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Blockmodeling and the Estimation of Evolutionary Architectural Growth in Major Defense Acquisition Programs

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Blockmodeling and the Estimation of Evolutionary Architectural Growth in Major Defense Acquisition Programs

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Agenda

- Purpose, Question, and Contribution
- Challenge and Opportunity in Pre-MS A Cost Analysis
- Mapping DoDAF to COSYSMO
- Leveraging SE and Exploiting the SV-3
- Estimating Unforeseen Architectural Growth in MDAPs
  - Microstructure
  - Macrostructure
- Simulating Growth and Estimating Cost
- "Blockmodeling" Beyond Architectural Communities
- Future work
- Questions

Overarching Purpose: To transform Model-Based System Engineering (MBSE) artifacts into computational knowledge that can be leveraged early in the system lifecycle when uncertainty is high and confidence is low

Focused Question: Can parametric cost estimation, in conjunction with DoD Architecture Framework (DoDAF) models, capture the monetary impact of architectural changes early in the system lifecycle?

Principal Contribution: A network science-based algorithm for estimating the cost of unforeseen architectural growth

Challenge and Opportunity in Pre-MS A Cost Analysis

We find ourselves in challenging times . . .

- Sequestration in 2013 + CRs =
  Reduced production +
  Hard modernization decisions +
  ··· + **Difficult cost planning**

. . . and times were already tough . . .

- 1997-2009: 47 MDAPs had cost overruns of at least 15%/30% over their current/baseline estimates

. . . especially early in the life cycle . . .

- ~ 28% of a system's baseline requirements will change

. . . as late adds carry substantial costs.

- 2014: 6 of 14 largest cost overruns due to new capabilities

But there is an appetite for change . . .

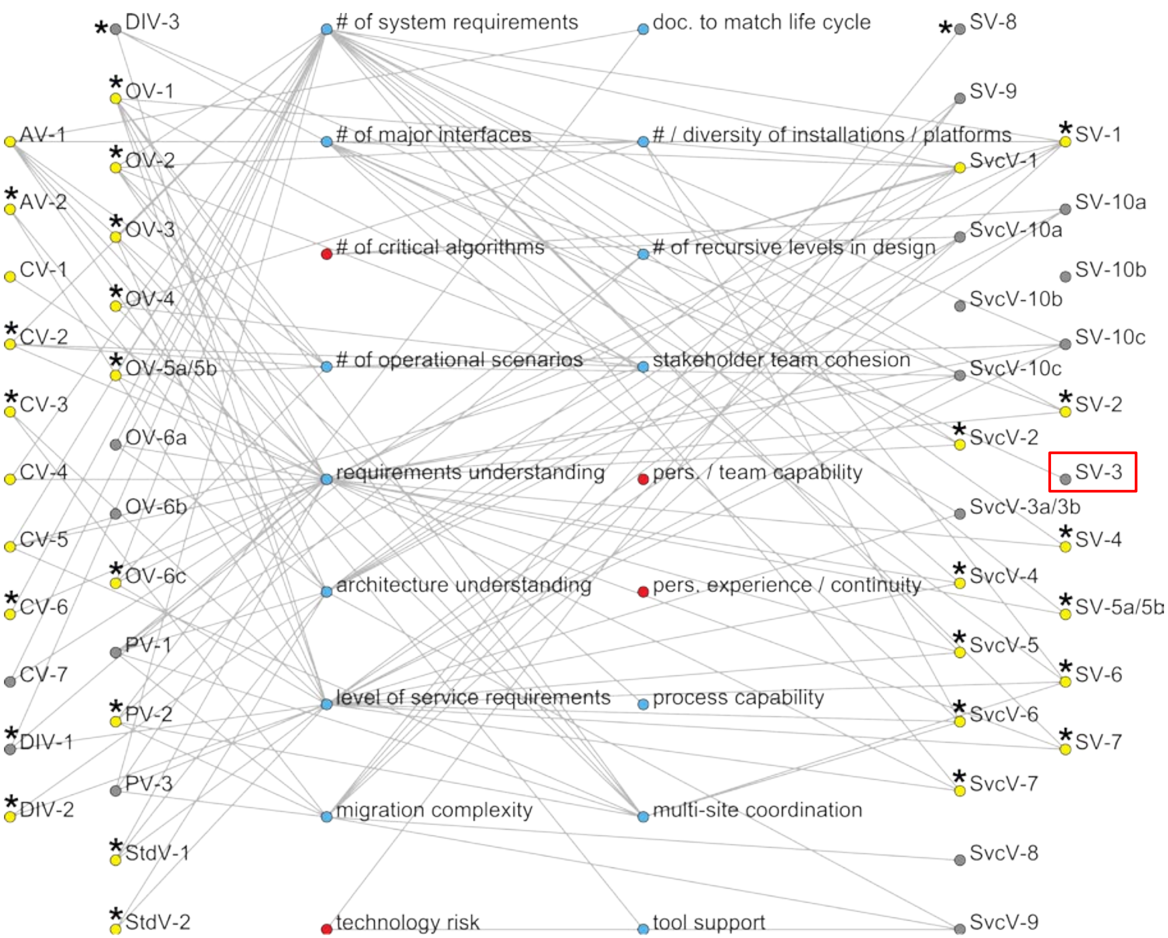
- *WSARA* (2009): Increased the rigor of Pre-MS A cost analysis (baseline shifted from MS B to MS A)
- *DoDI 5000.02* (2013): Mandated a draft CDD, with required DoDAF models, be submitted Pre-MS A

. . . and this presents an opportunity.

