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An optimal family of eighth-order simple-root finders with weight functions dependent on function-to-function ratios and their dynamics underlying extraneous fixed points

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ABSTRACT

We extend in this paper an optimal family of three-step eighth-order methods developed by Džunić et al. (2011) with higher-order weight functions employed in the second and third sub-steps and investigate their dynamics under the relevant extraneous fixed points among which purely imaginary ones are specially treated for the analysis of the rich dynamics. Their theoretical and computational properties are fully investigated along with a main theorem describing the order of convergence and the asymptotic error constant as well as proper choices of special cases. A wide variety of relevant numerical examples are illustrated to confirm the underlying theoretical development. In addition, this paper investigates the dynamics of selected existing optimal eighth-order iterative maps with the help of illustrative basins of attraction for various polynomials.

1. Introduction

Root-finding problems arise in many areas of natural and physical sciences, which include initial- and boundary-value problems, heat and fluid flow problems, electrostatics problems as well as problems associated with global positioning systems(GPS). We find it valuable to develop an efficient algorithm finding accurate numerical roots of the governing equation under consideration. It has been about half a century since Traub [1] performed in the 1960s the extensive analyses on qualitative as well as quantitative viewpoints of iterative methods locating numerical roots for nonlinear equations. A number of authors [2–7] have developed high-order multipoint methods to solve a given nonlinear equation in the form of $f(x) = 0$. In 2011, Džunić et al. [5] extensively investigated a family of optimal three-point methods for solving nonlinear equations using two parametric functions. In 2012, Petković et al. [8] collected and updated the state of the art of multipoint methods. They showed that many new methods are just special cases or reformulation of known methods. A numerical scheme is said to be optimal according to Kung–Traub's conjecture [9] that any multipoint method [8] without memory can attain its convergence order of at most 2^{r-1} for r functional evaluations with $r \in \mathbb{N}$. For the purpose of comparison, we employ several existing eighth-order methods in [3,4,7], being respectively presented by (1.1), (1.5), and (1.6).

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- Cordero–Torregrosa–Vassileva method (CTV)

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \\ z_n = y_n - \dfrac{1}{1-2s}\dfrac{f(y_n)}{f'(x_n)}, \\ x_{n+1} = w_n - \dfrac{f(z_n)}{f'(x_n)}\dfrac{3(\beta_2+\beta_3)(w_n-z_n)}{\beta_1(w_n-z_n)+\beta_2(y_n-x_n)+\beta_3(z_n-x_n)}, \end{cases} \tag{1.1}$$

where

$$s = \frac{f(y_n)}{f(x_n)}, \tag{1.2}$$

$$u = \frac{f(z_n)}{f(y_n)}, \tag{1.3}$$

$$w_n = z_n - \frac{f(z_n)}{f'(x_n)}\left(\frac{1-s}{1-2s} + \frac{u}{2(1-2u)}\right)^2, \tag{1.4}$$

and β_1, β_2, and β_3 are real parameters with $\beta_2+\beta_3 \neq 0$.

- Liu–Wang method (LW)

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \\ z_n = y_n - \dfrac{1}{1-2s}\cdot\dfrac{f(y_n)}{f'(x_n)}, \\ x_{n+1} = z_n - \left[\left(\dfrac{1-s}{1-2s}\right)^2 + \dfrac{u}{1-\alpha_1 u} + \dfrac{4su}{1+\alpha_2 su}\right]\cdot\dfrac{f(z_n)}{f'(x_n)}, \quad \alpha_1, \alpha_2 \in \mathbb{R}. \end{cases} \tag{1.5}$$

- Sharma–Arora method (SA)

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \\ z_n = y_n - \dfrac{f(y_n)}{2f[y_n,x_n]-f'(x_n)} = y_n - \dfrac{1}{1-2s}\cdot\dfrac{f(y_n)}{f'(x_n)}, \\ x_{n+1} = z_n - \dfrac{f[z_n,y_n]}{f[z_n,x_n]}\cdot\dfrac{f(z_n)}{2f[z_n,y_n]-f[z_n,x_n]} = z_n - \dfrac{(1-s)^2(1-u)}{(1-2s)(1-su)(1-2s-2u+3su)}\cdot\dfrac{f(z_n)}{f'(x_n)}, \\ \text{where } f[r,t] = \dfrac{f(r)-f(t)}{r-t}. \end{cases} \tag{1.6}$$

Definition 1 (*Error Equation, Asymptotic Error Constant, Order of Convergence*)**.** Let $x_0, x_1, \ldots, x_n, \ldots$ be a sequence of numbers converging to α. Let $e_n = x_n - \alpha$ for $n = 0, 1, 2, \ldots$. If constants $p \geq 1$, $c \neq 0$ exist in such a way that $e_{n+1} = c\, {e_n}^p + O(e_n^{p+1})$ called the *error equation*, then p and $\eta = |c|$ are said to be the *order of convergence* and the *asymptotic error constant*, respectively. It is easy to find $c = \lim_{n\to\infty} \frac{e_{n+1}}{{e_n}^p}$. Some authors call c itself the asymptotic error constant.

In this paper, our special attention is paid to the dynamics of a generic family of three-point eighth-order methods. To this end, we extend an optimal eighth-order family of iterative methods developed by Džunić et al. [5] with higher-order weight functions employed in the second and third sub-steps in the following form:

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \\ z_n = y_n - L_f(s)\cdot\dfrac{f(y_n)}{f'(x_n)}, \\ x_{n+1} = z_n - Q_f(s,u)\cdot\dfrac{f(z_n)}{f'(x_n)}, \end{cases} \tag{1.7}$$

where s and u are given by (1.2) and (1.3), respectively and $L_f : \mathbb{C} \to \mathbb{C}$ is a weight function being analytic [10] in a neighborhood of 0 and $Q_f : \mathbb{C}^2 \to \mathbb{C}$ is a weight function being holomorphic [11,12] in a neighborhood of $(0, 0)$. Note that (1.1), (1.5) and (1.6) are special cases of (1.7) with $L_f(s) = 1/(1-2s)$ and $Q_f(s,u)$ as shown in their respective equations.

It is interesting to see that (1.6) can be expressed by means of fifth-order rational weight function $Q_f(s,u)$ without using divided differences.

Our primary aim of this paper is not only to extend the existing optimal eighth-order methods by considering more generic forms of both weight functions $L_f(s)$ and $Q_f(s,u)$, but also to investigate their dynamics by means of basins of attractions behind the extraneous fixed points (to be described in Section 4) when applied to a wide variety of prototype polynomials.

Iterative methods for nonlinear equations in general require suitable initial guesses close to exact roots of the equations in order to guarantee the desired convergence. It is, however, a difficult matter to select such suitable initial guesses due to the chaotic nature inherent in many environments including computational precision, error bound, the exact root and the equation under consideration. The basin of attraction is the set of initial guesses leading to long-time behavior that approaches the attractors (e.g., periodic, quasi-periodic or chaotic behaviors of different types) under the action of the iterative function. Hence, convergence behavior of global character can be conveniently observed on the basin of attraction. The basic topological structure of such a basin of attraction as a region can vary greatly from system to system with a variety forms of weight functions. If both weight functions L_f and Q_f contain multiple parameters, then the basin of attraction of the iterative methods will vary depending on the selection of free parameters.

Our secondary aim of this paper is to choose proper parameters giving the basin of attraction with a larger region of convergence. The presence of extraneous fixed points may induce attractive, indifferent, repulsive as well as other chaotic orbits influencing the relevant dynamics of the iterative methods. Notice that the imaginary axis symmetrically divides the whole complex plane into two half planes. Since we observe the convergence behavior in the dynamical planes through the basins of attraction in the form of a square region centered at the origin, the resulting dynamics behind the extraneous fixed points on the symmetry (imaginary) axis is expected to be less influenced by the possible periodic or chaotic attractors. Thus, it would be preferable to choose free parameters in such a way that the extraneous fixed points should be located on the imaginary axis.

In Section 2, the main theorem regarding the convergence behavior is described with appropriate forms of two weight functions L_f and Q_f. Section 3 investigates some special cases of $Q_f(s,u)$. Section 4 fully discusses the extraneous fixed points among which purely imaginary ones are specially treated. Section 5 presents numerical experiments along with the illustration of the relevant dynamics and concluding remarks.

2. Main theorem

We will state in this section the main theorem without details of its proof describing the methodology and convergence behavior on iterative scheme (1.7), due to the detailed self-explanatory proof given in [8]. With more generic forms of $L_f(s)$ and $Q_f(s,u)$ introduced, the resulting error equation becomes more extended, which is remarked after the theorem below:

Theorem 2.1. *Assume that $f:\mathbb{C}\to\mathbb{C}$ has a simple root α and is analytic in a region containing α. Let $c_j=\frac{f^{(j)}(\alpha)}{j!f'(\alpha)}$ for $j=2,3,\ldots$. Let x_0 be an initial guess chosen in a sufficiently small neighborhood of α. Let $L_f:\mathbb{C}\to\mathbb{C}$ be analytic in a neighborhood of 0. Let $L_i=\frac{1}{i!}\frac{d^i}{ds^i}L_f(s)\big|_{(s=0)}$ for $0\le i\le 4$. Let $Q_f:\mathbb{C}^2\to\mathbb{C}$ be holomorphic in a neighborhood of $(0,0)$. Let $Q_{ij}=\frac{1}{i!j!}\frac{\partial^{i+j}}{\partial s^i\partial u^j}Q_f(s,u)\big|_{(s=0,u=0)}$ for $0\le i\le 4$ and $0\le j\le 2$. If $L_0=1, L_1=2, Q_{00}=1, Q_{01}=1, Q_{10}=2, Q_{20}=1+L_2, Q_{11}=4, Q_{30}=2L_2+L_3-4$ are satisfied, then iterative scheme (1.7) defines a family of eighth-order methods satisfying the error equation below: for $n=0,1,2,\ldots$,*

$$e_{n+1}=c_2\{c_3+c_2^2(L_2-5)\}\left[-c_2c_4+c_3^2(Q_{02}-1)+c_2^2c_3\phi_1+c_2^4\phi_2\right]e_n^8+O(e_n^9), \tag{2.1}$$

where $\phi_1=19-10Q_{02}+L_2(2Q_{02}-1)-Q_{21}$ and $\phi_2=-2L_3-L_4+L_2\{(L_2-10)Q_{02}-Q_{21}+9\}+5(5Q_{02}+Q_{21}-9)+Q_{40}$.

Remark 2.2. Since $L_f(s)$ is expanded up to fourth-order terms in s, and $Q_f(s,u)$ expanded up to fourth- and second-order terms in s and u, respectively, we obtain a more general form of the error equation shown in the above theorem. Indeed, we find that a special case of selected parameters with $L_2=2,\ L_3=L_4=Q_{30}=Q_{40}=0$ immediately yields the error equation claimed in [8].

3. Special cases of weight functions

As a result of Theorem 2.1, we easily find $L_f(s)$ and $Q_f(s,u)$ in the form of Taylor polynomials as follows.

$$\begin{cases} L_f(s) = 1+2s+L_2s^2+L_3s^3+L_4s^4+O(e^5),\\ Q_f(s,u) = 1+u+Q_{02}u^2+s(2+4u)+s^2(Q_{20}+Q_{21}u)+Q_{30}s^3+Q_{40}s^4+O(e^5), \end{cases} \tag{3.1}$$

where $Q_{20}=1+L_2$ and $Q_{30}=2L_2+L_3-4$, while $L_2, L_3, L_4, Q_{02}, Q_{21}, Q_{40}$ may be free parameters.

Table 1
Parameter values of $L_2, A_2, B_2, a_1, a_2, a_3, a_4, a_5, a_6, b_0, b_1, b_2, b_3, b_4, b_5, b_6$ for all cases.

Case	(L_2, A_2, B_2)	a_1	a_2	a_3	a_4	a_5	a_6	b_0	b_1	b_2	b_3	b_4	b_5	b_6
1A	(0, 0, 0)	−4	7	−6	0	0	0	1	−2	0	0	1	0	0
1B	(0, 4, 0)	−2	3	8	−1	0	0	1	0	0	0	0	0	0
1C	(4, −4, 0)	−2	−1	0	0	0	0	1	0	0	0	1	2	0
1D	(3, 0, 0)	−2	$\frac{3}{4}$	0	−1	0	0	1	0	$\frac{3}{4}$	0	0	0	0
1E	(3, 0, 0)	−2	0	0	−1	0	0	1	0	0	$-\frac{3}{2}$	0	0	0
1F	(5, 0, 0)	−4	0	0	0	0	0	1	−2	−2	$-\frac{11}{2}$	1	0	0
1G	(3, −3, 0)	−2	−2	$-\frac{5}{2}$	−1	0	0	1	0	0	0	0	0	0
1H	(5, 10, 2)	$-\frac{9}{2}$	5	$-\frac{3}{2}$	−1	$\frac{5}{2}$	0	1	$-\frac{5}{2}$	2	0	0	0	0
1I	(5, 10, 2)	−2	0	$-\frac{77}{8}$	−1	0	0	1	0	2	$-\frac{25}{8}$	0	0	0
1J	(5, 10, 2)	$-\frac{29}{10}$	$-\frac{1}{5}$	$-\frac{7}{10}$	−1	$\frac{19}{10}$	0	1	$-\frac{9}{10}$	0	0	0	1	0
2A1	(0, 0, −8)	−2	3	0	−1	0	3	1	0	0	0	0	0	0
2A2	(0, 0, −8)	−4	4	0	0	0	0	1	−2	−3	0	1	0	3
2A3	(0, 0, −8)	−2	3	0	0	0	3	1	0	0	0	1	2	0
2B1	(4, 8, 0)	−2	−1	0	−1	0	1	1	0	0	0	0	0	0
2B2	(4, 8, 0)	−2	0	−2	0	−2	0	1	0	1	0	1	0	1
2B3	(4, 8, 0)	−2	−1	0	0	0	0	1	0	0	0	1	2	2
2C1	(8, 16, 8)	$-\frac{9}{2}$	0	$-\frac{7}{2}$	−1	$\frac{5}{2}$	$-\frac{3}{2}$	1	$-\frac{5}{2}$	0	0	0	0	0
2C2	(8, 16, 8)	0	0	0	−2	0	0	1	2	9	44	−1	0	−18
2C3	(8, 16, 8)	$-\frac{26}{5}$	$\frac{7}{5}$	0	0	0	1	1	$-\frac{16}{5}$	0	0	1	$-\frac{6}{5}$	0
2D1	(12, 24, 16)	−2	−9	−48	0	−2	−9	1	0	0	0	1	0	0
2D2	(12, 24, 16)	0	0	0	0	−4	0	1	2	13	92	1	0	13
2D3	(12, 24, 16)	$-\frac{22}{3}$	$\frac{5}{3}$	0	0	0	−1	1	$-\frac{16}{3}$	0	0	1	$-\frac{10}{3}$	0
3Ai[a]	(4, 8, 0)	a_1	a_2	a_3	a_4	a_5	a_6	1	b_1	(10, 25, 35)	0	b_4	b_5	b_6
3Bi	(4, 8, 0)	a_1	a_2	a_3	a_4	a_5	a_6	1	b_1	$(3, \frac{2909}{684}, 6)$	0	b_4	b_5	b_6
3Ci	(4, 8, 0)	a_1	a_2	a_3	a_4	a_5	a_6	1	b_1	(1, 1.5, 2)	0	b_4	b_5	b_6
3Di	(4, 8, 0)	a_1	a_2	a_3	a_4	a_5	a_6	1	b_1	$(-1, 0, \frac{2}{11})$	0	b_4	b_5	b_6
3Ei	(4, 8, 0)	a_1	a_2	a_3	a_4	a_5	a_6	1	b_1	$(6, \frac{11\,347}{1444}, 18)$	0	b_4	b_5	b_6
3Fi	(4, 8, 0)	a_1	a_2	a_3	a_4	a_5	a_6	1	b_1	$(\frac{18}{7}, 3, 3.1)$	0	b_4	b_5	b_6
3Gi	(4, 8, 0)	a_1	a_2	a_3	a_4	a_5	a_6	1	b_1	(0.4, 0.5, 0.9)	0	b_4	b_5	b_6
3Hi	(4, 8, 0)	a_1	a_2	a_3	a_4	a_5	a_6	1	b_1	$(\frac{1531}{244}, 8, 10)$	0	b_4	b_5	b_6
3Ii	(4, 8, 0)	a_1	a_2	a_3	a_4	a_5	a_6	1	b_1	(1, 1.5, 2.5)	0	b_4	b_5	b_6
3Ji	(4, 8, 0)	a_1	a_2	a_3	a_4	a_5	a_6	1	b_1	$(3.5, 4, \frac{22}{5})$	0	b_4	b_5	b_6
4A	(0, 0, 0)	6	21	96	−2	1	4	−1	0	0	0	1	0	1
4B	(0, 4, 0)	−2	0	0	$-\frac{4}{3}$	−2	1	0	0	3	−20	3	1	−2
4C	(4, −4, 0)	0	0	0	−3	4	−1	3	3	6	13	−1	0	−1
4D	(3, 0, 0)	−4	$\frac{19}{2}$	0	0	−4	0	0	1	0	0	1	1	1
4E	(5, 0, 0)	$-\frac{29}{20}$	0	0	−1	0	1	0	−2	$\frac{129}{10}$	0	1	0	1
4F	(3, −3, 0)	−3	0	0	−2	−2	−1	0	0	2	0	1	−1	1

[a] **3Ai** = (**3A1**, **3A2**, **3A3**). For **Cases 3Ai–3Pi**, values for a_1–b_6 other than b_2 and $b_3 = 0$ are not explicitly displayed due to their lengthy expression dependent on b_2. Sub-cases of **Case 3** are listed in order with a triplet of b_2 values. In addition, for **Cases 4A–4F**, parameter values of a triplet (b_7, b_8, b_9) are to be separately listed in order with $(b_7, b_8, b_9) \in \{(0, 1, 0), (0, 1, 0), (0, 1, 0), (0, 0, 1), (1, 0, 4), (0, 0, 2)\}$.

