements of $\overline{\mathcal{D}}$, is representable while we hypothesize that the same functor, if built only out of elements of $\mathcal{D}$, is representable. In a Cauchy complete category, retracts of representable functors are themselves representable (). Thus, using the fact that every element of $\overline{\mathcal{D}}$ is a retract of an element of $\mathcal{D}$ (), we aim to show that those functors we want to be representable are retracts of those functors we know to be representable.

We use these results to justify our interest in $(\mathcal{M}, \overline{\mathcal{D}})$ over $(\mathcal{M}, \mathcal{D})$ since they imply that if there is a model in $(\mathcal{M}, \mathcal{D})$, there is also a model in $(\mathcal{M}, \overline{\mathcal{D}})$. We do not deny that there might be interesting models of the form $(\mathcal{M}, \mathcal{D})$ where $\mathcal{D} \subsetneq \overline{\mathcal{D}}$ (and indeed, 'syntactic' models will likely be of this form). However, this work is motivated by questions of the following sort: given a weak factorization system $(\mathcal{L}, \mathcal{R})$ in a category $\mathcal{C}$, how can one decide whether or not it carries the structure of *a* model? We show below, that to answer this question, it suffices to consider just the pair $(\mathcal{C}, \mathcal{R})$.

2.5.1 Basics.

Definition 2.5.1 ([Bor94, Def. 6.5.8])**.** A category $\mathcal{C}$ is *Cauchy complete* if every idempotent splits: that is, for every idempotent $e : C \to C$ in $\mathcal{C}$, there is a retract of C

$$R \xrightarrow{i} C \xrightarrow{r} R$$

such that $ir = e$.

Lemma 2.5.2. If a category $\mathcal{C}$ is Cauchy complete, then every slice $\mathcal{C}/X$ for $X \in \mathcal{C}$ is Cauchy complete.

Proof. Consider an idempotent $c \to c$ in a slice $\mathcal{C}/X$ which is represented by the following diagram in $\mathcal{C}$.

$$\begin{tikzcd} C \arrow[rr, "e"] \arrow[dr, "c"'] & & C \arrow[dl, "c"] \\ & X & \end{tikzcd} \tag{$*$}$$

Then the morphism $e : C \to C$ is an idempotent in $\mathcal{C}$. It splits into

$$R \xrightarrow{i} C \xrightarrow{r} R$$

such that $ir = e$. Since $cir = ce = c$, the following diagram commutes.

$$\begin{tikzcd} R \arrow[r, "i"] \arrow[dr, "ci"'] & C \arrow[r, "r"] \arrow[d, "c"] & R \arrow[dl, "ci"] \\ & X & \end{tikzcd}$$

This is a retraction in $\mathcal{C}/X$ which splits our original idempotent $(*)$. $\square$

We will make extensive use of the following two lemmas so we record them here.

Lemma 2.5.3. Consider a category $\mathcal{C}$, idempotents $e : C \to C$ and $f : D \to D$ in $\mathcal{C}$, and a morphism $c : C \to D$ making the following diagram commute.

$$\begin{tikzcd} C \arrow[r, "e"] \arrow[d, "c"] & C \arrow[d, "c"] \\ D \arrow[r, "f"] & D \end{tikzcd}$$

Then splittings of both $e : C \to C$ and $f : D \to D$ extend uniquely to a splitting of the idempotent $\langle e, f\rangle$ in $\mathcal{C}^2$.

$$\begin{tikzcd} R \arrow[r, "i"] \arrow[d, dashed, "sci"'] & C \arrow[r, "r"] \arrow[d, "c"] & R \arrow[d, dashed, "sci"] \\ S \arrow[r, "j"] & D \arrow[r, "s"] & S \end{tikzcd}$$

Moreover, if c is an isomorphism, then so is sci.

Proof. The proof is a straightforward diagram chase.

First, we claim that both squares in the diagram of the statement commute. For this we need that $j \circ sci = c \circ i$ and $s \circ c = sci \circ r$. We see that

$$\begin{aligned} jsci &= fci && \text{(since } js = f\text{)} \\ &= cei && \text{(since } ce = fc\text{)} \\ &= ciri && \text{(since } e = ir\text{)} \\ &= ci && \text{(since } ri = 1\text{)} \end{aligned}$$

and similarly

$$\begin{aligned} scir &= sce && \text{(since } ir = e\text{)} \\ &= sfc && \text{(since } ce = fc\text{)} \\ &= sjsc && \text{(since } f = js\text{)} \\ &= sc && \text{(since } sj = 1\text{)} \end{aligned}$$

so the diagram in the statement commutes.

Suppose there were another $x : R \to S$ making this diagram commute. Then we would have that $x = sjx = sci$.

Now suppose that c is an isomorphism with inverse c^{-1}. Using our splitting of f and e to split the idempotent $\langle f, e\rangle : c^{-1} \to c^{-1}$, we obtain a morphism $rc^{-1}j : S \to R$. Both 1 and the composition $rc^{-1}j \circ sci$ give splittings of the idempotent $\langle e, e\rangle : 1 \to 1$, so $1 = rc^{-1}j \circ sci$. Similarly, $sci \circ rc^{-1}j = 1$. Therefore, sci is an isomorphism. □

Corollary 2.5.4. *If $\mathcal{C}$ is Cauchy complete, then $\mathcal{C}^2$ is Cauchy complete.*

Corollary 2.5.5. *Splittings of idempotents are unique up to unique isomorphism.*

This result can be used to show that a splitting of idempotent $e : C \to C$ can be obtained as either the equalizer or the coequalizer of the diagram $1_C, e : C \rightrightarrows C$ [Bor94, Prop. 6.5.4]. Thus, the requirement that a category be Cauchy complete is weaker than requiring the existence of all equalizers or all coequalizers.

Lemma 2.5.6 ([Bor94, Lem. 6.5.6]). Consider a category $\mathcal{C}$, and a representable functor $\mathcal{C}(-, C) : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}et$ such that all idempotents $C \to C$ split. Then any retract of $\mathcal{C}(-, C)$ is itself representable.

In particular, if $\mathcal{C}$ is Cauchy complete, then any retract of any representable functor $\mathcal{C}^{\mathrm{op}} \to \mathcal{S}et$ is representable.

Proof. Consider such a $\mathcal{C}(-, C)$, and consider a retract of it as below.

$$F \xrightarrow{\iota} \mathcal{C}(-, C) \xrightarrow{\rho} F$$

Then $\iota\rho$ is an idempotent $\mathcal{C}(-, C) \to \mathcal{C}(-, C)$. Since the Yoneda embedding is full and faithful, we have $\iota\rho = \mathcal{C}(-, e)$ for some idempotent $e : C \to C$. But since this idempotent splits, we obtain a retraction of C

$$R \xrightarrow{i} C \xrightarrow{r} R$$

such that $ir = e$, and this produces a retraction of $\mathcal{C}(-, C)$.

$$\mathcal{C}(-, R) \xrightarrow{\mathcal{C}(-,i)} \mathcal{C}(-, C) \xrightarrow{\mathcal{C}(-,r)} \mathcal{C}(-, R)$$

But splittings of idempotents are unique by Lemma 2.5.5. Thus, $\mathcal{C}(-, R) \cong F$, and we conclude that F is representable. □

2.5.2 Categories of display maps.

Proposition 2.5.7. *Consider a Cauchy complete category $\mathcal{M}$. If $(\mathcal{M}, \mathcal{D})$ is a category of display maps, then $(\mathcal{M}, \overline{\mathcal{D}})$ is as well.*

Recall that $\overline{\mathcal{D}}$ denotes ${}^{\boxslash}(\mathcal{D}^{\boxslash})$ (Notation 2.3.5).

Proof. Since $\mathcal{D} \subseteq \overline{\mathcal{D}}$ and $\mathcal{D}$ contains all isomorphisms and morphisms to the terminal object, then $\overline{\mathcal{D}}$ does as well. Since $\overline{\mathcal{D}}$ is the right class of a lifting pair, it is stable under pullback (Lemma 1.1.5). It only remains to show that pullbacks of morphisms of $\overline{\mathcal{D}}$ exist.

Consider a morphism $d : X \to Y$ of $\overline{\mathcal{D}}$ and a morphism $\alpha : A \to Y$ of $\mathcal{M}$. By Corollary 2.3.7, d is a retract in $\mathcal{M}/Y$ of some $d' : X' \to Y$ in $\mathcal{D}$. Let P denote the pullback diagram category, and let $D, D' : P \to \mathcal{M}$ denote the following two pullback diagrams in $\mathcal{M}$.

$$\begin{tikzcd} & X \arrow[d, "d"] \\ A \arrow[r, "\alpha"] & Y \end{tikzcd} \qquad \begin{tikzcd} & X' \arrow[d, "d'"] \\ A \arrow[r, "\alpha"] & Y \end{tikzcd}$$

Let c denote the functor $\mathcal{M} \to [P, \mathcal{M}]$ which sends an object m of $\mathcal{M}$ to the constant functor $c_m : P \to \mathcal{M}$ at m.

Then since d is a retract of d' in $\mathcal{M}/Y$, the functor D is a retract of D' in $[P, \mathcal{M}]$, and thus the functor $\text{Nat}(c(-), D) : \mathcal{M} \to \mathcal{S}et$ is a retract of $\text{Nat}(c(-), D') : \mathcal{M} \to \mathcal{S}et$. Now since we assume that there is a limit of the pullback diagram D', the functor $\text{Nat}(c(-), D')$ is representable. Therefore, by Lemma 2.5.6, the functor $\text{Nat}(c(-), D)$ is also representable, and we conclude that D has a limit.

Therefore, assuming that pullbacks of morphisms of $\mathcal{D}$ exist, pullbacks of morphisms of $\overline{\mathcal{D}}$ exist. $\square$

2.5.3 Σ types.

Since right classes of lifting pairs are closed under composition, we see the following. Note that for this result, we only use the hypothesis that $\mathcal{M}$ is Cauchy complete to ensure, by Proposition 2.5.7, that $(\mathcal{M}, \overline{\mathcal{D}})$ is a category of display maps.

Proposition 2.5.8. *Consider a Cauchy complete category $\mathcal{M}$ and a category of display maps $(\mathcal{M}, \mathcal{D})$ which models Σ and* `Id` *types.*

Then $(\mathcal{M}, \overline{\mathcal{D}})$ is a category of display maps which models Σ types.

Proof. $\overline{\mathcal{D}}$ is closed under composition by Lemma 1.1.5, and this means that $(\mathcal{M}, \overline{\mathcal{D}})$ models Σ types. $\square$

2.5.4 `Id` types.

The result for `Id` types relies on machinery developed in the next chapter, so we relegate the bulk of its proof to the appendix.

Proposition 2.5.9. *Consider a Cauchy complete category $\mathcal{M}$ and a category of display maps $(\mathcal{M}, \mathcal{D})$ which models Σ and* `Id` *types.*

Then $(\mathcal{M}, \overline{\mathcal{D}})$ models `Id` *types.*

Proof. Note that since $(\mathcal{M}, \mathcal{D})$ models `Id` types on objects, so does $(\mathcal{M}, \overline{\mathcal{D}})$. Then by Proposition A.1.5, $(\mathcal{M}, \overline{\mathcal{D}})$ models `Id` types. $\square$

Note that in the preceding result, we did not use the hypothesis that $\mathcal{M}$ is Cauchy complete except to establish that $(\mathcal{M}, \overline{\mathcal{D}})$ is a display map category. In the next proposition, we provide an alternative result which does not rely on machinery developed in the next chapter but instead utilizes the Cauchy complete hypothesis. However, it only holds when the `Id` types are given 'functorially'.

Proposition 2.5.10. *Consider a Cauchy complete category $\mathcal{M}$. Suppose that $(\mathcal{M}, \mathcal{D})$ is a category of display maps which models Σ types and* `Id` *types. Suppose further that the identity types are given functorially in the sense that, in each slice $\{\mathcal{D}, \mathcal{M}\}_Y$, the assignment*

$$\begin{tikzcd} X \arrow[d, "d"] & & X \arrow[r, "r_d"] \arrow[dr, "d"'] & \mathtt{Id}(d) \arrow[r, "\epsilon_d"] \arrow[d, "\iota_d"] & X \times_Y X \arrow[dl, "d \times d"] \\ Y & \mapsto & & Y & \end{tikzcd}$$

is the object part of a functor $\{\mathcal{D}, \mathcal{M}\}_Y \to \{\mathcal{D}, \mathcal{M}\}_Y^{\mathbf{3}}$.

Then $(\mathcal{M}, \overline{\mathcal{D}})$ is a category of display maps which models `Id` *types (and these* `Id` *types are also given functorially).*

Proof. Fix a slice $\mathcal{M}/Y$ and an object $e \in \overline{\mathcal{D}}$ in this slice. We want to construct an `Id` type on e. There is a $d \in \mathcal{D}$ such that e is a retract of d (Corollary 2.3.7). Since we have an `Id` type on d, we have the following diagram in $\mathcal{M}/Y$ (where

i, s form the retraction and r_d, ϵ_d form the Id type on d).

$$\begin{tikzcd} e \arrow[d, "i"] & & e \times e \arrow[d, "i\times i"] \\ d \arrow[r, "r_d"] & \iota_d \arrow[r, "\epsilon_d"] & d \times d \arrow[d, "s\times s"] \\ e \arrow[from=u, "s"] & & e \times e \end{tikzcd} \qquad (*)$$

The factorization gives a morphism $\iota_{\langle is, is\times is\rangle} : \iota_d \to \iota_d$ making the following diagram commute.

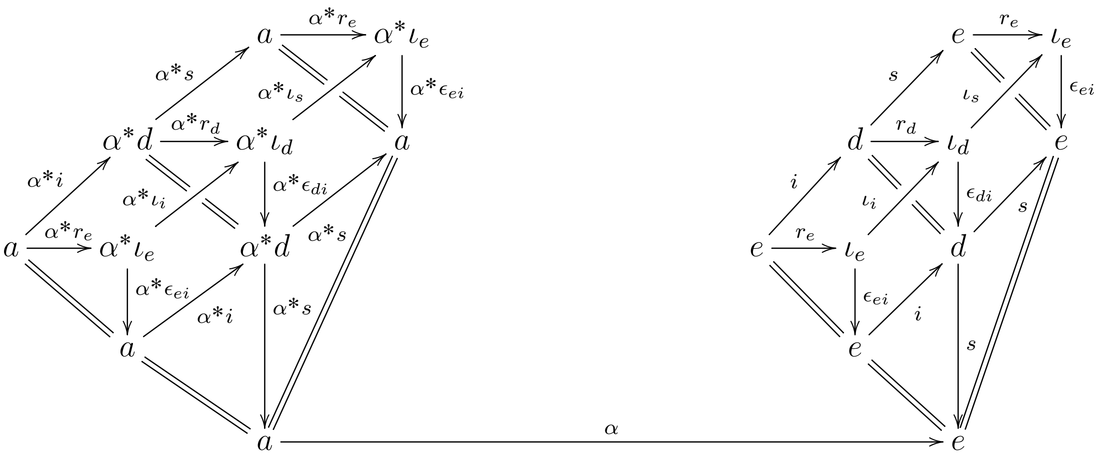
$$\begin{tikzcd} d \arrow[r, "r_d"] \arrow[d, "s"] & \iota_d \arrow[r, "\epsilon_d"] \arrow[dd, "\iota_{\langle is, is\times is\rangle}"] & d \times d \arrow[d, "s\times s"] \\ e \arrow[d, "i"] & & e \times e \arrow[d, "i\times i"] \\ d \arrow[r, "r_d"] & \iota_d \arrow[r, "\epsilon_d"] & d \times d \end{tikzcd}$$

Since $\langle is, is \times is\rangle$ is an idempotent and this factorization is given functorially, the morphism $\iota_{\langle is, is\times is\rangle}$ is also an idempotent. By Lemma 2.5.2, $\mathcal{M}/Y$ is Cauchy complete, so we can split the idempotent $\iota_{\langle is, is\times is\rangle}$. Then by Lemma 2.5.3, this extends to splittings of the rectangles in diagram (∗) above. This gives us the following commutative diagram.

$$\begin{tikzcd} e \arrow[r, "r_e"] \arrow[d, "i"] & \iota_e \arrow[r, "\epsilon_e"] \arrow[d, "\iota_i"] & e \times e \arrow[d, "i\times i"] \\ d \arrow[r, "r_d"] \arrow[d, "s"] & \iota_d \arrow[r, "\epsilon_d"] \arrow[d, "\iota_r"] & d \times d \arrow[d, "s\times s"] \\ e \arrow[r, "r_e"] & \iota_e \arrow[r, "\epsilon_e"] & e \times e \end{tikzcd}$$

Now we see that the morphism ϵ_e is in $\overline{\mathcal{D}}_Y$ since it is a retract of $\epsilon_d \in \mathcal{D}_Y$.

Now, we need to show that for any $\alpha : a \to e$, the pullback $\alpha^* r_e$ is in ${}^{\boxslash}\overline{\mathcal{D}}_Y$. Let ϵ_{xi} denote the composition $\pi_i \epsilon_x$ for $x = d, e$ and $i = 0, 1$. Since r_e is a retract of r_d, as shown in the following diagram, $\alpha^* r_e$ is a retract of $\alpha^* r_d$.



Since $\alpha^* r_d$ is in ${}^{\boxslash}\mathcal{D}_Y$ by hypothesis, and ${}^{\boxslash}\mathcal{D}$ is closed under retracts, we find that $\alpha^* r_e$ is in ${}^{\boxslash}\overline{\mathcal{D}}$.

Therefore, we have `Id` types for $(\mathcal{M}, \overline{D})$. $\square$

2.5.5 Π types.

Proposition 2.5.11. *Consider a Cauchy complete category of display maps $(\mathcal{M}, \mathcal{D})$ which models Σ types,* `Id` *types, and Π types. Then the category of display maps $(\mathcal{M}, \overline{\mathcal{D}})$ also models Π types.*

Proof. Recall that by Proposition 2.3.4, the `Id` types (on objects) generate a weak factorization structure $(\lambda, \rho, {}^{\boxslash}\mathcal{D}, \overline{\mathcal{D}})$ on $\mathcal{M}$.

Consider $f : X \to Y$ and $g : W \to X$ in $\overline{\mathcal{D}}$. We aim to obtain a Π type $\Pi_f g$.

Note that because

$$\rho(g) \times_Y \mathtt{Id}(Y) : (W \times_X \mathtt{Id}(X)) \times_Y \mathtt{Id}(Y) \to X \times_Y \mathtt{Id}(Y)$$

is a pullback of $\rho(g)$, it is in $\mathcal{D}$.

$$\begin{tikzcd}
(W \times_X \mathtt{Id}(X)) \times_Y \mathtt{Id}(Y) \arrow[r] \arrow[d, "\rho(g)\times_Y \mathtt{Id}(Y)"'] \arrow[dr, phantom, "\lrcorner", very near start] & W \times_X \mathtt{Id}(X) \arrow[d, "\rho(g)"] \\
X \times_Y \mathtt{Id}(Y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\
\mathtt{Id}(Y) \arrow[r, "\epsilon_0"] & Y
\end{tikzcd}$$

We will denote $\rho(g) \times_Y \mathtt{Id}(Y)$ as

$$M(\rho g) : M(f \circ \rho g) \to Mf$$

when it improves readability. (Note that the domain and codomain are indeed the middle objects of the factorizations of $f \circ \rho g$ and f, respectively.)

Since $M(\rho g)$ and ρf are in $\mathcal{D}$, we can form the Π type $\Pi_{\rho f} M(\rho g)$.

In Lemma 2.4.3, we demonstrated the following isomorphism for any $y : A \to Y$ in $\mathcal{M}$.

$$\mathcal{M}/Y(y, \Pi_{\rho f} M(\rho g)) \cong \mathcal{M}/Mf(\rho f^* y, M(\rho g))$$

This means that $\Pi_{\rho(f)} M(\rho g)$ represents the functor

$$\mathcal{M}/Mf(\rho f^* -, M(\rho g)) : \mathcal{M}/Y \to \mathcal{S}et.$$

We now show that $\mathcal{M}/X(f^* -, g)$ is a retract of this functor, so by Lemma 2.5.6, it will itself be representable.

Let i denote the natural transformation

$$\mathcal{M}/X(f^* -, g) \to \mathcal{M}/Mf(\rho f^* -, M(\rho g))$$

which at a morphism $z : Z \to Y$ in $\mathcal{M}$, takes a morphism $m : f^* z \to g$ in $\mathcal{M}/X$

$$\begin{tikzcd}
X \times_Y Z \arrow[dr, "f^*z"] \arrow[dd, "m"'] & \\
& X \\
W \arrow[ur, "g"] &
\end{tikzcd}$$

to the following morphism in $\mathcal{M}/Mf$

$$\begin{tikzcd}
X \times_{\epsilon_0} \mathtt{Id}(Y)_{\epsilon_1} \times Z \arrow[d, "a \times 1_{\mathtt{Id}Y} \times 1_Z"'] \arrow[ddr, "\rho f^* z"] & \\
\mathtt{Id}(X)_{(f\epsilon_0 \times f\epsilon_1)} \times_{(\epsilon_0 \times \epsilon_1)} \mathtt{Id}(Y)_{\epsilon_1} \times Z \arrow[d, "1_{\mathtt{Id}X} \times 1_{\mathtt{Id}Y} \times m(\epsilon_{1X} \times 1_Z)"'] \arrow[dr, "\epsilon_{X0} \times 1"] & \\
\mathtt{Id}(X)_{(f\epsilon_0 \times f\epsilon_1)} \times_{(\epsilon_0 \times \epsilon_1)} \mathtt{Id}(Y)_{\epsilon_1} \times W \arrow[d, "b \times 1_{\mathtt{Id}Y}"'] \arrow[r, "\epsilon_{X0} \times 1"] & X \times_Y \mathtt{Id}(Y) \\
W \times_{\epsilon_0} \mathtt{Id}(X)_{f\epsilon_1} \times_{\epsilon_0} \mathtt{Id}(Y) \arrow[ur, "M(\rho g)"] &
\end{tikzcd}$$

where a and b are given by solutions to the following lifting problems.

$$\begin{tikzcd}
X \arrow[r, "r"] \arrow[d, "\lambda(f)"'] & \mathtt{Id}(X) \arrow[d, "\epsilon_0 \times f\epsilon_1"] \\
X_f \times_{\epsilon_0} \mathtt{Id}(Y) \arrow[r, "1 \times \epsilon_1"] \arrow[ur, dashed, "a"] & X \times Y
\end{tikzcd}
\qquad
\begin{tikzcd}
W \arrow[r, "\lambda(g)"] \arrow[d, "rg \times 1"'] & W_g \times_{\epsilon_0} \mathtt{Id}(X) \arrow[d, "\rho(g)"] \\
\mathtt{Id}(X)_{\epsilon_1} \times_g W \arrow[r, "\epsilon_0"] \arrow[ur, dashed, "b"] & X
\end{tikzcd}$$

(The morphism $\epsilon_0 \times f\epsilon_1$ is in $\overline{\mathcal{D}}$ because it is the composition of $\epsilon_0 \times \epsilon_1 : \mathtt{Id}(X) \to X \times X$ with $1 \times f : X \times X \to X \times Y$. The morphism $rg \times 1$ is in ${}^{\boxslash}\mathcal{D}$ because it is one of the pullbacks of $r : X \to \mathtt{Id}(X)$ ensured to be in ${}^{\boxslash}\mathcal{D}$ by the definition of $\mathtt{Id}$ types.)

Then let r denote the natural transformation

$$\mathcal{M}/Mf(\rho f^* -, M(\rho g)) \to \mathcal{M}/X(f^* -, g)$$

which at a morphism $z : Z \to Y$ in $\mathcal{M}$, takes a morphism $n : \rho f^* z \to M(\rho g)$ in $\mathcal{M}/Mf$

$$\begin{tikzcd}
X \times_{\epsilon_0} \mathtt{Id}(Y)_{\epsilon_1} \times Z \arrow[dd, "n"] \arrow[dr, "\rho f^* z"] & \\
& X \times_{\epsilon_0} \mathtt{Id}(Y) \\
W \times_{\epsilon_0} \mathtt{Id}(X)_{\epsilon_1} \times_{\epsilon_0} \mathtt{Id}(Y) \arrow[ur, "M(\rho g)"'] &
\end{tikzcd}$$

to the following composition in $\mathcal{M}/X$

$$\begin{tikzcd}
X \times_Y Z \arrow[d, "1_X \times r_Y \times 1_Z"'] \arrow[ddr, "f^*z"] & \\
X \times_{\epsilon_0} \mathtt{Id}(Y)_{\epsilon_1} \times Z \arrow[d, "n"'] \arrow[dr, "\pi_X"] & \\
W \times_{\epsilon_0} \mathtt{Id}(X)_{\epsilon_1} \times_{\epsilon_0} \mathtt{Id}(Y) \arrow[d, "c"'] \arrow[r, "\epsilon_X 1"] & X \\
W \arrow[ur, "g"'] &
\end{tikzcd}$$

where c is a solution to the following lifting problem.

$$\begin{tikzcd}
W \arrow[r, equal] \arrow[d, "\lambda(g)"] & W \arrow[d, "g"] \\
W \times_{\epsilon_0} \mathtt{Id}(X) \arrow[r, "\rho(g)"] \arrow[ur, dashed, "c"] & X
\end{tikzcd}$$

Now we claim that

$$\mathcal{M}/X(f^*-, g) \xrightarrow{i} \mathcal{M}/Mf(\rho(f)^*-, M(\rho g)) \xrightarrow{r} \mathcal{M}/X(f^*-, g)$$

is a retract diagram. To that end, consider a morphism m of $\mathcal{M}/X(f^*z, g)$. Then $ri(m)$ is the following composition.

$$\begin{tikzcd}
X \times_Y Z \arrow[d, "1_X \times r_Y \times 1_Z"'] \arrow[dddr, "f^*z"] & \\
X \times_{\epsilon_0} \mathtt{Id}(Y)_{\epsilon_1} \times Z \arrow[d, "a \times 1_{\mathtt{Id}Y} \times 1_Z"'] \arrow[ddr, "\pi_X"] & \\
\mathtt{Id}(X)_{(f\epsilon_0 \times f\epsilon_1)} \times_{(\epsilon_0 \times \epsilon_1)} \mathtt{Id}(Y)_{\epsilon_1} \times Z \arrow[d, "1_{\mathtt{Id}X} \times 1_{\mathtt{Id}Y} \times m(\epsilon_{1X} \times 1_Z)"'] \arrow[dr, "\epsilon_X 1"] & \\
\mathtt{Id}(X)_{(f\epsilon_0 \times f\epsilon_1)} \times_{(\epsilon_0 \times \epsilon_1)} \mathtt{Id}(Y)_{\epsilon_1} \times W \arrow[d, "b \times 1_{\mathtt{Id}Y}"'] \arrow[r, "\epsilon_X 1"] & X \\
W \times_{\epsilon_0} \mathtt{Id}(X)_{f\epsilon_1} \times_{\epsilon_0} \mathtt{Id}(Y) \arrow[d, "c"'] \arrow[ur, "\epsilon_X 1"] & \\
W \arrow[uur, "g"'] &
\end{tikzcd}$$

The composition $a \circ (1_X \times r_Y) : X \to \mathtt{Id}(X)$ is r_X. Thus, the composite of the first three vertical morphisms in the above diagram is

$$r_X \times r_Y \times m : X \times_Y Z \to \mathtt{Id}(X)_{(f\epsilon_0 \times f\epsilon_1)} \times_{(\epsilon_0 \times \epsilon_1)} \mathtt{Id}(Y)_{\epsilon_1} \times W.$$

Moreover, $b \circ (r_X \times 1_W) : W \to W \times_{\epsilon_0} \mathtt{Id}(X)$ is $1_W \times r_X$ so the composite of the first four morphisms above is

$$m \times r_X \times r_Y : X \times_Y Z \to W \times_{\epsilon_0} \mathtt{Id}(X)_{f\epsilon_1} \times_{\epsilon_0} \mathtt{Id}(Y).$$

The composite $c \circ (1_W \times r_X) : W \to W$ is the identity, so the vertical composite above is m. Therefore, $ri(m) = m$, and i and r form a retract.

Now by Lemma 2.5.6, we can conclude that $\mathcal{M}/X(f^*-, g) : \mathcal{M}/Y \to \mathcal{S}et$ is representable by an object which we will denote by $\Pi_f g$. Furthermore, $\Pi_f g$ is a retract of $\Pi_{\rho f} M(\rho g)$. Since $\Pi_{\rho f} M(\rho g)$ is in $\mathcal{D}$, we can conclude that $\Pi_f g$ is in $\overline{\mathcal{D}}$, the retract closure of $\mathcal{D}$. Finally, by Proposition 2.4.3 we find that $(\mathcal{M}, \overline{\mathcal{D}})$ does in fact model Π types. $\square$

2.5.6 Summary.

Putting together Propositions 2.5.7, 2.5.8, 2.5.9, and 2.5.11 of this section, we get the following theorem.

Theorem 2.5.12. *Consider a Cauchy complete category $(\mathcal{M}, \mathcal{D})$ with display maps modeling Σ types, Π types, and* `Id` *types. Then $(\mathcal{M}, \overline{\mathcal{D}})$ is again a category with display maps modeling Σ types, Π types, and* `Id` *types.*

Proof. By Proposition 2.5.7, $(\mathcal{M}, \overline{\mathcal{D}})$ is a category with display maps. By Proposition 2.5.8, it models Σ types. By Proposition 2.5.9, it models `Id` types. By Proposition 2.5.11, it models Π types. $\square$

2.6 Weak factorization systems.

In the following chapters of this thesis, we consider the converse situation to that considered in the last sections. Given a category $\mathcal{C}$ with a weak factorization system $(\mathcal{L}, \mathcal{R})$, when does $(\mathcal{C}, \mathcal{R})$ carry the structure of a category with display maps modeling Σ types, `Id` types, and Π types?

We take the time here to record the most basic results towards an answer to this question in order to clarify the problem for the following chapters.

We will assume that our categories with weak factorization systems have all finite limits. This is usually the case in examples (to be precise, our examples will often come from Quillen model categories which are assumed to be complete and cocomplete). At the very least, we would need to assume that every pullback of every morphism of $\mathcal{R}$ exists and that there is a terminal object, but we will make the stronger assumption for simplicity.

First, we point out that any such weak factorization system gives a display map category modeling Σ types.

Proposition 2.6.1. *Consider a weak factorization system $(\mathcal{L}, \mathcal{R})$ on a category $\mathcal{C}$ with finite limits such that every object is fibrant. Then $(\mathcal{C}, \mathcal{R})$ is a display map category which models Σ types.*

Proof. By Lemma 1.1.5, every isomorphism is in $\mathcal{R}$, and $\mathcal{R}$ is stable under pullback. Thus, $(\mathcal{C}, \mathcal{R})$ is a display map category.

By the same proposition, $\mathcal{R}$ is closed under composition, so it models Σ types. □

Corollary 2.6.2. *Consider a weak factorization system $(\mathcal{L}, \mathcal{R})$ on a category $\mathcal{C}$ with finite limits. Let $\mathcal{C}_{\mathcal{F}}$ denote the full subcategory of $\mathcal{C}$ spanned by the fibrant objects. Then $(\mathcal{C}_{\mathcal{F}}, \mathcal{R} \cap \mathcal{C}_{\mathcal{F}})$ is a display map category which models Σ types.*

Proof. By Corollary 1.5.4, $(\mathcal{L} \cap \mathcal{C}_{\mathcal{F}}, \mathcal{R} \cap \mathcal{C}_{\mathcal{F}})$ is a weak factorization system on $\mathcal{C}_{\mathcal{F}}$. Then the statement follows from the proposition above. □

Secondly, we give a name to the situation where this weak factorization system gives a model of `Id` types. Such weak factorization systems will be studied in the next chapter.

Definition 2.6.3. Consider a weak factorization system $(\mathcal{L}, \mathcal{R})$ on a category $\mathcal{C}$ with finite limits. Say that $(\mathcal{L}, \mathcal{R})$ has an `Id`-*presentation* if $(\mathcal{C}, \mathcal{R})$ has a model of `Id` types of objects which presents (Def. 2.3.6) the weak factorization system $(\mathcal{L}, \mathcal{R})$.

Proposition 2.6.4. *Consider a weak factorization system $(\mathcal{L}, \mathcal{R})$ on a category $\mathcal{C}$ with finite limits. This weak factorization system has an* `Id`*-presentation if and only if $(\mathcal{C}, \mathcal{R})$ models* `Id` *types.*

Proof. If $(\mathcal{C}, \mathcal{R})$ models `Id` types, then, in particular, it models `Id` types on objects which generates the weak factorization system $({}^{\boxslash}\mathcal{R}, \overline{\mathcal{R}}) = (\mathcal{L}, \mathcal{R})$ by Proposition 2.3.4.

Proposition A.1.5 gives the converse. $\square$

Lastly, we disentangle the categorical requirements of modeling Π types from the more homotopical requirements. To model Π types is to have the existence of universal morphisms which, furthermore, are display maps. We leave the problem of finding such universal morphisms aside, and focus on the problem of when they are display maps. Many examples (e.g., simplicial sets, the topological topos, etc.) are already locally cartesian closed so these universal morphisms are already known to exist. In any case, it is not a question that categorical homotopy theory is well equipped to answer.

Definition 2.6.5. Say that a weak factorization system $(\mathcal{L}, \mathcal{R})$ on a category $\mathcal{C}$ with finite limits is *type theoretic* if

1. every object of $\mathcal{C}$ is fibrant, and

2. $\mathcal{L}$ is stable under pullback along $\mathcal{R}$.

Note that condition 2 above is often called the *Frobenius* condition [BG12].

We are particularly interested in this property because, as we will show in Theorem 3.4.4, weak factorization systems with `Id`-presentations are type theoretic, and we prove below that this implies that the universal objects $\Pi_f g$ defined below are in $\overline{\mathcal{D}}$.

Definition 2.6.6. Say that a weak factorization system $(\mathcal{L}, \mathcal{R})$ on $\mathcal{C}$ *models pre-*Π *types* if for every $g : W \to X$ and $f : X \to Y$ in $\mathcal{R}$, there is a morphism $\Pi_f g$ with codomain Y satisfying the universal property

$$i : \mathcal{C}/X(f^* y, g) \cong \mathcal{C}/Y(y, \Pi_f g)$$

natural in y.

In particular, any weak factorization system in a locally cartesian category models pre-Π types.

Proposition 2.6.7. *Consider a category $\mathcal{C}$ with finite limits. Consider also a weak factorization system $(\mathcal{L}, \mathcal{R})$ on $\mathcal{C}$ in which all objects of $\mathcal{C}$ are fibrant and which models pre-Π types.*

Then $(\mathcal{C}, \mathcal{R})$ models Π types if and only if $(\mathcal{L}, \mathcal{R})$ is type theoretic.

Lemma 1.1.7 says that a right adjoint preserves the right class of a weak factorization system if and only if the left adjoint preserves the left class. That is not exactly the situation here, but it is close enough that we imitate its proof.

Proof. Suppose that $(\mathcal{C}, \mathcal{R})$ models Π types. Let i_{yfg} denote the bijection

$$i_{yfg} : \mathcal{C}/Y(y, \Pi_f g) \cong \mathcal{C}/X(f^* y, g)$$

of Proposition 2.4.3. We need to show that $\mathcal{L}$ is stable under pullback along $\mathcal{R}$. To that end, consider a morphism ℓ of $\mathcal{L}$ and a morphism f of $\mathcal{R}$ such that $\textsc{cod}\,\ell = \textsc{cod}\, f$. To show that $f^*\ell$ is in $\mathcal{L}$, we must show that it has a λ-coalgebra structure for every factorization (λ, ρ) into $(\mathcal{L}, \mathcal{R})$.

$$\begin{tikzcd} {} \arrow[r, "\lambda(f^*\ell)"] \arrow[d, "f^*\ell"'] & {} \arrow[d, "\rho(f^*\ell)"] \\ {} \arrow[r, equal] \arrow[ur, dashed] & {} \end{tikzcd}$$

Consider the following lifting problem. It is the transpose of the above lifting problem under i^{-1}.

$$\begin{tikzcd} {} \arrow[r, "i^{-1}(\lambda(f^*\ell))"] \arrow[d, "\ell"'] & {} \arrow[d, "\Pi_f \rho(f^*\ell)"] \\ {} \arrow[r, equal] \arrow[ur, dashed, "\sigma"] & {} \end{tikzcd}$$

It has a solution σ since ℓ is in $\mathcal{L}$ and $\Pi_f \rho(f^*\ell)$ is in $\mathcal{R}$. Then $i(\sigma)$ gives us a solution to our original lifting problem.

Now suppose that $(\mathcal{L}, \mathcal{R})$ is type theoretic. We need to show that $\Pi_f g$ is in $\mathcal{R}$. Then by Proposition 2.4.3, the display map category $(\mathcal{C}, \mathcal{R})$ will model Π types. Let i_{yfg} denote the bijection

$$i_{yfg} : \mathcal{C}/Y(y, \Pi_f g) \cong \mathcal{C}/X(f^* y, g)$$

which defines the pre-Π type $\Pi_f g$.

The morphism $\Pi_f g$ is in $\mathcal{R}$ if and only if for every factorization (λ, ρ) into $(\mathcal{L}, \mathcal{R})$ there is a solution to the following lifting problem.

$$\begin{tikzcd} \cdot \arrow[r, equal] \arrow[d, "\lambda(\Pi_f g)"'] & \cdot \arrow[d, "\Pi_f g"] \\ \cdot \arrow[r, "\rho(\Pi_f g)"'] \arrow[ur, dashed] & \cdot \end{tikzcd}$$

Consider the following lifting problem. It is the transpose of the above lifting problem under i.

$$\begin{tikzcd} \cdot \arrow[r, "i(1)"] \arrow[d, "f^*\lambda(\Pi_f g)"'] & \cdot \arrow[d, "g"] \\ \cdot \arrow[r, "f^*\rho(\Pi_f g)"'] \arrow[ur, dashed, "\sigma"] & \cdot \end{tikzcd}$$

Since $f^*\lambda(\Pi_f g)$ is in $\mathcal{L}$ and g is in $\mathcal{R}$, there is a solution σ to this lifting problem. Then $i^{-1}\sigma$ is a solution to the original lifting problem. □

Corollary 2.6.8. *Consider a type theoretic weak factorization structure $(\mathcal{L}, \mathcal{R})$ on a locally cartesian closed category $\mathcal{C}$. Then $(\mathcal{C}, \mathcal{R})$ models Π types.*

We summarize the results of this section with the following proposition.

Proposition 2.6.9. *Consider a weak factorization system $(\mathcal{L}, \mathcal{R})$ on a category $\mathcal{C}$ with finite limits. Then $(\mathcal{C}, \mathcal{R})$ is a display map category modeling Σ types. If it has an* `Id`*-presentation, then $(\mathcal{C}, \mathcal{R})$ models* `Id` *types. If it is type theoretic and models pre-Π types, then $(\mathcal{C}, \mathcal{R})$ models Π types.*

Proof. This follows from Proposition 2.6.1, Proposition 2.6.7, and Proposition 2.6.4. □

2.7 Summary.

In this chapter, we have defined what it means for a display map category $(\mathcal{M}, \mathcal{D})$ to model Σ types, Π types, and `Id` types. We showed that if $\mathcal{M}$ is a Cauchy complete category and $(\mathcal{M}, \mathcal{D})$ models Σ types, Π types, and `Id` types, then so does $(\mathcal{M}, \overline{\mathcal{D}})$. Thus, if one wants to know whether a weak factorization system $(\mathcal{L}, \mathcal{R})$ on a category $\mathcal{C}$ underlies a display map category $(\mathcal{C}, \mathcal{D})$ (where $\overline{\mathcal{D}} = \mathcal{R}$) modeling Σ types, Π types, and `Id` types, it suffices to check just the display map category $(\mathcal{C}, \mathcal{R})$. Such a display map category models Σ types, Π

types, and `Id` types if it is type-theoretic, has an `Id`-presentation, and has pre-Π types. In the next chapter, we study weak factorization systems which are type theoretic – needed to model Π types – and those which have an `Id`-presentation – needed to model `Id` types. We will show that these properties of a weak factorization system are equivalent.

Chapter 3

Weak factorization systems in display map categories.

In this chapter, we study the weak factorization systems which form display map categories modeling Σ, $\mathtt{Id}$, and Π types. We saw in the previous chapter that for a weak factorization system $(\mathcal{L}, \mathcal{R})$ on a category $\mathcal{C}$ with finite limits to form a display map category $(\mathcal{C}, \mathcal{R})$ modeling Σ types, it is necessary and sufficient that every object of $\mathcal{C}$ is fibrant. Thus, all weak factorization systems in this section will be of this flavor. We also saw that for $(\mathcal{C}, \mathcal{R})$ to model $\mathtt{Id}$ types, the weak factorization system must have an $\mathtt{Id}$-presentation, and for it to model Π types (in addition to having pre-Π types) it should be type theoretic.

In the following sections, we will show that for a weak factorization system $(\mathcal{L}, \mathcal{R})$ on a category $\mathcal{C}$ with finite limits, the conditions that it (1) has an $\mathtt{Id}$-presentation and (2) is type theoretic are equivalent. To prove this, we also describe an algebraic structure (called a Moore relation structure) on a category $\mathcal{C}$ with finite limits which generates a weak factorization system of this flavor and, conversely, is always entailed by such a weak factorization system.

In the first section, we give a categorical analysis of the structure underlying an $\mathtt{Id}$-presentation. In the second, we define Moore relation structures in such categories and show that they generate type theoretic weak factorization systems. In the third section, we show that any type theoretic weak factorization system has an $\mathtt{Id}$-presentation. In the fourth section, we tie these threads together by showing that a weak factorization system has an $\mathtt{Id}$-presentation if and only if it is generated by a Moore relation system. A more precise description can be found below in Section 3.1.6.

To our knowledge, the material of this chapter is new, save that in Section 3.2.1 which overlaps considerably with that of [BG12]. However, it is the only section of this chapter that does not contribute directly to the proof of our main result, Theorem 3.5.1. This is because the notion of model that interests us is decidedly weaker than that under consideration in [BG12]. It is however informative to see that the approach here and in [BG12] are, at least at this point, congruent.

3.1 Relations and factorizations.

This section is largely intended to build vocabulary for the following sections. We discuss how a factorization on a category $\mathcal{C}$ can be generated from a suitable collection of internal relations

$$X \underset{\epsilon_1}{\overset{\epsilon_0}{\leftleftarrows}} \xrightarrow{\eta} \mathrm{R}(X)$$

in $\mathcal{C}$.

We have already seen one such collection of internal relations which generates a factorization on a category: `Id` types on objects in a display map category $(\mathcal{C}, \mathcal{D})$. This will be an example of what we discuss below. However, in defining `Id` types on objects, we already had a potential lifting pair $({}^{\boxslash}\mathcal{D}, \overline{\mathcal{D}})$ in mind and required certain morphisms to be in certain classes (${}^{\boxslash}\mathcal{D}$ or $\mathcal{D}$) accordingly. In this and the next section, we take a lifting-pair-agnostic approach justified by the fact (Corollary 1.4.5) that a factorization which is part of a weak factorization structure completely determines the lifting pair. Thus in this section, we describe how a collection of relations determines a factorization, and in the next section (3.2), we will characterize those collections of relations which determine factorizations underlying weak factorization structures.

3.1.1 Relations.

Definition 3.1.1. Let $\mathfrak{R}$ denote the diagram category generated by the graph

$$\begin{tikzcd} \bigcirc \arrow[r, "\eta" description] & \Psi \arrow[l, bend right, "\epsilon_0"'] \arrow[l, bend left, "\epsilon_1"] \end{tikzcd}$$

and the relations $\epsilon_0\eta = \epsilon_1\eta = 1_\bigcirc$. A *relation* on an object X of a category $\mathcal{C}$ is a functor $R : \mathfrak{R} \to \mathcal{C}$ such that $R(\bigcirc) = X$.

Let $\mathcal{C}^\mathfrak{R}$ denote the category of relations in $\mathcal{C}$, and let $\mathcal{C}^\bigcirc : \mathcal{C}^\mathfrak{R} \to \mathcal{C}$ denote the forgetful functor given by evaluation at $\bigcirc$.

Remark 3.1.2. What we have just defined could more descriptively be called an internal reflexive pseudo-relation. However, since all relations in this work will be of this type, we will call them relations.

We will sometimes use the following kind of relation.

Definition 3.1.3. A *monic relation* on an object X of a category $\mathcal{C}$ is a relation R on X such that the morphisms $R(\epsilon_0)$ and $R(\epsilon_1)$ are jointly monic.

A monic relation in $\mathcal{S}et$ is then just a reflexive relation, in the usual sense.

Example 3.1.4. On any object X of any category $\mathcal{C}$, there is a minimal monic relation on X given by the following diagram.

$$\begin{tikzcd} X \arrow[r, "1_X" description] & X \arrow[l, bend right, "1_X"'] \arrow[l, bend left, "1_X"] \end{tikzcd}$$

Example 3.1.5. On any object X of any category $\mathcal{C}$ with binary products, there is a maximal monic relation on X given by the following diagram

$$\begin{tikzcd} X \arrow[r, "\Delta_X" description] & X \times X \arrow[l, bend right, "\pi_0"'] \arrow[l, bend left, "\pi_1"] \end{tikzcd}$$

where π_0 and π_1 are the first and second projections $X \times X \to X$, respectively, and Δ_X is the diagonal $X \to X \times X$.

Example 3.1.6. Consider the category $\mathcal{T}$ of topological spaces. Let $I = [0, 1]$ denote the usual interval. For any space X, consider the internal hom X^I which we think of as the space of paths in X. Let $c : X \to X^I$ denote the continuous function which takes any point in X to the constant path at that point. For any $t \in I$, let $\mathsf{ev}_t : X^I \to X$ denote the continuous function which takes a path p in X to $p(t)$. Then the following is a relation on X in $\mathcal{T}$.

$$\begin{tikzcd} X \arrow[r, "c" description] & X^I \arrow[l, shift right=2, "\mathsf{ev}_0"'] \arrow[l, shift left=2, "\mathsf{ev}_1"] \end{tikzcd}$$

Example 3.1.7. Consider any category $\mathcal{C}$ with a terminal object $*$, objects X and I of $\mathcal{C}$, and morphisms $0, 1 : * \rightrightarrows I$ of $\mathcal{C}$ such that an internal hom X^I exists in $\mathcal{C}$. Let $! : I \to *$ denote the unique morphism to the terminal object.

This creates a relation on X whose image is the following diagram.

$$\begin{tikzcd} X \arrow[r, "X^!" description] & X^I \arrow[l, shift right=2, "X^0"'] \arrow[l, shift left=2, "X^1"] \end{tikzcd}$$

The preceding three examples can be seen as a special case of this example where $I = *$ for the minimal relation, $I = * + *$ for the maximal relation, and I is the interval $[0, 1]$ for the relation in $\mathcal{T}$.

Definition 3.1.8. Consider a morphism $f : X \to Y$ in a category $\mathcal{C}$ and relations $\mathbf{R}X$ on X and $\mathbf{R}Y$ on Y (whose images are illustrated below).

$$\begin{tikzcd} X \arrow[r, "\eta_X" description] & RX \arrow[l, shift right=2, "\epsilon_{0X}"'] \arrow[l, shift left=2, "\epsilon_{1X}"] \end{tikzcd} \qquad \begin{tikzcd} Y \arrow[r, "\eta_Y" description] & RY \arrow[l, shift right=2, "\epsilon_{0Y}"'] \arrow[l, shift left=2, "\epsilon_{1Y}"] \end{tikzcd}$$

Say that a natural transformation $\mathbf{R}f : \mathbf{R}X \to \mathbf{R}Y$ is a *lift of f* if it is sent to f via the forgetful functor $\mathcal{C}^{\bigcirc}$. That is, the natural transformation $\mathbf{R}f$ is a lift if it has the component $f : X \to Y$ at $\bigcirc$ and some component which we will denote $Rf : RX \to RY$ at Ψ, as illustrated below.

$$\begin{tikzcd} \mathrm{R}X \arrow[r, "Rf"] \arrow[d, shift right=4, "\epsilon_{0X}"'] \arrow[d, shift left=4, "\epsilon_{1X}"] & \mathrm{R}Y \arrow[d, shift right=4, "\epsilon_{0Y}"'] \arrow[d, shift left=4, "\epsilon_{1Y}"] \\ X \arrow[r, "f"] \arrow[u, "\eta_X" description] & Y \arrow[u, "\eta_Y" description] \end{tikzcd}$$

Example 3.1.9. Consider the relation of Example 3.1.7 defined on two objects, X and Y, of a category $\mathcal{C}$. Given a morphism $f : X \to Y$ we obtain a lift of f from the relation on X to that on Y as the natural transformation illustrated below.

$$\begin{tikzcd}
X^I \arrow[r, "f^I"] \arrow[d, shift right=3, "X^0"'] \arrow[d, shift left=3, "X^1"] & Y^I \arrow[d, shift right=3, "X^0"'] \arrow[d, shift left=3, "X^1"] \\
X \arrow[u, "X^!"] \arrow[r, "f"] & Y \arrow[u, "X^!"]
\end{tikzcd}$$

Definition 3.1.10. A *relation* $\mathbf{R}$ on a category $\mathcal{C}$ consists of a relation $\mathbf{R}(X)$ on each X in $\mathcal{C}$ and a lift $\mathbf{R}(f) : \mathbf{R}(X) \to \mathbf{R}(Y)$ of each $f : X \to Y$ in $\mathcal{C}$.

Remark 3.1.11. Note that we are abusing terminology by speaking of both relations on an object of a category and a relation on that category.

Example 3.1.12. Consider a category $\mathcal{C}$ in which there exist an object I and morphisms $0, 1 : * \to I$ such that the relation of Example 3.1.7 can be defined on any object X of $\mathcal{C}$. Denote this relation by $\mathbf{I}(X)$. Then in Example 3.1.9, we saw that any $f : X \to Y$ of $\mathcal{C}$ has a lift $\mathbf{I}(f) : \mathbf{I}(X) \to \mathbf{I}(Y)$. This forms a relation $\mathbf{I}$ on the category $\mathcal{C}$.

Now we define morphisms between relations on a given category $\mathcal{C}$. Given two relations $\mathbf{R}$, $\mathbf{R}'$ on $\mathcal{C}$, an (unnatural) transformation $\tau : \mathbf{R} \to \mathbf{R}'$ consists of a morphism $\tau(X)$ for each object X of $\mathcal{C}$ which makes the following diagram in $\mathcal{C}$ display a natural transformation of relations $\mathbf{R}(X) \to \mathbf{R}'(X)$.

$$\begin{tikzcd}
\mathbf{R}(X) \arrow[rr, "\tau(X)"] \arrow[dr, shift left=2, "\epsilon_0"] \arrow[dr, shift right=2, "\epsilon_1"'] & & \mathbf{R}'(X) \arrow[dl, shift right=2, "\epsilon'_1"'] \arrow[dl, shift left=2, "\epsilon'_0"] \\
& X \arrow[ul, "\eta"] \arrow[ur, "\eta'"] &
\end{tikzcd}$$

(To be clear, at each object X of $\mathcal{C}$, $\tau(X)$ is a natural transformation between the functors $\mathbf{R}(X)$ and $\mathbf{R}'(X) : \mathfrak{R} \to \mathcal{C}$. However, τ itself is not required to be natural: that is, no naturality square of the form $\mathbf{R}'(f)\tau(X) = \tau(Y)\mathbf{R}(f)$ is required to commute.)

Definition 3.1.13. Consider a category $\mathcal{C}$. Let $\mathfrak{Rel}^{00}_{\mathcal{C}}$ denote the category of relations $\mathbf{R}$ on $\mathcal{C}$ and transformations between them.

Notation 3.1.14. As in Notation 1.2.5, the superscript 00 of $\mathfrak{Rel}^{00}_{\mathcal{C}}$ signifies that the objects are not functorial and the morphisms are not natural. We will define variants below.

Now we begin to show the connection between relations and factorizations. This will be made more explicit in the following sections.

Proposition 3.1.15. *Let $\mathcal{C}$ be a category with binary products and a factorization (λ, ρ).*

Then there is a relation $\mathbf{R}(\lambda, \rho)$ on $\mathcal{C}$ which at each object X is given by the following diagram

$$\begin{tikzcd} X \arrow[r, "\lambda(\Delta_X)" description] & M(\Delta_X) \arrow[l, bend right, "\pi_1\rho(\Delta_X)"'] \arrow[l, bend left, "\pi_0\rho(\Delta_X)"] \end{tikzcd}$$

(where M denotes $\textsc{cod}\lambda = \textsc{dom}\rho$).

Proof. For every X in $\mathcal{C}$, we factorize the diagonal $\Delta_X : X \to X \times X$.

$$X \xrightarrow{\lambda(\Delta_X)} M(\Delta_X) \xrightarrow{\rho(\Delta_X)} X \times X$$

Rearranging this, we get the following relation on X

$$\begin{tikzcd} X \arrow[r, "\lambda(\Delta_X)" description] & M(\Delta_X) \arrow[l, bend right, "\pi_1\rho(\Delta_X)"'] \arrow[l, bend left, "\pi_0\rho(\Delta_X)"] \end{tikzcd}$$

which will be denoted by $\mathbf{R}(\lambda, \rho)(X)$.

For any morphism $f : X \to Y$, there is a square $\langle f, f \times f \rangle : \Delta_X \to \Delta_Y$. The factorization of this square gives a lift $\mathbf{R}(\lambda, \rho)(f) : \mathbf{R}(\lambda, \rho)(X) \to \mathbf{R}(\lambda, \rho)(Y)$ of f.

This defines a relation $\mathbf{R}(\lambda, \rho)$ on $\mathcal{C}$. $\square$

Now we define the functorial version of what we have been considering.

Definition 3.1.16. A *functorial relation* on a category $\mathcal{C}$ is a section of the functor $\mathcal{C}^{\bigcirc} : \mathcal{C}^{\mathfrak{R}} \to \mathcal{C}$.

Note that a functorial relation $\mathbf{R}$ on a category $\mathcal{C}$ is in particular a relation $\mathbf{R}$ on $\mathcal{C}$ with the additional requirements that the specified lifts $\mathbf{R}(f)$ of morphisms f of $\mathcal{C}$ respect identities and composition of morphisms (that is, $\mathbf{R}(1_X) = 1_{\mathbf{R}(X)}$ and $\mathbf{R}(g \circ f) = \mathbf{R}(g) \circ \mathbf{R}(f)$ for all objects X and all composable morphisms f, g of $\mathcal{C}$).

Definition 3.1.17. Let $\mathfrak{Rel}_{\mathcal{C}}^{10}$ denote the category of functorial relations on $\mathcal{C}$ and transformations between them.

Let $\mathfrak{Rel}_{\mathcal{C}}^{11}$ denote the category of sections of the functor $\mathcal{C}^{\bigcirc} : \mathcal{C}^{\mathfrak{R}} \to \mathcal{C}$. This is the category of functorial relations on $\mathcal{C}$ and natural transformations between them.

There are natural inclusions

$$\mathfrak{Rel}_{\mathcal{C}}^{11} \hookrightarrow \mathfrak{Rel}_{\mathcal{C}}^{10} \hookrightarrow \mathfrak{Rel}_{\mathcal{C}}^{00}.$$

The first inclusion above is the identity on objects, and the second is the identity on morphisms.

Proposition 3.1.18. *Let $\mathcal{C}$ be a category with binary products and a functorial factorization (λ, ρ).*

Then the relation $\mathbf{R}(\lambda, \rho)$ on $\mathcal{C}$ constructed in Proposition 3.1.15 is functorial.

Proof. We need to show that $\mathbf{R}(\lambda, \rho)(1_X) = 1_{\mathbf{R}(\lambda,\rho)X}$ and that $\mathbf{R}(\lambda, \rho)(g \circ f) = \mathbf{R}(\lambda, \rho)(g) \circ \mathbf{R}(\lambda, \rho)(f)$.

Recall that $\mathbf{R}(\lambda, \rho)(f)$ for any $f : X \to Y$ in $\mathcal{C}$ is obtained by factoring the square shown below on the left to get the diagram shown below on the right

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\Delta_X"'] & Y \arrow[d, "\Delta_Y"] \\ X \times X \arrow[r, "f \times f"'] & Y \times Y \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\lambda(\Delta_X)"'] & Y \arrow[d, "\lambda(\Delta_Y)"] \\ M(\Delta_X) \arrow[r, "M\langle f, f \times f \rangle"] \arrow[d, "\rho(\Delta_X)"'] & M(\Delta_Y) \arrow[d, "\rho(\Delta_Y)"] \\ X \times X \arrow[r, "f \times f"'] & Y \times Y \end{tikzcd}$$

and then rearranging that diagram to get the diagram shown below.

$$\begin{tikzcd} M(\Delta_X) \arrow[rr, "M\langle f, f \times f \rangle"] \arrow[d, shift right=3, "\pi_0\rho(\Delta_X)"'] \arrow[d, shift left=3, "\pi_1\rho(\Delta_X)"] & & M(\Delta_Y) \arrow[d, shift right=3, "\pi_0\rho(\Delta_Y)"'] \arrow[d, shift left=3, "\pi_1\rho(\Delta_Y)"] \\ X \arrow[u, "\lambda(\Delta_X)"'] \arrow[rr, "f"'] & & Y \arrow[u, "\lambda(\Delta_Y)"'] \end{tikzcd}$$

Then we can see that (λ, ρ) is functorial if and only if $M : \mathcal{C}^2 \to \mathcal{C}$ is functorial if and only if $\mathbf{R}(\lambda, \rho)$ is functorial. $\square$

3.1.2 Relational factorizations.

Definition 3.1.19. Let $\mathfrak{F}$ denote the diagram category generated by the graph

$$\begin{tikzcd} \bigcirc \arrow[r, "\lambda"'] & \Phi \arrow[l, bend right, "\kappa"'] \arrow[r, "\rho"'] & \odot \end{tikzcd}$$

and the equation $\kappa\lambda = 1_{\bigcirc}$. A *relational factorization* of a morphism $f : X \to Y$ in a category $\mathcal{C}$ is a functor $F : \mathfrak{F} \to \mathcal{C}$ such that $F(\rho\lambda) = f$.

Let $\mathcal{C}^{\mathfrak{F}}$ denote the category of relational factorizations, and let $\mathcal{C}^{\rho\lambda} : \mathcal{C}^{\mathfrak{F}} \to \mathcal{C}^{2}$ denote the forgetful functor which sends a factorization of f to f itself.

Example 3.1.20. Consider the relation of Example 3.1.12. Let $f : X \to Y$ be a morphism in the category. Let $X \times_Y Y^I$ denote the pullback

$$\begin{tikzcd} X \times_Y Y^I \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y^I \arrow[d, "Y^0"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

of $f : X \to Y$ and $Y^0 : Y^I \to Y$. Then

$$\begin{tikzcd} X \arrow[r, "1_X \times Y^! f"'] & X \times_Y Y^I \arrow[l, bend right, "\pi_X"'] \arrow[r, "Y^1 \pi_{Y^I}"] & Y \end{tikzcd}$$

is a relational factorization of f.

Proposition 3.1.21. *Consider a category $\mathcal{C}$ with a weak factorization structure $(\lambda, \rho, \mathcal{L}, \mathcal{R})$ and a terminal object $*$. There is a relational factorization of any morphism in $\mathcal{C}$ whose domain is fibrant.*

Proof. Consider a morphism $f : X \to Y$ such that X is fibrant. Factor f,

$$X \xrightarrow{\lambda} Mf \xrightarrow{\rho} Y$$

and observe that the following lifting problem has a solution.

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, "\lambda"'] & X \arrow[d] \\ Mf \arrow[r] \arrow[ur, dashed, "k"] & * \end{tikzcd}$$

Then the diagram

$$\begin{tikzcd} X \arrow[r, "\lambda"'] & Mf \arrow[l, bend right, "\kappa"'] \arrow[r, "\rho"'] & Y \end{tikzcd}$$

in $\mathcal{C}$ is a relational factorization of f. $\square$

Definition 3.1.22. Consider a category $\mathcal{C}$, morphisms $f : W \to X$ and $g : Y \to Z$ in $\mathcal{C}$ and a morphism $\langle \alpha, \beta \rangle : f \to g$ in $\mathcal{C}^2$ which is given by the following commutative square in $\mathcal{C}$.

$$\begin{tikzcd} W \arrow[r, "\alpha"] \arrow[d, "f"'] & Y \arrow[d, "g"] \\ X \arrow[r, "\beta"] & Z \end{tikzcd}$$

Consider also relational factorizations $F(f)$ on f and $F(g)$ on g whose components are illustrated below.

$$\begin{tikzcd} W \arrow[r, "\lambda f"'] & Mf \arrow[l, bend right, "\kappa f"'] \arrow[r, "\rho f"'] & X \end{tikzcd} \qquad \begin{tikzcd} Y \arrow[r, "\lambda g"'] & Mg \arrow[l, bend right, "\kappa g"'] \arrow[r, "\rho g"'] & Z \end{tikzcd}$$

Say that a natural transformation $F(\alpha, \beta) : F(f) \to F(g)$ is a *lift* of $\langle \alpha, \beta \rangle$ if $\mathcal{C}^{\rho\lambda} F(\alpha, \beta) = \langle \alpha, \beta \rangle$. That is, $F(\alpha, \beta)$ is a *lift* of $\langle \alpha, \beta \rangle$ if it has components $\alpha : W \to Y$ at $\bigcirc$, $\beta : X \to Z$ at $\odot$, and some $M\langle \alpha, \beta \rangle : Mf \to Mg$ at Φ making the following diagram display a natural transformation $F(f) \to F(g)$.

$$\begin{tikzcd} W \arrow[rr, "\alpha"] \arrow[d, "\lambda f"'] & & Y \arrow[d, "\lambda g"'] \\ Mf \arrow[u, bend right, "\kappa f"'] \arrow[rr, "M\langle\alpha,\beta\rangle"] \arrow[d, "\rho f"'] & & Mg \arrow[u, bend right, "\kappa g"'] \arrow[d, "\rho g"'] \\ X \arrow[rr, "\beta"] & & Z \end{tikzcd}$$

Example 3.1.23. Consider factorizations of two morphisms $f : X \to Y$ and $g : V \to W$ as obtained in Example 3.1.20, and a morphism $\langle \alpha, \beta \rangle : f \to g$. The following is a lift of $\langle \alpha, \beta \rangle$.

$$\begin{tikzcd}
X \arrow[r, "\alpha"] \arrow[d, shift right, "1_X \times Y^! f"'] & V \arrow[d, shift right, "1_V \times W^! g"'] \\
X \times_Y Y^I \arrow[r, "\alpha \times_\beta \beta^I"] \arrow[u, shift right, bend right, "\pi_X"'] \arrow[d, "Y^1 \pi_{Y^I}"'] & V \times_W W^I \arrow[u, shift right, bend right, "\pi_V"'] \arrow[d, "W^1 \pi_{W^I}"'] \\
Y \arrow[r, "\beta"] & W
\end{tikzcd}$$

Definition 3.1.24. A *relational factorization* on a category $\mathcal{C}$ consists of a relational factorization $F(f)$ of every morphism f of $\mathcal{C}$ and a lift $F(\alpha, \beta) : F(f) \to F(g)$ of every $\langle \alpha, \beta \rangle : f \to g$ in $\mathcal{C}^2$.