- DoDAF includes factors that influence system engineering (SE) effort (e.g., interfaces)
- COSYSMO estimates SE effort

DoDAF's models appear to map to COSYSMO's parameters

DoDAF models required Pre-MS A nearly span COSYSMO's drivers*



78% of COSYSMO's drivers map to DoDAF models submitted early in the life cycle

Legend

- DoDAF model X is relevant for rating COSYSMO driver Y
- Model required Pre-MS A (2012-2015)
* - Model required Pre-MS A (2015-)
- Pre-MS A model(s) maps to driver
- No Pre-MS A model maps to driver

AV - All (2 models)
CV - Capability (7 models)
DIV - Data and information (3 models)
OV - Operational (9 models)
PV - Project (3 models)
StdV - Standards (2 models)
SvcV - Services (13 models)
SV - Systems (13 models)

* Valerdi, R., Dabkowski, M., & Dixit, I. (2015). Reliability Improvement of Major Defense Acquisition Program Cost Estimates – Mapping DoDAF to COSYSMO. *Systems Engineering*, 18(5), 530-547. doi:10.1002/sys.21327

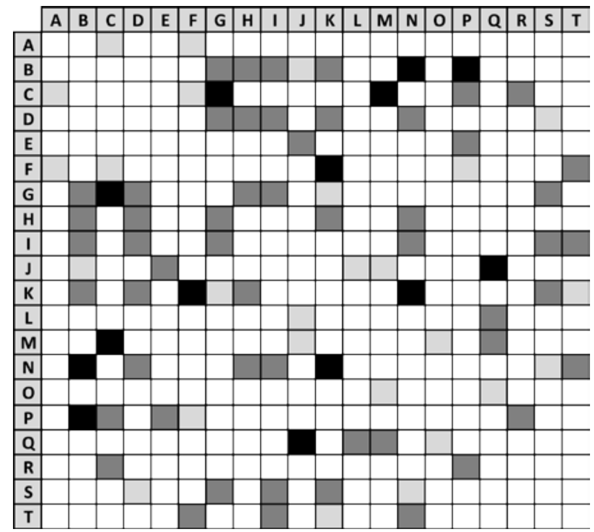
Leveraging SE and Exploiting the SV-3

- From the 2008 National Research Council report "Pre-Milestone A and Early-Phase Systems Engineering" . . .
  - The "application of SE to decisions made in the pre-Milestone A period is critical to avoiding (or at least minimizing) cost and schedule overruns" (p. 3)
  - 3 of the 6 primary drivers of cost growth addressable by SE are:
    1. Incomplete requirements at MS B,
    2. System complexity (via internal, architectural design), and
    3. External interface complexity (via network-centric operations or "systems of systems" constructs) (pp. 82-85)
- The SV-3 (or Systems-Systems Matrix) provides an abstraction of all 3, as requirements (however incomplete) drive the selection of subsystems (nodes) which are connected by interfaces (edges), both internal and external

Formally evaluating the SV-3 Pre-MS A and estimating its potential growth holds promise for minimizing cost overruns

Hypothetical SV-3

- 20 subsystems with 47 interfaces of varying complexity
- Without loss of generality, assume there are . . .
  - 200 easy, 200 nominal, and 50 difficult requirements
  - 5 difficult critical algorithms
- Using additional w_{ik} and EM_j data,* apply CER to obtain an initial estimate of PM_{NS}