Although various forms of weight functions $L_f(s)$ and $Q_f(s, u)$ are available, in this paper we will first limit ourselves to considering only low-order rational weight functions with real coefficients in the form below.

$$\begin{cases} L_f(s) = \dfrac{1 + B_1 s + B_2 s^2}{1 + A_1 s + A_2 s^2}, \\ Q_f(s, u) = \dfrac{b_0 + b_1 s + b_2 s^2 + b_3 s^3 + u(b_4 + b_5 s + b_6 s^2)}{1 + a_1 s + a_2 s^2 + a_3 s^3 + u(a_4 + a_5 s + a_6 s^2)}, \end{cases} \tag{3.2}$$

where $A_2, B_2, a_i, b_i \in \mathbb{R}$ and $B_1 = \frac{1}{2}(B_2 - A_2 - L_2) + 2,\ A_1 = B_1 - 2,\ b_0 = 1,\ b_1 = -2 - 2a_4 - a_5 + b_5,\ b_4 = 1 + a_4,\ a_1 = -4 - 2a_4 - a_5 + b_5,\ a_2 = 7 + 4a_4 + 2a_5 + b_2 - 2b_5 - L_2,\ a_3 = -6 - 6a_4 - 3a_5 - 2b_2 + b_3 + 3b_5 + A_2(2 - \frac{L_2}{2}) + (4 + 2a_4 + a_5 + \frac{B_2}{2} - b_5)L_2 - \frac{L_2^2}{2}$, with 10 free parameters $L_2, A_2, B_2, a_4, a_5, a_6, b_2, b_3, b_5, b_6$ being available.

In the current investigation, we will select 4 cases whose free parameters are suitably chosen for simplified forms of weight functions. The parameters chosen for each case are summarized in Table 1.

We first consider Case 1 for a rational weight function $Q_f(s, u)$ with $a_6 = b_6 = 0$ and investigate its sub-cases as follows.

Case 1: Cubic-order rational weight function $Q_f(s,u)$ with $a_6 = b_6 = 0$.

$$\begin{cases} L_f(s) = \dfrac{1+B_1 s + B_2 s^2}{1+A_1 s + A_2 s^2}, \\ Q_f(s,u) = \dfrac{b_0 + b_1 s + b_2 s^2 + b_3 s^3 + u(b_4 + b_5 s)}{1 + a_1 s + a_2 s^2 + a_3 s^3 + u(a_4 + a_5 s)}. \end{cases} \tag{3.3}$$

Case 1A: When $L_2 = A_2 = B_2 = a_4 = a_5 = b_5 = b_2 = b_3 = 0$,

$$\begin{cases} L_f(s) = 1 + 2s, \\ Q_f(s,u) = \dfrac{1 - 2s + u}{1 - 4s + 7s^2 - 6s^3}. \end{cases} \tag{3.4}$$

Case 1B: When $L_2 = B_2 = b_5 = b_2 = b_3 = a_5 = 0, A_2 = 4, a_4 = -1$,

$$\begin{cases} L_f(s) = \dfrac{1}{1 - 2s + 4s^2}, \\ Q_f(s,u) = \dfrac{1}{1 - 2s + 3s^2 + 8s^3 - u}. \end{cases} \tag{3.5}$$

Case 1C: When $L_2 = 4, A_2 = -4, B_2 = a_4 = a_5 = b_2 = b_3 = 0, b_5 = 2$,

$$\begin{cases} L_f(s) = \dfrac{1}{1 - 2s}, \\ Q_f(s,u) = \dfrac{1 + u + 2su}{1 - 2s - s^2}. \end{cases} \tag{3.6}$$

Case 1D: When $L_2 = 3, A_2 = B_2 = 0, b_5 = b_3 = a_5 = 0, a_4 = -1, b_2 = 3/4$,

$$\begin{cases} L_f(s) = \dfrac{2 + s}{2 - 3s}, \\ Q_f(s,u) = \dfrac{4 + 3s^2}{4 - 8s + 3s^2 - 4u}. \end{cases} \tag{3.7}$$

Case 1E: When $L_2 = 3, A_2 = B_2 = 0, b_5 = b_2 = a_5 = 0, a_4 = -1, b_3 = -3/2$,

$$\begin{cases} L_f(s) = \dfrac{2 + s}{2 - 3s}, \\ Q_f(s,u) = \dfrac{2 - 3s^3}{2(1 - 2s - u)}. \end{cases} \tag{3.8}$$

Case 1F: When $L_2 = 5, A_2 = B_2 = 0, b_5 = a_4 = a_5 = 0, b_2 = -2, b_3 = -11/2$,

$$\begin{cases} L_f(s) = \dfrac{2 - s}{2 - 5s}, \\ Q_f(s,u) = \dfrac{2 - 4s - 4s^2 - 11s^3 + 2u}{2(1 - 4s)}. \end{cases} \tag{3.9}$$

Case 1G: When $L_2 = 3, A_2 = -3, B_2 = 0, a_4 = -1, b_5 = a_5 = b_2 = b_3 = 0$,

$$\begin{cases} L_f(s) = \dfrac{1 + 2s}{1 - 3s^2}, \\ Q_f(s,u) = \dfrac{1}{1 - 2s - u}. \end{cases} \tag{3.10}$$

Note that this sub-case **1G** was already studied in [13].

The following three typical sub-cases **1H**, **1I**, **1J** are obtained with further constraints $L_2 = 5,\ Q_{02} = 1,\ Q_{21} = 14, Q_{40} = 2L_3 + L_4$, for which the corresponding error equation reduces to a simplified form of

$$e_{n+1} = -c_2^2 c_3 c_4\, e_n^8 + O(e_n^9).$$

For further simplicity of $L_f(s)$, we also take $A_2 = 10$ and $B_2 = 2$ in these three sub-cases. As a result, coefficients $b_1, b_4, b_5, a_1, a_2, a_3, a_4, a_5$ of $Q_f(s,u)$ can be expressed in terms of two parameters b_2 and b_3.

Case 1H: When $L_2 = 5, A_2 = 10, B_2 = 2, b_2 = 2, b_3 = 0;\ b_1 = \frac{1}{10}(-9 - 8b_2 - 8b_3), b_4 = 0, b_5 = 1 - \frac{b_2}{2}, a_1 = \frac{-29-8b_2-8b_3}{10}, a_2 = \frac{-1+13b_2+8b_3}{5}, a_3 = \frac{-7-4b_2+26b_3}{10}, a_4 = -1, a_5 = \frac{19+3b_2+8b_3}{10}$.

$$\begin{cases} L_f(s) = \dfrac{2-s}{2-5s}, \\ Q_f(s,u) = \dfrac{2-5s+4s^2}{2-9s+10s^2-3s^3-2u+5su}. \end{cases} \tag{3.11}$$

Case 1I: When $L_2 = 5, A_2 = 10, B_2 = 2, b_2 = 2, b_3 = -\frac{25}{8};\ b_1 = \frac{1}{10}(-9 - 8b_2 - 8b_3), b_4 = 0, b_5 = 1 - \frac{b_2}{2}, a_1 = \frac{(-29-8b_2-8b_3)}{10}, a_2 = \frac{(-1+13b_2+8b_3)}{5}, a_3 = \frac{1}{10}(-7 - 4b_2 + 26b_3), a_4 = -1, a_5 = \frac{19+3b_2+8b_3}{10}$.

$$\begin{cases} L_f(s) = \dfrac{2-s}{2-5s}, \\ Q_f(s,u) = \dfrac{-8-16s^2+25s^3}{-8+16s+77s^3+8u}. \end{cases} \tag{3.12}$$

Case 1J: When $L_2 = 5, A_2 = 10, B_2 = 2, b_2 = 0, b_3 = 0;\ b_1 = \frac{1}{10}(-9 - 8b_2 - 8b_3), b_4 = 0, b_5 = 1 - \frac{b_2}{2}, a_1 = \frac{-29-8b_2-8b_3}{10}, a_2 = \frac{-1+13b_2+8b_3}{5}, a_3 = \frac{-7-4b_2+26b_3}{10}, a_4 = -1, a_5 = \frac{19+3b_2+8b_3}{10}$.

$$\begin{cases} L_f(s) = \dfrac{2-s}{2-5s}, \\ Q_f(s,u) = \dfrac{-10+9s-10su}{-10+29s+2s^2+7s^3+10u-19su}. \end{cases} \tag{3.13}$$

As Case 2, we are going to deal with a generic rational weight function of up to cubic-order given by (3.2).

Case 2: Cubic-order rational weight function $Q_f(s,u)$ with free parameters $A_2, B_2, L_2, a_4, a_5, a_6, b_2, b_3, b_5, b_6$

$$\begin{cases} L_f(s) = \dfrac{1+B_1 s+B_2 s^2}{1+A_1 s+A_2 s^2}, \\ Q_f(s,u) = \dfrac{b_0+b_1 s+b_2 s^2+b_3 s^3+u(b_4+b_5 s+b_6 s^2)}{1+a_1 s+a_2 s^2+a_3 s^3+u(a_4+a_5 s+a_6 s^2)}. \end{cases} \tag{3.14}$$

The 10 free parameters $A_2, B_2, L_2, a_4, a_5, a_6, b_2, b_3, b_5, b_6$ with a_6 and b_6 non-vanishing simultaneously will be selected in such a way that the governing equations yielding extraneous fixed points of proposed map (1.7) might possess lower-degree polynomials. In view of the analysis on extraneous fixed points to be shown in the next section, we consider four sub-cases **Case 2A**, **Case 2B**, **Case 2C**, **Case 2D** with four values of $L_2 \in \{0, 4, 8, 12\}$ along with a selected pair of (A_2, B_2)-values giving simplified forms of $L_f(s)$ in order. Note that two parameters $a_2 = 7 + 4a_4 + 2a_5 + b_2 - 2b_5 - L_2$ and $a_3 = -6 - 6a_4 - 3a_5 - 2b_2 + b_3 + 3b_5 + A_2(2 - \frac{L_2}{2}) + (4 + 2a_4 + a_5 + \frac{B_2}{2} - b_5)L_2 - \frac{L_2^2}{2}$ are dependent upon L_2 and need to be explicitly displayed in each sub-case which contains 7 free parameters $a_4, a_5, a_6, b_2, b_3, b_5, b_6$.

Case 2A: $L_2 = 0, A_2 = 0, B_2 = -8, a_2 = 4a_4 + 2a_5 + b_2 - 2b_5 + 7, a_3 = -6a_4 - 3a_5 - 2b_2 + b_3 + 3b_5 - 6$.

$$\begin{cases} L_f(s) = 1+2s, \\ Q_f(s,u) = \dfrac{b_0+b_1 s+b_2 s^2+b_3 s^3+u(b_4+b_5 s+b_6 s^2)}{1+a_1 s+a_2 s^2+a_3 s^3+u(a_4+a_5 s+a_6 s^2)}. \end{cases} \tag{3.15}$$

Case 2A1: $a_4 = -1, a_5 = 0, a_6 = 3, b_2 = 0, b_3 = 0, b_5 = 0, b_6 = 0$.

$$Q_f(s,u) = \frac{1}{1-2s-u+3s^2(1+u)}.$$

Case 2A2: $a_4 = 0, a_5 = 0, a_6 = 0, b_2 = -3, b_3 = 0, b_5 = 0, b_6 = 3$.

$$Q_f(s,u) = \frac{1-2s+3s^2(u-1)+u}{(1-2s)^2}.$$

Case 2A3: $a_4 = 0, a_5 = 0, a_6 = 3, b_2 = 0, b_3 = 0, b_5 = 2, b_6 = 0$.

$$Q_f(s,u) = \frac{1+u+2su}{1-2s+3s^2(1+u)}.$$

Case 2B: $L_2 = 4, A_2 = 8, B_2 = 0, a_2 = 4a_4 + 2a_5 + b_2 - 2b_5 + 3, a_3 = 2a_4 + a_5 - 2b_2 + b_3 - b_5 + 2.$

$$\begin{cases} L_f(s) = \dfrac{1}{1-2s}, \\ Q_f(s,u) = \dfrac{b_0 + b_1 s + b_2 s^2 + b_3 s^3 + u(b_4 + b_5 s + b_6 s^2)}{1 + a_1 s + a_2 s^2 + a_3 s^3 + u(a_4 + a_5 s + a_6 s^2)}. \end{cases} \tag{3.16}$$

Case 2B1: $a_4 = -1, a_5 = 0, a_6 = 1, b_2 = 0, b_3 = 0, b_5 = 0, b_6 = 0.$

$$Q_f(s,u) = \frac{1}{1 - 2s + s^2(s-1) - u}.$$

Case 2B2: $a_4 = 0, a_5 = -2, a_6 = 0, b_2 = 1, b_3 = 0, b_5 = 0, b_6 = 1.$

$$Q_f(s,u) = \frac{(1+s^2)(1+u)}{1 - 2s^3 - 2s(1+u)}.$$

Case 2B3: $a_4 = 0, a_5 = 0, a_6 = 0, b_2 = 0, b_3 = 0, b_5 = 2, b_6 = 2.$

$$Q_f(s,u) = \frac{1 + u + 2su + 2s^2 u}{1 - 2s - s^2}.$$

Case 2C: $L_2 = 8, A_2 = 16, B_2 = 8, a_2 = 4a_4 + 2a_5 + b_2 - 2b_5 - 1, a_3 = 10a_4 + 5a_5 - 2b_2 + b_3 - 5b_5 - 6.$

$$\begin{cases} L_f(s) = \dfrac{1-2s}{1-4s}, \\ Q_f(s,u) = \dfrac{b_0 + b_1 s + b_2 s^2 + b_3 s^3 + u(b_4 + b_5 s + b_6 s^2)}{1 + a_1 s + a_2 s^2 + a_3 s^3 + u(a_4 + a_5 s + a_6 s^2)}. \end{cases} \tag{3.17}$$

