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Curve Shortening Flow for Spatial Random Permutations

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Deutsche Zusammenfassung

Die nachstehende Arbeit basiert auf physikalischen Beobachtungen, wie sie z.B. in der Thermodynamik oder für Magnetismus anzutreffen sind. Man stelle sich ein System vor, in dem zu Beginn ein Gebiet („Tröpfchen“) vorliegt, das vorwiegend durch eine relevante physikalische Größe dominiert wird. Dieses Gebiet sei wiederum eingebettet in einen Raum, in dem eine gegensätzliche physikalische Größe vorliegt, die mit dem ursprünglichen Gebiet derart interagiert, dass das Tröpfchen versucht seine Oberflächenspannung zu minimieren. Dies führt dazu, dass das Tröpfchen im Laufe der Zeit immer kleiner wird und irgendwann komplett aufgelöst wird. Hierbei ist weniger die zeitliche Größenordnung von Interesse, in der das Tröpfchen verschwindet, als vielmehr die Art und Weise wie dies geschieht. Letztere kann durch eine partielle Differentialgleichung beschrieben werden.

Die mathematische Beschreibung eines solchen (makroskopischen) Umstands durch mikroskopische, zufällige Systeme im Sinne der statistischen Mechanik ist äußerst komplex und nahm erst in den 1990er Jahren durch H. Spohn’s *Interface motion in models with stochastic dynamics* richtig an Fahrt auf. Ein Grundbaustein der darin angegebenen Methoden ist die Feststellung, dass die zeitliche Entwicklung der Oberfläche des Tröpfchens (im Wesentlichen) lokal durch hydrodynamische Grenzwerte von Partikelsystemen beschrieben werden kann. Lacoin, Simenhaus und Toninelli (*Zero-temperature 2D Ising model and anisotropic curve-shortening flow*) nutzten diesen Umstand im Jahre 2011 aus, um den Schrumpfungsprozess eines 2-dimensionalen Tröpfchen im Nulltemperatur-Ising Modell, welches den Ferromagnetismus von Festkörpern beschreibt, durch eine zugehörige partielle Differentialgleichung zu beschreiben.

Jene Arbeit bot Anlass und Hoffnung ein ähnliches Resultat auch für eine andere mikroskopische, zufällige Beschreibung herleiten zu können. Im Fokus dieser Dissertation stehen *zufällige räumliche Permutationen*, die erst seit wenigen Jahren mehr Aufmerksamkeit erhalten, obwohl ihnen bereits seit Mitte des letzten Jahrhunderts große Bedeutung (z.B. bezüglich Bose-Einstein-Kondensation) attestiert wurde.

Im Rahmen des vorliegenden Werks wird die Oberflächendynamik, welche sich mittels Glauber-Dynamik aus dem Permutations-Modell ergibt, in Partikelsysteme übersetzt und anschließend (für Teile der Oberfläche) im hydrodynamischen Grenzwert untersucht. Wesentliche Arbeitsschritte sind hierbei die Herleitung des stationären Maßes, welches keine Produkt-Form aufweist, sowie der Umgang mit der Tatsache, dass das Partikelsystem nicht vom Gradienten-Typ ist. Als Hauptresultat ergibt sich die hydrodynamische Gleichung des Partikelsystems.

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1. Introduction

The starting point of this Ph.D. project is set in an area of probability theory that developed mainly in very recent years. It deals with random permutations (on countable sets) which are chosen according to probability weights depending on the underlying spatial structure. Typically, permutations are weighed in such a way that long jumps are discouraged, in particular in the annealed version of the model. The Glauber dynamics with respect to such a probability measure produces a time-continuous Markov chain on the set of permutations.

For spatial random permutations, many problems have already been addressed, mainly regarding the lengths of permutation cycles. Another branch of investigation is the hydrodynamic (or thermodynamic) limit of the system, i.e. when both space and time are rescaled.

The motivation of such a scaling comes from statistical mechanics, where one aims to derive the macroscopical behaviour of a system from (random) microscopical dynamics. Typically, the equilibrium state of the system is described by observables (such as temperature, pressure or density), denoted by a parameter function f in time and space. In many models, it is reasonable to assume that locally, in every neighbourhood V_u of a point u, the system tries to obtain equilibrium according to the values of the observables $f(t, u)$ at time $t \in \mathbb{R}_+$ in u. The challenge is to give a description of how the parameter function $f(t, u)$ evolves, which is done by a system-specific partial differential equation, called hydrodynamic equation. The derivation of these PDEs constitutes a main part of this thesis.

In our setting, a convex domain $\mathcal{D} \subset [-1, 1]^2$ with smooth enough boundary $\gamma_0 = \partial\mathcal{D}$ is given, as well as the local (microscopical) dynamics, which are governed by the above mentioned Markov chain (with respect to an annealed Gibbs measure) for permutations on a 2-dimensional lattice. The situation can be thought of from a thermodynamic point of view as immersing a droplet of one phase into another phase, such that the interface between the two shrinks over time and the droplet eventually disappears. Mathematically, if we think of the underlying lattice as embedded in $[-1, 1]^2$ and if the lattice spacing tends to zero, γ_0 can be approximated by a long cycle of nearest-neighbour permutations. Even though the droplet $\mathcal{D}$ could locally increase its size, the dynamics will be such that the curve eventually shortens and shrinks to a singleton in the hydrodynamic limit. Macroscopically, such a behaviour can be observed for many physical models and is known as Lifshitz law. However, the precise nature of the curve shortening flow is of much bigger interest.

To put it differently, the question one would like to answer is

> "How exactly does the droplet's boundary evolve in the hydrodynamic limit?"

For the zero-temperature stochastic Ising model, recent papers [18, 19] have shown that it shrinks according to some mean-curvature motion. This basically means that the normal speed at a boundary point is given by the local mean curvature times a factor which depends on the particular lattice. The proof in [18] relies on two connections to interacting particle systems, for which the hydrodynamic equations are known and can be used to describe the interface locally [35]. Having to deal with two particle systems (instead of just one) pays credit to the lattice structure, such that we have a natural distinction between the local dynamics at the poles and away from the poles.

For planar random permutations, the matter is more complex. This is mainly owed to the fact that diagonal permutation jumps on the lattice are allowed. As a consequence, the arising interacting particle system that models the droplet's boundary evolution away from the poles, is of non-gradient type and has no stationary measure of product form. Still, with the help of a convenient toy-model, we will manage to derive the associated hydrodynamic equation.

1.1. Outline of this Thesis

This thesis is structured in the following way.

Chapter 2 gives an introduction to several topics regarding Markov/Feller processes, and in particular interacting particle systems. Concepts such as stationary measures, the hydrodynamic limit or local equilibrium measures are discussed in the context of two standard particle systems, namely the simple symmetric exclusion process in Section 2.2 on one hand and the zero-range process in Section 2.3 on the other hand. Apart from the convenience of having concrete examples to illustrate the above concepts for readers which are new to the field, the two interacting particle systems also play a fundamental role in the proof of mean-curvature droplet shrinking for the zero-temperature 2-dimensional Ising model. Furthermore, the exclusion process is used in a heuristic approach for the range-r exclusion process later on.

The Ising model is introduced in Chapter 3, which both gives a recapitulation of the curve shortening flow shown in [18, 19] and lays the groundwork for a similar result for spatial random permutations.

Chapter 4 constitutes the main body of this thesis. In Section 4.1 we connect the planar random permutations to a model of interacting particle systems. Due to its difficult character, we include the derivation of the hydrodynamic equation for a similar particle system, the range-r exclusion process, in Section 4.2. After defining the proper stochastic model, we state the hydrodynamic equation in Theorem 4.2.1. To prove the latter, we derive the stationary measures first (Subsection 4.2.3) and then move on to the proof in Subsection 4.2.4. Here, a well known heuristic approach for hydrodynamic equations (the so-called local equilibrium ansatz) is included in Remark 4.2.4. However, in Subsection 4.2.5, we show another (unusual) approach, that relies on the similarity to the simple symmetric exclusion process. We wrap up this particle system with some conclusive thoughts and connections (Subsection 4.2.6).

Afterwards, we are better prepared to deal with the AFP-model, i.e. with the particle system that models the spatial random permutation dynamics away from the poles. We proceed by defining the model in Subsection 4.3.1, deriving non-product stationary measures in Subsection 4.3.2 and stating the hydrodynamic equation in Subsection 4.3.3. Contrary to the previous section, due to its complexity with respect to the non-gradient property of the particle system, the martingale approach obtains his own subsection in 4.3.4. In the end, we prove the hydrodynamic behaviour in Subsection 4.3.5.

Finally, in the last Chapter 5, we give some outlook and address the current status regarding the dynamics at the poles.

2. Hydrodynamic Limits for Interacting Particle Systems

2.1. General Definitions

This chapter is devoted to a special class of continuous-time Markov Processes and their behaviour under appropriate scaling limits in space and time. The following content lays the groundwork for the techniques applied in Chapter 4. At first, we will revisit standard results on Feller processes and interacting particle systems in particular. In Sections 2.2 and 2.3 we introduce two specific examples that are not only used prominently by Lacoin, Simenhaus and Toninelli [19], but that also have characteristic similarities to our own model.

In 1970 Frank Spitzer published an influential article [34], in which he considered the dynamics of (finitely or infinitely many) indistinguishable particles on a countable set S. Up to that point, this particular setup had been mainly studied for the case that each particle moves independently of the others and according to some given transition function $P_t : S^2 \to [0,1],\ t \in \mathbb{R}_+$. We will come back to this in Section 2.3 when dealing with the zero range processes. The models analysed by F. Spitzer on the other hand have additional particle interactions superimposed, which complicate the individual movements on S. Many of them were motivated by open questions in Statistical Mechanics; the Ising Model [11], which is featured in Section 3.1, is an important example.