SV-3

Interface Complexity

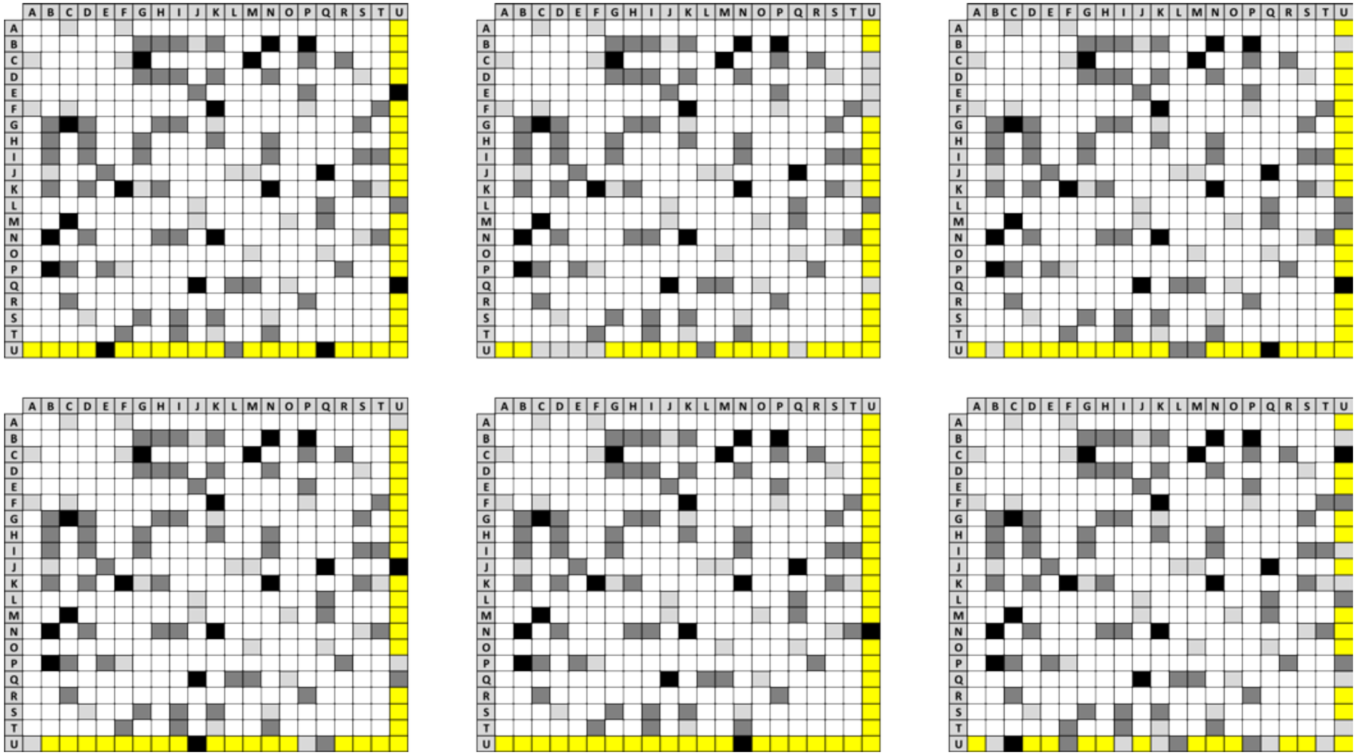
□ = Easy, ■ = Nominal, ■ = Difficult

$$PM_{NS} = 0.25 \cdot \left(\underbrace{(0.5 \times 200 + 1.0 \times 200 + 5.0 \times 50)}_{\text{requirements}} + \underbrace{(11.5 \times 5)}_{\text{algorithms}} + \underbrace{(1.1 \times 13 + 2.8 \times 27 + 6.3 \times 7)}_{\text{interfaces}} \right)^{1.06} \cdot 0.89 = 245.27$$

* Valerdi, R. (2008). *The Constructive Systems Engineering Cost Model (COSYSMO): Quantifying the Costs of Systems Engineering Effort in Complex Systems*. Saarbrücken, Germany: VDM Verlag.

What about inevitable, unforeseen change?

- This is Pre-MS A ⇒ <u>requirements will change</u>



If we add a new subsystem U to the existing architecture, how will it connect?

What will it cost?

The analytical task

Estimate the number of interfaces (by complexity level) U will generate

(Q1) How many subsystems should **U** connect to (**degree, *m***)?,

(Q2) If **U** connects to *m* subsystems, which *m* subsystems should it connect to (**adjacency**)?, and

(Q3) If **U** connects to a specific set of *m* subsystems, what should the complexity of these interfaces be (**weights**)?

Network Science – A mechanism for generating unforeseen architectural growth (microstructure)

Fundamental assumption: Current architecture foretells future architecture

Degree: Treat degree of **U** (M_U) as a random variable with PMF equal to the degree distribution of the existing system ("rich-by-birth") (Dorogovtsev & Mendes, 2003)

m	2	4	5	6	7	8
n_m	5	3	5	3	3	1
$P(M_U = m)$	0.25	0.15	0.25	0.15	0.15	0.05

Adjacency: Utilize Barabási–Albert (1999) preferential attachment (PA) model, where highly connected subsystems are more likely to interface with **U** ("rich-get-richer")

System (i)	A	B	C	D	E	F	G	H	I	J
d_i	2	7	6	6	2	5	7	5	6	5
p_i	0.021	0.074	0.064	0.064	0.021	0.053	0.074	0.053	0.064	0.053
System (i)	K	L	M	N	O	P	Q	R	S	T
d_i	8	2	4	7	2	5	4	2	5	4
p_i	0.085	0.021	0.043	0.074	0.021	0.053	0.043	0.021	0.053	0.043

$$p_i = \frac{d_i}{\sum_{j=1}^{N} d_j}$$

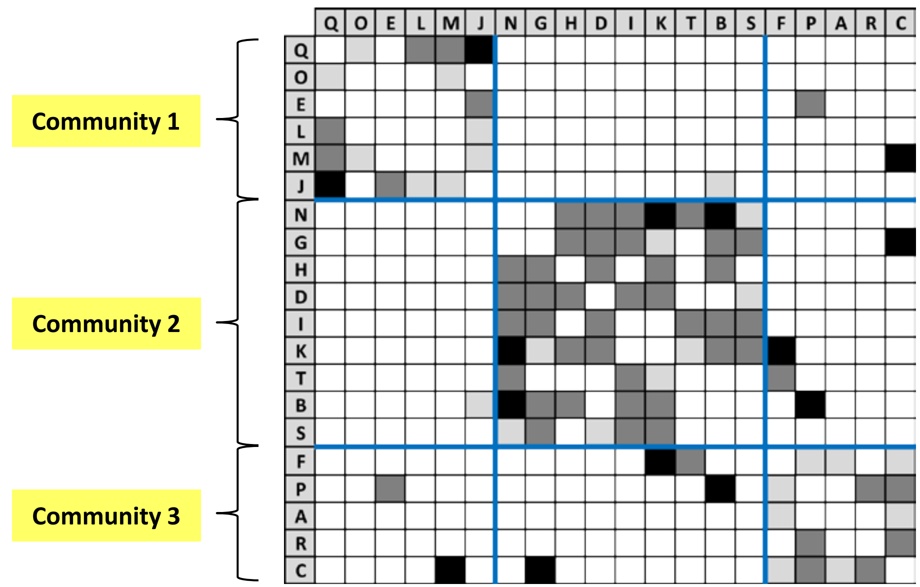
Weights: Model complexity of the interface between **U** and subsystem ***i*** (w_{iU}) as a random variable, where the pmf for w_{iU} is ***i***'s interface complexity distribution

Network Science – A mechanism for generating unforeseen architectural growth (macrostructure)

Fundamental assumption: Current architecture foretells future architecture

From *The Art of Systems Architecting*: "The most important aggregation and partitioning heuristics are to **minimize external coupling** and **maximize internal cohesion**"*

Architectural communities: Utilize Girvan-Newman (2002) to identify groups of subsystems such that the number of interfaces is sparse between and dense within groups



Intracommunity Community	Δ	Intercommunity Communities	Δ
1: {Q, O, E, L, M, J}	0.5333	1 and 2	0.0095
2: {N, G, H, D, I, K, T, B, S}	0.6944	1 and 3	0.0364
3: {F, P, A, R, C}	0.7000	2 and 3	0.0440

* Maier, M., & Rechtin, E. (2000). *The Art of Systems Architecting*. (2nd ed.). New York, NY: CRC Press.