Example 3.1.25. The previous example, 3.1.23, describes a relational factorization on a category $\mathcal{C}$ (where the internal hom X^I exists for every X in $\mathcal{C}$).

As we did for relations, we now define (unnatural) transformations between relational factorizations.

A transformation $\tau : F \to G$ between relational factorizations on a category $\mathcal{C}$ consists of a morphism $\tau(f) : F(f)\Phi \to G(f)\Phi$ which makes the following diagram in $\mathcal{C}$ display a natural transformation $F(f) \to G(f)$.

$$\begin{tikzcd}
 & X \arrow[dl, "F\lambda"'] \arrow[dr, "G\lambda"] & \\
F\Phi \arrow[ur, bend left, "F\kappa"] \arrow[rr, "\tau(f)"'] \arrow[dr, "F\rho"'] & & G\Phi \arrow[ul, bend right, "G\kappa"'] \arrow[dl, "G\rho"] \\
 & Y &
\end{tikzcd}$$

Definition 3.1.26. Let $\mathfrak{RelFact}^{00}_{\mathcal{C}}$ denote the category of relational factorizations on $\mathcal{C}$ and transformations between them.

Now we define the analogous functorial object.

Definition 3.1.27. A *functorial relational factorization* on a category $\mathcal{C}$ is a section of the forgetful functor $\mathcal{C}^{\rho\lambda} : \mathcal{C}^{\mathfrak{F}} \to \mathcal{C}^2$.

Definition 3.1.28. Let $\mathfrak{RelFact}^{10}_{\mathcal{C}}$ denote the category of functorial relational factorizations on $\mathcal{C}$ and transformations between them.

Let $\mathfrak{RelFact}_{\mathcal{C}}^{11}$ denote the category of sections of $\mathcal{C}^{\rho\lambda} : \mathcal{C}^{\mathfrak{F}} \to \mathcal{C}^{2}$. Its objects are functorial relational factorizations on $\mathcal{C}$, and its morphisms are natural transformations between them.

As before, there are natural inclusions

$$\mathfrak{RelFact}_{\mathcal{C}}^{11} \hookrightarrow \mathfrak{RelFact}_{\mathcal{C}}^{10} \hookrightarrow \mathfrak{RelFact}_{\mathcal{C}}^{00}$$

where the first is the identity on objects, and the second is the identity on morphisms.

3.1.3 Factorizations and relational factorizations.

In this section, we describe the relationship between the categories $\mathfrak{Fact}_{\mathcal{C}}^{ij}$ of factorizations on a category $\mathcal{C}$ and $\mathfrak{RelFact}_{\mathcal{C}}^{ij}$ of relational factorizations on $\mathcal{C}$ (for each ij).

In this section, we define functors

$$\mathbf{R} : \mathfrak{Fact}_{\mathcal{C}}^{ij} \leftrightarrows \mathfrak{RelFact}_{\mathcal{C}}^{ij} : \mathbf{U}$$

(for $ij = 00, 01, 11$). The functor $\mathbf{U}$ takes a relational factorization to its underlying factorization (described below).

In the first subsection below, we define the functors $\mathbf{U}$ and $\mathbf{R}$. In the second, we show that they produce a comonad $\mathbf{UR}$ on $\mathfrak{Fact}_{\mathcal{C}}^{11}$ (and a similar endofunctor on $\mathfrak{Fact}_{\mathcal{C}}^{ij}$ when $ij \neq 11$). In the third subsection below, we justify our interest in the category $\mathfrak{RelFact}_{\mathcal{C}}^{ij}$ by showing that for every weak factorization structure W in the image of $\mathbf{U}$, all objects are fibrant, and, furthermore, that for every weak factorization structure W in $\mathfrak{Fact}_{\mathcal{C}}^{ij}$, every object is fibrant in W if and only if $\mathbf{UR}(W) \simeq W$. Thus, the weak factorization structures that interest us in this chapter all underlie relational factorizations.

3.1.3.1 The functors.

In this section, we describe the functors between the categories $\mathfrak{Fact}_{\mathcal{C}}^{ij}$ and $\mathfrak{RelFact}_{\mathcal{C}}^{ij}$.

First of all, there is an obvious forgetful functor $\mathfrak{RelFact}_{\mathcal{C}}^{ij} \to \mathfrak{Fact}_{\mathcal{C}}^{ij}$. This arises from the inclusion $I : \mathfrak{Z} \hookrightarrow \mathfrak{F}$ which maps the morphism $0 \leqslant 1$ to λ and

$1 \leqslant 2$ to ρ. (In the diagram below, what appears to the right of $\mathfrak{I}$ is the generating graph of the category $\mathfrak{I}$, and what appears to the right of $\mathfrak{F}$ is the generating graph of the category $\mathfrak{F}$, described in Definition 3.1.19.)

$$\begin{tikzcd}
\mathfrak{I}: \arrow[d, "I"'] & 0 \arrow[r, "0\leqslant 1"] & 1 \arrow[r, "1\leqslant 2"] & 2 \\
\mathfrak{F}: & \bigcirc \arrow[r, "\lambda"'] & \Phi \arrow[l, bend right, "\kappa"'] \arrow[r, "\rho"] & \odot
\end{tikzcd}$$

Then any relational factorization $R(f) : \mathfrak{F} \to \mathcal{C}$ of a morphism f in a category $\mathcal{C}$ has an underlying factorization $R(f) \circ I : \mathfrak{I} \to \mathcal{C}$. Furthermore, this produces the following functor.

Proposition 3.1.29. *There is a forgetful functor*

$$\mathbf{U}^{ij} : \mathfrak{RelFact}_{\mathcal{C}}^{ij} \to \mathfrak{Fact}_{\mathcal{C}}^{ij}$$

(for $ij = 00, 10, 11$*) which at a relational factorization* R *in* $\mathfrak{RelFact}_{\mathcal{C}}^{ij}$ *and morphism* f *in* $\mathcal{C}$ *gives the underlying factorization* $\mathbf{U}(R)(f) = R(f) \circ I$.

These make the following diagram commute.

$$\begin{tikzcd}
\mathfrak{RelFact}_{\mathcal{C}}^{11} \arrow[r, "\mathbf{U}^{11}"] \arrow[d, hook] & \mathfrak{Fact}_{\mathcal{C}}^{11} \arrow[d, hook] \\
\mathfrak{RelFact}_{\mathcal{C}}^{10} \arrow[r, "\mathbf{U}^{10}"] \arrow[d, hook] & \mathfrak{Fact}_{\mathcal{C}}^{10} \arrow[d, hook] \\
\mathfrak{RelFact}_{\mathcal{C}}^{00} \arrow[r, "\mathbf{U}^{00}"] & \mathfrak{Fact}_{\mathcal{C}}^{00}
\end{tikzcd}$$

Proof. First, for any relational factorization R, we describe the factorization $\mathbf{U}(R)$ on $\mathcal{C}$. We set $\mathbf{U}(R)(f) = R(f) \circ I$ for any morphism f of $\mathcal{C}$ and $\mathbf{U}(R)\langle \alpha, \beta \rangle = R\langle \alpha, \beta \rangle \circ I$ for any square $\langle \alpha, \beta \rangle : f \to g$. Note that $\mathbf{U}(R)$ is functorial if R is functorial.

Consider any transformation $\tau : R \to S$ of relational factorizations. At a morphism f, this is a natural transformation $\tau(f) : R(f) \to S(f)$. Whiskering this with I, we get a natural transformation $\tau(f) \circ I : \mathbf{U}(R)(f) \to \mathbf{U}(S)(f)$. Thus, we set $\mathbf{U}(\tau)(f)$ to $\tau(f) \circ I$. Note that if τ is a natural transformation, then so is $\mathbf{U}(\tau)$.

Now let each $\mathbf{U}^{ij} : \mathfrak{RelFact}^{ij}_{\mathcal{C}} \to \mathfrak{Fact}^{ij}_{\mathcal{C}}$ for each $ij = 00, 10, 11$ be the restriction of $\mathbf{U}$ to $\mathfrak{RelFact}^{ij}_{\mathcal{C}}$. We have shown that its image is in $\mathfrak{Fact}^{ij}_{\mathcal{C}}$. □

Remark 3.1.30. When $ij = 11$, the categories $\mathfrak{RelFact}^{11}_{\mathcal{C}}$ and $\mathfrak{Fact}^{11}_{\mathcal{C}}$ are just the categories of sections of $\mathcal{C}^{\rho\lambda} : \mathcal{C}^{\mathfrak{F}} \to \mathcal{C}^{2}$ and $\mathcal{C}^{\delta_1} : \mathcal{C}^{\mathfrak{Z}} \to \mathcal{C}^{2}$, respectively. Then $\mathbf{U}^{11} : \mathfrak{RelFact}^{11}_{\mathcal{C}} \to \mathfrak{Fact}^{11}_{\mathcal{C}}$ is just postcomposition with $\mathcal{C}^{I} : \mathcal{C}^{\mathfrak{F}} \to \mathcal{C}^{\mathfrak{Z}}$.

Proposition 3.1.31. *Consider a category $\mathcal{C}$ with binary products. Then there is a functor*

$$\mathbf{R}^{ij} : \mathfrak{Fact}^{ij}_{\mathcal{C}} \to \mathfrak{RelFact}^{ij}_{\mathcal{C}}$$

(for $ij = 00, 10, 11$) which maps a factorization (λ, ρ) and a morphism $f : X \to Y$ to $\mathbf{R}(\lambda, \rho)(f)$ depicted by the following diagram.

$$\begin{tikzcd} X \arrow[r, "\lambda(1\times f)"'] & M(1\times f) \arrow[l, bend right, "\pi_X\rho(1\times f)"'] \arrow[r, "\pi_Y\rho(1\times f)"'] & Y \end{tikzcd}$$

These make the following diagram commute.

$$\begin{tikzcd} \mathfrak{Fact}^{11}_{\mathcal{C}} \arrow[r, "\mathbf{R}^{11}"] \arrow[d, hook] & \mathfrak{RelFact}^{11}_{\mathcal{C}} \arrow[d, hook] \\ \mathfrak{Fact}^{10}_{\mathcal{C}} \arrow[r, "\mathbf{R}^{10}"] \arrow[d, hook] & \mathfrak{RelFact}^{10}_{\mathcal{C}} \arrow[d, hook] \\ \mathfrak{Fact}^{00}_{\mathcal{C}} \arrow[r, "\mathbf{R}^{00}"] & \mathfrak{RelFact}^{00}_{\mathcal{C}} \end{tikzcd}$$

Proof. Consider a factorization (λ, ρ) and a morphism $f : X \to Y$ in $\mathcal{C}$. We use the factorization (λ, ρ) to factor $1 \times f : X \to X \times Y$ as shown below.

$$X \xrightarrow{\lambda(1\times f)} M(1 \times f) \xrightarrow{\rho(1\times f)} X \times Y$$

Then we obtain a relational factorization $\mathbf{R}(\lambda, \rho)(f)$ of f as shown below.

$$\begin{tikzcd} X \arrow[r, "\lambda(1\times f)"'] & M(1\times f) \arrow[l, bend right, "\pi_X\rho(1\times f)"'] \arrow[r, "\pi_Y\rho(1\times f)"'] & Y \end{tikzcd}$$

Consider any square $\langle \alpha, \beta \rangle : f \to g$ in $\mathcal{C}$. We set $\mathbf{R}(\lambda, \rho)\langle \alpha, \beta \rangle$ to the following lift of $\langle \alpha, \beta \rangle$.

$$\begin{tikzcd}
X \arrow[r, "\lambda(1\times f)"'] \arrow[d, "\alpha"] & M(1 \times f) \arrow[l, bend right, "\pi_X \rho(1\times f)"'] \arrow[r, "\pi_Y \rho(1\times f)"] \arrow[d, "M\langle \alpha, \alpha\times\beta\rangle"] & Y \arrow[d, "\beta"] \\
W \arrow[r, "\lambda(1\times g)"'] & M(1 \times g) \arrow[l, bend right, "\pi_W \rho(1\times g)"'] \arrow[r, "\pi_Z \rho(1\times g)"'] & Z
\end{tikzcd}$$

Now note that the relational factorization $\mathbf{R}(\lambda, \rho)$ is functorial if the factorization (λ, ρ) is.

Consider any transformation $\tau : (\lambda, \rho) \to (\lambda', \rho')$ of factorizations on $\mathcal{C}$ which consists of morphisms $\tau(f) : Mf \to M'f$ for each morphism f of $\mathcal{C}$ (where $M = \textsc{cod}\lambda = \textsc{dom}\rho$, $M' = \textsc{cod}\lambda' = \textsc{dom}\rho'$). Then the morphisms $\tau(1 \times f) : M(1\times f) \to M'(1\times f)$ assemble into a transformation $\mathbf{R}(\tau) : \mathbf{R}(\lambda, \rho) \to \mathbf{R}(\lambda', \rho')$. Moreover, this is natural when τ is natural.

Then let $\mathbf{R}^{ij}$ be the restriction of $\mathbf{R}$ to $\mathfrak{Fact}^{ij}_{\mathcal{C}}$. We have shown that its image is in $\mathfrak{RelFact}^{ij}_{\mathcal{C}}$. $\square$

3.1.3.2 The near-adjoint relationship.

We show in this section that $\mathbf{R}^{11}$ and $\mathbf{U}^{11}$ have a near-adjoint relationship.

The discussion in this section can be regarded as parenthetical. It is not necessary to understand the rest of the chapter, unlike the following section, 3.1.3.3, where we justify our interest in the $\mathfrak{RelFact}$ categories over the $\mathfrak{Fact}$ categories.

In this section, we restrict ourselves to the case when $ij = 11$ in order to evaluate the properties that these functors have. (When $ij \neq 11$ we expect the properties to be completely analogous, but more difficult to express without recourse to standard categorical language. For example, in Proposition 3.1.34 below, we define natural transformations η and ϵ. However, their naturality is inherited from the naturality of the morphisms in $\mathfrak{Fact}^{11}$.)

Notation 3.1.32. For the remainder of this section, we will omit the superscript 11.

Remark 3.1.33. Though we will show in this section that $\mathbf{R} : \mathfrak{Fact}_{\mathcal{C}} \to \mathfrak{RelFact}_{\mathcal{C}}$ is 'nearly' a right adjoint to $\mathbf{U} : \mathfrak{RelFact}_{\mathcal{C}} \to \mathfrak{Fact}_{\mathcal{C}}$, the functor $\mathbf{U}$ does in fact have a true right adjoint $\mathbf{L} : \mathfrak{Fact}_{\mathcal{C}} \to \mathfrak{RelFact}_{\mathcal{C}}$ when the ambient category $\mathcal{C}$ has binary products.

This functor $\mathbf{L}$ takes a factorization (λ, ρ) and a morphism $f : X \to Y$ to the following relational factorization.

$$\begin{tikzcd} X \arrow[r, "1_X \times \lambda(f)"'] & X \times M(f) \arrow[l, bend right, "\pi_X"'] \arrow[r, "\rho(f)\pi_{Mf}"'] & Y \end{tikzcd}$$

However, we are not interested in this functor. Our goal is to find a relational factorization R for any type theoretic weak factorization structure W such that the underlying factorization $\mathbf{U}(R)$ is a weak factorization structure equivalent to W. This is not satisfied by the functor $\mathbf{L}$ but will be by $\mathbf{R}$.

(To see that $\mathbf{L}$ does not satisfy this in general, we can consider the minimal monic relation $\mathbf{Min}$ on $\mathcal{S}et$ given in Example 3.1.4. We will see in Examples 3.2.4, 3.2.15, and 3.2.25 that this is strictly transitive, homotopical, and symmetric. Thus by Theorem 3.2.34, it generates a type theoretic weak factorization structure $\mathbf{UF}(\mathbf{Min})$ whose underlying factorization takes a function $f : X \to Y$ to the following simple factorization.

$$X \xrightarrow{1} X \xrightarrow{f} Y$$

Its left class is the class of bijections and its right class is the class of all functions. The factorization $\mathbf{ULUF}(\mathbf{Min})$ takes $f : X \to Y$ to

$$X \xrightarrow{\Delta} X \times X \xrightarrow{f\pi_1} Y$$

Since Δ is not in general a bijection, we see that $\mathbf{ULUF}(\mathbf{Min})$ is not a weak factorization structure equivalent to $\mathbf{UF}(\mathbf{Min})$.)

Proposition 3.1.34. *Consider a category $\mathcal{C}$ with binary products, and the functors $\mathbf{U} : \mathfrak{RelFact}_{\mathcal{C}} \leftrightarrows \mathfrak{Fact}_{\mathcal{C}} : \mathbf{R}$ defined above. There are natural transformations $\epsilon : \mathbf{UR} \to 1_{\mathfrak{Fact}_{\mathcal{C}}}$ and $\eta : \mathbf{R} \to \mathbf{RUR}$ making the following diagrams commute.*

$$\begin{tikzcd} \mathbf{R} \arrow[r, "\eta"] \arrow[dr, equal] & \mathbf{RUR} \arrow[d, "\mathbf{R}\epsilon"] \\ & \mathbf{R} \end{tikzcd} \qquad \begin{tikzcd} \mathbf{UR} \arrow[r, "\mathbf{U}\eta"] \arrow[dr, equal] & \mathbf{URUR} \arrow[d, "\epsilon"] \\ & \mathbf{UR} \end{tikzcd} \qquad \begin{tikzcd} \mathbf{R} \arrow[r, "\eta"] \arrow[d, "\eta"] & \mathbf{RUR} \arrow[d, "\eta"] \\ \mathbf{RUR} \arrow[r, "\mathbf{RU}\eta"] & \mathbf{RURUR} \end{tikzcd}$$

Proof. As before, given a morphism $f : X \to Y$, we will often consider the morphism $1 \times f : X \to X \times Y$.

Consider a factorization (λ, ρ) in $\mathfrak{Fact}_{\mathcal{C}}$. If (λ, ρ) takes a morphism $f : X \to Y$ to the diagram on the left, then $\mathbf{UR}(\lambda, \rho)$ takes f to the diagram on the right.

$$X \xrightarrow{\lambda_f} Mf \xrightarrow{\rho_f} Y \qquad\qquad X \xrightarrow{\lambda_{1\times f}} M(1\times f) \xrightarrow{\pi_Y \rho_{1\times f}} Y$$

Then we get a natural transformation $\epsilon_{(\lambda,\rho)} : \mathbf{UR}(\lambda, \rho) \to (\lambda, \rho)$ which at f has the following component.

$$\begin{tikzcd}
& M(1\times f) \arrow[dd, "M\langle 1_X, \pi_Y\rangle"] \arrow[dr, "\pi_Y \rho_{1\times f}"] & \\
X \arrow[ur, "\lambda_{1\times f}"] \arrow[dr, "\lambda_f"'] & & Y \\
& Mf \arrow[ur, "\rho_f"'] &
\end{tikzcd}$$

This assembles into a natural transformation $\epsilon : \mathbf{UR} \to 1$.

Now, the relational factorization $\mathbf{R}(\lambda, \rho)$ takes f to the following diagram,

$$\begin{tikzcd}
X \arrow[r, "\lambda_{1\times f}"'] & M(1\times f) \arrow[l, bend right, "\pi_X \rho_{1\times f}"'] \arrow[r, "\pi_Y \rho_{1\times f}"] & Y
\end{tikzcd}$$

and $\mathbf{RUR}(\lambda, \rho)$ takes f to the diagram below.

$$\begin{tikzcd}
X \arrow[r, "\lambda_{1\times 1\times f}"'] & M(1\times 1\times f) \arrow[l, bend right, "\pi_0 \rho_{1\times 1\times f}"'] \arrow[r, "\pi_Y \rho_{1\times 1\times f}"] & Y
\end{tikzcd}$$

Then we get a natural transformation $\eta_{(\lambda,\rho)} : \mathbf{R}(\lambda, \rho) \to \mathbf{RUR}(\lambda, \rho)$ which at f has the following component.

$$\begin{tikzcd}
& M(1\times f) \arrow[dd, "M\langle 1_X, \Delta_X \times 1_Y\rangle"] \arrow[dr, "\pi_Y \rho_{1\times f}"] \arrow[dl, bend right, "\pi_X \rho_{1\times f}"'] & \\
X \arrow[ur, "\lambda_{1\times f}"] \arrow[dr, "\lambda_{1\times 1\times f}"'] & & Y \\
& M(1\times 1\times f) \arrow[ur, "\pi_Y \rho_{1\times 1\times f}"'] \arrow[ul, bend right, "\pi_0 \rho_{1\times 1\times f}"'] &
\end{tikzcd}$$

This assembles into a natural transformation $\eta : \mathbf{R} \to \mathbf{RUR}$.

Now, to see that the diagrams in the statement commute, it suffices to check that they commute point-wise. Thus, consider again a factorization (λ, ρ) on $\mathcal{C}$ and a morphism $f : X \to Y$ of $\mathcal{C}$. Evaluated at (λ, ρ) and f, these become natural transformations between functors $\mathfrak{Z} \to \mathcal{C}$ or $\mathfrak{F} \to \mathcal{C}$ which are the identity at the objects 0 and 2 of $\mathfrak{Z}$, or the objects $\bigcirc, \odot$ of $\mathfrak{F}$ (i.e., the domain and codomain of the morphism being factorized). Thus, it suffices to check that the diagrams in the statement commute when evaluated not only at a factorization (λ, ρ) and a morphism f, but also at the middle object 1 of $\mathfrak{Z}$ or Φ of $\mathfrak{F}$. Evaluating, we find the following diagrams.

$$\begin{tikzcd}
M(1 \times f) \arrow[r, "M\langle 1_X, \Delta_X \times 1_Y \rangle"] \arrow[dr, equal] & M(1 \times 1 \times f) \arrow[d, "M\langle 1_X, \pi_1 \times \pi_Y \rangle"] \\
& M(1 \times f)
\end{tikzcd}
\qquad
\begin{tikzcd}
M(1 \times f) \arrow[r, "M\langle 1_X, \Delta_X \times 1_Y \rangle"] \arrow[dr, equal] & M(1 \times 1 \times f) \arrow[d, "M\langle 1_X, \pi_0 \times \pi_Y \rangle"] \\
& M(1 \times f)
\end{tikzcd}$$

$$\begin{tikzcd}
M(1 \times f) \arrow[r, "M\langle 1_X, \Delta_X \times 1_Y \rangle"] \arrow[d, "M\langle 1_X, \Delta_X \times 1_Y \rangle"] & M(1 \times 1 \times f) \arrow[d, "M\langle 1_X, \Delta_X \times \pi_1 \times \pi_Y \rangle"] \\
M(1 \times 1 \times f) \arrow[r, "M\langle 1_X, \pi_0 \times \Delta_X \times 1_Y \rangle"'] & M(1 \times 1 \times 1 \times f)
\end{tikzcd}$$

But these diagrams are wrapped in the functor M. Thus, to show that these diagrams commute, it suffices to remove the applications of M. But then note that every morphism is of the form $\langle 1_X, ? \rangle$. Thus, it suffices to show that these diagrams commute when we remove the applications of M and then project to the codomain $\langle 1_X, ? \rangle \mapsto ?$. Doing this, we find the following diagrams.

$$\begin{tikzcd}
X \times Y \arrow[r, "\Delta_X \times 1_Y"] \arrow[dr, equal] & X \times X \times Y \arrow[d, "\pi_1 \times \pi_Y"] \\
& X \times Y
\end{tikzcd}
\qquad
\begin{tikzcd}
X \times Y \arrow[r, "\Delta_X \times 1_Y"] \arrow[dr, equal] & X \times X \times Y \arrow[d, "\pi_0 \times \pi_Y"] \\
& X \times Y
\end{tikzcd}$$

$$\begin{tikzcd}
X \times Y \arrow[r, "\Delta_X \times 1_Y"] \arrow[d, "\Delta_X \times 1_Y"] & X \times X \times Y \arrow[d, "\Delta_X \times \pi_1 \times \pi_Y"] \\
X \times X \times Y \arrow[r, "\pi_0 \times \Delta_X \times 1_Y"'] & X \times X \times X \times Y
\end{tikzcd}$$

Now, we can easily see that these diagrams commute, and we conclude that the diagrams of the statement commute. $\square$

Corollary 3.1.35. *Consider a category $\mathcal{C}$ with binary products. Then* $(\mathbf{UR}, \epsilon, \mathbf{U}\eta)$ *is a comonad on* $\mathfrak{Fact}_{\mathcal{C}}$.

Proof. The following diagrams display the comonad laws, and are obtained from the diagrams of the above proposition by applying the functor $\mathbf{U}$ to the first and third.

$$\begin{tikzcd} \mathbf{UR} \arrow[r, "\mathbf{U}\eta"] \arrow[dr, equal] & (\mathbf{UR})^2 \arrow[d, "\mathbf{UR}\epsilon"] \\ & \mathbf{UR} \end{tikzcd} \qquad \begin{tikzcd} \mathbf{UR} \arrow[r, "\mathbf{U}\eta"] \arrow[dr, equal] & (\mathbf{UR})^2 \arrow[d, "\epsilon"] \\ & \mathbf{UR} \end{tikzcd} \qquad \begin{tikzcd} \mathbf{UR} \arrow[r, "\mathbf{U}\eta"] \arrow[d, "\mathbf{U}\eta"] & (\mathbf{UR})^2 \arrow[d, "\mathbf{U}\eta"] \\ (\mathbf{UR})^2 \arrow[r, "\mathbf{URU}\eta"] & (\mathbf{UR})^3 \end{tikzcd}$$

□

The coalgebras of this comonad are, in particular, factorizations (λ, ρ) equipped with morphisms $\gamma(f) : M(f) \to M(1 \times f)$ (natural in f) as depicted below.

$$\begin{tikzcd} & M(1 \times f) \arrow[dr, "\pi_Y \rho_{1\times f}"] & \\ X \arrow[ur, "\lambda_{1\times f}"] \arrow[dr, "\lambda_f"'] & & Y \\ & Mf \arrow[uu, "\gamma(f)"] \arrow[ur, "\rho_f"'] & \end{tikzcd}$$

Let $\mathcal{EM}(\mathbf{UR})$ denote the Eilenberg-Moore category of coalgebras of $\mathbf{UR}$, and let

$$\mathbf{U}' : \mathcal{EM}(\mathbf{UR}) \leftrightarrows \mathfrak{Fact}_{\mathcal{C}} : \mathbf{R}'$$

denote the associated adjunction. Then $\mathbf{R}, \mathbf{U}$ factors through $\mathbf{R}', \mathbf{U}'$ in the following way.

Given a coalgebra $\gamma : (\lambda, \rho) \to \mathbf{UR}(\lambda, \rho)$ as described above, we get the following composition

$$Mf \xrightarrow{\gamma(f)} M(1 \times f) \xrightarrow{\rho_{1\times f}} X \times Y \xrightarrow{\pi_X} X$$

which is natural in f, and this makes (λ, ρ) into the relational factorization with the following components at any morphism $f : X \to Y$.

$$\begin{tikzcd} X \arrow[r, shift right, "\lambda_f"'] & M(f) \arrow[l, shift right, "\pi_X \rho_{1\times f} \gamma(f)"'] \arrow[r, "\rho_f"] & Y \end{tikzcd}$$

This defines a functor $\mathbf{A} : \mathcal{EM}(\mathbf{UR}) \to \mathfrak{RelFact}_{\mathcal{C}}$ which relates the pair $\mathbf{U}, \mathbf{R}$ of functors to the adjunction $\mathbf{U}' \dashv \mathbf{R}'$ in the following way.

Proposition 3.1.36. *Consider a category $\mathcal{C}$ with binary products, and the following diagram of functors defined in the previous pages.*

$$\begin{tikzcd}
\mathfrak{RelFact}_{\mathcal{C}} \arrow[dr, shift left, "\mathbf{U}"] & \\
& \mathfrak{Fact}_{\mathcal{C}} \arrow[ul, shift left, "\mathbf{R}"] \arrow[dl, shift left, "\mathbf{R}'"] \\
\mathcal{EM}(\mathbf{UR}) \arrow[uu, "\mathbf{A}"] \arrow[ur, shift left, "\mathbf{U}'"] \arrow[ur, phantom, "\perp"] &
\end{tikzcd}$$

Then $\mathbf{AR}' = \mathbf{R}$ and $\mathbf{UA} = \mathbf{U}'$.

Proof. Consider a factorization (λ, ρ) on $\mathcal{C}$. Then for any $f : X \to Y$ in $\mathcal{C}$, $\mathbf{AR}'(\lambda, \rho)(f)$ is the relational factorization below on the left, and $\mathbf{R}(\lambda, \rho)(f)$ is the relational factorization below on the right.

$$\begin{tikzcd}
X \arrow[r, shift right, "\lambda_{1\times f}"'] & M(1 \times f) \arrow[l, shift right, "\pi_0\rho_{1\times 1\times f}M\langle 1_X, \Delta_X \times 1_Y\rangle"'] \arrow[r, "\pi_Y\rho_{1\times f}"] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, shift right, "\lambda_{1\times f}"'] & M(1 \times f) \arrow[l, shift right, "\pi_X\rho_{1\times f}"'] \arrow[r, "\pi_Y\rho_{1\times f}"] & Y
\end{tikzcd}$$

To show that these two relational factorizations coincide, we must check that $\pi_0\rho_{1\times 1\times f}M\langle 1_X, \Delta_X \times 1_Y\rangle = \pi_X\rho_{1\times f}$. This follows from the commutativity of the following diagram.

$$\begin{tikzcd}
M(1 \times f) \arrow[r, "\rho_{1\times f}"] \arrow[d, "M\langle 1_X, \Delta_X \times 1_Y\rangle"] & X \times Y \arrow[r, "\pi_X"] \arrow[d, "\Delta_X \times 1_Y"] & X \arrow[d, equal] \\
M(1 \times 1 \times f) \arrow[r, "\rho_{1\times 1\times f}"] & X \times X \times Y \arrow[r, "\pi_0"] & X
\end{tikzcd}$$

Thus, we have established that $\mathbf{AR}'(\lambda, \rho)(f) = \mathbf{R}(\lambda, \rho)(f)$.

Now consider a morphism $\langle \alpha, \beta\rangle : f \to g$ of $\mathcal{C}^2$. Both of the natural transformations $\mathbf{AR}'(\lambda, \rho)\langle \alpha, \beta\rangle$ and $\mathbf{R}(\lambda, \rho)\langle \alpha, \beta\rangle$ have components $\alpha, M\langle \alpha, \alpha \times \beta\rangle$, and β.

$$\begin{tikzcd}
X \arrow[r, shift right, "\lambda_{1\times f}"'] \arrow[d, "\alpha"'] & M(1 \times f) \arrow[l, shift right, "\pi_X\rho_{1\times f}"'] \arrow[r, "\pi_Y\rho_{1\times f}"] \arrow[d, "M\langle \alpha, \alpha\times\beta\rangle"] & Y \arrow[d, "\beta"] \\
V \arrow[r, shift right, "\lambda_{1\times g}"'] & M(1 \times g) \arrow[l, shift right, "\pi_V\rho_{1\times g}"'] \arrow[r, "\pi_W\rho_{1\times g}"] & W
\end{tikzcd}$$

Thus, we have $\mathbf{AR}'(\lambda,\rho)\langle\alpha,\beta\rangle = \mathbf{R}(\lambda,\rho)\langle\alpha,\beta\rangle$, and consequently we have shown that $\mathbf{AR}'(\lambda,\rho) = \mathbf{R}(\lambda,\rho)$.

Now consider a natural transformation $\tau : (\lambda,\rho) \to (\lambda',\rho')$. Viewed as a natural transformation of functors $\mathfrak{Z} \to \mathcal{C}$, it has three components: $1 : \textsc{dom} \to \textsc{dom}, g : M \to M'$, and $1 : \textsc{cod} \to \textsc{cod}$. And, viewed as natural transformations of functors $\mathfrak{F} \to \mathcal{C}$, both $\mathbf{AR}'\tau$ and $\mathbf{R}\tau$ have the components $1 : \textsc{dom} \to \textsc{dom}, g_{1\times f} : M(1\times -) \to M'(1\times -)$, and $1 : \textsc{dom}\times\textsc{cod} \to \textsc{dom}\times\textsc{cod}$.

$$\begin{tikzcd} & M(1\times f) \arrow[dd, "\tau(1\times f)"] \arrow[dr, "\rho_{1\times f}"] & \\ X \arrow[ur, "\lambda_{1\times f}"] \arrow[dr, "\lambda'_{1\times f}"'] & & Y \\ & M'(1\times f) \arrow[ur, "\rho'_{1\times f}"'] & \end{tikzcd}$$

Therefore, $\mathbf{AR}'\tau = \mathbf{R}\tau$, and we have shown that $\mathbf{AR}' = \mathbf{R}$.

Now consider a coalgebra $\gamma : (\lambda,\rho) \to \mathbf{UR}(\lambda,\rho)$. Both $\mathbf{U}'$ and $\mathbf{UA}$ take γ to (λ,ρ) and morphisms $\langle\alpha,\mathbf{UR}\alpha\rangle : \gamma \to \gamma'$ to α. Thus $\mathbf{UA} = \mathbf{U}'$. $\square$

3.1.3.3 Fibrant objects.

In this section, we justify our interest in the categories $\mathfrak{RelFact}$ over the categories $\mathfrak{Fact}$. Namely, we are interested only in weak factorization systems in which all objects are fibrant, and show below that these each have representative weak factorization structures in the image of $\mathbf{U} : \mathfrak{RelFact} \to \mathfrak{Fact}$.

Proposition 3.1.37. *Consider a category $\mathcal{C}$ with finite products and a weak factorization structure $(\lambda,\rho,\mathcal{L},\mathcal{R})$. Suppose that every object of $\mathcal{C}$ is fibrant. Then the factorization $\mathbf{UR}(\lambda,\rho)$ is a weak factorization structure equivalent to the original one $(\lambda,\rho,\mathcal{L},\mathcal{R})$.*

That is, $|\mathbf{UR}| \cong 1$ on the full subcategory of $|\mathfrak{Fact}_{\mathcal{C}}^{ij}|$ spanned by weak factorization structures in which all objects are fibrant.

Proof. Recall that the factorization $\mathbf{UR}(\lambda,\rho)$ takes a morphism $f : X \to Y$ to the following factorization.

$$X \xrightarrow{\lambda(1\times f)} M(1\times f) \xrightarrow{\pi_Y\rho(1\times f)} Y$$

The left factor is $\lambda(1 \times f)$, so it is in $\mathcal{L}$.

Note that since $! : X \to *$ is in $\mathcal{R}$ (which is stable under pullback), the projection $\pi_Y : X \times Y \to Y$ is in $\mathcal{R}$. Thus, the right factor, which is the composition $\pi_Y \rho(1 \times f)$, is also in $\mathcal{R}$.

By Corollary 1.4.5, we see that

$$(\lambda(\mathbf{1} \times -)\text{-}\mathsf{alg}, \pi\rho(\mathbf{1} \times -)\text{-}\mathsf{coalg}) = (\mathcal{L}, \mathcal{R}).$$

Therefore, the factorization $\mathbf{UR}(\lambda, \rho)$ is a weak factorization structure equivalent to $(\lambda, \rho, \mathcal{L}, \mathcal{R})$. □

Proposition 3.1.38. *Every factorization (λ, ρ) in the image of $\mathbf{U} : \mathfrak{RelFact}_{\mathcal{C}}^{ij} \to \mathfrak{Fact}_{\mathcal{C}}^{ij}$ has the property that every map to the terminal object has a ρ-algebra structure.*

In particular, consider a weak factorization structure $(\lambda, \rho, \mathcal{L}, \mathcal{R})$ whose factorization (λ, ρ) underlies a relational factorization on a category $\mathcal{C}$. Then every object of $\mathcal{C}$ is fibrant.

Proof. Consider an object X of $\mathcal{C}$ and a factorization (λ, ρ) in the image of $\mathbf{U}$. We want to show that $! : X \to *$ has a ρ-algebra structure. Thus, we need to show that the following lifting problem has a solution.

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, "\lambda(!)"'] & X \arrow[d] \\ M! \arrow[r] \arrow[ur, dashed] & * \end{tikzcd}$$

Since (λ, ρ) extends to a relational factorization, $\lambda(!)$ has a retraction $\kappa(!)$. This solves the lifting problem. □

Corollary 3.1.39. *Consider a weak factorization system $(\mathcal{L}, \mathcal{R})$ on a category $\mathcal{C}$ with finite products. Every object of $\mathcal{C}$ is fibrant if and only if $(\mathcal{L}, \mathcal{R})$ has a representative weak factorization structure $(\lambda, \rho, \mathcal{L}, \mathcal{R})$ where (λ, ρ) underlies a relational factorization on $\mathcal{C}$.*

Proof. The “if" statement is Proposition 3.1.38. The “only if" statement is Proposition 3.1.37. □

Since we are only considering weak factorization systems in which all objects are fibrant, we move from considering factorizations to considering relational factorizations. However, we do not introduce a variant of the notion of weak factorization system. Instead, we will extract a relational factorization from any weak factorization system by applying the functor $\mathbf{R}$.

3.1.4 Relational factorizations and relations.

In this section, we describe the relationship between the categories $\mathfrak{Rel}^{ij}$ and the categories $\mathfrak{RelFact}^{ij}$. In the first subsection, we describe functors between them. In the second subsection, we show that these functors form an adjunction when $ij = 11$.

3.1.4.1 The functors.

First, we describe the forgetful functor $\mathbf{V} : \mathfrak{RelFact} \to \mathfrak{Rel}$.

Proposition 3.1.40. *There are forgetful functors*

$$\mathbf{V}^{ij} : \mathfrak{RelFact}^{ij}_{\mathcal{C}} \to \mathfrak{Rel}^{ij}_{\mathcal{C}}$$

(for $ij = 00, 10, 11$) which at a relational factorization R in $\mathfrak{RelFact}^{ij}_{\mathcal{C}}$ and object X in $\mathcal{C}$ gives the following relation on X.

$$\begin{tikzcd} X \arrow[r, "R(1_X)\lambda" description] & \mathrm{R}(1_X)\Phi \arrow[l, bend right, "R(1_X)\kappa"'] \arrow[l, bend left, "R(1_X)\rho"] \end{tikzcd}$$

These make the following diagram commute.

$$\begin{tikzcd} \mathfrak{RelFact}^{11}_{\mathcal{C}} \arrow[r, "\mathbf{V}^{11}"] \arrow[d, hook] & \mathfrak{Rel}^{11}_{\mathcal{C}} \arrow[d, hook] \\ \mathfrak{RelFact}^{10}_{\mathcal{C}} \arrow[r, "\mathbf{V}^{10}"] \arrow[d, hook] & \mathfrak{Rel}^{10}_{\mathcal{C}} \arrow[d, hook] \\ \mathfrak{RelFact}^{00}_{\mathcal{C}} \arrow[r, "\mathbf{V}^{00}"] & \mathfrak{Rel}^{00}_{\mathcal{C}} \end{tikzcd}$$

Proof. Consider a relational factorization R on $\mathcal{C}$ which takes any morphism $f : X \to Y$ of $\mathcal{C}$ to the following diagram.

$$\begin{tikzcd}
X \arrow[r, "R(f)\lambda"'] & R(f)\Phi \arrow[l, bend right, "R(f)\kappa"'] \arrow[r, "R(f)\rho"'] & Y
\end{tikzcd}$$

We define the relation $\mathbf{V}(R)$ by setting $\mathbf{V}(R)(X)$ to the following relation on X

$$\begin{tikzcd}
X \arrow[r, "R(1_X)\lambda" description] & \mathrm{R}(1_X)\Phi \arrow[l, shift right=2, "R(1_X)\kappa"'] \arrow[l, shift left=2, "R(1_X)\rho"]
\end{tikzcd}$$

for any object X of $\mathcal{C}$ and setting $\mathbf{V}(R)(f)$ to the following lift of f

$$\begin{tikzcd}
X \arrow[r, "R(1_X)\lambda" description] \arrow[d, "f"] & \mathrm{R}(1_X)\Phi \arrow[l, shift right=2, "R(1_X)\kappa"'] \arrow[l, shift left=2, "R(1_X)\rho"] \arrow[d, "R\langle f,f\rangle\Phi"] \\
Y \arrow[r, "R(1_Y)\lambda" description] & \mathrm{R}(1_Y)\Phi \arrow[l, shift right=2, "R(1_Y)\kappa"'] \arrow[l, shift left=2, "R(1_Y)\rho"]
\end{tikzcd}$$

for any morphism $f : X \to Y$ of $\mathcal{C}$. Note that if the relational factorization R is functorial, then so is the relation $\mathbf{V}(R)$.

Consider a transformation $\tau : R \to R'$ between relational factorizations on $\mathcal{C}$. We define the transformation $\mathbf{V}(\tau) : \mathbf{V}(R) \to \mathbf{V}(R')$ at an object X of $\mathcal{C}$ to be given by the following diagram.

$$\begin{tikzcd}
R(1_X)\Phi \arrow[rr, "\tau(X)\Phi"] \arrow[dr, shift left=2, "R(1_X)\kappa"] \arrow[dr, shift right=2, "R(1_X)\rho"'] & & R'(1_X)\Phi \arrow[dl, shift right=2, "R'(1_X)\kappa"'] \arrow[dl, shift left=2, "R'(1_X)\rho"] \\
& X \arrow[ul, "R(1_X)\lambda" description] \arrow[ur, "R'(1_X)\lambda" description] &
\end{tikzcd}$$

Note that if τ is natural, then so is $\mathbf{V}(\tau)$.

Now, let each $\mathbf{V}^{ij} : \mathfrak{RelFact}^{ij}_{\mathcal{C}} \to \mathfrak{Rel}^{ij}_{\mathcal{C}}$ be the appropriate restriction of $\mathbf{V}$. $\square$

Now we construct a functor in the opposite direction. The construction is the same as that used to construct a factorization from an `Id`-presentation in Proposition 2.3.4.

Proposition 3.1.41. *Consider a category $\mathcal{C}$ with pullbacks. There are functors*

$$\mathbf{F}^{ij} : \mathfrak{Rel}^{ij}_{\mathcal{C}} \to \mathfrak{RelFact}^{ij}_{\mathcal{C}}$$

which takes a relation R on $\mathcal{C}$ and a morphism $f : X \to Y$ in $\mathcal{C}$ to the relational factorization $\mathbf{F}(R)(f)$

$$\begin{tikzcd} X \arrow[r, "1\times\eta f"'] & X\,{}_f\!\times_{\epsilon_0} RY \arrow[l, bend right, "\pi_X"'] \arrow[r, "\epsilon_1\pi_{RY}"] & Y \end{tikzcd}$$

when the relation RY on Y is denoted as follows.

$$\begin{tikzcd} Y \arrow[r, "\eta"] & \mathrm{R}Y \arrow[l, shift right=2, "\epsilon_0"'] \arrow[l, shift left=2, "\epsilon_1"] \end{tikzcd}$$

These functors make the following diagram commute.

$$\begin{tikzcd} \mathfrak{Rel}^{11}_{\mathcal{C}} \arrow[r, "\mathbf{F}^{11}"] \arrow[d, hook] & \mathfrak{RelFact}^{11}_{\mathcal{C}} \arrow[d, hook] \\ \mathfrak{Rel}^{10}_{\mathcal{C}} \arrow[r, "\mathbf{F}^{10}"] \arrow[d, hook] & \mathfrak{RelFact}^{10}_{\mathcal{C}} \arrow[d, hook] \\ \mathfrak{Rel}^{00}_{\mathcal{C}} \arrow[r, "\mathbf{F}^{00}"] & \mathfrak{RelFact}^{00}_{\mathcal{C}} \end{tikzcd}$$

Proof. Consider a relation R on $\mathcal{C}$. We define $\mathbf{F}(R)(f)$ to be the following relational factorization of f

$$\begin{tikzcd} X \arrow[r, "1\times\eta f"'] & X\,{}_f\!\times_{\epsilon_0} RY \arrow[l, bend right, "\pi_X"'] \arrow[r, "\epsilon_1\pi_{RY}"] & Y \end{tikzcd}$$

(when RY is denoted as in the statement) where the middle object is the pullback obtained in the following diagram.

$$\begin{tikzcd} X\,{}_{\eta f}\!\times_{\epsilon_0} RY \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & RY \arrow[d, "\epsilon_0"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

Denote this factorization by the following diagram.

$$\begin{tikzcd} X \arrow[r, "\lambda_f"'] & Mf \arrow[l, bend right, "\kappa_f"'] \arrow[r, "\rho_f"] & Y \end{tikzcd}$$

The relational factorization $\mathbf{F}(R)$ takes any square $\langle\alpha,\beta\rangle : f \to g$ in $\mathcal{C}$ to the following natural transformation $\mathbf{F}(R)\langle\alpha,\beta\rangle$.

$$\begin{tikzcd} X \arrow[r, "\lambda_f"'] \arrow[d, "\alpha"] & Mf \arrow[l, bend right, "\kappa_f"'] \arrow[r, "\rho_f"] \arrow[d, "M\langle\alpha,\beta\rangle"] & Y \arrow[d, "\beta"] \\ X' \arrow[r, "\lambda_g"'] & Mg \arrow[l, bend right, "\kappa_g"'] \arrow[r, "\rho_g"] & Y' \end{tikzcd}$$

where $M\langle\alpha,\beta\rangle : Mf \to Mg$ is $\alpha \times R\beta : X\,{}_{\eta f}{\times}_{\epsilon_0} RY \to X'\,{}_{\eta g}{\times}_{\epsilon_0} RY'$.

Consider a transformation $\tau : R \to R'$ which consists of components $\tau(X) : R(X) \to R'(X)$ on each object X of $\mathcal{C}$. Then the transformation $1_X \times \tau(Y) : X \times_Y RY \to X \times_Y R'Y$ for each $f : X \to Y$ in $\mathcal{C}$ assembles into a transformation $\mathbf{F}(\tau) : \mathbf{F}(R) \to \mathbf{F}(R')$.

If R is functorial, then $\mathbf{F}(R)$ will be functorial as well. If τ is a natural transformation, then so will be $\mathbf{F}(\tau)$.

Then let $\mathbf{F}^{ij}$ be the restriction of $\mathbf{F}$ to $\mathfrak{Rel}^{ij}_{\mathcal{C}} \to \mathfrak{RelFact}^{ij}_{\mathcal{C}}$. $\square$

3.1.4.2 The adjunction.

In this section, as in Section 3.1.3.2 above, we restrict ourselves to the case when the superscript ij in $\mathbf{F}^{ij} : \mathfrak{Rel}^{ij}_{\mathcal{C}} \to \mathfrak{RelFact}^{ij}_{\mathcal{C}}$ is 11. In this case, the functor $\mathbf{F}^{11}$ arises as a right Kan extension of $\mathbf{V}^{11}$. (As in Section 3.1.3.2, analogous results hold for $ij \neq 11$, but they are not readily expressible in the standard language of category theory.)

Lemma 3.1.42. Consider a category $\mathcal{C}$. The category $\mathfrak{Rel}^{11}_{\mathcal{C}}$ of functorial relations on $\mathcal{C}$ is isomorphic to the category of functors $E : \mathcal{C} \to \mathcal{C}^{\mathfrak{F}}$ making the following diagram commute.

$$\begin{tikzcd} \mathcal{C} \arrow[r, "E"] \arrow[dr, "\mathcal{C}^!"'] & \mathcal{C}^{\mathfrak{F}} \arrow[d, "\mathcal{C}^{\rho\lambda}"] \\ & \mathcal{C}^2 \end{tikzcd}$$

Proof. Consider a functorial relation on $\mathcal{C}$: it is a morphism $D : \mathcal{C} \to \mathcal{C}^{\mathfrak{R}}$ in the slice $\mathcal{C}at/\mathcal{C}$ as illustrated below.

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "D"] \arrow[dr, "1"'] & \mathcal{C}^{\mathfrak{R}} \arrow[d, "\mathcal{C}^{\circ}"] & & \mathcal{C} \arrow[r, "E"] \arrow[dr, "\mathcal{C}^!"'] & \mathcal{C}^{\mathfrak{F}} \arrow[d, "\mathcal{C}^{\rho\lambda}"] \\
& \mathcal{C} \arrow[rrr, "\mathcal{C}^!"] & & & \mathcal{C}^2
\end{tikzcd}$$

Under the adjunction $(C^!)^* \vdash \Sigma_{C^!} : \mathcal{C}at/\mathcal{C} \to \mathcal{C}at/\mathcal{C}^2$, we see that

$$\hom_{\mathcal{C}at/\mathcal{C}}(\mathcal{C}, (\mathcal{C}^!)^*\mathcal{C}^{\mathfrak{F}}) \cong \hom_{\mathcal{C}at/\mathcal{C}^2}(\Sigma_{\mathcal{C}^!}\mathcal{C}, \mathcal{C}^{\mathfrak{F}}).$$

And since $(\mathcal{C}^!)^*\mathcal{C}^{\mathfrak{F}} = \mathcal{C}^{\mathbb{1}} \times_{\mathcal{C}^2} \mathcal{C}^{\mathfrak{F}} \cong \mathcal{C}^{\mathbb{1}+_2\mathfrak{F}} \cong \mathcal{C}^{\mathfrak{R}}$ and $\Sigma_{\mathcal{C}^!}\mathcal{C} = \mathcal{C}$, we have that

$$\hom_{\mathcal{C}at/\mathcal{C}}(\mathcal{C}, \mathcal{C}^{\mathfrak{R}}) \cong \hom_{\mathcal{C}at/\mathcal{C}^2}(\mathcal{C}, \mathcal{C}^{\mathfrak{F}}).$$ □

Notation 3.1.43. For any category $\mathcal{C}$, we will denote the following objects of the comma category $\mathcal{C}at/\mathcal{C}^2$ by their domains: $\mathcal{C}$ will stand for $\mathcal{C}^! : \mathcal{C} \to \mathcal{C}^2$, $\mathcal{C}^{\mathfrak{F}}$ will stand for $\mathcal{C}^{\rho\lambda} : \mathcal{C}^{\mathfrak{F}} \to \mathcal{C}^2$, and $\mathcal{C}^2$ will stand for $1_{\mathcal{C}^2} : \mathcal{C}^2 \to \mathcal{C}^2$.

Theorem 3.1.44. *Let $\mathcal{C}$ be a category with a functorial relation R and with pullbacks of $RX\epsilon_0$ for every object X of $\mathcal{C}$. Then the right Kan extension of R along $\mathcal{C}^!$ in $\mathcal{C}at/\mathcal{C}^2$ is $\mathbf{F}(R)$.*

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "\mathcal{C}^!"] \arrow[d, "R"'] & \mathcal{C}^2 \arrow[dl, dashed] \\
\mathcal{C}^{\mathfrak{F}} &
\end{tikzcd}$$

Furthermore, $\mathbf{F}(R)\mathcal{C}^! \cong R$.

Proof. For any morphism $f : X \to Y$, we will denote the components of RY by

$$\begin{tikzcd}
Y \arrow[r, "\eta" description] & \mathrm{R}(Y) \arrow[l, bend right, "\epsilon_0"'] \arrow[l, bend left, "\epsilon_1"]
\end{tikzcd}$$

and the components of $\mathbf{F}(R)f$ by the following diagram.

$$\begin{tikzcd}
X \arrow[r, "1\times\eta f"'] & X \times_Y RY \arrow[l, bend right, "\pi_X"'] \arrow[r, "\epsilon_1\pi_{RY}"'] & Y
\end{tikzcd}$$

First note that at any object X of $\mathcal{C}$, $\mathbf{F}(R)\mathcal{C}^! X$ is isomorphic to

$$\begin{tikzcd} X \arrow[r, "\eta"'] & RX \arrow[l, bend right, "\epsilon_0"'] \arrow[r, "\epsilon_1"'] & X \end{tikzcd}$$

which is the relational factorization RX, and similarly we see that $\mathbf{F}(R)\mathcal{C}^! f \cong Rf$ for any morphism f of $\mathcal{C}$. Thus, $\mathbf{F}(R)\mathcal{C}^! \cong R$. (Note that we could choose pullbacks along ϵ_0 so that they preserve identities, and this case we would have $\mathbf{F}(R)\mathcal{C}^! = R$. For simplicity, we will assume, without loss of generality, that this is the case in the rest of this proof.)

We want to show that the description of $\mathbf{F}(R)$ in the proof of Proposition 3.1.41 is indeed the right Kan extension.

Suppose there is a commutative diagram of the following form.

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\mathcal{C}^!"] \arrow[d, "R"'] \arrow[dr, phantom, "\overset{\alpha}{\Leftarrow}"] & \mathcal{C}^2 \arrow[dl, "F"] \\ \mathcal{C}^{\mathfrak{F}} & \end{tikzcd}$$

We will construct a natural transformation $\beta : F \Rightarrow \mathbf{F}(R)$ such that $\beta \circ \mathcal{C}^! = \alpha$ and show that this β is unique.

Denote the components of Ff by the following diagram for any morphism $f : X \to Y$ of $\mathcal{C}$.

$$\begin{tikzcd} X \arrow[r, "\lambda_f"'] & Ff \arrow[l, bend right, "\kappa_f"'] \arrow[r, "\rho_f"'] & Y \end{tikzcd}$$

For every arrow $f : X \to Y$, we have the following diagrams in which all squares commute. (Both diagrams represent the same information, and each row of morphisms in the right-hand diagram is an unpacking of the corresponding object in the left-hand diagram.)

$$
\begin{tikzcd}
Ff \arrow[d, "F\langle f,1_Y\rangle"] \\
F\mathcal{C}^!(Y) = F(1_Y) \arrow[d, "\alpha_Y"] \\
RY
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, shift right, "\lambda_f"'] \arrow[d, "f"'] & Ff \arrow[l, shift right, bend right, "\kappa_f"'] \arrow[r, "\rho_f"'] \arrow[d, "F\langle f,1_Y\rangle"] & Y \arrow[d, equal] \\
Y \arrow[r, shift right, "\lambda_{1_Y}"'] \arrow[d, equal] & F1_Y \arrow[l, shift right, bend right, "\kappa_{1_Y}"'] \arrow[r, "\rho_{1_Y}"'] \arrow[d, "\alpha_Y"] & Y \arrow[d, equal] \\
Y \arrow[r, shift right, "\eta_Y"'] & RY \arrow[l, shift right, bend right, "\epsilon_{0Y}"'] \arrow[r, "\epsilon_{1Y}"'] & Y
\end{tikzcd}
$$

in $\mathcal{C}^{\mathfrak{F}}$ in $\mathcal{C}$

In the right-hand diagram, we can see arrows $\kappa_f : Ff \to X$ and $\alpha_Y \circ F\langle f, 1_Y\rangle : Ff \to RY$ which induce an arrow $\kappa_f \times \alpha_Y \circ F\langle f, 1_Y\rangle : Ff \to X \times_Y RY$ by the universal property of $X \times_Y RY$. All squares in the following diagram commute

$$
\begin{tikzcd}
X \arrow[r, shift right, "\lambda_f"'] \arrow[d, equal] & Ff \arrow[l, shift right, bend right, "\kappa_f"'] \arrow[r, "\rho_f"'] \arrow[d, "\kappa_f \times \alpha_Y \circ F\langle f,1_Y\rangle"] & Y \arrow[d, equal] \\
X \arrow[r, shift right, "1\times_Y \eta_Y f"'] & X \times_Y RY \arrow[l, shift right, bend right, "\pi_X"'] \arrow[r, "\epsilon_{1Y}\pi_{RY}"'] & Y
\end{tikzcd}
$$

so this depicts a transformation $\beta_f : Ff \Rightarrow \mathbf{F}(R)f$ of relational factorizations. Since κ_f, α_Y, and $F\langle f, 1_Y\rangle$ are all natural in f, this assembles into a natural transformation $\beta : F \Rightarrow \mathbf{F}(R)$.

Whiskering β with $\mathcal{C}^! : \mathcal{C} \to \mathcal{C}^{\mathbb{2}}$ gives a natural transformation with the following component at any object X of $\mathcal{C}$.

$$
\begin{tikzcd}
X \arrow[r, shift right, "\lambda_{1_X}"'] \arrow[d, equal] & F1_X \arrow[l, shift right, bend right, "\kappa_{1_X}"'] \arrow[r, "\rho_{1_X}"'] \arrow[d, "\alpha_X"] & X \arrow[d, equal] \\
X \arrow[r, shift right, "\eta"'] & RX \arrow[l, shift right, bend right, "\epsilon_0"'] \arrow[r, "\epsilon_1"'] & X
\end{tikzcd}
$$

Therefore, $\beta \circ \mathcal{C}^! = \alpha$.

To see the uniqueness of β, suppose there were another natural transformation $\gamma : F \Rightarrow \mathbf{F}(R)$ such that $\gamma \circ \mathcal{C}^! = \alpha$. For each morphism $f : X \to Y$ of $\mathcal{C}$, the components β_f and γ_f are completely determined by their only nontrivial components $\beta_{f,\Phi}, \gamma_{f,\Phi} : Ff \to X \times_Y RY$ (recall that Φ is the middle object of the relational factorization diagram $\mathfrak{F}$), so it is enough to show that these two are equal for every f. By the universal property of $X \times_Y RY$, it is then enough to show that these coincide when projected to X and when projected to RY. We have that $\pi_X \gamma_{f,\Phi} = \pi_X \beta_{f,\Phi}$ since both squares in the following diagram commute.

$$\begin{tikzcd}
X \arrow[d, equal] & Ff \arrow[l, bend right, "\kappa_f"'] \arrow[d, shift right=2, "\gamma_f \Phi"'] \arrow[d, shift left=2, "\beta_f \Phi"] \\
X & X \times_Y RY \arrow[l, "\pi_X"']
\end{tikzcd}$$

Now we want to show that $\pi_{RY} \gamma_{f,\Phi} = \pi_{RY} \beta_{f,\Phi}$. We have the equations

$$\mathbf{F}(R)\langle f, 1_Y \rangle \beta_f = \beta_{1_Y} F\langle f, 1_Y \rangle$$
$$\mathbf{F}(R)\langle f, 1_Y \rangle \gamma_f = \gamma_{1_Y} F\langle f, 1_Y \rangle$$

by considering the naturality of β and γ on the morphism $\langle f, 1_Y \rangle : f \to 1_Y$, and since $\beta_{1_Y} = \alpha_Y = \gamma_{1_Y}$, we can see that

$$\mathbf{F}(R)\langle f, 1_Y \rangle \beta_f = \mathbf{F}(R)\langle f, 1_Y \rangle \gamma_f.$$

Now the middle component of $\mathbf{F}(R)\langle f, 1_Y \rangle$ is $\mathbf{F}(R)\langle f, 1_Y \rangle \Phi = \pi_{RY}$, so by taking the component of the above equation at Φ, we can conclude that $\pi_{RY} \gamma_f \Phi = \pi_{RY} \beta_f \Phi$. Therefore, $\beta = \gamma$. $\square$

Corollary 3.1.45. *If a category $\mathcal{C}$ has pullbacks, then $\mathbf{F} : \mathfrak{Rel}_{\mathcal{C}}^{11} \hookrightarrow \mathfrak{RelFact}_{\mathcal{C}}^{11}$ is a reflective subcategory with reflector $\mathbf{V}$.*

Proof. Proposition 3.1.44 gives an adjunction $(\mathcal{C}^!)^* \dashv \mathbf{F}$ where $(\mathcal{C}^!)^* \mathbf{F} \cong 1$.

Both $(\mathcal{C}^!)^*$ and $\mathbf{V} : \mathfrak{RelFact}_{\mathcal{C}}^{11} \to \mathfrak{Rel}_{\mathcal{C}}^{11}$ take a relational factorization which at any $f : X \to Y$ of $\mathcal{C}$ gives the following diagram

$$\begin{tikzcd}
X \arrow[r, "\lambda(f)"'] & M(f) \arrow[l, bend right, "\kappa(f)"'] \arrow[r, "\rho(f)"'] & Y
\end{tikzcd}$$

to the relation which at any X of $\mathcal{C}$ gives the following diagram

$$\begin{tikzcd}
X \arrow[r, "\lambda(1_X)" description] & M(1_X) \arrow[l, shift right=3, "\kappa(1_X)"'] \arrow[l, shift left=3, "\rho(1_X)"]
\end{tikzcd}$$

and which at any $f : X \to Y$ of $\mathcal{C}$ gives the following natural transformation of diagrams.

$$\begin{tikzcd}
X \arrow[r, "\lambda(1_X)" description] \arrow[d, "f"] & M(1_X) \arrow[l, shift right=2, "\kappa(1_X)"'] \arrow[l, shift left=2, "\rho(1_X)"] \arrow[d, "M\langle f,f\rangle"] \\
Y \arrow[r, "\lambda(1_Y)" description] & M(1_Y) \arrow[l, shift right=2, "\kappa(1_Y)"'] \arrow[l, shift left=2, "\rho(1_Y)"]
\end{tikzcd}$$

Thus, $(\mathcal{C}^!)^* = \mathbf{V}$. $\square$

3.1.5 `Id`-presentations.

In Definition 2.6.3, we defined what it means for a weak factorization system $(\mathcal{L}, \mathcal{R})$ on a category $\mathcal{C}$ to have an `Id`-presentation. This consisted of a model of `Id` types on objects in the display map category $(\mathcal{C}, \mathcal{R})$ which presents the weak factorization system $(\mathcal{L}, \mathcal{R})$. Explicitly, this consists of a factorization of the diagonal

$$X \xrightarrow{r_X} \mathtt{Id}(X) \xrightarrow{\epsilon_X} X \times X$$

on every object X of $\mathcal{C}$ such that

1. ϵ_X is in $\mathcal{R}$ for every object X of $\mathcal{C}$
2. for every morphism $\alpha : A \to X$ in $\mathcal{M}$, the pullback $\alpha^* r_X$, as shown below, is in $\mathcal{L}$.

$$\begin{tikzcd}
 & \alpha^*\mathtt{Id}(X) \arrow[rr] \arrow[dd] & & \mathtt{Id}(X) \arrow[dd, "\pi_i \epsilon_X"] \\
A \arrow[ur, "\alpha^* r_X"] \arrow[rr] \arrow[dr, equal] & & X \arrow[ur, "r_X"] \arrow[dr, equal] & \\
 & A \arrow[rr, "\alpha"] & & X
\end{tikzcd}$$

for each $i = 0, 1$.