Interacting Particle Systems as Feller Processes

The introduction to the topic in this section follows closely the standard works by Thomas M. Liggett. While his book from 1985 „Interacting Particle Systems“ [23] is considered to be a standard reference, we will follow mainly his recent textbook [25], which has a wider range for its basic setup. We will restrict ourselves directly to a state space suited for particle systems, even though the Feller process theory originated from a more general point of view.

Consider the space $X = E^S$ for countable spaces S and E. We will call it *configuration space* and write $\eta \in X$ for a *particle configuration*, where we interpret S as the underlying lattice and $\eta(x) \in E$ as the number of particles at site $x \in S$. We will mainly encounter settings in which both S and E are finite sets. In this thesis, the lattice will be modelled either by the integers $\mathbb{Z}$ or by the finite ring $\mathbb{Z}/n\mathbb{Z}$. In [23], as well as some of the sections later on, only a finite number of particles is allowed per site. In this case, one can equip E with the discrete topology, i.e. simply the power set, and obtains a compact metric space (the choice of the metric is not important for compactness). By Tychonoff's theorem, the product topology[1] then makes X a compact metric space as well.

In this section, we consider both the case that X is a compact and a locally compact separable metric space. We write

$$C(X) = \begin{cases} \{f : X \to \mathbb{R} \mid f \text{ is continuous}\}, & \text{for } X \text{ compact,} \\ \{f : X \to \mathbb{R} \mid f \text{ is continuous and} & \text{for } X \text{ locally compact.} \\ \forall \varepsilon > 0 \ \exists K \subset X \text{ compact with } |f(\eta)| < \varepsilon \ \forall \eta \in K^C\}, & \end{cases}$$

The second property for functions in $C(X)$ when X is locally compact is referred to as „vanishing at ∞“. The uniform norm on $C(X)$ is given by

$$||f|| = \sup_{\eta \in X} |f(\eta)|, \tag{2.1}$$

which makes $C(X)$ a Banach space, i.e. a complete normed vector space.

Studying the time evolution of configurations, the canonical choice is the space $D([0,\infty), X)$ of càdlàg functions, i.e. functions $\eta : \mathbb{R}_+ \to X$ that are right continuous and have left limits everywhere. As we also have a space variable, we will put the time component in the lower index throughout, e.g. writing $\eta_s(x) \in X$ for the number of particles at time $s \in \mathbb{R}_+$ in $x \in S$. The projection map $p_s : D([0,\infty), X) \to X, (\eta_t)_{t\in\mathbb{R}_+} \mapsto \eta_s$ evaluates the configuration path at time $s \in \mathbb{R}_+$. The σ-algebra $\mathcal{F}_t$ will be the smallest σ-algebra, such that the projections p_s are measurable for all $s \leq t$. This gives a natural filtration $(\mathcal{F})_{t\in\mathbb{R}_+}$ on $D([0,\infty), X)$. Furthermore we set $\mathcal{F} = \bigcup_{t\in\mathbb{R}_+} \mathcal{F}_t$ and define the translation map θ by means of $p_u(\theta_s((\eta_t)_{t\in\mathbb{R}_+})) = \eta_{s+u}$ for all $s, u \in \mathbb{R}_+$.

Definition 2.1.1. *A family $\{\mathbb{P}^{\eta^0},\ \eta^0 \in X\}$ of probability measures on $D([0,\infty), X)$ is called* Feller process *if the following conditions hold:*

1.

$$\mathbb{P}^{\eta^0}((\eta_t)_{t\in\mathbb{R}_+} \in D([0,\infty), X) \ : \ \eta_0 = \eta^0) = 1 \quad \forall \eta^0 \in X, \tag{2.2}$$

2.

$$\eta^0 \mapsto \mathbb{E}^{\eta^0} f(\eta_t) \in C(X) \ \forall f \in C(X), \ t \in \mathbb{R}_+, \tag{2.3}$$

3.

$$\mathbb{E}^{\eta^0}(Y \circ \theta_s \mid \mathcal{F}_s) = \mathbb{E}^{\eta(s)} Y \tag{2.4}$$

$\mathbb{P}^{\eta^0}$-a.s. for all $\eta^0 \in X$ and all bounded measurable $Y : D([0,\infty), X) \to \mathbb{R}$.

[1] The *product topology* is the smallest topology, such that all projection maps $S \to E,\ x \mapsto \eta(x)$ are continuous

We used the shorthand notation $\mathbb{E}^\eta$ for the expectation with respect to the measure $\mathbb{P}^\eta$. The second condition is the *Feller property*, which expresses that the stochastic process at a later time $t > 0$ depends continuously on the starting configuration $\eta^0 \in X$. The last condition is called *Markov property.* It asserts that the evolution of a path $(\eta_t)_{t\in\mathbb{R}_+}$ from time $s \in \mathbb{R}_+$ onwards does only depend on the current configuration $\eta_s \in X$, but not on the past prior to s. A nice introduction to Markov processes both in discrete and continuous time is [28], although we will give some basic definitions and results in Appendix A.1, as well.

We can also interpret the expectation in (2.3) as a linear operator on $C(X)$. Related to that, we give another definition.

Definition 2.1.2. *A family of continuous linear operators $\{T(t),\ t \in \mathbb{R}_+\}$ on $C(X)$ is called* probability semigroup *if the following properties hold:*

1. $T(0)f = f$ *for all* $f \in C(X)$,
2. $\lim_{t\downarrow 0} T(t)f = f$ *for all* $f \in C(X)$,
3. $T(s+t)f = T(s)(T(t)f)$ *for all* $f \in C(X)$,
4. $T(t)f \geq 0$ *for all nonnegative* $f \in C(X)$,
5. $\begin{cases} T(t)\mathbb{1} = \mathbb{1}\ \forall t \in \mathbb{R}_+ & \textit{for } X \textit{ compact,} \\ \exists f_n \in C(X) : \sup_{n\in\mathbb{N}} ||f_n|| < \infty,\ T(t)f_n \to \mathbb{1} \textit{ ptw. } \forall t \in \mathbb{R}_+ & \textit{for } X \textit{ locally compact.} \end{cases}$

Here $\mathbb{1}$ is the constant function with value 1 on X. The third property is called *semigroup property.* Readers that are familiar with continuous-time Markov theory will recognise the connection to the Chapman-Kolmogorov equations in case the configuration space would be countable; see Equation (2.3) in [25].

As indicated above, given a Feller process, we can directly define a probability semigroup.

Theorem 2.1.1 (Liggett). *Let $\{P^{\eta^0},\ \eta^0 \in X\}$ be a Feller process on X. Define*

$$T(t)f(\eta^0) := \mathbb{E}^{\eta^0} f(\eta_t).$$

Then $\{T(t),\ t \in \mathbb{R}_+\}$ is a probability semigroup on $C(X)$.

Proof. See [25], Theorem 3.15. □

One major advantage in considering Feller processes (in comparison to continuous-time Markov processes) is the one-to-one correspondence between the definition via probability measures and the definition via probability semigroups (transition functions respectively). That is to say one can show the other direction to Theorem 2.1.1 as well; given a probability semigroup $\{T(t),\ t \in \mathbb{R}_+\}$, there exists a Feller process $(\eta_t)_{t\in\mathbb{R}_+}$ satisfying $\mathbb{E}^{\eta^0} f(\eta_t) = T(t)f(\eta^0)$ for all $\eta^0 \in X$, $t \in \mathbb{R}_+$ and $f \in C(X)$. We refer to Appendix A.2 for details.

When we introduce particular interacting particle systems in the following sections, we will not do so by explicitly giving the family of measures $\{\mathbb{P}^{\eta^0}, \eta^0 \in X\}$. Instead, we will use a special operator that encodes enough information to determine the Feller process uniquely in our setting.

Definition 2.1.3. *A linear operator L on $C(X)$ satisfying the conditions*

1. *the domain $\mathcal{D}(L)$ of L is dense in $C(X)$,*
2. *if $\lambda \geq 0$ and $f - \lambda L f = g$ for $f \in \mathcal{D}(L)$, then*
$$\inf_{\eta \in X} f(\eta) \geq \inf_{\eta \in X} g(\eta),$$
3. *$\{(I - \lambda L) f \ : \ f \in \mathcal{D}(I - \lambda L)\} = C(X)$, where $I : C(X) \to C(X)$ is the identity map,*
4. $$\begin{cases} \mathbb{1} \in \mathcal{D}(L) \ \wedge \ L\mathbb{1} \equiv 0 & \textit{for } X \textit{ compact,} \\ \textit{for small } \lambda > 0 \ \exists f_n \in \mathcal{D}(L) \textit{ s.t. } \sup_{n \in \mathbb{N}} \|f_n - \lambda L f_n\| < \infty & \textit{for } X \textit{ locally compact.} \\ \textit{and } f_n \to \mathbb{1}, f_n - \lambda L f_n \to \mathbb{1} \textit{ ptw.,} & \end{cases}$$

is called (probability) generator.

Note that the definition does not force L to be a bounded operator. This makes it a technical task to obtain the probability semigroup from a probability generator. However, given a probability generator L, one can define the operator

$$L_\varepsilon := L(I - \varepsilon L)^{-1}, \tag{2.5}$$

which approximates L, and then put

$$T_\varepsilon(t) := e^{tL_\varepsilon} = \sum_{n=0}^{\infty} \frac{t^n L_\varepsilon^n}{n!}. \tag{2.6}$$

This is well defined as can be seen once more in Appendix A.2.