Simulating Growth and Estimating Cost – Dabkowski et al. (2014)

For a specified number of iterations . . .

Preprocessing

1. Initialize the system as the current system
2. Use Girvan-Newman (2002) to identify architectural communities
3. Randomly assign **U** to community *k*

Intracommunity Growth

4. Generate a realization for $M_{\mathbf{U},intra}$ given **U** is assigned to community *k* (m_{intra})
5. Connect **U** to m_{intra} subsystems inside community *k* using the BA model
6. For each interface established in (5), assign complexity ($w_{i\mathbf{U},intra}$)

Intercommunity Growth

7. Generate a realization for $M_{\mathbf{U},inter}$ given **U** is assigned to community *k* (m_{inter})
8. Connect **U** to m_{inter} communities using the BA model
9. For each interface established in (8), assign complexity ($w_{i\mathbf{U},inter}$)

Cost Estimation

10. Estimate cost for augmented system using COSYSMO (PM_{NS}*)
11. Calculate additional cost of adding subsystem **U** ($PM_{NS}* - PM_{NS}$)
12. Store results and return to (3)

"Did I build the right model? ***Is it general enough?"***

Community detection may miss key macrostructure . . .

Hypothetical SV-3

Original representation

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T
A	0	1	0	0	0	0	1	0	1	0	0	0	0	0	1	0	1	0	0	0
B	1	0	0	0	0	0	1	0	1	0	0	0	0	0	1	0	1	0	0	0
C	1	1	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
D	1	1	0	0	1	0	1	1	1	1	0	0	1	1	1	0	1	1	1	1
E	1	1	0	1	0	0	1	1	1	1	0	0	1	1	1	0	1	1	1	1
F	1	1	1	1	1	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1
G	1	1	0	0	0	0	0	0	1	0	0	0	0	0	1	0	1	0	0	0
H	1	1	0	1	1	0	1	0	1	1	0	0	1	1	1	0	1	1	1	1
I	1	1	0	0	0	0	1	0	0	0	0	0	0	0	1	0	1	0	0	0
J	1	1	0	1	1	0	1	1	1	0	0	0	1	1	1	0	1	1	1	1
K	1	1	1	1	1	1	1	1	1	1	0	1	1	1	1	1	1	1	1	1
L	1	1	1	1	1	1	1	1	1	1	1	0	1	1	1	1	1	1	1	1
M	1	1	0	1	1	0	1	1	1	1	0	0	0	1	1	0	1	1	1	1
N	1	1	0	1	1	0	1	1	1	1	0	0	1	0	1	0	1	1	1	1
O	1	1	0	0	0	0	1	0	1	0	0	0	0	0	0	0	1	0	0	0
P	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	1	1	1	1
Q	1	1	0	0	0	0	1	0	1	0	0	0	0	0	1	0	0	0	0	0
R	1	1	0	1	1	0	1	1	1	1	0	0	1	1	1	0	1	0	1	1
S	1	1	0	1	1	0	1	1	1	1	0	0	1	1	1	0	1	1	0	1
T	1	1	0	1	1	0	1	1	1	1	0	0	1	1	1	0	1	1	1	0

Representation following permutation

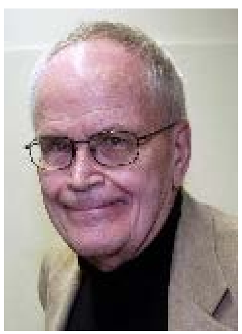
	I	Q	O	B	A	G	R	J	H	N	M	D	S	T	E	P	K	F	C	L
I	0	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Q	1	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
O	1	1	0	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
B	1	1	1	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
A	1	1	1	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
G	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
R	1	1	1	1	1	1	0	1	1	1	1	1	1	1	1	0	0	0	0	0
J	1	1	1	1	1	1	1	0	1	1	1	1	1	1	1	0	0	0	0	0
H	1	1	1	1	1	1	1	1	0	1	1	1	1	1	1	0	0	0	0	0
N	1	1	1	1	1	1	1	1	1	0	1	1	1	1	1	0	0	0	0	0
M	1	1	1	1	1	1	1	1	1	1	0	1	1	1	1	0	0	0	0	0
D	1	1	1	1	1	1	1	1	1	1	1	0	1	1	1	0	0	0	0	0
S	1	1	1	1	1	1	1	1	1	1	1	1	0	1	1	0	0	0	0	0
T	1	1	1	1	1	1	1	1	1	1	1	1	1	0	1	0	0	0	0	0
E	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0
P	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	1	1	1	1
K	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	1	1	1
F	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	1	1
C	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	1
L	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0

- $N = 20$ subsystems and $E = 251$ *directed* interfaces; relatively dense ($\Delta = 0.661$)
- Girvan-Newman (2002) identifies 6 architectural communities with a modularity of just 0.017

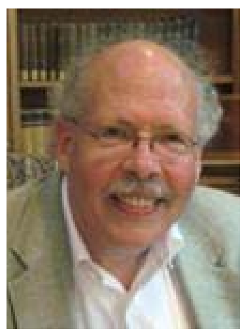
Girvan-Newman misses the indisputable, hierarchical clustering of subsystems!

. . . but blockmodeling does not.

- Blockmodeling
  - Partitions a network consisting of $i = 1, \cdots, N$ objects (i.e., the SV-3) into $k = 1, \cdots, P$ non-overlapping positions, where the positions generally abide the structure represented in a $(P \times P)$ image matrix such that $P \ll N$
  - Developed by computational sociologists at Harvard in the mid-1970's



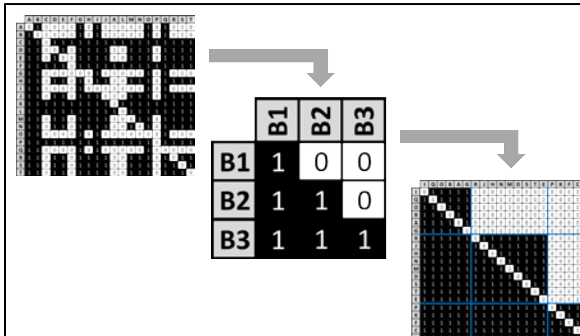
Harrison White



Scott Boorman

Ronald Breiger

Social structure from multiple networks. I. Blockmodels of roles and positions. *AJS*, 81(4), 730-780.