In this situation (when the weak factorization system $(\mathcal{L}, \mathcal{R})$ has an `Id`-presentation), the data of the `Id`-presentation partially defines a factorization into $(\mathcal{L}, \mathcal{R})$. Consider the following relation on any object X of $\mathcal{C}$ which we obtain by rearranging the factorization of the diagonal $X \to X \times X$.

$$\begin{tikzcd} X \arrow[r, "r_X"] & \mathtt{Id}(X) \arrow[l, shift right=2, "\pi_0\epsilon_X"'] \arrow[l, shift left=2, "\pi_1\epsilon_X"] \end{tikzcd}$$

For any morphism $f : X \to Y$ of $\mathcal{C}$, we can find a lift of f by solving the following lifting problem.

$$\begin{tikzcd} X \arrow[r, "rf"] \arrow[d, "r"'] & \mathtt{Id}(Y) \arrow[d, "\epsilon"] \\ \mathtt{Id}(X) \arrow[r, "(f\times f)\epsilon"'] \arrow[ur, dashed] & Y \times Y \end{tikzcd}$$

A solution to this lifting problem exists since $r \in \mathcal{L}$, $\epsilon \in \mathcal{R}$, and $\mathcal{L} \boxslash \mathcal{R}$. Now the relation on each object of $\mathcal{C}$ and chosen lift on each morphism assemble into a relation on $\mathcal{C}$ which we will denote by `Id`. Then Proposition 2.3.4 states that the factorization $\mathbf{UF}(\mathtt{Id})$ is a weak factorization structure with lifting pair $(\mathcal{L}, \mathcal{R})$.

Now we define what it means for a *relation* on $\mathcal{C}$ to be an `Id`-presentation of a weak factorization system.

Definition 3.1.46. Consider a category $\mathcal{C}$ with finite limits. Consider a relation R which takes an object X to the diagram below.

$$\begin{tikzcd} X \arrow[r, "\eta_X"] & \mathrm{R}(X) \arrow[l, shift right=2, "\epsilon_{1X}"'] \arrow[l, shift left=2, "\epsilon_{0X}"] \end{tikzcd}$$

Say that R is an `Id`-*presentation* if the factorization $\mathbf{UF}(R)$ is a weak factorization structure (whose lifting pair we will denote by $(\mathcal{L}, \mathcal{R})$) such that $\epsilon_{0X} \times \epsilon_{1X} : R(X) \to X \times X$ is in $\mathcal{R}$ and every pullback of η_X for $i = 0, 1$ and any morphism $f : W \to X$ as shown below is in $\mathcal{L}$

$$\begin{tikzcd} & f^*RX \arrow[rr] \arrow[dd] & & RX \arrow[dd, "\epsilon_{iX}"] \\ W \arrow[ur, "f^*\eta_X"] \arrow[rr] \arrow[dr, equal] & & X \arrow[ur, "\eta_X"] \arrow[dr, equal] & \\ & W \arrow[rr, "f"] & & X \end{tikzcd}$$

for every object X of $\mathcal{C}$.

In this case, we say that R is an `Id`-presentation *of* the weak factorization system $[\mathbf{UF}(R)]$ that it represents.

That is, a relation R on a category $\mathcal{C}$ with finite limits is an `Id`-presentation of a weak factorization system $(\mathcal{L}, \mathcal{R})$ just when the data $(r, R, \epsilon_0 \times \epsilon_1)$ (using the notation of the definition above) forms a model of `Id` types on objects in the display map category $(\mathcal{C}, \mathcal{R})$ which is an `Id`-presentation of $(\mathcal{L}, \mathcal{R})$.

Conversely, a model $(r, \mathtt{Id}, \epsilon)$ of `Id` types on objects in the display map category $(\mathcal{C}, \mathcal{R})$ is an `Id`-presentation of a weak factorization system $(\mathcal{L}, \mathcal{R})$ just when all (or, equivalently, one) of the relations `Id` on $\mathcal{C}$ which can be constructed from $(r, \mathtt{Id}, \epsilon)$ by adding lifts of morphisms are `Id`-presentations of $(\mathcal{L}, \mathcal{R})$.

Thus, we have the following fact.

Proposition 3.1.47. *Consider a weak factorization system $(\mathcal{L}, \mathcal{R})$ on a finitely complete category $\mathcal{C}$. There is a relation which is an* `Id`*-presentation of $(\mathcal{L}, \mathcal{R})$ if and only if the display map category $(\mathcal{C}, \mathcal{R})$ has a model of* `Id` *types which is an* `Id`*-presentation of $(\mathcal{L}, \mathcal{R})$.*

Note that in the statement of this proposition, the first object (the `Id`-presentation relation) contains exactly the same data as the second (the `Id`-presentation model of `Id` types) with the exception that lifts of morphisms are included explicitly as part of the *structure* of the relation but are only ensured to *exist* by the model of `Id` types. It will behoove us to carry along these lifts as structure in our categorical analysis. Thus, in this chapter, we restrict our analysis to relations which are `Id`-presentations though we ultimately are interested in models of `Id` types which are `Id`-presentations.

In Section 3.3.2, we will be able to simplify what is required for a relation R to be an `Id`-presentation.

3.1.6 Summary and prospectus.

Consider a category $\mathcal{C}$ with finite limits. In the preceding sections of this chapter, we have described the following diagram of categories and functors in which each square commutes.

$$\begin{tikzcd}
\mathfrak{Fact}_{\mathcal{C}}^{11} \arrow[r, shift left, "\mathbf{R}^{11}"] \arrow[d, hook] & \mathfrak{RelFact}_{\mathcal{C}}^{11} \arrow[l, shift left, "\mathbf{U}^{11}"] \arrow[r, shift right, "\mathbf{V}^{11}"'] \arrow[d, hook] & \mathfrak{Rel}_{\mathcal{C}}^{11} \arrow[l, shift right, "\mathbf{F}^{11}"'] \arrow[d, hook] \\
\mathfrak{Fact}_{\mathcal{C}}^{10} \arrow[r, shift left, "\mathbf{R}^{10}"] \arrow[d, hook] & \mathfrak{RelFact}_{\mathcal{C}}^{10} \arrow[l, shift left, "\mathbf{U}^{10}"] \arrow[r, shift right, "\mathbf{V}^{10}"'] \arrow[d, hook] & \mathfrak{Rel}_{\mathcal{C}}^{10} \arrow[l, shift right, "\mathbf{F}^{10}"'] \arrow[d, hook] \\
\mathfrak{Fact}_{\mathcal{C}}^{00} \arrow[r, shift left, "\mathbf{R}^{00}"] & \mathfrak{RelFact}_{\mathcal{C}}^{00} \arrow[l, shift left, "\mathbf{U}^{00}"] \arrow[r, shift right, "\mathbf{V}^{00}"'] & \mathfrak{Rel}_{\mathcal{C}}^{00} \arrow[l, shift right, "\mathbf{F}^{00}"']
\end{tikzcd}$$

We are interested in the relationship between type theoretic weak factorization structures and $\mathtt{Id}$-presentations of weak factorization systems. The former are a kind of factorization, so they naturally form full subcategories of the categories $\mathfrak{Fact}$. The latter are a kind of relation, so they naturally form full subcategories of the categories $\mathfrak{Rel}$.

Definition 3.1.48. Let $\mathfrak{ttWFS}_{\mathcal{C}}^{ij}$ be the full subcategory of $\mathfrak{WFS}_{\mathcal{C}}^{ij}$ spanned by type theoretic weak factorization structures on $\mathcal{C}$.

Definition 3.1.49. Let $\mathtt{Id}\mathfrak{Pres}_{\mathcal{C}}^{ij}$ denote the full subcategory of $\mathfrak{Rel}_{\mathcal{C}}^{ij}$ spanned by those relations R which are $\mathtt{Id}$-presentations.

Then we are interested in what relationship the subcategories $\mathfrak{ttWFS}_{\mathcal{C}}^{ij}$ and $\mathtt{Id}\mathfrak{Pres}_{\mathcal{C}}^{ij}$ have in the following diagram.

$$\begin{tikzcd}
\mathfrak{ttWFS}_{\mathcal{C}}^{11} \arrow[r, hook] \arrow[d, hook] & \mathfrak{Fact}_{\mathcal{C}}^{11} \arrow[r, shift left, "\mathbf{VR}"] \arrow[d, hook] & \mathfrak{Rel}_{\mathcal{C}}^{11} \arrow[l, shift left, "\mathbf{UF}"] \arrow[d, hook] & \mathtt{Id}\mathfrak{Pres}_{\mathcal{C}}^{11} \arrow[l, hook'] \arrow[d, hook] \\
\mathfrak{ttWFS}_{\mathcal{C}}^{10} \arrow[r, hook] \arrow[d, hook] & \mathfrak{Fact}_{\mathcal{C}}^{10} \arrow[r, shift left, "\mathbf{VR}"] \arrow[d, hook] & \mathfrak{Rel}_{\mathcal{C}}^{10} \arrow[l, shift left, "\mathbf{UF}"] \arrow[d, hook] & \mathtt{Id}\mathfrak{Pres}_{\mathcal{C}}^{10} \arrow[l, hook'] \arrow[d, hook] \\
\mathfrak{ttWFS}_{\mathcal{C}}^{00} \arrow[r, hook] & \mathfrak{Fact}_{\mathcal{C}}^{00} \arrow[r, shift left, "\mathbf{VR}"] & \mathfrak{Rel}_{\mathcal{C}}^{00} \arrow[l, shift left, "\mathbf{UF}"] & \mathtt{Id}\mathfrak{Pres}_{\mathcal{C}}^{00} \arrow[l, hook']
\end{tikzcd}$$

In the next section, 3.2, we describe structure on functorial relations R which will make the factorizations $\mathbf{UF}(R)$ type theoretic, algebraic weak factorization structures. We also describe structure on relations R which will make the factorizations $\mathbf{UF}(R)$ type theoretic weak factorization structures.

In Section 3.3, we show that given any type theoretic weak factorization structure $W \in \mathfrak{ttWFS}^{ij}$, the factorization $\mathbf{UFVR}(W)$ is again a type theoretic weak factorization structure in $\mathfrak{ttWFS}^{ij}$ equivalent to the original one, W.

In Section 3.4, we show that any relation R is in $\mathtt{Id}\mathfrak{Pres}^{ij}$ if and only if it has the structure described in Section 3.2, and that $\mathbf{VRUF}(R)$ is equivalent to R.

Putting these results together, we will have the following result.

Theorem 3.1.50. *The functors* $\mathbf{VR}$ *and* $\mathbf{UF}$ *described above restrict to functors shown below.*

$$\begin{tikzcd}
\mathfrak{ttWFS}^{11}_{C} \arrow[r, shift left, "\mathbf{VR}"] \arrow[d, hook] & \mathtt{Id}\mathfrak{Pres}^{11}_{C} \arrow[l, shift left, "\mathbf{UF}"] \arrow[d, hook] \\
\mathfrak{ttWFS}^{10}_{C} \arrow[r, shift left, "\mathbf{VR}"] \arrow[d, hook] & \mathtt{Id}\mathfrak{Pres}^{10}_{C} \arrow[l, shift left, "\mathbf{UF}"] \arrow[d, hook] \\
\mathfrak{ttWFS}^{00}_{C} \arrow[r, shift left, "\mathbf{VR}"] & \mathtt{Id}\mathfrak{Pres}^{00}_{C} \arrow[l, shift left, "\mathbf{UF}"]
\end{tikzcd}$$

Furthermore, when we apply the proset reflection, these give equivalences.

$$\begin{tikzcd}
|\mathfrak{ttWFS}^{10}_{C}| \arrow[r, shift left, "|\mathbf{VR}|"] \arrow[r, phantom, "\simeq"] \arrow[d, hook] & |\mathtt{Id}\mathfrak{Pres}^{10}_{C}| \arrow[l, shift left, "|\mathbf{UF}|"] \arrow[d, hook] \\
|\mathfrak{ttWFS}^{00}_{C}| \arrow[r, shift left, "|\mathbf{VR}|"] \arrow[r, phantom, "\simeq"] & |\mathtt{Id}\mathfrak{Pres}^{00}_{C}| \arrow[l, shift left, "|\mathbf{UF}|"]
\end{tikzcd}$$

In what follows, we will prove these results for the 00-flavored categories. But then since both squares in the following diagram commute,

$$\begin{tikzcd}
|\mathfrak{ttWFS}^{10}_{C}| \arrow[r, shift left, "|\mathbf{VR}|"] \arrow[d, hook] & |\mathtt{Id}\mathfrak{Pres}^{10}_{C}| \arrow[l, shift left, "|\mathbf{UF}|"] \arrow[d, hook] \\
|\mathfrak{ttWFS}^{00}_{C}| \arrow[r, shift left, "|\mathbf{VR}|"] & |\mathtt{Id}\mathfrak{Pres}^{00}_{C}| \arrow[l, shift left, "|\mathbf{UF}|"]
\end{tikzcd}$$

we see that an equivalence $|\mathfrak{ttWFS}^{00}_{C}| \simeq |\mathtt{Id}\mathfrak{Pres}^{00}_{C}|$ will restrict to an equivalence $|\mathfrak{ttWFS}^{10}_{C}| \simeq |\mathtt{Id}\mathfrak{Pres}^{10}_{C}|$.

Recall that the property of being type-theoretic is one of weak factorization systems (i.e., one representative weak factorization structure has it if and only if all do). Thus, the objects of $|\mathfrak{ttWFS}^{ij}_{C}|$ are really type theoretic weak factorization systems. Then we can interpret the above theorem by the following.

Theorem 3.1.51. *Any type theoretic weak factorization system has an* `Id`*-presentation, and, conversely, any* `Id`*-presentation generates a type theoretic weak factorization* *system. Thus, the properties of (1) being type theoretic and (2) having an* `Id`*-presentation are equivalent.*

3.2 Type theoretic weak factorization systems generated from relations.

In this section, we consider a category $\mathcal{C}$ with finite limits and a relation R. In the first subsection, we describe structure on R which will make $\mathbf{UF}(R)$ a type theoretic, algebraic weak factorization structure. We call this a *strict Moore relation structure*. In the second subsection we describe structure on R which will make $\mathbf{UF}(R)$ a type theoretic weak factorization structure. We call this structure a *Moore relation structure*.

In Section 3.4, we will show that any relation is an `Id`-presentation of a weak factorization system (i.e., an object of $\mathtt{Id}\mathfrak{Pres}_{\mathcal{C}}^{00}$) if and only if it has a Moore relation structure. Then the full subcategory of $\mathfrak{Rel}_{\mathcal{C}}^{00}$ spanned by Moore relation systems will coincide with $\mathtt{Id}\mathfrak{Pres}_{\mathcal{C}}^{00}$.

We originally defined the subcategory $\mathtt{Id}\mathfrak{Pres}_{\mathcal{C}}^{00}$ by referencing the functor $\mathbf{UF} : \mathfrak{Rel}_{\mathcal{C}}^{00} \to \mathfrak{Fact}_{\mathcal{C}}^{00}$. The description of Moore relation structures which follows describes this subcategory more directly, without making reference to $\mathbf{UF}$. Thus, it will be invaluable in connecting the category $\mathtt{Id}\mathfrak{Pres}_{\mathcal{C}}^{00}$ with the category $\mathfrak{tt}\mathfrak{WFS}_{\mathcal{C}}^{00}$, the goal of this chapter.

We are mostly interested in the (plain) Moore relation structures since these correspond to `Id`-presentations. These will be described in Section 3.2.2 below. However, first we describe strict Moore relation structures in Section 3.2.1. As mentioned in the introduction to this chapter, these have already been investigated in [BG12]. We mention these first because they have many natural examples, and are thus more readily understandable. By contrast, the only examples of non-strict Moore relation structures that we know of will come from the equivalence between them and type theoretic weak factorization systems.

3.2.1 Strict Moore relation systems.

In this subsection, we consider a functorial relation $\mathbf{R}$ which preserves pullbacks. For any object X in $\mathcal{C}$, denote the image of $\mathbf{R}X$ by

$$\begin{tikzcd} X \arrow[r, "\eta" description] & RX \arrow[l, shift right=2ex, "\epsilon_0"'] \arrow[l, shift left=2ex, "\epsilon_1"] \end{tikzcd} .$$

Note that the requirement that $\mathbf{R}$ preserves pullbacks is equivalent to the requirement that R does.

For any morphism $f : X \to Y$ of $\mathcal{C}$, denote the relational factorization $\mathbf{F}(\mathbf{R})f$ by the following diagram.

$$\begin{tikzcd} X \arrow[r, "\lambda_f"'] & Mf \arrow[l, bend right, "\kappa_f"'] \arrow[r, "\rho_f"'] & Y \end{tikzcd}$$

Recall that λ is a copointed endofunctor on $\mathcal{C}^2$, and ρ is a pointed endofunctor on $\mathcal{C}^2$.

In this section, we discuss the structure on $\mathbf{R}$ that will produce a comonad structure on $\lambda : \mathcal{C}^2 \to \mathcal{C}^2$ and a monad structure on $\rho : \mathcal{C}^2 \to \mathcal{C}^2$.

3.2.1.1 Strictly transitive functorial relations.

Definition 3.2.1. Say that a functorial relation $\mathbf{R} : \mathcal{C} \to \mathcal{C}^{\mathfrak{R}}$ is *strictly transitive* if there exists a natural transformation

$$\mu_X : RX {}_{\epsilon_1}\times_{\epsilon_0} RX \to RX$$

(natural in X) such that:

1. μ is a lift of the identity between the following functorial relations (that is, the following diagram commutes).

$$\begin{tikzcd} R {}_{\epsilon_1}\times_{\epsilon_0} R \arrow[rr, "\mu"] \arrow[d, shift right, "\epsilon_0\pi_0"'] \arrow[d, shift left, "\epsilon_1\pi_1"] & & R \arrow[d, shift right, "\epsilon_0"'] \arrow[d, shift left, "\epsilon_1"] \\ 1_{\mathcal{C}} \arrow[rr, equal] & & 1_{\mathcal{C}} \end{tikzcd} \tag{3.2.2}$$

2. $(1_{\mathcal{C}}, R, \epsilon_0, \epsilon_1, \eta, \mu)$ is an internal category in $[\mathcal{C}, \mathcal{C}]$ (that is, the following diagrams commute).

$$\begin{tikzcd} R \arrow[r, "\eta\times 1"] \arrow[dr, equal] & R_{\epsilon_1}\times_{\epsilon_0} R \arrow[d, "\mu"] & R \arrow[l, "1\times\eta"'] \arrow[dl, equal] \\ & R & \end{tikzcd} \qquad \begin{tikzcd} R_{\epsilon_1}\times_{\epsilon_0} R_{\epsilon_1}\times_{\epsilon_0} R \arrow[r, "1\times\mu"] \arrow[d, "\mu\times 1"] & R_{\epsilon_1}\times_{\epsilon_0} R \arrow[d, "\mu"] \\ R_{\epsilon_1}\times_{\epsilon_0} R \arrow[r, "\mu"] & R \end{tikzcd} \tag{3.2.3}$$

Note that if $\mathbf{R}$ is a monic relation, then the existence of μ with the commutativity of the diagram in (3.2.2) says that the relation $\mathbf{R}(X)$ on each object X of $\mathcal{C}$ is transitive, and the commutativity of the diagrams in (3.2.3) is automatic. Thus, the notion of transitivity here is a generalization of the usual one.

Example 3.2.4. Consider the minimal monic relation $\mathbf{Min}$ on any category $\mathcal{C}$ introduced in Example 3.1.4 which takes any object X to the following diagram.

$$\begin{tikzcd} X \arrow[r, "1_X" description] & X \arrow[l, shift right=2, "1_X"'] \arrow[l, shift left=2, "1_X"] \end{tikzcd} .$$

The morphism $1_X : X \to X$ for μ_X makes this relation strictly transitive.

Example 3.2.5. Consider the maximal monic relation $\mathbf{Max}$ on any category $\mathcal{C}$ with binary products introduced in Example 3.1.5 which takes any object X to the following diagram.

$$\begin{tikzcd} X \arrow[r, "\Delta" description] & X \times X \arrow[l, shift right=2, "\pi_0"'] \arrow[l, shift left=2, "\pi_1"] \end{tikzcd} .$$

The morphism $\pi_0 \times \pi_2 : X \times X \times X \to X \times X$ for μ_X makes this relation strictly transitive.

Example 3.2.6. More generally, consider the relation which takes any object X in $\mathcal{C}$ to the following diagram

$$\begin{tikzcd} X \arrow[r, "X^!" description] & \mathrm{X}^I \arrow[l, shift right=2, "X^0"'] \arrow[l, shift left=2, "X^1"] \end{tikzcd}$$

as in Example 3.1.7.

Suppose that there a morphism m making the following diagrams commute.

$$\begin{tikzcd} * \arrow[r, "i"] \arrow[d, "i"'] & I \arrow[d, "\iota_i"] \\ I \arrow[r, "m"] & I_1{+_0}I \end{tikzcd} \qquad \begin{tikzcd} & & I \\ I \arrow[urr, equal, bend left] \arrow[drr, equal, bend right] \arrow[r, "m"] & I_1{+_0}I \arrow[ur, "!+1_I"'] \arrow[dr, "1_I+!"] & \\ & & I \end{tikzcd} \qquad \begin{tikzcd} I \arrow[r, "m"] \arrow[d, "m"] & I_1{+_0}I \arrow[d, "m_1{+_0}I"] \\ I_1{+_0}I \arrow[r, "I_1{+_0}m"] & I_1{+_0}I_1{+_0}I \end{tikzcd}$$

Then taking $X^m : {X^I}_{\epsilon_1}{\times}_{\epsilon_0}X^I \to X^I$ for μ_X makes this relation strictly transitive.

For example, in the category $\mathcal{C}at$, there is such an m when I is $\mathbb{2}$ (i.e., the category generated by the graph $0 \to 1$) or the groupoid generated by the graph $0 \to 1$.

Example 3.2.7. Consider the category $\mathcal{T}$ of topological spaces. Let $\mathbb{R}^+$ denote the non-negative reals, and let ΓX denote the subspace of $X^{\mathbb{R}^+} \times \mathbb{R}^+$ consisting of pairs (p, r) such that p is constant on $[r, \infty)$. This is called the space of *Moore paths* in X, and it is functorial in X. We think of this as the space of paths in X of finite length.

There is a natural transformation $c : X \to \Gamma X$ which maps $x \in X$ to the constant path of length 0 at x. There are natural transformations ev_0, ev_∞ : $\Gamma X \to X$ which map a pair (p, r) to $p(0)$ and $p(r)$, respectively. These assemble into a functorial relation $\mathbf{\Gamma} : \mathcal{T} \to \mathcal{T}^{\mathfrak{R}}$.

There is also a natural transformation $\mu_X : \Gamma X_{ev_\infty}{\times}_{ev_0}\Gamma X \to \Gamma X$ which maps two paths to their concatenation. To be precise, it takes a pair $((p, r), (p', t'))$ such that $p(r) = p'(0)$ to the pair (q, s) where $s = r + r'$, $q|_{[0,r]} = p|_{[0,r]}$, and $q(x)|_{[r,\infty)} = p'(x - r)$. This makes $\mathbf{\Gamma}$ a strictly transitive functorial relation.

This will be revisited and generalized in Section 4.5.

Proposition 3.2.8. *Let $\mathbf{R} : \mathcal{C} \to \mathcal{C}^{\mathfrak{R}}$ be a strictly transitive functorial relation. Then the functor $\rho : \mathcal{C}^2 \to \mathcal{C}^2$ underlies a monad on $\mathcal{C}^2$ with unit and multiplication components at an object $f : X \to Y$ in $\mathcal{C}^2$ given by the following diagrams*

$$\begin{tikzcd} X \arrow[r, "\lambda_f"] \arrow[d, "f"] & Mf \arrow[d, "\rho_f"] \\ Y \arrow[r, equal] & Y \end{tikzcd} \qquad \begin{tikzcd} M\rho_f \arrow[r, "1\times\mu"] \arrow[d, "\rho_f^2"] & Mf \arrow[d, "\rho_f"] \\ Y \arrow[r, equal] & Y \end{tikzcd}$$

where $1 \times \mu : M\rho_f \to Mf$ *is the morphism*

$$1_X \times \mu_Y : X\,{}_f{\times}_{\epsilon_0} RY\,{}_{\epsilon_1}{\times}_{\epsilon_0} RY \longrightarrow X\,{}_f{\times}_{\epsilon_0} RY.$$

Proof. We have already seen that the unit square above commutes. The commutativity of the multiplication square above follows from the commutativity of (3.2.2).

The following diagram displays the unit axioms for the monad.

$$\begin{tikzcd}
X\,{}_f{\times}_{\epsilon_0} RY \arrow[r, "1\times 1\times\eta"] \arrow[dr, equal] & X\,{}_f{\times}_{\epsilon_0} RY\,{}_{\epsilon_1}{\times}_{\epsilon_0} RY \arrow[d, "1\times\mu"] & X\,{}_f{\times}_{\epsilon_0} RY \arrow[l, "1\times\eta f\times 1"'] \arrow[dl, equal] \\
& X\,{}_f{\times}_{\epsilon_0} RY &
\end{tikzcd}$$

Its commutativity follows from that of the left-hand diagram in (3.2.3).

This diagram displays the associativity axiom for the monad.

$$\begin{tikzcd}
X\,{}_f{\times}_{\epsilon_0} RY\,{}_{\epsilon_1}{\times}_{\epsilon_0} RY\,{}_{\epsilon_1}{\times}_{\epsilon_0} RY \arrow[r, "1\times 1\times\mu"] \arrow[d, "1\times\mu\times 1"] & X\,{}_f{\times}_{\epsilon_0} RY\,{}_{\epsilon_1}{\times}_{\epsilon_0} RY \arrow[d, "1\times\mu"] \\
X\,{}_f{\times}_{\epsilon_0} RY\,{}_{\epsilon_1}{\times}_{\epsilon_0} RY \arrow[r, "1\times\mu"] & X\,{}_f{\times}_{\epsilon_0} RY
\end{tikzcd}$$

Its commutativity follows from that of the right-hand diagram in (3.2.3). □

3.2.1.2 Strictly homotopical functorial relations.

Definition 3.2.9. Say that a functorial relation $\mathbf{R} : \mathcal{C} \to \mathcal{C}^{\mathfrak{R}}$ is *strictly homotopical* if there exist natural transformations

$$\delta_X : RX \to R^2X$$
$$\tau_X : X \times R(*) \to RX$$

(natural in X) such that:

1. δ is a lift of η between the following functors (that is, $\eta\eta = \delta\eta$ and $\eta\epsilon_0 = \epsilon_0\delta$ in the following diagram).

$$\begin{tikzcd} R \arrow[r, "\delta"] \arrow[d, shift right, "\epsilon_0"'] & R^2 \arrow[d, shift right, "\epsilon_0"'] \\ 1_{\mathcal{C}} \arrow[r, "\eta"] \arrow[u, shift right, "\eta"'] & R \arrow[u, shift right, "\eta"'] \end{tikzcd} \tag{3.2.10}$$

2. τ is a lift of the identity between the following functorial relations (that is, $\epsilon_i\tau = \pi$ and $\tau(1 \times \eta) = \eta$ in the following diagram).

$$\begin{tikzcd} 1_{\mathcal{C}} \times R(*) \arrow[r, "\tau"] \arrow[d, shift right=2, "\pi"'] \arrow[d, shift left=2, "\pi"] & R \arrow[d, shift right=2, "\epsilon_0"'] \arrow[d, shift left=2, "\epsilon_1"] \\ 1_{\mathcal{C}} \arrow[r, equal] \arrow[u, "1\times\eta"] & 1_{\mathcal{C}} \arrow[u, "\eta"] \end{tikzcd} \tag{3.2.11}$$

3. (R, ϵ_1, δ) is a comonad on $\mathcal{C}$ (that is, the following diagrams commute).

$$\begin{tikzcd} & R \arrow[dl, equal] \arrow[d, "\delta"] \arrow[dr, equal] & \\ R & R^2 \arrow[l, "\epsilon_1"] \arrow[r, "R\epsilon_1"'] & R \end{tikzcd} \qquad \begin{tikzcd} R \arrow[r, "\delta"] \arrow[d, "\delta"] & R^2 \arrow[d, "\delta"] \\ R^2 \arrow[r, "R\delta"] & R^3 \end{tikzcd} \tag{3.2.12}$$

4. τ is a strength for this comonad in the sense that the following diagrams commute.

$$\begin{tikzcd} 1_{\mathcal{C}} \times R* \arrow[d, "\tau"] \arrow[dr, "\pi_{R*}"] & \\ R \arrow[r, "R!"] & R* \end{tikzcd} \tag{3.2.13}$$

$$\begin{tikzcd} R \arrow[r, "\delta"] \arrow[d, "\epsilon_0 \times R!"] & R^2 \arrow[d, "R\epsilon_0"] \\ 1_{\mathcal{C}} \times R* \arrow[r, "\tau"] & R \end{tikzcd} \qquad \begin{tikzcd} 1_{\mathcal{C}} \times R* \arrow[r, "\tau\times\delta"] \arrow[d, "\tau"] & R(1_{\mathcal{C}} \times R*) \arrow[d, "R\tau"] \\ R \arrow[r, "\delta"] & R^2 \end{tikzcd} \tag{3.2.14}$$

The word *homotopical* is used to describe this functorial relation for the following reason. Suppose that we extract from the functorial relation $\mathbf{R}$ a notion of *homotopy equivalence* in the usual way: where two objects X and Y

are homotopic if there are morphisms $f : X \to Y$, $g : Y \to X$, $h : X \to RX$, $i : Y \to RY$ such that $\epsilon_0 h = gf$, $\epsilon_1 h = 1_X$, $\epsilon_0 i = fg$, and $\epsilon_1 i = 1_Y$. Then the data given in the above definition provide a homotopy between every X and RX.

Example 3.2.15. Consider the relation **Min** in Example 3.2.4. Then $1_X : X \to X$ for δ_X and τ_X make this relation strictly homotopical.

Example 3.2.16. Consider the relation **Max** in Example 3.2.5. Then

$$\pi_0 \times \pi_0 \times \pi_0 \times \pi_1 : X \times X \to X \times X \times X \times X$$

for δ_X and $\Delta : X \to X \times X$ for τ_X make this relation strictly homotopical.

Example 3.2.17. More generally, consider the relation in Example 3.2.6.

Suppose that there is a morphism d making the following diagrams commute.

$$\begin{tikzcd} I \arrow[r, "1!\times I"] \arrow[dr, equal] & I^2 \arrow[d, "d"] & I \arrow[l, "I\times 1!"'] \arrow[dl, equal] \\ & I & \end{tikzcd} \qquad \begin{tikzcd} I \arrow[r, "0!\times I"] \arrow[dr, "0!"'] & I^2 \arrow[d, "d"] & I \arrow[l, "I\times 0!"'] \arrow[dl, "0!"] \\ & I & \end{tikzcd} \qquad \begin{tikzcd} I^3 \arrow[r, "I\times d"] \arrow[d, "d\times I"] & I^2 \arrow[d, "d"] \\ I^2 \arrow[r, "d"] & I \end{tikzcd}$$

Then taking $X^d : X^I \to (X^I)^I$ for δ_X and $X^! : X \to X^I$ for τ_X makes this relation strictly homotopical.

For example, in the category $\mathcal{C}at$, there is such a d when I is $\mathbb{2}$ (i.e., the category generated by the graph $I : 0 \to 1$) or the groupoid generated by the graph $I : 0 \to 1$. Let the following diagram denote the graph $(I : 0 \to 1)^2$.

$$\begin{tikzcd} 00 \arrow[r, "0I"] \arrow[d, "I0"] & 01 \arrow[d, "I1"] \\ 10 \arrow[r, "1I"] & 11 \end{tikzcd}$$

Then in either case, d is generated by sending $0I$ and $I0$ to the identity morphism on 0, and $I1$ and $1I$ to $I : 0 \to 1$.

Example 3.2.18. Consider the functorial relation $\mathbf{\Gamma}$ on topological spaces described in Example 3.2.7.

There is a natural transformation $\delta_X : \Gamma X \to \Gamma^2 X$ which takes a pair (p, r) to the standard path from $c(p(0))$ to (p, r). To be precise, it maps (p, r) to (q, r) where $q(t) = (p_t, t) \in \Gamma X$ and $p_t|_{[0,t]} = p|_{[0,t]}$ for each $t \in \mathbb{R}^+$.

There is a natural transformation $\tau_X : X \times \Gamma(*) \to \Gamma X$. The space $\Gamma(*)$ is isomorphic to $\mathbb{R}^+$, so it maps a pair $(x, r) \in X \times \mathbb{R}^+$ to the constant path at x of length r.

These natural transformations make $\mathbf{\Gamma}$ into a strictly homotopical functorial relation.

In the following lemma, we record a natural transformation $\tilde{\tau}$ whose existence is equivalent to that of τ, but which will make the proof of the following proposition clearer.

Lemma 3.2.19. Consider a strictly homotopical functorial relation as above. For any $f : X \to Y$, let $\tilde{\tau}_f : X\,{}_f{\times}_{\epsilon_0} RY \to RX$ be the composite

$$X\,{}_f{\times}_{\epsilon_0} RY \xrightarrow{1 \times R!} X \times R* \xrightarrow{\tau} RX.$$

It makes the following diagrams commute.

$$\begin{tikzcd} & X\,{}_f{\times}_{\epsilon_0} RY \arrow[dl, "\pi"'] \arrow[d, "\tilde{\tau}"] \arrow[dr, "\pi"] & \\ X & RX \arrow[l, "\epsilon_0"] \arrow[r, "\epsilon_1"'] & X \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "1 \times \eta f"] \arrow[dr, "\eta"'] & X\,{}_f{\times}_{\epsilon_0} RY \arrow[d, "\tilde{\tau}"] \\ & RX \end{tikzcd} \tag{3.2.20}$$

$$\begin{tikzcd} X\,{}_f{\times}_{\epsilon_0} RY \arrow[r, "\delta \pi_{RY}"] \arrow[d, "\tilde{\tau}"'] & R^2Y \arrow[d, "R\epsilon_0"] \\ RX \arrow[r, "Rf"] & RY \end{tikzcd} \qquad \begin{tikzcd} X\,{}_f{\times}_{\epsilon_0} RY \arrow[r, "\tilde{\tau} \times \delta"] \arrow[d, "\tilde{\tau}"'] & R(X\,{}_f{\times}_{\epsilon_0} RY) \arrow[d, "R\tilde{\tau}"] \\ RX \arrow[r, "\delta"] & R^2X \end{tikzcd} \tag{3.2.21}$$

Proof. The commutativity of these diagrams is equivalent to that of the corresponding diagrams in (3.2.11), (3.2.13), and (3.2.14). □

Proposition 3.2.22. *Let $\mathbf{R} : \mathcal{C}^2 \to \mathcal{C}^{\mathfrak{R}}$ be a strictly homotopical functorial relation. Then the functor $\lambda : \mathcal{C}^2 \to \mathcal{C}^2$ underlies a comonad on $\mathcal{C}^2$ where the components of the counit and comultiplication at each object $f : X \to Y$ in $\mathcal{C}^2$ are given by the following diagrams*

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, "\lambda_f"'] & X \arrow[d, "f"] \\ Mf \arrow[r, "\rho_f"] & Y \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, equal] \arrow[d, "\lambda_f"'] & X \arrow[d, "\lambda_f^2"] \\ Mf \arrow[r, "1 \times \tilde{\tau} \times \delta"] & M\lambda_f \end{tikzcd}$$

where the morphism $1 \times \tilde{\tau} \times \delta$ *is the composition*

$$X{}_f{\times}_{\epsilon_0} RY \xrightarrow{1_X \times \tilde{\tau}_f \times \delta_Y} X{}_{\lambda_f}{\times}_{\epsilon_0} (RX{}_{Rf}{\times}_{R\epsilon_0} R^2Y) \cong X{}_{\lambda_f}{\times}_{\epsilon_0} R(X{}_f{\times}_{\epsilon_0} RY).$$

Proof. We have already seen that the counit square commutes. To define $1_X \times \tilde{\tau}_f \times \delta_Y$ we make use of the commutativity of (3.2.10) and the left hand sides of (3.2.20) and (3.2.21). The commutativity of the comultiplication square above is given by the commutativity of (3.2.10) and the right-hand diagram of (3.2.20).

The following diagrams display the comonad axioms. The commutativity of

$$\begin{tikzcd}
X{}_f{\times}_{\epsilon_0} RY & X{}_{\lambda_f}{\times}_{\epsilon_0} R(X{}_f{\times}_{\epsilon_0} RY) \arrow[l, "1\times R(\epsilon_1\pi_{RY})"'] \arrow[r, "\epsilon_1\pi_{R(X\times RY)}"] & X{}_f{\times}_{\epsilon_0} RY \\
& X{}_f{\times}_{\epsilon_0} RY \arrow[ul, equal] \arrow[u, "1\times\tilde{\tau}\times\delta"] \arrow[ur, equal] &
\end{tikzcd}$$

follows from the commutativity of the left-hand diagrams in (3.2.12) and (3.2.20), and the commutativity of

$$\begin{tikzcd}
X{}_f{\times}_{\epsilon_0} RY \arrow[r, "1\times\tilde{\tau}_f\times\delta_Y"] \arrow[d, "1\times\tilde{\tau}_f\times\delta_Y"] & X{}_{\lambda f}{\times}_{\epsilon_0} R(X{}_f{\times}_{\epsilon_0} RY) \arrow[d, "1\times\tilde{\tau}_\lambda\times\delta_{X\times RY}"] \\
X{}_{\lambda f}{\times}_{\epsilon_0} R(X{}_f{\times}_{\epsilon_0} RY) \arrow[r, "1\times R(1\times\tilde{\tau}_f\times\delta_Y)"] & X{}_{\lambda^2 f}{\times}_{\epsilon_0} R(X{}_{\lambda f}{\times}_{\epsilon_0} R(X{}_f{\times}_{\epsilon_0} RY))
\end{tikzcd}$$

follows from the right-hand diagrams in (3.2.12) and (3.2.21). □

3.2.1.3 Strictly symmetric functorial relations.

Definition 3.2.23. Say that a functorial relation $\mathbf{R} : \mathcal{C} \to \mathcal{C}^{\mathfrak{R}}$ is *strictly symmetric* if there exists a natural isomorphism

$$\iota_X : RX \to RX$$

(natural in X) which is a lift of the identity between the following functorial relations.

$$\begin{tikzcd} R \arrow[r, "\iota"] \arrow[d, shift right=2, "\epsilon_0"'] \arrow[d, shift left=2, "\epsilon_1"] & R \arrow[d, shift right=2, "\epsilon_1"'] \arrow[d, shift left=2, "\epsilon_0"] \\ 1_{\mathcal{C}} \arrow[u, "\eta"] \arrow[r, equal] & 1_{\mathcal{C}} \arrow[u, "\eta"] \end{tikzcd} \tag{3.2.24}$$

(That is, $\iota\eta = \eta$, $\epsilon_0\iota = \epsilon_1$, and $\epsilon_1\iota = \epsilon_0$).

If $\mathbf{R}$ is a monic relation, then the definition of *strictly symmetric* given here coincides with the usual definition of *symmetric*.

Example 3.2.25. Consider the relation $\mathbf{Min}$ in Example 3.2.4. Then $1_X : X \to X$ for ι_X makes this relation strictly symmetric.

Example 3.2.26. Consider the relation $\mathbf{Max}$ in Example 3.2.5. Then the twist $\pi_1 \times \pi_0 : X \times X \to X \times X$ for ι_X makes this relation strictly symmetric.

Example 3.2.27. More generally, consider the relation in Example 3.2.6.

Suppose that there an isomorphism i making the following diagrams commute.

$$\begin{tikzcd} * \arrow[d, "n"] \arrow[dr, "n-1"] & \\ I \arrow[r, "i"] & I \end{tikzcd}$$

for $n \in \mathbb{Z}/2$. Then taking $X^i : X^I \to X^I$ for ι_X makes this relation strictly symmetric.

For example, in the category $\mathcal{C}at$, there is such an i when I is the groupoid generated by the graph $I : 0 \to 1$.

Example 3.2.28. Consider the functorial relation $\mathbf{\Gamma}$ on topological spaces described in Example 3.2.7.

There is a natural transformation $\iota_X : \Gamma X \to \Gamma X$ which takes a pair (p, r) to the pair (q, r) where $q(t) = p(r - t)$ on $[0, r]$.

This makes $\mathbf{\Gamma}$ into a strictly symmetric functorial relation.

Lemma 3.2.29. Consider a strictly symmetric, strictly transitive functorial relation $\mathbf{R} : \mathcal{C} \to \mathcal{C}^{\mathfrak{R}}$, and denote the factorization $\mathbf{UF}(\mathbf{R})$ by (λ, ρ). Then for every object X of $\mathcal{C}$, the morphism

$$RX \xrightarrow{\epsilon_0 \times \epsilon_1} X \times X$$

has a ρ-algebra structure.

Proof. We need to show that there is a solution to the following lifting problem.

$$\begin{tikzcd}
RX \arrow[r, equal] \arrow[d, "\lambda(\epsilon_0\times\epsilon_1)"'] & RX \arrow[d, "\epsilon_0\times\epsilon_1"] \\
RX_{(\epsilon_0\times\epsilon_1)}\times_{\epsilon_0}R(X\times X) \arrow[r, "\rho(\epsilon_0\times\epsilon_1)"] \arrow[ur, dashed] & X\times X
\end{tikzcd}$$

We will do this by finding two lifts a and b as illustrated below.

$$\begin{tikzcd}
RX \arrow[rr, equal] \arrow[dr, "\eta\epsilon_0\times 1\times\eta\epsilon_1"] \arrow[dd, "1\times\eta(\epsilon_0\times\epsilon_1)"'] & & RX \arrow[dd, "\epsilon_0\times\epsilon_1"] \\
& RX_{\epsilon_0}\times_{\epsilon_0}RX_{\epsilon_1}\times_{\epsilon_0}RX \arrow[ur, dashed, "b"] \arrow[dr, "\epsilon_1\pi_0\times\epsilon_1\pi_2"] & \\
RX_{(\epsilon_0\times\epsilon_1)}\times_{\epsilon_0}R(X\times X) \arrow[ur, dashed, "a"] \arrow[rr, "\epsilon_1\pi_1"'] & & X\times X
\end{tikzcd} \qquad (*)$$

Let $u : R(X\times X)\to RX\times RX$ denote the universal morphism induced by the universal property of $RX\times RX$. It makes the following diagram commute.

$$\begin{tikzcd}
RX \arrow[r, "1\times\eta\epsilon_0\times\eta\epsilon_1"] \arrow[d, "1\times\eta(\epsilon_0\times\epsilon_1)"'] & RX_{(\epsilon_0\times\epsilon_1)}\times_{(\epsilon_0\times\epsilon_0)}(RX\times RX) \arrow[d, "(\epsilon_1\times\epsilon_1)\pi_{(RX\times RX)}"] \\
RX_{(\epsilon_0\times\epsilon_1)}\times_{\epsilon_0}R(X\times X) \arrow[ur, "1\times u"] \arrow[r, "\epsilon_1\pi_{R(X\times X)}"] & X\times X
\end{tikzcd}$$

Note that the outside square of this diagram is isomorphic to the lower-left portion of diagram $(*)$. Therefore, $1\times u$ is the lift a that we seek.

Now we let $b : RX_{\epsilon_0}\times_{\epsilon_0}RX_{\epsilon_1}\times_{\epsilon_0}RX\to RX$ be the following composite.

$$RX_{\epsilon_0}\times_{\epsilon_0}RX_{\epsilon_1}\times_{\epsilon_0}RX \xrightarrow{1\times\mu} RX_{\epsilon_0}\times_{\epsilon_0}RX \xrightarrow{\iota\times 1} RX_{\epsilon_1}\times_{\epsilon_0}RX \xrightarrow{\mu} RX.$$

This b makes the upper right-hand portion of the above diagram commute.

Therefore, we have found a lift in the original diagram, and shown that $\epsilon_0\times\epsilon_1$ has a ρ-algebra structure. $\square$

Theorem 3.2.30. *Consider a strictly symmetric functorial relation* $\mathbf{R} : \mathcal{C}\to\mathcal{C}^{\mathfrak{R}}$ *such that the factorization* $\mathbf{UF}(\mathbf{R})$ *represents a weak factorization system* $(\mathcal{L},\mathcal{R})$

and such that every morphism

$$RX \xrightarrow{\epsilon_0 \times \epsilon_1} X \times X$$

is in $\mathcal{R}$. Then the class $\mathcal{L}$ is stable under pullback along $\mathcal{R}$.

Proof. Consider the following pullback

$$\begin{tikzcd}
A \times_Y X \arrow[r] \arrow[d, "\pi_X"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "\ell"] \\
X \arrow[r, "r"] & Y
\end{tikzcd}$$

where r is in $\mathcal{R}$, and ℓ is in $\mathcal{L}$.

The morphism π_X is in $\mathcal{L}$ if and only if there is a solution to the following lifting problem.

$$\begin{tikzcd}
A \times_Y X \arrow[r, "\lambda_{\pi_X}"] \arrow[d, "\pi_X"'] & A_{\ell}\times_{r\epsilon_0} RX \arrow[d, "\rho_{\pi_X}"] \\
X \arrow[r, equal] \arrow[ur, dashed, "s"] & X
\end{tikzcd} \tag{$*$}$$

We will construct such a lift.

Since ℓ is in $\mathcal{L}$, there is a lift a in the following square.

$$\begin{tikzcd}
A \arrow[r, "\lambda_\ell"] \arrow[d, "\ell"'] & A_{\ell}\times_{\epsilon_0} RY \arrow[d, "\rho_\ell"] \\
Y \arrow[r, equal] \arrow[ur, dashed, "a"] & Y
\end{tikzcd}$$

Since r is in $\mathcal{R}$, the morphism $r \times 1_X : X \times X \to Y \times X$ is in $\mathcal{R}$ (as it is a pullback of r), and then the morphism $r\epsilon_0 \times \epsilon_1 : RX \to Y \times X$ is in $\mathcal{R}$ (as it is the composition of $\epsilon_0 \times \epsilon_1 \in \mathcal{R}$ and $r \times 1_X \in \mathcal{R}$). Thus, there is a lift in the following square.

$$\begin{tikzcd}
X \arrow[r, "\eta"] \arrow[d, "\lambda_r"'] & RX \arrow[d, "r\epsilon_0 \times \epsilon_1"] \\
X_{r}\times_{\epsilon_0} RY \arrow[r, "\epsilon_1 \times 1"] \arrow[ur, dashed, "b"] & Y \times X
\end{tikzcd}$$

Now let s be the following composition.

$$X \xrightarrow{ar \times 1_X} A_{\ell}\times_{\epsilon_0} RY\,{}_{\epsilon_1}\!\times_{r} X \xrightarrow{\pi_X \times \iota_Y \times \pi_A} X_{r}\times_{\epsilon_0} RY\,{}_{\epsilon_1}\!\times_{\ell} A \xrightarrow{\pi_A \times b} A_{\ell}\times_{r\epsilon_0} RX$$

This makes the diagram $(*)$ commute. $\square$

Corollary 3.2.31. *Consider a strictly symmetric, strictly transitive relation* $\mathbf{R}$ *on* $\mathcal{C}$ *such that the factorization* $\mathbf{UF}(\mathbf{R})$ *represents a weak factorization system* $(\mathcal{L}, \mathcal{R})$.

Then $(\mathcal{L}, \mathcal{R})$ *is type theoretic.*

Proof. By the previous two results, we know that $\mathcal{L}$ is stable under pullback along $\mathcal{R}$. By Proposition 3.1.38, every object is fibrant. Thus, $(\mathcal{L}, \mathcal{R})$ is type theoretic. $\square$

3.2.1.4 Summary.

We now have the following theorem.

Theorem 3.2.32. *Consider a category* $\mathcal{C}$ *with finite limits and a strictly transitive, strictly homotopical functorial relation* $\mathbf{R} : \mathcal{C} \to \mathcal{C}^{\mathfrak{R}}$. *Then the functorial factorization* $\mathbf{UF}(\mathbf{R})$ *is an algebraic weak factorization structure on* $\mathcal{C}$.

Proof. Let (λ, ρ) denote the functorial factorization of the statement. By Proposition 3.2.22, λ underlies a comonad, and by Proposition 3.2.8, ρ underlies a monad. Thus by Theorem 1.7.2, (λ, ρ) is an algebraic weak factorization structure on $\mathcal{C}$. $\square$

Definition 3.2.33. A *Moore relation structure* on a category $\mathcal{C}$ with finite limits is a functorial relation $\mathbf{R}$ together with the structure described in the definitions of strictly transitive, strictly homotopical, and strictly symmetric.

A *strict Moore relation system* on a category $\mathcal{C}$ with finite limits is a functorial relation $\mathbf{R}$ which is strictly transitive, strictly homotopical, and strictly symmetric (i.e., a relation for which a strict Moore relation structure exists).

Then we have the following theorem.

Theorem 3.2.34. *Consider a category* $\mathcal{C}$ *with finite limits and a strict Moore relation system* $\mathbf{R} : \mathcal{C} \to \mathcal{C}^{\mathfrak{R}}$. *Then the functorial factorization* $\mathbf{UF}(\mathbf{R})$ *is a type theoretic, algebraic weak factorization structure on* $\mathcal{C}$.

Proof. By the previous theorem, $\mathbf{UF}(\mathbf{R})$ is an algebraic weak factorization structure on $\mathcal{C}$. By Proposition 3.2.31, it is type theoretic. $\square$

Example 3.2.35. Consider the relation $\mathbf{Min}$ on any category $\mathcal{C}$ from Examples 3.2.4, 3.2.15, and 3.2.25. Then this generates a type theoretic, algebraic weak factorization structure on $\mathcal{C}$ whose factorization of a morphism $f : X \to Y$ is

$$X \xrightarrow{1_X} X \xrightarrow{f} Y,$$

whose left class consists of all isomorphisms, and whose right class consists of all morphisms.

Example 3.2.36. Consider the relation $\mathbf{Max}$ on any category $\mathcal{C}$ with binary products from Examples 3.2.5, 3.2.16, and 3.2.26. Then this generates a type theoretic, algebraic weak factorization structure on $\mathcal{C}$ whose factorization of a morphism $f : X \to Y$ is

$$X \xrightarrow{1_X \times f} X \times Y \xrightarrow{\pi_Y} Y,$$

whose left class consists of split monomorphisms, and whose right class consists of retracts of product projections.

Example 3.2.37. Consider the relation $\mathbf{\Gamma}$ on the category $\mathcal{T}$ of topological spaces from Examples 3.2.7, 3.2.18, and 3.2.28. Then this generates a type theoretic, algebraic weak factorization structure on $\mathcal{C}$ whose factorization of a morphism $f : X \to Y$ is

$$X \xrightarrow{1_X \times cf} X \times_Y \Gamma Y \xrightarrow{\pi_Y} Y,$$

whose left class consists of trivial Hurewicz cofibrations, and whose right class consists of Hurewicz fibrations (This weak factorization system was first described in [Str72] while this particular weak factorization structure was originally described in [May75].)

3.2.2 Moore relation systems.

In this section, we describe the minimal structure that a relation $\mathbf{R}$ on a category $\mathcal{C}$ with finite limits needs to have so that $\mathbf{UF}(\mathbf{R})$ is a type theoretic weak factorization structure. The minimality will be justified by Corollary 3.4.8, and though we do not give any examples in this section, many can be obtained from that corollary.

In what follows, we define what it means for a relation to be *transitive*, *homotopical*, and *symmetric*. Note that while the properties required of a transitive relation can be easily seen to be a subset of the properties required of a *strictly* transitive relation, the definitions of homotopical and symmetric given below differ more significantly from their strict predecessors.

In what follows, let (λ, ρ) denote the factorization $\mathbf{UF}(\mathbf{R})$, and let M denote $\textsc{cod}\lambda = \textsc{dom}\rho$.

3.2.2.1 Transitive relations.

Definition 3.2.38. Say that a relation $\mathbf{R}$ on $\mathcal{C}$ is *transitive* if there exists a morphism

$$\mu_X : RX\,{}_{\epsilon_1}\!\times_{\epsilon_0} RX \to RX$$

for every object X of $\mathcal{C}$ such that the following diagrams commute.

$$\begin{tikzcd} RX\,{}_{\epsilon_1}\!\times_{\epsilon_0} RX \arrow[r, "\mu"] \arrow[d, shift right, "\epsilon_0\pi_0"'] \arrow[d, shift left, "\epsilon_1\pi_1"] & RX \arrow[d, shift right, "\epsilon_0"'] \arrow[d, shift left, "\epsilon_1"] \\ X \arrow[r, equal] & X \end{tikzcd} \qquad \begin{tikzcd} RX \arrow[r, "1\times\eta"] \arrow[dr, equal] & RX\,{}_{\epsilon_1}\!\times_{\epsilon_0} RX \arrow[d, "\mu"] \\ & RX \end{tikzcd} \tag{3.2.39}$$

Non-example 3.2.40. Now we can see why the relation $\mathbf{\Gamma}$ on the category $\mathcal{T}$ of topological spaces is more useful than the relation $\mathbf{I}$ on $\mathcal{T}$ sending every space X to

$$\begin{tikzcd} X \arrow[r, "X^!" description] & X^I \arrow[l, shift right=2, "X^0"'] \arrow[l, shift left=2, "X^1"] \end{tikzcd}$$

(where I is the usual interval $[0, 1]$).

Suppose that this relation is transitive with a $\mu : X^I{}_{X^1}\!\times_{X^0} X^I \to X^I$ of the form $X^m : X^{[0,2]} \to X^{[0,1]}$. Then m would have to make the following diagrams commute for $i = 0, 1$

$$\begin{tikzcd} * \arrow[d, "i"] \arrow[dr, "i*2"] & \\ I \arrow[r, "m"] & {[0,2]} \end{tikzcd} \qquad \begin{tikzcd} I \arrow[r, "m"] \arrow[dr, equal, bend right] & {[0,2]} \arrow[d, "s"] \\ & I \end{tikzcd}$$

where s is the surjection which maps $[0, 1]$ onto $[0, 1]$ identically and $[1, 2]$ onto the point $\{1\}$. These diagrams say that $m(0) = 0$, $m(1) = 2$, and $sm = 1$. But there is no such continuous function (if there were, $m^{-1}(1, 2]$ would be a nonempty open set in I sent to $\{1\}$ by $sm = 1_I$).

Proposition 3.2.41. *Consider a transitive relation* $\mathbf{R}$ *on* $\mathcal{C}$ *as above. Then for every morphism* f *of* $\mathcal{C}$*, the morphism* ρ_f *has a* ρ*-algebra structure given by*

$$\begin{tikzcd} M\rho_f \arrow[r, "1\times\mu"] \arrow[d, "\rho_f^2"] & Mf \arrow[d, "\rho_f"] \\ Y \arrow[r, equal] & Y \end{tikzcd}$$

where $1 \times \mu : M\rho_f \to Mf$ *is the morphism*

$$1_X \times \mu_Y : X\,{}_f{\times}_{\epsilon_0} RY\,{}_{\epsilon_1}{\times}_{\epsilon_0} RY \longrightarrow X\,{}_f{\times}_{\epsilon_0} RY.$$

Proof. The commutativity of the square in the statement follows from the commutativity of the left-hand diagram of (3.2.39).

It remains to check that the composition of the point with the algebra structure, $(1 \times \mu) \circ \lambda\rho(f)$, is the identity.

$$\begin{tikzcd} X\,{}_f{\times}_{\epsilon_0} RY \arrow[r, "1\times 1\times\eta\epsilon_1"] \arrow[dr, equal] & X\,{}_f{\times}_{\epsilon_0} RY\,{}_{\epsilon_1}{\times}_{\epsilon_0} RY \arrow[d, "1\times\mu"] \\ & X\,{}_f{\times}_{\epsilon_0} RY \end{tikzcd}$$

The commutativity of this diagram follows from that of the right-hand diagram in (3.2.39). $\square$

As for the strictly transitive relations of the last section, when a relation $\mathbf{R}$ is monic, our definition of transitivity and the usual definition coincide.

3.2.2.2 Homotopical relations.

The definition of *transitive* could immediately be seen to be a weaker version of the definition of strictly transitive. This is not the case for the definition of *homotopical*.

Definition 3.2.42. Say that a relation $\mathbf{R}$ on $\mathcal{C}$ is *homotopical* if for each object X of $\mathcal{C}$, there exists an object $R^{\square}X$ of $\mathcal{C}$ with morphisms

$$\begin{tikzcd} X \arrow[r, "\eta"] & R^{\square}X \arrow[r, shift left=2, "\epsilon_0"] \arrow[r, "\epsilon_1" description] \arrow[r, shift right=2, "\zeta"'] & RX \end{tikzcd}$$

$$\delta_X : RX \to R^{\square}X,$$

and for every morphism $f : X \to Y$, a morphism

$$\tau_f : X\, {}_{\eta f}{\times}_{\zeta} R^{\square}Y \to R(X\, {}_{f}{\times}_{\epsilon_0} RY)$$

which make the following diagrams commute.

$$\begin{tikzcd}
RX & R^{\square}X \arrow[l, "\epsilon_i"'] \arrow[r, "\zeta"] & RX \\
& X \arrow[ul, "\eta"] \arrow[u, "\eta"] \arrow[ur, "\eta"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
R^{\square}Y \arrow[r, "\epsilon_i"] \arrow[d, "\zeta"] & RX \arrow[d, "\epsilon_0"] \\
RX \arrow[r, "\epsilon_i"] & X
\end{tikzcd}
\tag{3.2.43}$$

$$\begin{tikzcd}
X \arrow[r, "\eta"] \arrow[dr, "\eta"'] & RX \arrow[d, "\delta"] \\
& R^{\square}X
\end{tikzcd}
\qquad
\begin{tikzcd}
RX \arrow[r, "\epsilon_0"] \arrow[d, "\delta"] & X \arrow[d, "\eta"] \\
R^{\square}X \arrow[r, shift left, "\epsilon_0"] \arrow[r, shift right, "\zeta"'] & RX
\end{tikzcd}
\qquad
\begin{tikzcd}
RX \arrow[d, "\delta"] \arrow[dr, equal] & \\
R^{\square}X \arrow[r, "\epsilon_1"] & RX
\end{tikzcd}
\tag{3.2.44}$$

$$\begin{tikzcd}
X \arrow[r, "1\times\eta f"] \arrow[dr, "\eta(1\times\eta f)"'] & X\, {}_{\eta f}{\times}_{\zeta} R^{\square}Y \arrow[d, "\tau"] \\
& R(X\, {}_{f}{\times}_{\epsilon_0} RY)
\end{tikzcd}
\qquad
\begin{tikzcd}
X\, {}_{\eta f}{\times}_{\zeta} R^{\square}Y \arrow[d, "\tau"] \arrow[dr, "1\times\epsilon_i"] & \\
R(X\, {}_{f}{\times}_{\epsilon_0} RY) \arrow[r, "\epsilon_i"] & X\, {}_{f}{\times}_{\epsilon_0} RY
\end{tikzcd}
\tag{3.2.45}$$

where i ranges over $0, 1$.

Example 3.2.46. The object $R^{\square}X$ will often (as in Proposition 3.4.6) be the middle object of the factorization of the morphism $\eta : RX \to R^{\times 4}X$ where $R^{\times 4}X$ is the limit of the diagram below on the left and $\eta : RX \to R^{\times 4}X$ is induced by the cone below on the right

$$\begin{tikzcd}
X & RX \arrow[l, "\epsilon_0"'] \arrow[r, "\epsilon_1"] & X \\
RX \arrow[u, "\epsilon_0"] \arrow[d, "\epsilon_1"] & & RX \arrow[u, "\epsilon_0"'] \arrow[d, "\epsilon_1"'] \\
X & RX \arrow[l, "\epsilon_0"'] \arrow[r, "\epsilon_1"] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
X & RX \arrow[l, "\epsilon_0"'] \arrow[r, "\epsilon_1"] & X \\
RX \arrow[u, "\epsilon_0"] \arrow[d, "\epsilon_1"] \arrow[r, equal] & RX \arrow[u, "\eta\epsilon_0"] \arrow[d, "\eta\epsilon_1"] \arrow[r, equal] & RX \arrow[u, "\epsilon_0"] \arrow[d, "\epsilon_1"] \\
X & RX \arrow[l, "\epsilon_0"'] \arrow[r, "\epsilon_1"] & X
\end{tikzcd}$$

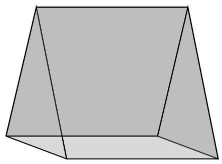
In the category of topological spaces, this might look like the following. (We use the relation $\mathbf{I}$ here, though we ultimately are interested in the relation $\mathbf{\Gamma}$. This is because the description involving $\mathbf{I}$ is much easier to write down but still provides intuition to think about $\mathbf{\Gamma}$.)

Let $\mathbf{I}$ denote the functorial relation on topological spaces which takes any space X to the relation

$$\begin{tikzcd} X \arrow[r, "X^!" description] & \mathrm{X}^I \arrow[l, shift right=2ex, "X^0"'] \arrow[l, shift left=2ex, "X^1"] \end{tikzcd}$$

as described in Example 3.1.6.

Let $\delta(I \times I)$ denote the boundary of the unit square $I \times I$. Let S denote the mapping cylinder of the continuous function $\delta(I \times I) \to I$ which maps (x, y) to x. That is, S is the quotient of $I \times \delta(I \times I)$ obtained by identifying the point $(1, x, y)$ with the point $(1, x, y')$ for any $(x, y), (x, y')$ in $\delta(I \times I)$.



Then let $I^\square X$ denote the space X^S of all continuous functions from S into X. The morphism $\eta : X \to I^\square X$ is the precomposition with the map $S \to *$. The projections $\epsilon_i, \zeta_i : I^\square X \to X^I$ are the precompositions of the inclusions of I into each of the bottom edges in the illustration above.

There is a continuous function $S \to I$ which takes the bottom edges associated to ϵ_0 and ζ_0 and the top vertex above their intersection to the point $0 \in I$ and maps the top edge and the edges associated to ϵ_1 and ζ_1 each homeomorphically onto I. Precomposition with this continuous function is the morphism $\delta_X : X^I \to I^\square X$.

There is a homotopy equivalence $h : S \to I^2$ which commutes with the projections to $I^{\times 4}$. Then the composition

$$X\, {}_{\eta f}\!\times_{\zeta_0} I^\square Y \xhookrightarrow{\eta \times h} X^I\, {}_{f^I}\!\times_{\epsilon_0^I} Y^{I \times I} \cong (X\, {}_f\!\times_{\epsilon_0} Y^I)^I$$

is the morphism τ_f.