Theorem 2.1.2 (Liggett). *For $f \in C(X)$,*

$$T(t) f = \lim_{\varepsilon \downarrow 0} T_\varepsilon(t) f$$

exists uniformly on bounded time intervals. It defines a probability semigroup in the sense of Definition 2.1.1, whose probability generator is L, i.e.

$$L f = \lim_{t \downarrow 0} \frac{T(t) f - f}{t} \tag{2.7}$$

on

$$\mathcal{D}(L) = \{f \in C(X) \ : \ \textit{the (strong) limit in (2.7) exists}\}.$$

Proof. See Theorem 3.24. in [25] and Theorem A.2.1. □

From the semigroup, we directly get the finite-dimensional distributions of the process via

$$\mathbb{E}^{\eta^0} \prod_{i=1}^{n} f_i(\eta_{t_i}) = T_{t_1} f_1 T_{t_2 - t_1} f_2 \dots T_{t_n - t_{n-1}} f_n(\eta^0)$$

and indeed one can show (see Theorem A.2.2) that there is a Feller process $(\eta_t)_{t \in \mathbb{R}_+}$ satisfying

$$\mathbb{E}^{\eta^0} f(\eta_t) = T(t) f(\eta^0),$$

for $\eta^0 \in X,\ t \in \mathbb{R}_+$ and $f \in C(X)$.

Generators of Interacting Particle Systems

For the upcoming processes, configurations η of the space $X = E^S$ (introduced above) change in time by means of particle jumps from one site to another. We will not have to deal with *spin systems* for which transitions involve only one site at a time, e.g. by flipping the value $\eta(x)$ at site $x \in S$ from 0 to 1 or vice versa, without changing the remaining values $\eta(y)$ for $y \neq x$. Instead, whenever a particle jump occurs, the amount of particles $\eta(x)$ decreases by one unit, and the amount at another site $y \in S$ increases by one. The new configuration obtained is denoted by $\eta^{x,y}$, always given that such a configuration is permitted according to the state space. The stochastic process is implicitly described by the *transition rates* (or *jump rates*) $q(x, y, \eta)$ at which particle jumps from $x \in S$ to $y \in S$ occur for the configuration $\eta \in X$. It is assumed to be nonnegative, uniformly bounded and continuous in $\eta \in X$ (w.r.t. to the topology given above). In our cases, the jump rate q can be written in terms of a product

$$q(x, y, \eta) = c(x) \cdot p(x, y) \cdot \mathrm{excl}(x, y, \eta), \tag{2.8}$$

where $c(x)$ is called the *leaving-rate* of a particle at site x, $p(x, y)$ is called the *transition probability* (or *elementary jump probability*) from x to y and $\mathrm{excl}(x, y, \eta)$ is an *exclusion rule* that rejects jumps whenever the new configuration would not belong to the state space. For finite state space X Liggett [23] gives the intuitive interpretation

$$\mathbb{P}^{\eta^0}(\eta_t = \eta^{x,y}) = q(x, y, \eta^0) \cdot t + o(t).$$

We define $C_0(X)$ as the set of *cylinder functions*, i.e. $f \in C_0(X)$ if there exists a finite subset $S_f \subset S$ with

$$f(\eta) = f(\xi) \quad \forall \eta, \xi \in X \text{ with } \eta(x) = \xi(x)\ \forall x \in S_f.$$

Thus, cylinder functions depend only on a finite number of lattice sites. The probability generators L in later sections will be defined by

$$Lf(\eta) = \sum_{x,y \in S} q(x, y, \eta) \left(f(\eta^{x,y}) - f(\eta) \right) \tag{2.9}$$

for $f \in C_0(X)$ or subsets thereof. The restriction to $C_0(X)$ is important for the convergence of the series in (2.9), in case of an infinite lattice. Another potential problem with this definition is the behaviour of the transition function, which might lead to an infinite norm $||Lf||$ (see (2.1)) in the Banach space $C(X)$. The literature gives strong results for sufficient conditions on q to guarantee that the linear operator in (2.9) indeed fulfils the criteria of a probability generator according to Definition 2.1.3. For a compact space X we refer to Theorem I.3.9. in [23]. Fortunately, the particle systems considered here are of *finite range*, i.e. there exists $R > 0$ such that

- for all $x \in S$ there exists $S_x \subset S$ with $|S_x| \leq R$ such that $q(x, y, \eta) = q(x, y, \eta_{S_x})$ for all $\eta, \eta_{S_x} \in X$ with $\eta(z) = \eta_{S_x}(z)$ $\forall z \in S_x$,
- for all $y \in S$ there exists $S_y \subset S$ with $|S_y| \leq R$ such that $q(x, y, \eta) = q(x, y, \eta_{S_y})$ for all $\eta, \eta_{S_y} \in X$ with $\eta(z) = \eta_{S_y}(z)$ $\forall z \in S_y$,
- for all $x \in S$ there exists $S_x \subset S$ with $|S_x| \leq R$ such that $|\{y \in S \ : \ q(x, y, \eta) > 0\}| \leq R$ for all $\eta \in X$.

For most of the appearing interacting particle systems, this quality is more than enough to ensure that (2.9) gives rise to a probability generator that is uniquely defined by the values on $C_0(X)$.

The generator of a Feller process $(\eta_t)_{t\in\mathbb{R}_+}$ enables us to define an important martingale

$$M_t := f(\eta_t) - \int_0^t Lf(\eta_s)\,\mathrm{d}s,$$

which will be fundamental for the approach ("martingale approach") used to derive the hydrodynamic limits in Chapter 4, see [25, Theorem 3.32] or Lemma A.1.1 for a more general version, that we will refer to frequently throughout this thesis.

Stationary Measures

Stationary (or invariant) measures are one of the main tools to describe the behaviour of interacting particle systems in the considered scaling limits. Prior to the definition, we introduce the distribution of the random variable $\eta_t \in X$, given that the starting distribution is μ. It is denoted by $\mu T(t)$ and is uniquely determined by

$$\mathbb{E}^{\mu} f(\eta_t) = \int \mathbb{E}^{\eta^0} f(\eta_t)\mu(\mathrm{d}\eta^0), \quad f \in C(X),$$

due to Riesz's Representation Theorem.

Definition 2.1.4. *Given a Feller process on X with probability semigroup $T(t)$, a probability measure μ is called* stationary *if $\mu T(t) = \mu$ $\forall t \in \mathbb{R}_+$, i.e.*

$$\int T(t)f\,\mathrm{d}\mu = \int f\,\mathrm{d}\mu \quad \forall f \in C(X),\ \forall t \in \mathbb{R}_+. \tag{2.10}$$

Roughly speaking, when a process has a stationary distribution, it is in some kind of equilibrium. More precisely, its distribution will remain the same at all later times. A stationary measure must not be unique, see e.g. subsection 2.2.2. Therefore, we introduce the notation $\mathcal{I}$ for the set of stationary measures of a Feller process. When X is compact, one can show that $\mathcal{I} \neq \emptyset$. When we consider finite state spaces later on, the following will be a helpful result regarding uniqueness.

Proposition 2.1.1 (**Liggett**). *The stationary measure for an irreducible recurrent Markov chain is unique up to constant multiples.*

Proof. See Proposition 2.61. in [25]. □

When we proof stationarity of measures, we will make use of the following theorem. Recall that a linear subspace $D \subset \mathcal{D}(L)$ is a *core* for the generator L if L is the closure of its restriction to D, i.e. if $L = (L\big|_D)$.

Theorem 2.1.3 (**Liggett**). *Suppose D is a core for the generator L. Then a probability measure μ on X is stationary for the corresponding process if and only if*

$$\int Lf \,\mathrm{d}\mu = 0 \quad \textit{for all } f \in D.$$

Proof. See Theorem 3.37 in [25]. □

2.2. Simple Exclusion Processes

The well known and extensively studied exclusion process was introduced to the mathematical community in the aforementioned influential work by F. Spitzer [34] as one example for interacting particle systems. It models the motion of indistinguishable particles on a lattice, such that individual transitions affect only two sites. At every point in time, there is at most one particle at each site and a particle jumps to another site according to some transition probability, respecting some exclusion rule (in particular that the targeted site is vacant). The following introduction is orientated on Section 2.2 in [14], where the transition probability is modelled to be translation-invariant, of finite range and irreducible. A simple and convincing way to think of exclusion processes is to imagine that every single particle is a random walker on the lattice, with independent exponentially distributed waiting times before each (attempted) jump. Generally the random walkers are independent of one another, except for the fact that they cannot occupy the same lattice site at the same time. If this is about to happen, the causing jump is oppressed and the particle that tried to jump remains in his place instead.

In this standard setting, there is no sudden death or birth of particles and the particle number (or particle density) is conserved in time. It is the evolution of this local quantity that helps to understand the upcoming droplet shrinking for the Ising model. The connection is obtained via the hydrodynamic limit of the nearest-neighbour *simple symmetric exclusion* *process* (SSEP), which is an exclusion process with nearest-neighbour jumps only, a basic exclusion rule and a symmetric transition probability, i.e. given a 1-dimensional (horizontally aligned) lattice, the probability of jumping to the left is the same as jumping to the right.

In the following subsection we will formalise this stochastic process. Then, the family of stationary measures for exclusion processes is presented, followed by a subsection about particular starting measures, which are close to the stationary ones. At last we present the hydrodynamic equation in Subsection 2.2.4.

Standard references to the subject include [14, 23, 24, 25].