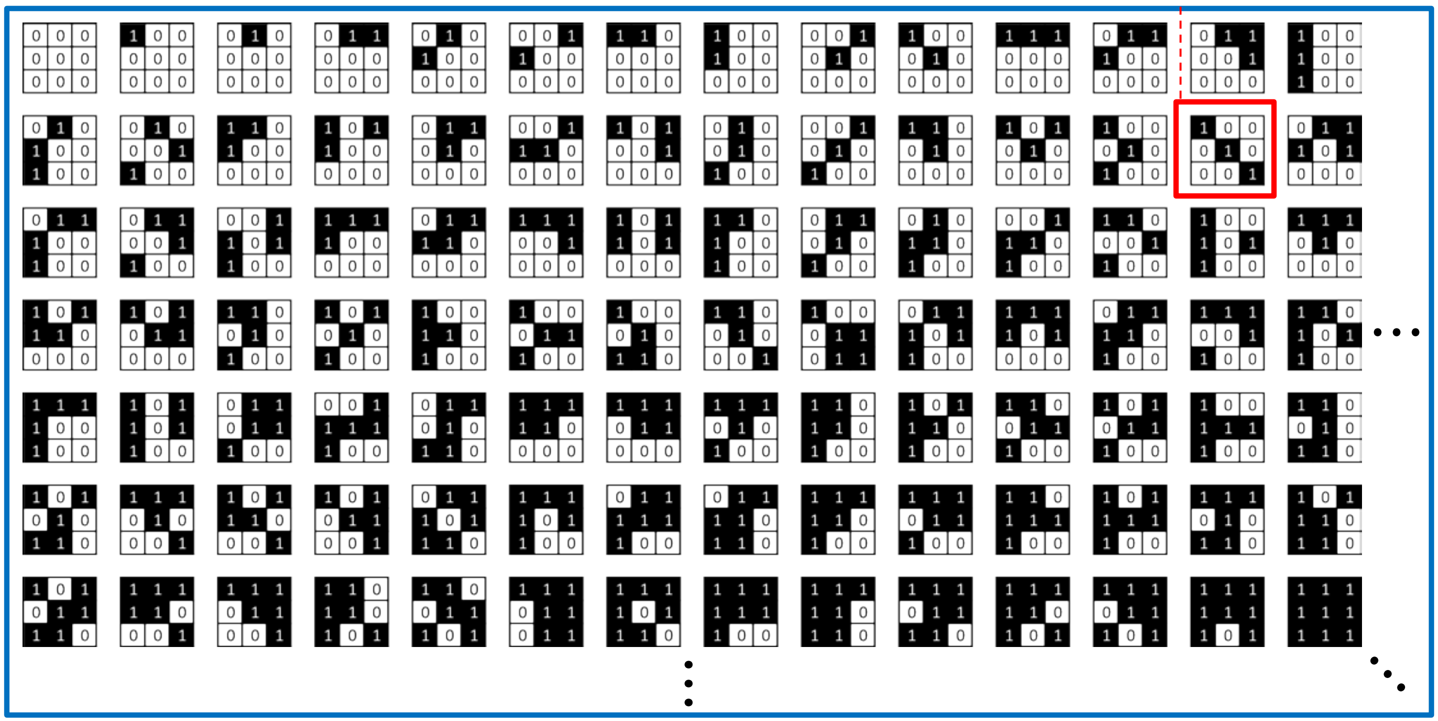
Pajek recovers the SV-3's hierarchical structure exactly!

  - Integrated into popular network analysis software (i.e., Pajek via Doreian, Batagelj, and Ferligoj's (2005) direct approach) azengineering

Blockmodeling is the natural generalization of community detection . . .

Blockmodeling can find

Community detection can find



Of the 512 possible (3×3) binary image matrices, community detection can find a partition for 1 – the identity; blockmodeling can accommodate all 512!

. . . but blockmodeling is not a panacea.

- Issue #1: Blockmodeling (BM) problems are NP-hard ⇒ time to find globally optimal solutions can *explode* as the # of subsystems/positions ↑

  Consequence #1: BM normally applies heuristics versus exact methods ⇒ better fitting image matrices and partitions may exist

- Issue #2: Exact methods largely confined to *confirmatory* fitting (image matrix is pre-specified) ⇒ exact *exploratory* fitting procedures are lacking

  Consequence #2: An SV-3's macrostructure is not "known in advance" ⇒ available exact BM methods are ill-suited for the task at hand

- Issue #3: Majority of BM heuristics and all exact methods focus on single one-/two-mode networks ⇒ BM multiple relations is an open problem

  Consequence #3: SV-3s are often mixed-mode networks ⇒ new methods are required to accommodate all SV-3s

WANTED . . . An Efficient Exact Method for Blockmodeling Mixed-Mode SV-3s

Given:

A mixed-mode SV-3

1-mode portion | **2-mode portion**

Internal Subsystems		Internal Subsystems										External Subsystems						
		I1	I2	I3	I4	I5	I6	I7	I8	I9	I10	E1	E2	E3	E4	E5	E6	E7
	I1	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
	I2	1	0	0	0	0	1	1	0	1	1	0	0	0	0	0	0	0
	I3	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
	I4	1	0	1	0	0	0	0	0	0	0	1	1	0	0	0	0	0
	I5	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0
	I6	0	1	0	0	0	0	0	1	0	0	0	1	0	0	0	0	0
	I7	0	1	0	0	0	0	0	1	0	0	0	1	1	0	0	0	0
	I8	0	0	0	0	0	1	1	0	1	1	0	1	0	0	0	0	0
	I9	0	1	0	0	0	0	0	1	0	1	0	0	0	1	0	1	1
	I10	0	1	0	0	0	0	0	1	1	0	0	0	0	0	1	1	1