Now we can provide some intuition as to why we have switched from considering $R^2 X$ to $R^\square X$. In a space $\Gamma^2 X$, the lengths of the sides are coupled (e.g., for any $\gamma \in \Gamma^2 X$, $\Gamma\epsilon_0\gamma$ has the same length as $\Gamma\epsilon_1\gamma$) but this is not the case for $\Gamma^\square X$. In particular, the middle diagram of 3.2.44 could not be satisfied if $\Gamma^\square X = \Gamma^2 X$. To explain this from a slightly different perspective, when we obtain $R^\square X$ in this way, the morphism $R^{\square}X \to R^{\times 4}X$ is in the right class of the weak factorization system, giving it better behavior than $R^2X \to R^{\times 4}X$.

This intuition will be given mathematical content when we extract this structure from any type theoretic weak factorization structure in Proposition 3.4.6.

Proposition 3.2.47. *Let* $\mathbf{R}$ *be a homotopical relation on* $\mathcal{C}$*. Then for every morphism* $f : X \to Y$ *in* $\mathcal{C}$*, the morphism* λ_f *has a coalgebra structure given by*

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, "\lambda_f"] & X \arrow[d, "\lambda_f^2"] \\ Mf \arrow[r, "1\times\tau\delta"] & M\lambda_f \end{tikzcd}$$

where $1 \times \tau\delta : Mf \to M\lambda_f$ *is*

$$1_X \times \tau_f\delta_Y : X\,{}_f{\times}_{\epsilon_0} RY \to X\,{}_{1\times\eta f}{\times}_{\epsilon_0} R(X\,{}_f{\times}_{\epsilon_0} RY).$$

Proof. The morphism $1_X \times \tau_f\delta_Y$ in the statement is induced from the morphisms $\pi_X : X\,{}_f{\times}_{\epsilon_0} RY \to X$ and $\tau_f(1 \times \delta_Y) : X\,{}_f{\times}_{\epsilon_0} RY \to R(X\,{}_f{\times}_{\epsilon_0} RY)$ by the universal property of the pullback $X\,{}_{1\times\eta f}{\times}_{\epsilon_0} R(X\,{}_f{\times}_{\epsilon_0} RY)$ because the following diagram commutes.

$$\begin{tikzcd} & & X \arrow[drr, "1\times\eta f"] & & \\ X\,{}_f{\times}_{\epsilon_0} RY \arrow[urr, "1_X"] \arrow[rrrr, "1\times\eta\epsilon_0"] \arrow[dr, "1\times\delta"'] & & & & X\,{}_f{\times}_{\epsilon_0} RY \\ & X\,{}_{\eta f}{\times}_{\zeta} R^{\square}Y \arrow[urrr, "1\times\epsilon_0"] \arrow[r, "\tau"'] & R(X\,{}_f{\times}_{\epsilon_0} RY) \arrow[urr, "\epsilon_0"'] & & \end{tikzcd}$$

The upper triangle commutes by the properties of the pullback in its domain. The lower left-hand triangle commutes because of the commutativity of the middle diagram in (3.2.44). The lower right-hand triangle commutes because of the commutativity of the right-handle diagram in (3.2.45)

The coalgebra square in the statement can be written more explicitly as

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, "1\times\eta f"] & X \arrow[d, "1\times\eta(1\times\eta f)"] \\ X\,{}_f{\times}_{\epsilon_0} RY \arrow[r, "1\times\tau\delta"] & X\,{}_{1\times\eta f}{\times}_{\epsilon_0} R(X\,{}_f{\times}_{\epsilon_0} RY) \end{tikzcd}$$

The commutativity of this square follows from the commutativity of the outside of the following diagram by the universal property of the pullback in the lower right-hand corner.

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d, "1\times\eta f"] & X \arrow[r, equal] \arrow[d, "1\times\eta f"] & X \arrow[d, "1\times\eta(1\times\eta f)"] \\
X\,{}_f{\times}_{\epsilon_0}RY \arrow[r, "1\times\delta"] & X\,{}_{\eta f}{\times}_{\zeta}R^{\square}Y \arrow[r, "1\times\tau"] & X\times R(X\,{}_f{\times}_{\epsilon_0}RY)
\end{tikzcd}$$

The left-hand square above commutes because the left-hand diagram of (3.2.44) commutes. The right-hand square commutes because the left-hand diagram of (3.2.45) commutes.

Now it remains to check that the copoint composed with the coalgebra is the identity.

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d, "\lambda_f"] & X \arrow[r, equal] \arrow[d, "\lambda_f^2"] & X \arrow[d, "\lambda_f"] \\
Mf \arrow[r, "1\times\tau\delta"] \arrow[rr, equal, bend right] & M\lambda_f \arrow[r, "\rho\lambda_f"] & Mf
\end{tikzcd}$$

We have already seen that the two squares in this diagram commute. The composition $(\rho\lambda_f)(1\times\tau\delta)$ is equal to the composition of the top and right sides of the diagram below.

$$\begin{tikzcd}
X\,{}_f{\times}_{\epsilon_0}RY \arrow[r, "1\times\delta"] \arrow[drr, equal, bend right=10] & X\,{}_{\eta f}{\times}_{\zeta_0}R^{\square}Y \arrow[r, "1\times\tau"] \arrow[dr, "1\times\epsilon_1"] & X\times R(X\,{}_f{\times}_{\epsilon_0}RY) \arrow[d, "\epsilon_1\pi_1"] \\
 & & X\,{}_f{\times}_{\epsilon_0}RY
\end{tikzcd}$$

The commutativity of the left-hand triangle above follows from the commutativity of the right-hand diagram in (3.2.44). The commutativity of the right-hand triangle above follows from the commutativity of the right-hand diagram in (3.2.45). $\square$

3.2.2.3 Symmetric relations.

Definition 3.2.48. Say that a relation $\mathbf{R}$ on $\mathcal{C}$ is *symmetric* if there exist morphisms

$$\nu_X : RX_{\epsilon_0}{\times}_{\epsilon_0}RX \to RX$$

for every object X of $\mathcal{C}$ such that the following diagrams commute.

$$\begin{tikzcd} RX \,{}_{\epsilon_0}\!\times_{\epsilon_0} RX \arrow[r, "\nu"] \arrow[d, shift right, "\epsilon_1\pi_0"'] \arrow[d, shift left, "\epsilon_1\pi_1"] & RX \arrow[d, shift right, "\epsilon_0"'] \arrow[d, shift left, "\epsilon_1"] \\ X \arrow[r, equal] & X \end{tikzcd} \qquad \begin{tikzcd} RX \arrow[r, "\eta\times 1"] \arrow[dr, equal] & RX \,{}_{\epsilon_0}\!\times_{\epsilon_0} RX \arrow[d, "\nu"] \\ & RX \end{tikzcd} \tag{3.2.49}$$

This might look very different from the strict symmetry defined previously. But notice that if one takes $\iota_X : RX \to RX$ to be the following composite,

$$RX \xrightarrow{1\times\eta\epsilon_0} RX \,{}_{\epsilon_0}\!\times_{\epsilon_0} RX \xrightarrow{\nu} RX$$

then ι is a lift of the identity, as displayed in the following diagram.

$$\begin{tikzcd} RX \arrow[r, "\iota"] \arrow[d, shift right=2, "\epsilon_0"'] \arrow[d, shift left=2, "\epsilon_1"] & RX \arrow[d, shift right=2, "\epsilon_1"'] \arrow[d, shift left=2, "\epsilon_0"] \\ X \arrow[u, "\eta" description] \arrow[r, equal] & X \arrow[u, "\eta" description] \end{tikzcd}$$

(That is, $\iota\eta = \eta$, $\epsilon_0\iota = \epsilon_1$, and $\epsilon_1\iota = \epsilon_0$).

Thus, ν begets a more familiar symmetry, ι. However, we need the full strength of the morphism ν to prove the following lemma.

Lemma 3.2.50. Consider a symmetric, transitive relation $\mathbf{R}$ on $\mathcal{C}$ and denote the factorization $\mathbf{UF}(\mathbf{R})$ by (λ, ρ). Then for every object X of $\mathcal{C}$, the morphism

$$RX \xrightarrow{\epsilon_0\times\epsilon_1} X \times X$$

has a ρ-algebra structure.

Remark 3.2.51. Note that the following proof for this Lemma is identical to that for the strict version (Lemma 3.2.50) except that here we define b to be $\nu(1\times\mu)$ instead of $\mu(\iota\times 1)(1\times\mu)$.

Proof. We need to show that there is a solution to the following lifting problem.

$$\begin{tikzcd} RX \arrow[rr, equal] \arrow[d, "\lambda(\epsilon_0\times\epsilon_1)"'] & & RX \arrow[d, "\epsilon_0\times\epsilon_1"] \\ RX \,{}_{(\epsilon_0\times\epsilon_1)}\!\times_{\epsilon_0} R(X\times X) \arrow[rru, dashed] \arrow[rr, "\rho(\epsilon_0\times\epsilon_1)"] & & X\times X \end{tikzcd}$$

We will do this by finding two lifts a and b as illustrated below.

$$\begin{tikzcd}
RX \arrow[rr, equal] \arrow[dd, "1\times\eta(\epsilon_0\times\epsilon_1)"'] \arrow[dr, "\eta\epsilon_0\times 1\times\eta\epsilon_1"] & & RX \arrow[dd, "\epsilon_0\times\epsilon_1"] \\
& RX\,{}_{\epsilon_0}\!\times_{\epsilon_0} RX\,{}_{\epsilon_1}\!\times_{\epsilon_0} RX \arrow[ur, dashed, "b"] \arrow[dr, "\epsilon_1\pi_0\times\epsilon_1\pi_2"] & \\
RX\,{}_{(\epsilon_0\times\epsilon_1)}\!\times_{\epsilon_0} R(X\times X) \arrow[rr, "\epsilon_1\pi"'] \arrow[ur, dashed, "a"] & & X\times X
\end{tikzcd} \qquad (*)$$

Let $u : R(X \times X) \to RX \times RX$ denote the universal morphism induced by the universal property of $RX \times RX$. It makes the following diagram commute.

$$\begin{tikzcd}
RX \arrow[r, "1\times\eta\epsilon_0\times\eta\epsilon_1"] \arrow[d, "1\times\eta(\epsilon_0\times\epsilon_1)"'] & RX\,{}_{(\epsilon_0\times\epsilon_1)}\!\times_{(\epsilon_0\times\epsilon_0)} (RX\times RX) \arrow[d, "(\epsilon_1\times\epsilon_1)\pi_{(RX\times RX)}"] \\
RX\,{}_{(\epsilon_0\times\epsilon_1)}\!\times_{\epsilon_0} R(X\times X) \arrow[r, "\epsilon_1\pi_{R(X\times X)}"] \arrow[ur, "1\times u"] & X\times X
\end{tikzcd}$$

Note that the outside square of this diagram is isomorphic to the lower-left triangle of diagram $(*)$. Therefore, $1 \times u$ is the lift a that we seek.

Now we let $b : RX\,{}_{\epsilon_0}\!\times_{\epsilon_0} RX\,{}_{\epsilon_1}\!\times_{\epsilon_0} RX \to RX$ be the following composite.

$$RX\,{}_{\epsilon_0}\!\times_{\epsilon_0} RX\,{}_{\epsilon_1}\!\times_{\epsilon_0} RX \xrightarrow{1\times\mu} RX\,{}_{\epsilon_0}\!\times_{\epsilon_0} RX \xrightarrow{\nu} RX.$$

This b makes the upper right-hand portion of the above diagram commute.

Therefore, we have found a lift in the original diagram and shown that $\epsilon_0 \times \epsilon_1$ has a ρ-algebra structure. $\square$

Theorem 3.2.52. *Consider a symmetric relation* $\mathbf{R}$ *on* $\mathcal{C}$ *such that the factorization* $\mathbf{UF}(\mathbf{R})$ *represents a weak factorization system* $(\mathcal{L}, \mathcal{R})$ *and such that every morphism*

$$RX \xrightarrow{\epsilon_0\times\epsilon_1} X \times X$$

is in $\mathcal{R}$*. Then the class* $\mathcal{L}$ *is stable under pullback along* $\mathcal{R}$*.*

Proof. The proof for this is identical to that for Theorem 3.2.30. $\square$

Corollary 3.2.53. *Consider a transitive, symmetric relation* $\mathbf{R}$ *on* $\mathcal{C}$ *such that the factorization* $\mathbf{UF}(\mathbf{R})$ *represents a weak factorization system* $(\mathcal{L}, \mathcal{R})$*.*

Then $(\mathcal{L}, \mathcal{R})$ *is type theoretic.*

Proof. By the previous two results, we know that $\mathcal{L}$ is stable under pullback along $\mathcal{R}$. By Proposition 3.1.38, every object is fibrant. Thus, $(\mathcal{L}, \mathcal{R})$ is type theoretic. □

3.2.2.4 Summary.

Now we have the following theorem.

Theorem 3.2.54. *Consider a transitive and homotopical relation* $\mathbf{R}$ *on a category* $\mathcal{C}$ *with finite limits. Then* $\mathbf{UF}(\mathbf{R})$ *is a weak factorization structure.*

Proof. Let (λ, ρ) denote the factorization of the statement. By Proposition 3.2.47, every morphism in the image of λ has a λ-coalgebra structure, and by 3.2.41, every morphism in the image of ρ has a ρ-algebra structure. Then (λ, ρ) is weakly algebraic, so Proposition 1.4.7 says that (λ, ρ) is a weak factorization structure. □

Definition 3.2.55. A *Moore relation structure* on $\mathcal{C}$ is a relation $\mathbf{R}$ together with the structure given in the definitions of transitive, homotopical, and symmetric.

A *Moore relation system* on $\mathcal{C}$ is a relation $\mathbf{R}$ together which is transitive, homotopical, and symmetric (i.e., a relation for which there exists a Moore relation structure).

Now we have the following theorem.

Theorem 3.2.56. *Consider a Moore relation system* $\mathbf{R}$ *on a category* $\mathcal{C}$ *with finite limits. Then* $\mathbf{UF}(\mathbf{R})$ *is a type theoretic weak factorization structure.*

Proof. By the previous theorem, $\mathbf{UF}(\mathbf{R})$ is a weak factorization structure. Then by Corollary 3.2.53, $\mathbf{UF}(\mathbf{R})$ is type theoretic. □

3.3 Finding relations to generate type theoretic weak factorization systems.

In this section, we consider a type theoretic weak factorization structure W on a category $\mathcal{C}$ with finite limits. In the first section, 3.3.1, we show that the factorization $\mathbf{UFVR}(W)$ is again a weak factorization structure equivalent to W

(Corollary 3.3.6). In the second section, 3.3.2, we show that the relation $\mathbf{VR}(W)$ is an `Id`-presentation of $[\mathbf{UFVR}(W)] = [W]$ (Theorem 3.3.10). Combining these two results, we will have shown that any type theoretic weak factorization system has an `Id`-presentation.

3.3.1 The main result.

Consider any type theoretic weak factorization structure W on a category $\mathcal{C}$ with finite limits. Let $(\mathcal{L}, \mathcal{R})$ denote the lifting pair underlying W.

Our aim in this section is to show that $\mathbf{UFVR}(W)$ is equivalent to W. However, we prove a slightly more general result which will become useful later (in Lemma 3.3.9, Proposition 3.4.6, and Proposition A.1.5).

To that end, consider any relation R with the following components at each object X of $\mathcal{C}$

$$\begin{tikzcd} X \arrow[r, "\eta_X" description] & \mathrm{R}X \arrow[l, shift right=2, "\epsilon_{0X}"'] \arrow[l, shift left=2, "\epsilon_{1X}"] \end{tikzcd}$$

such that each $\eta_X : X \to RX$ is in $\mathcal{L}$ and each $\epsilon_X = \epsilon_{0X} \times \epsilon_{1X} : RX \to X \times X$ is in $\mathcal{R}$. (We have in mind the relation $\mathbf{VR}(W)$ for our main result.)

Now let (λ, ρ) denote the factorization $\mathbf{UF}(R)$. Recall that for morphism $f : X \to Y$, this factorization gives

$$X \xrightarrow{\lambda_f} Mf \xrightarrow{\rho_f} Y$$

where Mf is the pullback $X \times_Y RY$, where λ_f is $1_X \times \eta_Y f$, and ρ_f is $\epsilon_{1Y}\pi_{RY}$ (as described in Proposition 3.1.41).

Now we show that $\mathbf{UF}(R)$ is a weak factorization structure equivalent to W. For this, we need to show that (1) λ-$\mathsf{alg} = \mathcal{L}$, (2) ρ-$\mathsf{coalg} = \mathcal{R}$, (3) $\lambda(f) \in \mathcal{L}$, and (4) $\rho(f) \in \mathcal{R}$ for every morphism f of $\mathcal{C}$. These facts are all relatively straightforward to show except (3) that $\lambda(f) \in \mathcal{L}$ which appears as Proposition 3.3.4.

The hypothesis that W is type theoretic is integral to the proof below. In Lemma 3.3.1, where we show fact (4), we need every object in W to be fibrant. In Lemma 3.3.2, which is used to show fact (3) in Proposition 3.3.4, we need $\mathcal{L}$ to be stable under pullback along $\mathcal{R}$.

Lemma 3.3.1. For any morphism f of $\mathcal{C}$, the morphism $\rho(f)$ is in $\mathcal{R}$.

Proof. Note first that $\pi_Y : X \times Y \to Y$ and $1_X \times \epsilon_1 : X {}_f{\times}_{\epsilon_0} RY \to X \times Y$ are in $\mathcal{R}$ because they are pullbacks of morphisms hypothesized to be in $\mathcal{R}$.

$$\begin{tikzcd}
X \times Y \arrow[r] \arrow[d, "\pi_Y"'] \arrow[dr, phantom, very near start, "\lrcorner"] & X \arrow[d, "!"] \\
Y \arrow[r, "!"] & *
\end{tikzcd}
\qquad
\begin{tikzcd}
X {}_f{\times}_{\epsilon_0} RY \arrow[r] \arrow[d, "1_X \times \epsilon_1"'] \arrow[dr, phantom, very near start, "\lrcorner"] & RY \arrow[d, "\epsilon_0 \times \epsilon_1"] \\
X \times Y \arrow[r, "f \times 1_Y"] & Y \times Y
\end{tikzcd}$$

Since $\rho(f)$ is the composition of these two maps, it is also in $\mathcal{R}$. □

Lemma 3.3.2. For any morphism f in $\mathcal{R}$, the morphism $\lambda(f)$ is in $\mathcal{L}$,

Proof. The morphism λ_f is a pullback of $\eta \in \mathcal{L}$ along $f \in \mathcal{R}$,

$$\begin{tikzcd}
& X {}_f{\times}_{\epsilon_0} RY \arrow[rr] \arrow[dd] & & RY \arrow[dd, "\epsilon_0"] \\
X \arrow[ur, "\lambda_f"] \arrow[rr] \arrow[dr, equal] & & Y \arrow[ur, "\eta"'] \arrow[dr, equal] & \\
& X \arrow[rr, "f"] & & Y
\end{tikzcd}$$

and since $(\mathcal{L}, \mathcal{R})$ is type theoretic, $\mathcal{L}$ is stable under pullback along $\mathcal{R}$. □

Proposition 3.3.3. *There are equalities $\mathcal{L} = \lambda$-alg and $\mathcal{R} = \rho$-coalg.*

Proof. Consider a morphism f in ρ-coalg. Because f is a ρ-coalgebra, it is a retract of $\rho(f)$. By Lemma 3.3.1, $\rho(f)$ is in $\mathcal{R}$. Since $\mathcal{R}$ is closed under retracts, f is in $\mathcal{R}$.

Now consider a morphism f in $\mathcal{R}$. Since λ_f is in $\mathcal{L}$ by Lemma 3.3.2, λ_f has the left lifting property against f. Therefore, f has a ρ-coalgebra structure and is in ρ-coalg.

Thus, $\mathcal{R} = \rho$-coalg.

Now consider $\ell \in \mathcal{L}$. Since ℓ has the left lifting property against $\mathcal{R}$, it has the left lifting property against ρ_ℓ in particular (Lemma 3.3.1). Thus it has a λ-coalgebra structure, and so is in λ-alg.

Now suppose that $\ell \in \lambda$-alg. Since ℓ has a λ-coalgebra structure and any $r \in \mathcal{R} = \rho$-coalg has an ρ-algebra structure, ℓ has the left-lifting property against any such r (Proposition 1.4.3). Thus, ℓ is in ${}^{\boxslash}\mathcal{R} = \mathcal{L}$.

Therefore, $\mathcal{L} = \lambda$-alg. □

Now we have established that $(\lambda, \rho) = (\mathcal{L}, \mathcal{R})$ and is thus a lifting pair. It only remains to be seen that (λ, ρ) is truly a factorization *into* this lifting pair. We have already showed that any $\rho(f)$ is in $\mathcal{R}$, and we now show that $\lambda(f)$ is in $\mathcal{L}$ for any morphism f of $\mathcal{C}$.

Proposition 3.3.4. *For any morphism f of $\mathcal{C}$, the morphism $\lambda(f)$ is in $\mathcal{L}$.*

Proof. We need to show that $\lambda(f)$ has a λ-coalgebra structure, or that, equivalently, there is a solution to the following lifting problem.

$$\begin{tikzcd}
X \arrow[r, "\lambda^2 f"] \arrow[d, "\lambda_f"] & X {}_{\lambda f}\times_{\epsilon_0} R(X {}_f\times_{\epsilon_0} RY) \arrow[d, "\rho\lambda f"] \\
X {}_f\times_{\epsilon_0} RY \arrow[r, equal] \arrow[ur, dashed] & X {}_f\times_{\epsilon_0} RY
\end{tikzcd}$$

First we define a new morphism $\mu : RY {}_{\epsilon_1}\times_{\epsilon_0} RY \to RY$. Note that $\eta\epsilon_0 \times 1 : RY \to RY {}_{\epsilon_1}\times_{\epsilon_0} RY$ is in $\mathcal{L}$ since it is a pullback of a morphism in $\mathcal{L}$ along a morphism in $\mathcal{R}$, as shown below.

$$\begin{tikzcd}
& RY {}_{\epsilon_1}\times_{\epsilon_0} RY \arrow[rr] \arrow[dd] \arrow[dr, phantom, "\lrcorner", very near start] & & RY \arrow[dd, "\epsilon_1"] \\
RY \arrow[ur, "\eta\epsilon_0\times 1"] \arrow[rr] \arrow[dr, equal] \arrow[dr, phantom, "\lrcorner", very near start] & & Y \arrow[ur, "\eta"] \arrow[dr, equal] & \\
& RY \arrow[rr, "\epsilon_0"] & & Y
\end{tikzcd}$$

Then, we define μ to be a solution to the following lifting problem.

$$\begin{tikzcd}
RY \arrow[r, equal] \arrow[d, "\eta\epsilon_0\times 1"'] & RY \arrow[d, "\epsilon_0\times\epsilon_1"] \\
RY {}_{\epsilon_1}\times_{\epsilon_0} RY \arrow[r, "\epsilon_0\pi_0\times\epsilon_1\pi_1"] \arrow[ur, dashed, "\mu"] & Y \times Y
\end{tikzcd}$$

Now, we refer to figure Figure 3.1 on page 133. Since $\rho(f)$ is in $\mathcal{R}$, we know that $\lambda\rho(f)$ is in $\mathcal{L}$. Therefore, there is a lift σ as illustrated in the figure.

Let $\sigma' : X_f\times_{\epsilon_0} RY \to R(X_f\times_{\epsilon_0} RY)$ be the composite $R(1_X \times \mu)\sigma(1_X \times \eta f \times 1_{RY})$ – that is the composite from the bottom left to top right of the diagram in Figure 3.1. Then a rearrangement of Figure 3.1 produces the commutative

$$
\begin{tikzcd}
X \arrow[r, "1\times\eta f"'] \arrow[d, "\lambda_f = 1\times\eta f"] \arrow[rrr, bend left, "\eta\lambda_f = \eta(1_X\times\eta f)"] & X\,{}_f\!\times_{\epsilon_0} RY \arrow[r, "\eta(1_{X\times RY}\times\eta\epsilon_1)"'] \arrow[d, "\lambda\rho_f = 1_{X\times RY}\times\eta\epsilon_1"] & R(X\,{}_f\!\times_{\epsilon_0} RY\,{}_{\epsilon_1}\!\times_{\epsilon_0} RY) \arrow[r, "R(1_X\times\mu)"'] \arrow[d, "\epsilon_0\times\epsilon_1"] & R(X\,{}_f\!\times_{\epsilon_0} RY) \arrow[d, "\epsilon_0\times\epsilon_1"] \\
X\,{}_f\!\times_{\epsilon_0} RY \arrow[r, "1_X\times\eta f\times 1_{RY}"] \arrow[rrr, bend right, "(1_X\times\eta f)\times(1_{X\times RY})"'] & X\,{}_f\!\times_{\epsilon_0} RY\,{}_{\epsilon_1}\!\times_{\epsilon_0} RY \arrow[r, "(1_{X\times RY}\times\eta\epsilon_1)\times(1_{X\times RY\times RY})"] \arrow[ur, dashed, "\sigma"] & (X\,{}_f\!\times_{\epsilon_0} RY\,{}_{\epsilon_1}\!\times_{\epsilon_0} RY)^2 \arrow[r, "(1_X\times\mu)^2"] & (X\,{}_f\!\times_{\epsilon_0} RY)^2
\end{tikzcd}
$$

Figure 3.1: Lifting diagram

diagram below, and $1_X \times \sigma'$ is our desired lift.

$$
\begin{tikzcd}
X \arrow[r, "\lambda\lambda_f=", "1_X\times\eta(1_X\times\eta f)"'] \arrow[d, "\lambda_f=1\times\eta f"'] & X\ {}_{\lambda_f}\!\times_{\epsilon_0} R(X\ {}_f\!\times_{\epsilon_0} RY) \arrow[d, "\rho\lambda_f=\epsilon_1\pi_{R(X\times RY)}"] \\
X\ {}_f\!\times_{\epsilon_0} RY \arrow[ur, "1_X\times\sigma'"] \arrow[r, equal] & X\ {}_f\!\times_{\epsilon_0} RY
\end{tikzcd}
$$

Therefore, λ_f is in $\mathcal{L}$. $\square$

We put the preceding results together into the following theorems.

Theorem 3.3.5. *Consider a type theoretic weak factorization structure W on a category $\mathcal{C}$ with finite limits. Consider a relation R on $\mathcal{C}$ which has components*

$$
\begin{tikzcd}
X \arrow[r, "\eta_X" description] & \mathrm{R}X, \arrow[l, bend right, "\epsilon_{0X}"'] \arrow[l, bend left, "\epsilon_{1X}"]
\end{tikzcd}
$$

at each object X of $\mathcal{C}$ such that each η_X is in the left class and each $\epsilon_{0X} \times \epsilon_{1X} : R(X) \to X \times X$ is in the right class of this weak factorization structure.

Then the factorization $\mathbf{UF}(R)$ is a weak factorization structure equivalent to W.

Proof. Let (λ, ρ) denote the factorization $\mathbf{UF}(R)$. By Lemma 3.3.1 and Proposition 3.3.4, (λ, ρ) is weakly algebraic. Thus, by Proposition 1.4.7, it is a weak factorization structure. By Proposition 3.3.3, it is equivalent to W. $\square$

The following corollary is the main result of this section.

Corollary 3.3.6. *Consider a type theoretic weak factorization structure W on a category $\mathcal{C}$ with finite limits. The factorization $\mathbf{UFVR}(W)$ is a weak factorization structure equivalent to W.*

Proof. We need to show that the relation $\mathbf{VR}(W)$ can be substituted for R in the statement of the previous theorem, 3.3.5. Let (λ^W, ρ^W) denote the factorization of W. Then (in the notation of the previous theorem, 3.3.5) η_X is $\lambda^W(\Delta_X)$ and ϵ_X is $\rho^W(\Delta_X)$, so these are in the left and right class, respectively, as required. $\square$

The following corollary will become a useful technical device (in Proposition 3.4.6) and is the reason that we proved Theorem 3.3.5 in more generality than needed for Corollary 3.3.6.

Corollary 3.3.7. *Consider a type theoretic weak factorization structure W on a category $\mathcal{C}$ with finite limits. Consider a relation $R : \mathfrak{R} \to \mathcal{C}$ on an object Y of $\mathcal{C}$ with the following components*

$$\begin{tikzcd} Y \arrow[r, "\eta_Y"] & \mathrm{R}Y, \arrow[l, shift right=2ex, "\epsilon_{0Y}"'] \arrow[l, shift left=2ex, "\epsilon_{1Y}"] \end{tikzcd}$$

such that η_Y is in the left class and $\epsilon_{0Y} \times \epsilon_{1Y} : RY \to Y \times Y$ is in the right class of W. (Note that R is a relation just on Y, not on the whole of $\mathcal{C}$.)

Then for any morphism $f : X \to Y$ of $\mathcal{C}$, in the following factorization

$$X \xrightarrow{1\times\eta f} X \times_{\epsilon_0} RY \xrightarrow{\epsilon_1\pi_{RY}} Y$$

the morphism $1 \times \eta f$ is in the left class, and $\epsilon_1\pi_{RY}$ is in the right class of W.

Proof. Consider the relation $\mathbf{VR}(W)$. We construct a new relation S which coincides with $\mathbf{VR}(W)$ everywhere except at Y. So set $S(X) = \mathbf{VR}(W)(X)$ for every object $X \neq Y$ and set $S(Y) = R$. Then a lift of any morphism with domain or codomain Y can be extracted from the weak factorization structure W. That is, a lift of any morphism $f : X \to Y$ can be obtained as a solution to the following lifting problem.

$$\begin{tikzcd} X \arrow[r, "\eta f"] \arrow[d, "\eta"'] & \mathtt{Id}(Y) \arrow[d, "\epsilon_0\times\epsilon_1"] \\ \mathtt{Id}(X) \arrow[r, "f\epsilon_0\times f\epsilon_1"'] \arrow[ur, dashed] & Y \times Y \end{tikzcd}$$

A lift of any morphism $g : Y \to Z$ can be obtained analogously.

The relation S satisfies the hypotheses of Theorem 3.3.5 so $\mathbf{UF}(S)$ is a weak factorization structure equivalent to W. But $\mathbf{UF}(S)$ sends a morphism $f : X \to Y$ to the factorization in the statement. Thus $1 \times \eta f$ is in the left class and $\epsilon_1\pi_{RY}$ is in the right class of W. $\square$

Example 3.3.8. Given any Cisinski model structure $(\mathcal{C}, \mathcal{F}, \mathcal{W})$ on a topos $\mathcal{M}$ ([Cis06]), we claim that the weak factorization system $(\mathcal{C} \cap \mathcal{W} \cap \mathcal{M}_{\mathcal{F}}, \mathcal{F} \cap \mathcal{M}_{\mathcal{F}})$

restricted to the full subcategory $\mathcal{M}_{\mathcal{F}}$ of fibrant objects (Corollary 1.5.4) is type theoretic.

Note that if $f : X \to Y$ is in $\mathcal{F} \cap \mathcal{M}_{\mathcal{F}}$ and $g : Z \to Y$ is in $\mathcal{M}_{\mathcal{F}}$, then the pullback square

$$\begin{tikzcd} X \times_Z Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Z \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

is contained in $\mathcal{M}_{\mathcal{F}}$ (since g^*f is a fibration, the composition $! \circ g^*f : X \times_Z Y \to Z \to *$ is a fibration). Thus, any pullback along a fibration in $\mathcal{M}_{\mathcal{F}}$ exists and coincides with that in $\mathcal{M}$.

For this weak factorization system to be type theoretic, all its objects must be fibrant, which we have satisfied by construction, and $\mathcal{C} \cap \mathcal{W} \cap \mathcal{M}_{\mathcal{F}}$ must be stable under pullback along $\mathcal{F} \cap \mathcal{M}_{\mathcal{F}}$.

In a Cisinski model structure, $\mathcal{C}$ is precisely the class of monomorphisms, so it is stable under pullback (along all morphisms) in $\mathcal{M}$. Then, in particular, it is stable under pullback along $\mathcal{F} \cap \mathcal{M}_{\mathcal{F}}$ in $\mathcal{M}_{\mathcal{F}}$.

A standard result of model category theory says that $\mathcal{W} \cap \mathcal{M}_{\mathcal{F}}$ is stable under pullback in $\mathcal{M}_{\mathcal{F}}$ (see [Bro73], §1, Example 1 and §4, Lemma 1).

We conclude that $\mathcal{C} \cap \mathcal{W} \cap \mathcal{M}_{\mathcal{F}}$ is stable under pullback along $\mathcal{F} \cap \mathcal{M}_{\mathcal{F}}$, and the weak factorization system $(\mathcal{C} \cap \mathcal{W} \cap \mathcal{M}_{\mathcal{F}}, \mathcal{F} \cap \mathcal{M}_{\mathcal{F}})$ is type theoretic.

Thus, we find many examples of type theoretic weak factorization systems, including those in the categories of Kan complexes ([Qui67]), quasicategories ([Joy08]), and fibrant cubical sets ([Cis06]).

3.3.2 `Id`-presentations.

In this section, we can now clarify what it means for a relation R to be an `Id`-presentation.

We have the following simplifying result.

Lemma 3.3.9. Consider a relation R on a category $\mathcal{C}$ such that $\mathbf{UF}(R)$ is a type theoretic weak factorization structure. Denote the components of $R(X)$ for any object X of $\mathcal{C}$ by the following diagram.

$$\begin{tikzcd} X \arrow[r, "\eta_X" description] & RX \arrow[l, shift right=2, "\epsilon_{0X}"'] \arrow[l, shift left=2, "\epsilon_{1X}"] \end{tikzcd}$$

Then R is an `Id`-presentation of the weak factorization system $[\mathbf{UF}(R)]$ if and only if $\epsilon_{0X} \times \epsilon_{1X} : RX \to X \times X$ is in the right class for each object X.

Proof. Let $(\lambda, \rho, \mathcal{L}, \mathcal{R})$ denote the weak factorization structure $\mathbf{UF}(R)$.

Suppose that R is an `Id`-presentation. Then by definition, we must have that each $\epsilon_{0X} \times \epsilon_{1X} : RX \to X \times X$ is in $\mathcal{R}$.

Conversely, suppose that each $\epsilon_{0X} \times \epsilon_{1X} : RX \to X \times X$ is in $\mathcal{R}$. Then it remains to show that each $f^*\eta_Y$, as displayed in the diagram $(*)$ below, is in $\mathcal{L}$.

$$\begin{tikzcd}
& f^*RY \arrow[rr] \arrow[dd] & & RY \arrow[dd, "\epsilon_{iY}"] \\
X \arrow[ur, "f^*\eta_Y"] \arrow[rr] \arrow[dr, equal] & & Y \arrow[ur, "\eta_Y"'] \arrow[dr, equal] & \\
& X \arrow[rr, "f"] & & Y
\end{tikzcd} \tag{$*$}$$

Note that when $i = 0$ in the diagram $(*)$ above, the morphism $f^*\eta_Y$ is isomorphic to λ_f (i.e., it has the same universal property as $1_X \times \eta_Y f : X \to X \times_Y RY$). Thus, it must be in $\mathcal{L}$.

There is an involution I on $\mathfrak{Rel}_{\mathcal{C}}^{00}$ which sends $R(X)\epsilon_i$ to $R(X)\epsilon_{i+1}$ for any $R \in \mathfrak{Rel}_{\mathcal{C}}^{00}$, $X \in \mathcal{C}$, and $i \in \mathbb{Z}/2$ (and keeps all else constant). Then IR satisfies the hypotheses of Theorem 3.3.5, so it generates an equivalent weak factorization system which we will denote by $(-\lambda, -\rho, \mathcal{L}, \mathcal{R})$. Now when $i = 1$, the morphism $f^*\eta_Y$ in the diagram $(*)$ is isomorphic to $-\lambda_f$, so it is in $\mathcal{L}$.

Therefore, every $f^*\eta_Y$ in the diagram $(*)$ is in $\mathcal{L}$, so R is an `Id`-presentation of this weak factorization system. $\square$

Now combining Corollary 3.3.6 with this lemma, 3.3.9, we see the following.

Theorem 3.3.10. *Consider a type theoretic weak factorization structure W on a category $\mathcal{C}$ with finite limits. The relation $\mathbf{VR}(W)$ is an `Id`-presentation of the weak factorization system $[W]$.*

Thus, every type theoretic weak factorization system has an `Id`-presentation.

Proof. By Corollary 3.3.6, $\mathbf{VR}(W)$ generates a type theoretic weak factorization structure $\mathbf{UFVR}(W)$ equivalent to W. For any object X of $\mathcal{C}$, the morphism

$$\mathbf{VR}(W)X\epsilon_0 \times \mathbf{VR}(W)X\epsilon_1 : \mathbf{VR}(W)X\Phi \to X \times X$$

is the right factor of the morphism $\Delta(X)$ in the factorization W. Thus, it is in the right class of the weak factorization system. Then this is an `Id`-presentation of $[W]$ by Lemma 3.3.9. $\square$

Corollary 3.3.11. *The functor* $\mathbf{VR} : \mathfrak{Fact}_{\mathcal{C}}^{00} \to \mathfrak{Rel}_{\mathcal{C}}^{00}$ *restricts to a functor* $\mathbf{VR} : \mathfrak{ttWFS}_{\mathcal{C}}^{00} \to \mathtt{Id}\mathfrak{Pres}_{\mathcal{C}}^{00}$*, and the composition* $|\mathbf{UFVR}| : |\mathfrak{ttWFS}_{\mathcal{C}}^{00}| \to |\mathfrak{ttWFS}_{\mathcal{C}}^{00}|$ *is isomorphic to the identity functor.*

Proof. By the previous theorem, 3.3.10, all objects in the image of $\mathbf{VR} : \mathfrak{ttWFS}_{\mathcal{C}}^{00} \to \mathfrak{Rel}_{\mathcal{C}}^{00}$ are `Id`-presentations. Thus, this functor restricts to $\mathbf{VR} : \mathfrak{ttWFS}_{\mathcal{C}}^{00} \to \mathtt{Id}\mathfrak{Pres}_{\mathcal{C}}^{00}$.

By the previous theorem again, for any object $W \in \mathfrak{ttWFS}_{\mathcal{C}}^{00}$, we have that W is equivalent to $\mathbf{UFVR}(W)$. Thus, they are isomorphic as objects of $|\mathfrak{ttWFS}_{\mathcal{C}}^{00}|$. Since $|\mathfrak{ttWFS}_{\mathcal{C}}^{00}|$ is a proset, these isomorphisms assemble into a natural transformation $1 \cong |\mathbf{UFVR}|$. $\square$

Example 3.3.12. Consider Example 3.3.8. Then given a Cisinski model structure $(\mathcal{C}, \mathcal{F}, \mathcal{W})$ on a topos $\mathcal{M}$, the weak factorization system $(\mathcal{C} \cap \mathcal{W} \cap \mathcal{M}_{\mathcal{F}}, \mathcal{F} \cap \mathcal{M}_{\mathcal{F}})$ has an `Id`-presentation. In particular, the weak factorization systems in the category of Kan complexes, the category of quasicategories, and that of cubical sets have `Id`-presentations.

3.4 Relations which generate type theoretic weak factorization systems.

In this section, we tie up the preceding sections by showing that a relation R on a category $\mathcal{C}$ with finite limits is a Moore relation system if and only if it is an `Id`-presentation.

We can immediately see from our previous results that any relation R which underlies a Moore relation system is an `Id`-presentation of the weak factorization system it generates.

Proposition 3.4.1. *Consider a Moore relation system* R *on a category* $\mathcal{C}$ *with finite limits. Then* R *is an* `Id`*-presentation.*

Proof. By Theorem 3.2.56, R generates a type theoretic weak factorization structure $\mathbf{UF}(R)$. By Lemma 3.2.50, every $RX\epsilon_0 \times RX\epsilon_1$ is in the right class. Then by Lemma 3.3.9, this is a `Id`-presentation of $[\mathbf{UF}(R)]$. $\square$

Now we prove the converse: that any Id-presentation is a Moore relation system.

In the following results, we consider a category $\mathcal{C}$ with finite limits and a relation $\mathbf{R}$ on $\mathcal{C}$ which at an object X gives the following diagram.

$$\begin{tikzcd} X \arrow[r, "\eta_X" description] & RX, \arrow[l, bend right, "\epsilon_{0X}"'] \arrow[l, bend left, "\epsilon_{1X}"] \end{tikzcd}$$

We let (λ, ρ) denote the factorization $\mathbf{UF}(\mathbf{R})$.

Proposition 3.4.2. *Suppose that* $\mathbf{R}$ *is an* Id*-presentation of a weak factorization system. Then* $\mathbf{R}$ *is transitive.*

Proof. For any object X of $\mathcal{C}$, we let μ_X be a solution to the following lifting problem.

$$\begin{tikzcd} RX \arrow[r, equal] \arrow[d, "\lambda_{\epsilon_1}"'] & RX \arrow[d, "\epsilon_0 \times \epsilon_1"] \\ RX\,{}_{\epsilon_1}\!\times_{\epsilon_0} RX \arrow[r, "\epsilon_0\pi_0 \times \epsilon_1\pi_1"'] \arrow[ur, dashed, "\mu_X"] & X \times X \end{tikzcd}$$

This makes $\mathbf{R}$ into a transitive relation. ☐

Proposition 3.4.3. *Suppose that* $\mathbf{R}$ *is an* Id*-presentation of a weak factorization system. Then* $\mathbf{R}$ *is symmetric.*

Proof. For any object X of $\mathcal{C}$, we let ν_X be a solution to the following lifting problem (where $\tau : RX_{\epsilon_0}\times_{\epsilon_0}RX \to RX_{\epsilon_0}\times_{\epsilon_0}RX$ is the standard twist involution).

$$\begin{tikzcd} RX \arrow[r, equal] \arrow[d, "\tau\lambda_{\epsilon_0}"'] & RX \arrow[d, "\epsilon_0 \times \epsilon_1"] \\ RX\,{}_{\epsilon_0}\!\times_{\epsilon_0} RX \arrow[r, "\epsilon_1\pi_0 \times \epsilon_1\pi_1"'] \arrow[ur, dashed, "\nu_X"] & X \times X \end{tikzcd}$$

This makes $\mathbf{R}$ into a symmetric relation. ☐

Theorem 3.4.4. *Suppose that a relation* $\mathbf{R}$ *on a category* $\mathcal{C}$ *with finite limits is an* Id*-presentation. Then* $\mathbf{UF}(\mathbf{R})$ *is a type theoretic weak factorization structure.*

Proof. By Proposition 3.4.3, $\mathbf{R}$ is symmetric. Then by Proposition 3.1.38 and Theorem 3.2.52, $\mathbf{UF}(\mathbf{R})$ is type theoretic. ☐

Corollary 3.4.5. *Consider a category $\mathcal{C}$ with finite limits. The functor* $\mathbf{UF}$: $\mathfrak{Rel}^{00}_{\mathcal{C}} \to \mathfrak{Fact}^{00}_{\mathcal{C}}$ *restricts to a functor* $\mathbf{UF} : \mathtt{Id}\mathfrak{Pres}^{00}_{\mathcal{C}} \to \mathfrak{ttWFS}^{00}_{\mathcal{C}}$.

Proof. The previous theorem, 3.4.4, tells us that every object in the image of $\mathbf{UF} : \mathtt{Id}\mathfrak{Pres}^{00}_{\mathcal{C}} \to \mathfrak{Fact}^{00}_{\mathcal{C}}$ is in the full subcategory $\mathfrak{ttWFS}^{00}_{\mathcal{C}}$. Thus, this functor restricts to $\mathbf{UF} : \mathtt{Id}\mathfrak{Pres}^{00}_{\mathcal{C}} \to \mathfrak{ttWFS}^{00}_{\mathcal{C}}$. □

Proposition 3.4.6. *Suppose that* $\mathbf{R}$ *is an* $\mathtt{Id}$*-presentation of a weak factorization system. Then* $\mathbf{R}$ *is homotopical.*

Proof. Let $R^{\times 4}X$ denote the limit of the following diagram in $\mathcal{C}$.

$$\begin{tikzcd}
X & RX \arrow[l, "\epsilon_0"'] \arrow[r, "\epsilon_1"] & X \\
RX \arrow[u, "\epsilon_0"] \arrow[d, "\epsilon_1"'] & & RX \arrow[u, "\epsilon_0"'] \arrow[d, "\epsilon_1"] \\
X & RX \arrow[l, "\epsilon_0"'] \arrow[r, "\epsilon_1"] & X
\end{tikzcd}$$

There is a morphism $u : RX \to R^{\times 4}X$ which is induced by the following cone.

$$\begin{tikzcd}
X & RX \arrow[l, "\epsilon_0"'] \arrow[r, "\epsilon_1"] & X \\
RX \arrow[u, "\epsilon_0"] \arrow[d, "\epsilon_1"'] \arrow[r, equal] & RX \arrow[u, "\eta\epsilon_0"] \arrow[d, "\eta\epsilon_1"'] \arrow[r, equal] & RX \arrow[u, "\epsilon_0"'] \arrow[d, "\epsilon_1"] \\
X & RX \arrow[l, "\epsilon_0"'] \arrow[r, "\epsilon_1"] & X
\end{tikzcd}$$

Now we factor $u : RX \to R^{\times 4}X$.

$$RX \xrightarrow{\lambda_u} Mu \xrightarrow{\rho_u} R^{\times 4}X$$

Let $R^{\square}X$ denote Mu, and let the following diagram denote the cone corresponding to ρ_u.

$$\begin{tikzcd}
X & RX \arrow[l, "\epsilon_0"'] \arrow[r, "\epsilon_1"] & X \\
RX \arrow[u, "\epsilon_0"] \arrow[d, "\epsilon_1"'] & R^{\square}X \arrow[l, "\zeta_0"'] \arrow[r, "\zeta_1"] \arrow[u, "\epsilon_0"] \arrow[d, "\epsilon_1"'] & RX \arrow[u, "\epsilon_0"'] \arrow[d, "\epsilon_1"] \\
X & RX \arrow[l, "\epsilon_0"'] \arrow[r, "\epsilon_1"] & X
\end{tikzcd}$$

Note that the object $R^{\square}X$ is defined to be the pullback $RX\,{}_u{\times}_{\epsilon_0} R(R^{\times 4}X)$.

Now we let $\delta_X : RX \to R^{\square}X$ be a solution to the following lifting problem.

$$\begin{tikzcd} X \arrow[r, "\lambda_u \eta"] \arrow[d, "\eta"'] & R^{\square}X \arrow[d, "\rho_u", "\epsilon_0\times\epsilon_1\times\zeta_0\times\zeta_1"'] \\ RX \arrow[r, "\eta\epsilon_0\times 1\times\eta\epsilon_0\times 1"'] \arrow[ur, dashed, "\delta_X"] & R^{\times 4}X \end{tikzcd}$$

For any $f : X \to Y$, we need to find a solution to the following lifting problem in order to define $\tau_f : X\,{}_{\eta f}{\times}_{\zeta} R^{\square}Y \to R(X\,{}_f{\times}_{\epsilon_0} RY)$.

$$\begin{tikzcd} X \arrow[r, "\eta(1\times\eta f)"] \arrow[d, "1\times\lambda_u\eta f"'] & R(X\,{}_f{\times}_{\epsilon_0} RY) \arrow[d, "\epsilon_0\times\epsilon_1"] \\ X\,{}_{\eta f}{\times}_{\zeta_0} R^{\square}Y \arrow[r, "(1_X\times\epsilon_0)\times(1_X\times\epsilon_1)"'] \arrow[ur, dashed, "\tau_f"] & (X\,{}_f{\times}_{\epsilon_0} RY)\times(X\,{}_f{\times}_{\epsilon_0} RY) \end{tikzcd}$$

Since $\mathbf{R}$ is an `Id`-presentation of $(\mathcal{L}, \mathcal{R})$, we know that the right hand map above, $\epsilon_0 \times \epsilon_1$, is in $\mathcal{R}$. Thus, we need to show that $1 \times \lambda_u \eta f$ is in $\mathcal{L}$.

To see this, first observe that $\zeta_0 \times \zeta_1 : R^{\times 4}Y \to RY \times RY$ is in $\mathcal{R}$, since it is given by the following pullback.

$$\begin{tikzcd} R^{\times 4}Y \arrow[r, "\epsilon_0\times\epsilon_1"] \arrow[d, "\zeta_0\times\zeta_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & RY\times RY \arrow[d, "\epsilon_0\times\epsilon_1\times\epsilon_0\times\epsilon_1"] \\ RY\times RY \arrow[r, "\epsilon_0\times\epsilon_0\times\epsilon_1\times\epsilon_1"] & Y\times Y\times Y\times Y \end{tikzcd}$$

The right-hand map in the above diagram is in $\mathcal{R}$ since it is the product of two maps in $\mathcal{R}$, and thus its pullback, $\zeta_0 \times \zeta_1$, is also in $\mathcal{R}$. Then the composition $(\zeta_0 \times \zeta_1)\rho_u : R^{\square}Y \to RY \times RY$, which we also denote by $\zeta_0 \times \zeta_1$, is in $\mathcal{R}$.

Thus, the following is a factorization of the diagonal Δ_{RY} into $(\mathcal{L}, \mathcal{R})$.

$$RY \xrightarrow{\lambda_u} R^{\square}Y \xrightarrow{\zeta_0\times\zeta_1} RY \times RY$$

By Corollary 3.3.7, in the following factorization of $\eta f : X \to RY$,

$$X \xrightarrow{1\times\lambda_u\eta f} X\,{}_{\eta f}{\times}_{\zeta_0} R^{\square}Y \xrightarrow{\zeta_1} RY$$

the morphism $1 \times \lambda_u \eta f$ is in $\mathcal{L}$.

Thus, we obtain a lift τ_f as above.

Then τ and δ make $\mathbf{R}$ into a homotopical relation where the diagram

$$X \xrightarrow{\eta} R^{\square}X \overset{\epsilon_0}{\underset{\zeta}{\overset{\epsilon_1}{\rightrightarrows}}} RX$$

of Definition 3.2.42 is given by the diagram

$$X \xrightarrow{\lambda_u \eta} R^{\square}X \overset{\epsilon_0}{\underset{\zeta_0}{\overset{\epsilon_1}{\rightrightarrows}}} RX$$

that we have defined here. $\square$

Thus, we have the following theorem.

Theorem 3.4.7. *Consider a relation* $\mathbf{R}$ *on* $\mathcal{C}$*. It is an* `Id`*-presentation of a weak factorization system if and only if it is a Moore relation system.*

Proof. By Proposition 3.4.1, a Moore relation system is an `Id`-presentation of the weak factorization system it generates.

By Propositions 3.4.2, 3.4.3, and 3.4.6, an `Id`-presentation of a weak factorization system is a Moore relation system. $\square$

Now we can restate Theorem 3.3.10 in the following way.

Corollary 3.4.8. *Consider a type theoretic weak factorization structure* W*. Then the relation* $\mathbf{VR}(W)$ *is a Moore relation system which generates the weak factorization system represented by* W*.*

In particular, every type theoretic weak factorization system can be generated by a Moore relation system.

Proof. This is Theorem 3.3.10 with 'Moore relation system' substituted for '`Id`-presentation' as justified by Theorem 3.4.7. $\square$

Example 3.4.9. Consider Example 3.3.8. Then given a Cisinski model structure $(\mathcal{C}, \mathcal{F}, \mathcal{W})$ on a topos $\mathcal{M}$, the weak factorization system $(\mathcal{C} \cap \mathcal{W} \cap \mathcal{M}_{\mathcal{F}}, \mathcal{F} \cap \mathcal{M}_{\mathcal{F}})$ is generated by a Moore relation system. In particular, the weak factorization systems in the category of Kan complexes, the category of quasicategories, and that of cubical sets are generated by Moore relation systems.

To conclude this section, we show that $|\mathbf{VRUF}| : |\mathtt{Id}\mathfrak{Pres}_{\mathcal{C}}^{00}| \to |\mathtt{Id}\mathfrak{Pres}_{\mathcal{C}}^{00}|$ is isomorphic to the identity functor. We have shown that $|\mathbf{UFVR}| : |\mathfrak{tt}\mathfrak{WFS}_{\mathcal{C}}^{00}| \to |\mathfrak{tt}\mathfrak{WFS}_{\mathcal{C}}^{00}|$ is also isomorphic to the identity functor (Corollary 3.3.11). Thus, this will show that $|\mathbf{VR}|$ and $|\mathbf{UF}|$ form an equivalence $|\mathtt{Id}\mathfrak{Pres}_{\mathcal{C}}^{00}| \simeq |\mathfrak{tt}\mathfrak{WFS}_{\mathcal{C}}^{00}|$.

Proposition 3.4.10. *The functor* $|\mathbf{VRUF}| : |\mathtt{Id}\mathfrak{Pres}_{\mathcal{C}}^{00}| \to |\mathtt{Id}\mathfrak{Pres}_{\mathcal{C}}^{00}|$ *is isomorphic to the identity functor.*

Proof. We need to provide an equivalence between any R in $\mathtt{Id}\mathfrak{Pres}_{\mathcal{C}}^{00}$ and $\mathbf{VRUF}(R)$. Since $|\mathtt{Id}\mathfrak{Pres}_{\mathcal{C}}^{00}|$ is a proset, this will automatically assemble into a natural isomorphism $1 \cong |\mathbf{VRUF}|$.

As usual, let RX be denoted by the following diagram for any R in $\mathtt{Id}\mathfrak{Pres}_{\mathcal{C}}^{00}$ and any X in $\mathcal{C}$.

$$\begin{tikzcd} X \arrow[r, "\eta_X" description] & RX \arrow[l, shift right=2ex, "\epsilon_{0X}"'] \arrow[l, shift left=2ex, "\epsilon_{1X}"] \end{tikzcd}$$

Then $\mathbf{VRUF}(RX)$ gives the following diagram

$$\begin{tikzcd} X \arrow[r, "\lambda(\Delta_X)" description] & X\,{}_{\Delta}\!\times_{\epsilon_0} R(X \times X) \arrow[l, shift right=2ex, "\pi_0\rho(\Delta_X)"'] \arrow[l, shift left=2ex, "\pi_1\rho(\Delta_X)"] \end{tikzcd}$$

where (λ, ρ) denotes the factorization $\mathbf{UF}(R)$.

Now a morphism $R \to \mathbf{VRUF}(R)$ consists of a natural transformation $R(X) \to \mathbf{VRUF}(R)(X)$ at each X, as displayed below, which, in turn, consists of the identity on X and a morphism $\tau_X : RX \to X\,{}_{\Delta}\!\times_{\epsilon_0} R(X \times X)$.

$$\begin{tikzcd} X \arrow[r, "\eta_X" description] \arrow[d, equal] & RX \arrow[l, shift right=2ex, "\epsilon_{0X}"'] \arrow[l, shift left=2ex, "\epsilon_{1X}"] \arrow[d, dashed, "\tau_X"] \\ X \arrow[r, "\lambda(\Delta_X)" description] & X\,{}_{\Delta}\!\times_{\epsilon_0} R(X \times X) \arrow[l, shift right=2ex, "\pi_0\rho(\Delta_X)"'] \arrow[l, shift left=2ex, "\pi_1\rho(\Delta_X)"] \end{tikzcd}$$

But we can obtain the morphism τ_X as a lift in the diagram below.

$$\begin{tikzcd}
X \arrow[r, "\lambda(\Delta_X)"] \arrow[d, "\eta"'] & X\, {}_{\Delta}{\times}_{\epsilon_0} R(X \times X) \arrow[d, "\rho(\Delta_X)"] \\
RX \arrow[r, "\epsilon_0 \times \epsilon_1"'] \arrow[ur, dashed, "\tau_X"] & X \times X
\end{tikzcd}$$

since η is in $\mathcal{L}$ and $\rho(\Delta_X)$ is in $\mathcal{R}$.

Similarly, we can get a morphism $\mathbf{VRUF}(R) \to R$ by solving the following lifting problem for each object X.

$$\begin{tikzcd}
X \arrow[r, "\eta"] \arrow[d, "\lambda(\Delta_X)"'] & RX \arrow[d, "\epsilon_0 \times \epsilon_1"] \\
X\, {}_{\Delta}{\times}_{\epsilon_0} R(X \times X) \arrow[r, "\rho(\Delta_X)"'] \arrow[ur, dashed] & X \times X
\end{tikzcd}$$

These lifts exist since $\lambda(\Delta_X)$ is in $\mathcal{L}$ and $\epsilon_0 \times \epsilon_1$ is in $\mathcal{R}$. $\square$

3.5 Summarizing theorems.

In Section 3.1, we defined the following diagram of categories,

$$\begin{tikzcd}
\mathfrak{ttWFS}_{\mathcal{C}}^{11} \arrow[r, hook] \arrow[d, hook] & \mathfrak{Fact}_{\mathcal{C}}^{11} \arrow[r, shift left, "\mathbf{R}"] \arrow[d, hook] & \mathfrak{RelFact}_{\mathcal{C}}^{11} \arrow[l, shift left, "\mathbf{U}"] \arrow[r, shift right, "\mathbf{V}"'] \arrow[d, hook] & \mathfrak{Rel}_{\mathcal{C}}^{11} \arrow[l, shift right, "\mathbf{F}"'] \arrow[d, hook] & \mathtt{Id}\mathfrak{Pres}_{\mathcal{C}}^{11} \arrow[l, hook'] \arrow[d, hook] \\
\mathfrak{ttWFS}_{\mathcal{C}}^{10} \arrow[r, hook] \arrow[d, hook] & \mathfrak{Fact}_{\mathcal{C}}^{10} \arrow[r, shift left, "\mathbf{R}"] \arrow[d, hook] & \mathfrak{RelFact}_{\mathcal{C}}^{10} \arrow[l, shift left, "\mathbf{U}"] \arrow[r, shift right, "\mathbf{V}"'] \arrow[d, hook] & \mathfrak{Rel}_{\mathcal{C}}^{10} \arrow[l, shift right, "\mathbf{F}"'] \arrow[d, hook] & \mathtt{Id}\mathfrak{Pres}_{\mathcal{C}}^{10} \arrow[l, hook'] \arrow[d, hook] \\
\mathfrak{ttWFS}_{\mathcal{C}}^{00} \arrow[r, hook] & \mathfrak{Fact}_{\mathcal{C}}^{00} \arrow[r, shift left, "\mathbf{R}"] & \mathfrak{RelFact}_{\mathcal{C}}^{00} \arrow[l, shift left, "\mathbf{U}"] \arrow[r, shift right, "\mathbf{V}"'] & \mathfrak{Rel}_{\mathcal{C}}^{00} \arrow[l, shift right, "\mathbf{F}"'] & \mathtt{Id}\mathfrak{Pres}_{\mathcal{C}}^{00} \arrow[l, hook']
\end{tikzcd}$$

and showed that the functors in the top row enjoy certain universal properties.

In Section 3.2, we defined (strict) Moore relation systems. We showed that strict Moore relation systems generate type theoretic, algebraic weak factorization systems, and that Moore relation systems generate type theoretic weak factorization systems.

In Sections 3.3 and 3.4, we showed the following theorem.

Theorem 3.5.1. *Consider a category $\mathcal{C}$ with finite limits. The functors* $\mathbf{VR}$ *and* $\mathbf{UF}$ *described above restrict to functors shown below.*

$$\begin{tikzcd}
\mathfrak{ttWFS}_{\mathcal{C}}^{11} \arrow[r, shift left, "\mathbf{VR}"] \arrow[d, hook] & \mathtt{Id}\mathfrak{Pres}_{\mathcal{C}}^{11} \arrow[l, shift left, "\mathbf{UF}"] \arrow[d, hook] \\
\mathfrak{ttWFS}_{\mathcal{C}}^{10} \arrow[r, shift left, "\mathbf{VR}"] \arrow[d, hook] & \mathtt{Id}\mathfrak{Pres}_{\mathcal{C}}^{10} \arrow[l, shift left, "\mathbf{UF}"] \arrow[d, hook] \\
\mathfrak{ttWFS}_{\mathcal{C}}^{00} \arrow[r, shift left, "\mathbf{VR}"] & \mathtt{Id}\mathfrak{Pres}_{\mathcal{C}}^{00} \arrow[l, shift left, "\mathbf{UF}"]
\end{tikzcd}$$

Furthermore, when we apply the proset functor, these give equivalences.

$$\begin{tikzcd}
|\mathfrak{ttWFS}_{\mathcal{C}}^{10}| \arrow[r, shift left, "|\mathbf{VR}|"] \arrow[r, phantom, "\simeq"] \arrow[d, hook] & |\mathtt{Id}\mathfrak{Pres}_{\mathcal{C}}^{10}| \arrow[l, shift left, "|\mathbf{UF}|"] \arrow[d, hook] \\
|\mathfrak{ttWFS}_{\mathcal{C}}^{00}| \arrow[r, shift left, "|\mathbf{VR}|"] \arrow[r, phantom, "\simeq"] & |\mathtt{Id}\mathfrak{Pres}_{\mathcal{C}}^{00}| \arrow[l, shift left, "|\mathbf{UF}|"]
\end{tikzcd}$$

Proof. The fact that $\mathbf{VR}$ and $\mathbf{UF}$ restrict to functors $\mathfrak{ttWFS}_{\mathcal{C}}^{00} \leftrightarrows \mathtt{Id}\mathfrak{Pres}_{\mathcal{C}}^{00}$ is proven in Corollary 3.3.11 and Corollary 3.4.4. Then consider an object or morphism X of $\mathfrak{ttWFS}_{\mathcal{C}}^{ij}$. We have just seen that $\mathbf{VR}(X) \in \mathtt{Id}\mathfrak{Pres}_{\mathcal{C}}^{00}$, and we know $\mathbf{VR}(X) \in \mathfrak{Rel}_{\mathcal{C}}^{ij}$ (see Propositions 3.1.31 and 3.1.40). Since $\mathtt{Id}\mathfrak{Pres}_{\mathcal{C}}^{ij}$ is the intersection of $\mathtt{Id}\mathfrak{Pres}_{\mathcal{C}}^{00}$ and $\mathfrak{Rel}_{\mathcal{C}}^{ij}$ in $\mathfrak{Rel}_{\mathcal{C}}^{00}$, we see that $\mathbf{VR}(X) \in \mathtt{Id}\mathfrak{Pres}_{\mathcal{C}}^{ij}$. Therefore, $\mathbf{VR}$ restricts to a functor $\mathfrak{ttWFS}_{\mathcal{C}}^{ij} \to \mathtt{Id}\mathfrak{Pres}_{\mathcal{C}}^{ij}$ for $ij = 00, 10, 11$. Similarly, $\mathbf{UF}$ restricts to a functor $\mathtt{Id}\mathfrak{Pres}_{\mathcal{C}}^{ij} \to \mathfrak{ttWFS}_{\mathcal{C}}^{ij}$ for $ij = 00, 10, 11$.