2.2.1. Construction

For simplicity, we restrict ourselves to the case of a one-dimensional torus $\mathbb{T}_N := \mathbb{Z}/N\mathbb{Z}$. A configuration of a simple exclusion process then takes values in $X := \{0,1\}^{\mathbb{T}_N}$. In spirit of the (lonely) random walker on $\mathbb{Z}$, we neglect the continuous time-component for a moment and think of a single particle that jumps freely from site to site according to some Markov chain with transition function p, which is just the elementary jump probability mentioned earlier. This measure is assumed to be

- *translation-invariant* (or *homogeneous* in Markov chain terminology), i.e. it is the same function for all sites a particle tries to leave from,
- of *finite range* (cf. definition for jump rates on page 10 combined with (2.8)) and
- *irreducible*, in the sense that the Markov chain reaches every site of the lattice (eventually) no matter what the starting point is.

By the first property $p : \mathbb{Z} \to [0,1]$ only depends on one variable, which is the length (and direction) of the jump. Since p is a probability distribution, there should obviously hold $\sum_{z\in\mathbb{Z}} p(z) = 1$. If there further holds $p(z) = p(-z)$ for all $z \in \mathbb{Z}$, then we call p *symmetric*. We did not prohibit jumps of length bigger than the torus size N, so the elementary jump probability of going from $x \in \mathbb{T}_N$ to $x + z \in \mathbb{T}_N$ is given by $p^N(z) := \sum_{y\in\mathbb{Z}} p(z + yN)$ for all $x \in \mathbb{T}_N$. We are now able to define the generator of a simple exclusion process by means of

$$(L_N f)(\eta) := \sum_{x\in\mathbb{T}_N} \sum_{z\in\mathbb{T}_N} \eta(x)(1 - \eta(x+z))p^N(z)\big(f(\eta^{x,x+z}) - f(\eta)\big), \tag{2.11}$$

for $f \in C(X)$ where

$$\eta^{x,y}(z) = \begin{cases} \eta(z) & \text{if } z \neq x, y, \\ \eta(x) - 1 & \text{if } z = x, \\ \eta(y) + 1 & \text{if } z = y \end{cases}$$

is the new configuration obtained after letting a particle of the configuration η jump from $x \in \mathbb{T}_N$ to $y \in \mathbb{T}_N$, given that there is a particle present at x and given y is vacant. It is well known [23] that the linear operator L_N is indeed a probability generator in the sense of Definition 2.1.3 and gives rise to a Feller process $(\eta_t)_{t\in\mathbb{R}_+}$ on X with probability semigroup $S_N(t) = \exp(tL_N)$.

In view of (2.11), we can directly read out the exclusion rule in form of $\eta(x)\cdot(1-\eta(x+z))$ and the leaving rate $c(x) = 1$. This means that a particle waits for an exponentially distributed random time with parameter 1 to jump according to the jump probability p, but always respecting the exclusion rule. Regarding the latter, we will see later on that more complicated systems are possible, for which the jump rate $q(x, y, \eta)$ does not only depend on the configuration's values $\eta(x)$ and $\eta(y)$, but also on other values. The term *simple* within SSEP illustrates the very basic form of an exclusion given above. Furthermore, we speak of a *nearest-neighbour* SSEP when a particle can jump only to his directly neighbouring sites, such that in dimension 1 there holds $p(x, x-1) = p(x, x+1) = \frac{1}{2}$ for all $x \in \mathbb{T}_N$.

2.2.2. Stationary Measures

Simple exclusion processes possess not only one, but a whole family of stationary measures. The reason for this variety lies in the presence of a conserved quantity for the system, i.e. a macroscopical quantity that does not change over time. As T. Liggett described in [23]: "The existence of a conserved quantity tends to break up the state space [...] into classes determined by the value of this quantity, and then there tends to be an invariant measure for each of its possible values". In the present case, this quantity is given by the amount of particles (or the particle density), which leads to a family of stationary measures of particularly convenient and simple form.

Definition 2.2.1. *For $0 \le \alpha \le 1$, the* Bernoulli product measure *of parameter α, denoted by ν_α^N, is a measure on X with the properties of*

- *being translation-invariant, i.e. $\tau_x\nu_\alpha^N = \nu_\alpha^N$ for all $x \in \mathbb{T}_N$, where $\{\tau_x\}_{x\in\mathbb{T}_N}$ is the translation group,*
- *being* product*, i.e. the random variables $\{\eta(x)\}_{x\in\mathbb{T}_N}$ are independent, and*
- *having marginals*

$$\nu_\alpha^N(\eta(x) = 1) = \alpha = 1 - \nu_\alpha^N(\eta(x) = 0), \quad \forall x \in \mathbb{T}_N.$$

It is no hard task to show that Bernoulli product measures are indeed stationary for simple exclusion processes [14, Proposition 2.2.2]. The parameter α is directly connected to the conserved quantity, the average particle density, as can be seen by

$$\mathbb{E}_{\nu_\alpha^N}(\eta(x)) = \alpha\cdot 1 + (1-\alpha)\cdot 0 = \alpha,$$

which holds at all sites $x \in \mathbb{T}_N$.

As the total number of particles $\sum_{x\in\mathbb{T}_N}\eta(x)=K$ remains constant, it seems natural to consider measures that are concentrated on these subsets (indexed by K). Regarding the Bernoulli product measure ν_α^N, the conditional measure

$$\nu_{N,K}(\cdot) := \nu_\alpha^N\left(\cdot \;\middle|\; \sum_{x\in\mathbb{T}_N}\eta(x)=K\right)$$

is still stationary on the subset

$$\Omega_{N,K} := \{\eta\in X \;:\; \sum_{x\in\mathbb{T}_N}\eta(x)=K\}$$

and does not depend on the parameter α. In fact, it is simply the uniform distribution on $\Omega_{N,K}$, which is something that we will use in a similar fashion for two different interacting particle systems in Chapter 4. As mentioned in [14, Remark 2.2.4], for the simple exclusion process it is possible to calculate directly that for all finite subsets $Z\subset\mathbb{Z}$, for all sequences $\{\varepsilon_x \;:\; x\in Z\}$ with values in $\{0,1\}$ and for all $0\le\alpha\le 1$,

$$\lim_{N\to\infty}\nu_\alpha\left(\eta(x)=\varepsilon_x,\ x\in Z \;\middle|\; \sum_{y\in\mathbb{T}_N}\eta(y)=\lfloor\alpha_0 N\rfloor\right)=\nu_{\alpha_0}\left(\eta(x)=\varepsilon_x,\ x\in Z\right), \tag{2.12}$$

where $\lfloor\cdot\rfloor$ is the floor-function. We take this opportunity to introduce some more terminology. Conditional stationary measures like $\nu_{N,K}$ concentrated on a fixed number of particles are called *canonical measures*, whereas weak limits of canonical measures (as $N\to\infty$) in the sense of Equation (2.12) are called *grand-canonical measures*. Whenever the finite dimensional marginals of canonical measures converge to the same marginals of the grand-canonical measures as $N\to\infty$ and $\frac{K}{N}\to\alpha_0$ uniformly on compact sets for α_0, we speak of an *equivalence of ensembles*[2].

This equivalence offers a choice of what stationary measure to work with in order to derive the hydrodynamic behaviour for the SSEP (and also the zero-range process in the next section). Given the simple nature of the grand-canonical measures (here the Bernoulli product measure ν_α^N), it is convenient to choose them. However, in Chapter 4, we will go the other way.

2.2.3. Equilibrium Formulations

We briefly take the opportunity and introduce some more forms of equilibrium.

Definition 2.2.2. *A sequence of probability measures $(\mu^N)_{N\ge1}$ on $\{0,1\}^{\mathbb{T}_N}$ is a* local equilibrium of profile *$\rho_0:\mathbb{T}\to\mathbb{R}_+$ if*

$$\tau_{\lfloor uN\rfloor}\mu^N \xrightarrow[N\to\infty]{w} \nu_{\rho_0(u)}$$

for all continuity points u of $\rho_0(\cdot)$.

[2] The term "ensemble" is often used in statistical mechanics as a substitute for "probability distribution".

It is well known that the definition of such weak limits is not an issue, despite formally being defined on different spaces for each N. We can divide up into periods on $\{0,1\}^{\mathbb{Z}}$ both the configurations in Ω_N and the measures thereon, such that for all $N \in \mathbb{N}$ everything is defined on the same space, see for example Remark 1.6 in Chapter 2.2 of [14].

If the local equilibrium property still remains true at later times $t \in \mathbb{R}_+$, we use the next definition.

Definition 2.2.3. *The local equilibrium $(\mu^N)_{N\geq 1}$ with respect to the profile ρ_0 is* conserved *by the time renormalisation θ_N if there exists a function $\rho : \mathbb{R}_+ \times \mathbb{T} \to \mathbb{R}_+$ such that*

$$S^N(t\theta_N)\tau_{\lfloor uN \rfloor}\mu^N \to^w \nu_{\rho(t,u)}, \tag{2.13}$$

for all $t \geq 0$ and all continuity points u of $\rho(t,\cdot)$.

Having local equilibrium starting measures is quite a restrictive initial condition, as we demand weak convergence in every continuity point of ρ_0. Certainly for continuous functions $G : \mathbb{T} \to \mathbb{R}$ a local equilibrium sequence implies the following convergence in (2.14), where we merely expect the spatial means to converge. The empirical measure at time $t = 0$ is denoted by π_0^N (cf. bottom of this page).