Find:

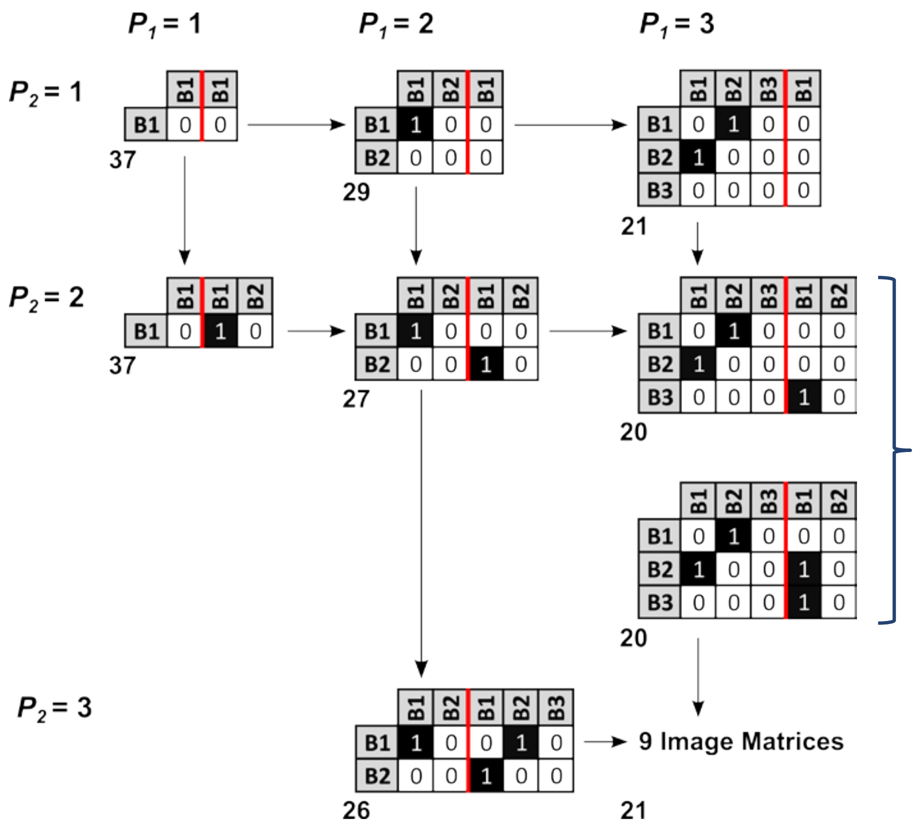
The globally optimal mixed-mode image matrix and corresponding partition with three or fewer internal and external subsystem positions

Internal Subsystems		Internal Subsystems			External Subsystems		
		B1	B2	B3	B1	B2	B3
	B1	0 or 1	0 or 1	0 or 1	0 or 1	0 or 1	0 or 1
	B2	0 or 1	0 or 1	0 or 1	0 or 1	0 or 1	0 or 1
	B3	0 or 1	0 or 1	0 or 1	0 or 1	0 or 1	0 or 1

Idea: Leverage the results in Brusco and Steinley's (2009) paper "Integer programs for one- and two-mode blockmodeling based on prespecified image matrices for structural and regular equivalence"

Globally Optimal IMs and Partition

- Formulated a series of IPs using C++; solved in IBM's ILOG CPLEX



Internal Subsystems		Internal Subsystems										External Subsystems						
		I1	I6	I7	I9	I10	I2	I8	I3	I4	I5	E2	E1	E3	E4	E5	E6	E7
	I1	0	0	0	0	0	1	0	0	1	0	0	0	0	0	0	0	0
	I6	0	0	0	0	0	1	1	0	0	0	1	0	0	0	0	0	0
	I7	0	0	0	0	0	1	1	0	0	0	1	0	1	0	0	0	0
	I9	0	0	0	0	1	1	1	0	0	0	0	0	0	1	0	1	1
	I10	0	0	0	1	0	1	1	0	0	0	0	0	0	0	1	1	1
	I2	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
	I8	0	1	1	1	1	0	0	0	0	0	1	0	0	0	0	0	0
	I3	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0
	I4	1	0	0	0	0	0	0	1	0	0	1	1	0	0	0	0	0
	I5	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0

Partition of internal subsystems driven by "outside" interfaces

Generalizing Dabkowski et al. (2014) via Blockmodeling

For a specified number of iterations . . .

Preprocessing

1. Initialize the system as the current system
2. Build an optimal set of $\{P(M = m), \beta\}$ pairs
3. Use Dabkowski-Fan-Breiger (2015; 2016) to identify an optimal P-position image matrix and partition of subsystems

Growth

4. Randomly select a member from the optimal set of $\{P(M = m), \beta\}$ pairs
5. Generate a realization for the incoming subsystem's (**X**'s) number of interfaces using $P(M = m)$; **if the IM and partition suggest a compelling, underlying architectural structure**, use *Connection Option A*; otherwise, use *Connection Option B*

Connection Option A (use blockmodel)