The fact that $|\mathbf{VR}|$ and $|\mathbf{UF}|$ give an equivalence $|\mathfrak{ttWFS}_{\mathcal{C}}^{00}| \simeq |\mathtt{Id}\mathfrak{Pres}_{\mathcal{C}}^{00}|$ is proven in Corollary 3.3.11 and Proposition 3.4.10. Since both squares in the following diagram commute,

$$\begin{tikzcd}
|\mathfrak{ttWFS}_{\mathcal{C}}^{10}| \arrow[r, shift left, "|\mathbf{VR}|"] \arrow[r, phantom, "\simeq"] \arrow[d, hook] & |\mathtt{Id}\mathfrak{Pres}_{\mathcal{C}}^{10}| \arrow[l, shift left, "|\mathbf{UF}|"] \arrow[d, hook] \\
|\mathfrak{ttWFS}_{\mathcal{C}}^{00}| \arrow[r, shift left, "|\mathbf{VR}|"] \arrow[r, phantom, "\simeq"] & |\mathtt{Id}\mathfrak{Pres}_{\mathcal{C}}^{00}| \arrow[l, shift left, "|\mathbf{UF}|"]
\end{tikzcd}$$

we see that this restricts to an equivalence $|\mathfrak{ttWFS}_{\mathcal{C}}^{10}| \simeq |\mathtt{Id}\mathfrak{Pres}_{\mathcal{C}}^{10}|$. □

We then interpret this in the following theorem.

Theorem 3.5.2. *Consider a category $\mathcal{C}$ with finite limits. The following properties of any weak factorization system $(\mathcal{L}, \mathcal{R})$ on $\mathcal{C}$ are equivalent:*

1. *it has an* `Id`*-presentation;*

2. *it is type theoretic;*

3. *it is generated by a Moore relation system;*

4. *$(\mathcal{C}, \mathcal{R})$ is a display map category modeling Σ and* `Id` *types.*

Proof. The equivalence between (1) and (3) appears as Theorem 3.4.7.

That (2) implies (1) is Theorem 3.3.10.

That (3) implies (2) is Theorem 3.2.56.

By Proposition 2.6.1, (2) implies that $(\mathcal{C}, \mathcal{R})$ is a display map category modeling Σ types. By Proposition 3.1.47, (1) is equivalent to $(\mathcal{C}, \mathcal{R})$ modeling `Id` types on objects. Then by Proposition A.1.5, this is equivalent to $(\mathcal{C}, \mathcal{R})$ modeling `Id` types. Thus, the combination of (1) and (2) is equivalent to (4). □

Theorem 3.5.3. *Consider a category $\mathcal{C}$ with finite limits and a weak factorization system $(\mathcal{L}, \mathcal{R})$ satisfying the equivalent statements of the preceding theorem, 3.5.2.*

If $(\mathcal{C}, \mathcal{R})$ models pre-Π types, then it models Π types. In particular, if $\mathcal{C}$ is locally cartesian closed, then $(\mathcal{C}, \mathcal{R})$ models Π types.

Proof. By the previous theorem, 3.5.2, $(\mathcal{L}, \mathcal{R})$ is type theoretic. Therefore, by Proposition 2.6.7, $(\mathcal{C}, \mathcal{R})$ models Π types. □

Chapter 4

Convenient categories of topological spaces.

In this chapter, we find type theoretic weak factorization systems in convenient categories of topological spaces. In the sections leading up to the final one, we generalize the construction of compactly generated weak Hausdorff spaces to obtain a large class of convenient categories of topological spaces. In the final section, we then construct strict Moore relation structures in these categories and in the topological topos. By Theorem 3.2.34, this will generate type theoretic weak factorization systems in these categories.

In the first section, we review the construction of a coreflective hull of a subcategory. The results of this section belong to folklore. In the second section, we apply the results of the first section to the category of topological spaces. Accounts of the results of this section (with the exception of Proposition 4.1.22, which we believe to be original) abound; we were most influenced by [Vog71]. In the third section, we generalize the construction of the weak Hausdorff reflection. We believe that the results of this and the following sections are original in their generality. In the fourth section, we characterize exponentiable morphisms in these categories, generalizing the results of [Lew85]. In the last section, we construct strict Moore relation structures in categories which contain a certain subcategory of the category of topological spaces. This produces models of Σ and `Id` types in most of the categories defined in this chapter and in the topological topos.

4.1 The coreflective hull of a subcategory.

In this section, we consider a category $\mathcal{C}$ which may not be cartesian closed but which contains a collection $\mathcal{B}$ of exponentiable objects. Here we recount how one can use these exponentiable objects to obtain a cartesian closed 'approximation' – precisely, a coreflective subcategory – of $\mathcal{C}$.

In the sections that follow, we will apply this theory to the case where $\mathcal{C}$ is the category $\mathcal{T}$ of topological spaces. Letting $\mathcal{B}$ be the category of compact Hausdorff spaces, we recover the classical recipe for the category of compactly generated spaces. Letting $\mathcal{B}$ be the category spanned by the disks $\{D_n = [0,1]^n \mid n \in \mathbb{N}\}$, we obtain Δ-generated spaces.

Now fix a bicomplete, concrete category $\mathcal{C}$ whose underlying set functor $U : \mathcal{C} \to \mathcal{S}et$ is represented by the terminal object and preserves colimits. We will assume that $\mathcal{C}$ is equipped with a choice of colimit for each diagram.

Also fix a full subcategory $I : \mathcal{B} \hookrightarrow \mathcal{C}$. We require that the colimits displayed in (4.1.1) exist in $\mathcal{C}$, which is not guaranteed by the cocompleteness of $\mathcal{C}$ unless $\mathcal{B}$ is small. In practice, either $\mathcal{B}$ will be small or $\mathcal{B}$ and $\mathcal{C}$ will satisfy the hypotheses of Proposition 4.1.3.

4.1.1 The comonad.

Consider the left Kan extension of I along I.

$$\begin{tikzcd} \mathcal{B} \arrow[r, "I"] \arrow[d, "I"'] & \mathcal{C} \\ \mathcal{C} \arrow[ur, dashed, "\mathsf{Lan}_I I"'] & \end{tikzcd}$$

This left Kan extension can be defined point-wise as

$$\mathsf{Lan}_I I(X) := \mathrm{colim}_{f \in \mathcal{B} \downarrow X} \mathrm{DOM}_X(f) \tag{4.1.1}$$

where $\mathcal{B} \downarrow X$ is the comma category whose objects are morphisms $f : B \to X$ of $\mathcal{C}$ such that $B \in \mathcal{B}$ and where $\mathrm{DOM}_X : \mathcal{B} \downarrow X \to \mathcal{C}$ is the functor which sends an object $f : B \to X$ in $\mathcal{B} \downarrow X$ to its domain B in $\mathcal{C}$.

We will denote $\mathsf{Lan}_I I(X) = \mathrm{colim}_{f \in \mathcal{B} \downarrow X} \mathrm{DOM}_X(f)$ by $\widehat{X}$ to simplify notation.

There is a canonical cocone with vertex X over the diagram DOM_X given by the natural transformation whose component at $f : B \to X$ in D_X is f itself. This induces a universal morphism

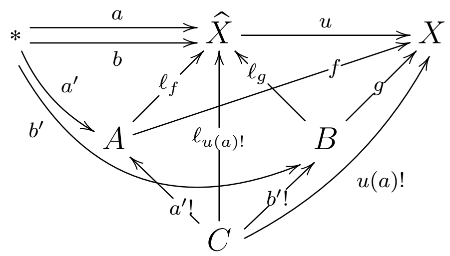
$$u_X : \widehat{X} \to X$$

natural in X.

Proposition 4.1.2. *For every object X of $\mathcal{C}$, the morphism $u_X : \widehat{X} \to X$ is a monomorphism.*

Proof. Since U is faithful, it reflects monomorphisms. We show that Uu_X is an injection. Since U preserves colimits, Uu_X is the universal arrow induced by the cocone on $U\text{DOM}_X$ with vertex UX.

Suppose there are two distinct points $a, b \in U\widehat{X}$. These are monomorphisms $a, b : * \to \widehat{X}$ in $\mathcal{C}$. Each of these points must be in the image of some set in the cocone under $U\widehat{X}$. That is, there are some objects $f : A \to X, g : B \to X$ in $\mathcal{B} \downarrow X$ and points $a' : * \to A,\ b' : * \to B$ such that $\ell_f a' = a,\ \ell_g b' = b$ (where $\ell_f : \text{DOM}_X(f) \to \widehat{X}$ is the appropriate leg of the colimiting cocone).



Now suppose that $u(a) = u(b)$. Consider any $C \in \mathcal{B}$ such that $UC \neq \varnothing$. Let $! : C \to *$ denote the terminal map. Then $u(a)! : C \to X$ is in $\mathcal{B} \downarrow X$, and there are morphisms $a'! : u(a)! \to f$ and $b'! : u(a)! \to g$ as shown in the diagram above.

Then we have the equalities $\ell_f a'! = \ell_{u(a)!} = \ell_g b'! : C \to \hat{X}$. Substituting the equalities $\ell_f(a') = a, \ell_g(b') = b$, this gives $a! = b!$. Since C is nonempty, $U! : C \to *$ is a surjection, and we conclude that $a = b$.

Therefore, Uu is an injection, and we conclude that u is a monomorphism $\square$

Proposition 4.1.3. *Suppose that $\mathcal{C}$ is well-powered and that $U : \mathcal{C} \to \mathcal{S}et$ is an isofibration with small fibers. Then the colimit displayed in* (4.1.1) *exists.*

Proof. For any cardinal κ, let $\mathcal{S}et_\kappa$ denote the full subcategory of $\mathcal{S}et$ consisting of those sets of cardinality less than or equal to κ. This is an essentially small category. Since U is an isofibration with small fibers, the full subcategory $\mathcal{B}_\kappa := (UI)^{-1}\mathcal{S}et_\kappa$ of $\mathcal{B}$ is essentially small (since it is equivalent to the preimage of the skeleton of $\mathcal{S}et_\kappa$ which is small).

Now since $\mathcal{C}$ is cocomplete, each

$$\widehat{X_\kappa} := \mathrm{colim}_{f \in \mathcal{B}_\kappa \downarrow X} \mathrm{DOM}_\kappa(f)$$

(where DOM_κ is again the domain functor) is a colimit of a small diagram, and so exists in $\mathcal{C}$. There is a monomorphism $u_\kappa : \widehat{X_\kappa} \hookrightarrow X$ by the preceding proposition.

The inclusions of the subcategories $\mathcal{B}_\kappa \hookrightarrow \mathcal{B}_\lambda$ for $\kappa \leqslant \lambda$ induce inclusions of the diagrams $d_{\kappa \leqslant \lambda} : \mathrm{DOM}_\kappa \hookrightarrow \mathrm{DOM}_\lambda$, and so we obtain universal morphisms $u_{\kappa \leqslant \lambda} : \widehat{X_\kappa} \to \widehat{X_\lambda}$ such that for any $\iota \leqslant \kappa \leqslant \lambda$, we have $u_{\kappa \leqslant \lambda} u_{\iota \leqslant \kappa} = u_{\iota \leqslant \lambda}$ and $u_\lambda u_{\kappa \leqslant \lambda} = u_\kappa$. From this last equation, we see that each $u_{\kappa \leqslant \lambda}$ is a monomorphism.

Since $\mathcal{C}$ is well-powered, X has only a set of subobjects. Thus, this chain of monomorphisms must eventually be constant. Let $\widehat{X}$ denote the object it converges to.

Now since for each κ, we have $\mathcal{C}(\widehat{X}_\kappa, Z) \cong \mathrm{Nat}(\mathrm{DOM}_\kappa, c_Z)$, we have the following isomorphism.

$$\mathcal{C}(\mathrm{colim}_\kappa \widehat{X}_\kappa, Z) \cong \lim_\kappa \mathcal{C}(\widehat{X}_\kappa, Z) \cong \lim_\kappa \mathrm{Nat}(\mathrm{DOM}_\kappa, c_Z) \cong \mathrm{Nat}(\mathrm{colim}_\kappa \mathrm{DOM}_\kappa, c_Z)$$

Since $\mathrm{colim}_\kappa \widehat{X}_\kappa \cong \widehat{X}$ and $\mathrm{colim}_\kappa \mathrm{DOM}_\kappa = \mathrm{DOM} : \mathcal{B} \downarrow X \to \mathcal{C}$, we see that $\widehat{X}$ is the colimit of DOM. □

Proposition 4.1.4. *For any X in $\mathcal{C}$, the monomorphism $\widehat{u_X} : \widehat{\widehat{X}} \to \widehat{X}$ is an isomorphism.*

Proof. Consider the functor

$$u_* : \mathcal{B} \downarrow \widehat{X} \to \mathcal{B} \downarrow X$$

given by postcomposition with u_X. Since $\widehat{X}$ is the colimit of the diagram $\mathrm{DOM}_X : \mathcal{B} \downarrow X \to \mathcal{C}$, there is an inclusion

$$i : \mathcal{B} \downarrow X \to \mathcal{B} \downarrow \widehat{X}$$

such that $u_* i = 1$. Thus, u_* is surjective. To see that u_* is injective on objects, consider two objects $f, g \in \mathcal{B} \downarrow \widehat{X}$. Since u is a monomorphism, $u_* f = u_* g$ implies $f = g$. To see that u_* is injective on morphisms, recall that the hom-sets of $\mathcal{B} \downarrow \widehat{X}$ and $\mathcal{B} \downarrow X$ are subsets of the hom-sets of $\mathcal{B}$, and u_* acts identically on these.

Therefore, u_* is an isomorphism. This induces an isomorphism between $\textsc{dom}_{\widehat{X}}$ and $\textsc{dom}_X$, and the isomorphism induced between the colimits is $\widehat{u_X}$. $\square$

4.1.2 Coreflective subcategories.

In this section, we consider a more general situation in order to show how the functor $\widehat{\ }: \mathcal{C} \to \mathcal{C}$ picks out a coreflective subcategory of $\mathcal{C}$.

Fix a category $\mathcal{A}$, an endofunctor $M : \mathcal{A} \to \mathcal{A}$, and a natural transformation $\alpha : M \to 1_{\mathcal{A}}$, such that for each $A \in \mathcal{A}$, α_A is a monomorphism and $M\alpha_A$ is an isomorphism.

Lemma 4.1.5. The natural transformations $M\alpha$ and $\alpha_M : M^2 \to M$ are equal.

Proof. For any $A \in \mathcal{A}$, there is the following naturality square.

$$\begin{tikzcd} M^2A \arrow[r, "M\alpha_A"] \arrow[d, "\alpha_{MA}"] & MA \arrow[d, "\alpha_A"] \\ MA \arrow[r, "\alpha_A"] & A \end{tikzcd}$$

Since α_A is monic, we get that $\alpha_{MA} = M\alpha_A$. $\square$

Theorem 4.1.6. *The functor $M : \mathcal{A} \to \mathcal{A}$ is an idempotent comonad with counit α and comultiplication $M\alpha^{-1}$.*

Proof. The isomorphism α_M gives an isomorphism $M^2 \cong M$.

The comonad axioms follow immediately from Lemma 4.1.5.

$$\begin{tikzcd} M & M^2 \arrow[l, "M\alpha"'] \arrow[r, "\alpha_M"] & M \\ & M \arrow[u, "M\alpha^{-1}"] \arrow[ul, equal] \arrow[ur, equal] & \end{tikzcd} \qquad \begin{tikzcd} M \arrow[r, "M\alpha^{-1}"] \arrow[d, "M\alpha^{-1}"'] & M^2 \arrow[d, "M\alpha_M{}^{-1}"] \\ M^2 \arrow[r, "M^2\alpha^{-1}"] & M^3 \end{tikzcd}$$

$\square$

Definition 4.1.7. Let $\mathcal{A}_M$ denote the full subcategory of $\mathcal{A}$ spanned by those objects A for which α_A is an isomorphism.

Proposition 4.1.8. *The subcategory $\mathcal{A}_M$ is closed under all colimits of $\mathcal{A}$.*

Proof. Let $D : \mathcal{D} \to \mathcal{A}_M$ be a diagram whose colimit exists in $\mathcal{A}$. Let ℓ denote the colimiting cocone $D \to \mathrm{colim}D$. Then $M\ell$ is a cocone $MD \to M\mathrm{colim}D$. The natural transformation α^{-1} gives a natural transformation $\alpha^{-1}D : D \to MD$. This induces a universal morphism $\beta : \mathrm{colim}D \to M\mathrm{colim}D$ such that the left hand square below commutes.

$$\begin{tikzcd}
\mathrm{colim}D \arrow[r, "\beta"] & M\mathrm{colim}D \arrow[r, "\alpha_{\mathrm{colim}D}"] & \mathrm{colim}D \\
D \arrow[u, "\ell"] \arrow[r, "\alpha_D^{-1}"] \arrow[rr, equal, bend right] & MD \arrow[u, "M\ell"] \arrow[r, "\alpha_D"] & D \arrow[u, "\ell"]
\end{tikzcd}$$

Since α is a natural transformation, the right hand square above commutes. Since the outer square above commutes, the universal property of $\mathrm{colim}D$ forces $\alpha_{\mathrm{colim}D}\beta = 1_{\mathrm{colim}D}$. But since α is monic, we can conclude that β is its inverse.

Therefore, $\mathrm{colim}D$ is in $\mathcal{A}_M$. □

Theorem 4.1.9. *The category $\mathcal{A}_M$ is isomorphic to the Eilenberg-Moore category $\mathcal{A}^M$ of coalgebras of M via the forgetful functor $\mathcal{A}^M \to \mathcal{A}$.*

In other words, every coalgebra of M is of the form $\alpha_A^{-1} : A \to MA$, and thus every coalgebra of M is isomorphic to a free coalgebra.

Proof. For every $A \in \mathcal{A}_M$, the morphism $\alpha_A^{-1} : A \to MA$ gives A the structure of a coalgebra. Let

$$V : \mathcal{A}_M \to \mathcal{A}^M$$

denote the functor which sends every A to this coalgebra (A, α_A^{-1}) and every f to its naturality square for α^{-1}.

For every (A, c) in $\mathcal{A}^M$, the counit axiom for this coalgebra is the equation $\alpha_A c = 1_A$. Since α_A is monic, we see that $\alpha_A c \alpha_A = \alpha_A$ implies $c\alpha_A = 1_A$. Thus, c must be α_A^{-1}. Moreover, we see that any morphism $(A, c) \to (B, d)$ must be a naturality square for α^{-1}. Thus there is a forgetful functor

$$F : \mathcal{A}^M \to \mathcal{A}_M$$

sending each $(A, c) = (A, \alpha_A^{-1})$ to A, and this forgetful functor is the inverse of V. □

Proposition 4.1.10. *$\mathcal{A}_M$ is a coreflective subcategory of $\mathcal{A}$ whose coreflection is given by $M : \mathcal{A} \to \mathcal{A}_M$.*

This coincides with the Eilenberg-Moore adjunction $\mathcal{A}^M \leftrightarrows \mathcal{A}$, but we give a proof of the adjunction in this special case.

Proof. Let $J : \mathcal{A}_M \hookrightarrow \mathcal{A}$ denote the inclusion.

The natural isomorphism $\alpha^{-1} : 1_{\mathcal{A}_M} \to MJ$ gives the unit, and the natural transformation $\alpha : JM \to 1_{\mathcal{A}}$ gives the counit. The triangle equalities follow immediately from Lemma 4.1.5.

$$\begin{tikzcd} M \arrow[r, "\alpha_M^{-1}"] \arrow[dr, equal] & MJM \arrow[d, "M\alpha"] \\ & M \end{tikzcd} \qquad \begin{tikzcd} J \arrow[r, "\alpha^{-1}"] \arrow[dr, equal] & JMJ \arrow[d, "\alpha"] \\ & J \end{tikzcd}$$

□

Corollary 4.1.11. *The functor $M : \mathcal{A} \to \mathcal{A}_M$ preserves limits.*

Proof. By the previous proposition, M is a right adjoint. □

4.1.3 The coreflective hull.

Now we apply the general results of the previous section to our setting.

Definition 4.1.12. Let $\widehat{\mathcal{B}}$ denote the full subcategory of $\mathcal{C}$ spanned by those objects X of $\mathcal{C}$ for which $u_X : \widehat{X} \to X$ is an isomorphism. Call this subcategory the *coreflective hull* of $\mathcal{B}$ in $\mathcal{C}$.

Note that the subcategory $\widehat{\mathcal{B}}$ contains the subcategory $\mathcal{B}$ of $\mathcal{C}$. This is because for any object B of $\mathcal{B}$, the object $1_B : B \to B$ of $\mathcal{B} \downarrow B$ is terminal, and so the colimit $\operatorname{colim}\text{DOM}_B$ is isomorphic to B via u_B.

The general results of the previous section give the following.

Theorem 4.1.13. *The endofunctor $\widehat{\ }$ is an idempotent comonad on $\mathcal{C}$. This induces an adjunction*

$$\begin{tikzcd} \widehat{\mathcal{B}} \arrow[r, hook, shift left] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, shift left, "\widehat{\ }"] \end{tikzcd}$$

displaying $\widehat{\mathcal{B}}$ as a coreflective subcategory of $\mathcal{C}$.

Moreover, $\widehat{\mathcal{B}}$ is closed under colimits of $\mathcal{C}$, and the coreflection $\widehat{\ }$ preserves limits. In particular, $\widehat{\mathcal{B}}$ is bicomplete.

Proof. This follows from Theorem 4.1.6, Proposition 4.1.10, Proposition 4.1.8, and Corollary 4.1.11. □

Proposition 4.1.14. *For any subcategory $\mathcal{B}'$ of $\mathcal{C}$ such that $\mathcal{B} \subseteq \mathcal{B}' \subseteq \widehat{\mathcal{B}}$, the coreflective hulls of $\mathcal{B}$ and $\mathcal{B}'$, and their respective coreflections, coincide.*

Proof. Because coreflective hulls are closed under colimits, we have that $\widehat{\mathcal{B}} \subseteq \widehat{\mathcal{B}'} \subseteq \widehat{\widehat{\mathcal{B}}}$. Moreover, since each object of $\widehat{\widehat{\mathcal{B}}}$ is a colimit in $\mathcal{C}$ of a diagram in $\widehat{\mathcal{B}}$, we see that $\widehat{\widehat{\mathcal{B}}} \subseteq \widehat{\mathcal{B}}$. Therefore, $\widehat{\mathcal{B}} = \widehat{\mathcal{B}'} = \widehat{\widehat{\mathcal{B}}}$.

Since then the inclusions of $\widehat{\mathcal{B}}$ and $\widehat{\mathcal{B}'}$ coincide, their right adjoints, the coreflections, coincide as well. □

4.1.4 Including the terminal object.

In this section, we consider a condition on $\mathcal{B}$ which, in some sense, says that $\mathcal{B}$ is not trivial.

Lemma 4.1.15. The terminal object $*$ of $\mathcal{C}$ is in $\widehat{\mathcal{B}}$ if and only if the image $U\mathcal{B}$ contains a nonempty set.

Proof. Recall that the set $U\widehat{X}$ is the colimit of $U\textsc{dom}_X$ for every X in $\mathcal{C}$. Then $U\widehat{X}$ is empty if and only if every set in the image of $U\textsc{dom}_X$ is empty.

Suppose that the terminal object $*$ of $\mathcal{C}$ is in $\widehat{\mathcal{B}}$. Then $U(\widehat{*}) \cong *$ is nonempty, so $U\mathcal{B}$ contains a nonempty set.

Suppose that $U\mathcal{B}$ contains a nonempty set. Then $U(\widehat{*})$ is nonempty (meaning there is a point $p : * \to \widehat{*}$), and the morphism $u_* : \widehat{*} \to *$ is monic. Then from the equation $u_* p = 1_*$, we see that u_* is an isomorphism. Thus, the terminal object is in $\widehat{\mathcal{B}}$. □

Proposition 4.1.16. *Suppose the terminal object $*$ of $\mathcal{C}$ is in $\widehat{\mathcal{B}}$. Then every $u_X : \widehat{X} \to X$ is an epimorphism.*

Proof. Let $\mathcal{B}_\bullet$ be the full subcategory of $\mathcal{C}$ spanned by the objects of $\mathcal{B}$ and the terminal object. We have the inclusions $\mathcal{B} \subseteq \mathcal{B}_\bullet \subseteq \widehat{\mathcal{B}}$, so we can assume without loss of generality that $\mathcal{B} = \mathcal{B}_\bullet$ by Corollary 4.1.14.

Since U is faithful, it reflects epimorphisms. Thus we show that $Uu_X : U\widehat{X} \to UX$ is surjective. Since U preserves coproducts, this is the universal arrow $\operatorname{colim} U\textsc{dom}_X \to UX$.

Consider any point $x \in UX$. This is a morphism $x : * \to X$. Then x is in $\textsc{dom}_X$ and there is a leg of the colimiting cocone $\ell_x : U(*) \to U\widehat{X}$ such that $(Uu_X)\ell_x = Ux$. Thus, the image of ℓ_x is a point y such that $Uu_X(y) = x$. Therefore, Uu_X is surjective. $\square$

Corollary 4.1.17. *Suppose the terminal object $*$ of $\mathcal{C}$ is in $\widehat{\mathcal{B}}$. Then the underlying function Uu_X of $u_X : \widehat{X} \to X$ is a bijection for every object X of $\mathcal{C}$.*

4.1.5 The cartesian closure of the coreflective hull.

In this section, we show that $\widehat{\mathcal{B}}$ is cartesian closed under certain conditions on $\mathcal{B}$.

Notation 4.1.18. For clarity, we will use $\hat{\times}$ to denote the product in $\widehat{\mathcal{B}}$, and $\times$ to denote the product in $\mathcal{C}$. Note that for all X, Y, there is an isomorphism

$$\widehat{X \times Y} \cong \widehat{X} \hat{\times} \widehat{Y}.$$

We need the following lemma to prove the main theorem of this section. Note that the hypothesis that $\mathcal{B}$ is closed under binary products will not in general be satisfied by the subcategories $\mathcal{B}$ that we will consider in the following sections.

Lemma 4.1.19. Suppose that for any $A, B \in \mathcal{B}$, the product $A \times B$ is in $\mathcal{B}$. Then for any objects X, Y of $\mathcal{C}$, there is an isomorphism

$$\widehat{X \times Y} \cong \operatorname{colim}(\textsc{dom}_X \times \textsc{dom}_Y)$$

where

$$\textsc{dom}_X \times \textsc{dom}_Y : (\mathcal{B} \downarrow X) \times (\mathcal{B} \downarrow Y) \to \mathcal{C}$$

is the diagram which maps an object (f, g) to $\textsc{dom}_X f \times \textsc{dom}_Y g$.

Proof. The object $\widehat{X \times Y}$ is itself a colimit, and the isomorphism of the statement can be written more explicitly as

$$\operatorname{colim}_{\mathcal{B}\downarrow(X\times Y)} \textsc{dom}_{X\times Y} \cong \operatorname{colim}_{(\mathcal{B}\downarrow X)\times(\mathcal{B}\downarrow Y)} \textsc{dom}_X \times \textsc{dom}_Y.$$

There is a functor

$$\iota : (\mathcal{B} \downarrow X) \times (\mathcal{B} \downarrow Y) \hookrightarrow \mathcal{B} \downarrow (X \times Y)$$

which maps a pair $(f : B \to X, g : C \to Y)$ to $f \times g : B \times C \to X \times Y$. This makes the following diagram commute.

$$\begin{tikzcd} (\mathcal{B} \downarrow X) \times (\mathcal{B} \downarrow Y) \arrow[rr, "\iota"] \arrow[dr, "\mathrm{DOM}_X \times \mathrm{DOM}_Y"'] & & \mathcal{B} \downarrow (X \times Y) \arrow[dl, "\mathrm{DOM}_{X \times Y}"] \\ & \mathcal{C} & \end{tikzcd}$$

We claim that ι is a final functor. To that end, consider the comma category $(f \times g) \downarrow \iota$ for any $f \times g : B \to X \times Y$ in $\mathcal{B} \downarrow (X \times Y)$. It is nonempty since it contains the following object.

$$\begin{tikzcd} B \arrow[rr, "\Delta_B"] \arrow[dr, "f \times g"'] & & B \times B \arrow[dl, "f \times g"] \\ & X \times Y & \end{tikzcd}$$

It is connected since every object $(c \times d) : (f \times g) \to (h \times i)$ as in the following diagram

$$\begin{tikzcd} & C \times D \arrow[dd, "h \times i"] & \\ B \arrow[ur, "c \times d"] \arrow[rr, "\Delta_B"] \arrow[dr, "f \times g"'] & & B \times B \arrow[ul, dashed, "c \times d"'] \arrow[dl, "f \times g"] \\ & X \times Y & \end{tikzcd}$$

is connected to Δ_B by the arrow $c \times d : \Delta_B \to (c \times d)$ as displayed above.

Therefore, $\iota : (\mathcal{B} \downarrow X) \times (\mathcal{B} \downarrow Y) \hookrightarrow \mathcal{B} \downarrow (X \times Y)$ is a final functor, and by [ML98, IX.3, Thm. 1], we can conclude that it induces an isomorphism

$$\operatorname{colim} \iota : \operatorname{colim}_{(\mathcal{B}\downarrow X)\times(\mathcal{B}\downarrow Y)} \mathrm{DOM}_X \times \mathrm{DOM}_Y \cong \operatorname{colim}_{\mathcal{B}\downarrow(X\times Y)} \mathrm{DOM}_{X\times Y}. \qquad \square$$

Definition 4.1.20. Say that $\mathcal{B}$ *generates its products* if any finite product (taken in $\mathcal{C}$) of objects of $\mathcal{B}$ lies in $\widehat{\mathcal{B}}$.

Recall that an object X of $\mathcal{C}$ is *exponentiable* if the functor $\mathcal{C}(- \times X, Y)$ is representable for every object Y of $\mathcal{C}$.

Theorem 4.1.21. *Suppose that $\mathcal{B}$ generates its products and contains only exponentiable objects of $\mathcal{C}$. Then $\widehat{\mathcal{B}}$ is cartesian closed.*

Proof. By Corollary 4.1.11, $\widehat{\mathcal{B}}$ inherits all limits from $\mathcal{C}$.

Let $\overline{\mathcal{B}}$ denote the closure of $\mathcal{B}$ under finite products of $\mathcal{C}$. Since finite products of exponentiable objects are exponentiable themselves, $\overline{\mathcal{B}}$ verifies the hypotheses of both this theorem and the preceding lemma, 4.1.19. We have that $\mathcal{B} \subseteq \overline{\mathcal{B}} \subseteq \widehat{\mathcal{B}}$ and so by Corollary 4.1.14, the coreflective hulls $\widehat{\mathcal{B}}$ and $\widehat{\overline{\mathcal{B}}}$ coincide. Thus, we can and will assume without loss of generality that $\mathcal{B} = \overline{\mathcal{B}}$.

We need to show that for any objects $X, Y, Z \in \mathcal{C}$ there is an object $\widehat{Z}^{\widehat{Y}}$ such that

$$\mathcal{C}(\widehat{X} \hat{\times} \widehat{Y}, \widehat{Z}) \cong \mathcal{C}(\widehat{X}, \widehat{Z}^{\widehat{Y}}).$$

Define

$$\widehat{Z}^{\widehat{Y}} := \widehat{\lim_{f \in \mathcal{B} \downarrow Y} \widehat{Z}^{\mathrm{DOM}_Y f}}$$

where, for each f, $\widehat{Z}^{\mathrm{DOM}_Y f}$ is the object in $\mathcal{C}$ which represents $\mathcal{C}(- \times \mathrm{DOM}_Y f, \widehat{Z})$.

Now we note the following chain of bijections.

$$\begin{aligned}
\mathcal{C}(\widehat{X}, \widehat{Z}^{\widehat{Y}}) &= \mathcal{C}(\mathrm{colim}_{g \in \mathcal{B} \downarrow X} \mathbf{DOM}_X g, \widehat{\lim_{f \in \mathcal{B} \downarrow Y} \widehat{Z}^{\mathrm{DOM}_Y f}}) \\
&\cong \mathcal{C}(\mathrm{colim}_{g \in \mathcal{B} \downarrow X} \mathbf{DOM}_X g, \lim_{f \in \mathcal{B} \downarrow Y} \widehat{Z}^{\mathrm{DOM}_Y f}) \\
&\cong \lim_{f \in \mathcal{B} \downarrow Y} \lim_{g \in \mathcal{B} \downarrow X} \mathcal{C}(\mathrm{DOM}_X g, \widehat{Z}^{\mathrm{DOM}_Y f}) \\
&\cong \lim_{f \in \mathcal{B} \downarrow Y} \lim_{g \in \mathcal{B} \downarrow X} \mathcal{C}(\mathrm{DOM}_X g \times \mathrm{DOM}_Y f, \widehat{Z}) \\
&\cong \mathcal{C}(\mathrm{colim}_{f \in \mathcal{B} \downarrow Y} \mathrm{colim}_{g \in \mathcal{B} \downarrow X} \mathbf{DOM}_X g \times \mathrm{DOM}_Y f, \widehat{Z}) \\
&\cong \mathcal{C}(\widehat{X} \hat{\times} \widehat{Y}, \widehat{Z})
\end{aligned}$$

where the first equality is given by substituting the definitions of $\widehat{X}$ and $\widehat{Z}^{\widehat{Y}}$, the second bijection follows from the adjunction of Proposition 4.1.10, the third and fifth follow from the commutativity of colimits and limits with hom-sets, the forth from the adjunction defining $\widehat{Z}^{\text{DOM}_Y f}$, and the sixth from Lemma 4.1.19. $\square$

Proposition 4.1.22. *Suppose that $\mathcal{B}$ generates its products and contains only exponentiable objects of $\mathcal{C}$. Then for any $B \in \mathcal{B}$ and $C \in \widehat{\mathcal{B}}$, the product $B \times C$ (taken in $\mathcal{C}$) is in $\widehat{\mathcal{B}}$.*

Proof. Consider $B \in \mathcal{B}$ and $C \in \widehat{\mathcal{B}}$. Then we have the following chain of isomorphisms

$$\begin{aligned} B \times C &\cong B \times \operatorname{colim}_{f \in \mathcal{B} \downarrow C} \text{DOM}_C f \\ &\cong \operatorname{colim}_{f \in \mathcal{B} \downarrow C} B \times \text{DOM}_C f \end{aligned}$$

where the first follows from the fact that $C \cong \widehat{C} = \operatorname{colim}_{f \in \mathcal{B} \downarrow C} \text{DOM}_C f$ and the second follows from the fact that B is exponentiable so $B \times -$ is a left adjoint. Since each $B \times \text{DOM}_C f$ is in $\widehat{\mathcal{B}}$ by hypothesis, and $\widehat{\mathcal{B}}$ is closed under colimits by Proposition 4.1.8, we conclude that $\operatorname{colim}_{f \in \mathcal{B} \downarrow C} B \times \text{DOM}_C f$, and thus $B \times C$, is in $\widehat{B}$. $\square$

4.2 Coreflections of topological spaces.

Now we focus on the situation where the ambient category is the category $\mathcal{T}$ of topological spaces. This is a bicomplete, well-powered concrete category whose underlying functor is an isofibration with both left and right adjoints and small fibers. Thus, it satisfies all the hypotheses placed on $\mathcal{C}$ in the preceding section.

4.2.1 The coreflection.

In this section, we consider any full subcategory $\mathcal{B}$ of $\mathcal{T}$ which contains the terminal object $*$. Recall from the preceding section (Corollary 4.1.17) that for every space X, the underlying function of $u_X : \widehat{X} \to X$ is a bijection. Thus, $\widehat{X}$ and X have the same underlying set, and $\widehat{X}$ has a stronger topology than X. We now describe this topology.

Proposition 4.2.1. *For each topological space X, let $\widetilde{X}$ be the topological space whose underlying set is that of X and whose open sets are those subsets V such that* *for every $f \in \mathcal{B} \downarrow X$, the subset $f^{-1}V$ is open in $\mathrm{DOM}_X(f)$. This defines a functor* $\widetilde{\ } : \mathcal{T} \to \widehat{\mathcal{B}}$ *which is naturally isomorphic to* $\widehat{\ } : \mathcal{T} \to \widehat{\mathcal{B}}$.

Proof. First note that the description of $\widetilde{X}$ defines a topology on $U(X)$. Since each $f^{-1}\varnothing$ is open in $\mathrm{DOM}_X(f)$, the empty subset is open in $\widetilde{X}$. Similarly, $\widetilde{X}$ is open in $\widetilde{X}$. For any finite collection of opens $V_1, ..., V_n$, since each $f^{-1}V_i$ is open in $\mathrm{DOM}_X(f)$, then $\cap_{i=1}^n f^{-1}V_i = f^{-1} \cap_{i=1}^n V_i$ is open in $\mathrm{DOM}(f)$ for every f, and therefore $\cap_{i=1}^n V_i$ is open in $\widetilde{X}$. Similarly, for any collection of opens $\{V_i\}_i$, its union $\cup_i V_i$ is open in $\widetilde{X}$.

The topology on $\widetilde{X}$ is defined so that the canonical cocone $Uc : U\mathrm{DOM}_X \to UX$ in $\mathcal{S}et$ lifts to a cocone $\mathrm{DOM}_X \to \widetilde{X}$ in $\mathcal{T}$ and that, furthermore, it is the strongest such topology that can be placed on UX. In other words, it is initial amongst cocones $d : \mathrm{DOM}_X \to Y$ such that $Ud \cong Uc$. But we know that the colimit $\widehat{X}$ has this property (Corollary 4.1.17). Thus $\widetilde{X}$ has the defining universal property of $\widehat{X}$. □

Corollary 4.2.2. *A subset V of $\widehat{X}$ is open if and only if for every $f \in \mathcal{B} \downarrow X$, the subset $f^{-1}V$ is open in $\mathrm{DOM}_X(f)$.*

Corollary 4.2.3. *A subset C of $\widehat{X}$ is closed if and only if for every $f \in \mathcal{B} \downarrow X$, the subset $f^{-1}C$ is closed in $\mathrm{DOM}_X(f)$.*

From now on, we will use the more concrete specification of $\widetilde{X}$ for $\widehat{X}$.

4.2.2 Examples.

We are interested in cartesian closed $\widehat{\mathcal{B}}$. We require from now on that $\mathcal{B}$ contains only exponentiable objects of $\mathcal{T}$ and generates its products. Then the results of Section 4.1.5 (except Lemma 4.1.19) will apply so $\widehat{\mathcal{B}}$ will be cartesian closed.

This section contains examples of subcategories $\mathcal{B}$ satisfying these hypotheses.

Proposition 4.2.4. *Consider the full subcategory $\mathcal{K}$ of $\mathcal{T}$ spanned by compact Hausdorff spaces. Then $\mathcal{K}$ contains only exponentiable objects of $\mathcal{T}$ and generates its products.*

Proof. Compact Hausdorff spaces are exponentiable by [Fox45, Thm. 1].

Now we claim that that $\mathcal{K}$ generates its products. First of all, the terminal space $*$ is compact Hausdorff, so it contains nullary products. Now consider a product $K \times L$ of compact Hausdorff spaces. It is compact by Tychonoff's theorem ([Kel75, Ch. 5, Thm. 13]) and is Hausdorff by [Kel75, Ch. 3, Thm. 5]. □

The category $\widehat{\mathcal{K}}$ is usually called the category of *compactly generated* spaces.

Proposition 4.2.5. *Consider the full subcategory $\mathcal{E}$ of $\mathcal{T}$ spanned by exponentiable spaces of $\mathcal{T}$. Then $\mathcal{E}$ contains only exponentiable objects of $\mathcal{T}$ and generates its products.*

Proof. Note that $*$ is exponentiable as the identity functor is right adjoint to $- \times *$ (also the identity functor). Therefore, $\mathcal{E}$ contains nullary products.

To see that $\mathcal{E}$ contains its binary products, consider two exponentiable spaces E and F of $\mathcal{T}$. Consider also any two spaces X, Y of $\mathcal{T}$. Then we have the following chain of isomorphisms natural in X and Y.

$$\begin{aligned} \hom(X \times (E \times F), Y) &\cong \hom(X \times E, Y^F) \\ &\cong \hom(X, (Y^F)^E) \end{aligned}$$

Therefore, $(-^F)^E$ is a right adjoint to $- \times (E \times F)$, so $E \times F$ is exponentiable. □

Proposition 4.2.6. *Consider the full subcategory $\mathcal{D}$ of $\mathcal{T}$ spanned by just the interval $I = [0,1]$ of $\mathcal{T}$. Then $\mathcal{D}$ contains only exponentiable objects of $\mathcal{T}$ and generates its products.*

To prove this, we first need the following lemma.

Lemma 4.2.7. Any locally path-connected metric space is in $\widehat{\mathcal{D}}$.

Proof. Consider a locally path-connected metric space X. Let C denote a closed subset of $\widehat{X}$: that is, a subset for which the preimage $f^{-1}C$ under any continuous function $f : I \to X$ is closed. We need to show that C is already closed in X. Then by Corollary 4.2.3, we will be able to conclude that $X \cong \widehat{X}$ and that X is in $\widehat{\mathcal{D}}$.

Let $x \in X$ be a limit point of C. This means there is a sequence $\{x_i\}_{i \in \mathbb{N}}$ in C converging to x.

Let $B_{x,\epsilon}$ denote the open ball $B_{x,\epsilon}$ around x of radius ϵ.

Now we can inductively define a sequence $A_0 \supseteq A_1 \supseteq A_2 \supseteq \ldots$ of path-connected open neighborhoods of x such that each A_i is contained in $B_{x,1/i}$. Since X is locally path-connected, we can find a path-connected open neighborhood of x to be A_0. Then let A_{i+1} be a path-connected open neighborhood of x contained in $A_i \cap B_{x,1/i}$.

Now for each i, let $j(i)$ denote the least natural number j such that x_j is contained in A_i. This exists since each open A_i contains some open ball $B_{x,1/k}$ which contains some x_ℓ. Moreover, since each $x_{j(i)}$ is contained in the open ball $B_{x,1/i}$, the sequence $\{x_{j(i)}\}_{i\in\mathbb{N}}$ also converges to x.

Now one can construct a function $f : I \to X$ by setting $f(0)$ to x, $f(\frac{1}{i})$ to $x_{j(i)}$, and $f|_{[\frac{1}{i+1},\frac{1}{i}]}$ to a path from x_{i+1} to x_i in A_i (for $i \in \mathbb{N}^+$). We claim that this function is continuous. On $(0, 1]$, it is a piecewise continuous function. To see that is continuous at 0, consider an open ball $B_{x,1/i}$ around x. This contains A_i, and $f^{-1}(A_i)$ contains $[0, 1/i)$ by construction. Then the open ball $[0, 1/i)$ around the point 0 is contained in the preimage of $f^{-1}B_{x,1/i}$. Therefore, f is continuous, and so is an object of $\mathcal{D} \downarrow X$.

Then we see that $f^{-1}C$ contains the sequence $\{1/t\}_t$, and $f^{-1}C$ contains 0 if and only if $x \in C$. But 0 is the limit point of $\{1/t\}_t$, and since $f^{-1}C$ must be closed, it must contain 0. Therefore, the limit point x must be in C, and so C must contain all its limit points. In other words C is closed. □

Proof of Proposition 4.2.6. First of all, I is compact Hausdorff, so by Proposition 4.2.4, $\mathcal{D}$ contains only exponentiable objects of $\mathcal{T}$.

Since any finite product I^n is a locally path-connected metric space, it is in $\widehat{\mathcal{D}}$ by Lemma 4.2.7. □

The category $\widehat{\mathcal{D}}$ is usually called the category of Δ-generated spaces.

We will find the following lemma useful.

Lemma 4.2.8. Let $\mathcal{D}'$ denote the full subcategory of $\mathcal{T}$ spanned just by $\mathbb{R}^+$, the non-negative real numbers. Then the subcategories $\widehat{\mathcal{D}}$ and $\widehat{\mathcal{D}'}$ of $\mathcal{T}$ coincide.

Proof. By Lemma 4.2.7, $\mathbb{R}^+$ is in $\widehat{\mathcal{D}}$, so $\widehat{\mathcal{D}'} \subseteq \widehat{\mathcal{D}}$.

It only remains to show that $I \in \widehat{\mathcal{D}'}$.

To that end, consider a subset $U \subseteq I$ such that $f^{-1}U$ is open for every $f : \mathbb{R}^+ \to I$. There are homeomorphisms $f : \mathbb{R}^+ \to [0, 1)$ and $g : \mathbb{R}^+ \to (0, 1]$ (where $f(t) = t/(t + 1)$, $f^{-1}(t) = t/(1 - t)$, $g(t) = 1/(t + 1)$, and $g^{-1}(t) =$

$(1-t)/t)$. Since $f^{-1}U$ is open in $\mathbb{R}^+$, then $ff^{-1}U = U \cap [0,1)$ is open in $[0,1)$. Similarly, $gg^{-1}U = U \cap (0,1]$ is also open in $(0,1]$. Since $[0,1)$ and $(0,1]$ are open in I, then $U \cap [0,1)$ and $U \cap (0,1]$ are open in I. We conclude that $U = (U \cap [0,1)) \cup (U \cap (0,1])$ is open in I.

Therefore, $\widehat{I} \cong I$ (where $\widehat{}$ here denotes the coreflection $\mathcal{T} \to \widehat{\mathcal{D}'}$), so we can conclude that I is in $\widehat{\mathcal{D}'}$.

Therefore, $\widehat{\mathcal{D}} \subseteq \widehat{\mathcal{D}'}$. $\square$

4.2.3 The topology of products in $\widehat{\mathcal{B}}$.

Recall from the previous section (Proposition 4.1.22) that for any $B \in \mathcal{B}$ and $C \in \widehat{\mathcal{B}}$ the product $B \times C$ taken in $\mathcal{T}$ coincides with the product $B \hat{\times} C$ taken in $\widehat{\mathcal{B}}$.

Proposition 4.2.9. *For any product $X \hat{\times} Y$ in $\widehat{\mathcal{B}}$, the projections $\pi_X : X \hat{\times} Y \to X$ and $\pi_Y : X \hat{\times} Y \to Y$ are open maps.*

Proof. The projection $\pi_X : X \times Y \to X$ is an open map. We check that it remains open under the strengthened topology of $X \hat{\times} Y$.

Consider an open $U \subseteq X \hat{\times} Y$, a space B of $\mathcal{B}$, and a continuous function $f : B \to X$. We need to show that $f^{-1}\pi_X U$ is open in B. Since the following diagram is a pullback square,

$$\begin{tikzcd} B \hat{\times} Y \arrow[r, "f \hat{\times} 1"] \arrow[d, "\pi_B"] \arrow[dr, phantom, "\lrcorner", very near start] & X \hat{\times} Y \arrow[d, "\pi_X"] \\ B \arrow[r, "f"] & X \end{tikzcd}$$

we have the following Beck-Chevalley equation:

$$f^{-1}\pi_X(U) = \pi_B(f \times 1_Y)^{-1}(U).$$

Since $B \hat{\times} Y \cong B \times Y$, the projection π_B is open. Thus, $\pi_B(f \times 1_Y)^{-1}(U)$ and hence $f^{-1}\pi_X(U)$ are open. $\square$

Definition 4.2.10. Say that a space X in $\widehat{\mathcal{B}}$ is *locally in $\mathcal{B}$* if every point $x \in X$ has a neighborhood N_x which is in $\mathcal{B}$ with the subspace topology.

Proposition 4.2.11. *Consider spaces X and Y in $\widehat{\mathcal{B}}$ such that X is locally in $\mathcal{B}$. Then the products of X and Y in $\mathcal{T}$ and in $\widehat{\mathcal{B}}$ coincide:*

$$X \times Y \cong X \hat{\times} Y.$$

Proof. Recall that $\widehat{X \times Y} \cong X \hat{\times} Y$. We show that an open set of $\widehat{X \times Y}$ is already open in $X \times Y$.

Fix such an open set $U \subseteq \widehat{X \times Y}$ and a point $(x, y) \in U$. Since X is locally in $\mathcal{B}$, there is a neighborhood N of x which is in $\mathcal{B}$ with the subspace topology. Let M denote an open neighborhood of x contained in N.

We show that there are open neighborhoods V of x and W of y such that $V \times W \subseteq U$.

By Proposition 4.1.22, the product $N \times Y$ is in $\widehat{\mathcal{B}}$. Let ι denote the following inclusion.

$$N \times Y \cong \widehat{N \times Y} \hookrightarrow \widehat{X \times Y}.$$

Then $\iota^{-1}(U) = U \cap (N \times Y)$ must be open in $N \times Y$. Thus, there is an open (in N) neighborhood V' of x and an open neighborhood W of y such that $V' \times W \subseteq U \cap (N \times Y)$.

Let $V := V' \cap M$. Then V is an open (in both N and X) neighborhood of x such that $V \times W \subseteq U \cap (N \times Y) \subseteq U$.

Now we see that every point of U is contained in an open (in $X \times Y$) neighborhood contained in U. Therefore, U is open in $X \times Y$.

Therefore, $\widehat{X \times Y}$ has the same topology as $X \times Y$. □

4.2.4 The topology of mapping spaces in $\widehat{\mathcal{B}}$.

For $X, Y \in \widehat{\mathcal{B}}$, let X^Y denote the representing object of $\widehat{\mathcal{B}}(- \times Y, X)$ which was defined in Proposition 4.1.21 as

$$X^Y := \widehat{\lim_{f \in \mathcal{B} \downarrow Y} \widehat{X}^{\mathrm{DOM}_Y f}}.$$

Its underlying set is $\widehat{\mathcal{B}}(*, X^Y) \cong \widehat{\mathcal{B}}(Y, X) = \mathcal{T}(Y, X)$.

Proposition 4.2.12. *Consider spaces $X, Y \in \widehat{\mathcal{B}}$ and a closed subset $C \subseteq X$. Then the subset $C^Y \subseteq X^Y$ consisting of all those maps $Y \to X$ whose image is in C is a closed subset of X^Y.*

Proof. Consider the counit $\epsilon : X^Y \hat{\times} Y \to X$ of the defining adjunction. It is given by mapping a pair (f, y) to $f(y)$. For any $y \in Y$, let $\epsilon_y : X^Y \to X$ denote the restriction of ϵ to $X^Y \times \{y\}$. Let C_y denote the preimage $\epsilon_y^{-1}C$ which is a closed subset of X^Y. Since C^X is the intersection $\cap_{y \in Y} C_y$, it is closed in X^Y. □

4.2.5 Closed and open subspaces.

In certain cases, $\mathcal{B}$ 'generates its closed or open subspaces' (defined below). We record some consequences of these properties here.

Definition 4.2.13. A *closed subspace* of a space X is a closed subset of X with the subspace topology. Say that $\mathcal{B}$ *generates its closed subspaces* if every closed subspace of every space of $\mathcal{B}$ is in $\widehat{\mathcal{B}}$.

Analogously, an *open subspace* of a space X is a open subset of X with the subspace topology. Say that $\mathcal{B}$ *generates its open subspaces* if every open subspace of every space of $\mathcal{B}$ is in $\widehat{\mathcal{B}}$.

Proposition 4.2.14. *$\mathcal{B}$ generates its closed subspaces if and only if $\widehat{\mathcal{B}}$ generates its closed subspaces. Analogously, $\mathcal{B}$ generates its open subspaces if and only if $\widehat{\mathcal{B}}$ generates its open subspaces.*

Proof. It is clear that if $\widehat{\mathcal{B}}$ generates its closed subspaces, then $\mathcal{B}$ generates its closed subspaces. We show the other direction.

Suppose that $\mathcal{B}$ generates its closed subspaces.

Consider a closed subspace C of a space X in $\widehat{\mathcal{B}}$. We want to show that C is in $\widehat{\mathcal{B}}$, and to that end we show that any closed subset of $\widehat{C}$ is already closed in C.

Let D be a closed subset of $\widehat{C}$. For every $f : B \to X$ in $\mathcal{B} \downarrow X$, the subset $f^{-1}C$ is closed in B. Let f_C denote the restriction of f to $f^{-1}C \to C$. By hypothesis, $f^{-1}C$ is in $\widehat{\mathcal{B}}$, so we obtain a continuous function $\widehat{f_C} : f^{-1}C \to \widehat{C}$. Then $\widehat{f_C}^{-1}D$ must be closed in $f^{-1}C$, and since $f^{-1}C$ is closed in B, we conclude that $f^{-1}D$ is closed in B.

Therefore, D is closed in X, and so is closed in C. We conclude that $\widehat{C} \cong C$.

The proof that $\mathcal{B}$ generates its open subspaces if and only if $\widehat{\mathcal{B}}$ generates its open subspaces is exactly the same as this proof with 'closed' replaced by 'open'. □

Example 4.2.15. Consider the class $\mathcal{K}$ of compact Hausdorff spaces. Since closed subspaces of Hausdorff spaces are Hausdorff, and closed subspaces of compact spaces are compact, $\mathcal{K}$ generates its closed subspaces.

Example 4.2.16. Consider the class $\mathcal{D}$ whose only object is the interval $I = [0, 1]$. We claim that $\mathcal{D}$ generates its open subspaces.

Consider an open subspace U of I. By Lemma 4.2.7, it suffices to show that U is locally path-connected. For this, we need to show that for any open neighborhood V of a point t, there exists a path-connected open neighborhood W of t contained in V. But in this case, the open set V contains an open ball $B_{t,\epsilon}$ around t which is locally path connected.

4.3 The weak Hausdorff reflection.

We again focus on the situation where the ambient category is $\mathcal{T}$ and $\mathcal{B}$ has the properties we required in the previous section. In this section, we construct a reflective subcategory of 'weak Hausdorff' spaces of $\widehat{\mathcal{B}}$ which remains bicomplete and cartesian closed.

4.3.1 The reflection.

Definition 4.3.1. Say that a space $X \in \widehat{\mathcal{B}}$ is *weak Hausdorff* if the image of the diagonal $\Delta_X : X \to X \hat{\times} X$ is a closed set in $X \hat{\times} X$. Let $\widehat{\mathcal{B}}_{\mathsf{H}}$ denote the full subcategory of $\widehat{\mathcal{B}}$ spanned by its weak Hausdorff spaces.

We construct a functor $H : \widehat{\mathcal{B}} \to \widehat{\mathcal{B}}_{\mathsf{H}}$. To do this, for each space X of $\mathcal{B}$, we will take a quotient of X which identifies $\mathrm{Im}\Delta_X$ with a closed subset of $X \hat{\times} X$.

Consider equivalence relations on the space X

$$\begin{tikzcd} X \arrow[r, "\eta"] & \mathrm{R} \arrow[l, shift right=2, "\epsilon_0"'] \arrow[l, shift left=2, "\epsilon_1"] \end{tikzcd}$$

such that ϵ_0 and ϵ_1 are jointly monic. These can be described as monomorphisms

$$f : R \xhookrightarrow{\epsilon_0 \hat{\times} \epsilon_1} X \hat{\times} X$$

which obey the usual axioms for equivalence relations: $(x, x) \in R$, $(x, y) \in R$ implies $(y, x) \in R$, and $(x, y), (y, z) \in R$ implies $(x, z) \in R$ for every $x, y, z \in X$. Say that such an equivalence relation is *closed* if the image of the monomorphism

$$R \xhookrightarrow{\epsilon_0 \hat{\times} \epsilon_1} X \hat{\times} X$$

is a closed subset of $X \hat{\times} X$.

Lemma 4.3.2. For every space X of $\widehat{\mathcal{B}}$, there is a minimal closed equivalence relation $M(X)$ on X, and this induces a functorial relation on $\widehat{\mathcal{B}}$.

$$M : \widehat{\mathcal{B}} \to \widehat{\mathcal{B}}^{\mathfrak{R}}$$

Proof. Let $\mathcal{E}_X$ denote the set of subsets of $X \hat{\times} X$ which are closed equivalence relations on X. Since $X \hat{\times} X$ itself is a closed equivalence relation of X, the set $\mathcal{E}_X$ is nonempty.

Let $m(X)$ denote the intersection of $\mathcal{E}_X$. We claim that this is a closed equivalence relation. First of all, it is a closed subset of $X \hat{\times} X$ as the intersection of such. It is straightforward to check that it is an equivalence relation:

1. It is reflexive: Fix $x \in X$. Since (x, x) is in every element of $\mathcal{E}_X$, it is also in the intersection.

2. It is symmetric: Fix $x, y \in X$. If (x, y) is in $m(X)$, then it is in every element of $\mathcal{E}_X$. Thus, (y, x) is in every element of $\mathcal{E}_X$ and so is in $m(X)$.

3. It is transitive: Fix $x, y, z \in X$. If (x, y) and (y, z) are in $m(X)$, then they are in every element of $\mathcal{E}_X$. Thus, (x, z) is in every element of $\mathcal{E}_X$ and so is in $m(X)$.

Let $M(X) := \widehat{m(X)}$, and let $\mu_X : M(X) \to X \hat{\times} X$ denote the coreflection of the inclusion $m(X) \hookrightarrow X \hat{\times} X$. Let $\mu_0, \mu_1 : M(X) \to X$ denote the composition of μ with each projection to X. Note that $M(X)$ is a closed equivalence relation on X. The image of the diagonal $\Delta : X \to X \hat{\times} X$ falls within $M(X)$, so let $\delta : X \to M(X)$ denote the restriction of Δ.

Now we show that M extends to a functor. Consider a continuous function $f : X \to Y$. The preimage $(f \hat{\times} f)^{-1}m(Y)$ is a closed subset of $X \hat{\times} X$. It is straightforward to check that it is an equivalence relation:

1. It is reflexive: Fix $x \in X$. Since (fx, fx) is in $m(Y)$, then (x, x) is in $(f \hat{\times} f)^{-1}m(Y)$.
2. It is symmetric: Fix $x, y \in X$. If (x, y) is in $(f \hat{\times} f)^{-1}m(Y)$, then (fx, fy) and hence (fy, fx) is in $m(Y)$. Thus (y, x) is in $(f \hat{\times} f)^{-1}m(Y)$.
3. It is transitive: Fix $x, y, z \in X$. If (x, y) and (y, z) are in $(f \hat{\times} f)^{-1}m(Y)$, then (fx, fy) and (fy, fz) and hence (fx, fz) are in $m(Y)$. Thus (x, z) is in $(f \hat{\times} f)^{-1}m(Y)$.

Since $(f \hat{\times} f)^{-1}m(Y)$ is a closed equivalence relation on X, the minimal one $m(X)$ is contained in it. Therefore, the image $(f \hat{\times} f)m(X)$ is contained in $m(Y)$, and so $f \hat{\times} f$ restricts to a continuous function $M(f) : M(X) \to M(Y)$ making the following diagram into a lift of f.

$$\begin{tikzcd} X \arrow[r, "\delta" description] \arrow[d, "f"'] & M(X) \arrow[l, shift right=2ex, "\mu_0"'] \arrow[l, shift left=2ex, "\mu_1"] \arrow[d, "M(f)"] \\ Y \arrow[r, "\delta" description] & M(Y) \arrow[l, shift right=2ex, "\mu_0"'] \arrow[l, shift left=2ex, "\mu_1"] \end{tikzcd}$$

Therefore, M extends to a functor $\widehat{\mathcal{B}} \to \widehat{\mathcal{B}}^{\mathfrak{R}}$. $\square$

Now let H be the composite of M with the colimit functor.

$$H := \widehat{\mathcal{B}} \xrightarrow{M} \widehat{\mathcal{B}}^{\mathfrak{R}} \xrightarrow{\text{colim}} \widehat{\mathcal{B}}$$

For every X, the space $H(X)$ is the coequalizer of the following diagram.

$$\begin{tikzcd} M(X) \arrow[r, shift left, "\mu_0"] \arrow[r, shift right, "\mu_1"'] & X \end{tikzcd}$$

Let v denote the universal natural epimorphism in the coequalizer cocone.

$$\begin{tikzcd} M(X) \arrow[r, shift left, "\mu_0"] \arrow[r, shift right, "\mu_1"'] & X \arrow[r, two heads, "v_X"] & H(X) \end{tikzcd}$$

Note that $\widehat{\mathcal{B}}$ is closed under colimits in $\mathcal{T}$ (Theorem 4.1.13) so this is also the coequalizer in $\mathcal{T}$. Thus $H(X)$ has the underlying set $X/m(X)$ endowed with the quotient topology.

This quotient is X itself if and only if $m(X)$ is the minimal equivalence relation $\mathrm{Im}\Delta_X$ which is the case if and only if X is weak Hausdorff. Thus, the epimorphism v_X is an isomorphism if and only if X is in $\widehat{\mathcal{B}}_{\mathsf{H}}$.

Now we show that H preserves finite products. For this, we need a lemma.

Lemma 4.3.3. Consider two reflexive coequalizer diagrams $C : \mathfrak{R} \to \mathcal{C}$ and $D : \mathfrak{R} \to \mathcal{C}$ in a cartesian closed category $\mathcal{C}$.

If their colimits exist, there is an isomorphism

$$\mathrm{colim}\ C \times \mathrm{colim}\ D \cong \mathrm{colim}(C \times D)$$

where $C \times D : \mathfrak{R} \to \mathcal{C}$ is the diagram which maps an object r in $\mathfrak{R}$ to $C(r) \times D(r)$ in $\mathcal{C}$.

Proof. Because $\mathcal{C}$ is cartesian closed, the product preserves colimits in each variable. Thus, there is the following chain of isomorphisms.

$$\begin{aligned}\mathrm{colim}_{c\in\mathfrak{R}} Cc \times \mathrm{colim}_{d\in\mathfrak{R}} Dd &\cong \mathrm{colim}_{d\in\mathfrak{R}}(\mathrm{colim}_{c\in\mathfrak{R}} Cc \times Dd)\\ &\cong \mathrm{colim}_{d\in\mathfrak{R}}\mathrm{colim}_{c\in\mathfrak{R}}(Cc \times Dd)\\ &\cong \mathrm{colim}_{(c,d)\in\mathfrak{R}\times\mathfrak{R}} Cc \times Dd.\end{aligned}$$

Thus, it remains to be seen that the colimit of the diagram $C \times D : \mathfrak{R} \times \mathfrak{R} \to \mathcal{C}$ which maps $(c, d) \in \mathfrak{R} \times \mathfrak{R}$ to $Cc \times Dd$ is isomorphic to the colimit of the diagram $C \times D : \mathfrak{R} \to \mathcal{C}$ which maps a $c \in \mathfrak{R}$ to $Cc \times Dc$.