Case 2C1: $a_4 = -1, a_5 = 5/2, a_6 = -3/2, b_2 = 0, b_3 = 0, b_5 = 0, b_6 = 0.$

$$Q_f(s,u) = \frac{2 - 5s}{2 - 9s - 7s^3 - (1-s)(2-3s)u}.$$

Case 2C2: $a_4 = -2, a_5 = 0, a_6 = 0, b_2 = 9, b_3 = 44, b_5 = 0, b_6 = -18.$

$$Q_f(s,u) = \frac{1 + 2s + 44s^3 + 9s^2(1-2u) - u}{1 - 2u}.$$

Case 2C3: $a_4 = 0, a_5 = 0, a_6 = 1, b_2 = 0, b_3 = 0, b_5 = -6/5, b_6 = 0.$

$$Q_f(s,u) = \frac{5(1+u) - 2s(8+3u)}{5 - 26s + s^2(7+5u)}.$$

Case 2D: $L_2 = 12, A_2 = 24, B_2 = 16, a_2 = 4a_4 + 2a_5 + b_2 - 2b_5 - 5, a_3 = 18a_4 + 9a_5 - 2b_2 + b_3 - 9b_5 - 30.$

$$\begin{cases} L_f(s) = \dfrac{1-4s}{1-6s}, \\ Q_f(s,u) = \dfrac{b_0 + b_1 s + b_2 s^2 + b_3 s^3 + u(b_4 + b_5 s + b_6 s^2)}{1 + a_1 s + a_2 s^2 + a_3 s^3 + u(a_4 + a_5 s + a_6 s^2)}. \end{cases} \tag{3.18}$$

Case 2D1: $a_4 = 0, a_5 = -2, a_6 = -9, b_2 = 0, b_3 = 0, b_5 = 0, b_6 = 0.$

$$Q_f(s,u) = \frac{1+u}{1 - 48s^3 - 2s(1+u) - 9s^2(1+u)}.$$

Case 2D2: $a_4 = 0, a_5 = -4, a_6 = 0, b_2 = 13, b_3 = 92, b_5 = 0, b_6 = 13.$

$$Q_f(s,u) = \frac{1 + 2s + 92s^3 + u + 13s^2(1+u)}{1 - 4su}.$$

Case 2D3: $a_4 = 0, a_5 = 0, a_6 = -1, b_2 = 0, b_3 = 0, b_5 = -10/3, b_6 = 0.$

$$Q_f(s,u) = \frac{2s(8+5u) - 3(1+u)}{-3 + 22s + s^2(3u-5)}.$$

As **Case 3**, in the current study, we begin by an extensive investigation of **Case 2B**, an appropriate selection of whose free parameters leads us to purely imaginary extraneous fixed points. To this end, instead of random selection of 7 free parameters $a_4, a_5, a_6, b_2, b_3, b_5, b_6$, we will seek feasible relationships among the free parameters by imposing some constraints on simplifying the numerator of the resulting expression $F(t)$ to be described in (4.3). Further detailed analysis of such relationships will be discussed later in Section 4.

Case 3: A special case of **Case 2B** leading to purely imaginary extraneous fixed points.

$$\begin{cases} L_f(s) = \dfrac{1}{1-2s}, \\ Q_f(s,u) = \dfrac{b_0 + b_1 s + b_2 s^2 + b_3 s^3 + u(b_4 + b_5 s + b_6 s^2)}{1 + a_1 s + a_2 s^2 + a_3 s^3 + u(a_4 + a_5 s + a_6 s^2)}, \end{cases} \tag{3.19}$$

where $b_1 = -2 - 2a_4 - a_5 + b_5, b_4 = 1 + a_4, a_1 = -4 - 2a_4 - a_5 + b_5, a_2 = 3 + 4a_4 + 2a_5 + b_2 - 2b_5, a_3 = 2 + 2a_4 + a_5 - 2b_2 + b_3 - b_5$. A possible combination of free parameters leading to purely imaginary extraneous fixed points will be described in Section 4 with $L_2 = 4, B_2 = 8, A_2 = 0$. We will determine 6 free parameters $a_4, a_5, a_6, b_3, b_5, b_6$ in terms of a remaining single parameter b_2. According to the possible values of $\sigma, r_6 \in \{0, 1, 2, 3, 4, 5, 6\}$ to be extensively discussed in Section 4, we are able to consider a number of cases as well as their sub-cases with selected pairs of (σ, r_6) as described below.

Case 3A: $(\sigma, r_6) = (2, 1)$.

$$a_4 = \frac{3226 - 103b_2}{399}, \qquad a_5 = \frac{-101 + 2b_2}{3}, \qquad a_6 = \frac{593 + 4b_2}{21},$$
$$b_3 = 0, \qquad b_5 = \frac{10(-911 + 2b_2)}{399}, \qquad b_6 = \frac{124(10 + b_2)}{133};$$
$$\frac{53}{8} < b_2 < \frac{319}{8}.$$

Case 3A1: $b_2 = 10$.
Case 3A2: $b_2 = 25$.
Case 3A3: $b_2 = 35$.

Case 3B: $(\sigma, r_6) = (2, 2)$.

$$a_4 = \frac{-398 + 113b_2}{55}, \qquad a_5 = -\frac{3(-749 + 194b_2)}{55}, \qquad a_6 = \frac{-2909 + 684b_2}{55},$$
$$b_3 = 0, \qquad b_5 = -\frac{18(-7 + 2b_2)}{5}, \qquad b_6 = \frac{4(-70 + 17b_2)}{11};$$
$$\frac{49}{24} < b_2 < \frac{53}{8}.$$

Case 3B1: $b_2 = 3$.
Case 3B2: $b_2 = \frac{2909}{684}$.
Case 3B3: $b_2 = 6$.

Case 3C: $(\sigma, r_6) = (2, 3)$.

$$a_4 = -\frac{58}{9} + \frac{5b_2}{3}, \qquad a_5 = \frac{161 + 6b_2}{9}, \qquad a_6 = -3(1 + 4b_2), \qquad b_3 = 0,$$
$$b_5 = \frac{2(35 + 6b_2)}{9}, \qquad b_6 = -\frac{28(-2 + 3b_2)}{9};$$
$$\frac{19}{24} < b_2 < \frac{49}{24}.$$

Case 3C1: $b_2 = 1$.
Case 3C2: $b_2 = 1.5$.
Case 3C3: $b_2 = 2$.

Case 3D: $(\sigma, r_6) = (2, 4)$.

$$a_4 = -\frac{3(10 + 37b_2)}{23}, \qquad a_5 = 1 + 22b_2, \qquad a_6 = \frac{83 - 468b_2}{23}, \qquad b_3 = 0, \qquad b_5 = \frac{2(5 + 122b_2)}{23},$$
$$b_6 = -\frac{4(-2 + 11b_2)}{23};$$
$$-\frac{9}{8} < b_2 < \frac{19}{24}.$$

Case 3D1: $b_2 = -1$.
Case 3D2: $b_2 = 0$.
Case 3D3: $b_2 = \frac{2}{11}$.

Case 3E: $(\sigma, r_6) = (3, 1)$.

$$a_4 = \frac{-742 + 123b_2}{509}, \quad a_5 = \frac{6489 - 898b_2}{509}, \quad a_6 = \frac{-11347 + 1444b_2}{509}, \quad b_3 = 0,$$
$$b_5 = -\frac{2(-1803 + 386b_2)}{509}, \quad b_6 = \frac{4(-1570 + 263b_2)}{509};$$
$$\frac{51}{16} < b_2 < \frac{611}{32}.$$

Case 3E1: $b_2 = 6$.
Case 3E2: $b_2 = \frac{11347}{1444}$.
Case 3E3: $b_2 = 18$.

Case 3F: $(\sigma, r_6) = (3, 2)$.

$$a_4 = -6 + \frac{5b_2}{3}, \quad a_5 = 5 + \frac{2b_2}{3}, \quad a_6 = 25 - 12b_2, \quad b_3 = 0, \quad b_5 = -2 + \frac{4b_2}{3}, \quad b_6 = 24 - \frac{28b_2}{3};$$
$$2.56066 \le b_2 < 3.1875.$$

Case 3F1: $b_2 = \frac{18}{7}$.
Case 3F2: $b_2 = 3$.
Case 3F3: $b_2 = 3.1$.

Case 3G: $(\sigma, r_6) = (3, 3)$.

$$a_4 = \frac{5(-2 + b_2)}{3}, \quad a_5 = \frac{23 + 2b_2}{3}, \quad a_6 = 1 - 12b_2, \quad b_3 = 0, \quad b_5 = \frac{2(5 + 2b_2)}{3},$$
$$b_6 = -\frac{4(-2 + 7b_2)}{3};$$
$$0.3125 < b_2 \le 0.93934.$$

Case 3G1: $b_2 = 0.4$.
Case 3G2: $b_2 = 0.5$.
Case 3G3: $b_2 = 0.9$.

Case 3H: $(\sigma, r_6) = (4, 1)$.

$$a_4 = \frac{-346 + 47b_2}{137}, \quad a_5 = \frac{1497 - 218b_2}{137}, \quad a_6 = \frac{-1531 + 244b_2}{137},$$
$$b_3 = 0, \quad b_5 = -\frac{6(-113 + 30b_2)}{137}, \quad b_6 = -\frac{4(-130 + 43b_2)}{137};$$
$$\frac{21}{8} < b_2 < \frac{1095}{104}.$$

Case 3H1: $b_2 = \frac{1531}{244}$.
Case 3H2: $b_2 = 8$.
Case 3H3: $b_2 = 10$.

Case 3I: $(\sigma, r_6) = (4, 2)$.

$$a_4 = -2 + \frac{b_2}{7}, \quad a_5 = 9 - \frac{6b_2}{7}, \quad a_6 = -11 + \frac{12b_2}{7},$$
$$b_3 = 0, \quad b_5 = 6 - \frac{12b_2}{7}, \quad b_6 = -8 + \frac{20b_2}{7};$$
$$\frac{7}{8} < b_2 < \frac{21}{8}.$$

Case 3I1: $b_2 = 1$.
Case 3I2: $b_2 = 1.5$.
Case 3I3: $b_2 = 2.5$.

Case 3J: $(\sigma, r_6) = (5, 1)$,

$$a_4 = \frac{-10 + b_2}{7}, \quad a_5 = \frac{145}{21} - \frac{6b_2}{7}, \quad a_6 = \frac{-227 + 36b_2}{21},$$
$$b_3 = 0, \quad b_5 = -\frac{2(-25 + 6b_2)}{7}, \quad b_6 = \frac{4(-22 + 5b_2)}{7};$$
$$3.2622 \le b_2 < 5.47917.$$

Case 3J1: $b_2 = 3.5$.
Case 3J2: $b_2 = 4$.
Case 3J3: $b_2 = \frac{22}{5}$.

As a last case, we now consider $Q_f(s, u)$ being different from that of (3.2) in the form of the sum and product of univariate weight functions. Due to its inherent complicated algebraic structure, we will not attempt to locate purely imaginary extraneous fixed points in this case.

Case 4: Sum and product of univariate weight functions
$Q_f(s, u) = w_1(s) + w_2(u) + w_3(s) \times w_4(u)$.

$$\begin{cases} L_f(s) = \dfrac{1 + B_1 s + B_2 s^2}{1 + A_1 s + A_2 s^2}, \\ Q_f(s, u) = \dfrac{b_0 + b_1 s + b_2 s^2 + b_3 s^3}{1 + a_1 s + a_2 s^2 + a_3 s^3} + \dfrac{b_4 + b_5 u}{1 + a_4 u} + \left(\dfrac{b_6 + b_7 u}{1 + a_5 u}\right) \cdot \left(\dfrac{b_8 + b_9 s}{1 + a_6 s}\right), \end{cases} \tag{3.20}$$

where $b_0 = 1 - b_4 - b_6 b_8$, $b_1 = 2 + a_6 b_6 b_8 - a_1(-1 + b_4 + b_6 b_8) - b_6 b_9$, $b_2 = 1 - a_2(-1 + b_4 + b_6 b_8) + a_6 b_6(-a_6 b_8 + b_9) + a_1(2 + a_6 b_6 b_8 - b_6 b_9) + L_2$, $b_3 = -4 + 2a_2 + a_2 a_6 b_6 b_8 + a_6^3 b_6 b_8 - a_3(-1 + b_4 + b_6 b_8) - (a_2 + a_6^2) b_6 b_9 + 2L_2 + a_1(1 + a_6 b_6(-a_6 b_8 + b_9) + L_2) + \frac{1}{2}(A_2(L_2 - 4) + L_2(L_2 - B_2))$, $a_4 = \frac{(b_5 - 1)}{b4} + \frac{4b_8}{b4(-a_6 b_8 + b_9)}$, $a_5 = \frac{b_7}{b6} - \frac{4}{b6(-a_6 b_8 + b_9)}$.

Case 4A: When $L_2 = A_2 = B_2 = 0$, $b_5 = b_7 = b_9 = 0$, $b_4 = b_6 = b_8 = 1$, $a_6 = 4$, $a_1 = 6$, $a_2 = 21$, $a_3 = 96$,

$$\begin{cases} L_f(s) = 1 + 2s, \\ Q_f(s, u) = -\dfrac{1}{1 + 6s + 21s^2 + 96s^3} + \dfrac{1}{1 - 2u} + \dfrac{1}{1 + 4s + u + 4su}. \end{cases} \tag{3.21}$$

Case 4B: When $L_2 = B_2 = 0$, $A_2 = 4$, $b_7 = b_9 = 0$, $b_5 = b_8 = a_6 = 1$, $b_6 = -2$, $b_4 = 3$, $a_1 = -2$, $a_2 = a_3 = 0$,

$$\begin{cases} L_f(s) = \dfrac{1}{1 - 2s + 4s^2}, \\ Q_f(s, u) = \dfrac{(3 - 20s)s^2}{1 - 2s} + \dfrac{2}{(1 + s)(-1 + 2u)} + \dfrac{9 + 3u}{3 - 4u}. \end{cases} \tag{3.22}$$

Case 4C: When $L_2 = 4$, $A_2 = -4$, $B_2 = 0$, $b_5 = b_7 = b_9 = 0$, $b_8 = 1$, $a_6 = b_4 = b_6 = -1$, $a_1 = a_2 = a_3 = 0$,

$$\begin{cases} L_f(s) = \dfrac{1}{1 - 2s}, \\ Q_f(s, u) = 3 + 3s + 6s^2 + 13s^3 + \dfrac{1}{3u - 1} + \dfrac{1}{(s - 1)(4u + 1)}. \end{cases} \tag{3.23}$$

Case 4D: When $L_2 = 3$, $A_2 = B_2 = 0$, $b_4 = b_5 = b_6 = b_9 = 1$, $b_7 = b_8 = a_6 = 0$, $a_1 = -4$, $a_2 = 19/2$, $a_3 = 0$,

$$\begin{cases} L_f(s) = \dfrac{2 + s}{2 - 3s}, \\ Q_f(s, u) = 1 + s\left(\dfrac{2}{2 - 8s + 19s^2} + \dfrac{1}{1 - 4u}\right) + u. \end{cases} \tag{3.24}$$

Case 4E: When $L_2 = 5$, $A_2 = B_2 = 0$, $b_9 = 4$, $b_4 = b_6 = b_7 = a_6 = 1$, $a_1 = -29/20$, $b_5 = b_8 = a_2 = a_3 = 0$,

$$\begin{cases} L_f(s) = \dfrac{2 - s}{2 - 5s}, \\ Q_f(s, u) = -\dfrac{2s(-20 + 129s)}{(-20 + 29s)} + \dfrac{1}{1 - u} + \dfrac{4s(1 + u)}{(1 + s)}. \end{cases} \tag{3.25}$$

Case 4F: When $L_2 = 3$, $A_2 = -3$, $B_2 = 0$, $b_4 = b_6 = 1$, $b_9 = 2$, $b_5 = a_6 = -1$, $a_1 = -3$, $b_7 = b_8 = a_2 = a_3 = 0$,

$$\begin{cases} L_f(s) = \dfrac{1 + 2s}{1 - 3s^2}, \\ Q_f(s, u) = -\dfrac{4s^2}{1 + 3(2s - 1)} + \dfrac{2s}{(s - 1)(2u - 1)} + \dfrac{u - 1}{2u - 1}. \end{cases} \tag{3.26}$$

Despite the availability of rich sub-cases considered thus far, we typically select cases **1A–1F**, **1G**, **2A1**, **2B1**, **2C1**, **2C3**, **2D1** as well as **3A2**, **3B2**, **3C1**, **3C2**, **3D2**, **3D3 3E2**, **3F1**, **3F3**, **3G2**, **3H1**, **3I1**, **3I3**, **3J3**, **4A**, **4C**, **4F**, whose extraneous fixed points are listed in Table 3 together with those extraneous fixed points of existing methods **SA**, **CTV**, **LW**.