Definition 2.2.4. *A sequence $(\mu^N)_{N\geq 1}$ of probability measures on $\{0,1\}^{\mathbb{T}_N}$ is* associated to a profile *$\rho_0 : \mathbb{T} \to \mathbb{R}_+$ if for every continuous function $G : \mathbb{T} \to \mathbb{R}$, and for every $\delta > 0$ we have*

$$\lim_{N\to\infty} \mu^N\Big(\Big|\int_{\mathbb{T}} G(u)\pi_0^N(\mathrm{d}u) - \int_{\mathbb{T}} G(u)\rho_0(u)\,\mathrm{d}u\Big| > \delta\Big) = 0. \tag{2.14}$$

2.2.4. Hydrodynamic Equation

The hydrodynamic equations play a crucial role in this thesis. As explained earlier, they allow to describe the evolution of a macroscopic observable (like the particle density in this case) by means of an appropriate scaling limit in space and time. Particularly the hydrodynamic equation for the SSEP, which is simply the heat equation, was used [19] in order to derive the curve shortening flow for the zero-temperature Ising model. We will now state the hydrodynamic limit result for the SSEP.

Theorem 2.2.1 (Kipnis/Landim, Theorem 4.2.1)**.** *Let $\rho_0 : \mathbb{T} \to [0,1]$ be an initial density profile and let μ^N be the sequence of Bernoulli product measures of slowly varying parameter associated to the profile ρ_0, i.e.*

$$\mu^N\{\eta \ : \ \eta(x) = 1\} = \rho_0\left(\frac{x}{N}\right), \quad x \in \mathbb{T}_N.$$

Then, for every $t > 0$, the sequence of random measures

$$\pi_t^N(\mathrm{d}u) = \frac{1}{N}\sum_{x\in\mathbb{T}_N} \eta_t(x)\delta_{x/N}(\mathrm{d}u)$$

converges in probability to the absolutely continuous measure $\pi_t(\mathrm{d}u) = \rho(t,u)\,\mathrm{d}u$ *whose density is the solution of the heat equation:*

$$\begin{cases} \partial_t \rho & = \partial_x^2 \rho, \\ \rho(0,\cdot) & = \rho_0(\cdot). \end{cases} \tag{2.15}$$

The proof, as given for example in [20], is both standard by now and also highly non-trivial. In order to derive the hydrodynamic equations for two particle systems in Chapter 4, we will adapt the proof to our needs.

2.3. Zero Range Processes

2.3.1. Construction

The zero-range process is an interacting particle system without restrictions on the maximal number of particles per lattice site, i.e. the state space is given by $\mathbb{N}^{\mathbb{T}_N}$. Many different scenarios are possible, but we will only consider the standard case, in which the process can be described by two parameters, namely the translation-invariant transition probability p^N and a rate function $g : \mathbb{N} \to \mathbb{R}_+$, vanishing at 0, which describes the rate at which a particle leaves a site in dependence of the number of particles present there. The generator of the zero-range process is then given by

$$L_N f(\eta) = \sum_{x \in \mathbb{T}_N} \sum_{z \in \mathbb{T}_N} p^N(z) g(\eta(x)) (f(\eta^{x,x+z}) - f(\eta)). \tag{2.16}$$

Here, $\eta^{x,y}$ is the configuration obtained from η by letting a particle jump from x to y. Clearly some technical requirements are necessary for the function g in order to guarantee that it is indeed possible to define a process by means of (2.16). Since we only want to give a light introduction here, we refer to the literature for details.

2.3.2. Hydrodynamic Equation

As the last part of the introductory second chapter we would like to state another hydrodynamic equation. Once more we refer to [14], this time to Chapter 5 therein. Let the family of stationary measures $\{\nu_\alpha\}_\alpha$ be parameterised by the expected number of particles per site and define the expected value of the jump rate by

$$\Phi(\alpha) := \mathbb{E}_{\nu_\alpha}(g(\eta(0))).$$

Theorem 2.3.1 (Kipnis/Landim, Theorem 5.1.1)**.** *Assume the jump rate* $g(\cdot)$ *to increase at least linearly: there exists a positive constant* a_0 *such that* $g(k) \geq a_0 k$ *for all* $k \geq 0$*. Let* $\rho_0 : \mathbb{T} \to \mathbb{R}_+$ *be an integrable function with respect to the Lebesgue measure. Let* μ^N *be a sequence of probability measures on* $\mathbb{N}^{\mathbb{T}_N}$ *associated to the profile* ρ_0 *and for which there* *exist positive constants K_0, K_1 and $\alpha^\star$ such that the relative entropy of μ^N with respect to $\nu_{\alpha^\star}$ is bounded by $K_0 N$:*

$$H(\mu^N \mid \nu_{\alpha^\star}) \leq K_0 N$$

and

$$\limsup_{N\to\infty} \mathbb{E}_{\mu^N}\left(\frac{1}{N}\sum_{x\in\mathbb{T}_N}\eta(x)^2\right) \leq K_1.$$

Then, for every $t \leq T$, for every continuous function $G : \mathbb{T} \to \mathbb{R}$ and for every $\delta > 0$,

$$\lim_{N\to\infty} \mathbb{P}_{\mu^N}\left(\left|\frac{1}{N}\sum_{x\in\mathbb{T}_N} G\left(\frac{x}{N}\right)\eta_t(x) - \int_{\mathbb{T}} G(u)\rho(t,u)\,\mathrm{d}u\right| > \delta\right) = 0,$$

where $\rho(t,u)$ is the unique weak solution of the non-linear heat equation

$$\begin{cases} \partial_t \rho & = \frac{1}{2}\partial_x^2\left(\Phi(\rho)\right), \\ \rho(0,\cdot) & = \rho_0(\cdot). \end{cases}$$

In comparison to the SSEP, the proof is even more complex here. This has to do with the fact that a certain equation cannot be closed directly in terms of the empirical measure, which is a crucial part of the proof. We will deal with this problem extensively for two other processes in Chapter 4.

3. Curve Shortening Flows

We have mentioned before how recent results for the curve-shortening flow in the Ising model in [18, 19] have motivated this thesis. In this chapter, we sum up those results from 2011 and 2013 in Section 3.1 in order to prepare for our own treatment of the droplet shrinking with spatial random permutations, which is initiated in Section 3.2.

3.1. 2-dim. Zero-Temperature Ising Model

We recapitulate in this section the fundamental results obtained and methods applied in Lacoin's, Simenhaus' and Toninelli's article about the curve shortening flow for a zero-temperature 2-dimensional Ising model [18]. In fact, the same authors produced a similar article [19] two years later, where the results have been slightly stronger. The biggest change therein was the treatment of a more general initial droplet in $\mathbb{R}^2$ and not necessarily a convex one. For simplicity, we will stick to the first article.

3.1.1. Dynamics

Set $\mathbb{Z}^\star := \mathbb{Z} + \frac{1}{2}$ and consider the zero-temperature stochastic Ising model on $(\mathbb{Z}^\star)^2$, i.e. the continuous-time Markov chain $(\sigma(t))_{t\geq 0}$ on the spin-configuration space $\Omega := \{-1,1\}^{(\mathbb{Z}^\star)^2}$, where we put $\sigma(t) := (\sigma_x(t))_{x\in(\mathbb{Z}^\star)^2}$.

Transitions of the chain occur in the following way. For each site $x \in (\mathbb{Z}^\star)^2$, σ_x is updated independently of the others. When σ_x is updated, it adapts the spin value of the majority of its neighbours. In case that there is no majority, σ_x chooses its spin with probability $1/2$ each.

Consider now a compact, simply connected subset $\mathcal{D} \subset [-1,1]^2$ whose boundary is a closed smooth curve. For $L \in \mathbb{N}$ we start with a deterministic configuration

$$\sigma_x(0) = \begin{cases} -1 & \text{if } x \in (\mathbb{Z}^\star)^2 \cap L\mathcal{D}, \\ +1 & \text{otherwise.} \end{cases}$$

Put further

$$\mathcal{C}_x := x + \left[-\frac{1}{2}, \frac{1}{2}\right]^2$$

and

$$\mathcal{A}_L(t) := \bigcup_{\{x:\ \sigma_x(t)=-1\}} \mathcal{C}_x,$$

which is the droplet's evolution under the given Ising dynamics and the given starting configuration. In the hydrodynamic limit, the boundary of this random subset was conjectured to evolve deterministically according to some partial differential equation. More precisely, with $\gamma(t, L)$ describing the boundary of the rescaled set

$$\frac{1}{L}\mathcal{A}_L(L^2 t),$$

it was conjectured in [35] that, in the limit $L \to \infty$, $\gamma(t, L)$ should converge to a deterministic curve $\gamma(t)$ with evolution $(\gamma(t))_{t\geq 0}$ such that the normal speed at a point $x \in \gamma(t)$ is given by the curvature at x multiplied with

$$a(\theta_x) := \frac{1}{2(|\cos(\theta_x)| + |\sin(\theta_x)|)^2},$$

where $a(\theta_x)$ is a factor which enters the equation due to the geometry of the lattice and the parameter θ_x is the slope of the outwards directed normal to $\gamma(t)$ at x. As a consequence, $\gamma(t)$ should shrink to a singleton in finite time (Lifshitz law).

3.1.2. Surface Evolution

Consider a strictly convex, smooth curve $\gamma = \partial\mathcal{D} \subset \mathbb{R}^2$, which can be expressed by

$$\mathcal{D} = \bigcap_{0\leq\theta\leq 2\pi} \{x \in \mathbb{R}^2 \ : \ x \cdot v(\theta) \leq h(\theta)\},$$

where $\cdot$ is the scalar product, $\nu(\theta)$ is the unit vector with angle $\theta \in [0, 2\pi]$ towards the horizontal axis and

$$h(\theta) = \sup\{x \cdot v(\theta),\ x \in \gamma\}$$

is the support function which uniquely describes the curve γ. Owing to this parameterisation, it is possible to write the curve shortening evolution by means of the PDE

$$\begin{cases} \partial_t h(\theta, t) & = -a(\theta)k(\theta, t), \\ h(\theta, 0) & = h(\theta), \end{cases} \tag{3.1}$$

where the time-derivative is taken for constant θ and, for a convex curve γ, $k(\theta) \geq 0$ is the curvature at the point $x(\theta) \in \gamma$, which is characterised by having an outward directed normal which forms an anticlockwise angle θ between itself and the horizontal axis.