Let $\delta : \mathfrak{R} \to \mathfrak{R} \times \mathfrak{R}$ denote the diagonal embedding. Then we have the following commutative diagram.

$$\begin{tikzcd} \mathfrak{R} \arrow[rr, "\delta"] \arrow[dr, "C\times D"'] & & \mathfrak{R}\times\mathfrak{R} \arrow[dl, "C\times D"] \\ & \mathcal{C} & \end{tikzcd}$$

We claim that δ is a final functor.

The category $\mathfrak{R}$ is generated by the graph

$$\begin{tikzcd} \bigcirc \arrow[r, "\eta" description] & \Psi \arrow[l, shift right=2ex, "\epsilon"'] \arrow[l, shift left=2ex, "\zeta"] \end{tikzcd}$$

and the relations $\epsilon\eta = \zeta\eta = 1_{\bigcirc}$. Thus, the category $\mathfrak{R} \times \mathfrak{R}$ is generated by the graph

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\Psi\bigcirc \arrow[r, "1_\Psi\eta" description] \arrow[d, shift right=3ex, "\epsilon 1_{\bigcirc}"'] \arrow[d, shift left=3ex, "\zeta 1_{\bigcirc}"] & \Psi\Psi \arrow[l, shift right=2ex, "1_\Psi\epsilon"'] \arrow[l, shift left=2ex, "1_\Psi\zeta"] \arrow[d, shift right=3ex, "\epsilon 1_\Psi"'] \arrow[d, shift left=3ex, "\zeta 1_\Psi"] \arrow[dl, shift right=2ex, "\epsilon\epsilon"'] \arrow[dl, shift left=2ex, "\zeta\zeta"] \\
\bigcirc\bigcirc \arrow[u, "\eta 1_{\bigcirc}" description] \arrow[r, "1_{\bigcirc}\eta" description] \arrow[ur, "\eta\eta" description] & \bigcirc\Psi \arrow[u, "\eta 1_\Psi" description] \arrow[l, shift right=2ex, "1_{\bigcirc}\epsilon"'] \arrow[l, shift left=2ex, "1_{\bigcirc}\zeta"]
\end{tikzcd}$$

and the appropriate relations.

The comma category $(\bigcirc\bigcirc \downarrow \delta)$ is isomorphic to the slice $\bigcirc/\mathfrak{R}$ which has an initial object, $1_{\bigcirc}$. Thus, this comma category and, dually, $(\Psi\Psi \downarrow \delta)$ are nonempty and connected.

The slice $(\Psi\bigcirc \downarrow \delta)$ is generated by the graph

$$\begin{tikzcd} \epsilon 1_{\bigcirc} & 1_\Psi\eta \arrow[l, shift right=0.5ex, "\epsilon\zeta"'] \arrow[l, shift left=0.5ex, "\epsilon\epsilon"] \arrow[r, shift left=0.5ex, "\zeta\epsilon"] \arrow[r, shift right=0.5ex, "\zeta\zeta"'] & \zeta 1_{\bigcirc} \end{tikzcd}$$

so it is nonempty and connected.

The slice $(\bigcirc\Psi \downarrow \delta)$ is isomorphic to $(\Psi\bigcirc \downarrow \delta)$ via the twist functor $\mathfrak{R} \times \mathfrak{R} \to \mathfrak{R} \times \mathfrak{R}$ so it is also nonempty and connected.

Therefore, δ is a final functor, and by [ML98, IX.3, Thm. 1], we can conclude that it induces the desired isomorphism. □

Proposition 4.3.4. *The reflector H preserves all finite products.*

Proof. We prove that H preserves the terminal object and binary products.

Consider $*$, the one point space. It is weak Hausdorff, so it is in $\widehat{\mathcal{B}}_{\mathsf{H}}$, and $H(*) = *$. It is the terminal object of both $\widehat{\mathcal{B}}$ and $\widehat{\mathcal{B}}_{\mathsf{H}}$, so H preserves the terminal object.

Now consider objects X, Y in $\widehat{\mathcal{B}}$. We claim that the subset $m(X) \times m(Y)$ of $(X \hat{\times} Y)^2$ is the minimal closed equivalence relation $m(X \hat{\times} Y)$ on $X \hat{\times} Y$.

For any $x \in X$, let ι_x denote the inclusion

$$Y^2 \cong (\{x\} \hat{\times} Y)^2 \hookrightarrow (X \hat{\times} Y)^2.$$

Consider the preimage $\iota_x^{-1} m(X \hat{\times} Y)$. This is closed in Y^2, and it is straightforward to check that it is an equivalence relation on Y. Therefore, it must contain $m(Y)$. Similarly, for any $y \in Y$, we define $\iota_y : X^2 \hookrightarrow (X \hat{\times} Y)^2$ and find that $m(X) \subseteq \iota_y^{-1} m(X \hat{\times} Y)$.

This means that for every $x_1 \sim x_2 \in m(X)$ and $y_1 \sim y_2 \in m(Y)$, we have

$$(x_1, y_1) \sim (x_1, y_2) \in m(X \hat{\times} Y)$$
$$(x_1, y_2) \sim (x_2, y_2) \in m(X \hat{\times} Y)$$

and by transitivity, we find that

$$(x_1, y_1) \sim (x_2, y_2) \in m(X \hat{\times} Y).$$

Therefore, $m(X \hat{\times} Y) = m(X) \times m(Y)$.

By the preceding lemma, 4.3.3, we see that the product of the coequalizers $H(X) \hat{\times} H(Y)$ is isomorphic to the coequalizer of the following diagram.

$$M(X) \hat{\times} M(Y) \underset{\mu_1}{\overset{\mu_0}{\rightrightarrows}} X \hat{\times} Y$$

But we have just shown that $M(X) \hat{\times} M(Y) \cong M(X \hat{\times} Y)$ as subobjects of $X \hat{\times} Y$, so this is the coequalizer of the following diagram

$$M(X \hat{\times} Y) \underset{\mu_1}{\overset{\mu_0}{\rightrightarrows}} X \hat{\times} Y$$

which is $H(X \hat{\times} Y)$.

Therefore, $H(X) \hat{\times} H(Y) \cong H(X \hat{\times} Y)$. □

Proposition 4.3.5. *For every space X of $\widehat{\mathcal{B}}$, the space $H(X)$ is weak Hausdorff.*

Proof. We need to show that the image of the diagonal $\Delta_{HX} : H(X) \to H(X) \hat{\times} H(X)$ is closed.

By Lemma 4.3.3 and Proposition 4.3.4, we have $H(X)\hat{\times}H(X) \cong H(X\hat{\times}X)$ and the following commutative diagram.

$$\begin{tikzcd}
& H(X)\hat{\times}H(X) \arrow[dd, "\cong"] & \\
H(X) \arrow[ur, "\Delta"] \arrow[dr, "H\Delta"'] & & X\hat{\times}X \arrow[ul, "v\times v"'] \arrow[dl, "v"] \\
& H(X\hat{\times}X) &
\end{tikzcd}$$

Thus, $\mathrm{Im}\Delta$ is closed in $H(X)\hat{\times}H(X)$ if and only if $v^{-1}\mathrm{Im}H\Delta$ is closed in $X\hat{\times}X$ if and only if $(v \times v)^{-1}\mathrm{Im}\Delta$ is closed in $X\hat{\times}X$. But this last subset is

$$\{(x, y) \in X\hat{\times}X \ |vx = vy\} = m(X)$$

which is closed in $X\hat{\times}X$. We conclude that $\mathrm{Im}\Delta_{HX}$ is closed in $H(X)\hat{\times}H(X)$. □

Corollary 4.3.6. *The functor H is idempotent.*

Proof. Since $m(X) = \mathrm{Im}\Delta_X$ just when X is weak Hausdorff, we have $H(X) \cong X$ just in this case. Then by the preceding theorem, $H^2(X) \cong H(X)$. □

Proposition 4.3.7. *H preserves exponentials of the objects of $\widehat{\mathcal{B}}_{\mathsf{H}}$.*

Proof. For any $Z \in \widehat{\mathcal{B}}$, we have the following isomorphisms.

$$\begin{aligned}
\widehat{\mathcal{B}}(Z, X^Y\hat{\times}X^Y) &\cong \widehat{\mathcal{B}}(Z, X^Y) \times \widehat{\mathcal{B}}(Z, X^Y) \\
&\cong \widehat{\mathcal{B}}(Z\hat{\times}Y, X) \times \widehat{\mathcal{B}}(Z\hat{\times}Y, X) \\
&\cong \widehat{\mathcal{B}}(Z\hat{\times}Y, X\hat{\times}X) \\
&\cong \widehat{\mathcal{B}}(Z, (X\hat{\times}X)^Y)
\end{aligned}$$

Using the Yoneda lemma, we then see a natural isomorphism $X^Y\hat{\times}X^Y \cong (X\hat{\times}X)^Y$ under which the subset $\mathrm{Im}(\Delta_{X^Y})$ corresponds to $\mathrm{Im}(\Delta_X)^Y$.

If X is weak Hausdorff, then $\mathrm{Im}(\Delta_X)$ is closed in $X\hat{\times}X$. Thus by Proposition 4.2.12, $\mathrm{Im}(\Delta_X)^Y$ is a closed subset of $(X\hat{\times}X)^Y$. Therefore, $\mathrm{Im}(\Delta_{X^Y})$ is closed in $X^Y\hat{\times}X^Y$, and we see that X^Y is weak Hausdorff. □

Corollary 4.3.8. *$\widehat{\mathcal{B}}_{\mathsf{H}}$ is cartesian closed.*

Proof. Since H preserves finite products (Proposition 4.3.4) and exponentials (Proposition 4.3.7), $\widehat{\mathcal{B}}_{\mathsf{H}}$ inherits exponentials from $\widehat{\mathcal{B}}$. □

Proposition 4.3.9. *The functor $H : \widehat{\mathcal{B}} \to \widehat{\mathcal{B}}$ is an idempotent monad with unit v and multiplication Hv. The full subcategory $\widehat{\mathcal{B}}_{\mathsf{H}}$ is isomorphic to the Eilenberg-Moore category of algebras of H.*

This generates an adjunction

$$\begin{tikzcd} \widehat{\mathcal{B}}_{\mathsf{H}} \arrow[r, hook, shift left] \arrow[r, phantom, "\top"] & \widehat{\mathcal{B}} \arrow[l, shift left, "H"] \end{tikzcd}$$

which displays $\widehat{\mathcal{B}}_{\mathsf{H}}$ as a reflective subcategory of $\widehat{\mathcal{B}}$.

Moreover, $\widehat{\mathcal{B}}_{\mathsf{H}}$ is closed under limits of $\widehat{\mathcal{B}}$, and H preserves colimits.

Proof. This follows from the dual of the results in section 4.1.2. □

4.3.2 The topology of pullbacks.

Proposition 4.3.10. *For any $X, Y \in \widehat{\mathcal{B}}$, $Z \in \widehat{\mathcal{B}}_{\mathsf{H}}$, and morphisms $f : X \to Z$ and $g : Y \to Z$, the canonical inclusion of the pullback $X \hat{\times}_Z Y$ into $X \hat{\times} Y$ has a closed image.*

Proof. The pullback can also be obtained as the preimage of $\mathrm{Im}\Delta_Z$ under $f \hat{\times} g$.

$$\begin{tikzcd} X \hat{\times}_Z Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Z \arrow[d, "\Delta"] \\ X \hat{\times} Y \arrow[r, "f \hat{\times} g"] & Z \hat{\times} Z \end{tikzcd}$$

Since $\mathrm{Im}\Delta_Z$ is a closed subset of $Z \hat{\times} Z$, then $X \hat{\times}_Z Y$ is a closed subset of $X \hat{\times} Y$.
□

Corollary 4.3.11. *Suppose that $\widehat{\mathcal{B}}$ generates its closed subspaces. Then for any $X, Y, Z \in \widehat{\mathcal{B}}$ such that $Z \in \widehat{\mathcal{B}}_{\mathsf{H}}$, and any morphisms $f : X \to Z$ and $g : Y \to Z$, the pullback $X \hat{\times}_Z Y$ has the subspace topology as a subspace of $X \hat{\times} Y$.*

Proof. Since $X \times_Z Y$ is a closed subset of $X \hat{\times} Y$ by the preceding proposition, it is in $\widehat{\mathcal{B}}$ with the subspace topology. □

Proposition 4.3.12. *Consider an open map $g : Y \to Z$ such that for any $f : X \to Z$, the pullback $X \hat{\times}_Z Y$ has the subspace topology as a subspace of $X \hat{\times} Y$. Then any pullback of g in $\widehat{\mathcal{B}}_{\mathsf{H}}$ is open.*

Proof. Consider such a pullback square in $\widehat{\mathcal{B}}_{\mathsf{H}}$.

$$\begin{tikzcd} X \hat{\times}_Z Y \arrow[r] \arrow[d, "\pi_X"] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & Z \end{tikzcd}$$

We need to show that for any open set U in $X \hat{\times}_Z Y$, the image $\pi_X U$ is open in X. It suffices to show that for any $w : W \to X$ in $\mathcal{B} \downarrow X$, the subset $w^{-1}\pi_X U$ is open in W.

$$\begin{tikzcd} W \hat{\times}_Z Y \arrow[r, "\pi_{X \times Y}"] \arrow[d, "\pi_W"] \arrow[dr, phantom, "\lrcorner", very near start] & X \hat{\times}_Z Y \arrow[r] \arrow[d, "\pi_X"] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "g"] \\ W \arrow[r, "w"] & X \arrow[r, "f"] & Z \end{tikzcd}$$

Since Beck-Chevalley holds for the left-hand pullback square above, we have that $w^{-1}\pi_X U = \pi_W \pi^{-1}_{X \times_Z Y} U$. Since $W \hat{\times} Y \cong W \times Y$ by Proposition 4.1.22 and $W \hat{\times}_Z Y$ has the subspace topology of $W \hat{\times} Y$ by hypothesis, $W \hat{\times}_Z Y \cong W \times_Z Y$. Since pullbacks of open maps are open in $\mathcal{T}$, the projection π_W in particular is open. Thus $w^{-1}\pi_X U = \pi_W \pi^{-1}_{X \times_Z Y} U$ is open. $\square$

Corollary 4.3.13. *Suppose that $\widehat{\mathcal{B}}$ generates its closed subspaces. Then the pullback of any open map in $\widehat{\mathcal{B}}_{\mathsf{H}}$ is open.*

Proof. This follows immediately from Corollary 4.3.11 and Proposition 4.3.12. $\square$

4.3.3 Pushouts.

Here, we record the calculation of some pushouts for future reference. Let $\hat{\cup}$ denote pushouts of inclusions in $\widehat{\mathcal{B}}_{\mathsf{H}}$ and $\cup$ denote pushouts of inclusions in $\mathcal{T}$.

Proposition 4.3.14. *Consider a space X in $\widehat{\mathcal{B}}_{\mathsf{H}}$ and a subset $Y \subset X$. The pushouts $X \hat{\cup}_{\hat{Y}} X$ and $X \hat{\cup}_{\hat{\overline{Y}}} X$ are isomorphic.*

Proof. Let $i : \hat{Y} \hookrightarrow X$ denote the inclusion of Y into X, and let $\overline{i} : \hat{\overline{Y}} \hookrightarrow X$ denote the inclusion of the closure of Y into X. Let P and $\overline{P}$ represent the following diagrams in $\widehat{\mathcal{B}}_{\mathsf{H}}$.

$$\begin{tikzcd} \widehat{Y} \arrow[r, "i"] \arrow[d, "i"] & X \\ X & \end{tikzcd} \qquad \begin{tikzcd} \widehat{\overline{Y}} \arrow[r, "\bar{i}"] \arrow[d, "\bar{i}"] & X \\ X & \end{tikzcd}$$

$$P \qquad\qquad \overline{P}$$

For any $B \in \widehat{\mathcal{B}}_{\mathsf{H}}$, let $B!$ denote the constant diagram at B.

There is a function $\mathrm{Nat}(\overline{P}, B!) \to \mathrm{Nat}(P, B!)$ induced by the inclusion $P \hookrightarrow \overline{P}$. Now we show that this has an inverse $\mathrm{Nat}(P, B!) \to \mathrm{Nat}(\overline{P}, B!)$.

Consider an element $f \in \mathrm{Nat}(P, B!)$. It is given by two morphisms $f_0, f_1 : X \to Z$ such that $f_0 i = f_1 i$, or, equivalently, such that the preimage of the diagonal $(f_0 i \times f_1 i)^{-1}\mathrm{Im}\Delta_Z$ includes the diagonal $\mathrm{Im}\Delta_Y$ in $Y \times Y$. Then the preimage of the diagonal $(f_0\bar{i} \times f_1\bar{i})^{-1}\mathrm{Im}\Delta_Z$ in $\overline{Y} \times \overline{Y}$ is a closed subset containing $\mathrm{Im}\Delta_Y$. Since the closure of $\mathrm{Im}\Delta_Y$ in $\overline{Y} \times \overline{Y}$ is $\mathrm{Im}\Delta_{\overline{Y}}$, this preimage contains $\mathrm{Im}\Delta_{\overline{Y}}$. In other words, $f_0\bar{i} = f_1\bar{i}$.

Therefore, $\mathrm{Nat}(P, B!) \cong \mathrm{Nat}(\overline{P}, B!)$ for any $B \in \widehat{\mathcal{B}}_{\mathsf{H}}$, and we conclude that the colimits of P and $\overline{P}$ are the same. □

Proposition 4.3.15. *Consider a space X in $\widehat{\mathcal{B}}_{\mathsf{H}}$ and a closed subset $Y \subseteq X$. Then the pushout $X \cup_{\widehat{Y}} X$ is in $\widehat{\mathcal{B}}_{\mathsf{H}}$.*

Proof. First note that $X \cup_{\widehat{Y}} X$ is in $\widehat{\mathcal{B}}$ since $\widehat{\mathcal{B}}$ is closed under colimits (Theorem 4.1.13).

Note that $X \cup_{\widehat{Y}} X$ is the coequalizer of the following diagram

$$\widehat{Y} \underset{\iota_1}{\overset{\iota_0}{\rightrightarrows}} X \cup X$$

where ι_0 and ι_1 are the inclusions of $\widehat{Y}$ into each copy of X. Let c denote the universal morphism $X \cup X \to X \cup_{\widehat{Y}} X$.

Since $\widehat{\mathcal{B}}$ is a cartesian closed category, products preserve colimits in each variable. Thus, the product $(X \cup_{\widehat{Y}} X) \hat{\times} (X \cup_{\widehat{Y}} X)$ is the colimit of the following diagram.

$$\begin{tikzcd}
(X \cup X) \hat{\times} \widehat{Y} \arrow[r, shift left, "1 \hat{\times} \iota_0"] \arrow[r, shift right, "1 \hat{\times} \iota_1"'] & (X \cup X) \hat{\times} (X \cup X) \\
\widehat{Y} \hat{\times} \widehat{Y} \arrow[u, shift left, "\iota_0 \hat{\times} 1"] \arrow[u, shift right, "\iota_1 \hat{\times} 1"'] \arrow[ur, shift left, "\iota_0 \hat{\times} \iota_0"] \arrow[ur, shift right, "\iota_1 \hat{\times} \iota_1"'] \arrow[r, shift left, "1 \hat{\times} \iota_0"] \arrow[r, shift right, "1 \hat{\times} \iota_1"'] & \widehat{Y} \hat{\times} (X \cup X) \arrow[u, shift left, "\iota_0 \hat{\times} 1"] \arrow[u, shift right, "\iota_1 \hat{\times} 1"']
\end{tikzcd}$$

In particular, the induced map

$$c \hat{\times} c : (X \cup X) \hat{\times} (X \cup X) \to (X \cup_{\widehat{Y}} X) \hat{\times} (X \cup_{\widehat{Y}} X)$$

is a quotient map.

Consider the image of the diagonal

$$\Delta : X \cup_{\widehat{Y}} X \to (X \cup_{\widehat{Y}} X) \hat{\times} (X \cup_{\widehat{Y}} X).$$

We want to show that it is closed. It is closed if and only if its preimage $(c \hat{\times} c)^{-1} \mathrm{Im}\Delta$ is closed in $(X \cup X) \hat{\times} (X \cup X)$.

Let the superscripts 0 and 1 distinguish between copies of X and its elements in the union $X \cup X = X^0 \cup X^1$. Now the preimage is the subset

$$\begin{aligned}
&\{(x^i, x^j) \in (X^0 \cup X^1) \hat{\times} (X^0 \cup X^1) \mid cx^i = cx^j \} \\
&\cong \{(x^0, x^0) \in X^0 \hat{\times} X^0\} \cup \{(x^1, x^1) \in X^1 \hat{\times} X^1\} \\
&\qquad \cup \{(y^0, y^1) \in X^0 \hat{\times} X^1 \mid y \in Y\} \cup \{(y^1, y^0) \in X^1 \hat{\times} X^0 \mid y \in Y\} \\
&= \mathrm{Im}\Delta_{X^0} \cup \mathrm{Im}\Delta_{X^1} \cup \big((Y^0 \times Y^1) \cap \mathrm{Im}\Delta_X\big) \cup \big((Y^1 \times Y^0) \cap \mathrm{Im}\Delta_X\big).
\end{aligned}$$

Since X is weak Hausdorff, $\mathrm{Im}\Delta_{X^0}$ is closed in each copy of $X \hat{\times} X$. Since Y is closed in X, the product $Y \times Y$ is a closed subset of $X \hat{\times} X$, and so each $Y^i \times Y^{i+1} \cap \Delta_X$ is closed in $X^i \hat{\times} X^{i+1}$. Thus, the preimage is a closed subset.

We conclude that the image of the diagonal in closed in $(X \cup_{\widehat{Y}} X) \hat{\times} (X \cup_{\widehat{Y}} X)$, and $X \cup_{\widehat{Y}} X$ is weak Hausdorff. $\square$

Corollary 4.3.16. *Consider a space X in $\widehat{\mathcal{B}}_{\mathsf{H}}$ and a subspace $Y \subseteq X$. Then there is an isomorphism*

$$X \hat{\cup}_{\widehat{Y}} X \cong X \cup_{\hat{\widehat{Y}}} X.$$

Proof. By the Proposition 4.3.14, we have $X \hat{\cup}_{\hat{Y}} X \cong X \hat{\cup}_{\hat{\hat{Y}}} X$. By Proposition 4.3.15, we have $X \hat{\cup}_{\hat{\hat{Y}}} X \cong X \cup_{\hat{\hat{Y}}} X$. □

4.4 Exponentiable morphisms in the weak Hausdorff reflection.

In this section, we describe exponentiable morphisms in $\widehat{\mathcal{B}}$ and $\widehat{\mathcal{B}}_{\mathsf{H}}$. Recall that a morphism $f : X \to Y$ of a category $\mathcal{C}$ is *exponentiable* if the functor $\mathcal{C}/Y(- \times f, g) : \mathcal{C}/Y \to \mathcal{S}et$ is representable for every $g \in \mathcal{C}/Y$: i.e., if it is exponentiable as an object of $\mathcal{C}/Y$.

4.4.1 The Sierpinski space

Let S denote the Sierpinski space. Its underlying set is $\{0, \epsilon\}$, and its open sets are $\varnothing, \{\epsilon\}, \{0, \epsilon\}$.

There is a bijection

$$\mathcal{T}(X, S) \cong \{C \subseteq X \mid C \text{ is closed}\}$$

which takes a continuous function $f : X \to S$ to $f^{-1}0$. Then there is also a bijection

$$\widehat{\mathcal{B}}(X, \widehat{S}) \cong \{C \subseteq X \mid C \text{ is closed}\}$$

which arises from the adjunction associated to $\widehat{\mathcal{B}} \hookrightarrow \mathcal{T}$.

Lemma 4.4.1. Suppose that there is some space X of $\mathcal{B}$ whose open and closed sets do not coincide. Then the Sierpinski space is in $\widehat{\mathcal{B}}$ but not $\widehat{\mathcal{B}}_{\mathsf{H}}$.

Proof. Consider the topology given to S by $\widehat{\ }$ as described in Proposition 4.2.1. It is a refinement of the topology of S, so either it is that of S or it adds $\{0\}$ to the open sets. For it to add $\{0\}$ to the open sets, $f^{-1}0$ would have to be open for every f in $\mathcal{B} \downarrow S$. But for $f^{-1}0$ to be open for every $f : Y \to S$, every closed set in every Y must be open. Since we hypothesize that this is not the case, $\{0\}$ cannot be open in $\widehat{S}$, and so $\widehat{S} \cong S$.

To see that S is not in $\widehat{\mathcal{B}}_{\mathsf{H}}$, consider the product $S \times S$. It has the underlying set

$$\{(0,0),(0,\epsilon),(\epsilon,0),(\epsilon,\epsilon)\},$$

and its only nontrivial open set is $\{(\epsilon,\epsilon)\}$. If S were in $\widehat{\mathcal{B}}_{\mathsf{H}}$, then the diagonal $\{(0,0),(\epsilon,\epsilon)\}$ would have to be closed in $\widehat{S \times S}$. Consider a closed, non-open subset of a space X of $\widehat{\mathcal{B}}$. Let $f : X \to S \times S$ map C to $(0,\epsilon)$ and C^c to (ϵ,ϵ). We see that $f^{-1}\{(0,0),(\epsilon,\epsilon)\} = C^c$ is not closed, so $\{(0,0),(\epsilon,\epsilon)\}$ cannot be open in $\widehat{S \times S}$. Therefore, S is not weak Hausdorff. $\square$

Corollary 4.4.2. *The following are equivalent.*

1. *S is not in $\widehat{\mathcal{B}}$.*
2. *$\widehat{S}$ is discrete.*
3. *The open and closed sets of each space in $\widehat{\mathcal{B}}$ coincide.*

Proof. We saw in the proof above that (1) implies (3).

To see that (3) implies (2), consider the space $\widehat{S}$ whose topology is the same or stronger than S. Since $\{\epsilon\}$ is open in $\widehat{S}$, it must also be closed if (3) holds. Then $\widehat{S}$ is discrete.

If (2) holds, then $\widehat{S} \ncong S$ so S is not in $\widehat{\mathcal{B}}$. Thus, (2) implies (1). $\square$

Now for any $X \in \widehat{\mathcal{B}}$, let $\widetilde{X}$ be defined as the following pushout in $\widehat{\mathcal{B}}$.

$$\begin{tikzcd}
X \cong X \hat{\times} \{\epsilon\} \arrow[r, hook] \arrow[d] & X \hat{\times} \widehat{S} \arrow[d] \\
* \arrow[r] & \widetilde{X} \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

Lemma 4.4.3. Suppose that S is in $\widehat{\mathcal{B}}$.

Then the space $\widetilde{X}$ has the underlying set $X \cup \{\epsilon\}$. The space X is a closed subspace of $\widetilde{X}$, and the nontrivial open sets of $\widetilde{X}$ are all those $U \cup \{\epsilon\}$ such that U is an open subset of X.

Proof. First of all, since $\widehat{\mathcal{B}}$ is closed under colimits of $\mathcal{T}$, the pushout square in $\widehat{\mathcal{B}}$ defining $\widetilde{X}$ is also a pushout square in $\mathcal{T}$. Since the underlying set functor

$U : \mathcal{T} \to \mathcal{S}et$ preserves colimits and limits, we see that the underlying set of $\widetilde{X}$ is the following pushout

$$\begin{tikzcd} U(X) \times \{\epsilon\} \arrow[r, hook] \arrow[d] & UX \times US \arrow[d] \\ * \arrow[r] & U\widetilde{X} \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

And calculating this pushout, we see that $U\widetilde{X} \cong X \cup \{\epsilon\}$.

The open sets of $\widetilde{X}$ are those $V \subseteq \widetilde{X}$ such that the preimage of V under the projection $\pi : X \hat{\times} S \to \widetilde{X}$ is open.

Consider subsets of $\widetilde{X}$ which contain ϵ. They are of the form $U \cup \{\epsilon\}$ where $U \subseteq X$. Their preimage under π is

$$(U \times S) \cup (X \times \{\epsilon\}) \tag{$*$}$$

If U is open in X, then ($*$) is open in $X \times S$, so it is also open in $\widehat{X \times S} \cong X \hat{\times} S$. Conversely, if ($*$) is open in $X \hat{\times} S$, then the preimage of ($*$) under the inclusion $X \cong X \hat{\times} \{0\} \hookrightarrow X \hat{\times} \widehat{S}$, which is U, is also open. Therefore, a subset of the form $U \cup \{\epsilon\}$ (where $U \subseteq X$) is open in $\widetilde{X}$ if and only if U is open in X.

Now consider subsets of $\tilde{X}$ which do not contain ϵ. They are of the form U where $U \subseteq X$. Then the preimage $\pi^{-1}U$ is $U \times \{0\}$. If this were open in $X \hat{\times} \widehat{S}$, then $\{0\}$ would be open in $\widehat{S}$ (since projections are open maps, Proposition 4.2.9). Then $\widehat{S}$ would be discrete, a contradiction by Corollary 4.4.2. $\square$

Now consider a morphism $f : Y \to \widetilde{X}$ in $\widehat{\mathcal{B}}$. There is a bijection

$$\widehat{\mathcal{B}}(Y, \widetilde{X}) \cong \{f : C \to X \mid C \text{ is closed in } X\}$$

where each C has the subspace topology. (Note that C might not be in $\widehat{\mathcal{B}}$, but if $\mathcal{B}$ generates its closed subspaces, it will be by Proposition 4.2.14.) In one direction, this bijection sends $g : Y \to \widetilde{X}$ to its restriction to $g^{-1}X$. In the other direction, it sends a $f : C \to X$ to $f' : Y \to \widetilde{X}$ which coincides with f on C and sends C^c to ϵ.

4.4.2 The representing morphism.

In this section, we follow [Lew85] closely in order to generalize it.

Consider morphisms $p : X \to Z$ and $q : Y \to Z$ in $\widehat{\mathcal{B}}$ where $Z \in \widehat{\mathcal{B}}_{\mathsf{H}}$.

Let $G(q) \subset Y \hat{\times} Z$ denote the graph of q. It is the preimage of Δ_Z under the map $q \times 1_Z : Y \hat{\times} Z \to Z \hat{\times} Z$.

$$\begin{tikzcd} G(q) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Z \arrow[d, "\Delta"] \\ Y \hat{\times} Z \arrow[r, "q \times 1"] & Z \hat{\times} Z \end{tikzcd}$$

Since Z is weak Hausdorff, the subset $\mathrm{Im}\Delta_Z \subset Z \hat{\times} Z$ and thus its preimage $G(q) \subset Y \hat{\times} Z$ are closed. Thus, the projection $G(q) \hookrightarrow Y \hat{\times} Z \xrightarrow{\pi_Z} Z$ corresponds to a continuous function $g_q : Y \hat{\times} Z \to \tilde{Z}$. We take its adjunct $\overline{g_q} : Z \to \tilde{Z}^Y$. This function maps $z \in Z$ to the function $\overline{g_q}(z) : Y \to \tilde{Z}$ which sends $q^{-1}(z)$ to z and everything else to ϵ.

Now define p^q to be the following pullback in $\widehat{\mathcal{B}}$.

$$\begin{tikzcd} X^q \arrow[r] \arrow[d, "p^q"'] \arrow[dr, phantom, "\lrcorner", very near start] & \tilde{X}^Y \arrow[d, "\tilde{p}^Y"] \\ Z \arrow[r, "\overline{g_q}"] & \tilde{Z}^Y \end{tikzcd}$$

Proposition 4.4.4 ([Day72, Thm. 3.4])**.** *Suppose that for every object $z : A \to Z$ in $\widehat{\mathcal{B}}/Z$, the pullback $A \hat{\times}_Z Y$ has the subspace topology as a subspace of $A \hat{\times} Y$. Then the object p^q in $\widehat{\mathcal{B}}/Z$ represents the functor $\widehat{\mathcal{B}}/Z(- \times q, p) : \widehat{\mathcal{B}}/Z \to \mathcal{S}et$.*

Proof. Consider a morphism $a : A \to Z$ in $\widehat{\mathcal{B}}$. Then we see the following isomorphisms.

$$\begin{aligned} \widehat{\mathcal{B}}/Z(a, p^q) &\cong \{f : A \to \tilde{X}^Y \mid \tilde{p}^Y f = \overline{g_q} a\} \\ &\cong \{f : A \hat{\times} Y \to \tilde{X} \mid \tilde{p} f = g_q(a \times 1_Y)\} \\ &\cong \widehat{\mathcal{B}}/Z(a \times q, p) \end{aligned}$$

The first isomorphism above follows from the definition of p^q as a pullback. The second follows from the adjunction $(-) \hat{\times} Y \dashv (-)^Y$.

To see the third, recall that functions $f : A \hat{\times} Y \to \tilde{X}$ are in bijection with functions $f|_{f^{-1}X} : f^{-1}X \to X$ (where $f^{-1}X$ has the subspace topology and might not be in $\widehat{\mathcal{B}}$). Since the following diagram commutes,

$$\begin{tikzcd} A\hat{\times}Y \arrow[r, "f"] \arrow[d, "a\times 1"] & \tilde{X} \arrow[d, "\tilde{p}"] \\ Z\hat{\times}Y \arrow[r, "g_q"] & \tilde{Z} \end{tikzcd}$$

we see that $f^{-1}X = (\tilde{p}f)^{-1}Z = (g_q(a \times 1))^{-1}Z$. A point $(\alpha, y) \in A\hat{\times}Y$ is mapped into $Z \subset \tilde{Z}$ by $g_q(a \times 1)$ if and only if $a(\alpha) = q(y)$ so the preimage $f^{-1}X$ is the subset $A \times_Z Y$. Then we have

$$\left\{ f : A\hat{\times}Y \to \tilde{X} \;\middle|\; \tilde{p}f = g_q(a \times 1_Y) \right\} \cong \left\{ f : A \times_Z Y \to X \;\middle|\; pf = \pi_Z \right\}$$

where the $A \times_Z Y$ above has the subspace (of $A\hat{\times}Y$) topology. Since we hypothesized that this space is $A\hat{\times}_Z Y$, we see that the above is isomorphic to

$$\left\{ f : A\hat{\times}_Z Y \to X \;\middle|\; pf = \pi_Z \right\},$$

and this is $\widehat{\mathcal{B}}/Z(a \times q, p)$. □

Proposition 4.4.5. *Suppose that for every $z : A \to Z$ in $\widehat{\mathcal{B}}/Z$, the pullback $A\hat{\times}_Z Y$ has the subspace topology as a subspace of $A\hat{\times}Y$. If X is in $\widehat{\mathcal{B}}_{\mathsf{H}}$, and q is an open map, then p^q is in $\widehat{\mathcal{B}}_{\mathsf{H}}$.*

Proof. In what follows, let $\textsc{dom} : \widehat{\mathcal{B}}/Z \to \widehat{\mathcal{B}}$ denote the domain projection.

We must show that the domain X^q of p^q is in $\widehat{\mathcal{B}}_{\mathsf{H}}$.

Let $\epsilon : (-)^q \times q \to (-)$ denote the counit of the adjunction $(-) \times q \dashv (-)^q$, and let e denote $\textsc{dom}\epsilon$. We will consider the component $\epsilon_{p\times p} : (p \times p)^q \times q \to p \times p$ which is illustrated below.

$$\begin{tikzcd} (X \times_Z X)^q \times_Z Y \arrow[rr, "e_{p\times p}"] \arrow[dr, "(p\times p)^q \times q"'] & & X \times_Z X \arrow[dl, "p\times p"] \\ & Z & \end{tikzcd}$$

Now consider the complement U of the image of the diagonal Δ_X in $X \times_Z X = \textsc{dom}(p \times p)$. Since X is weak Hausdorff, it is open. Then the preimage $e^{-1}_{p\times p}U$ is the open subset

$$\left\{z \in Z,\ f : q^{-1}z \to (p \times p)^{-1}z,\ y \in q^{-1}z \;\middle|\; \pi_0 f(y) \neq \pi_1 f(y)\right\}.$$

Consider also the projection $\pi_{(p\times p)^q} : (p\times p)^q \times q \to (p\times p)^q$ which is illustrated below.

$$\begin{tikzcd}
(X \times_Z X)^q \times_Z Y \arrow[rr, "\pi_{(X\times_Z X)^q}"] \arrow[dr, "(p\times p)^q \times q"'] & & (X \times_Z X)^q \arrow[dl, "p\times p"] \\
& Z &
\end{tikzcd}$$

In $\widehat{\mathcal{B}}$, the morphism $\textsc{dom}\pi_{(p\times p)^q} = \pi_{(X\times_Z X)^q}$ is obtained as a pullback of $q : Y \to Z$ and is therefore open by Proposition 4.3.12. Therefore, it takes the open set $e^{-1}_{p\times p}U$ to an open set which is the following.

$$\begin{aligned}
\pi_{(X\times_Z X)^q} e^{-1}_{p\times p}U &= \left\{z \in Z,\ f : q^{-1}z \to (p \times p)^{-1}z \;\middle|\; \exists y \in q^{-1}z : \pi_0 f(y) \neq \pi_1 f(y)\right\} \\
&= \left\{z \in Z,\ f : q^{-1}z \to (p \times p)^{-1}z \;\middle|\; \pi_0 f \neq \pi_1 f\right\}
\end{aligned}$$

The complement of this set, $\pi_{(X\times_Z X)^q} e^{-1}_{p\times p}U$, in $(X \times_Z X)^q \cong X^q \times_Z X^q$ is the diagonal Δ_{X^q} which is therefore closed in $X^q \times_Z X^q$. Since $X^q \times_Z X^q$ is closed in $X^q \times X^q$ by Proposition 4.3.10, so is the diagonal. Therefore, X^q is weak Hausdorff. $\square$

Proposition 4.4.6. *If the functor $\widehat{\mathcal{B}}_{\mathsf{H}}/Z(- \times q, p) : \widehat{\mathcal{B}}_{\mathsf{H}}/Z \to \widehat{\mathcal{B}}_{\mathsf{H}}/Z$ is representable for every p, then q is an open map.*

Proof. Suppose that q is not open.

If $\widehat{\mathcal{B}}_{\mathsf{H}}/Z(- \times q, p) : \widehat{\mathcal{B}}_{\mathsf{H}}/Z \to \widehat{\mathcal{B}}_{\mathsf{H}}/Z$ were representable for every p, then $- \times q$ would be a left adjoint and would preserve colimits. We show this does not preserve colimits.

Let U be an open set in Y such that qU is not open. Let $(qU)^c$ denote the complement of qU, and $\overline{(qU)^c}$ its closure.

We want to show that pulling back along q does not preserve the pushout $\widehat{\overline{(qU)^c}} \mathbin{\hat{\cup}}_{\widehat{(qU)^c}} \widehat{\overline{(qU)^c}}$.

This pushout is

$$\widehat{\overline{(qU)^c}} \cup_{\widehat{(qU)^c}} \widehat{\overline{(qU)^c}} \;\cong\; \widehat{\overline{(qU)^c}}$$

by Corollary 4.3.16.

Now consider the following pushout.

$$q^*\widehat{\overline{(qU)^c}} \mathbin{\hat{\cup}}_{q^*\widehat{(qU)^c}} q^*\widehat{\overline{(qU)^c}} \tag{$*$}$$

We want to show that this is not $q^*\widehat{\overline{(qU)^c}} = q^{-1}\widehat{\overline{(qU)^c}}$. First note that this pushout $(*)$ is isomorphic to

$$q^{-1}\widehat{\overline{(qU)^c}} \cup_{\widehat{q^{-1}(qU)^c}} q^{-1}\widehat{\overline{(qU)^c}}$$

by Corollary 4.3.16. Since the underlying set functor preserves colimits in $\mathcal{T}$, it suffices to show that $q^{-1}\overline{(qU)^c}$ contains a point not in $\overline{q^{-1}(qU)^c}$.

Since qU is not open, there is a $u \in U$ such that $qu \in \overline{(qU)^c}$. Then $u \in q^{-1}\overline{(qU)^c}$. Since $q^{-1}(qU)^c$ is contained in the closed U^c, the closure $\overline{q^{-1}(qU)^c}$ is as well, and thus $u \notin \overline{q^{-1}(qU)^c}$. Since the underlying set functor $\mathcal{T} \to \mathcal{S}et$ preserves colimits, we see that the pushout

$$q^{-1}\overline{(qU)^c} \cup_{\overline{q^{-1}(qU)^c}} q^{-1}\overline{(qU)^c}$$

is not isomorphic to $q^{-1}\overline{(qU)^c}$.

Therefore, q^* does not preserve this colimit. □

Now we summarize these results in the following theorem.

Theorem 4.4.7. *Let $q : Y \to Z$ be a map in $\widehat{\mathcal{B}}_{\mathsf{H}}$ such that for every $z : A \to Z$ in $\widehat{\mathcal{B}}/Z$, the pullback $A \hat{\times}_Z Y$ has the subspace topology as a subspace of $A \hat{\times} Y$.*

Then q is exponentiable if and only if q is an open map.

Proof. Suppose that q is exponentiable. By Proposition 4.4.6, the morphism q must be open.

Now suppose that q is open. By Proposition 4.4.4, we have an isomorphism

$$\widehat{\mathcal{B}}/Z(z \times q, p) \cong \widehat{\mathcal{B}}/Z(z, p^q)$$

for any $p, z \in \widehat{\mathcal{B}}/Z$. Then by Proposition 4.4.5, we have that p^q is in $\widehat{\mathcal{B}}_{\mathsf{H}}/Z$. By Proposition 4.3.9, $\widehat{\mathcal{B}}_{\mathsf{H}}$ is closed under limits of $\widehat{\mathcal{B}}$, so $z \times q$ is in $\widehat{\mathcal{B}}_{\mathsf{H}}/Z$. Thus since $\widehat{\mathcal{B}}_{\mathsf{H}}/Z$ is a full subcategory of $\widehat{\mathcal{B}}/Z$ the isomorphism above restricts to an isomorphism

$$\widehat{\mathcal{B}}_{\mathsf{H}}/Z(z \times q, p) \cong \widehat{\mathcal{B}}_{\mathsf{H}}/Z(z, p^q)$$

for any $p, z \in \widehat{\mathcal{B}}_{\mathsf{H}}/Z$, and p^q represents $\widehat{\mathcal{B}}_{\mathsf{H}}/Z(- \times q, p)$. □

Corollary 4.4.8. *Suppose that $\widehat{\mathcal{B}}$ (or, equivalently, $\mathcal{B}$) generates its closed subspaces. Let $p : X \to Z$ and $q : Y \to Z$ be maps in $\widehat{\mathcal{B}}_{\mathsf{H}}$.*

Then the functor $\widehat{\mathcal{B}}_{\mathsf{H}}/Z(- \times q, p)$ is representable if and only if q is an open map.

Proof. By Corollary 4.3.11, for every $z : A \to Z$ in $\widehat{\mathcal{B}}/Z$, the pullback $A \hat{\times}_Z Y$ has the subspace topology as a subspace of $A \hat{\times} Y$. Then the preceding theorem applies. □

4.5 Moore relation structures in convenient categories of topological spaces.

In this section, we construct strict Moore relation structures in many of the categories constructed above in this chapter and in the topological topos. By the results of the previous chapter, we will then obtain a construction of type theoretic, algebraic weak factorization systems which generalizes that of the weak factorization system consisting of trivial Hurewicz cofibrations and Hurewicz fibrations in the category of compactly generated weak Hausdorff spaces.

4.5.1 The setting.

We will construct a strict Moore relation structure in any finitely complete category which includes a key fragment of the category $\mathcal{T}$ of topological spaces.

Let $\mathcal{R}$ denote the full subcategory of $\mathcal{T}$ spanned by a terminal object $*$, the real numbers $\mathbb{R}$, the nonnegative real numbers $\mathbb{R}^+$, and the product $\mathbb{R}^+ \times \mathbb{R}^+$.

In this section, we consider any finitely complete category $\mathcal{C}$ for which there is a full embedding $\mathcal{R} \hookrightarrow \mathcal{C}$ which preserves the terminal object, the product $\mathbb{R}^+ \times \mathbb{R}^+$, and the pushout $\mathbb{R} \cong \mathbb{R}^+ \cup_0 \mathbb{R}^+$ and which takes $\mathbb{R}^+$ to an exponentiable object in $\mathcal{C}$.

The examples which we have in mind for $\mathcal{C}$ are closely related to $\mathcal{T}$. Thus, we think of the objects of $\mathcal{C}$ as topological spaces or generalizations of them.

Remark 4.5.1. In the following construction, we will often describe the points $\hom(*, X)$ of an object $X \in \mathcal{C}$, and we will describe a morphism by its function on points. Many of the categories in which this construction will be applied (e.g., subcategories of $\mathcal{T}$) are well-pointed. For these special cases, the results of this section would have much simpler proofs (since it would be much easier to prove that diagrams commute by examining the points). Here we do not assume that $\mathcal{C}$ is well-pointed, but it will be illuminating to have this description.

We make use of the following morphisms of $\mathcal{R}$, and so give them the following names.

$$\begin{aligned}
0 &: * \to \mathbb{R}^+ & \mathsf{add} &: \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+ \\
&\quad * \mapsto 0 & &\quad (r,s) \mapsto r+s \\
\mathsf{min} &: \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+ & \mathsf{add}_+ &: \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}^+ \\
&\quad (s,t) \mapsto \mathsf{min}(s,t) & &\quad (r,s) \mapsto \mathsf{max}(0, r+s) \\
\mathsf{min}_3 &: \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+ & \mathsf{sub} &: \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R} \\
&\quad (r,s,t) \mapsto \mathsf{min}(r, \mathsf{min}(s,t)) & &\quad (r,s) \mapsto r-s \\
\mathsf{mid} &: \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R} & \mathsf{sub}_+ &: \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+ \\
&\quad (r,s,t) \mapsto \mathsf{max}(-r, \mathsf{min}(s,t)) & &\quad (r,s) \mapsto \mathsf{max}(0, s-r)
\end{aligned}$$

Remark 4.5.2. Note that the assumption that $\mathcal{R}$ is a full subcategory of topological spaces is stronger than necessary, since we only make use of the above morphisms, their composites, and those associated to the limits $*$, $\mathbb{R}^+ \times \mathbb{R}^+$ and the colimit $\mathbb{R}^+ \cup_0 \mathbb{R}^+$.

Furthermore, one can see that the morphisms above form a fragment of the totally-ordered group structure on $\mathbb{R}$. Thus, one might describe a category $\mathcal{G}$ axiomatizing (this part of) the structure of an internal totally-ordered group. Then one would expect Moore relation structures to arise from embeddings of $\mathcal{G}$. However, we will not consider such a general theory here.

The rest of this section, 4.5, is devoted to proving the following theorem.

Theorem 4.5.3. *Let $\mathcal{C}$ be a finitely complete category which contains an embedding $R : \mathcal{R} \hookrightarrow \mathcal{C}$ which*

1. *preserves the terminal object, the product $\mathbb{R}^+ \times \mathbb{R}^+$, and the pushout $\mathbb{R}^+ \cup_0 \mathbb{R}^+$ and*

2. *takes $\mathbb{R}^+$ to an exponentiable object in $\mathcal{C}$.*

Then $\mathcal{C}$ is equipped with a functorial relation $\mathbf{\Gamma}(R)$ which is a strict Moore relation system.

Example 4.5.4. Consider any full subcategory $\mathcal{B} \subseteq \mathcal{T}$ such that $\mathcal{B}$ contains only exponentiable objects and generates its own products. Then we claim that the coreflective hull $\widehat{\mathcal{B}}$ satisfies the hypotheses of Theorem 4.5.3 if $\mathbb{R}^+$ is in $\widehat{\mathcal{B}}$.

First of all, since $\mathbb{R}^+$ is in $\widehat{\mathcal{B}}$, we have that $\widehat{\mathcal{D}'} \subseteq \widehat{\mathcal{B}}$ (where $\widehat{\mathcal{D}'}$ is the full subcategory of $\mathcal{T}$ spanned by $\mathbb{R}^+$). Then by Lemma 4.2.8 and Lemma 4.2.7, we see that all locally path-connected metric spaces are in $\widehat{\mathcal{B}}$. This includes $*$, $\mathbb{R}$, $\mathbb{R}^+$, and $\mathbb{R}^+ \times \mathbb{R}^+$, so there is a natural embedding $R : \mathcal{R} \hookrightarrow \widehat{\mathcal{B}}$.

We know that $*$ is the terminal object in $\widehat{\mathcal{B}}$. Since $\mathbb{R}^+ \times \mathbb{R}^+$ is in $\widehat{\mathcal{B}}$, it is the product in $\widehat{\mathcal{B}}$. Since $\widehat{\mathcal{B}}$ is closed under colimits in $\mathcal{T}$, we also have that $\mathbb{R}$ is the pushout $\mathbb{R}^+ \cup_0 \mathbb{R}^+$. Therefore, R preserves these (co)limits.

Example 4.5.5. Consider a $\mathcal{B} \subseteq \mathcal{T}$ satisfying the hypotheses of the previous example, 4.5.4. Then we claim that the weak Hausdorff coreflection $\widehat{\mathcal{B}}_{\mathsf{H}}$ also satisfies the hypotheses of Theorem 4.5.3 if $\mathbb{R}^+$ is in $\widehat{\mathcal{B}}$.

Since $*$, $\mathbb{R}$, $\mathbb{R}^+$, and $\mathbb{R}^+ \times \mathbb{R}^+$ are actually Hausdorff, they are also weak Hausdorff in $\widehat{\mathcal{B}}$, and thus they are in $\widehat{\mathcal{B}}_{\mathsf{H}}$. Therefore, there is a natural embedding $R : \mathcal{R} \hookrightarrow \widehat{\mathcal{B}}$.

Since $\widehat{\mathcal{B}}_{\mathsf{H}}$ is closed under limits taken in $\widehat{\mathcal{B}}$, we see that $*$ is the terminal object and $\mathbb{R}^+ \times \mathbb{R}^+$ is the product of $\mathbb{R}^+$ with itself. Note that $\mathbb{R}^+ \hat{\cup}_0 \mathbb{R}^+ \cong H(\mathbb{R}^+ \cup_0 \mathbb{R}^+) \cong \mathbb{R}$. Therefore R preserves these (co)products.

Example 4.5.6. Consider the topological topos $\mathcal{E}$ ([Joh79]). Let $\mathcal{F}$ denote the full subcategory of $\mathcal{T}$ spanned by sequential spaces. Note that $\mathcal{R}$ is a subcategory of $\mathcal{F}$. There is a full embedding of $\mathcal{F} \hookrightarrow \mathcal{E}$ which preserves all limits (as it is a right adjoint, [Joh79, p. 254]), and which preserves many colimits, including $\mathbb{R} \cong \mathbb{R}^+ \cup_0 \mathbb{R}^+$ ([Joh79, Thm. 6.2]). Therefore, the full embedding of $\mathcal{R}$ into $\mathcal{E}$ preserves the limits $*$ and $\mathbb{R}^+ \times \mathbb{R}^+$ and the colimit $\mathbb{R} \cong \mathbb{R}^+ \cup_0 \mathbb{R}^+$.

This example is of particular interest. Since $\mathcal{E}$ is a topos, it is locally cartesian closed. Then $\mathcal{E}$ with the right class of the weak factorization system $\mathbf{UF\Gamma}(R)$ is a display map category modeling pre-Π types. Since $\mathbf{UF\Gamma}(R)$ is type theoretic, it also models Π types by Proposition 2.6.7. Thus, this is a display map category modeling Σ, Π, and `Id` types.

4.5.2 The functorial relation.

In this subsection, we will define a functorial relation $\mathbf{\Gamma}$ on $\mathcal{C}$ with the following components.

$$1_{\mathcal{C}} \underset{\epsilon_1}{\overset{\epsilon_0}{\leftleftarrows}} \xrightarrow{\eta} \Gamma$$

First consider the object $X^{\mathbb{R}^+}$ for any object $X \in \mathcal{C}$. We think of $X^{\mathbb{R}^+}$ as the space of infinite-length paths in X since its points are morphisms $\mathbb{R}^+ \to X$. We denote the counit of the adjunction defining $X^{\mathbb{R}^+}$ by ev and we denote the unit by h.

Let $\overline{\mathsf{min}} : X^{\mathbb{R}^+} \times \mathbb{R}^+ \to X^{\mathbb{R}^+}$ denote the transpose of the following composition.

$$X^{\mathbb{R}^+} \xrightarrow{X^{\mathsf{min}}} X^{\mathbb{R}^+ \times \mathbb{R}^+}$$

It takes a point (p, r) in $X^{\mathbb{R}^+} \times \mathbb{R}^+$ to the point $p(\mathsf{min}(t, r))$ in $X^{\mathbb{R}^+}$.

Remark 4.5.7. In this section, the letter t will be reserved to denote a variable in $\mathbb{R}^+$. If f is a morphism with domain $\mathbb{R}^+$, we will let $f(t)$ denote $f = \lambda t.f(t)$.

Definition 4.5.8. For any object X of $\mathcal{C}$, let $\Gamma(X)$ denote the subobject of $X^{\mathbb{R}^+} \times \mathbb{R}^+$ obtained as the following equalizer.

$$\Gamma(X) \xrightarrow{i} X^{\mathbb{R}^+} \times \mathbb{R}^+ \underset{\pi_0}{\overset{\overline{\mathsf{min}}}{\rightrightarrows}} X^{\mathbb{R}^+}$$

Call $\Gamma(X)$ the space of *Moore paths* of X.

The underlying set of $\Gamma(X)$ then consists of those points (p, r) such that p is constant on $[r, \infty)$.

Note that Γ is an endofunctor on $\mathcal{C}$.

Lemma 4.5.9. The endofunctor $\Gamma : \mathcal{C} \to \mathcal{C}$ preserves pullbacks.

Proof. The functor $(-)^{\mathbb{R}^+} \times \mathbb{R}^+ : \mathcal{C} \to \mathcal{C}/\mathbb{R}^+$ is a composition of right adjoints, so it preserves pullbacks. The forgetful functor $\mathcal{C}/\mathbb{R}^+ \to \mathcal{C}$ also preserves pullbacks, so the composition $(-)^{\mathbb{R}^+} \times \mathbb{R}^+ : \mathcal{C} \to \mathcal{C}$ does as well. Thus for any pullback $X \times_Z Y$, the following two equalizer diagrams are isomorphic.

$$\begin{tikzcd}
\Gamma(X \times_Z Y) \arrow[r, "i"] \arrow[d, phantom, "\wr\|"] & (X \times_Z Y)^{\mathbb{R}^+} \times \mathbb{R}^+ \arrow[r, shift left, "\overline{\mathsf{min}}"] \arrow[r, shift right, "\pi_0"'] \arrow[d, phantom, "\wr\|"] & (X \times_Z Y)^{\mathbb{R}^+} \arrow[d, phantom, "\wr\|"] \\
? \arrow[r] & (X^{\mathbb{R}^+} \times \mathbb{R}^+) \times_{Z^{\mathbb{R}^+} \times \mathbb{R}^+} (Y^{\mathbb{R}^+} \times \mathbb{R}^+) \arrow[r, shift left, "\overline{\mathsf{min}}"] \arrow[r, shift right, "\pi_0"'] & X^{\mathbb{R}^+} \times_{Z^{\mathbb{R}^+}} Y^{\mathbb{R}^+}
\end{tikzcd}$$

Note that the second equalizer diagram above is an equalizer of pullbacks. Since limits commute, we can compute the pullback object $?$ to be $\Gamma(X) \times_{\Gamma(Z)} \Gamma(Y)$. Therefore, we see that $\Gamma(X \times_Z Y) \cong \Gamma(X) \times_{\Gamma(Z)} \Gamma(Y)$. □

Notation 4.5.10. For any object Y in $\mathcal{C}$, let $! : Y \to *$ denote the map from Y to the terminal object. Then $0! : Y \to \mathbb{R}^+$ will denote the composite

$$Y \xrightarrow{!} * \xrightarrow{0} \mathbb{R}^+.$$

Let $\epsilon_0 : \Gamma(X) \to X$ denote the composite

$$\Gamma(X) \overset{i}{\hookrightarrow} X^{\mathbb{R}^+} \times \mathbb{R}^+ \xrightarrow{X^{\mathbb{R}^+} \times 0!} X^{\mathbb{R}^+} \times \mathbb{R}^+ \xrightarrow{\mathsf{ev}} X,$$

and let $\epsilon_1 : \Gamma(X) \to X$ denote the composite

$$\Gamma(X) \overset{i}{\hookrightarrow} X^{\mathbb{R}^+} \times \mathbb{R}^+ \xrightarrow{\mathsf{ev}} X.$$

Then ϵ_0 maps a point $(p, r) \in \Gamma(X)$ to $p(0) \in X$, and ϵ_1 maps (p, r) to $p(r) \in X$.

We record the following equations for later use.

Lemma 4.5.11. The composite

$$X \xrightarrow{X^! \times 0!} X^{\mathbb{R}^+} \times \mathbb{R}^+ \xrightarrow{\mathsf{ev}} X$$

is the identity on X, and the composite

$$X \times \mathbb{R}^+ \xrightarrow{X^! \times \mathbb{R}^+} X^{\mathbb{R}^+} \times \mathbb{R}^+ \xrightarrow{\mathsf{ev}} X$$

is the projection to X.

Proof. The second composite in the statement is the transpose of $X^! : X \to X^{\mathbb{R}^+}$. Therefore, it equal to π_X.

The first composite above is equal to the following composite,

$$X \xrightarrow{1\times 0!} X \times \mathbb{R}^+ \xrightarrow{X^! \times 1} X^{\mathbb{R}^+} \times \mathbb{R}^+ \xrightarrow{\mathsf{ev}} X,$$

and we have just shown that $\mathsf{ev}(X^! \times 1) = \pi_X$, so this is equal to 1_X. □

Now let $\eta : X \to \Gamma(X)$ denote the morphism induced by the universal property of $\Gamma(X)$ as illustrated below.

$$\begin{tikzcd}
X \arrow[d, dashed, "\eta"'] \arrow[dr, "X^! \times 0!"] \arrow[drr, bend left, "X^!"] & & \\
\Gamma(X) \arrow[r, "i"] & X^{\mathbb{R}^+} \times \mathbb{R}^+ \arrow[r, shift left, "\overline{\mathsf{min}}"] \arrow[r, shift right, "\pi_0"'] & X^{\mathbb{R}^+}
\end{tikzcd} \tag{4.5.12}$$

It takes a point $x \in X$ to the pair $(c(x), 0) \in \Gamma(X)$ where $c(x)$ is the constant path at x.

Lemma 4.5.13. The diagram (4.5.12) displays a cone over the equalizer diagram which defines $\Gamma(X)$. This induces the morphism η.

Proof. We need to show that $\overline{\mathsf{min}}(X^! \times 0!) = X^!$. The transpose of $\overline{\mathsf{min}}(X^! \times 0!)$ is the composite

$$X \times \mathbb{R}^+ \xrightarrow{X^! \times 0! \times 1} X^{\mathbb{R}^+} \times \mathbb{R}^+ \times \mathbb{R}^+ \xrightarrow{1 \times \mathsf{min}} X^{\mathbb{R}^+} \times \mathbb{R}^+ \xrightarrow{\mathsf{ev}} X$$

which becomes

$$X \times \mathbb{R}^+ \xrightarrow{X^! \times 0!} X^{\mathbb{R}^+} \times \mathbb{R}^+ \xrightarrow{\mathsf{ev}} X$$

since $\mathsf{min}(0! \times 1) = 0! : \mathbb{R}^+ \to \mathbb{R}^+$. Then Lemma 4.5.11 tells us that the above composite is π_X, so it is the transpose of $X^!$. Therefore, $\overline{\mathsf{min}}(X^! \times 0!) = X^!$. □

Proposition 4.5.14. *The morphisms $\eta, \epsilon_0, \epsilon_1$ assemble into natural transformations which form a functorial relation $\mathbf{\Gamma}$ on $\mathcal{C}$ which has the following components.*

$$\begin{tikzcd}
1_{\mathcal{C}} \arrow[r, "\eta" description] & \Gamma \arrow[l, shift right=2, "\epsilon_0"'] \arrow[l, shift left=2, "\epsilon_1"]
\end{tikzcd} .$$

Proof. We need to show that $\epsilon_0\eta = 1$ and $\epsilon_1\eta = 1$. Substituting the definitions of ϵ_0 and ϵ_1, we find that these equations are equivalent to

$$\mathsf{ev}(1 \times 0!)i\eta = 1 \qquad \mathsf{ev}i\eta = 1,$$

and then substituting the equation $i\eta = X^! \times 0!$, we find that these two equations are equivalent to $\mathsf{ev}(X^! \times 0!) = 1$ which holds by Lemma 4.5.11. □

4.5.3 The symmetry.

We want to find a natural transformation

$$\iota_X : \Gamma X \to \Gamma X$$

which takes a path to its 'reverse' path. To be precise, it should map a pair (p, r) to the pair (p', r) where the path $p'|_{[0,r]}$ is the reverse of the path $p|_{[0,r]}$. To be more precise, it should map (p, r) to $(p\ \mathsf{max}(0, r - t), r)$.

For any space $X \in \mathcal{C}$, let $\overline{\mathsf{sub}_+} : X^{\mathbb{R}^+} \times \mathbb{R}^+ \to X^{\mathbb{R}^+}$ be the transpose of the composite

$$X^{\mathbb{R}^+} \xrightarrow{X^{\mathsf{sub}_+}} X^{\mathbb{R}^+ \times \mathbb{R}^+}.$$

It takes a pair (p, r) to $p(\mathsf{max}(0, r - t))$.

We will restrict $\overline{\mathsf{sub}_+} \times 1 : X^{\mathbb{R}^+} \times \mathbb{R}^+ \to X^{\mathbb{R}^+} \times \mathbb{R}^+$ to a morphism $\Gamma X \to \Gamma X$ to obtain ι.

Lemma 4.5.15. The composite

$$X^{\mathbb{R}^+} \times \mathbb{R}^+ \xrightarrow{\overline{\mathsf{sub}_+} \times 1} X^{\mathbb{R}^+} \times \mathbb{R}^+ \xrightarrow{\overline{\mathsf{sub}_+} \times 1} X^{\mathbb{R}^+} \times \mathbb{R}^+$$

is equal to $\overline{\mathsf{min}} \times \mathbb{R}^+$ and is an idempotent.

Proof. Consider the morphism

$$\begin{tikzcd} X^{\mathbb{R}^+} \times \mathbb{R}^+ \arrow[rr, "\overline{\mathsf{sub}_+} \times \mathbb{R}^+"] \arrow[dr, "\pi"'] & & X^{\mathbb{R}^+} \times \mathbb{R}^+ \arrow[dl, "\pi"] \\ & \mathbb{R}^+ & \end{tikzcd}$$

in the slice $\mathcal{C}/\mathbb{R}^+$. We will denote it by σ.