4. Extraneous fixed points and their dynamics

We in this section will devote ourselves to investigating the extraneous fixed points [14] of iterative map (1.7) and relevant dynamics associated with their basins of attraction. The dynamics underlying basins of attraction was initiated by Stewart [15] and followed by works of Amat et al. e.g. [16,17], Scott et al. [18], Chun et al. [19], Chun–Neta [20], Chicharro et al. [21], Cordero et al. [22], Neta et al. [23,24], Argyros-Magreñan [25], Geum et al. [26] and Andreu et al. [27]

We usually locate a zero α of a nonlinear equation $f(x) = 0$ by means of a fixed point ξ of iterative methods of the form

$$x_{n+1} = R_f(x_n), \quad n = 0, 1, \ldots, \tag{4.1}$$

where R_f is the iteration function under consideration. In general, R_f might possess other fixed points $\xi \neq \alpha$. Such fixed points are called the *extraneous fixed points* of the iteration function R_f. It is well known that extraneous fixed points may result in attractive, indifferent or repulsive cycles as well as other periodic orbits influencing the dynamics underlying the basins of attraction. Exploration of such dynamics as well as discovery of its complicated behavior gives us a valuable motivation of the current analysis. In connection with proposed methods (1.7), we obtain a more specific form of iterative maps (4.1) as follows:

$$x_{n+1} = R_f(x_n) = x_n - \frac{f(x_n)}{f'(x_n)} H_f(x_n), \tag{4.2}$$

where $H_f(x_n) = 1 + s \cdot [L_f(s) + u \cdot Q_f(s, u)]$ can be regarded as a weight function of the classical Newton's method. It is obvious that α is a fixed point of R_f. The points $\xi \neq \alpha$ for which $H_f(\xi) = 0$ are extraneous fixed points of R_f.

For ease of analysis of the relevant dynamics, we limit ourselves to considering only combinations of weight functions $L_f(s)$ and $Q_f(s, u)$ in the form of low-order rational functions as described by (3.2). Other types of combinations have empirically shown poor convergence as indicated in the existing studies by [28–32]. A special attention will be paid to cases **1A–1J**, **1G**, **2A1**, **2B1**, **2C1**, **2C3**, **2D1** including some sub-cases of **Case 3** as well as cases **4A**–**4F** in order to pursue further properties of extraneous fixed points and relevant dynamics associated with their basins of attraction. The existence of such extraneous fixed points would affect the global iteration dynamics, which was demonstrated for simple zeros via König functions and Schröder functions [14] applied to a family of functions $\{f_k(x) = x^k - 1, k \geq 2\}$ according to the joint work of Vrscay and Gilbert published in 1988. Especially the presence of attractive cycles induced by the extraneous fixed points of R_f may alter the basins of attraction due to the trapped sequence $\{x_n\}$. Even in the case of repulsive or indifferent fixed points, an initial value x_0 chosen near a desired root may converge to another unwanted remote root. Indeed, these aspects of the Schröder functions were observed in an application to the same family of functions $\{f_k(x) = x^k - 1, k \geq 2\}$.

For simplified dynamics related to the extraneous fixed points underlying the basins of attraction for iterative maps (4.2), we first choose a simple quadratic polynomial from the family of functions $\{f_k(x) = x^k - 1, k \geq 2\}$. By closely following the works of Chun et al. [28,29] and Neta et al. [23,24,32], we then construct $H_f(x_n) = 1 + s \cdot [L_f(s) + u \cdot Q_f(s, u)]$ in (4.2). We now apply a prototype quadratic polynomial $f(z) = (z^2 - 1)$ to $H_f(x_n)$ and construct $H(z)$, with a change of a variable $t = z^2$, in the form of

$$H(z) = \Lambda(t) \cdot \Gamma(t), \tag{4.3}$$

where both $\Lambda(t)$ and $\Gamma(t)$ are rational functions of t, with $\Lambda(t)$ independent of coefficients of $L_f(s)$ or $Q_f(s, u)$ and with $\Gamma(t)$ that may possess the extraneous fixed points H. Since H is a rational function, it would be preferable for us to deal with the underlying dynamics of iterative map (4.2) on the Riemann sphere [33] where points "0(zero)" and "∞" can be treated as the desired extraneous fixed points. If such points arise, we are interested in only the finite extraneous fixed point 0 under which the relevant dynamics can be described in a region containing the origin by investigating the attractor basins associated with iterative map (4.2).

Indeed, the extraneous fixed points ξ of H can be found from the roots t of $\Gamma(t)$ via relation below:

$$\xi = \begin{cases} t^{\frac{1}{2}}, & \text{if } t \neq 0, \\ 0(\text{double root}), & \text{if } t = 0. \end{cases} \tag{4.4}$$

Consequently, $\Gamma(t) = 0$ can be solved by annihilating its numerator with known polynomial root-finding methods.

4.1. Purely imaginary extraneous fixed points

We now pay a special attention to the dynamics underlying purely imaginary extraneous fixed points of iterative map (4.2). One should be aware that the boundary of two basins of attraction of two roots for the prototype quadratic polynomial $f(z) = z^2 - 1$ is the imaginary axis of the complex plane. Hence it is worth to explore how the extraneous fixed points on the imaginary axis influence the dynamical behavior of iterative map (4.2). It is our important task to find a possible combination of L_f and Q_f in **Case 3** leading to purely imaginary extraneous fixed points, whose investigation was done by Chun et al. [28]. As a preliminary task, we first describe the following lemma regarding the negative real roots of a quadratic equation, which would play a role in determining the desired purely imaginary extraneous fixed points in connection with the prototype quadratic polynomial $f(z) = z^2 - 1$. The following lemma holds according to the analysis of [34].

Table 2
$\mathscr{A}(t)$, $\frac{F(t;a_4,a_5,\ldots,b_6)}{\Omega(t;a_4,a_5,\ldots,b_6)}$ and ν(the degree of F) for the selected values of L_2.

L_2	$\mathscr{A}(t)$	$\frac{F(t;a_4,a_5,\ldots,b_6)}{\Omega(t;a_4,a_5,\ldots,b_6)}$	ν[a]
0	$256\,t^5$	$\frac{-b_6+(4b_5+19b_6)t+\cdots+(14\,096a_4+16(9169-2072a_5+550a_6+804b_2+377b_3)+70\,852b_5+625b_6)t^{11}}{a_6-2(2a_5+5a_6)t+\cdots+(352+48a_4-76a_5+25a_6+32b_2+16b_3+176b_5)t^6}$	11
4	$16\,t(1+t)^2$	$\frac{b_3+(-16-16a_4-8a_5+12b_2-12b_3+8b_5-b_6)t+\cdots+(384+32a_4-88a_5+24a_6+52b_2+25b_3+188b_5+b_6)t^8}{-2-2a_4-a_5+2b_2-b_3+b_5+(14+18a_4+9a_5+a_6+2b_2+b_3-9b_5)t+\cdots+(14+2a_4-3a_5+a_6+2b_2+b_3+7b_5)t^5}$	8
8	$256\,t^3$	$\frac{-b_6+(16b_3+4b_5+35b_6)t+\cdots+(16+16a_4+4b_5+b_6)t^{11}}{a_6-2(-48+80a_4+42a_5+9a_6-16b_2+8b_3-40b_5)t+\cdots+(16a_4+4a_5+a_6)t^6}$	11
12	$16\,t^3(t-3)^2$	$\frac{-b_6+(9b_3+4b_5+31b_6)t+\cdots+4(-104+296a_4+52a_5+16a_6+12b_2+5b_3+28b_5+4b_6)t^{11}}{a_6+(270-162a_4-85a_5-16a_6+18b_2-9b_3+81b_5)t+\cdots+(-50+66a_4+17a_5+4a_6+2b_2+b_3-b_5)t^6}$	11

[a] ν represents the degree of $F(t)$.

Lemma 4.1. *Let $q(x) = ax^2 + bx + c$ be a quadratic equation with real coefficients $a \neq 0, b, c$ satisfying $b^2 - 4ac \geq 0$. Let r_1 and r_2 be the two roots of $q(x) = 0$. Then both roots $r_1 < 0$ and $r_2 < 0$ hold if and only if all three coefficients a, b, c have the same sign.*

To begin the detailed study regarding the purely imaginary extraneous fixed points, we now consider **Case 2** described by (3.14) to discuss another selection of 10 free parameters $L_2, A_2, B_2, a_4, a_5, a_6, b_2, b_3, b_4, b_6$ for simplified weight functions Q_f. Applying $f(z) = (z^2 - 1)$ yields

$$\begin{cases} s = \dfrac{1}{4}\left(1 - \dfrac{1}{z^2}\right), \\ L_f = \dfrac{B_2 + 2(-4 + A_2 - 2B_2 + L_2)z^2 + (24 - 2A_2 + 3B_2 - 2L_2)z^4}{A_2 + (-2B_2 + 2L_2)z^2 + (-A_2 + 2(8 + B_2 - L_2))z^8}. \end{cases} \tag{4.5}$$

Besides, we are able to express Q_f in terms of z, A_2, B_2, L_2 and free parameters $a_4, a_5, \ldots, b_6$ with the use of

$$u = \frac{1}{16}\left(1 - \frac{1}{z^2}\right)^2 \cdot \frac{B_2^2 - 4B_2(4 + A_2 + 2B_2 - L_2)z^2 + \sigma_2 z^4 + \sigma_3 z^6 + (2A_2 - 5(8 + B_2) + 6L_2)^2 z^4}{A_2 + 2(B_2 - L_2)z^2 + (A_2 - 2(8 + B_2 - L_2))z^4}, \tag{4.6}$$

where $\sigma_2 = 2(2A_2^2 + 19B_2^2 + B_2(56 - 18L_2) + 2(-4 + L_2)^2 - 2A_2(-8 + 3B_2 + 2L_2))$ and $\sigma_3 = -4(96 + 2A_2^2 + 2B_2(62 + 7B_2) + A_2(-9B_2 + 4(-8 + L_2)) - 64L_2 - 23B_2L_2 + 10L_2^2)$. Although such lengthy expression of Q_f is not explicitly shown here, the simplified second-order form of L_f will greatly reduce the complexity of Q_f as well as the desired H_f given by (4.3). To fulfill the simplification, we annihilate the coefficients of quartic-order terms in L_f by setting $24 - 2A_2 + 3B_2 - 2L_2 = 0$ and $-A_2 + 2(8 + B_2 - L_2) = 0$, from which two coefficients $A_2 = 2L_2$ and $B_2 = 2(-4 + L_2)$ are found and give us the desired weight function

$$L_f = \frac{4 - L_2 + (L_2 - 12)z^2}{-L_2 + (L_2 - 8)z^2}. \tag{4.7}$$

Substituting these two coefficients A_2, B_2 into both L_f and Q_f, we are finally able to express $H_f = 1 + s \cdot (L_f + u \cdot Q_f)$ in terms of z, L_2 and 7 free parameters $a_4, a_5, a_6, b_2, b_3, b_5, b_6$. In view of the form of L_f, we consider four special values of $L_2 \in \{0, 4, 8, 12\}$. As can be seen in Table 1, the explicit form of the relevant $H(z)$ given by (4.3) becomes

$$H(z) = \frac{1}{\mathscr{A}(t)} \cdot \frac{F(t; a_4, a_5, \ldots, b_6)}{\Omega(t; a_4, a_5, \ldots, b_6)}, \tag{4.8}$$

where F and Ω are polynomials in $t = z^{1/2}$ with no common factors and with their coefficients dependent on free parameters $a_4, a_5, a_6, b_2, b_3, b_5, b_6$, while $\mathscr{A}$ is a polynomial in t with its coefficients free of parameters $a_4, a_5, a_6, b_2, b_3, b_5, b_6$ (see Table 2).

The cases when $L_2 \in \{0, 8, 12\}$ have degrees of no less than 11, which can yield quartic-degree polynomials at lowest without any spare free parameters after imposing some constraints on 7 coefficients of $F(t)$. Certainly such cases show lack of our freedom of selecting free parameters. Therefore, in these cases we rather choose 7 free parameters arbitrarily as already done in Section 3.

On the other hand, the case of $L_2 = 4$ (highlighted in yellow, being already named as **Case 2B**) gives us more freedom to select several parameters among 7 free parameters. Due to the fact that F defines a polynomial of degree 8, a generic second-order polynomial factor can be induced with its coefficients involving one of 7 free parameters by annihilating six coefficients in an appropriate manner.

We now investigate **Case 2B** described by (3.16) in more detail to discuss purely imaginary extraneous fixed points. From now on, **Case 3** will exclusively refer to the case whose extraneous fixed points are all purely imaginary in connection with

Table 3
Extraneous fixed points ξ and their stability for selected cases.