6a. Randomly assign **X** to position *k*,

6b. Model assignment of **X**'s m interfaces to positions as a random $(1 \times P)$ vector $\boldsymbol{C}$, where $\boldsymbol{C}$ follows a Multinomial($m, \boldsymbol{p}$) distribution and $\boldsymbol{p}$ is the $(1 \times P)$ vector of multinomial probabilities given by

$$\frac{\text{\# interfaces in block } (k, l) \text{ of the partitioned, permuted SV−3}}{\text{\# interfaces in blocks } (k, \bullet) \text{ of the partitioned, permuted SV−3}};$$

generate a feasible realization for $\boldsymbol{C}$

6c. For $l = 1, \cdots, P$, attach **X** to $\boldsymbol{c}_l$ subsystems inside position l using attachment probabilities $p_i = {d_i}^{\beta} / \sum_{j=1}^{N} {d_j}^{\beta}$

6d. For each interface established in (6c), assign complexity ($w_{i\mathbf{X}}$)

Connection Option B (do not use blockmodel)

6a. Attach **X** to m subsystems using attachment probabilities $p_i = {d_i}^{\beta} / \sum_{j=1}^{N} {d_j}^{\beta}$

6b. For each interface established in (6a), assign complexity ($w_{i\mathbf{X}}$)

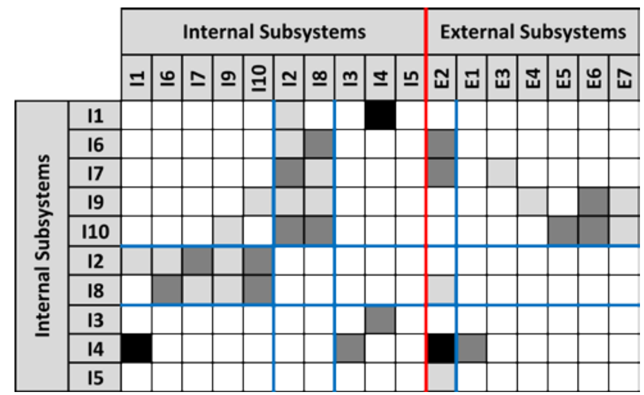
Cost Estimation (same as Dabkowski et. al (2014), goto Step 4)

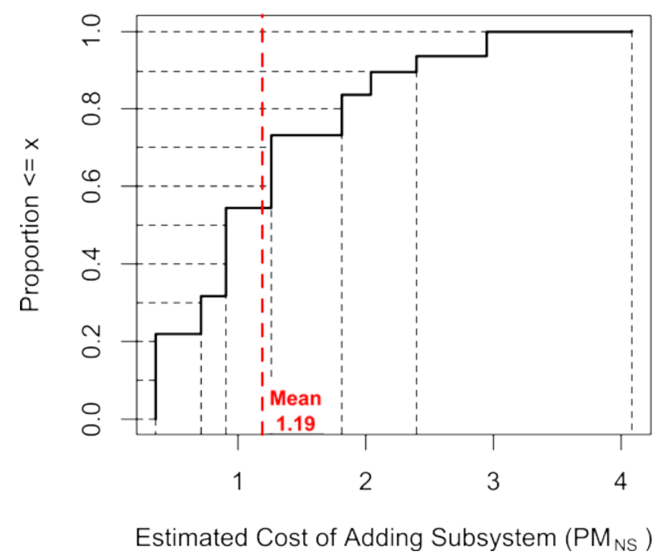
Estimating the Cost of Architectural Growth

- Assumed the following:
  - $A = 0.25; E = 1.06; \prod_{j=1}^{14} EM_j = 0.89$
  - Requirements: 75 easy, 50 nominal, 10 difficult
  - Internal interfaces: 6 easy, 5 nominal, 1 difficult
  - External interfaces: 6 easy, 6 nominal, 1 difficult

  $\Rightarrow$ 59.24 PM_{NS} of SE effort required
- Coded algorithm in R
- Ran 10,000 iterations

Expected cost to connect an additional subsystem to the SV-3's internal subsystems is (1.177, 1.206) PM_{NS}





Future Work

- Gather additional data for further validation and refinement
  - Secure sponsored research to weight SV-3s by interface complexity
  - Work with PMs to obtain multiple snapshots of SV-3s over time
- Explore additional connection options (e.g., model the probability that subsystem **X** is assigned to position *k* as a function of position *k*'s size)
- Modify algorithm to address external architectural growth
- Investigate the evolution of non-DoD architectures (e.g., open-source software architectures, non-militarized space systems, etc.)

Questions

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PRESENTATION TITLE: Blockmodeling and the Estimation of Evolutionary Architectural Growth in Major Defense Acquisition Programs

IP formulation

For all possible mixed-mode image matrices solve . . .

Data

$\boldsymbol{S}^{(1)}$: $(N_1 \times N_1)$ 1-mode portion of the SV-3

$\boldsymbol{S}^{(2)}$: $(N_1 \times N_2)$ 2-mode portion of the SV-3

$\left(\boldsymbol{B}_1^{(r)} | \boldsymbol{B}_2^{(q)}\right)$: Binary $(P_1 \times P_1 | P_1 \times P_2)$ mixed-mode image matrix (IM), where $\boldsymbol{B}_1^{(r)}$ and $\boldsymbol{B}_2^{(q)}$ are the 1- and 2-mode portions

Indices

$i, j \in \mathcal{N}_1 \quad m \in \mathcal{N}_2$

$p, l \in \mathcal{P}_1 \quad k \in \mathcal{P}_2$

Decision variables

When using image matrix $\left(\boldsymbol{B}_1^{(r)} | \boldsymbol{B}_2^{(q)}\right)$:

$x_{i,p}^{(r,q)} = 1 \Rightarrow$ internal subsystem i assigned to internal position p

$y_{m,k}^{(r,q)} = 1 \Rightarrow$ external subsystem m assigned to external position k