First note the following chain of isomorphisms hold for any object $\zeta : Z \to \mathbb{R}^+$ in $\mathcal{C}/\mathbb{R}^+$

$$\mathcal{C}/\mathbb{R}^+(\zeta, \pi_{X^{\mathbb{R}^+}}) \cong \mathcal{C}(Z, X^{\mathbb{R}^+}) \cong \mathcal{C}(Z \times \mathbb{R}^+, X) \cong \mathcal{C}/\mathbb{R}^+(\zeta \times \pi_{\mathbb{R}^+}, \pi_X) \cong \mathcal{C}/\mathbb{R}^+(\zeta, \pi_X^{\pi_{\mathbb{R}^+}})$$

where $\pi_?$ denotes the product projection $? \times \mathbb{R}^+ \to \mathbb{R}^+$. Thus, $\pi_{X^{\mathbb{R}^+}} \cong \pi_X^{\pi_{\mathbb{R}^+}}$, and furthermore, this isomorphism takes any morphism

$$\begin{tikzcd} X^{\mathbb{R}^+} \times \mathbb{R}^+ \arrow[rr, "\overline{\alpha} \times \mathbb{R}^+"] \arrow[dr, "\pi_{X^{\mathbb{R}^+}}"'] & & X^{\mathbb{R}^+} \times \mathbb{R}^+ \arrow[dl, "\pi_{X^{\mathbb{R}^+}}"] \\ & \mathbb{R}^+ & \end{tikzcd}$$

(where $\overline{\alpha} : X^{\mathbb{R}^+} \times \mathbb{R}^+ \to X^{\mathbb{R}^+}$ is defined as the transpose of some $X^\alpha : X^{\mathbb{R}^+} \to X^{\mathbb{R}^+ \times \mathbb{R}^+}$) to the morphism $\pi_X^{\alpha \times \mathbb{R}^+} : \pi_X^{\pi_{\mathbb{R}^+}} \to \pi_X^{\pi_{\mathbb{R}^+}}$.

Then σ is isomorphic to $\pi_X^{\mathsf{sub}_+ \times \mathbb{R}^+}$, and σ^2 is isomorphic to $\pi_X^{(\mathsf{sub}_+ \times \mathbb{R}^+)^2}$. Since $(\mathsf{sub}_+ \times \mathbb{R}^+)^2 : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+ \times \mathbb{R}^+$ maps any (r, s) to $(\mathsf{min}(r, s), s)$, it is equal to $\mathsf{min} \times \mathbb{R}^+$. Thus, we see that $\pi_X^{(\mathsf{sub}_+ \times \mathbb{R}^+)^2} = \pi_X^{\mathsf{min} \times \mathbb{R}^+}$ and the underlying morphism of σ^2 is equal to $\overline{\mathsf{min}} \times \mathbb{R}^+$.

Similarly, σ^3 is isomorphic to $(X \times \mathbb{R}^+)^{(\mathsf{sub}_+ \times \mathbb{R}^+)^3}$, and $(\mathsf{sub}_+ \times \mathbb{R}^+)^3 = \mathsf{sub}_+ \times \mathbb{R}^+$. Thus, $(X \times \mathbb{R}^+)^{(\mathsf{sub}_+ \times \mathbb{R}^+)^3} = (X \times \mathbb{R}^+)^{\mathsf{sub}_+ \times \mathbb{R}^+}$ and $\sigma^3 = \sigma$. We conclude that $\sigma^4 = \sigma^2$ so that σ^2 is an idempotent. $\square$

To split the idempotent $\overline{\mathsf{min}} \times \mathbb{R}^+$, one takes the equalizer of the following diagram

$$\begin{tikzcd} X^{\mathbb{R}^+} \times \mathbb{R}^+ \arrow[r, shift left, "\overline{\mathsf{min}} \times \mathbb{R}^+"] \arrow[r, shift right, "1"'] & X^{\mathbb{R}^+} \times \mathbb{R}^+ \end{tikzcd}$$

which was defined to be $i : \Gamma(X) \hookrightarrow X^{\mathbb{R}^+} \times \mathbb{R}^+$ in the last section. Now since the following diagram is a morphism of equalizer diagrams,

$$\begin{tikzcd} X^{\mathbb{R}^+} \times \mathbb{R}^+ \arrow[r, shift left, "\overline{\mathsf{min}} \times \mathbb{R}^+"] \arrow[r, shift right, "1"'] \arrow[d, "\overline{\mathsf{sub}}_+ \times \mathbb{R}^+"'] & X^{\mathbb{R}^+} \times \mathbb{R}^+ \arrow[d, "\overline{\mathsf{sub}}_+ \times \mathbb{R}^+"] \\ X^{\mathbb{R}^+} \times \mathbb{R}^+ \arrow[r, shift left, "\overline{\mathsf{min}} \times \mathbb{R}^+"] \arrow[r, shift right, "1"'] & X^{\mathbb{R}^+} \times \mathbb{R}^+ \end{tikzcd}$$

it induces a morphism $\iota : \Gamma(X) \to \Gamma(X)$. This takes a point (p, r) to $p(\mathsf{max}(0, r - t))$.

Since $(\mathsf{sub}_+ \times \mathbb{R}^+)^2 = \overline{\mathsf{min}} \times \mathbb{R}^+$, its induced endomorphism on $\Gamma(X)$, that is, ι^2, is the identity.

Proposition 4.5.16. *The following diagrams commute, making* $\mathbf{\Gamma}$ *a strictly symmetric functorial relation.*

$$\begin{tikzcd} R \arrow[r, "\iota"] & R \\ 1_{\mathcal{C}} \arrow[u, "\eta"] \arrow[r, equal] & 1_{\mathcal{C}} \arrow[u, "\eta"'] \end{tikzcd} \qquad \begin{tikzcd} R \arrow[r, "\iota"] \arrow[d, "\epsilon_0"'] & R \arrow[d, "\epsilon_0"'] \\ 1_{\mathcal{C}} \arrow[r, equal] & 1_{\mathcal{C}} \end{tikzcd} \qquad \begin{tikzcd} R \arrow[r, "\iota"] \arrow[d, "\epsilon_1"'] & R \arrow[d, "\epsilon_1"'] \\ 1_{\mathcal{C}} \arrow[r, equal] & 1_{\mathcal{C}} \end{tikzcd}$$

Proof. To see that $\iota\eta = \eta$, it suffices to show that $(\overline{\mathsf{sub}}_+ \times \mathbb{R}^+)(X^! \times 0!) : X \to X^{\mathbb{R}^+} \times \mathbb{R}^+$ is $X^! \times 0!$ since the following diagram commutes.

$$\begin{tikzcd} & \Gamma X \arrow[r, "\iota"] \arrow[d, hook] & \Gamma X \arrow[d, hook] \\ X \arrow[ur, "\eta"] \arrow[r, "X^! \times 0!"'] & X^{\mathbb{R}^+} \times \mathbb{R}^+ \arrow[r, "\overline{\mathsf{sub}} \times \mathbb{R}^+"'] & X^{\mathbb{R}^+} \times \mathbb{R}^+ \end{tikzcd}$$

Since $X^! \times 0!$ is the following composition

$$X \xrightarrow{X \times 0!} X \times \mathbb{R}^+ \xrightarrow{X^! \times \mathbb{R}^+} X^{\mathbb{R}^+} \times \mathbb{R}^+,$$

we see that $X^! \times \mathbb{R}^+$ underlies $\pi_X^{! \times \mathbb{R}^+}$ and $\overline{\mathsf{sub}}_+ \times \mathbb{R}^+$ underlines $\pi_X^{\mathsf{sub}_+ \times \mathbb{R}^+}$ in $\mathcal{C}/\mathbb{R}^+$, using the notation of the proof of Lemma 4.5.15. Their composition is $\pi_X^{\mathsf{sub}_+ ! \times \mathbb{R}^+} = \pi_X^{! \times \mathbb{R}^+}$, so the composition of $X^! \times \mathbb{R}^+$ and $\overline{\mathsf{sub}}_+ \times \mathbb{R}^+$ is $X^! \times \mathbb{R}^+$. Therefore, $(\overline{\mathsf{sub}}_+ \times \mathbb{R}^+)(X^! \times 0!) = X^! \times 0!$.

To show that $\epsilon_1\iota = \epsilon_0$, it suffices to show that $\mathsf{ev}(\overline{\mathsf{sub}}_+ \times \mathbb{R}^+) = \mathsf{ev}(X^{\mathbb{R}^+} \times 0!)$. Note that $\mathsf{ev} \times \mathbb{R}^+$ underlies π_X^{Δ} and $\mathsf{ev}(X^{\mathbb{R}^+} \times 0!) \times \mathbb{R}^+$ underlies $\pi_X^{0! \times \mathbb{R}^+}$. Then we see that $(\mathsf{ev} \times \mathbb{R}^+)(\overline{\mathsf{sub}}_+ \times \mathbb{R}^+)$ underlies $\pi_X^{(\mathsf{sub}_+ \times \mathbb{R}^+)\Delta} = \pi_X^{0! \times \mathbb{R}^+}$ so it is equal to $(\mathsf{ev} \times \mathbb{R}^+)(X^{\mathbb{R}^+} \times 0!)$. Therefore, $\mathsf{ev}(\overline{\mathsf{sub}}_+ \times \mathbb{R}^+) = \mathsf{ev}(X^{\mathbb{R}^+} \times 0!)$.

That $\epsilon_0\iota = \epsilon_1$ follows analogously. □

4.5.4 The transitivity.

We want a 'concatenation' natural transformation

$$\mu_X : \Gamma X {}_{\epsilon_1}\!\times_{\epsilon_0} \Gamma X \to \Gamma X$$

which maps a pair of pairs $((p_1, r_1), (p_2, r_2))$ where $p_1(r_1) = p_2(0)$ to the pair $(q, r_1 + r_2)$ where q coincides with p_1 on $[0, r_1]$ and with $p_2(t - r_1)$ on $[r_1, \infty)$.

We will define μ to be the composition of three isomorphisms followed by a projection.

$$\Gamma X {}_{\epsilon_1}\!\times_{\epsilon_0} \Gamma X \xrightarrow[\cong]{\iota \times 1} \Gamma X {}_{\epsilon_0}\!\times_{\epsilon_0} \Gamma X \xrightarrow[\cong]{\alpha} X^{\mathbb{R}} \underset{\cdot}{\times} \mathbb{R}^+ \underset{\cdot}{\times} \mathbb{R}^+ \xrightarrow[\cong]{\beta} X^{\mathbb{R}^+} \underset{\cdot}{\times} \mathbb{R}^+ \underset{\cdot}{\times} \mathbb{R}^+ \xrightarrow{\pi} \Gamma X$$

The first isomorphism $\iota \times 1$ was defined in the last section.

The isomorphism α is an isomorphism of two limits with the same universal property. Let $\overline{\mathsf{mid}} : X^{\mathbb{R}} \times \mathbb{R}^+ \times \mathbb{R}^+ \to X^{\mathbb{R}}$ denote the transpose of the following morphism.

$$X^{\mathbb{R}} \xrightarrow{X^{\mathsf{mid}}} X^{\mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+}$$

It takes a point (p, s, t) to a path p' which coincides with p on $[-s, t]$ and is constant on $(-\infty, -s] \cup [t, \infty)$. Let $X^{\mathbb{R}} \underset{\cdot}{\times} \mathbb{R}^+ \underset{\cdot}{\times} \mathbb{R}^+$ be the object obtained as the following equalizer.

$$X^{\mathbb{R}} \underset{\cdot}{\times} \mathbb{R}^+ \underset{\cdot}{\times} \mathbb{R}^+ \longrightarrow X^{\mathbb{R}} \times \mathbb{R}^+ \times \mathbb{R}^+ \overset{\overline{\mathsf{mid}}}{\underset{\pi}{\rightrightarrows}} X^{\mathbb{R}}$$

Its points are triples (p, s, t) such that the path $p : \mathbb{R} \to X$ is constant on $(\infty, -s]$ and also on $[t, \infty)$.

Lemma 4.5.17. The objects $\Gamma X {}_{\epsilon_0}\!\times_{\epsilon_0} \Gamma X$ and $X^{\mathbb{R}} \underset{\cdot}{\times} \mathbb{R}^+ \underset{\cdot}{\times} \mathbb{R}^+$ have the same universal property.

Proof. Consider the diagram defining the $\Gamma X {}_{\epsilon_0}\!\times_{\epsilon_0} \Gamma X$ which is illustrated below.

$$\begin{tikzcd}
X^{\mathbb{R}^+} \times \mathbb{R}^+ \arrow[d, shift right, "\overline{\mathsf{min}}"'] \arrow[d, shift left, "\pi"] & & X^{\mathbb{R}^+} \times \mathbb{R}^+ \arrow[d, shift right, "\overline{\mathsf{min}}"'] \arrow[d, shift left, "\pi"] \\
X^{\mathbb{R}^+} \arrow[dr, "\epsilon_0"'] & & X^{\mathbb{R}^+} \arrow[dl, "\epsilon_0"] \\
& X &
\end{tikzcd}$$

Using the hypothesis that $\mathbb{R} \cong \mathbb{R}^+ \cup_0 \mathbb{R}^+$, we can add the pullback $X^{\mathbb{R}} \cong X^{\mathbb{R}^+}{}_{\epsilon_0}\times_{\epsilon_0} X^{\mathbb{R}^+}$ to this diagram without changing the limit.

$$\begin{tikzcd}
X^{\mathbb{R}^+} \times \mathbb{R}^+ \arrow[d, shift right, "\overline{\mathsf{min}}"'] \arrow[d, shift left, "\pi"] & & X^{\mathbb{R}^+} \times \mathbb{R}^+ \arrow[d, shift right, "\overline{\mathsf{min}}"'] \arrow[d, shift left, "\pi"] \\
X^{\mathbb{R}^+} \arrow[dr, "\epsilon_0"'] & X^{\mathbb{R}} \arrow[l, "X^{j_-}"'] \arrow[r, "X^{j_+}"] \arrow[d, "\mathsf{ev}_0"] & X^{\mathbb{R}^+} \arrow[dl, "\epsilon_0"] \\
& X &
\end{tikzcd}$$

We let $j_- : \mathbb{R}^+ \to \mathbb{R}$ denote the injection which sends x to $-x$, and let $j_+ : \mathbb{R}^+ \to \mathbb{R}$ denote the injection sending x to x.

Similarly, we can add the pullback $X^{\mathbb{R}} \times \mathbb{R}^+ \times \mathbb{R}^+ \cong (X^{\mathbb{R}^+} \times \mathbb{R}^+)_{\pi\epsilon_0}\times_{\pi\epsilon_0}(X^{\mathbb{R}^+} \times \mathbb{R}^+)$ to this diagram without changing its limit. (Note that $\epsilon_0 \min = \pi\epsilon_0 : X^{\mathbb{R}^+} \times \mathbb{R}^+ \to X$.)

$$\begin{tikzcd}
X^{\mathbb{R}^+} \times \mathbb{R}^+ \arrow[d, shift right, "\overline{\mathsf{min}}"'] \arrow[d, shift left, "\pi"] & X^{\mathbb{R}} \times \mathbb{R}^+ \times \mathbb{R}^+ \arrow[l, "X^{j_-} \times \pi_1"'] \arrow[r, "X^{j_+} \times \pi_2"] \arrow[d, shift right, "\overline{\mathsf{mid}}"'] \arrow[d, shift left, "\pi"] & X^{\mathbb{R}^+} \times \mathbb{R}^+ \arrow[d, shift right, "\overline{\mathsf{min}}"'] \arrow[d, shift left, "\pi"] \\
X^{\mathbb{R}^+} \arrow[dr, "\epsilon_0"'] & X^{\mathbb{R}} \arrow[l, "X^{j_-}"'] \arrow[r, "X^{j_+}"] \arrow[d, "\mathsf{ev}_0"] & X^{\mathbb{R}^+} \arrow[dl, "\epsilon_0"] \\
& X &
\end{tikzcd} \tag{$*$}$$

The arrows $\overline{\mathsf{mid}}, \pi : X^{\mathbb{R}} \times \mathbb{R}^+ \times \mathbb{R}^+ \to X^{\mathbb{R}}$ in the diagram above are induced by the universal property of the pullback $X^{\mathbb{R}}$.

To see that $\overline{\mathsf{mid}} : X^{\mathbb{R}} \times \mathbb{R}^+ \times \mathbb{R}^+ \to X^{\mathbb{R}}$ is really induced by the universal property of the pullback $X^{\mathbb{R}}$, we must show first that $X^{j_-}\overline{\mathsf{mid}} = \overline{\mathsf{min}}(X^{j_-} \times \pi_1)$, and for this it suffices to show that the following square commutes.

$$\begin{tikzcd}
X^{\mathbb{R}^+} \times \mathbb{R}^+ \times \mathbb{R}^+ \arrow[d, "\overline{\mathsf{min}} \times \mathbb{R}^+ \times \mathbb{R}^+"'] & X^{\mathbb{R}} \times \mathbb{R}^+ \times \mathbb{R}^+ \arrow[l, "X^{j_-} \times \mathbb{R}^+ \times \mathbb{R}^+"'] \arrow[d, "\overline{\mathsf{mid}} \times \mathbb{R}^+ \times \mathbb{R}^+"] \\
X^{\mathbb{R}^+} \times \mathbb{R}^+ \times \mathbb{R}^+ & X^{\mathbb{R}} \times \mathbb{R}^+ \times \mathbb{R}^+ \arrow[l, "X^{j_-} \times \mathbb{R}^+ \times \mathbb{R}^+"']
\end{tikzcd}$$

This square lives over $\mathbb{R}^+ \times \mathbb{R}^+$ and so, as in Lemma 4.5.15, we see that it underlies the following square in $\mathcal{C}/(\mathbb{R}^+ \times \mathbb{R}^+)$.

$$
\begin{tikzcd}
\pi_X^{\pi_{\mathbb{R}^+}} \arrow[d, "\pi_X^{\mathsf{min}}"'] & \pi_X^{\pi_{\mathbb{R}}} \arrow[l, "\pi_X^{j_-}"'] \arrow[d, "\pi_X^{\mathsf{mid}}"] \\
\pi_X^{\pi_{\mathbb{R}^+}} & \pi_X^{\pi_{\mathbb{R}}} \arrow[l, "\pi_X^{j_-}"']
\end{tikzcd}
$$

And since $j_-\mathsf{mid} = \mathsf{min} j_-$, this diagram commutes, and we conclude that $X^{j_-}\overline{\mathsf{mid}} = \overline{\mathsf{min}}(X^{j_-} \times \pi_1)$.

Similarly, we can show that $X^{j_+}\overline{\mathsf{mid}} = \overline{\mathsf{min}}(X^{j_+} \times \pi_2)$.

Now we have established that $\Gamma X_{\epsilon_0}\!\times_{\epsilon_0}\Gamma X$ is the limit of diagram $(*)$. To see that $X^{\mathbb{R}} \underset{\cdot}{\times} \mathbb{R}^+ \underset{\cdot}{\times} \mathbb{R}^+$ is also the limit of this diagram, note that the inclusion of the following subdiagram

$$
\begin{tikzcd}
X^{\mathbb{R}} \times \mathbb{R}^+ \times \mathbb{R}^+ \arrow[d, shift right, "\overline{\mathsf{mid}}"'] \arrow[d, shift left, "\pi"] \\
X^{\mathbb{R}} \arrow[d, "\mathsf{ev}_0"] \\
X
\end{tikzcd}
$$

is initial. Therefore, by [ML98, IX.3, Thm. 1], the limit of the subdiagram and the limit of the diagram coincide, and we see that $\Gamma X_{\epsilon_0}\!\times_{\epsilon_0}\Gamma X$ and $X^{\mathbb{R}} \underset{\cdot}{\times} \mathbb{R}^+ \underset{\cdot}{\times} \mathbb{R}^+$ have the same universal property. $\square$

To define the isomorphism β, first let $X^{\mathbb{R}^+} \underset{\cdot}{\times} \mathbb{R}^+ \underset{\cdot}{\times} \mathbb{R}^+$ be the object obtained as the following equalizer.

$$
\begin{tikzcd}
X^{\mathbb{R}^+} \underset{\cdot}{\times} \mathbb{R}^+ \underset{\cdot}{\times} \mathbb{R}^+ \arrow[r] & X^{\mathbb{R}^+} \times \mathbb{R}^+ \times \mathbb{R}^+ \arrow[r, shift left, "\overline{\mathsf{min}}(1\times\mathsf{add})"] \arrow[r, shift right, "\pi"'] & X^{\mathbb{R}^+}.
\end{tikzcd}
$$

Its points are triples (p, s, t) such that the path $p : \mathbb{R}^+ \to X$ is constant on $[s + t, \infty)$.

Lemma 4.5.18. The two squares in the diagram below commute.

$$\begin{tikzcd}
X^{\mathbb{R}} \times \mathbb{R}^+ \times \mathbb{R}^+ \arrow[r, shift left, "\overline{\mathsf{mid}}\times 1\times 1"] \arrow[r, shift right, "1"'] \arrow[d, "\overline{\mathsf{sub}}\times 1\times 1"] & X^{\mathbb{R}} \times \mathbb{R}^+ \times \mathbb{R}^+ . \arrow[d, "\overline{\mathsf{sub}}\times 1\times 1"] \\
X^{\mathbb{R}^+} \times \mathbb{R}^+ \times \mathbb{R}^+ \arrow[r, shift left, "\overline{\mathsf{min}}(1\times\mathsf{add})\times 1\times 1"] \arrow[r, shift right, "1"'] & X^{\mathbb{R}^+} \times \mathbb{R}^+ \times \mathbb{R}^+
\end{tikzcd}$$

This induces a morphism $\beta : X^{\mathbb{R}} \mathbin{\underset{\cdot}{\times}} \mathbb{R}^+ \mathbin{\underset{\cdot}{\times}} \mathbb{R}^+ \to X^{\mathbb{R}^+} \mathbin{\underset{\cdot}{\times}} \mathbb{R}^+ \mathbin{\underset{\cdot}{\times}} \mathbb{R}^+$.

The two squares in the diagram below also commute.

$$\begin{tikzcd}
X^{\mathbb{R}^+} \times \mathbb{R}^+ \times \mathbb{R}^+ \arrow[r, shift left, "\overline{\mathsf{min}}(1\times\mathsf{add})\times 1\times 1"] \arrow[r, shift right, "1"'] \arrow[d, "\overline{\mathsf{add}}_+\times 1\times 1"] & X^{\mathbb{R}^+} \times \mathbb{R}^+ \times \mathbb{R}^+ \arrow[d, "\overline{\mathsf{add}}_+\times 1\times 1"] \\
X^{\mathbb{R}} \times \mathbb{R}^+ \times \mathbb{R}^+ \arrow[r, shift left, "\overline{\mathsf{mid}}\times 1\times 1"] \arrow[r, shift right, "1"'] & X^{\mathbb{R}} \times \mathbb{R}^+ \times \mathbb{R}^+ .
\end{tikzcd}$$

This induces a morphism $\beta^{-1} : X^{\mathbb{R}^+} \mathbin{\underset{\cdot}{\times}} \mathbb{R}^+ \mathbin{\underset{\cdot}{\times}} \mathbb{R}^+ \to X^{\mathbb{R}} \mathbin{\underset{\cdot}{\times}} \mathbb{R}^+ \mathbin{\underset{\cdot}{\times}} \mathbb{R}^+$ which is the inverse of β.

Proof. We need to show that the following square commutes.

$$\begin{tikzcd}
X^{\mathbb{R}} \times \mathbb{R}^+ \times \mathbb{R}^+ \arrow[r, "\overline{\mathsf{mid}}\times 1\times 1"] \arrow[d, "\overline{\mathsf{sub}}\times 1\times 1"] & X^{\mathbb{R}} \times \mathbb{R}^+ \times \mathbb{R}^+ . \arrow[d, "\overline{\mathsf{sub}}\times 1\times 1"] \\
X^{\mathbb{R}^+} \times \mathbb{R}^+ \times \mathbb{R}^+ \arrow[r, "\overline{\mathsf{min}}(1\times\mathsf{add})\times 1\times 1"] & X^{\mathbb{R}^+} \times \mathbb{R}^+ \times \mathbb{R}^+
\end{tikzcd}$$

This diagram lives naturally over $\mathbb{R}^+ \times \mathbb{R}^+$. As in Lemma 4.5.15, it underlies a diagram isomorphic to

$$\begin{tikzcd}
\pi_X^{\pi_{\mathbb{R}}} \arrow[r, "\pi_X^{\mathsf{mid}\times 1\times 1}"] \arrow[d, "\pi_X^{\mathsf{sub}\times 1\times 1}"'] & \pi_X^{\pi_{\mathbb{R}}} . \arrow[d, "\pi_X^{\mathsf{sub}\times 1\times 1}"] \\
\pi_X^{\pi_{\mathbb{R}^+}} \arrow[r, "\pi_X^{\mathsf{min}(1\times\mathsf{add})\times 1\times 1}"] & \pi_X^{\pi_{\mathbb{R}^+}}
\end{tikzcd} \tag{$*$}$$

in $\mathcal{C}/(\mathbb{R}^+ \times \mathbb{R}^+)$. Since the functions $(\mathsf{mid} \times 1 \times 1)(\mathsf{sub} \times 1 \times 1)$ and $(\mathsf{sub} \times 1 \times 1)$ $(\mathsf{min}(1 \times \mathsf{add}) \times 1 \times 1)$ are both functions $\mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+$ which map (q, r, s) to $(\mathsf{min}(q - r, s), r, s)$, we see that these diagrams commute.

Thus, the first diagram in the statement displays a natural transformation of the diagrams defining $X^{\mathbb{R}} \mathbin{\underset{\cdot}{\times}} \mathbb{R}^+ \mathbin{\underset{\cdot}{\times}} \mathbb{R}^+$ and $X^{\mathbb{R}^+} \mathbin{\underset{\cdot}{\times}} \mathbb{R}^+ \mathbin{\underset{\cdot}{\times}} \mathbb{R}^+$ and so induces a morphism β between them.

Similarly, the diagram

$$\begin{tikzcd}
X^{\mathbb{R}^+} \times \mathbb{R}^+ \times \mathbb{R}^+ \arrow[r, "\overline{\mathsf{min}}(1\times\mathsf{add})\times 1\times 1"] \arrow[d, "\overline{\mathsf{add}}_+\times 1\times 1"] & X^{\mathbb{R}^+} \times \mathbb{R}^+ \times \mathbb{R}^+ \arrow[d, "\overline{\mathsf{add}}_+\times 1\times 1"] \\
X^{\mathbb{R}} \times \mathbb{R}^+ \times \mathbb{R}^+ \arrow[r, "\overline{\mathsf{mid}}\times 1\times 1"] & X^{\mathbb{R}} \times \mathbb{R}^+ \times \mathbb{R}^+.
\end{tikzcd}$$

underlies a diagram isomorphic to the following diagram

$$\begin{tikzcd}
\pi_X^{\pi_{\mathbb{R}^+}} \arrow[r, "\pi_X^{\mathsf{min}(1\times\mathsf{add})\times 1\times 1}"] \arrow[d, "\pi_X^{\mathsf{add}_+\times 1\times 1}"'] & \pi_X^{\pi_{\mathbb{R}^+}} \arrow[d, "\pi_X^{\mathsf{add}_+\times 1\times 1}"] \\
\pi_X^{\pi_{\mathbb{R}}} \arrow[r, "\pi_X^{\mathsf{mid}\times 1\times 1}"] & \pi_X^{\pi_{\mathbb{R}}}
\end{tikzcd} \tag{$**$}$$

in $\mathcal{C}/(\mathbb{R}^+ \times \mathbb{R}^+)$. Since $(\mathsf{add}_+ \times 1 \times 1)(\mathsf{mid} \times 1 \times 1)$ and $(\mathsf{min}(1 \times \mathsf{add}) \times 1 \times 1)(\mathsf{add}_+ \times 1 \times 1)$ both send $(q, r, s) \in \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+$ to $\mathsf{min}(\mathsf{max}(0, q + r), r + s)$, this diagram commutes. Thus, the second diagram in the statement is a natural transformation between the diagrams defining $X^{\mathbb{R}^+} \mathbin{\underset{\cdot}{\times}} \mathbb{R}^+ \mathbin{\underset{\cdot}{\times}} \mathbb{R}^+$ and $X^{\mathbb{R}} \mathbin{\underset{\cdot}{\times}} \mathbb{R}^+ \mathbin{\underset{\cdot}{\times}} \mathbb{R}^+$ and so induces the morphism β^{-1}.

Now, composing the diagrams $(*)$ and $(**)$ we see that $\beta\beta^{-1} = 1$ since $(\mathsf{add}_+ \times 1 \times 1)(\mathsf{sub} \times 1 \times 1)$ is the identity on $\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+$.

$$\begin{tikzcd}
\pi_X^{\pi_{\mathbb{R}^+}} \arrow[r, "\pi_X^{\mathsf{min}(1\times\mathsf{add})\times 1\times 1}"] \arrow[d, "\pi_X^{\mathsf{add}_+\times 1\times 1}"] \arrow[dd, bend right, "1"'] & \pi_X^{\pi_{\mathbb{R}^+}} \arrow[d, "\pi_X^{\mathsf{add}_+\times 1\times 1}"'] \arrow[dd, bend left, "1"] \\
\pi_X^{\pi_{\mathbb{R}}} \arrow[r, "\pi_X^{\mathsf{mid}\times 1\times 1}"] \arrow[d, "\pi_X^{\mathsf{sub}\times 1\times 1}"] & \pi_X^{\pi_{\mathbb{R}}} \arrow[d, "\pi_X^{\mathsf{sub}\times 1\times 1}"'] \\
\pi_X^{\pi_{\mathbb{R}^+}} \arrow[r, "\pi_X^{\mathsf{min}(1\times\mathsf{add})\times 1\times 1}"] & \pi_X^{\pi_{\mathbb{R}^+}}
\end{tikzcd}$$

Composing the diagrams $(**)$ and $(*)$ we obtain the following commuting diagram

$$\begin{tikzcd}
\pi_X^{\pi_{\mathbb{R}}} \arrow[rr, "\pi_X^{\mathsf{mid}\times 1\times 1}"] \arrow[d, "\pi_X^{\mathsf{sub}\times 1\times 1}"] \arrow[dd, bend right, swap, "\pi_X^{\mathsf{max}'\times 1\times 1}"] & & \pi_X^{\pi_{\mathbb{R}}} \arrow[d, swap, "\pi_X^{\mathsf{sub}\times 1\times 1}"] \arrow[dd, bend left, "\pi_X^{\mathsf{max}'\times 1\times 1}"] \\
\pi_X^{\pi_{\mathbb{R}^+}} \arrow[rr, "\pi_X^{\mathsf{min}(1\times\mathsf{add})\times 1\times 1}"] \arrow[d, "\pi_X^{\mathsf{add}_+\times 1\times 1}"] & & \pi_X^{\pi_{\mathbb{R}^+}} \arrow[d, swap, "\pi_X^{\mathsf{add}_+\times 1\times 1}"] \\
\pi_X^{\pi_{\mathbb{R}}} \arrow[rr, "\pi_X^{\mathsf{mid}\times 1\times 1}"] & & \pi_X^{\pi_{\mathbb{R}}}
\end{tikzcd}$$

where $\mathsf{max}' : \mathbb{R}\times\mathbb{R}^+ \to \mathbb{R}\times\mathbb{R}^+$ takes (r, s) to $(\mathsf{max}(r, -s), s)$.

Thus, $\beta^{-1}\beta$ is induced by the following diagram.

$$\begin{tikzcd}
X^{\mathbb{R}} \mathbin{\dot\times} \mathbb{R}^+ \mathbin{\dot\times} \mathbb{R}^+ \arrow[r, "j"] \arrow[d, dashed, "\beta^{-1}\beta"] & X^{\mathbb{R}} \times \mathbb{R}^+ \times \mathbb{R}^+ \arrow[r, shift left, "\overline{\mathsf{mid}}\times 1\times 1"] \arrow[r, shift right, swap, "1"] \arrow[d, "\overline{\mathsf{max}'}\times 1\times 1"] & X^{\mathbb{R}} \times \mathbb{R}^+ \times \mathbb{R}^+ \arrow[d, "\overline{\mathsf{max}'}\times 1\times 1"] \\
X^{\mathbb{R}} \mathbin{\dot\times} \mathbb{R}^+ \mathbin{\dot\times} \mathbb{R}^+ \arrow[r, "j"] & X^{\mathbb{R}} \times \mathbb{R}^+ \times \mathbb{R}^+ \arrow[r, shift left, "\overline{\mathsf{mid}}\times 1\times 1"] \arrow[r, shift right, swap, "1"] & X^{\mathbb{R}} \times \mathbb{R}^+ \times \mathbb{R}^+
\end{tikzcd}$$

Note that $(\mathsf{max}' \times 1 \times 1)(\mathsf{mid} \times 1 \times 1) = \mathsf{mid} \times 1 \times 1$. Thus, we have the following equation.

$$j\beta^{-1}\beta = (\overline{\mathsf{max}'} \times 1 \times 1)j = (\overline{\mathsf{max}'} \times 1 \times 1)(\overline{\mathsf{mid}} \times 1 \times 1)j = (\overline{\mathsf{mid}} \times 1 \times 1)j = j$$

But j is a monomorphism, so $\beta^{-1}\beta = 1$.

Therefore, β is an isomorphism $X^{\mathbb{R}} \mathbin{\dot\times} \mathbb{R}^+ \mathbin{\dot\times} \mathbb{R}^+ \cong X^{\mathbb{R}^+} \mathbin{\dot\times} \mathbb{R}^+ \mathbin{\dot\times} \mathbb{R}^+$. $\square$

We let ν be the following composite of isomorphisms.

$$\Gamma X\,{}_{\epsilon_1}\!\times_{\epsilon_0}\Gamma X \xrightarrow[\cong]{\iota\times 1} \Gamma X\,{}_{\epsilon_0}\!\times_{\epsilon_0}\Gamma X \xrightarrow[\cong]{\alpha} X^{\mathbb{R}} \mathbin{\dot\times} \mathbb{R}^+ \mathbin{\dot\times} \mathbb{R}^+ \xrightarrow[\cong]{\beta} X^{\mathbb{R}^+} \mathbin{\dot\times} \mathbb{R}^+ \mathbin{\dot\times} \mathbb{R}^+$$

Lemma 4.5.19. The following diagrams commute.

$$\begin{tikzcd}
X^{\mathbb{R}^+} \mathbin{\dot\times} \mathbb{R}^+ \mathbin{\dot\times} \mathbb{R}^+ \arrow[r, "\pi_0\nu^{-1}"] \arrow[d, hook] & \Gamma(X) \arrow[d, hook] \\
X^{\mathbb{R}^+} \times \mathbb{R}^+ \times \mathbb{R}^+ \arrow[r, "\overline{\mathsf{min}}\times\pi_1"] & X^{\mathbb{R}^+} \times \mathbb{R}^+
\end{tikzcd}
\qquad
\begin{tikzcd}
X^{\mathbb{R}^+} \mathbin{\dot\times} \mathbb{R}^+ \mathbin{\dot\times} \mathbb{R}^+ \arrow[r, "\pi_1\nu^{-1}"] \arrow[d, hook] & \Gamma(X) \arrow[d, hook] \\
X^{\mathbb{R}^+} \times \mathbb{R}^+ \times \mathbb{R}^+ \arrow[r, "\overline{\mathsf{add}}\times\pi_2"] & X^{\mathbb{R}^+} \times \mathbb{R}^+
\end{tikzcd}$$

Proof. Expanding the definition of $\pi_0\nu^{-1}$, we obtain the following commutative diagram.

$$
\begin{tikzcd}
X^{\mathbb{R}^+} \underset{\cdot}{\times} \mathbb{R}^+ \underset{\cdot}{\times} \mathbb{R}^+ \arrow[r, "\beta^{-1}"'] \arrow[d, hook] \arrow[rrr, bend left, "\pi_0\nu^{-1}"] & X^{\mathbb{R}} \underset{\cdot}{\times} \mathbb{R}^+ \underset{\cdot}{\times} \mathbb{R}^+ \arrow[r, "\pi_0\alpha^{-1}"'] \arrow[d, hook] & \Gamma(X) \arrow[r, "\iota"'] \arrow[d, hook] & \Gamma(X) \arrow[d, hook] \\
X^{\mathbb{R}^+} \times \mathbb{R}^+ \times \mathbb{R}^+ \arrow[r, "\overline{\mathsf{add}}_+ \times 1 \times 1"] \arrow[rrr, bend right, "\overline{\mathsf{min}} \times \pi_1"'] & X^{\mathbb{R}} \times \mathbb{R}^+ \times \mathbb{R}^+ \arrow[r, "X^{j_-} \times \pi_1"] & X^{\mathbb{R}^+} \times \mathbb{R}^+ \arrow[r, "\overline{\mathsf{sub}}_+"] & X^{\mathbb{R}^+} \times \mathbb{R}^+
\end{tikzcd}
$$

Expanding $\pi_1\nu^{-1}$, we obtain the following commutative diagram.

$$
\begin{tikzcd}
X^{\mathbb{R}^+} \underset{\cdot}{\times} \mathbb{R}^+ \underset{\cdot}{\times} \mathbb{R}^+ \arrow[r, "\beta^{-1}"'] \arrow[d, hook] \arrow[rr, bend left, "\pi_1\nu^{-1}"] & X^{\mathbb{R}} \underset{\cdot}{\times} \mathbb{R}^+ \underset{\cdot}{\times} \mathbb{R}^+ \arrow[r, "\pi_1\alpha^{-1}"'] \arrow[d, hook] & \Gamma(X) \arrow[d, hook] \\
X^{\mathbb{R}^+} \times \mathbb{R}^+ \times \mathbb{R}^+ \arrow[r, "\overline{\mathsf{add}}_+ \times 1 \times 1"] \arrow[rr, bend right, "\overline{\mathsf{add}} \times \pi_2"'] & X^{\mathbb{R}} \times \mathbb{R}^+ \times \mathbb{R}^+ \arrow[r, "X^{j_+} \times \pi_2"] & X^{\mathbb{R}^+} \times \mathbb{R}^+
\end{tikzcd}
$$

$\square$

Now we can define the projection π to be the morphism induced by the following morphism of diagrams.

$$
\begin{tikzcd}
X^{\mathbb{R}^+} \underset{\cdot}{\times} \mathbb{R}^+ \underset{\cdot}{\times} \mathbb{R}^+ \arrow[r] \arrow[d, dashed, "\pi"] & X^{\mathbb{R}^+} \times \mathbb{R}^+ \times \mathbb{R}^+ \arrow[r, shift left, "\overline{\mathsf{min}}(1 \times \mathsf{add}) \times 1 \times 1"] \arrow[r, shift right, "1"'] \arrow[d, "1 \times \mathsf{add}"] & X^{\mathbb{R}^+} \times \mathbb{R}^+ \times \mathbb{R}^+ \arrow[d, "1 \times \mathsf{add}"] \\
\Gamma(X) \arrow[r] & X^{\mathbb{R}^+} \times \mathbb{R}^+ \arrow[r, shift left, "\overline{\mathsf{min}} \times 1"] \arrow[r, shift right, "1"'] & X^{\mathbb{R}^+} \times \mathbb{R}^+
\end{tikzcd}
$$

Let μ be the following composite.

$$\Gamma X \,{}_{\epsilon_1}\!\times_{\epsilon_0} \Gamma X \xrightarrow[\cong]{\nu} X^{\mathbb{R}^+} \underset{\cdot}{\times} \mathbb{R}^+ \underset{\cdot}{\times} \mathbb{R}^+ \xrightarrow{\pi} \Gamma X$$

Proposition 4.5.20. *The natural transformation μ makes the relation $\boldsymbol{\Gamma}$ into a strictly transitive relation.*

Proof. We must show that the following diagrams commute.

$$
\begin{tikzcd}
\Gamma X \,{}_{\epsilon_1}\!\times_{\epsilon_0} \Gamma X \arrow[r, "\mu"] \arrow[d, shift right, "\epsilon_0\pi_0"'] \arrow[d, shift left, "\epsilon_1\pi_1"] & \Gamma X \arrow[d, shift right, "\epsilon_0"'] \arrow[d, shift left, "\epsilon_1"] \\
X \arrow[r, equal] & X
\end{tikzcd}
$$

$$
\begin{tikzcd}
\Gamma X \arrow[r, "\eta\times 1"] \arrow[dr, equal] & \Gamma X \,{}_{\epsilon_1}\!\times_{\epsilon_0} \Gamma X \arrow[d, "\mu"] & \Gamma X \arrow[l, "1\times\eta"'] \arrow[dl, equal] \\
& \Gamma X &
\end{tikzcd}
\qquad
\begin{tikzcd}
\Gamma X \,{}_{\epsilon_1}\!\times_{\epsilon_0} \Gamma X \,{}_{\epsilon_1}\!\times_{\epsilon_0} \Gamma X \arrow[r, "1\times\mu"] \arrow[d, "\mu\times 1"] & \Gamma X \,{}_{\epsilon_1}\!\times_{\epsilon_0} \Gamma X \arrow[d, "\mu"] \\
\Gamma X \,{}_{\epsilon_1}\!\times_{\epsilon_0} \Gamma X \arrow[r, "\mu"] & \Gamma X
\end{tikzcd}
$$

To show that $\epsilon_0\mu = \epsilon_0\pi_0$ in the first diagram above, we show that the following diagram commutes.

$$
\begin{tikzcd}
\Gamma X \,{}_{\epsilon_1}\!\times_{\epsilon_0} \Gamma X \arrow[r, "\nu", "\cong"'] \arrow[rr, bend left, "\mu"] \arrow[d, hook] \arrow[dd, bend right=60, "\epsilon_0\pi_0"'] & X^{\mathbb{R}^+} \underset{\cdot}{\times} \mathbb{R}^+ \underset{\cdot}{\times} \mathbb{R}^+ \arrow[r, "\pi"] \arrow[d, hook] & \Gamma X \arrow[d, hook] \arrow[dd, bend left=60, "\epsilon_0"] \\
(X^{\mathbb{R}^+} \times \mathbb{R}^+)\,{}_{\mathsf{ev}}\!\times_{\mathsf{ev}_0} (X^{\mathbb{R}^+} \times \mathbb{R}^+) \arrow[d, "\mathsf{ev}_0\pi_0"] & X^{\mathbb{R}^+} \times \mathbb{R}^+ \times \mathbb{R}^+ \arrow[l, "(\overline{\mathsf{min}}\times\pi_1)\times(\overline{\mathsf{add}}\times\pi_2)"] \arrow[r, "1\times\mathsf{add}"] \arrow[d, "\mathsf{ev}_0"] & X^{\mathbb{R}^+} \times \mathbb{R}^+ \arrow[d, "\mathsf{ev}_0"] \\
X \arrow[r, equal] & X \arrow[r, equal] & X
\end{tikzcd}
$$

We know that the upper left-hand square in the diagram above commutes by the preceding lemma. The upper right-hand square commutes by the definition of π, and the bottom right-hand square also commutes. Thus, we only need to show that the bottom left-hand square commutes. But we see that

$$
\mathsf{ev}_0\pi_0((\overline{\mathsf{min}} \times \pi_1) \times (\overline{\mathsf{add}} \times \pi_2)) = \mathsf{ev}_0(\overline{\mathsf{min}} \times \pi_1) = \mathsf{ev}_0
$$

so this square commutes.

Similarly, to show that $\epsilon_1\mu = \epsilon_1\pi_1$, it suffices to show that the following square commutes.

$$
\begin{tikzcd}
(X^{\mathbb{R}^+} \times \mathbb{R}^+)\,{}_{\mathsf{ev}}\!\times_{\mathsf{ev}_0} (X^{\mathbb{R}^+} \times \mathbb{R}^+) \arrow[d, "\mathsf{ev}\pi_1"] & X^{\mathbb{R}^+} \times \mathbb{R}^+ \times \mathbb{R}^+ \arrow[l, "(\overline{\mathsf{min}}\times\pi_1)\times(\overline{\mathsf{add}}\times\pi_2)"] \arrow[d, "\mathsf{ev}(1\times\mathsf{add})"] \\
X \arrow[r, equal] & X
\end{tikzcd}
$$

But we see that

$$\mathsf{ev}\pi_1((\overline{\mathsf{min}} \times \pi_1) \times (\overline{\mathsf{add}} \times \pi_2)) = \mathsf{ev}(\overline{\mathsf{add}} \times \pi_2) = \mathsf{ev}(1 \times \mathsf{add})$$

so this square commutes.

Now we show that $\mu(\eta \times 1) = 1$ in the second diagram above. First note that

$$X^{\mathbb{R}^+} \times 0! \times \mathbb{R}^+ : X^{\mathbb{R}^+} \times \mathbb{R}^+ \to X^{\mathbb{R}^+} \times \mathbb{R}^+ \times \mathbb{R}^+$$

restricts to a morphism $\gamma : \Gamma X \to X^{\mathbb{R}^+} \underset{\cdot}{\times} \mathbb{R}^+ \underset{\cdot}{\times} \mathbb{R}^+$. We will show that $\eta \times 1 = \nu^{-1}\gamma$ since post-composing this equation with μ gives us $\mu(\eta \times 1) = \pi\gamma = 1$. Now, since $\eta \times 1$ is the restriction of $(X^! \times 0!) \times 1$, and $\nu^{-1}\gamma$ is the restriction of $((\overline{\mathsf{min}} \times \pi_1) \times (\overline{\mathsf{add}} \times \pi_2))(X^{\mathbb{R}^+} \times 0! \times \mathbb{R}^+)$ as shown below,

$$\begin{tikzcd}
\Gamma X \arrow[r, "\eta\times 1"] \arrow[d, hook] & \Gamma X {}_{\epsilon_1}\times_{\epsilon_0} \Gamma X \arrow[d, hook] \\
X^{\mathbb{R}^+} \times \mathbb{R}^+ \arrow[r, "(X^!\times 0!)\times 1"] & (X^{\mathbb{R}^+} \times \mathbb{R}^+)_{\mathsf{ev}}\times_{\mathsf{ev}_0}(X^{\mathbb{R}^+} \times \mathbb{R}^+)
\end{tikzcd}$$

$$\begin{tikzcd}
\Gamma X \arrow[r, "\gamma"] \arrow[d, hook] & X^{\mathbb{R}^+} \underset{\cdot}{\times} \mathbb{R}^+ \underset{\cdot}{\times} \mathbb{R}^+ \arrow[r, "\nu^{-1}"] \arrow[d, hook] & \Gamma X {}_{\epsilon_1}\times_{\epsilon_0} \Gamma X \arrow[d, hook] \\
X^{\mathbb{R}^+} \times \mathbb{R}^+ \arrow[r, "X^{\mathbb{R}^+}\times 0! \times \mathbb{R}^+"] & X^{\mathbb{R}^+} \times \mathbb{R}^+ \times \mathbb{R}^+ \arrow[r, "(\overline{\mathsf{min}}\times\pi_1)\times(\overline{\mathsf{add}}\times\pi_2)"] & (X^{\mathbb{R}^+} \times \mathbb{R}^+)_{\mathsf{ev}}\times_{\mathsf{ev}_0}(X^{\mathbb{R}^+} \times \mathbb{R}^+)
\end{tikzcd}$$

we see that it suffices to show that

$$(X^! \times 0!) \times 1 = ((\overline{\mathsf{min}} \times \pi_1) \times (\overline{\mathsf{add}} \times \pi_2))(X^{\mathbb{R}^+} \times 0! \times \mathbb{R}^+).$$

But $(\overline{\mathsf{min}} \times \pi_1)(X^{\mathbb{R}^+} \times 0!) = X^! \times 0!$, and $(\overline{\mathsf{add}} \times \pi_2)(X^{\mathbb{R}^+} \times 0!) = 1$, so we are done.

The equation $\mu(1 \times \eta) = 1$ follows analogously.

Now we must show that $\mu(1 \times \mu) = \mu(\mu \times 1)$. Let $\epsilon_0 : X^{\mathbb{R}^+} \underset{\cdot}{\times} \mathbb{R}^+ \underset{\cdot}{\times} \mathbb{R}^+ \to X$ denote the composite

$$X^{\mathbb{R}^+} \underset{\cdot}{\times} \mathbb{R}^+ \underset{\cdot}{\times} \mathbb{R}^+ \xrightarrow{\pi} X^{\mathbb{R}^+} \xrightarrow{\mathsf{ev}_0} X,$$

and let $\epsilon_1 : X^{\mathbb{R}^+} \mathbin{\underset{\cdot}{\times}} \mathbb{R}^+ \mathbin{\underset{\cdot}{\times}} \mathbb{R}^+ \to X$ denote the composite

$$X^{\mathbb{R}^+} \mathbin{\underset{\cdot}{\times}} \mathbb{R}^+ \mathbin{\underset{\cdot}{\times}} \mathbb{R}^+ \xrightarrow{X^{\mathbb{R}^+} \times \mathsf{add}} X^{\mathbb{R}^+} \times \mathbb{R}^+ \xrightarrow{\mathsf{ev}} X.$$

Note that $X^{\mathbb{R}^+} \mathbin{\underset{\cdot}{\times}} \mathbb{R}^+ \mathbin{\underset{\cdot}{\times}} \mathbb{R}^+$ is isomorphic to the object of the following equalizer.

$$\begin{tikzcd}
\Gamma X \mathbin{\underset{\cdot}{\times}} (\mathbb{R}^+ \times \mathbb{R}^+) \arrow[r, hook] & \Gamma X \times \mathbb{R}^+ \times \mathbb{R}^+ \arrow[r, shift left, "\mathsf{add}(\pi_1 \times \pi_2)"] \arrow[r, shift right, "\pi_1\pi_0"'] & \mathbb{R}^+
\end{tikzcd}$$

Also let $X^{\mathbb{R}^+} \mathbin{\underset{\cdot}{\times}} \mathbb{R}^+ \mathbin{\underset{\cdot}{\times}} \mathbb{R}^+ \mathbin{\underset{\cdot}{\times}} \mathbb{R}^+$ be obtained by the following equalizer.

$$\begin{tikzcd}
X^{\mathbb{R}^+} \mathbin{\underset{\cdot}{\times}} \mathbb{R}^+ \mathbin{\underset{\cdot}{\times}} \mathbb{R}^+ \mathbin{\underset{\cdot}{\times}} \mathbb{R}^+ \arrow[r, hook] & X^{\mathbb{R}^+} \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \arrow[r, shift left, "\overline{\mathsf{min}}\ \mathsf{add}(\pi_1 \times \pi_2 \times \pi_3)"] \arrow[r, shift right, "\pi_0"'] & X^{\mathbb{R}^+}
\end{tikzcd}$$

where $\mathsf{add} : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ denotes the addition of three real numbers. There is an evident projection $\pi : X^{\mathbb{R}^+} \mathbin{\underset{\cdot}{\times}} \mathbb{R}^+ \mathbin{\underset{\cdot}{\times}} \mathbb{R}^+ \mathbin{\underset{\cdot}{\times}} \mathbb{R}^+ \to \Gamma X$ induced by add. Then we have the following diagram

$$\begin{tikzcd}
& \Gamma X\,{}_{\epsilon_1}\!\times_{\epsilon_0}(X^{\mathbb{R}^+} \mathbin{\underset{\cdot}{\times}} \mathbb{R}^+ \mathbin{\underset{\cdot}{\times}} \mathbb{R}^+) \arrow[dr, "\cong"] & \\
\Gamma X\,{}_{\epsilon_1}\!\times_{\epsilon_0}\Gamma X\,{}_{\epsilon_1}\!\times_{\epsilon_0}\Gamma X \arrow[ur, "1\times\nu"] \arrow[d, "\nu\times 1"] & & (\Gamma X\,{}_{\epsilon_1}\!\times_{\epsilon_0}\Gamma X) \mathbin{\underset{\cdot}{\times}} (\mathbb{R}^+ \times \mathbb{R}^+) \arrow[d, "\nu\times 1"] \\
(X^{\mathbb{R}^+} \mathbin{\underset{\cdot}{\times}} \mathbb{R}^+ \mathbin{\underset{\cdot}{\times}} \mathbb{R}^+)\,{}_{\epsilon_1}\!\times_{\epsilon_0}\Gamma X \arrow[d, "\cong"] & & (X^{\mathbb{R}^+} \mathbin{\underset{\cdot}{\times}} \mathbb{R}^+ \mathbin{\underset{\cdot}{\times}} \mathbb{R}^+) \mathbin{\underset{\cdot}{\times}} (\mathbb{R}^+ \times \mathbb{R}^+) \arrow[d, "\cong"] \\
(\mathbb{R}^+ \times \mathbb{R}^+) \mathbin{\underset{\cdot}{\times}} (\Gamma X\,{}_{\epsilon_1}\!\times_{\epsilon_0}\Gamma X) \arrow[dr, "1\times\nu"] & & X^{\mathbb{R}^+} \mathbin{\underset{\cdot}{\times}} \mathbb{R}^+ \mathbin{\underset{\cdot}{\times}} \mathbb{R}^+ \mathbin{\underset{\cdot}{\times}} \mathbb{R}^+ \\
& (\mathbb{R}^+ \times \mathbb{R}^+) \mathbin{\underset{\cdot}{\times}} (X^{\mathbb{R}^+} \mathbin{\underset{\cdot}{\times}} \mathbb{R}^+ \mathbin{\underset{\cdot}{\times}} \mathbb{R}^+) \arrow[ur, "\cong"] &
\end{tikzcd}$$

where the unlabeled isomorphisms are between limits with the same universal property. Now, suppressing these isomorphisms, we see the following diagram.

$$\begin{tikzcd}
 & \Gamma X {}_{\epsilon_1}\!\times_{\epsilon_0} \Gamma X \arrow[ddd, bend left=60, "\mu"] \\
\Gamma X {}_{\epsilon_1}\!\times_{\epsilon_0} \Gamma X {}_{\epsilon_1}\!\times_{\epsilon_0} \Gamma X \arrow[r, "1\times\nu"] \arrow[ur, bend left, "1\times\mu"] \arrow[d, "\nu\times 1"] \arrow[dd, bend right=60, "\mu\times 1"'] & (\Gamma X {}_{\epsilon_1}\!\times_{\epsilon_0} \Gamma X) \mathbin{\dot{\times}} (\mathbb{R}^+ \times \mathbb{R}^+) \arrow[u, "\pi"] \arrow[d, "\nu\times 1"] \\
(\mathbb{R}^+ \times \mathbb{R}^+) \mathbin{\dot{\times}} (\Gamma X {}_{\epsilon_1}\!\times_{\epsilon_0} \Gamma X) \arrow[r, "1\times\nu"] \arrow[d, "\pi"] & X^{\mathbb{R}^+} \mathbin{\dot{\times}} \mathbb{R}^+ \mathbin{\dot{\times}} \mathbb{R}^+ \mathbin{\dot{\times}} \mathbb{R}^+ \arrow[d, "\pi"] \\
\Gamma X {}_{\epsilon_1}\!\times_{\epsilon_0} \Gamma X \arrow[r, bend right=10, "\mu"] & \Gamma(X)
\end{tikzcd}$$

The four outside tiles in the above diagram commute, and so it remains to be seen that $(1 \times \nu)(\nu \times 1) = (\nu \times 1)(1 \times \nu)$. Since each of these morphisms is an isomorphism, we will show that the following diagram commutes.

$$\begin{tikzcd}
\Gamma X {}_{\epsilon_1}\!\times_{\epsilon_0} \Gamma X {}_{\epsilon_1}\!\times_{\epsilon_0} \Gamma X & (\Gamma X {}_{\epsilon_1}\!\times_{\epsilon_0} \Gamma X) \mathbin{\dot{\times}} (\mathbb{R}^+ \times \mathbb{R}^+) \arrow[l, "1\times\nu^{-1}"'] \\
(\mathbb{R}^+ \times \mathbb{R}^+) \mathbin{\dot{\times}} (\Gamma X {}_{\epsilon_1}\!\times_{\epsilon_0} \Gamma X) \arrow[u, "\nu^{-1}\times 1"] & X^{\mathbb{R}^+} \mathbin{\dot{\times}} \mathbb{R}^+ \mathbin{\dot{\times}} \mathbb{R}^+ \mathbin{\dot{\times}} \mathbb{R}^+ \arrow[l, "\pi_1\times\pi_2\times\nu^{-1}"] \arrow[u, "\nu^{-1}\times\pi_2\times\pi_3"']
\end{tikzcd}$$

But since ν^{-1} is the restriction of

$$(\overline{\mathsf{min}} \times \pi_1) \times (\overline{\mathsf{add}} \times \pi_2) : X^{\mathbb{R}^+} \times \mathbb{R}^+ \times \mathbb{R}^+ \to (X^{\mathbb{R}^+} \times \mathbb{R}^+) \times (X^{\mathbb{R}^+} \times \mathbb{R}^+),$$

we see that both composites above are the restriction of

$$\begin{aligned}
&(\overline{\mathsf{min}}(\pi_0 \times \pi_1) \times \pi_1) \times (\overline{\mathsf{min}}(\overline{\mathsf{add}}(\pi_0 \times \pi_1) \times \pi_2) \times \pi_2) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \times (\overline{\mathsf{add}}(\pi_0 \times \mathsf{add}(\pi_1 \times \pi_2)) \times \pi_3) : \\
&X^{\mathbb{R}^+} \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to (X^{\mathbb{R}^+} \times \mathbb{R}^+) \times (X^{\mathbb{R}^+} \times \mathbb{R}^+) \times (X^{\mathbb{R}^+} \times \mathbb{R}^+),
\end{aligned}$$

and thus the diagram above commutes. $\square$

4.5.5 The homotopy.

We now want to find a natural transformation

$$\delta_X : \Gamma X \to \Gamma^2 X$$

which maps a pair (p, r) to the pair (q, r) where $q(t) = (p_t, t) \in \Gamma X$ and $p_t|_{[0,t]} = p|_{[0,t]}$. Then δ_X will give for all $(p, t) \in \Gamma X$, the standard homotopy from the constant path $(c_{p(0)}, 0)$ to (p, t).

We also want to find a natural transformation

$$\tau_X : X \times \Gamma * \to \Gamma X$$

which maps a pair $(x, (p, r))$ to $(c(x), r)$.

Let $\delta_0 : X^{\mathbb{R}^+} \times \mathbb{R}^+ \to (X^{\mathbb{R}^+} \times \mathbb{R}^+)^{\mathbb{R}^+}$ denote the transpose of the following composite.

$$X^{\mathbb{R}^+} \times \mathbb{R}^+ \times \mathbb{R}^+ \xrightarrow{1 \times \mathsf{min}} X^{\mathbb{R}^+} \times \mathbb{R}^+ \xrightarrow{\overline{\mathsf{min}} \times 1} X^{\mathbb{R}^+} \times \mathbb{R}^+$$

The morphism δ_0 takes a point (p, r) to $(p_t, \mathsf{min}(r, t)) : \mathbb{R}^+ \to X^{\mathbb{R}^+} \times \mathbb{R}^+$ (recall that we are using t as a variable, Remark 4.5.7) where the path p_t is given by $p_t(s) = p\mathsf{min}(s, r, t)$.

Lemma 4.5.21. There is a restriction δ of $\delta_0 \times 1$ as shown below.

$$\begin{tikzcd}
\Gamma X \arrow[rr, "\delta"] \arrow[d, hook] & & \Gamma^2 X \arrow[d, hook] \\
X^{\mathbb{R}^+} \times \mathbb{R}^+ \arrow[r, "\delta_0 \times 1"] & (X^{\mathbb{R}^+} \times \mathbb{R}^+)^{\mathbb{R}^+} \times \mathbb{R}^+ &
\end{tikzcd}$$

Proof. First, we show that $\delta_0 \times 1$ restricts to a map $\delta_1 : X^{\mathbb{R}^+} \times \mathbb{R}^+ \to (\Gamma X)^{\mathbb{R}^+} \times \mathbb{R}^+$. Since $-^{\mathbb{R}^+} \times \mathbb{R}^+$ preserves equalizers, we see that the morphism δ_1 would be induced if the following diagram displayed a cone over the equalizer determining $(\Gamma X)^{\mathbb{R}^+} \times \mathbb{R}^+$.

$$\begin{tikzcd}
X^{\mathbb{R}^+} \times \mathbb{R}^+ \arrow[d, dashed, "\delta_1"] \arrow[dr, "\delta_0 \times 1"] & & \\
(\Gamma X)^{\mathbb{R}^+} \times \mathbb{R}^+ \arrow[r, hook] & (X^{\mathbb{R}^+} \times \mathbb{R}^+)^{\mathbb{R}^+} \times \mathbb{R}^+ \arrow[r, shift left, "(\overline{\mathsf{min}} \times 1)^{\mathbb{R}^+}"] \arrow[r, shift right, "\pi_0"'] & (X^{\mathbb{R}^+} \times \mathbb{R}^+)^{\mathbb{R}^+}
\end{tikzcd}$$

To see that this is actually a cone, we must show that $(\overline{\mathsf{min}} \times 1)^{\mathbb{R}^+}(\delta_0) = \delta_0$. But note that $(\overline{\mathsf{min}} \times 1)^{\mathbb{R}^+}(\delta_0)$ is the transpose of the following composite.

$$X^{\mathbb{R}^+} \times \mathbb{R}^+ \times \mathbb{R}^+ \xrightarrow{1 \times \mathsf{min}} X^{\mathbb{R}^+} \times \mathbb{R}^+ \xrightarrow{\overline{\mathsf{min}} \times 1} X^{\mathbb{R}^+} \times \mathbb{R}^+ \xrightarrow{\overline{\mathsf{min}} \times 1} X^{\mathbb{R}^+} \times \mathbb{R}^+$$

And since $(\overline{\mathsf{min}} \times 1)(\overline{\mathsf{min}} \times 1) = (\overline{\mathsf{min}} \times 1)$, this is the transpose of δ_0.