Case	ξ	No. of ξ
1A	$\pm0.549428\pm0.895494i,\ \pm0.558197\pm0.476595i,\ \pm0.459252\pm0.208066i,\ \pm0.455608,\ \pm0.342642\pm0.185031i$	18
1B	$\pm0.572046\pm1.03078i,\ \pm0.609678\pm0.462502i,\ \pm0.498122\pm0.508812i,\ \pm0.526338\pm0.380357i,\ \pm0.504029$	18
1C	$\pm0.537618\pm1.32538i,\ \pm0.380324\pm0.801195i,\ \pm0.354143i,\ \pm0.311856$	12
1D	$\pm2.17331i,\ \pm0.327121\pm0.587002i,\ \pm0.332521\pm0.324986i,\ \pm0.343352,\ \pm0.163886\pm0.246072i$	16
1E	$\left(\pm2.44418i,\ \pm0.366123\pm0.599852i,\ \pm0.513406i,\ \pm0.385693\pm0.249839i,\ \pm0.414739,\ \pm0.184477\pm0.238685i\right)$	18
1F	$\left(\pm8.4711i,\ \pm1.83039i,\ \pm0.404361\pm1.1344i,\ \pm0.893068i,\ \pm0.317259\pm0.36722i,\ \pm0.374099\pm0.108666i,\ \pm0.20014\right)$	20
1G	$\pm2.41868i,\ \pm0.308077\pm0.658553i,\ \pm0.570582,\ \pm0.547435,\ \pm0.540697,\ \pm0.310588i$	14
1H	$\pm2.74844i,\ \pm1.03355i,\ \pm0.30536\pm0.735255i,\ \pm0.188225\pm0.222285i,\ \pm0.228333$	14
1I	$\left(\pm3.32629i,\ \pm0.38279\pm1.05547i,\ \pm0.823764i,\ \pm0.544849\pm0.272033i,\ \pm0.218001\pm0.165235i,\ \pm0.217402\right)$	18
1J	$\left(\pm2.68374i,\ \pm1.49394i,\ \pm0.350059\pm1.08981i,\ \pm0.678353i,\ \pm0.286711\pm0.22839i,\ \pm0.240835,\pm0.220365i,\ \pm0.126975\right)$	20
2A1	$\pm0.442037\pm1.04276i,\ \pm0.478056\pm0.300391i,\ \pm0.514061\pm0.140131i,\ \pm0.419885\pm0.188045i$	16
2B1	$\pm2.5775i,\ \pm0.28956\pm0.794825i,\ \pm0.45147i,\ \pm0.303403$	10
2C1	$\pm3.59332,\ \pm2.63027i,\ \pm1.60145\pm1.16519i,\ \pm0.417526\pm0.327678i,\ \pm0.328874,\ \pm0.130214$	16
2C3	$\left(\pm4.28873,\ \pm2.72647\pm1.71352i,\ \pm2.22183,\pm1.64363i,\ \pm0.237655\pm0.332434i,\ \pm0.300555,\ \pm0.259423\pm0.114063i\right)$	20
2D1	$\pm2.11279,\ \pm1.71957,\ \pm1.70779,\ \pm1.56624\pm0.288202i,\ \pm0.6784\pm0.189179i,\ \pm0.385912,\ \pm0.125124$	18
3A2	$\pm1.32868i,\ \pm0.278833i$	4
3B2	$\pm1.29391i,\ \pm0.488498i,\ 0(double)$	6
3C1	$\pm1.24541i,\ \pm0.359088i,\ \boxed{0}\ (quadruple)$	8
3C2	$\pm1.66022i,\ \pm0.688790i,\ \boxed{0}\ (quadruple)$	8
3D2	$\pm2.15785i,\ \pm0.843864i,\ \boxed{0}\ (sextuple)$	10
3D3	$\pm2.38604i,\ \pm0.937146i,\ \boxed{0}\ (sextuple)$	10
3E2	$\pm i,\ \pm0.729708i,\ \pm0.236140i$	6
3F1	$\pm\mathbf{1}i,\ \pm0.829574i,\ \pm0.665013i,\ 0(double)$	8(**2**)
3F3	$\pm i,\ \pm3.72092i,\ \pm0.393411i,\ 0(double)$	8
3G2	$\pm i,\ \pm2.41421i,\ \pm0.414214i,\ 0(quadruple)$	10
3H1	$\pm\boxed{i}\ (double),\pm1.16737i,\ \pm0.220055i$	8
3I1	$\pm\boxed{i}\ (double),\pm1.79075i,\ \pm0.154879i,\ 0(double)$	10
3I3	$\pm\boxed{i}\ (double),\pm6.45664i,\ \pm0.558426i,\ 0(double)$	10
3J3	$\pm\boxed{i}\ (triple),\pm1.42896i,\ \pm0.257255i$	10
4A	$\left(\pm\mathbf{2.44361}i,\ \pm0.398228\pm0.207352i,\ \pm\mathbf{0.448447}\pm\mathbf{0.287632}i,\ \pm\mathbf{0.400622}\pm\mathbf{0.181815}i,\ \pm0.640194\pm0.415578i,\ \pm\mathbf{0.658881},\ \pm0.776714\pm0.294264i,\ \pm\mathbf{0.719238},\ \pm0.753684\right)$	28(**14**)
4B	$\left(\pm\mathbf{7.48865}i,\ \pm0.400261\pm1.14448i,\ \pm0.607591\pm0.673767i,\ \pm0.473386\pm0.490478i,\ \pm0.499763\pm0.495657i,\ \pm0.50246\pm0.495814i,\ \pm0.305438,\ \pm\mathbf{0.416286},\ \pm0.588053\pm0.371073i,\ \pm0.581994\pm0.0864181i\right)$	34(**4**)
4C	$\left(\pm\mathbf{4.06812}i,\ \pm0.571955\pm0.933907i,\ \pm\mathbf{0.580408}i,\ \pm0.267356i,\ \pm\mathbf{0.677407}\pm\mathbf{0.700235}i,\ \pm\mathbf{0.432144}\pm\mathbf{0.378015}i,\ \pm\mathbf{0.461984}\pm\mathbf{0.111271}i\right)$	22(**16**)
4D	$\left(\pm0.356736\pm0.916153i,\ \pm\mathbf{0.403627}\pm\mathbf{0.504148}i,\ \pm0.138638\pm0.277396i,\ \pm\mathbf{0.185092}\pm\mathbf{0.244852}i,\ \pm0.174271\pm0.165477i,\ \pm\mathbf{0.764157}\pm\mathbf{0.662963}i,\ \pm0.49578\pm0.0315927i,\ \pm4.19349\right)$	30(**12**)
4E	$\left(\pm\mathbf{1.9569}i,\ \pm0.370871\pm1.37357i,\ \pm0.308741\pm1.15681i,\ \pm\mathbf{0.734706}i,\ \pm0.395223i,\ \pm\mathbf{0.362074}\pm\mathbf{0.528924}i,\ \pm0.0709408i,\ \pm\mathbf{0.163974},\ \pm0.200115,\pm0.427141\pm0.180876i,\ \pm0.485609\right)$	30(**10**)
4F	$\left(\pm2.23679i,\ \pm1.55521i,\ \pm\mathbf{0.996549}i,\ \pm0.200713\pm0.63981i,\ \pm\mathbf{0.231673}\pm\mathbf{0.239224}i,\ \pm\mathbf{0.217517},\ \pm0.54455,\pm\mathbf{0.547048},\pm\mathbf{0.55837},\pm\mathbf{0.598399}\right)$	24(**14**)
SA	$\pm\mathbf{2.74748}i,\ \pm\mathbf{1.19175}i,\ \pm0.57735i,\ \pm\mathbf{0.176327}i$	8(**6**)
CTV	$\left(\pm0.111081\pm2.34413i,\ \pm0.458316\pm1.31875i,\ \pm0.37733\pm0.868071i,\ \pm0.125467\pm0.598632i,\ \pm\mathbf{0.122706}\pm\mathbf{0.449599}i,\ \pm\mathbf{0.252528}\pm\mathbf{0.123626}i\right)$	24(**8**)
LW	$\left(\pm2.39114i,\ \pm0.430878\pm1.12787i,\ \pm\mathbf{0.783628}i,\ \pm\mathbf{0.208823}\pm\mathbf{0.339322}i,\ \pm\mathbf{0.443429}\pm\mathbf{0.404034}i,\ \pm\mathbf{0.298401},\ \pm4.29517\right)$	20(**12**)

In the table, bold-face values represent attractive extraneous fixed points, while framed-values indifferent extraneous fixed points.
Besides, the bold-face numbers in the parentheses of the last column indicate the number of attractive extraneous fixed points.

Case 2B. With the aid of symbolic operation of Mathematica, we are able to obtain $F(t)$ and $\Omega(t)$ as follows:

$$F(t) = \sum_{i=0}^{8} \beta_i\, t^i, \tag{4.9}$$

where $\beta_0 = b_3$, $\beta_1 = -16-16a_4-8a_5+12b_2-12b_3+8b_5-b_6$, $\beta_2 = 16+48a_4+24a_5+8a_6+92b_2-20b_3-20b_5+7b_6$, $\beta_3 = 928+864a_4+408a_5+76b_2+28b_3-464b_5-21b_6$, $\beta_4 = 3488+1792a_4+760a_5-56a_6-132b_2+62b_3-732b_5+35b_6$, $\beta_5 = 5168+432a_4+360a_5+64a_6-188b_2-20b_3-248b_5-35b_6$, $\beta_6 = 4560-1616a_4-696a_5+24a_6-12b_2-68b_3+692b_5+21b_6$, $\beta_7 = 1856-1536a_4-760a_5-64a_6+100b_2+4b_3+576b_5-7b_6$, $\beta_8 = 384+32a_4-88a_5+24a_6+52b_2+25b_3+188b_5+b_6$.

$$\Omega(t) = \sum_{i=0}^{5} \omega_i\, t^i, \tag{4.10}$$

where $\omega_0 = -2-2a_4-a_5+2b_2-b_3+b_5$, $\omega_1 = 14+18a_4+9a_5+a_6+2b_2+b_3-9b_5$, $\omega_2 = 2(34+18a_4+7a_5-2a_6-2b_2+b_3-9b_5)$, $\omega_3 = 2(50+6a_4+5a_5+3a_6-2b_2-b_3+b_5)$, $\omega_4 = 62-66a_4-29a_5-4a_6+2b_2-b_3+17b_5$, $\omega_5 = 14+2a_4-3a_5+a_6+2b_2+b_3+7b_5$.

It is interesting for us to observe that none of ω_i $(0 \le i \le 5)$ contains coefficient b_6.

We now seek the desired extraneous fixed points ξ of map (4.2) when applied to $f(z) = z^2 - 1$ by locating the roots t of $F(t) = 0$ and using the relation $\xi = t^{1/2}$. To make all such extraneous fixed points ξ purely imaginary, we further require that t should be real and negative. If the degree of $F(t)$ is higher than 3 or 4, then algebraic complexity would hinder our ability of imposing conditions on coefficients of $F(t)$ to make all its roots real. Hence it is an important task to reduce the degree to 2 or 3, preferably 2 for easier treatment in the current study. To this end, we first inspect 9 coefficients $\beta_i (0 \le i \le 8)$ for their linear independency on 7 parameters $a_4, a_5, a_6, b_2, b_3, b_5, b_6$. Suppose that we annihilate all the 9 coefficients $\beta_i (0 \le i \le 8)$, which give a set of linear relations in matrix–vector form below:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -16 & -8 & 0 & 12 & -12 & 8 & -1 \\ 48 & 24 & 8 & 92 & -20 & -20 & 7 \\ 864 & 408 & 76 & 0 & 28 & -464 & -21 \\ 1792 & 760 & -56 & -132 & 62 & -732 & 35 \\ 432 & 360 & 64 & -188 & -20 & -248 & -35 \\ -1616 & -696 & 24 & -12 & -68 & 692 & 21 \\ -1536 & -760 & -64 & 100 & 4 & 576 & -7 \\ 32 & -88 & 24 & 52 & 25 & 188 & 1 \end{pmatrix} \begin{pmatrix} a_4 \\ a_5 \\ a_6 \\ b_2 \\ b_3 \\ b_5 \\ b_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 16 \\ -16 \\ -928 \\ -3488 \\ -5168 \\ -4560 \\ -1856 \\ -384 \end{pmatrix}. \tag{4.11}$$

The rank of the coefficient matrix on the left side of (4.11) is found to be 7 by elementary row operations. This fact allows us to independently annihilate up to any 7 out of 9 coefficients. To extract a one-parameter family of quadratic polynomial from $F(t)$, we consider a specific form of $F(t)$ as follows:

$$F(t) = (1+t)^{\sigma} t^{r_6} (\beta_{r_6} + \beta_{r_7} t + \beta_{r_8} t^2), \tag{4.12}$$

where $0 \le \sigma \le 6$, $0 \le r_6 \le 6$ are integers satisfying a constraint $0 \le \sigma + r_6 \le 6$ (to maintain $F(t)$ as a polynomial of degree of at most 8) and r_6, r_7, r_8 are three consecutive integers. Thus, given a value of σ, we are ready to annihilate $(6-\sigma)$ coefficients in order to obtain the desired quadratic polynomial. One way of doing so is to let the $(6-\sigma)$ consecutive coefficients vanish, say, starting with a known subscript index $i (0 \le i \le 9-\sigma)$ as follows:

$$\beta_i = \beta_{i+1} = \beta_{i+2} = \beta_{i+3} = \cdots = \beta_{i+5-\sigma} = 0, \tag{4.13}$$

where $\beta_{i+j} = \beta_{r_j}$ with $i + j \equiv r_j (\text{mod} \ \ (9-\sigma))$ for $0 \le i+j \le 8-\sigma$. In view of linear independency from (4.11), we conveniently solve (4.13) for $a_4, a_5, a_6, b_3, b_5, b_6$ in terms of parameter b_2, when annihilating some coefficients β_i. Substituting these $a_4, a_5, a_6, b_3, b_5, b_6$ into $F(t)$ and simplifying $H(z)$ in (4.8) after canceling out common factors of $\mathscr{A}(t)$, we immediately obtain the following with $t = z^2$:

$$H(z) = \frac{(1+t)^{\sigma-2} t^{r_6-1} (\beta_{r_6} + \beta_{r_7} t + \beta_{r_8} t^2)}{16\, \Omega(t)}, \tag{4.14}$$

provided that $\sigma \ge 2$ and $r_6 \ge 1$. Let us denote the numerator of (4.14) by $\Phi(t) = (1+t)^{\sigma-2} t^{r_6-1} (\beta_{r_6} + \beta_{r_7} t + \beta_{r_8} t^2)$. Then the desired extraneous fixed points can be found from the roots of $\Phi(t) = 0$. One should note that four coefficients β_{r_6}, β_{r_7}, β_{r_8} contain only one parameter b_2. It is clear that the value of $t = -1$ gives purely imaginary extraneous fixed points $\pm i$ with multiplicity of $(\sigma - 2)$. In addition, $t = 0$ gives an extraneous fixed point 0 with multiplicity of $2(\gamma_6 - 1)$. We are interested in other desired extraneous fixed points from the roots of the quadratic equation denoted by

$$\psi(t) = \beta_{r_6} + \beta_{r_7} t + \beta_{r_8} t^2 \tag{4.15}$$

on the right side of (4.14). Note that the discriminant $\mathscr{D}$ of $\psi(t)$ can be expressed in terms of parameter b_2. We denote a set

$$\mathbf{D} = \{b_2 \in \mathbb{R} : \mathscr{D} \geq 0\}. \tag{4.16}$$

We further denote a set

$$\mathbf{B} = \{b_2 \in \mathbb{R} : \beta_{r_6}\beta_{r_7} > 0 \text{ and } \beta_{r_7}\beta_{r_8} > 0\} \tag{4.17}$$

whose elements make all three coefficients $\beta_{r_6}, \beta_{r_7}, \beta_{r_8}$ have the same sign. We now use Lemma 4.1 to locate all two negative roots of $\psi(t) = 0$ for purely imaginary extraneous fixed points. After a lengthy algebraic process, we are able to find coefficients $a_4, a_5, a_6, b_3, b_5, b_6$ of $Q_f(s, u)$ and $\Phi(t)$, in addition to the desired set $\mathbf{D} \cap \mathbf{B}$ containing b_2-values for which purely imaginary extraneous fixed points can be located.

In view of the numerator $\Phi(t)$ of $H(z)$ in (4.14), we select integer pairs of (σ, r_6) satisfying $\sigma \geq 2, r_6 \geq 1, 0 \leq \sigma + r_6 \leq 6$, to make $H(z)$ free of poles at $t = 0$ (*i.e.*, $z = 0$) and $t = -1$ (*i.e.*, $z = \pm i$). Such a pole-free $H(z)$ would give rise to better convergence behavior. Consequently, 10 such desired pairs of (σ, r_6) are explicitly given by (2, 1), (2, 2), (2, 3), (2, 4), (3, 1), (3, 2), (3, 3), (4, 1), (4, 2) and (5, 1).

Based on these pairs of (σ, r_6), we classify the sub-cases of **Case 3** into 10 sub-cases ranging from **Cases 3A** through **3J** as follows. The determined coefficients $a_4, a_5, a_6, b_3, b_5, b_6$ have been already shown in each of the sub-cases of **Case 3** of Section 3. Thus, they will not be repeatedly shown here. We list here only the desired sets **D**, **B** and $\mathbf{D} \cap \mathbf{B}$ as well as function $\Phi(t)$, besides the three selected values of b_2 for its three sub-subcases of each sub-case. The sub-subcases are denoted by sub-case numbers at the end of which sequential Arabic numerals are appended such as **Case 3A1, Case 3A2**, $\cdots$, etc.

Case 3A: $(\sigma, r_6) = (2, 1)$.

(1) $\Phi(t) = 32\left[53 - 8b_2 + (-374 - 64b_2)t + 9(-319 + 8b_2)t^2\right]$.
(2) $\mathbf{D} = \mathbb{R}$, $\mathbf{B} = \{b_2 : \frac{53}{8} < b_2 < \frac{319}{8}\}$.
(3) $\mathbf{D} \cap \mathbf{B} = \{b_2 : \frac{53}{8} < b_2 < \frac{319}{8}\}$.

The three sub-subcases **Cases 3A1, 3A2, 3A3** are identified with $b_2 \in \{10, 25, 35\}$ in order.

Case 3B: $(\sigma, r_6) = (2, 2)$.