Theorem 2.1 in [18] constitutes the main theorem in their paper. It states that under some reasonable assumptions on the boundary $\gamma = \partial\mathcal{D}$ of the initial droplet $\mathcal{D}$, the support function $h(\theta, t)$ associated to the flow of (convex) curves $(\gamma(t))_{t\geq 0}$ with $\gamma(0) = \gamma$ solves the PDE (3.1).

3.1.3. Local Interface Dynamics

The proof of Theorem 2.1 in [18] is both elegant and technical, as there are many specific problems that occur along the way. Not the least of them is the fact that the interface which has to be controlled over time is not quite the graph of a function. Consequently, the idea of the proof is to consider the dynamics in two different areas, where individually the boundary evolution can be described in the hydrodynamic limit as a graph which (locally) behaves like the solution of (3.1). Afterwards, both results are "glued together" such that the whole droplet and its interface motion is considered after all. The areas mentioned above are determined to be the points *away from the poles* on the one hand and *at the poles* on the other hand. Here, in context of convex droplets, a pole means a point where the tangent to the deterministic curve $\gamma(t)$ is either horizontal or vertical.

Local Dynamics away from the Poles

The part of the interface away from the poles will also be the one that we will study in Section 4.3 for another stochastic dynamics.

For the Ising model and positive natural numbers M, N, define the state space of nearest-neighbour directed paths of length $L := M + N$ with M steps up and N steps down by

$$\Omega_{M,N} = \{(h_x)_{x\in\{0,\dots,M+N\}} \in \mathbb{Z}^{M+N+1} \ : \ |h_{x+1} - h_x| = 1, \ h_0 = 0; \ h_{M+N} = M - N\}.$$

Given $h \in \Omega_{M,N}$ and $x \in \{1, \dots, L-1\}$, denote by $h^{(x)}$ the path with a corner "flipped" at x defined by $h_y^{(x)} = h_y$ for all $y \neq x$ and

$$h_x^{(x)} := \begin{cases} h_x - 2 & \text{if } h_{x\pm1} = h_x - 1, \\ h_x + 2 & \text{if } h_{x\pm1} = h_x + 1, \\ h_x & \text{if } |h_{x+1} - h_{x-1}| = 2. \end{cases}$$

The dynamics is defined by the generator

$$Lf(h) := \frac{1}{2} \sum_{x=1}^{L-1} (f(h^{(x)}) - f(h)),$$

for functions $f : \Omega_{M,N} \to \mathbb{R}$.

This dynamics is in one-to-one correspondence to the SSEP on a finite interval, which can be seen by placing a particle at $x = 0, \dots, M + N - 1$ whenever $h_{x+1} - h_x = 1$. The SSEP and its hydrodynamic evolution are widely studied (cf. Section 2.2) and thus it is reasonable to assume that in the limit $M, N \to \infty$, a properly rescaled version of h should satisfy the heat equation. In [18, Theorem 3.2] this statement is formalised.

Local Dynamics around the Poles

For the auxiliary stochastic dynamics around the poles, define the state space of functions

$$\Omega_L := \{h : \{-L, \dots, L+1\} \to \mathbb{Z}\}.$$

Additionally, for $x \in \{-L+1, \dots, L\}$ and $h \in \Omega_L$, set $h^{+,x}$ (resp. $h^{-,x}$) to be the configuration such that $h_y^{+,x} = h_y$ if $y \neq x$ and $h_x^{+,x} = h_x + 1$ (resp. $h_x^{-,x} = h_x - 1$). The generator

$$Lf(h) := \frac{1}{2} \sum_{x=-L+1}^{L} c^{+,x}(h)(f(h^{+,x}) - f(h)) + c^{-,x}(h)(f(h^{-,x}) - f(h))$$

with initial condition $h^0 \in \Omega_L$ and

$$\begin{cases} c^{+,x}(h) := \mathbb{1}_{\{h_{x+1} > h_x\}} + \mathbb{1}_{\{h_{x-1} > h_x\}}, \\ c^{-,x}(h) := \mathbb{1}_{\{h_{x+1} < h_x\}} + \mathbb{1}_{\{h_{x-1} < h_x\}}, \end{cases}$$

defines a continuous-time Markov chain $(h(t))_{t\geq 0}$ on Ω_L.

By interpreting the gradients $\eta_x := h_{x+1} - h_x$ as particles on a lattice site x, one recognises immediately the connection to a zero-range process. However, it does not coincide with the standard ZRP, as the gradient might be less than 0. Hence, in this case, one obtains a special class of zero-range processes with two different types of particles A and B that annihilate each other immediately when they try to occupy the same site. It was shown in Theorem 3.4 of [18] that the properly scaled height function fulfils a PDE which, in some sense, is very close to the heat equation. The concrete nature of this similarity is quantified in [18, Corollary 3.5], where the rescaled height function which corresponds to the curve shortening flow (around the poles) is with high probability enclosed between two classical solutions to the heat equation for marginal time and space differences.

In the end this means that both auxiliary interface dynamics (at the poles and around them) basically shrink just as a solution to the heat equation would do. This corresponds to the evolution of the family of support functions $(h(\theta, t))_{t\geq 0}$ (for the associated family of curves $(\gamma(t))_{t\geq 0}$) according to Equation (3.1).

3.2. Spatial Random Permutations

We have seen in the previous section how a microscopical stochastic system (in that case a simple Ising model) could be used to derive the Lifshitz law and motion by mean curvature for a macroscopical droplet's boundary. The latter two properties are considered to be natural phenomenons when a droplet of one phase is immersed into another phase under conditions such that the droplet eventually disappears [22]. Certainly the Ising model is not the only microscopic stochastic model that leads to the desired result, however it is probably the easiest. That being said, starting from the influential work of Spohn [35], it

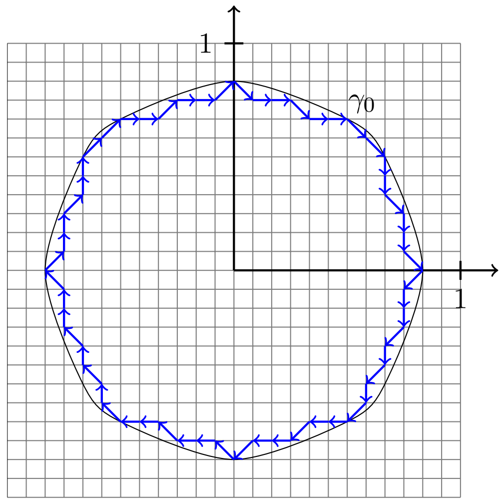

Figure 3.1.: Approximation of a droplet's boundary γ_0 with a cycle of nearest-neighbour permutation jumps

still took 18 years for a rigorous proof (in a simplified setting with respect to the initial droplet) [18]. The starting point of this Ph.D. project was to derive the mean curvature motion for a local stochastic dynamics which is given by a simple version of spatial random permutations, which we will introduce in the following.

The research field dealing with spatial random permutations flourished only in very recent years, even though its importance in physics, in particular for Bose-Einstein condensation, had already been emphasised by Feynman [9] and Penrose/Onsager [30]. A rich overview to the topic can be found in [4], and prior to that we refer to [36]. If available, we also recommend the lecture notes by V. Betz [5].

We consider the finite volume model. Let Λ be a bounded open domain in $\mathbb{R}^d$ and let V denote its volume with respect to the Lebesgue measure. The state space is given by

$$\Omega_{\Lambda,M} := \Lambda^M \times \mathcal{S}_M$$

where $\mathcal{S}_M$ is the symmetric group of permutations on $\{1, \dots, M\}$. We interpret an element $(x_1, \dots, x_M) \times \pi \in \Omega_{\Lambda,M}$ as a *spatial random permutation* by mapping $x_j \mapsto x_{\pi(j)}$ for all $j \in \{1, \dots, M\}$. The focus lies on probability measures on $\mathcal{S}_M$ that discourage long permutation jumps. This can be achieved for example by Gaussian weights of the type

$$\exp\Big\{ -\alpha \sum_{i=1}^{M} |x_i - x_{\pi(i)}|^2 \Big\},$$

which get smaller when the *energy*

$$H((x_1,\dots,x_M),\pi) := \sum_{i=1}^{M} |x_i - x_{\pi(i)}|^2 \tag{3.2}$$

increases. Such a model is appealing from a researcher's point of view since there is intrinsic competition between the energy, which is clearly minimal for the identical permutation $x_i \mapsto x_{\pi(i)} = x_i$, and the *entropy*, which favors many and long jumps.
In the hydrodynamic limit, i.e. for $|\Lambda|, M \to \infty$ with fixed density $\rho := M/|\Lambda|$, there are many (partially still unsolved) questions regarding permutation cycles, e.g. if there exist infinite cycles, if there are macroscopic cycles (cf. [5]) or questions regarding geometry and evolution (under some dynamics), which will be the focus of attention in the next subsection.