Now, we show that δ_1 restricts to a morphism $\delta : \Gamma X \to \Gamma^2 X$. To do this, we must show that the following is a morphism of equalizer diagrams.

$$\begin{tikzcd}
\Gamma X \arrow[r, hook] \arrow[d, dashed, "\delta"] & X^{\mathbb{R}^+} \times \mathbb{R}^+ \arrow[r, shift left, "\overline{\mathsf{min}} \times 1"] \arrow[r, shift right, "1"'] \arrow[d, "\delta_1"] & X^{\mathbb{R}^+} \times \mathbb{R}^+ \arrow[d, "\delta_1"] \\
\Gamma^2 X \arrow[r, hook] & (\Gamma X)^{\mathbb{R}^+} \times \mathbb{R}^+ \arrow[r, shift left, "\overline{\mathsf{min}} \times 1"] \arrow[r, shift right, "1"'] & (\Gamma X)^{\mathbb{R}^+} \times \mathbb{R}^+
\end{tikzcd}$$

We must show that $\delta_1(\overline{\mathsf{min}} \times 1) = (\overline{\mathsf{min}} \times 1)\delta_1$. To do this, we note that in the diagram below, the bottom square commutes, and the marked arrows are monomorphisms. Thus, the top square commutes if and only if the outside commutes.

$$\begin{tikzcd}
X^{\mathbb{R}^+} \times \mathbb{R}^+ \arrow[r, "\overline{\mathsf{min}} \times 1"] \arrow[d, "\delta_1"] \arrow[dd, bend right=60, "\delta_0 \times 1"'] & X^{\mathbb{R}^+} \times \mathbb{R}^+ \arrow[d, "\delta_1"] \arrow[dd, bend left=60, "\delta_0 \times 1"] \\
(\Gamma X)^{\mathbb{R}^+} \times \mathbb{R}^+ \arrow[r, "\overline{\mathsf{min}} \times 1"] \arrow[d, hook] & (\Gamma X)^{\mathbb{R}^+} \times \mathbb{R}^+ \arrow[d, hook] \\
(X^{\mathbb{R}^+} \times \mathbb{R}^+)^{\mathbb{R}^+} \times \mathbb{R}^+ \arrow[r, "\overline{\mathsf{min}} \times 1"] & (X^{\mathbb{R}^+} \times \mathbb{R}^+)^{\mathbb{R}^+} \times \mathbb{R}^+
\end{tikzcd}$$

Now we need to show that $(\delta_0 \times 1)(\overline{\mathsf{min}} \times 1) = (\overline{\mathsf{min}} \times 1)(\delta_0 \times 1)$. Note that both of these composites are isomorphic to morphisms of the following form.

$$X^{\mathbb{R}^+} \times \mathbb{R}^+ \to X^{\mathbb{R}^+ \times \mathbb{R}^+} \times (\mathbb{R}^+)^{\mathbb{R}^+} \times \mathbb{R}^+$$

Thus, we show that the compositions of $(\delta_0 \times 1)(\overline{\mathsf{min}} \times 1)$ and $(\overline{\mathsf{min}} \times 1)(\delta_0 \times 1)$ with projections to $X^{\mathbb{R}^+ \times \mathbb{R}^+}$, $(\mathbb{R}^+)^{\mathbb{R}^+}$, and $\mathbb{R}^+$ coincide in each case. Composing $(\delta_0 \times 1)(\overline{\mathsf{min}} \times 1)$ and $(\overline{\mathsf{min}} \times 1)(\delta_0 \times 1)$ with the projection to $(\mathbb{R}^+)^{\mathbb{R}^+}$ gives $\overline{\mathsf{min}}\pi_1$ in both cases. Composing with the projection to $\mathbb{R}^+$ gives π_1 in both cases. Thus, it remains to check that their projections to $X^{\mathbb{R}^+ \times \mathbb{R}^+}$ are equal. Let

$\mathsf{min}_3 : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ denote the function which takes the minimum of three real numbers. Then the projection of δ_0 to $X^{\mathbb{R}^+\times\mathbb{R}^+}$ is $\overline{\mathsf{min}_3}$. Thus, it suffices to prove that the following diagram commutes.

$$\begin{tikzcd}
X^{\mathbb{R}^+} \times \mathbb{R}^+ \arrow[r, "\overline{\mathsf{min}}\times 1"] \arrow[d, "\overline{\mathsf{min}_3}\times 1"] & X^{\mathbb{R}^+} \times \mathbb{R}^+ \arrow[d, "\overline{\mathsf{min}_3}\times 1"] \\
X^{\mathbb{R}^+\times\mathbb{R}^+} \times \mathbb{R}^+ \arrow[r, "\overline{\mathsf{min}}\times 1"] & X^{\mathbb{R}^+\times\mathbb{R}^+} \times \mathbb{R}^+
\end{tikzcd}$$

But, as in the proof of Lemma 4.5.15, this underlies the following diagram in $\mathcal{C}/\mathbb{R}^+$.

$$\begin{tikzcd}
\pi_X^{\pi_{\mathbb{R}^+}} \arrow[r, "\pi_X^{\mathsf{min}\times 1}"] \arrow[d, "\pi_X^{\mathsf{min}_3\times 1}"'] & \pi_X^{\pi_{\mathbb{R}^+}} \arrow[d, "\pi_X^{\mathsf{min}_3\times 1}"] \\
\pi_X^{\pi_{\mathbb{R}^+\times\mathbb{R}^+}} \arrow[r, "\pi_X^{1\times\mathsf{min}(\pi_1\times\pi_2)\times 1}"] & \pi_X^{\pi_{\mathbb{R}^+\times\mathbb{R}^+}}
\end{tikzcd}$$

But since $(\mathsf{min}_3\times 1)(1\times\mathsf{min}(\pi_1\times\pi_2)\times 1)$ and $(\mathsf{min}\times 1)(\mathsf{min}_3\times 1) : \mathbb{R}^+\times\mathbb{R}^+\times\mathbb{R}^+$ both map (q, r, s) to $(\mathsf{min}_3(q, r, s), s)$, we see that this diagram commutes. Therefore, δ_1 restricts to a morphism $\delta : \Gamma X \to \Gamma^2 X$. $\square$

To define $\tau : X \times \Gamma(*) \to \Gamma(X)$, first note that in the following equalizer diagram, there is an isomorphism $*^{\mathbb{R}^+} \cong *$.

$$\begin{tikzcd}
\Gamma(*) \arrow[r, "i"] & *^{\mathbb{R}^+} \times \mathbb{R}^+ \arrow[r, shift left, "\overline{\mathsf{min}}"] \arrow[r, shift right, "\pi_0"'] & *^{\mathbb{R}^+}
\end{tikzcd}$$

Thus, $\overline{\mathsf{min}} = \pi_0$, and we have the isomorphisms $\Gamma(*) \cong *^{\mathbb{R}^+} \times \mathbb{R}^+ \cong \mathbb{R}^+$. Now let τ_0 be the following composite.

$$X \times \Gamma(*) \cong X \times \mathbb{R}^+ \xrightarrow{X^! \times 1} X^{\mathbb{R}^+} \times \mathbb{R}^+$$

This factors through $\Gamma(X)$, so we obtain $\tau : X \times \Gamma(*) \to \Gamma(X)$.

Lemma 4.5.22. The morphism τ_0 constitutes a cone over the equalizer defining $\Gamma(X)$. Thus, we obtain a morphism $\tau : X \times \Gamma(*) \to \Gamma(X)$.

$$\begin{tikzcd}
X \times \mathbb{R}^+ \arrow[d, dashed, "\tau"'] \arrow[dr, "X^! \times 1"] \arrow[drr, bend left, "X^! \pi_0"] & & \\
\Gamma(X) \arrow[r, "i"] & X^{\mathbb{R}^+} \times \mathbb{R}^+ \arrow[r, shift left, "\overline{\mathsf{min}}"] \arrow[r, shift right, "\pi_0"'] & X^{\mathbb{R}^+}
\end{tikzcd}$$

Proof. We need to show that $\overline{\mathsf{min}}(X^! \times 1) = X^!\pi_0 : X \times \mathbb{R}^+ \to X^{\mathbb{R}^+}$. The transpose of $\overline{\mathsf{min}}(X^! \times 1)$ is the composite

$$X \times \mathbb{R}^+ \times \mathbb{R}^+ \xrightarrow{X^! \times 1 \times 1} X^{\mathbb{R}^+} \times \mathbb{R}^+ \times \mathbb{R}^+ \xrightarrow{1 \times \mathsf{min}} X^{\mathbb{R}^+} \times \mathbb{R}^+ \xrightarrow{\mathsf{ev}} X$$

which is equal to

$$X \times \mathbb{R}^+ \times \mathbb{R}^+ \xrightarrow{1 \times \mathsf{min}} X \times \mathbb{R}^+ \xrightarrow{X^! \times 1} X^{\mathbb{R}^+} \times \mathbb{R}^+ \xrightarrow{\mathsf{ev}} X.$$

Then Lemma 4.5.11 tells us that the above composite is π_X, so it is the transpose of $X^!\pi_0$. Therefore, $\overline{\mathsf{min}}(X^! \times 1) = X^!\pi_0$. $\square$

Proposition 4.5.23. *The morphisms δ and τ make the relation $\mathbf{\Gamma}$ strictly homotopical.*

Proof. We must show that the following diagrams commute.

$$\begin{tikzcd}
\Gamma X \arrow[r, "\delta"] \arrow[d, shift right, "\epsilon_0"'] & \Gamma^2 X \arrow[d, shift right, "\epsilon_0"'] \\
X \arrow[r, "\eta"] \arrow[u, shift right, "\eta"'] & \Gamma X \arrow[u, shift right, "\eta"']
\end{tikzcd}
\qquad
\begin{tikzcd}
X \times \Gamma * \arrow[r, "\tau"] \arrow[d, shift right=2, "\pi"'] \arrow[d, shift left=2, "\pi"] & \Gamma X \arrow[d, shift right=2, "\epsilon_0"'] \arrow[d, shift left=2, "\epsilon_1"] \\
X \arrow[u, "1 \times \eta"] \arrow[r, equal] & X \arrow[u, "\eta"]
\end{tikzcd}
\qquad (1)$$

$$\begin{tikzcd}
& \Gamma X \arrow[dl, equal] \arrow[d, "\delta"] \arrow[dr, equal] & \\
\Gamma X & \Gamma^2 X \arrow[l, "\epsilon_1"] \arrow[r, "\Gamma\epsilon_1"'] & \Gamma X
\end{tikzcd}
\qquad
\begin{tikzcd}
\Gamma X \arrow[r, "\delta"] \arrow[d, "\delta"] & \Gamma^2 X \arrow[d, "\delta"] \\
\Gamma^2 X \arrow[r, "\Gamma\delta"] & \Gamma^3 X
\end{tikzcd}
\qquad (2)$$

$$\begin{tikzcd}
X \times \Gamma * \arrow[d, "\tau"] \arrow[dr, "\pi_{\Gamma *}"] & \\
\Gamma X \arrow[r, "\Gamma !"] & \Gamma *
\end{tikzcd}
\qquad (3)$$

$$\begin{tikzcd}
\Gamma X \arrow[r, "\delta"] \arrow[d, "\epsilon_0 \times \Gamma !"] & \Gamma^2 X \arrow[d, "\Gamma\epsilon_0"] \\
X \times \Gamma * \arrow[r, "\tau"] & \Gamma X
\end{tikzcd}
\qquad
\begin{tikzcd}
X \times \Gamma * \arrow[r, "\tau \times \delta"] \arrow[d, "\tau"] & \Gamma(X \times \Gamma *) \arrow[d, "\Gamma\tau"] \\
\Gamma X \arrow[r, "\delta"] & \Gamma^2 X
\end{tikzcd}
\qquad (4)$$

First of all, it suffices to show that the following diagrams commute since they restrict to the diagrams above.

$$
\begin{tikzcd}
X^{\mathbb{R}^+} \times \mathbb{R}^+ \arrow[r, "\delta_0 \times 1"] \arrow[d, shift right, "\mathsf{ev}_0"'] & (X^{\mathbb{R}^+} \times \mathbb{R}^+)^{\mathbb{R}^+} \times \mathbb{R}^+ \arrow[d, shift right, "\mathsf{ev}_0"'] \\
X \arrow[r, "X^! \times 0!"] \arrow[u, shift right, "X^! \times 0!"'] & X^{\mathbb{R}^+} \times \mathbb{R}^+ \arrow[u, shift right, "X^! \times 0!"']
\end{tikzcd}
\qquad
\begin{tikzcd}
X \times \mathbb{R}^+ \arrow[r, "X^! \times 1"] \arrow[d, shift right=2, "\pi"'] \arrow[d, shift left=2, "\pi"] & X^{\mathbb{R}^+} \times \mathbb{R}^+ \arrow[d, shift right=2, "\mathsf{ev}_0"'] \arrow[d, shift left=2, "\mathsf{ev}"] \\
X \arrow[r, equal] \arrow[u, "1 \times 0!"] & X \arrow[u, "X^! \times 0!"]
\end{tikzcd}
\tag{1$'$}
$$

$$
\begin{tikzcd}
& X^{\mathbb{R}^+} \times \mathbb{R}^+ \arrow[dl, "\overline{\mathsf{min}} \times 1"'] \arrow[d, "\delta_0 \times 1"] \arrow[dr, "\overline{\mathsf{min}} \times 1"] & \\
X^{\mathbb{R}^+} \times \mathbb{R}^+ & (X^{\mathbb{R}^+} \times \mathbb{R}^+)^{\mathbb{R}^+} \times \mathbb{R}^+ \arrow[l, "\mathsf{ev}"] \arrow[r, "\mathsf{ev}^{\mathbb{R}^+} \times \mathbb{R}^+"'] & X^{\mathbb{R}^+} \times \mathbb{R}^+
\end{tikzcd}
$$

$$
\begin{tikzcd}
X^{\mathbb{R}^+} \times \mathbb{R}^+ \arrow[r, "\delta_0 \times 1"] \arrow[d, "\delta_0 \times 1"] & (X^{\mathbb{R}^+} \times \mathbb{R}^+)^{\mathbb{R}^+} \times \mathbb{R}^+ \arrow[d, "\delta_0 \times 1"] \\
(X^{\mathbb{R}^+} \times \mathbb{R}^+)^{\mathbb{R}^+} \times \mathbb{R}^+ \arrow[r, "(\delta_0 \times 1)^{\mathbb{R}^+} \times \mathbb{R}^+"] & ((X^{\mathbb{R}^+} \times \mathbb{R}^+)^{\mathbb{R}^+} \times \mathbb{R}^+)^{\mathbb{R}^+} \times \mathbb{R}^+
\end{tikzcd}
\tag{2$'$}
$$

$$
\begin{tikzcd}
X \times \mathbb{R}^+ \arrow[d, "X^! \times 1"] \arrow[dr, "\pi"] & \\
X^{\mathbb{R}^+} \times \mathbb{R}^+ \arrow[r, "\pi"] & \mathbb{R}^+
\end{tikzcd}
\tag{3$'$}
$$

$$
\begin{tikzcd}
X^{\mathbb{R}^+} \times \mathbb{R}^+ \arrow[r, "\delta_0 \times 1"] \arrow[d, "\mathsf{ev}_0 \times 1"] & (X^{\mathbb{R}^+} \times \mathbb{R}^+)^{\mathbb{R}^+} \times \mathbb{R}^+ \arrow[d, "(\mathsf{ev}_0)^{\mathbb{R}^+} \times \mathbb{R}^+"] \\
X \times \mathbb{R}^+ \arrow[r, "X^! \times 1"] & X^{\mathbb{R}^+} \times \mathbb{R}^+
\end{tikzcd}
\quad
\begin{tikzcd}
X \times \mathbb{R}^+ \arrow[r, "X^! \times \delta_0 \times 1"] \arrow[d, "X^! \times 1"] & (X \times \mathbb{R}^+)^{\mathbb{R}^+} \times \mathbb{R}^+ \arrow[d, "(X^! \times 1)^{\mathbb{R}^+} \times \mathbb{R}^+"'] \\
X^{\mathbb{R}^+} \times \mathbb{R}^+ \arrow[r, "\delta_0 \times 1"] & (X^{\mathbb{R}^+} \times \mathbb{R}^+)^{\mathbb{R}^+} \times \mathbb{R}^+
\end{tikzcd}
\tag{4$'$}
$$

We have shown that the right-hand diagram in (1$'$) commutes in Lemma 4.5.11. The left hand diagram in (1$'$) is composed of two diagrams.

$$
\begin{tikzcd}
X^{\mathbb{R}^+} \times \mathbb{R}^+ \arrow[r, "\delta_0 \times 1"] \arrow[d, "\mathsf{ev}_0"'] & (X^{\mathbb{R}^+} \times \mathbb{R}^+)^{\mathbb{R}^+} \times \mathbb{R}^+ \arrow[d, "\mathsf{ev}_0"'] \\
X \arrow[r, "X^! \times 0!"] & X^{\mathbb{R}^+} \times \mathbb{R}^+
\end{tikzcd}
\quad
\begin{tikzcd}
X^{\mathbb{R}^+} \times \mathbb{R}^+ \arrow[r, "\delta_0 \times 1"] & (X^{\mathbb{R}^+} \times \mathbb{R}^+)^{\mathbb{R}^+} \times \mathbb{R}^+ \\
X \arrow[r, "X^! \times 0!"] \arrow[u, "X^! \times 0!"] & X^{\mathbb{R}^+} \times \mathbb{R}^+ \arrow[u, "X^! \times 0!"]
\end{tikzcd}
\tag{1$''$}
$$

To show that the diagram on the left above commutes, we consider its projections to $X^{\mathbb{R}^+}$ and X, which are depicted below.

$$\begin{tikzcd} X^{\mathbb{R}^+} \times \mathbb{R}^+ \arrow[r, "\overline{\mathsf{min}_3}"] \arrow[d, "X^{0!}"'] & X^{\mathbb{R}^+ \times \mathbb{R}^+} \arrow[d, "X^{1\times 0!}"'] \\ X \arrow[r, "X^!"] & X^{\mathbb{R}^+} \end{tikzcd} \qquad \begin{tikzcd} \mathbb{R}^+ \arrow[r, "\overline{\mathsf{min}}"] \arrow[d, "!"'] & (\mathbb{R}^+)^{\mathbb{R}^+} \arrow[d, "(\mathbb{R}^+)^{0!}"'] \\ * \arrow[r, "0"] & \mathbb{R}^+ \end{tikzcd}$$

The composite $X^{1\times 0!}\overline{\mathsf{min}_3}$ is the transpose of

$$X^{\mathbb{R}^+} \times \mathbb{R}^+ \times \mathbb{R}^+ \xrightarrow{1\times 0!} X^{\mathbb{R}^+} \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \xrightarrow{1\times \mathsf{min}_3} X^{\mathbb{R}^+} \times \mathbb{R}^+ \xrightarrow{\mathsf{ev}} X$$

which is the transpose of $X^{0!}$, so the left-hand diagram above commutes. Similarly, we can see the right-hand diagram above commutes.

To show that the right-hand diagram of (1″) commutes, we also consider its projects to $X^{\mathbb{R}^+\times\mathbb{R}^+}$, $X^{\mathbb{R}^+}$, and X which are displayed below.

$$\begin{tikzcd} X^{\mathbb{R}^+} \times \mathbb{R}^+ \arrow[r, "\overline{\mathsf{min}_3}"] & X^{\mathbb{R}^+ \times \mathbb{R}^+} \\ X \arrow[r, "X^{0!}"] \arrow[u, "X^{0!}\times 0!"'] & X^{\mathbb{R}^+} \arrow[u, "X^{1\times 0!}"'] \end{tikzcd} \quad \begin{tikzcd} \mathbb{R}^+ \arrow[r, "\overline{\mathsf{min}}"] & (\mathbb{R}^+)^{\mathbb{R}^+} \\ * \arrow[r, "0"] \arrow[u, "0"'] & \mathbb{R}^+ \arrow[u, "(\mathbb{R}^+)^!"'] \end{tikzcd} \quad \begin{tikzcd} \mathbb{R}^+ \arrow[r, "1"] & \mathbb{R}^+ \\ * \arrow[r] \arrow[u, "0"'] & * \arrow[u, "0!"'] \end{tikzcd}$$

Since these diagrams commute, we see that all the diagrams of (1) commute.

It is immediately clear that (3′), and thus also (3), commutes.

To see that the remaining diagrams commute, notice that the morphism

$$\delta_0 \times 1 : X^{\mathbb{R}^+} \times \mathbb{R}^+ \to (X^{\mathbb{R}^+} \times \mathbb{R}^+)^{\mathbb{R}^+} \times \mathbb{R}^+$$

in $\mathcal{C}$ underlies the morphism

$$\pi_X^{\mathsf{min}_3 \times 1} \times \overline{\mathsf{min} \times 1} : \pi_X^{\pi_{\mathbb{R}^+}} \to \pi_X^{\pi_{\mathbb{R}^+ \times \mathbb{R}^+}} \times \pi_{\mathbb{R}^+}^{\pi_{\mathbb{R}^+}}$$

(using the notation of the proof of Lemma 4.5.15) in $\mathcal{C}/\mathbb{R}^+$ where $\overline{\mathsf{min} \times 1} : 1_{\mathbb{R}^+} \to \pi_{\mathbb{R}^+}^{\pi_{\mathbb{R}^+}}$ is the transpose of

$$\mathsf{min} \times 1 : \pi_{\mathbb{R}^+} \to \pi_{\mathbb{R}^+}.$$

The morphism

$$X^! \times 1 : X \times \mathbb{R}^+ \to X^{\mathbb{R}^+} \times \mathbb{R}^+$$

in $\mathcal{C}$ underlies the morphism

$$\pi_X^! : \pi_X \to \pi_X^{\pi_{\mathbb{R}^+}}$$

in $\mathcal{C}/\mathbb{R}^+$.

Thus, it suffices to show that the following diagrams commute in $\mathcal{C}/\mathbb{R}^+$.

$$\begin{tikzcd}
 & \pi_X^{\pi_{\mathbb{R}^+}} \arrow[dl, "\pi_X^{\mathsf{min}\times 1}"'] \arrow[d, "\pi_X^{\mathsf{min}_3\times 1}\times\overline{\mathsf{min}\times 1}"] \arrow[dr, "\pi_X^{\mathsf{min}\times 1}"] & \\
\pi_X^{\pi_{\mathbb{R}^+}} & \pi_X^{\pi_{\mathbb{R}^+\times\mathbb{R}^+}} \times \pi_{\mathbb{R}^+}^{\pi_{\mathbb{R}^+}} \arrow[l, "\pi_X^{1\times\Delta}"] \arrow[r, "\mathsf{ev}^{\pi_{\mathbb{R}^+}}"'] & \pi_X^{\pi_{\mathbb{R}^+}}
\end{tikzcd}$$

(2″)

$$\begin{tikzcd}
\pi_X^{\pi_{\mathbb{R}^+}} \arrow[r, "\pi_X^{\mathsf{min}_3\times 1}\times\overline{\mathsf{min}\times 1}"] \arrow[d, "\pi_X^{\mathsf{min}_3\times 1}\times\overline{\mathsf{min}\times 1}"] & \pi_X^{\pi_{\mathbb{R}^+\times\mathbb{R}^+}} \times \pi_{\mathbb{R}^+}^{\pi_{\mathbb{R}^+}} \arrow[d, "1\times\pi_X^{1\times\mathsf{min}_3\times 1}\times\pi_{\mathbb{R}^+}^{\mathsf{min}_3\times 1}\times\overline{\mathsf{min}\times 1}"] \\
\pi_X^{\pi_{\mathbb{R}^+\times\mathbb{R}^+}} \times \pi_{\mathbb{R}^+}^{\pi_{\mathbb{R}^+}} \arrow[r, "\pi_X^{\mathsf{min}_3\times 1\times 1}\times(\overline{\mathsf{min}\times 1\times 1})\times 1"] & \pi_X^{\pi_{\mathbb{R}^+\times\mathbb{R}^+\times\mathbb{R}^+}} \times \pi_{\mathbb{R}^+}^{\pi_{\mathbb{R}^+\times\mathbb{R}^+}} \times \pi_{\mathbb{R}^+}^{\pi_{\mathbb{R}^+}}
\end{tikzcd}$$

$$\begin{tikzcd}
\pi_X^{\pi_{\mathbb{R}^+}} \arrow[r, "\pi_X^{\mathsf{min}_3\times 1}\times\overline{\mathsf{min}\times 1}"] \arrow[d, "\pi_X^{0!\times 1}"] & \pi_X^{\pi_{\mathbb{R}^+\times\mathbb{R}^+}} \times \pi_{\mathbb{R}^+}^{\pi_{\mathbb{R}^+}} \arrow[d, "(\pi_X^{0!\times 1})^{\pi_{\mathbb{R}^+}}"] \\
\pi_X \arrow[r, "\pi_X^!"] & \pi_X^{\pi_{\mathbb{R}^+}}
\end{tikzcd}$$

(4″)

$$\begin{tikzcd}
\pi_X \arrow[r, "\pi_X^!\times\overline{\mathsf{min}\times 1}"] \arrow[d, "\pi_X^!"] & \pi_X^{\pi_{\mathbb{R}^+}} \times \pi_{\mathbb{R}^+}^{\pi_{\mathbb{R}^+}} \arrow[d, "\pi_X^{!\times 1}\times 1"] \\
\pi_X^{\pi_{\mathbb{R}^+}} \arrow[r, "\pi_X^{\mathsf{min}_3\times 1}\times\overline{\mathsf{min}\times 1}"] & \pi_X^{\pi_{\mathbb{R}^+\times\mathbb{R}^+}} \times \pi_{\mathbb{R}^+}^{\pi_{\mathbb{R}^+}}
\end{tikzcd}$$

To show that the first diagram in (2″) commutes, we first observe that the composite

$$\mathbb{R}^+ \times \mathbb{R}^+ \xrightarrow{1\times\Delta} \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \xrightarrow{\mathsf{min}_3\times 1} \mathbb{R}^+ \times \mathbb{R}^+$$

is equal to $\mathsf{min} \times 1 : \mathbb{R}^+ \times \mathbb{R}^+$, so the left-hand triangle commutes. To see that the right-hand triangle in this diagram commutes, note that the composite $\mathsf{ev}^{\pi_{\mathbb{R}^+}}(\pi_X^{\mathsf{min}_3\times 1} \times \overline{\mathsf{min} \times 1})$ is equal to the following composite (where here ev

denotes the counit $\pi_X^{\pi_{\mathbb{R}^+}} \times \pi_{\mathbb{R}^+} \to \pi_X$).

$$\pi_X^{\pi_{\mathbb{R}^+}} \xrightarrow{\pi_X^{\mathsf{min}_3 \times 1} \times \overline{1}} \pi_X^{\pi_{\mathbb{R}^+ \times \mathbb{R}^+}} \times \pi_{\mathbb{R}^+}^{\pi_{\mathbb{R}^+}} \xrightarrow{\pi_X^{\mathsf{min} \times 1 \times 1} \times 1} \pi_X^{\pi_{\mathbb{R}^+ \times \mathbb{R}^+}} \times \pi_{\mathbb{R}^+}^{\pi_{\mathbb{R}^+}} \xrightarrow{\mathsf{ev}^{\pi_{\mathbb{R}^+}}} \pi_X^{\pi_{\mathbb{R}^+}} \qquad (**)$$

Since the composite

$$\pi_X^{\pi_{\mathbb{R}^+ \times \mathbb{R}^+}} \xrightarrow{1 \times \overline{1}} \pi_X^{\pi_{\mathbb{R}^+ \times \mathbb{R}^+}} \times \pi_{\mathbb{R}^+}^{\pi_{\mathbb{R}^+}} \to \pi_X^{\pi_{\mathbb{R}^+}}$$

is π_X^{Δ}, the composite $(**)$ is equal to

$$\pi_X^{\pi_{\mathbb{R}^+}} \xrightarrow{\mathsf{min}_3 \times 1} \pi_X^{\pi_{\mathbb{R}^+ \times \mathbb{R}^+}} \xrightarrow{\pi_X^{\Delta}} \pi_X^{\pi_{\mathbb{R}^+}},$$

and this is equal to $\pi_X^{\mathsf{min} \times 1}$. Thus, the right-triangle in the first diagram in $(2'')$ commutes.

Since min is associative, the second diagram in $(2'')$ commutes.

Since $(\mathsf{min}_3 \times 1)(0! \times 1) = 0! \times 1 : \pi_{\mathbb{R}^+} \to \pi_{\mathbb{R}^+}$, the first diagram of $(4'')$ commutes.

To see that the second diagram of $(4'')$ commutes, we first consider the projections to $\pi_X^{\pi_{\mathbb{R}^+ \times \mathbb{R}^+}}$ and $\pi_{\mathbb{R}^+}^{\pi_{\mathbb{R}^+}}$ separately, as depicted below.

$$\begin{tikzcd}
\pi_X \arrow[r, "\pi_X^!"] \arrow[d, "\pi_X^!"] & \pi_X^{\pi_{\mathbb{R}^+}} \arrow[d, "\pi_X^{! \times 1}"] \\
\pi_X^{\pi_{\mathbb{R}^+}} \arrow[r, "\pi_X^{\mathsf{min}_3 \times 1}"] & \pi_X^{\pi_{\mathbb{R}^+ \times \mathbb{R}^+}}
\end{tikzcd}
\qquad
\begin{tikzcd}
* \arrow[r, "\overline{\mathsf{min} \times 1}"] \arrow[d] & \pi_{\mathbb{R}^+}^{\pi_{\mathbb{R}^+}} \arrow[d, "1"] \\
* \arrow[r, "\overline{\mathsf{min} \times 1}"] & \pi_{\mathbb{R}^+}^{\pi_{\mathbb{R}^+}}
\end{tikzcd}$$

Since these commute, the second diagram of $(4'')$, and thus (4), commutes. $\square$

4.5.6 Summary.

Now we have proven the following theorem and corollary.

Theorem 4.5.24. *Let $\mathcal{C}$ be a finitely complete category which contains an embedding $R : \mathcal{R} \hookrightarrow \mathcal{C}$ which*

1. *preserves the terminal object, the product $\mathbb{R}^+ \times \mathbb{R}^+$, and the pushout $\mathbb{R}^+ \cup_0 \mathbb{R}^+$ and*

2. *takes $\mathbb{R}^+$ to an exponentiable object in $\mathcal{C}$.*

Then $\mathcal{C}$ is equipped with a functorial relation $\mathbf{\Gamma}(R)$ which is a strict Moore relation system.

Proof. In Proposition 4.5.16, we showed that $\mathbf{\Gamma}(R)$ is strictly symmetric. In Proposition 4.5.20, we showed that $\mathbf{\Gamma}(R)$ is strictly transitive. In Proposition 4.5.23, we showed that $\mathbf{\Gamma}(R)$ is strictly homotopical. Therefore, $\mathbf{\Gamma}(R)$ is a strict Moore relation system. □

Theorem 4.5.25. *Consider an embedding $R : \mathcal{R} \hookrightarrow \mathcal{C}$ satisfying the hypotheses of the preceding theorem, 4.5.24. Then $\mathbf{UF\Gamma}(R)$ is an algebraic, type theoretic weak factorization system on $\mathcal{C}$. Furthermore, $\mathcal{C}$ with the right class of $\mathbf{UF\Gamma}(R)$ is a display map category which models Σ and `Id` types.*

Proof. By the preceding theorem, 4.5.24, $\mathbf{\Gamma}(R)$ is a strict Moore relation system. By Theorem 3.2.34, $\mathbf{UF\Gamma}(R)$ is then an algebraic, type theoretic weak factorization system on $\mathcal{C}$. By Theorem 3.5.2, $\mathcal{C}$ with the right class of $\mathbf{UF\Gamma}(R)$ is a display map category which models Σ and `Id` types. □

Corollary 4.5.26. *Consider a subcategory $\mathcal{B}$ of the category $\mathcal{T}$ of topological spaces which (1) generates its products, (2) contains only exponentiable objects of $\mathcal{T}$, and (3) contains $\mathbb{R}^+$.*

Then $\widehat{\mathcal{B}}$ and $\widehat{\mathcal{B}}_{\mathsf{H}}$ contain the subcategory $\mathcal{R}$ of $\mathcal{T}$, and the embedding produces a strict Moore relation system $\mathbf{\Gamma}(R)$ in both $\widehat{\mathcal{B}}$ and $\widehat{\mathcal{B}}_{\mathsf{H}}$. Moreover, this generates the structure of display map category which models Σ and `Id` types in both $\widehat{\mathcal{B}}$ and $\widehat{\mathcal{B}}_{\mathsf{H}}$.

Proof. This follows from Theorem 4.5.25 and Examples 4.5.4 and 4.5.5. □

This generalizes the weak factorization system consisting of trivial Hurewicz cofibrations and Hurewicz fibrations in the categories of compactly generated or compactly generated weak Hausdorff spaces.

Proposition 4.5.27. *When $\mathcal{B}$ is the category of compact Hausdorff spaces, then $\mathbf{UF\Gamma}(R)$ is the weak factorization system consisting of trivial Hurewicz cofibrations and Hurewicz fibrations in $\widehat{\mathcal{B}}$ or $\widehat{\mathcal{B}}_{\mathsf{H}}$.*

Proof. We claim that our factorization coincides with that defined on page 7 of [BR13].

For any map $f : X \to Y$ in $\widehat{\mathcal{B}}$ or $\widehat{\mathcal{B}}_{\mathsf{H}}$, the factorization $\mathbf{UF\Gamma}(R)$ takes f to the following.

$$X \xrightarrow{1\times\eta f} X \times_Y \Gamma Y \xrightarrow{\epsilon_1 \pi_{\Gamma Y}} Y$$

The space ΓY is the subspace of $X^{\mathbb{R}^+} \times \mathbb{R}^+$ consisting of those (p, t) such that $p : \mathbb{R}^+ \to Y$ is constant on $[t, \infty)$. Then the space $X \times_Y \Gamma Y$ is the subspace of $X \times \Gamma Y$ consisting of those (x, p, t) such that $p(0) = f(x)$ and p is constant on $[t, \infty)$. The left factor $1 \times \eta f$ maps a point x to $(x, c_{fx}, 0)$ where c_{fx} is the constant function $p : \mathbb{R}^+ \to Y$ at $f(x)$. The right factor $\epsilon_1 \pi_{\Gamma Y}$ takes (x, p, t) to $p(t)$.

This matches the description of the factorization given in [BR13] which factors any map in $\widehat{\mathcal{B}}$ or $\widehat{\mathcal{B}}_{\mathsf{H}}$ into a trivial Hurewicz cofibration followed by a Hurewicz fibration. $\square$

Corollary 4.5.28. *Consider the topological topos $\mathcal{E}$. It contains the subcategory $\mathcal{R}$ of $\mathcal{T}$, and the embedding produces a strict Moore relation system $\mathbf{\Gamma}$ in $\mathcal{E}$. Moreover, this generates the structure of a display map category which models Σ, `Id`, and Π types in $\mathcal{E}$.*

Proof. Let R denote the embedding $\mathcal{R} \hookrightarrow \mathcal{E}$. By Theorem 4.5.25, $\mathbf{UF\Gamma}(R)$ is an algebraic, type theoretic weak factorization system on $\mathcal{E}$ which generates the structure of a display map category modelling Σ and `Id` types.

Since $\mathcal{E}$ is topos, it is locally cartesian closed. Then by Theorem 3.5.3, the display map category on $\mathcal{E}$ also models Π types. $\square$

Further work.

The first three chapters of this thesis tell a complete and coherent story. We set out to understand those weak factorization systems which underlie display map categories modelling Σ and $\mathtt{Id}$ (and Π) types, and we accomplished this in Theorem 3.5.2.

The fourth chapter, however, is only the beginning of an investigation into the possibility of models of Π types in convenient categories of topological spaces. The possibility of such models is not as far-fetched as one might assume, but ultimately we still expect it to be impossible. Here, we summarize the nexus of the problem.

Consider the results of Section 4.4. Theorem 4.4.7 tells us that if $\widehat{\mathcal{B}}_{\mathsf{H}}$ is to have pre-Π types, then the morphisms of the right class (which we will call fibrations) of the weak factorization system (of Corollary 4.5.26) must be open. Now by examining the point-set topology of the basic fibrations $\epsilon : \Gamma(Y) \to Y \times Y$, we find that a necessary condition for any fibration to be open is that its base space (i.e., codomain) is locally path-connected. By employing the weak factorization system, we find that a sufficient condition for a fibration to be open is that its base space is in $\widehat{\mathcal{D}}$, the coreflective hull of the subcategory spanned by the interval I. Thus many fibrations (those whose base space is in $\mathcal{D}$, or in particular is a CW complex) in $\widehat{\mathcal{K}}_{\mathsf{H}}$, the category of compactly generated weak Hausdorff spaces, are exponentiable, and so Π types exist along them. Thus, one might naively hope that in $\widehat{\mathcal{D}}_{\mathsf{H}}$, or a similar category, all fibrations would be exponentiable.

On the other hand, for the results of Section 4.4 to apply, we need that pullbacks $X \times_Z Y$ (when one of the morphisms is a fibration) have the subspace topology of $X \times Y$. This is the case when the generating subcategory $\widehat{\mathcal{B}}$ generates its closed subspaces (Corollary 4.3.11), but is not the case when $\widehat{\mathcal{B}}$ contains only locally path-connected spaces.

Thus, in the quest to find Π types in a category $\widehat{\mathcal{B}}_{\mathsf{H}}$, we encounter a real tension between these two requirements: that $\widehat{\mathcal{B}}$ generates its closed subspaces and that $\widehat{\mathcal{B}}$ contains only locally path-connected spaces. This tension might be dissolved by showing the results of Section 4.4 hold without the hypothesis on the topology of pullbacks. However, in future work we hope to show that these two inconsistent requirements present a real obstruction to the existence of Π types in a category $\widehat{\mathcal{B}}_{\mathsf{H}}$.

Appendix A

Generating $\mathtt{Id}$ types.

A.1 $\mathtt{Id}$ types from $\mathtt{Id}$ types on objects.

Consider a category of display maps $(\mathcal{M}, \mathcal{D})$ which models Σ types and $\mathtt{Id}$ types of objects.

The $\mathtt{Id}$ types of objects generate a weak factorization structure on $\mathcal{M}$ (Proposition 2.3.4). We will denote this weak factorization structure by $(\lambda, \rho, {}^{\boxslash}\mathcal{D}, \overline{\mathcal{D}})$.

Consider $\{\mathcal{D}, \mathcal{M}\}_Y$ (the full subcategory of $\mathcal{M}/Y$ spanned by $\mathcal{D}$) for any object Y of $\mathcal{M}$. Let $\textsc{dom}_Y : \{\mathcal{D}, \mathcal{M}\}_Y \to \mathcal{M}$ denote the domain functor. Recall that the weak factorization structure on $\mathcal{M}$ induces a weak factorization structure $(\lambda_Y, \rho_Y, {}^{\boxslash}\mathcal{D}_Y, \overline{\mathcal{D}}_Y)$ on $\{\mathcal{D}, \mathcal{M}\}_Y$ (Corollary 1.5.5). The factorization (λ_Y, ρ_Y) takes a morphism $\alpha : f \to g$ of $\{\mathcal{D}, \mathcal{M}\}_Y$ to the following.

$$
\begin{tikzcd}
W \arrow[r, "\lambda\textsc{dom}(\alpha)"] \arrow[dr, "f"'] & W \times_X \mathtt{Id}(X) \arrow[r, "\rho\textsc{dom}(\alpha)"] \arrow[d, "g\rho\textsc{dom}(\alpha)" description] & X \arrow[dl, "g"] \\
& Y &
\end{tikzcd}
$$

The left class ${}^{\boxslash}\mathcal{D}_Y$ is $\textsc{dom}_Y^{-1}({}^{\boxslash}\mathcal{D})$, and the right class $\overline{\mathcal{D}}_Y$ is $\textsc{dom}_Y^{-1}\overline{\mathcal{D}}$.

Lemma A.1.1. The class ${}^{\boxslash}\mathcal{D}_Y$ is stable under pullback along $\mathcal{D}_Y$ in the weak factorization structure $(\lambda_Y, \rho_Y, {}^{\boxslash}\mathcal{D}_Y, \overline{\mathcal{D}}_Y)$ on $\{\mathcal{D}, \mathcal{M}\}_Y$.

Proof. Since all pullbacks along $\mathcal{D}$ exist, $\textsc{dom}_Y : \{\mathcal{D}, \mathcal{M}\}_Y \to \mathcal{M}$ preserves pullbacks along $\mathcal{D}_Y = \textsc{dom}_X^{-1}\mathcal{D}$. Since the weak factorization system $({}^{\boxslash}\mathcal{D}, \overline{\mathcal{D}})$ on $\mathcal{M}$ has an $\mathtt{Id}$-presentation, ${}^{\boxslash}\mathcal{D}$ is stable under pullback along $\mathcal{D}$ (Theorem 3.5.3). Thus ${}^{\boxslash}\mathcal{D}_Y = \textsc{dom}_Y^{-1}{}^{\boxslash}\mathcal{D}$ is stable under pullback along $\mathcal{D}_Y = \textsc{dom}_Y^{-1}\mathcal{D}$. $\square$

Note that in Section 3.3, though we made the hypothesis that the weak factorization system is type theoretic and the category has all pullbacks, we only used the weaker hypothesis that there is a subclass $\mathcal{D}$ of the right class which contains each ϵ_0, ϵ_1 (using the notation of Section 3.3) such that (1) pullbacks of $\mathcal{D}$ exist, (2) $\mathcal{D}$ is stable under pullbacks, (3) the left class is stable under pullback along $\mathcal{D}$, (4) $\mathcal{D}$ contains every morphism to the terminal object, and (5) $\mathcal{D}$ is closed under composition. Call a weak factorization system satisfying this more convoluted hypothesis *$\mathcal{D}$-type theoretic*. Then we have the following analogues of Theorem 3.3.10 and Corollary 3.3.7.

Proposition A.1.2. *Consider a $\mathcal{D}$-type theoretic weak factorization structure W on a category $\mathcal{C}$. Then $\mathbf{VR}(W)$ is an* `Id`*-presentation of W.*

Proposition A.1.3. *Consider a $\mathcal{D}$-type theoretic weak factorization structure W on a category $\mathcal{C}$. Consider a relation $R : \mathfrak{R} \to \mathcal{C}$ with the following components*

$$\begin{tikzcd} Y \arrow[r, "\eta_Y" description] & \mathrm{R}Y, \arrow[l, shift right=2, "\epsilon_{0Y}"'] \arrow[l, shift left=2, "\epsilon_{1Y}"] \end{tikzcd}$$

such that η_Y is in the left class of W and $\epsilon_{0Y} \times \epsilon_{1Y} : RY \to Y \times Y$ is in $\mathcal{D}$. (Note that R is a relation just on Y, not on the whole of $\mathcal{C}$.)

Then for any morphism $f : X \to Y$ of $\mathcal{C}$, in the following factorization

$$X \xrightarrow{1 \times \eta f} X \times_{\epsilon_0} RY \xrightarrow{\epsilon_1 \pi_{RY}} Y$$

the morphism $1 \times \eta f$ is in the left class of W, and $\epsilon_1 \pi_{RY}$ is in $\mathcal{D}$.

Remark A.1.4. Note that we have introduced this more complicated hypothesis to avoid assuming that all pullbacks of $\overline{\mathcal{D}}$ exist. If $\mathcal{M}$ had all pullbacks or were Cauchy complete (Proposition 2.5.7), we would not have had to introduce the notion of $\mathcal{D}$-type theoretic, and we could have used the original Theorem 3.3.10 and Corollary 3.3.7.

Since the weak factorization structure $(\lambda_Y, \rho_Y, {}^{\boxslash}\mathcal{D}_Y, \overline{\mathcal{D}}_Y)$ on $\{\mathcal{D}, \mathcal{M}\}_Y$ is $\mathcal{D}_Y$-type theoretic, it has an `Id`-presentation $\mathbf{VR}(\lambda_Y, \rho_Y, {}^{\boxslash}\mathcal{D}_Y, \overline{\mathcal{D}}_Y)$ which at each

$f : X \to Y$ gives the following relation (depicted as a diagram in $\mathcal{M}$).

$$\begin{tikzcd}
X \arrow[r, "\lambda(\Delta_f)" description] \arrow[dr, bend right, "f"'] & X_\Delta \times_{\epsilon_0} \mathtt{Id}(X \times_Y X) \arrow[l, shift right=2, "\pi_0\rho(\Delta_f)"'] \arrow[l, shift left=2, "\pi_1\rho(\Delta_f)"] \arrow[d, bend left, "(f\times f)\rho(\Delta_f)"] \\
& Y
\end{tikzcd}$$

Now we show that the collection of these $\mathtt{Id}$-presentations gives a model of $\mathtt{Id}$ types in the display map category $(\mathcal{M}, \mathcal{D})$.

Proposition A.1.5. *Consider a category of display maps $(\mathcal{M}, \mathcal{D})$ which models Σ types and $\mathtt{Id}$ types of objects. Then $(\mathcal{M}, \mathcal{D})$ models $\mathtt{Id}$ types.*

Proof. We need to show that for every object $f : X \to Y$ of every category $\{\mathcal{D}, \mathcal{M}\}_Y$, we can find a factorization of the diagonal $\Delta : f \to f \times f$ with the properties required by the definition (2.3.1) of $\mathtt{Id}$-types. By Lemma A.1.1, we know that $(\lambda_Y, \rho_Y, {}^{\boxslash}\mathcal{D}_Y, \overline{\mathcal{D}}_Y)$ is a $\mathcal{D}$-type theoretic weak factorization structure on $\{\mathcal{D}, \mathcal{M}\}_Y$. We use this factorization to factorize the diagonal.

$$f \xrightarrow{\lambda_Y(\Delta_f)} \mathtt{Id}_Y(\Delta_f) \xrightarrow{\rho_Y(\Delta_f)} f \times f$$

We know that $\rho_Y(\Delta_f)$ is in $\mathcal{D}_Y$ since its underlying morphism in $\mathcal{M}$ is $\rho(\textsc{dom}\Delta_f)$, which is in $\mathcal{D}$ by Proposition 2.3.4.

Now we need to check that pullbacks (those required by the definition of $\mathtt{Id}$ types) of $\lambda_Y(\Delta_f)$ are in ${}^{\boxslash}\mathcal{D}_Y$. It suffices to show that the underlying morphisms in $\mathcal{M}$ of those pullbacks are in ${}^{\boxslash}\mathcal{D}$.

Recall that the underlying morphisms of $\lambda_Y(\Delta_f)$ and $\rho_Y(\Delta_f)$ in $\mathcal{M}$ are

$$X \xrightarrow{\lambda(\Delta_f)} X_\Delta \times_{\epsilon_0} \mathtt{Id}(X \times_Y X) \xrightarrow{\rho(\Delta_f)} X \times_Y X$$

where we are abusively denoting the diagonal $X \to X \times_Y X$ by Δ_f in $\mathcal{M}$.

Thus, we need to check that a pullback $\alpha^*\lambda(\Delta_f)$ (as shown below) of $\lambda(\Delta_f)$ along any morphism $\alpha : A \to X$ for $i = 0, 1$ is in ${}^{\boxslash}\mathcal{D}$.

$$\begin{tikzcd}
& X_\Delta \times_{\epsilon_0} \mathtt{Id}(X \times_Y X)_{\pi_i\epsilon_1} \times_\alpha A \arrow[rr] \arrow[dd] & & X_\Delta \times_{\epsilon_0} \mathtt{Id}(X \times_Y X) \arrow[dd, "\pi_i\rho(\Delta_f)"] \\
A \arrow[ur, "\alpha^*\lambda(\Delta_f)"] \arrow[rr] \arrow[dr, equal] & & X \arrow[ur, "\lambda(\Delta_f)"] \arrow[dr, equal] & \\
& A \arrow[rr, "\alpha"] & & X
\end{tikzcd}$$

To do this, first note the pullback $(\Delta_f\pi_X)^*r$ of r (as displayed below) is in ${}^{\boxslash}\mathcal{D}$ by hypothesis,

$$\begin{tikzcd}
& A \times_Y (X_\Delta \times_{\epsilon_0} \mathtt{Id}(X \times_Y X)) \arrow[rr] \arrow[dd] & & \mathtt{Id}(X \times_Y X) \arrow[dd, "\epsilon_0"] \\
A \times_Y X \arrow[ur, "(\Delta_f\pi_X)^*r"] \arrow[rr] \arrow[dr, equal] & & X \times_Y X \arrow[ur, "r"] \arrow[dr, equal] & \\
& A \times_Y X \arrow[rr, "\Delta_f\pi_X"] & & X \times_Y X
\end{tikzcd}$$

and it is isomorphic to $(f\alpha)^*\lambda(\Delta_f)$.

Since the morphism $\rho(\Delta_f)$ is in $\mathcal{D}$, its pullback $(f\alpha)^*\rho(\Delta_f)$ is also in $\mathcal{D}$.

Thus the following diagram

$$\begin{tikzcd}
A \times_Y X \arrow[r, "(f\alpha)^*\lambda(\Delta_f)"] \arrow[dr, "\pi_A"'] & A \times_Y (X_\Delta \times_{\epsilon_0} \mathtt{Id}(X \times_Y X)) \arrow[r, "(f\alpha)^*\rho(\Delta_f)"] \arrow[d, "\pi_A"] & (A \times_Y X) \times_A (A \times_Y X) \arrow[dl, "\pi_A"] \\
& A &
\end{tikzcd}$$

depicts a factorization of the diagonal $\Delta : A \times_Y X \to (A \times_Y X) \times_A (A \times_Y X)$ into the pair $({}^{\boxslash}\mathcal{D}_A, \mathcal{D}_A)$ in the category $\{\mathcal{D}, \mathcal{M}\}_A$.

By Proposition A.1.3, this gives a factorization of the morphism $1_A \times \alpha : A \to A \times_Y X$ into $({}^{\boxslash}\mathcal{D}_A, \mathcal{D}_A)$.

$$A \xrightarrow{\lambda'(1_A \times \alpha)} A \times_{A \times_Y X} (A \times_Y (X_\Delta \times_{\epsilon_0} \mathtt{Id}(X \times_Y X))) \xrightarrow{\rho'(1_A \times \alpha)} A \times_Y X$$

Note that the middle object is isomorphic to $X_{\Delta}\times_{\epsilon_0}\mathtt{Id}(X \times_Y X) \times_{\pi_0\epsilon_1} A$ and then $\lambda'(1_A \times \alpha)$ is isomorphic to

$$A \xrightarrow{\alpha\times r\Delta_f\alpha\times 1_A} X_{\Delta}\times_{\epsilon_0}\mathtt{Id}(X \times_Y X) \times_{\pi_0\epsilon_1} A$$

which is isomorphic to the morphism $\alpha^*\lambda(\Delta_f)$.

Since $\lambda'(1_A \times \alpha)$ is in ${}^{\boxslash}\mathcal{D}_A = \textsc{dom}_A^{-1}\mathcal{D}$, the morphism $\alpha^*\lambda(\Delta_f)$ is in ${}^{\boxslash}\mathcal{D}$. $\square$

A.2 Id types from strong Id types.

Definition A.2.1 ([BG12, Def. 2.2.4]). Consider a category of display maps $(\mathcal{M}, \mathcal{D})$ which models Σ types. We say that it *models strong* $\mathtt{Id}$ *types* if for every $f : X \to Y$ in $\mathcal{D}$, the diagonal $\Delta_f : f \to f \times f$ in $\{\mathcal{D}, \mathcal{M}\}_Y$ has a factorization $\Delta_f = \epsilon_f r_f$

$$\begin{tikzcd}
X \arrow[r, "r_f"] \arrow[dr, "f"'] & \mathtt{Id}(f) \arrow[r, "\epsilon_f"] \arrow[d, "\iota_f"] & X \times_Y X \arrow[dl, "f\times f"] \\
& Y &
\end{tikzcd}$$

in $\{\mathcal{D}, \mathcal{M}\}_Y$ such that

1. ϵ_f is in $\mathcal{D}$,
2. every pullback of r_f as shown below is in ${}^{\boxslash}\mathcal{D}$, and

$$\begin{tikzcd}
& \alpha^*\mathtt{Id}(f) \arrow[rr] \arrow[dd] & & \mathtt{Id}(f) \arrow[dd, "\iota_f"] \\
\alpha^* f \arrow[ur, "\alpha^* r_f"] \arrow[rr] \arrow[dr] & & X \arrow[ur, "r_f"] \arrow[dr, "f"] & \\
& A \arrow[rr, "\alpha"] & & Y
\end{tikzcd}$$

3. every pullback of r_f along a display map is in ${}^{\boxslash}\mathcal{D}$.

First of all, we know that if $(\mathcal{M}, \mathcal{D})$ models $\mathtt{Id}$ types (Definition 2.3.1), then they are strong.

Proposition A.2.2. *Consider a category of display maps $(\mathcal{M}, \mathcal{D})$ which models Σ types and* $\mathtt{Id}$ *types. Then it models strong* $\mathtt{Id}$ *types.*

Proof. First of all note that a pullback of r_f of the form

$$
\begin{tikzcd}
& \alpha^*\mathtt{Id}(f) \arrow[rr] \arrow[dd] & & \mathtt{Id}(f) \arrow[dd, "\iota_f"] \\
\alpha^* f \arrow[ur, "\alpha^* r_f"] \arrow[rr] \arrow[dr] & & X \arrow[ur, "r_f"] \arrow[dr, "f"] & \\
& A \arrow[rr, "\alpha"] & & Y
\end{tikzcd}
$$

can be obtained as a pullback of the following form.

$$
\begin{tikzcd}
& \alpha^*\mathtt{Id}(f) \arrow[rr] \arrow[dd] & & \mathtt{Id}(f) \arrow[dd, "\epsilon_1"] \\
\alpha^* f \arrow[ur, "\alpha^* r_f"] \arrow[rr] \arrow[dr] & & X \arrow[ur, "r_f"] \arrow[dr, equal] & \\
& f^*A \arrow[rr, "f^*\alpha"] & & X
\end{tikzcd}
$$

Thus, the pullback $\alpha^* r_f$ is in ${}^{\boxslash}\mathcal{D}$.

By Theorem 3.4.4, we know that the weak factorization system generated by the Id types is type-theoretic. In particular, pullbacks of $r_f \in {}^{\boxslash}\mathcal{D}$ along morphisms of $\mathcal{D}$ are in ${}^{\boxslash}\mathcal{D}$. □

Now, we emulate the proof of Lemma 11 in [GG08] to show that strong Id types give Id types on objects. One could also use the proof of Theorem 3.3.10 to show this.

Proposition A.2.3 ([GG08, Lem. 11]). *Consider a category of display maps $(\mathcal{M}, \mathcal{D})$ which models Σ types and strong* Id *types. Then it models* Id *types on objects.*

Proof. Consider a object Y of $\mathcal{M}$. We need to show that the pullback $1 \times r_Y f$ of r_Y (shown below) is in ${}^{\boxslash}\mathcal{D}$ for $i = 0, 1$. We will focus on the case $i = 0$. The proof of the other is case is analogous.

$$
\begin{tikzcd}
& X \times_Y \mathtt{Id}(Y) \arrow[rr] \arrow[dd] & & \mathtt{Id}(Y) \arrow[dd, "\pi_i \epsilon_Y"] \\
X \arrow[ur, "1\times r_Y f"] \arrow[rr] \arrow[dr, equal] & & Y \arrow[ur, "r_Y"] \arrow[dr, equal] & \\
& X \arrow[rr, "f"] & & Y
\end{tikzcd}
$$

That is, we need to find a solution to the lifting problem on the left below for any $c \in \mathcal{D}$.

$$\begin{tikzcd}
X \arrow[r, "x"] \arrow[d, "1\times rf"'] & C \arrow[d, "c"] \\
X \times_Y \mathtt{Id}(Y) \arrow[r, "y"] \arrow[ur, dashed] & D
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[rr, "x"] \arrow[d, "1\times rf"'] & & \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & C \arrow[d, "c"] \\
X \times_Y \mathtt{Id}(Y) \arrow[rr, equal] \arrow[urr, dashed] & & X \times_Y \mathtt{Id}(Y) \arrow[r, "y"] & D
\end{tikzcd}$$

Pulling back c along y as shown on the right above, we see that it is only necessary to find a lift of $\alpha^* r_f$ against a display map with codomain $X \times_Y \mathtt{Id}(Y)$. Thus, we seek a solution for the following lifting problem.

$$\begin{tikzcd}
X \arrow[r, "x"] \arrow[d, "1\times rf"'] & C \arrow[d, "c"] \\
X \times_Y \mathtt{Id}(Y) \arrow[r, equal] \arrow[ur, dashed] & X \times_Y \mathtt{Id}(Y)
\end{tikzcd}$$

Let $C \times_{\epsilon_0} \mathtt{Id}(Y)$ denote the object obtained in the following pullback. (As usual, we let $\epsilon_i : \mathtt{Id}(Y) \to X$ denote the composition $\pi_i \epsilon_f$ for $i = 0, 1$.)

$$\begin{tikzcd}
C \times_{\epsilon_0} \mathtt{Id}(Y) \arrow[r] \arrow[d, "c\times \mathtt{Id}(Y)"'] \arrow[dr, phantom, very near start, "\lrcorner"] & C \arrow[d, "c"] \\
X \times_Y \mathtt{Id}(Y) \,{}_{\epsilon_1}\!\times_{\epsilon_0} \mathtt{Id}(Y) \arrow[r, "\pi"] & X \times_Y \mathtt{Id}(Y)
\end{tikzcd}$$

Let μ denote the solution to the following lifting problem.

$$\begin{tikzcd}
\mathtt{Id}(Y) \arrow[r, equal] \arrow[d, "1\times r\epsilon_1"'] & \mathtt{Id}(Y) \arrow[d, "\epsilon_Y"] \\
\mathtt{Id}(Y)_{\epsilon_1}\!\times_{\epsilon_0} \mathtt{Id}(Y) \arrow[r, "\epsilon_0\pi_0 \times \epsilon_1\pi_1"] \arrow[ur, dashed, "\mu"] & Y \times Y
\end{tikzcd}$$

(The left-hand map above is a pullback of r_f along the display map ϵ_1, so it is in ${}^{\boxslash}D$. Then the lift μ exists.)

Then consider the following commutative diagram.

$$\begin{tikzcd}
X \arrow[r, "x"] \arrow[d, "1\times rf"] & C \arrow[rr, equal] \arrow[d, "1\times r\epsilon_1 c"] & & C \arrow[d, "c"] \\
X \times_Y \mathtt{Id}(Y) \arrow[r, "x\times r\epsilon_1"'] & C \times_{\epsilon_0} \mathtt{Id}(Y) \arrow[r, "c\times \mathtt{Id}(Y)"'] \arrow[urr, dashed, "\ell"] & X \times_Y \mathtt{Id}(Y) \,{}_{\epsilon_1}\!\times_{\epsilon_0} \mathtt{Id}(Y) \arrow[r, "1\times\mu"'] & X \times_Y \mathtt{Id}(Y)
\end{tikzcd}$$

The lift ℓ above exists since the left-hand morphism $1 \times r\epsilon_1 c$ is a pullback of r_Y along the display map $\epsilon_1 c : C \to Y$, and so is in ${}^{\boxslash}\mathcal{D}$.

Now the composite along the bottom of the above diagram is equal to $1 \times \mu(r\epsilon_0 \times 1)$, so we get the commutative diagram below (where ℓ' is the composite $\ell(x \times r\epsilon_1)$).

$$\begin{tikzcd}
X \arrow[r, "x"] \arrow[d, "1\times rf"'] & C \arrow[d, "c"] \\
X \times_Y \mathtt{Id}(Y) \arrow[ur, "\ell'"] \arrow[r, "1\times\mu(r\epsilon_0\times 1)"] & X \times_Y \mathtt{Id}(Y)
\end{tikzcd}$$

Now this diagram is the one we sought save the bottom morphism. In what follows, we correct this to the identity on $X \times_Y \mathtt{Id}(Y)$.

Consider the following lifting problem in $\{\mathcal{D}, \mathcal{M}\}_Y$ which has a solution m.

$$\begin{tikzcd}
Y \arrow[r, "r(Y)"] \arrow[d, "r(Y)"'] & \mathtt{Id}(Y) \arrow[r, "r(\epsilon_0)"] & \mathtt{Id}(\epsilon_0) \arrow[d, "\epsilon(\epsilon_0)"] \\
\mathtt{Id}(Y) \arrow[urr, dashed, "m"] \arrow[rr, "\mu(r\epsilon_0\times 1)\times 1"'] \arrow[dr, "\epsilon_0"'] & & \mathtt{Id}(Y)_{\epsilon_0}\times_{\epsilon_0}\mathtt{Id}(Y) \arrow[dl, "\epsilon_0"] \\
& Y &
\end{tikzcd}$$

Pulling this back along $f : X \to Y$, we get the following diagram in $\{\mathcal{D}, \mathcal{M}\}_X$.

$$\begin{tikzcd}
X \arrow[r, "1\times rf"] \arrow[d, "1\times rf"'] & X \times_Y \mathtt{Id}(Y) \arrow[r, "1\times r(\epsilon_0)"] & X \times_Y \mathtt{Id}(\epsilon_0) \arrow[d, "1\times\epsilon(\epsilon_0)"] \\
X \times_Y \mathtt{Id}(Y) \arrow[urr, dashed, "1\times m"] \arrow[rr, "1\times\mu(r\epsilon_0\times 1)\times 1"'] \arrow[dr, "\pi_X"'] & & X \times_Y \mathtt{Id}(Y)_{\epsilon_0}\times_{\epsilon_0}\mathtt{Id}(Y) \arrow[dl, "\pi_X"] \\
& X &
\end{tikzcd}$$

A solution to the following lifting problem exists since $1 \times r(\epsilon_0)$ is one of pullbacks of $r(\epsilon_0)$ hypothesized to be in ${}^{\boxslash}\mathcal{D}$ (where π_X is the projection $X \times_Y \mathtt{Id}(Y) \to X$).

$$\begin{tikzcd}
X \times_Y \mathtt{Id}(Y) \arrow[r, "r(\pi_X)"] \arrow[d, "1\times r(\epsilon_0)"'] & \mathtt{Id}(\pi_X) \arrow[d, "\epsilon(\pi_X)"] \\
X \times_Y \mathtt{Id}(\epsilon_0) \arrow[ur, dashed, "n"] \arrow[r, "1\times\epsilon(\epsilon_0)"'] & X \times_Y \mathtt{Id}(Y)_{\epsilon_0}\times_{\epsilon_0}\mathtt{Id}(Y)
\end{tikzcd}$$

We also get a solution to the following lifting problem.

$$\begin{tikzcd}
C \arrow[rr, equal] \arrow[d, "1\times r(\pi_X)"'] & & C \arrow[d, "c"] \\
C \times_{(X \times_Y \mathtt{Id}(Y))} \mathtt{Id}(\pi_X) \arrow[rr, "\epsilon_1(\pi_X)"'] \arrow[urr, dashed, "p"] & & X \times_Y \mathtt{Id}(Y)
\end{tikzcd}$$

Now, putting all of these lifts together, the composite

$$X \times_Y \mathtt{Id}(Y) \xrightarrow{\ell' \times n(1\times m)} C \times_{(X \times_Y \mathtt{Id}(Y))} \mathtt{Id}(\pi_X) \xrightarrow{p} C$$

is the lift we sought. □

Now using our earlier result, we see that strong Id types are Id types.

Corollary A.2.4. *Consider a category of display maps $(\mathcal{M}, \mathcal{D})$ which models Σ types and strong* Id *types. Then it models* Id *types.*

Proof. The preceding proposition says that $(\mathcal{M}, \mathcal{D})$ models Id types on objects. Then Proposition A.1.5 says that $(\mathcal{M}, \mathcal{D})$ models Id types. □

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