(1) $\Phi(t) = 32\,t\left[49 - 24b_2 - 2(59 + 16b_2)t + 7(-53 + 8b_2)t^2\right]$.
(2) $\mathbf{D} = \mathbb{R}$, $\mathbf{B} = \{b_2 : \frac{49}{24} < b_2 < \frac{53}{8}\}$.
(3) $\mathbf{D} \cap \mathbf{B} = \{b_2 : \frac{49}{24} < b_2 < \frac{53}{8}\}$.

The three sub-subcases **Cases 3B1, 3B2, 3B3** are identified with $b_2 \in \{3, \frac{2909}{684}, 6\}$ in order.

Case 3C: $(\sigma, r_6) = (2, 3)$.

(1) $\Phi(t) = 32\,t^2\left[19 - 24b_2 - 42t + (-49 + 24b_2)t^2\right]$.
(2) $\mathbf{D} = \mathbb{R}, \mathbf{B} = \{b_2 : \frac{19}{24} < b_2 < \frac{49}{24}\}$.
(3) $\mathbf{D} \cap \mathbf{B} = \{b_2 : \frac{19}{24} < b_2 < \frac{49}{24}\}$.

The three sub-subcases **Cases 3C1, 3C2, 3C3** are identified with $b_2 \in \{1, 1.5, 2\}$ in order.

Case 3D: $(\sigma, r_6) = (2, 4)$.

(1) $\Phi(t) = 32\,t^3\left[-7(9 + 8b_2) + (-102 + 32b_2)t + (-19 + 24b_2)t^2\right]$.
(2) $\mathbf{D} = \mathbb{R}$, $\mathbf{B} = \{b_2 : -\frac{9}{8} < b_2 < \frac{19}{24}\}$.
(3) $\mathbf{D} \cap \mathbf{B} = \{b_2 : -\frac{9}{8} < b_2 < \frac{19}{24}\}$.

The three sub-subcases **Cases 3D1, 3D2, 3D3** are identified with $b_2 \in \{-1, 0, \frac{2}{11}\}$ in order.

Case 3E: $(\sigma, r_6) = (3, 1)$.

(1) $\Phi(t) = 16(1 + t)\left[51 - 16b_2 + (154 - 208b_2)t + 7(-611 + 32b_2)t^2\right]$.
(2) $\mathbf{D} = \mathbb{R}, \mathbf{B} = \{b_2 : \frac{51}{16} < b_2 < \frac{611}{32}\}$.
(3) $\mathbf{D} \cap \mathbf{B} = \{b_2 : \frac{51}{16} < b_2 < \frac{611}{32}\}$.

The three sub-subcases **Cases 3E1, 3E2, 3E3** are identified with $b_2 \in \{6, \frac{11347}{1444}, 18\}$ in order.
Observe that this sub-case yields additional extraneous fixed points $\pm i$.

Case 3F: $(\sigma, r_6) = (3, 2)$.

(1) $\Phi(t) = 16t(1 + t)\left[-3 + (30 - 16b_2)t + (-51 + 16b_2)t^2\right]$.
(2) $\mathbf{D} = \{b_2 : b_2 \leq 0.43934 \text{ or } b_2 \geq 2.56066\}, \mathbf{B} = \{b_2 : \frac{15}{8} < b_2 < \frac{51}{16}\}$.
(3) $\mathbf{D} \cap \mathbf{B} = \{b_2 : 2.56066 \leq b_2 < 3.1875\}$.

The three sub-subcases **Cases 3F1, 3F2, 3F3** are identified with $b_2 \in \{\frac{18}{7}, 3, 3.1\}$ in order.
Observe that this sub-case yields additional extraneous fixed points $\pm i$.

Case 3G: $(\sigma, r_6) = (3, 3)$.

(1) $\Phi(t) = 16\,t^2(1+t)\big[-5 + 16b_2 + (26 - 16b_2)t + 3t^2\big]$.
(2) $\mathbf{D} = \{b_2 : b_2 \le 0.93934 \text{ or } b_2 \ge 3.06066\}$, $\mathbf{B} = \{b_2 : \frac{5}{16} < b_2 < \frac{13}{8}\}$.
(3) $\mathbf{D} \cap \mathbf{B} = \{b_2 : 0.3125 < b_2 \le 0.93934\}$.

The three sub-subcases **Cases 3G1, 3G2, 3G3** are identified with $b_2 \in \{0.4, 0.5, 0.9\}$ in order.
Observe that this sub-case yields additional extraneous fixed points $\pm i$.

Case 3H: $(\sigma, r_6) = (4, 1)$.

(1) $\Phi(t) = 8(1+t)^2\big[21 - 8b_2 + (-22 - 96b_2)t + (-1095 + 104b_2)t^2\big]$.
(2) $\mathbf{D} = \mathbb{R}, \mathbf{B} = \{b_2 : \frac{21}{8} < b_2 < \frac{1095}{104}\}$.
(3) $\mathbf{D} \cap \mathbf{B} = \{b_2 : \frac{21}{8} < b_2 < \frac{1095}{104}\}$.

The three sub-subcases **Cases 3H1, 3H2, 3H3** are identified with $b_2 \in \{\frac{1531}{244}, 8, 10\}$ in order.
Observe that this sub-case yields additional extraneous fixed points $\pm i$ of multiplicity 2.

Case 3I: $(\sigma, r_6) = (4, 2)$.

(1) $\Phi(t) = 8t(1+t)^2\big[7 - 8b_2 - 42t + (-21 + 8b_2)t^2\big]$.
(2) $\mathbf{D} = \mathbb{R}$, $\mathbf{B} = \{b_2 : \frac{7}{8} < b_2 < \frac{21}{8}\}$.
(3) $\mathbf{D} \cap \mathbf{B} = \{b_2 : \frac{7}{8} < b_2 < \frac{21}{8}\}$.

The three sub-subcases **Cases 3I1, 3I2, 3I3** are identified with $b_2 \in \{1, 1.5, 2.5\}$ in order.
Observe that this sub-case yields additional extraneous fixed points $\pm i$ of multiplicity 2.

Case 3J: $(\sigma, r_6) = (5, 1)$.

(1) $\Phi(t) = 4(1+t)^3\big[-7 + (102 - 48b_2)t + (-263 + 48b_2)t^2\big]$.
(2) $\mathbf{D} = \{b_2 : b_2 \le 0.404464 \text{ or } b_2 \ge 3.2622\}$, $\mathbf{B} = \{b_2 : \frac{17}{8} < b_2 < \frac{263}{48}\}$.
(3) $\mathbf{D} \cap \mathbf{B} = \{b_2 : 3.2622 \le b_2 < 5.47917\}$.

The three sub-subcases **Cases 3J1, 3J2, 3J3** are identified with $b_2 \in \{3.5, 4, \frac{22}{5}\}$ in order.
Observe that this sub-case yields additional extraneous fixed points $\pm i$ of multiplicity 3.

4.2. Stability of extraneous fixed points and dynamics behind the polynomials

As a result of the case studies pursued thus far for $f(z) = z^2 - 1$, we include in Table 3 the desired purely imaginary extraneous fixed points in typical sub-cases of **Case 3**. With the help of Mathematica [35], by direct computation of absolute values of multipliers $R_f'(\xi)$ for iterative map (4.2) with $f(z) = z^2 - 1$, we find that most of the extraneous fixed points ξ of H in each of the listed cases in Table 3 are found to be repulsive. Among all the listed extraneous fixed points, the ones for **Cases 3C1, 3C2, 3D2, 3D3, 3H1, 3I1, 3I3, 3J3** are found to be indifferent and highlighted by framed-values, while 98 of them are found to be attractive and highlighted in bold-face in **Cases 3F1**, **4A**–**4F** and **SA, CTV, LW**. Interestingly all of the extraneous fixed points of cases **4A**–**4F** and existing methods **SA**, **CTV**, **LW** are found to be nearly indifferent but none of them is indifferent, i.e., they are all found to be hyperbolic [36]. Since absolute values of their multipliers are so close to 1, they have been computed at the expense of 128 precision digits to determine their stability.

In case that $f(z)$ is a generic polynomial rather than $z^2 - 1$, it would be certainly interesting to investigate the dynamics underlying the relevant extraneous fixed points. However, due to the increased algebraic complexity, we would encounter difficulties in describing the dynamics underlying the extraneous fixed points. An effective way of exploring such dynamics is to illustrate basins of attraction under iterative map (4.2) with $f(z)$ as a generic polynomial. We will illustrate the basins of attraction to pursue the dynamics of the iterative map R_p of the form

$$z_{n+1} = R_p(z_n) = z_n - \frac{p(z_n)}{p'(z_n)} H_p(z_n), \tag{4.18}$$

for a generic polynomial $p(z_n)$ and a weight function $H_p(z_n)$. Indeed, basins of attraction for the fixed points or the extraneous fixed points as well as their attracting periodic orbits would reflect complex dynamics whose illustrative description will be made for various polynomials in the latter part of Section 5.

Before closing this section, we denote iterative maps in Table 3 corresponding to cases **1A**–**1J**, **2A1**, **2B1**, **2C3**, **2D1** as well as all **3A2**, **3B2**, **3C1**, **3C2**, **3D2**, **3D3 3E2**, **3F1**, **3F3**, **3G2**, **3H1**, **3I1**, **3I3**, **3J3**, **4A**–**4F** respectively by **W1A**–**W1F**, **W2A1**, **W2B1**, **W2C3**, **W2D1** and **W3A2**, **W3B2**, **W3C1**, **W3C2**, **W3D2**, **W3D3 W3E2**, **W3F1**, **W3F3**, **W3G2**, **W3H1**, **W3I1**, **W3I3**, **W3J3**, **W4A**–**W4F** with W-prefixed for later use. In addition, we identify map **CTV** for method (1.1) with $\beta_1 = \beta_3 = 0,\ \beta_2 = 1$, map **LW** for method (1.5) with $\alpha_1 = 5,\ \beta_2 = -7$ and map **SA** for method (1.6).

5. Numerical experiments and complex dynamics

In this section, we first deal with computational aspects of proposed methods (1.7) for a variety of test functions in comparison with other existing methods; then we discuss the dynamics underlying extraneous fixed points based on iterative maps (4.18) by illustrating the relevant basins of attraction. Selected cases **1A**, **1B**, **1C**, **1G**, **2A1**, **2B1**, **2C3**, **2D1** as well as **3A2**, **3B2**, **3C1**, **3C2**, **3D2**, **3D3 3E2**, **3F1**, **3F3**, **3G2**, **3H1**, **3I1**, **3I3**, **3J3**, **4A**, **4C**, **4F** have been implemented to verify the theoretical convergence. Later on in this section, the complex dynamics will be explored along with illustrated basins of attraction of selected rational iterative maps **W1A**, **W1B**, **W1C**, **W1G**, **W2A1**, **W2B1**, **W2C3**, **W2D1** as well as **W3A2**, **W3B2**, **W3C1**, **W3C2**, **W3D2**, **W3D3 W3E2**, **W3F1**, **W3F3**, **W3G2**, **W3H1**, **W3I1**, **W3I3**, **W3J3**, **W4A**, **W4C**, **W4F** and existing three methods **CTV**, **LW**, **SA**.

A number of numerical experiments have been implemented with Mathematica programming to confirm the developed theory. Throughout these experiments, we have maintained 160 digits of minimum number of precision, via Mathematica command $\$MinPrecision = 160$, to achieve the specified accuracy. In case that α is not exact, it is replaced by a more accurate value which has more number of significant digits than the preassigned number $\$MinPrecision = 160$.

Definition 2 (*Computational Convergence Order*)**.** Assume that theoretical asymptotic error constant $\eta = \lim_{n\to\infty} \frac{|e_n|}{|e_{n-1}|^p}$ and convergence order $p \geq 1$ are known. Define $p_n = \frac{\log|e_n/\eta|}{\log|e_{n-1}|}$ as the computational convergence order. Note that $\lim_{n\to\infty} p_n = p$.

Remark 5.1. Note that p_n requires knowledge at two points x_n, x_{n-1}, while the usual COC (computational order of convergence) $\frac{\log(|x_n-x_{n-1}|/|x_{n-1}-x_{n-2}|)}{\log(|x_{n-1}-x_{n-2}|/|x_{n-2}-x_{n-3}|)}$ does require knowledge at four points $x_n, x_{n-1}, x_{n-2}, x_{n-3}$. Hence p_n can be handled with a less number of working precision digits than the usual COC whose number of working precision digits is at least p times as large as that of p_n.

Computed values of x_n are accurate with up to $\$MinPrecision$ significant digits. If α has the same accuracy of $\$MinPrecision$ as that of x_n, then $e_n = x_n - \alpha$ would be nearly zero and hence computing $|e_{n+1}|/e_n^p|$ would unfavorably break down. To clearly observe the convergence behavior, we desire α to have more significant digits that are Φ digits higher than \$MinPrecision. To supply such α, a set of following Mathematica commands are used:

$$sol = FindRoot[f(x), \{x, x_0\}, PrecisionGoal \to \Phi + \$MinPrecision, WorkingPrecision \to 2 * \$MinPrecision];$$
$$\alpha = sol[[1, 2]]$$

In this experiment, we assign $\Phi = 16$. As a result, the numbers of significant digits of x_n and α are found to be 160 and 176, respectively. Nonetheless, we list both of them with up to 15 significant digits for proper readability. The error bound $\varepsilon = \frac{1}{2} \times 10^{-128}$ is assigned to satisfy $|x_n - \alpha| < \varepsilon$.

Iterative methods (1.7) associated with case numbers are identified by W-prefixed names. Typical methods with cases **1D**, **2A2**, **3A1**, **3B1** are respectively identified by **W1D**, **W2A2**, **W3A1**, **W3B1**. These four typical methods have been successfully implemented with test functions F_1–F_4 below:

$$\begin{cases} \textbf{W1D}: F_1(x) = \cos\left(\frac{\pi x}{2}\right) + x^2 - 4,\ \alpha \approx -2.22250743480067, \\ \textbf{W2A2}: F_2(x) = \cos(x^2 - 1) + \log(x^2 - 3\pi) + 1,\ \alpha = \sqrt{3\pi + 1}, \\ \textbf{W3A1}: F_3(x) = \cos^{-1}(x - 1) + e^{x^2} - 5,\ \alpha \approx 1.12632039674987, \\ \textbf{W3B1}: F_4(x) = x^3 - \log(1 + \sin x),\ \alpha = 0, \\ \text{where } \log z (z \in \mathbb{C}) \text{ represents a principal analytic branch such that } -\pi < Im(\log z) \leq \pi. \end{cases}$$

Table 4 clearly confirms eighth-order convergence. The values of computational asymptotic error constant agree up to 10 significant digits with η. It appears that the computational convergence order well approaches 8.

Table 5 lists additional test functions to ensure the convergence behavior of proposed scheme (1.7).

In Table 6, we compare numerical errors $|x_n - \alpha|$ of proposed methods **W1A**, **W1B**, **W1C**, **W1G**, **W2A1**, **W2B1**, **W2C3**, **W2D1**, **W3A2**, **W3B2**, **W3C1**, **W3C2**, **W3D2**, **W3D3 W3E2**, **W3F1**, **W3F3**, **W3G2**, **W3H1**, **W3I1**, **W3I3**, **W3J3**, **W4A**, **W4C**, **W4F** with those of methods **CTV**, **LW** and **SA**. The least errors within the prescribed error bound are highlighted in bold face. Although we are limited to the selected current experiments, within two iterations, a strict comparison shows that Methods **W3E2**, **W3J3**, **W3B2**, **W2C3**, **W3F3**, **W4F** display slightly better convergence for test functions $f_1, f_2, f_3, f_4, f_5, f_6$, respectively.

In view of a close inspection of the asymptotic error constant $\eta(\theta_i, L_f, Q_f) = \frac{|x_{n+1}-\alpha|}{|x_n-\alpha|^8}$, we should be aware that the local convergence is dependent on the function $f(x)$, an initial value x_0, the zero α itself and the weight functions L_f and Q_f. Accordingly, for a given set of test functions, the convergence of one method is hardly expected to be better than the others.