3.2.1. Phase Boundaries for Large Cycles in the Hydrodynamic Limit

In the rest of this thesis, we will concentrate on planar random permutations, i.e. spatial random permutations in 2 dimensions. For $N \in \mathbb{R}$, let

$$\Lambda_N := \mathbb{Z}^2 \cap [-N,N]^2$$

with $M := |\Lambda_N|$ and let $(x_i)_{i=1,\dots,M}$ denote the points in Λ_N. This way $\rho = 1$. For a permutation $\pi \in \mathcal{S}_M$ we assign a probability measure by

$$\mathbb{P}(\pi) := \frac{1}{Z_N(\alpha)} e^{-\alpha \sum\limits_{i=1}^{M} |x_i - x_{\pi(i)}|^2},$$

where

$$Z_N(\alpha) := \sum_{\pi \in \mathcal{S}_M} e^{-\alpha \sum\limits_{i=1}^{M} |x_i - x_{\pi(i)}|^2}$$

is the normalisation constant depending on the *inverse temperature* value $\alpha > 0$. We observe that for a permutation $\pi_0 = \mathrm{id}$ we have

$$\mathbb{P}(\pi_0) = \frac{1}{Z_N(\alpha)},$$

which constitutes the highest probability possible (and equivalently the lowest energy possible). The expected jump length only depends on the parameter α. At infinite temperature ($\alpha = 0$), the energy H in (3.2) has no significance and all permutations are equally likely, i.e. $\mathbb{P}$ is the uniform distribution on $\mathcal{S}_M$. As $\alpha > 0$ increases, it becomes less likely to have a permutation with long jumps, i.e. with sinking temperature long jumps are

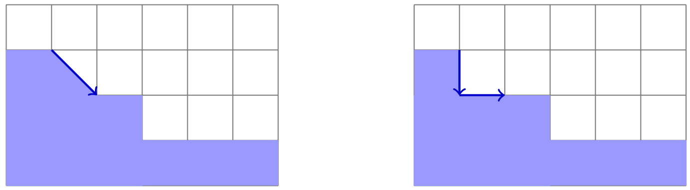

Figure 3.2.: Energy comparison between a diagonal permutation jump on the left and only horizontal/vertical jumps on the right. The blue filled area represents $\mathcal{A}_N(t)$ for some $N \in \mathbb{N}, t \in \mathbb{R}_+$.

penalised and $\mathbb{P}$ concentrates more and more on $\pi = \mathrm{id}$. Similarly to the Ising-model, we will consider an annealed version of spatial random permutations. In this case, given

$$\mathbb{P}(\pi) := \mathbb{1}_{\mathrm{id}}(\pi), \tag{3.3}$$

we can create the *Glauber dynamics* for $\mathbb{P}$ (cf. [21, Chapter 3.3]), i.e. a continuous-time Markov chain with stationary distribution defined by (3.3).

As discussed before, our goal is to describe the surface dynamics for a 2-dimensional droplet, whose boundary is governed by local stochastic dynamics according to the Glauber dynamics for spatial random permutations. The droplet shall be of convex form and of reasonable regularity in the sense that its boundary γ_0 should be possible to approximate by a single cycle of nearest-neighbour permutation jumps in the hydrodynamic limit. To be more concrete, we specify the following three points.

- We can think of the droplet as a simply connected, smooth enough, convex domain $\mathcal{D} \subset [-1,1]^2$ with boundary $\gamma_0 := \partial\mathcal{D}$ and Λ_N as being embedded in $[-1,1]^2$, such that the lattice spacing equals $1/N$ (tending to 0 as $N \to \infty$)[1]. For $N \in \mathbb{N}$, we will write $\mathcal{A}_N(0)$ for the domain in $[-1,1]^2$ whose boundary is determined by an approximation of γ_0 with a suitable cyclic permutation[2]. Under the Glauber dynamics, the cyclic approximation at later stages is random and evolves in time, which consequently will also be the case for the enclosed domain, whose evolution will be denoted by $(\mathcal{A}_N(t))_{t\in\mathbb{R}_+}$. In order to observe a (macroscopical) evolution[3] of $\mathcal{A}_N(t)$ for $N \to \infty$, we have to accelerate the Glauber dynamics by a factor of N^2, which is a characteristic feature of hydrodynamic scaling. The boundary of the random set $\mathcal{A}_N(N^2t)$ will be written as $\gamma_t(N)$.

- In the embedded lattice, a permutation $\pi \in \mathcal{S}_M$ is said to be of *nearest-neighbour*

[1] We will slightly abuse the notation and still refer to the embedded lattice as Λ_N whenever it seems appropriate.

[2] except for the jump lengths (see next bullet point), the details of the approximation are not important here, one might think for example of the biggest convex cycle contained in $\mathcal{D}$

[3] which in our case means shrinking to a point

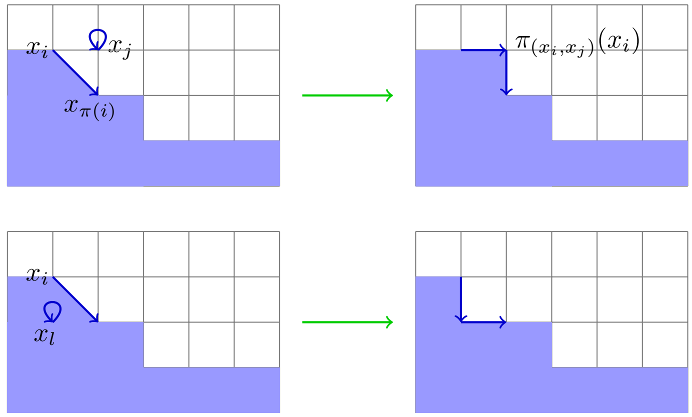

Figure 3.3.: Possible permutation jumps under Glauber dynamics away from poles. On the left: Two different choices of pairs ((x_i, x_j) and $(x_i, x_l) \in \Lambda_N$) for a permutation $\pi \in \mathcal{S}_M$. On the right: the new permutation $\pi_{(x_i,x_j)} \in \mathcal{S}_M$ (respectively $\pi_{(x_i,x_l)} \in \mathcal{S}_M$ on the bottom) after a Markov jump.

type if for all $i = 1, \ldots, M$ there holds

$$|x_i - x_{\pi(i)}|^2 \leq \frac{2}{N^2}.$$

Three scenarios are conceivable. For a site x_i with $x_{\pi(i)} = x_i$, the quadratic distance is obviously 0. When x_i is mapped to a horizontal or vertical direct neighbour, the euclidean distance is $1/N$ and thus $|x_i - x_{\pi(i)}|^2 = \frac{1}{N^2} \leq \frac{2}{N^2}$. The third possibility is the characteristic property in our model (compared to the Ising model), namely the diagonal mapping of points to the next (diagonal) neighbour. By the Pythagorean Theorem we have in this case

$$|x_i - x_{\pi(i)}|^2 = \frac{2}{N^2}. \tag{3.4}$$

- Even though the single cycle represents the droplet's boundary, keep in mind that the probability measure $\mathbb{P}$ lives on the whole $\mathcal{S}_M$, where (assuming periodic boundaries of Λ_N) $M = 4N^2$. Our starting permutation thus has a single cycle and all other sites are mapped to themselves.

As an example for an initial domain one could potentially take the whole unit square $[-1, 1]^2$ or a somewhat nicer droplet like the one illustrated (by means of its border γ_0) in Figure 3.1, where we also added an approximating cycle of nearest-neighbour jumps.

Given that the stochastic modelling seems to be very close to the Ising model (cf. Section 3.1), the conjecture that motivated this Ph.D. project was that this system should also

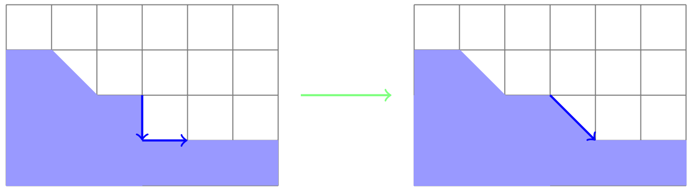

Figure 3.4.: Another permutation jump under Glauber dynamics away from poles, creating a diagonal mapping of a site

exhibit a mean-curvature type droplet shrinking, just as in [18, 19]. That is to say, $\gamma_t(N)$ should converge (for $N \to \infty$) at every point in time towards a *deterministic* curve γ_t which solves a partial differential equation similar to the one in (3.1).

In the annealed version of the model that we consider, i.e. with $\alpha \gg 0$, the Glauber dynamics will not allow energy-increasing jumps. At this point it is notable that both permutations (assuming they are the same at all other sites of the lattice) on the left and on the right of Figure 3.2 possess the same energy value H, since the energy of the two mappings which are illustrated by blue arrows on the right side add up to $2/N^2$ just as in (3.4) for the diagonal arrow on the left side. In other words, the Glauber dynamics has no preference whether or not to use a diagonal jump in the cycle. However, given that $\mathbb{P}(\pi) = \mathbb{1}_{\mathrm{id}}(\pi)$ is the stationary distribution, the dynamics will try to decrease its surface tension by jumping with probability 1 from a permutation $\pi \in \mathcal{S}_M$ to $\pi_{(x_i,x_j)} \in \mathcal{S}_M$, where

$$\pi_{(x_i,x_j)}(x_k) := \begin{cases} x_{\pi(i)} & \text{for k=j,} \\ \pi(x_j) & \text{for k=i,} \\ \pi(x_k) & \text{else,} \end{cases} \tag{3.5}$$

whenever the pair of distinct sites (x_i, x_j) is chosen[4] and

$$H((x_1,\ldots,x_M),\pi_{(x_i,x_j)}) \leq H((x_1,\ldots,x_M),\pi). \tag{3.6}$$

This way $\pi_{(x_i,x_j)}$ is the new permutation obtained from $\pi \in \mathcal{S}_M$ by swapping the targets of sites x_i and x_j under π. Inequality (3.6) implies that such a jump of the Markov chain can only happen for neighbouring sites, i.e. for $|x_i - x_j| \leq \sqrt{2}/N$.[5]

Just like the approach in [18], it is reasonable to treat the dynamics at the poles and the dynamics away from the poles separately. The poles melt off similarly to the treatment with zero-range processes in the Ising model. Still, there are technical difficulties and we are yet to prove a rigorous result. Consequently, we will deal with it in the last chapter.