Inconsistencies between one-mode IM and partition of internal subsystems | Inconsistencies between two-mode IM and partitions of internal and external subsystems

$$\min \sum_{p,l=1}^{P_1} \sum_{\substack{i,j=1 \\ i \neq j}}^{N_1} w_{i,j,p,l}^{(r,q)} \left(b_{p,l}^{(r)} + s_{i,j}^{(1)} - 2 b_{p,l}^{(r)} s_{i,j}^{(1)}\right) + \sum_{i=1}^{N_1} \sum_{m=1}^{N_2} \sum_{p=1}^{P_1} \sum_{k=1}^{P_2} z_{i,m,p,k}^{(r,q)} \left(b_{p,k}^{(q)} + s_{i,m}^{(2)} - 2 b_{p,k}^{(q)} s_{i,m}^{(2)}\right)$$

s.t.

$$\sum_{p=1}^{P_1} x_{i,p}^{(r,q)} = 1, \quad \forall\, i \in \mathcal{N}_1$$

$$\sum_{k=1}^{P_2} y_{m,k}^{(r,q)} = 1, \quad \forall\, m \in \mathcal{N}_2$$

Each internal / external subsystem assigned to a single internal / external position

$$\sum_{i=1}^{N_1} x_{i,p}^{(r,q)} \geq 1, \quad \forall\, p \in \mathcal{P}_1$$

$$\sum_{m=1}^{N_2} y_{m,k}^{(r,q)} \geq 1, \quad \forall\, k \in \mathcal{P}_2$$

No "empty" internal / external positions

$$x_{i,p}^{(r,q)} \in \{0,1\}, \quad \forall\, i \in \mathcal{N}_1, p \in \mathcal{P}_1$$

$$y_{m,k}^{(r,q)} \in \{0,1\}, \quad \forall\, m \in \mathcal{N}_2, k \in \mathcal{P}_2$$

Restrict decision variables to be binary

$$w_{i,j,p,l}^{(r,q)} \leq x_{i,p}^{(r,q)}, \quad \forall\, i,j \in \mathcal{N}_1, p,l \in \mathcal{P}_1$$

$$w_{i,j,p,l}^{(r,q)} \leq x_{j,l}^{(r,q)}, \quad \forall\, i,j \in \mathcal{N}_1, p,l \in \mathcal{P}_1$$

$$w_{i,j,p,l}^{(r,q)} \geq x_{i,p}^{(r,q)} + x_{j,l}^{(r,q)} - 1, \quad \forall\, i,j \in \mathcal{N}_1, p,l \in \mathcal{P}_1$$

$$w_{i,j,p,l}^{(r,q)} \geq 0, \quad \forall\, i,j \in \mathcal{N}_1, p,l \in \mathcal{P}_1$$

Linearization constraints for $w_{i,j,p,l}^{(r,q)} = x_{i,p}^{(r,q)} x_{j,l}^{(r,q)}$

$$z_{i,m,p,k}^{(r,q)} \leq x_{i,p}^{(r,q)}, \quad \forall\, i \in \mathcal{N}_1, m \in \mathcal{N}_2, p \in \mathcal{P}_1, k \in \mathcal{P}_2$$

$$z_{i,m,p,k}^{(r,q)} \leq y_{m,k}^{(r,q)}, \quad \forall\, i \in \mathcal{N}_1, m \in \mathcal{N}_2, p \in \mathcal{P}_1, k \in \mathcal{P}_2$$

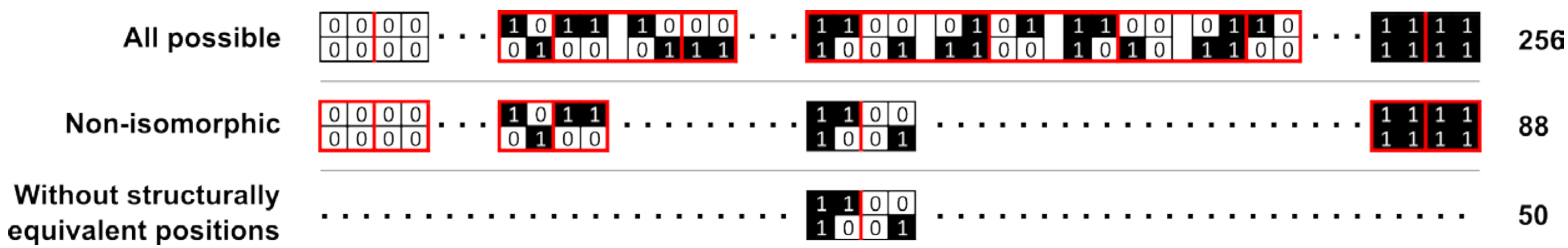
$$z_{i,m,p,k}^{(r,q)} \geq x_{i,p}^{(r,q)} + y_{m,k}^{(r,q)} - 1, \quad \forall\, i \in \mathcal{N}_1, m \in \mathcal{N}_2, p \in \mathcal{P}_1, k \in \mathcal{P}_2$$

$$z_{i,m,p,k}^{(r,q)} \geq 0, \quad \forall\, i \in \mathcal{N}_1, m \in \mathcal{N}_2, p \in \mathcal{P}_1, k \in \mathcal{P}_2.$$

Linearization constraints for $z_{i,m,p,k}^{(r,q)} = x_{i,p}^{(r,q)} y_{m,k}^{(r,q)}$

Two crucial structural observations

- Observation #1: Some image matrices are created “equal”
- Observation #2: Some positions are created “equal”



# of positions (P_1, P_2)	All possible IMs	Non-isomorphic IMs	IMs without structurally equivalent positions	Percentage of all possible IMs to fit
(1, 1)	4	4	4	100.00%
(1, 2)	8	6	2	25.00%
(2, 1)	64	36	32	50.00%
(2, 2)	256	88	50	19.53%
(2, 3)	1024	172	36	3.52%
(3, 1)	4096	752	688	16.80%
(3, 2)	32768	3272	2424	7.40%
(3, 3)	262144	10704	4912	1.87%
Total IMs to fit	300364	15034	8148	2.71%

Two order of magnitude reduction in the number of IMs to fit