With p as the order of convergence and d as the number of functional or derivative evaluations per iteration, the efficiency index [1] defined by $EI = p^{\frac{1}{d}}$ is found to be $8^{1/4} \approx 1.68179$ for the proposed methods (1.7), which evidently show a better performance than that of classical Newton's method. Weight functions L_f and Q_f dependent on two function-to-function ratios $\left[\frac{f(y_n)}{f(x_n)}\right]$ and $\left[\frac{f(w_n)}{f(x_n)}\right]$ undoubtedly contribute to establishing eighth-order convergence.

Table 4
Convergence for test functions $F_1(x) - F_4(x)$ with typically selected methods **W1D**, **W2A2**, **W3A1**, **W3B1**.

MT	F	n	x_n	$\|F(x_n)\|$	$\|x_n - \alpha\|$	$\|e_n/e_{n-1}^8\|$	η	p_n
W1D	F_1	0	-2.12	0.487887	0.102507			
		1	-2.22250743512851	1.633×10^{-9}	3.278×10^{-10}	0.02689144381	0.01786270393	7.82040
		2	-2.22250743480067	1.187×10^{-77}	2.383×10^{-78}	0.01786270390		8.00000
		3	-2.22250743480067	0.0×10^{-159}	0.0×10^{-159}			
W2A2	F_2	0	3.2	0.187272	0.0287425			
		1	3.22874247630269	1.751×10^{-8}	2.711×10^{-9}	5822.046718	385.6014409	7.23519
		2	3.22874247359082	7.283×10^{-66}	1.127×10^{-66}	385.6015782		8.00000
		3	3.22874247359082	0.0×10^{-160}	0.0×10^{-159}			
W3A1	F_3	0	1.2	0.590134	0.0736796			
		1	1.12632050629632	7.670×10^{-7}	1.095×10^{-7}	126.1304876	60.99176928	7.72141
		2	1.12632039674987	8.856×10^{-54}	1.264×10^{-54}	60.99180149		8.00000
		3	1.12632039674987	0.0×10^{-159}	0.0×10^{-159}			
W3B1	F_4	0	0.02	0.0197933	0.02			
		1	-2.077×10^{-15}	2.077×10^{-15}	2.077×10^{-15}	0.08116479822	0.07687289562	7.98611
		2	$-2.670.797 \times 10^{-119}$	2.670×10^{-119}	2.670×10^{-119}	0.07687289562		8.00000
		3	0.0×10^{-368}	0.0×10^{-368}	0.0×10^{-354}			

MT = method.

Table 5
Additional test functions $f_i(x)$ with zeros α and initial guesses x_0.

i	$f_i(x)$	α	x_0
1	$1 + \sin(x^2) - x$	1.58601884916183	1.65
2	$4x - \pi - \cos(2x) \cdot \log(4x^2 + x + 1)$	$\frac{\pi}{4}$	0.7
3	$x^2 + e^x + \sin(x^3 - x + 3) - 2$	-1.12126661425687	-1.07
4	$\cos(\frac{\pi x}{6}) + \frac{1}{x^3 - x + 1} - \frac{1}{25}$	3	2.75
5	$(x-1)^2 + \frac{3}{16} + e^{\frac{1}{x^3}} \cdot \log[(x-1)^4 + \frac{247}{256}]$	$\frac{1+i\sqrt{3}}{4}$	$0.98 + 0.4i$
6	$\log(x^2) - (x^2 + x - 1)\sqrt{x} + 2x^3 - 1$	1	1.12

Here $\log z$ ($z \in \mathbb{C}$) represents a principal analytic branch with $-\pi \le Im(\log z) < \pi$.

Proper initial values generally influence the convergence behavior of iterative methods. To guarantee the convergence of Newton-like iterative map (4.18) with a weight function $H_p(z)$, it requires good initial values close to zero α. It is, however, not a simple task to determine how close the initial values are to zero α, since initial values are generally sensitive to computational precision, error bound and the given function $f(x)$ under consideration. One effective way of selecting stable initial values would be directly using visual basins of attraction. Since the area of convergence can be seen on the basins of attraction, it would be reasonable to say that a method having a larger area of convergence implies a more stable method. It is no doubt for us to employ a quantitative analysis for measuring the size of area of convergence.

To this end, we present Tables 7–9 featuring a statistical data giving the average number of iterations per point, CPU time (in seconds) and number of points requiring 40 iterations. In the following examples, we take a 6 by 6 square centered at the origin and containing all the zeros of the given functions. We then take 601 × 601 equally spaced points in the square as initial points for the iterative methods. We color the point based on the root it converged to. This way we can figure out if the method converged within the maximum number of iteration allowed and if it converged to the root closer to the initial point.

We now are ready to discuss the complex dynamics of selected iterative maps in Table 3 including **W1B, W1C, W1E, W1G, W2B1, W2C3, W2D1**, and **W3A2, W3B2, W3C1, W3C2, W3D2, W3D3 W3E2, W3F1, W3F3, W3G2, W3H1, W3I1, W3I3, W3J3, W4A, W4C, W4F, SA, CTV, LW** when applied to various polynomials $p_k(z),\ k \in \mathbb{N}$.

Example 1. As a first example, we have taken a quadratic polynomial with all real roots:

$$p_1(z) = (z^2 - 1). \tag{5.1}$$

Clearly the roots are ±1. Basins of attraction for **W1B–W4F** are given in the top seven rows of Fig. 1. The last row presents the basins of attraction for **SA**, **CTV** and **LW**. It is clear that the worst methods are **W3A2** and **W4C**. Consulting Tables 7–9, we find that the method **SA** uses the least number of iterations per point on average, it also uses the least amount of CPU time and has the least number of black points. The worst is **W3A2** with 16.75 iterations per point on average and the highest number of black points (133,923). The next is **W4C** using 3.92 iterations per point on average. All other methods use no more than 3.1 iterations per point. In the following examples we will not show **W3A2**.

Table 6
Comparison of $|x_n - \alpha|$ for selected methods applied to various test functions.

Method	$\|x_n - \alpha\|$	$f(x);\ x_0$					
		$f_1;\ 1.65$	$f_2;\ 0.7$	$f_3;\ -1.07$	$f_4;\ 2.75$	$f_5;\ 0.98+0.4i$	$f_6;\ 1.12$
W1A	$\|x_1 - \alpha\|$	1.28e−8	1.03e−8	1.16e−9	1.10e−11	6.23e−9	5.60e−10
	$\|x_2 - \alpha\|$	9.81e−62	1.89e−64	4.13e−71	4.09e−94	1.00e−63	2.39e−75
W1B	$\|x_1 - \alpha\|$	2.80e−9	4.886e−10	1.39e−10	1.00e−11	1.11e−9	3.43e−9
	$\|x_2 - \alpha\|$	4.12e−68	6.01e−76	7.89e−80	1.19e−94	2.10e−70	6.62e−69
W1C	$\|x_1 - \alpha\|$	2.37e−9	1.21e−9	1.81e−10	4.16e−12	1.09e−9	1.11e−10
	$\|x_2 - \alpha\|$	1.31e−68	1.40e−72	4.20e−78	8.37e−98	2.31e−70	1.64e−73
W1G	$\|x_1 - \alpha\|$	1.98e−9	2.07e−9	3.42e−10	5.14e−12	8.28e−10	4.54e−9
	$\|x_2 - \alpha\|$	5.79e−69	7.22e−71	5.95e−76	3.36e−97	7.97e−72	8.93e−69
W2A1	$\|x_1 - \alpha\|$	1.00e−8	1.28e−8	1.06e−9	1.08e−11	6.32e−9	3.13e−8
	$\|x_2 - \alpha\|$	1.49e−62	9.09e−64	9.11e−72	2.73e−94	8.54e−64	4.11e−61
W2B1	$\|x_1 - \alpha\|$	9.43e−10	6.57e−10	1.80e−10	3.81e−12	2.61e−10	8.26e−10
	$\|x_2 - \alpha\|$	5.11e−72	3.28e−75	2.52e−78	2.67e−98	2.85e−76	8.02e−75
W2C3	$\|x_1 - \alpha\|$	1.30e−9	6.12e−10	7.86e−11	2.07e−11	2.76e−10	1.61e−7
	$\|x_2 - \alpha\|$	4.97e−71	1.55e−75	9.66e−82	**6.47e−101**	3.01e−76	8.65e−55
W2D1	$\|x_1 - \alpha\|$	2.40e−7	1.21e−7	7.18e−9	2.18e−11	4.64e−8	9.19e−6
	$\|x_2 - \alpha\|$	3.14e−50	5.17e−55	1.23e−64	3.36e−92	4.29e−56	3.69e−39
W3A2	$\|x_1 - \alpha\|$	7.77e−10	1.64e−10	1.59e−10	3.82e−12	1.28e−10	1.74e−8
	$\|x_2 - \alpha\|$	6.89e−73	4.51e−80	3.72e−79	3.27e−98	9.33e−79	4.03e−63
W3B2	$\|x_1 - \alpha\|$	2.39e−10	3.61e−10	5.84e−11	3.93e−12	3.35e−10	2.05e−9
	$\|x_2 - \alpha\|$	5.69e−77	1.41e−77	**3.51e−82**	5.27e−98	8.34e−75	2.79e−71
W3C1	$\|x_1 - \alpha\|$	4.04e−9	2.06e−9	6.20e−10	3.43e−12	1.52e−9	8.25e−10
	$\|x_2 - \alpha\|$	1.37e−66	1.69e−70	2.78e−73	5.38e−100	5.02e−69	1.41e−74
W3C2	$\|x_1 - \alpha\|$	2.29e−9	9.63e−10	3.97e−10	3.63e−12	7.59e−10	7.01e−10
	$\|x_2 - \alpha\|$	6.17e−69	2.24e−73	5.59e−75	5.35e−99	1.14e−71	3.86e−75
W3D2	$\|x_1 - \alpha\|$	6.13e−11	1.21e−10	6.84e−11	3.91e−12	2.49e−10	1.05e−9
	$\|x_2 - \alpha\|$	1.28e−82	8.83e−82	6.73e−82	3.52e−98	3.03e−76	9.57e−74
W3D3	$\|x_1 - \alpha\|$	9.54e−10	5.23e−10	2.22e−10	3.79e−12	2.55e−10	1.01e−9
	$\|x_2 - \alpha\|$	3.44e−72	7.02e−76	2.63e−77	1.97e−98	5.90e−76	7.13e−74
W3E2	$\|x_1 - \alpha\|$	1.53e−11	2.23e−10	1.07e−10	3.89e−12	1.56e−10	3.64e−9
	$\|x_2 - \alpha\|$	**2.85e−87**	3.66e−80	8.91e−82	3.97e−98	7.98e−78	3.98e−69
W3F1	$\|x_1 - \alpha\|$	2.50e−9	1.30e−9	3.49e−10	3.64e−12	7.70e−10	3.14e−10
	$\|x_2 - \alpha\|$	2.23e−68	2.31e−72	1.23e−75	1.28e−98	9.27e−72	1.40e−78
W3F3	$\|x_1 - \alpha\|$	3.22e−10	3.16e−10	1.46e−10	3.86e−12	1.21e−10	6.38e−10
	$\|x_2 - \alpha\|$	3.15e−76	4.54e−78	3.70e−79	3.10e−98	**2.80e−79**	4.05e−76
W3G2	$\|x_1 - \alpha\|$	1.089e−9	5.49e−10	2.47e−10	3.77e−12	3.17e−10	9.51e−10
	$\|x_2 - \alpha\|$	9.13e−72	1.22e−75	7.18e−77	1.70e−98	4.35e−75	4.30e−74
W3H1	$\|x_1 - \alpha\|$	1.36e−9	7.24e−10	2.17e−10	3.76e−12	3.69e−10	2.39e−9
	$\|x_2 - \alpha\|$	9.24e−71	1.03e−74	1.22e−77	2.45e−98	9.03e−75	1.04e−70
W3I1	$\|x_1 - \alpha\|$	1.05e−10	1.22e−10	1.07e−10	3.90e−12	1.65e−10	8.01e−10
	$\|x_2 - \alpha\|$	3.63e−80	2.29e−82	4.46e−80	3.01e−98	3.97e−78	9.97e−75
W3I3	$\|x_1 - \alpha\|$	2.09e−9	1.10e−9	3.12e−10	3.68e−12	6.34e−10	2.42e−10
	$\|x_2 - \alpha\|$	4.48e−69	5.19e−73	4.47e−76	1.50e−98	1.60e−72	1.08e−79
W3J3	$\|x_1 - \alpha\|$	1.12e−10	3.90e−12	1.00e−10	3.90e−12	1.37e−10	8.66e−10
	$\|x_2 - \alpha\|$	3.88e−80	**8.78e−95**	1.35e−80	3.52e−98	1.97e−78	9.81e−75
W4A	$\|x_1 - \alpha\|$	1.35e−10	2.07e−9	2.92e−10	9.64e−12	9.38e−10	2.12e−8
	$\|x_2 - \alpha\|$	3.20e−78	2.41e−71	3.66e−76	6.12e−95	1.31e−71	1.40e−62
W4C	$\|x_1 - \alpha\|$	3.88e−9	5.44e−9	3.57e−10	4.32e−12	2.72e−9	7.87e−9
	$\|x_2 - \alpha\|$	3.41e−66	3.26e−67	7.16e−76	5.95e−98	2.71e−67	5.66e−66
W4F	$\|x_1 - \alpha\|$	2.92e−9	1.78e−9	4.60e−10	3.57e−12	1.17e−9	5.01e−11
	$\|x_2 - \alpha\|$	9.84e−68	3.71e−71	1.41e−74	8.87e−99	3.69e−70	**1.07e−84**
SA	$\|x_1 - \alpha\|$	9.09e−10	4.69e−10	2.13e−10	3.79e−12	2.28e−10	6.80e−10
	$\|x_2 - \alpha\|$	2.04e−72	2.72e−76	1.73e−77	2.08e−98	2.14e−76	2.09e−75
CTV	$\|x_1 - \alpha\|$	4.19e−10[a]	3.19e−10	1.00e−10	3.92e−12	2.40e−10	9.02e−10
	$\|x_2 - \alpha\|$	5.11e−75	1.65e−78	3.07e−80	3.00e−98	9.57e−77	3.22e−74
LW	$\|x_1 - \alpha\|$	1.10e−9	1.02e−9	1.51e−10	3.97e−12	4.88e−10	1.47e−9
	$\|x_2 - \alpha\|$	4.16e−71	4.77e−74	1.81e−78	1.64e−98	3.97e−74	3.61e−72

[a] 4.19e−10 denotes 4.19×10^{-10}.

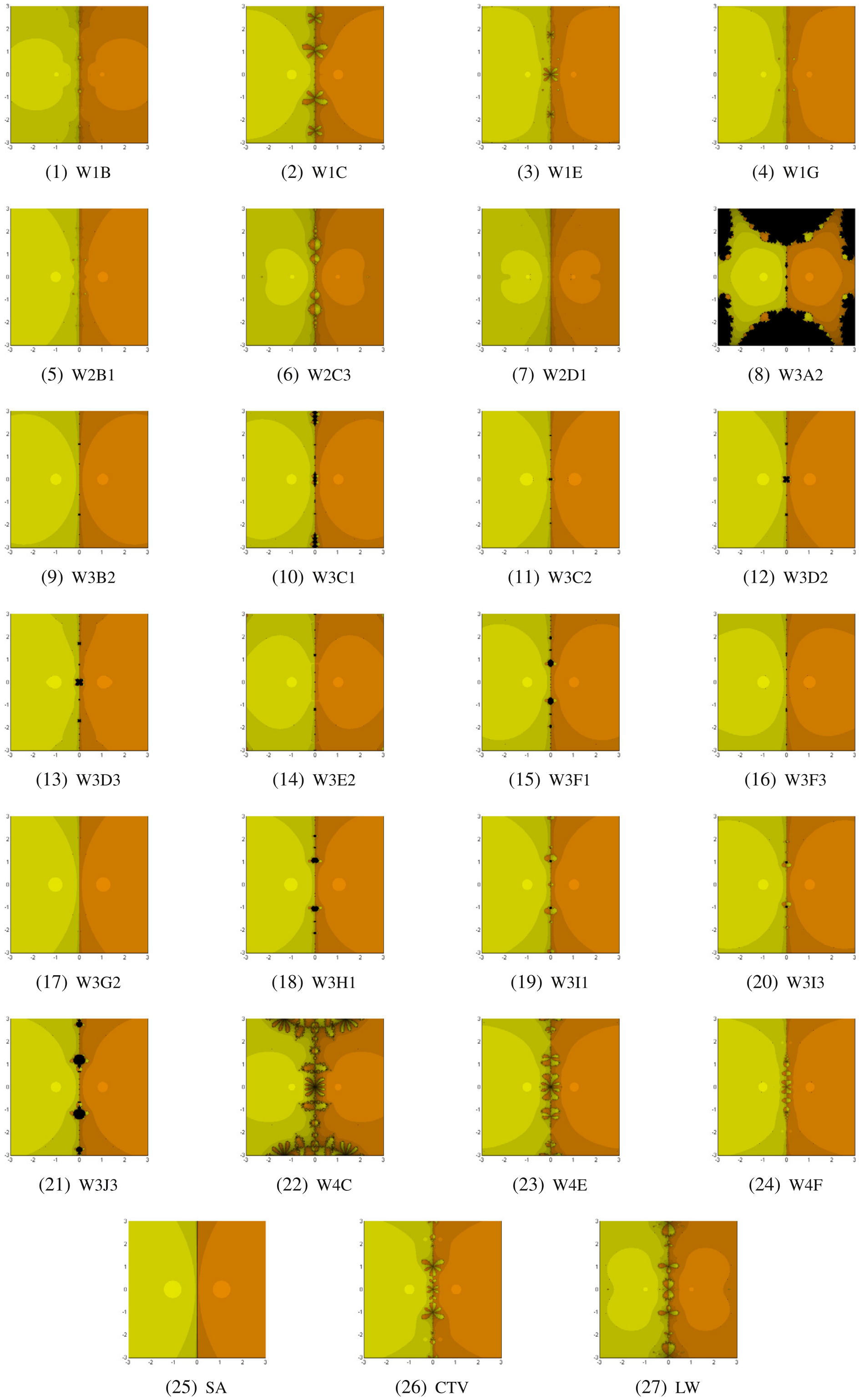

Fig. 1. The top row for **W1B** (left), **W1C** (center left), **W1E** (center right) and **W1G** (right). The second row for **W2B1** (left), **W2C3** (center left), **W2D1** (center right) and **W3A2** (right). The third row for **W3B2** (left), **W3C1** (center left), **W3C2** (center right) and **W3D2** (right). The fourth row for **W3D3** (left), **W3E2** (center left), **W3F1** (center right) and **W3F3** (right). The fifth row for **W3G2** (left), **W3H1** (center left), **W3I1** (center right) and **W3I3** (right). The sixth row for **W3J3** (left), **W4C** (center left), **W4E** (center right) and **W4F** (right). The bottom row for **SA8** (left), **CTV** (center) and **LW** (right) for the roots of the polynomial $(z^2 - 1)$.

Example 2. In our second example, we have taken a cubic polynomial:

$$p_2(z) = (z^3 + 4z^2 - 10). \tag{5.2}$$

We consult the tables to find that the method with the fewest number of iterations on average is **SA** with 2.46 iteration followed by **W3D2** (2.58), **W3G2** (2.62), **W3D3** and **W3F3** (2.63). The worst is **W4C** with 5.14 iterations. All the others require between 2.71 and 3.69. In terms of CPU time in seconds, the fastest is **SA** and the slowest is **W4C**. The method **W3E2** has the most black points and **W3C1**, **W3C2**, **W3B2** and **W3F3** have less than 10 black points. In the following examples we will remove from further consideration the method **W3E2** because it has the highest number of black points.

Example 3. As a third example, we have taken another cubic polynomial:

$$p_3(z) = (z^3 - z). \tag{5.3}$$

Now all the roots are real. Based on Table 7 we see that again **SA** has the lowest number of iteration per point on average followed by **W3D2** and **W3F3**. The fastest method is again **SA** (231.116 s) and the slowest are **W3H1** (690.975 s) and **W4C** (479.204 s). Most of the methods have no black points except **W3H1** with 21210, **W4C** with 402, **W4E** with 174 and **W1E** with 4 black points. We will eliminate **W3H1** because of the number of black points and the computational cost.

Example 4. As a fourth example, we have taken a quartic polynomial:

$$p_4(z) = (z^4 - 1). \tag{5.4}$$

We consult the tables and find that **SA** has the lowest number of iterations per point (2.98) and **W4C** has the highest number (13.94). The fastest is again **SA** using 293.968 s and the slowest is **W4C** using more than 4 times that, i.e., 1388.628 s. **W4C** has also the highest number of black points (73,077) and therefore will be eliminated from the rest of the examples. The next highest is **W4E** with 20,833 points. Many of the methods have 1201 black points.

Example 5. As a fifth example, we have taken a quintic polynomial:

$$p_5(z) = (z^5 - 1). \tag{5.5}$$

Upon examining Table 7, we find that SA using only 3.03 iteration per point on average and W3F1 using 3.21 iterations. The worst in this sense are W3I3 with 27.39 iterations and W2B1 with 20.58 iterations. In terms of CPU time (see Table 8), SA was the fastest using 338.74 s followed by W1B (441.03 s). The slowest are W2B1 with 2680.41 s and W3I3 with 3627.944 s. This is not surprising, since these methods required the highest number of iterations per point on average and have the most black points (see Table 9). We will exclude these two from our last example. There are seven methods with only one black point, one method each with 2, 5, 8 and 9 black points.

Example 6. As a last example, we have taken a sextic polynomial with complex coefficients:

$$p_6(z) = z^6 - \frac{1}{2}z^5 + \frac{11(i+1)}{4}z^4 - \frac{3i+19}{4}z^3 + \frac{5i+11}{4}z^2 - \frac{i+11}{4}z + \frac{3}{2} - 3i. \tag{5.6}$$

Based on Table 7 we find that **SA** requires the least number of iterations per point followed by **W3D2** and **W3D3** and the worst is **W4F** (8.46 iterations) followed by **W4E** with 7.58 iterations. In terms of CPU time, we find in Table 8 that **SA** is the fastest and **W4F** and **LW** are the slowest. **W4F** has the highest number of black points (48747) followed by **W4E** and **W3F1**. There are 6 methods with no black points, namely **SA**, **W3F3**, **W3D3**, **W3D2**, **W1G** and **W1B**. Method **W3B2** has only one black point and **W3I1** has 3 black points.

6. Conclusions

In summary, we find that **SA** is best method overall. The worst in terms of the number of black points and the number of iterations per point is **W4E** and in terms of CPU time is **LW**. Of course this is excluding the methods eliminated along the way, namely **W2B1**, **W3A2**, **W3E2**, **W3H1**, **W3I3** and **W4C**. To summarize the results of the 6 examples, we have averaged the results in Tables 7–9 across examples. Based on Table 7 we find that **SA** uses the least number of iterations per point (2.69 on average) followed closely by **W3D2** (2.88) and **W3D3** (2.97). All other methods use more than 3 iterations per point on average. The method requiring the highest number of iterations per point is **W4E** (5.52). The fastest method is **SA** (468.810 s) followed by **W1G** (521.381 s). The slowest is **LW** (1070.164 s). As for the number of black points (see Table 9) we find that **SA** has the lowest number (309 points) followed by **W1G** (329 points).

We conclude the current study as follows. Convergence order of proposed methods (1.7) has been improved with the introduction of weight functions expressed in terms of function-to-function ratios. Computational aspects through a variety of test equations in a number of selected cases well agree with the developed theory, verifying the convergence order and asymptotic error constants. To determine what type of initial values of the proposed methods chosen near the zero α must

Table 7
Average number of iterations per point for each example (1–6).

Map	Example						
	1	2	3	4	5	6	Average
W1B	2.81	3.47	3.50	4.38	4.55	4.22	3.82
W1C	2.61	3.23	3.54	4.39	4.36	4.46	3.76
W1E	2.42	3.08	3.18	4.87	5.02	4.18	3.79
W1G	2.37	2.88	3.02	3.45	3.39	3.24	3.06
W2B1	2.32	2.74	2.90	3.82	20.58	–	–
W2C3	3.08	3.40	3.73	4.65	4.78	4.58	4.03
W2D1	3.10	3.41	3.74	4.41	4.59	4.34	3.93
W3A2	16.75	–	–	–	–	–	–
W3B2	2.35	2.83	3.11	3.55	3.59	3.42	3.14
W3C1	2.68	3.00	2.89	3.85	4.12	3.93	3.41
W3C2	2.29	2.99	3.39	4.05	4.34	4.08	3.52
W3D2	2.36	2.58	2.77	3.25	3.25	3.09	2.88
W3D3	2.38	2.63	2.81	3.40	3.39	3.24	2.97
W3E2	2.60	3.49	–	–	–	–	–
W3F1	2.61	2.80	3.21	3.37	3.21	5.95	3.52
W3F3	2.45	2.63	2.76	3.35	3.58	3.75	3.09
W3G2	2.22	2.62	2.82	3.84	4.34	3.50	3.22
W3H1	2.47	3.06	5.85	–	–	–	–
W3I1	2.38	2.80	3.03	3.59	3.72	3.52	3.17
W3I3	2.40	2.71	3.21	3.26	27.39	–	–
W3J3	2.97	3.03	3.12	4.25	4.34	4.08	3.63
W4C	3.92	5.14	4.95	13.94	–	–	–
W4E	2.69	3.69	3.82	7.18	8.16	7.58	5.52
W4F	2.36	2.90	3.08	3.86	4.47	8.46	4.19
SA	2.16	2.46	2.62	2.98	3.03	2.89	2.69
CTV	2.43	3.20	3.47	5.17	5.83	5.07	4.19
LW	2.99	3.58	3.72	5.56	6.05	5.09	4.50

Table 8
CPU time (in seconds) required for each example (1–6) using a Dell Multiplex-990.

Map	Example						
	1	2	3	4	5	6	Average
W1B	239.072	454.400	385.385	512.775	591.727	1816.273	666.605
W1C	232.098	422.170	396.320	525.521	568.967	1912.244	676.220
W1E	215.718	409.472	351.985	574.487	658.683	1775.463	664.301
W1G	202.942	371.829	334.326	399.815	441.030	1378.346	521.381
W2B1	205.376	373.575	321.237	452.107	2680.41	–	–
W2C3	274.421	471.341	434.385	567.735	654.424	1979.278	730.264
W2D1	287.463	470.078	425.290	540.887	629.402	1886.068	706.531
W3A2	1500.371	–	–	–	–	–	–
W3B2	229.945	404.464	382.203	460.001	515.084	1499.123	581.803
W3C1	264.453	423.902	344.232	497.175	574.364	1720.581	637.451
W3C2	225.547	416.865	397.459	502.900	601.836	1777.803	653.735
W3D2	233.471	363.794	333.873	420.844	474.461	1359.580	531.004
W3D3	235.406	381.391	333.218	438.877	482.027	1411.949	547.145
W3E2	253.049	489.562	–	–	–	–	–
W3F1	251.505	394.136	391.297	424.479	453.246	2555.062	744.954
W3F3	237.277	368.724	329.006	420.907	510.763	1658.446	587.520
W3G2	214.049	367.149	339.006	480.717	611.024	1531.945	590.648
W3H1	239.586	439.798	690.975	–	–	–	–
W3I1	239.445	406.133	374.715	464.633	521.699	1533.521	590.024
W3I3	233.097	389.222	375.635	417.880	3627.944	–	–
W3J3	292.596	435.025	374.371	530.590	621.664	1772.811	671.176
W4C	288.056	601.898	479.204	1388.628	–	–	–
W4E	206.218	445.726	383.497	736.792	980.373	3123.624	979.372
W4F	190.072	360.284	319.022	435.103	563.522	3534.063	900.344
SA	152.381	291.300	231.116	293.968	338.740	1253.358	468.810
CTV	231.333	503.244	435.929	687.200	873.527	3220.391	991.937
LW	257.308	573.101	443.620	712.519	915.648	3518.789	1070.164

be given for their ensured convergence, we have not only carefully investigated the extraneous fixed points of the proposed maps applied to a polynomial $f(z) = (z^2 - 1)$ motivated by the earlier work of Vrscay and Gilbert [14], but also extensively illustrated relevant complex dynamics of selected methods from **Cases 1–4** and existing methods **CTV, LW, SA** behind the basins of attraction for a wide variety of polynomials $p_k(z)$. Among those methods selected from **Case 3**, two methods **W3D2**

Table 9
Number of points requiring 40 iterations for each example (1–6).

Map	Example						
	1	2	3	4	5	6	Average
W1B	737	96	0	1 201	8	0	340.3
W1C	757	27	0	1 233	306	794	519.5
W1E	767	53	4	5 733	9 520	1 784	2976.8
W1G	721	53	0	1 201	1	0	329.3
W2B1	749	48	0	1 201	168 265	–	–
W2C3	761	10	0	1 201	66	142	363.3
W2D1	997	95	0	1 201	32	64	398.2
W3A2	133 923	–	–	–	–	–	–
W3B2	893	8	0	1 201	1	1	350.7
W3C1	3 281	5	0	1 201	2	18	751.2
W3C2	999	6	0	1 201	15	54	379.2
W3D2	1 625	24	0	1 201	1	0	475.2
W3D3	2 023	12	0	1 201	1	0	539.5
W3E2	1 073	6196	–	–	–	–	–
W3F1	2 335	89	0	1 201	1	19 678	3884
W3F3	961	8	0	1 201	1	0	361.8
W3G2	749	24	0	1 545	3 911	17	1041
W3H1	2 283	3472	21 210	–	–	–	–
W3I1	907	15	0	1 201	5	3	355.2
W3I3	885	61	0	1 201	238 158	–	–
W3J3	6 269	10	0	1 201	9	22	1251.8
W4C	737	684	402	73 077	–	–	–
W4E	757	300	174	20 833	32 922	22 131	12852.8
W4F	731	66	0	2 217	3 563	48 747	9220.7
SA	601	54	0	1 201	1	0	309.5
CTV	605	64	0	4 765	9 846	2 308	2931.3
LW	799	123	0	2 893	5 866	777	1743

and **W3D3** performed reasonably well in view of the average number of iterations and CPU time. It seems that a method will not perform well unless its extraneous fixed points are on the imaginary axis.

We have tried to find connection between location and multiplicity of the extraneous fixed points (see Table 3) and the performance of the methods. In fact, the list of the fastest eight methods in order is found to be: {**SA, W1G, W3D2, W3D3, W3B2, W3F3, W3I1** and **W3G2**}. These same 8 methods also use the least number of iterations per point on average (not in the same order though). Five of those 8 methods are also amongst the 8 having the least number of black points. In view of Table 3, we find that except **W1G** all of these 8 methods have only purely imaginary extraneous fixed points (EFPs). These may mean that purely imaginary extraneous fixed points are necessary but not sufficient to guarantee a better performance. Besides, it is interesting to note that **W3F1** has the highest CPU time (744.954 s) but its purely imaginary extraneous fixed points $\pm i$ are found to be attractive.

In the current study, we have focused on the dynamics of methods with purely imaginary extraneous fixed points that are found from the roots of quadratic equation $\psi(t) = 0$ in (4.15). As a future study of the dynamics behind the purely imaginary extraneous fixed points to be found from the roots of a cubic equation $\psi(t) = 0$, we will continue to pursue other possible combinations of parameters $a_4, a_5, a_6, b_2, b_3, b_5, b_6$ described by (4.9). Hopefully, some of such combinations would give different weight functions $Q_f(s, u)$ enhancing the relevant dynamical behavior as desired.

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