[4] by this expression we refer to the graphical construction of the Glauber dynamics, cf. [21] and page 29 in Section 4.1

[5] We would like to add that a permutation $\pi \in \mathcal{S}_M$ is a bijective map after all and since the Markov chains must remain within its original state space, there cannot be too extravagant changes for $\gamma_t(N)$ anyways.

On the other hand, away from poles, we are able to translate the random permutations dynamics to an associated particle system (Section 4.1) and derive the hydrodynamic limit in Section 4.3.

Before we proceed, we will take a closer look at the Markov jumps from permutation to permutation that can occur away from poles. Suppose in Figure 3.3 that the Glauber dynamics randomly chose the pair of sites (x_i, x_j) ((x_i, x_l) respectively on the bottom). With probability 1 the cycle then changes to the top right one (bottom right one).

Remark 3.2.1. *In foresight of Section 4.1 we note that from the perspective of the diagonal blue arrow, given that it is "selected" to move, each of the two outcomes* $\pi_{(x_i,x_j)}$ *and* $\pi_{(x_i,x_l)}$ *(top and bottom) is equally likely to occur, i.e. with probability* $1/2$ *each. Vice versa, we can see in Figure 3.4 what happens when (so to speak) a neighbouring pair of horizontal and vertical arrows is chosen by the dynamics. After a change of targets for the corresponding sites, a new diagonal arrow arises for the new permutation with probability* 1.

4. Dynamics away from Poles

This chapter constitutes the main part of this thesis, as it deals with the hydrodynamic behaviour of nearest-neighbour permutations forming a boundary in the way described in Section 3.2. There are certainly different approaches to do so, but given the procedure of [18, 19] (cf. Section 3.1), motivated by [35], and the fact that interacting particle systems and their scaling limits have been studied intensively for decades (whereas the field of random permutations is relatively new), it seems convenient to make use of this huge bulk of knowledge. Therefor we will describe the dynamics away from the poles by means of this special class of Markov chains, which is done in Section 4.1. However, the arising interacting particle system turns out to be relatively difficult to handle, for reasons discussed in the introduction. As an intermediate step, we generate a toy-model, which is in some sense similar to the original model and contains one of its particular difficulties (namely non-product stationary measures). Since it is also an interesting model on its own with applications in various fields, Section 4.2 is devoted entirely to this stochastic process, which will be referred to as *range-r exclusion process* (RrEP).

4.1. Connections to Particle Systems

For simplicity, we will concentrate on the top rightward detail of the droplet $\mathcal{D}$, which means that its boundary has a negative slope just as for example in Figure 3.4. Of course, due to the symmetry of the lattice, it does not matter for the evolution if we would consider the top left, bottom left or bottom right detail instead.

The Glauber dynamics with respect to the stationary measure $\mathbb{P} = \mathbb{1}_{\text{id}}$ on the set of permutations $\pi \in \mathcal{S}_M$ might be considered to be slightly inconvenient. It seems particularly unattractive given the fact that the vast majority of sites (namely the ones that neither belong to the large cycle which forms $\gamma_t(N)$, nor to its "belt" of nearest-neighbours) is not influenced by the dynamics at all. This way, it would be difficult to arrange a connection to an IPS on a 1-dimensional lattice, which is our goal in this section. Instead, in view of Remark 3.2.1 and the fact that energy-increasing jumps are discarded, we boil the dynamics down to the essential part.

Let x, y, z be sites in the embedded version of Λ_N. For each v, with v being

- either a diagonal jump of length $\sqrt{2}/N$ within the cycle that forms $\gamma_t(N)$,
- or a pair of jumps $(\{x \mapsto y\}, \{y \mapsto z\})$ within the cycle that forms $\gamma_t(N)$ such that $|x - z| = \sqrt{2}/N$,

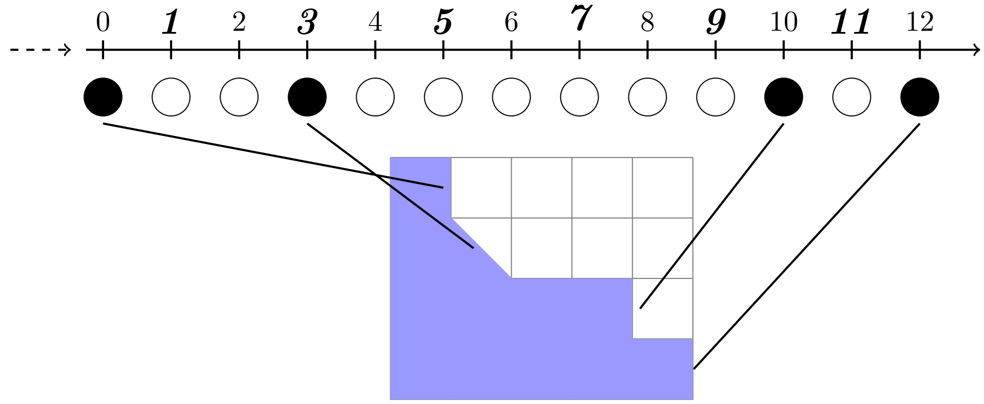

Figure 4.1.: Mapping between an interacting particle system in 1 dimension and an interface between two phases (blue and white) that stems from nearest-neighbour planar random permutations away from poles.

consider a family (τ_v) of independent exp-(1) distributed random variables ("Poisson clocks"). The diagonal jump/pair of jumps $\bar{v} := \text{argmin}\{\tau_v\}$ is the chosen one in this setup and changes the boundary $\gamma_t(N)$ according to the following rule:

- If $\bar{v}$ is a pair $(\{x \mapsto y\}, \{y \mapsto z\})$ with $|x - z| = \sqrt{2}/N$, then the cycle that forms $\gamma_t(N)$ updates with probability 1 by replacing $\bar{v}$ with $\tilde{v} := \{x \mapsto z\}$.
- If $\bar{v} := \{w \mapsto \bar{w}\}$ is a diagonal jump of the cycle that forms $\gamma_t(N)$, then the boundary updates by replacing $\bar{v}$ with $\tilde{v}$, which is one of the two possible pairs of lattice jumps of summed length $2/N$ that have the same starting/endpoint as the original diagonal jump $\bar{v}$. The probability is $1/2$ each.

Afterwards, this procedure repeats itself with Poisson-clocks reset and a set $\{v\}$ that has changed one of its elements, namely $\bar{v}$ has been excluded and $\tilde{v}$ has been included.

Consider now a symmetric interacting particle system with nearest-neighbour hopping on a 1-dimensional periodic lattice $\mathbb{T}_N := \mathbb{Z}/N\mathbb{Z}$ such that at each site $x \in \mathbb{T}_N$ there can be at most 1 particle at a time. So far, this resembles the state space for the SSEP, for which we learned in Section 3.1 that there exists a mapping which connects the particle system to the interface dynamics of the zero-temperature Ising model away from poles. We recall that the presence/absence of a particle corresponded to a vertical/horizontal element of the interface between phases. However, if we scroll back to the Figures that described the random cycle evolution away from poles for permutations at the end of Section 3.2, we recognise that we still have to implement the diagonal jumps.

We do so by introducing two different kinds of sites, in a way which can be seen in Figure 4.1. Here, a diagonal jump corresponds to a particle sitting on an odd-numbered site. Vertical interface elements correspond to a particle's presence at an even site, whereas horizontal interface elements correspond to a vacant even site. It is important to note that a diagonal jump binds both a horizontal and a vertical jump at the same time. In other words, a particle on an odd site can be interpreted as indetermined. If it would sit one spot to the left, the interface in the associated area would first move downwards and then rightwards (assuming that the boundary $\gamma_t(N)$ is given by a cycle of clock-wise directed permutation jumps). If it would sit one spot to the right, the interface would first move to the right and then down. From this we conclude that two particles can never sit next to each other. As a matter of fact, it is not even possible that two particles sit on neighbouring odd sites (for example on sites 3 and 5). With the interpretation given above, particles at sites 3 and 5 are both indetermined and influence the same site 4 (where Schrödingers cat is even more confused than usual). On the other hand, there is obviously no problem with having particles sitting at sites 2 and 4 at the same time, so there is a distinct difference with respect to the exclusion rule for even and odd sites. To have a bijective map between the two systems, one must restrict the state space of the particle system accordingly. The technical definitions follow in Section 4.3.

The dynamics of the particle system translates smoothly from the interface motion. For the latter, we can apply the simple construction at the beginning of this section and obtain that particles can jump both to the left and to the right with equal probability, given that the exclusion rules are obeyed.

We would also like to mention that a density of 1/2 for the SSEP in the Ising model corresponds to a density of 1/4 in this model. For example, every second odd site could be occupied, corresponding to an interface which consists of diagonal jumps only. Also, the maximal density is achieved for $\rho = 1/2$, when every even site is occupied. No particle could move in this case, just as for $\rho = 1$ in the SSEP.

In the following, we will refer to this IPS as AFP-process to merit its derivation from the curve shortening flow *away from poles*.

The larger exclusion rules are one part of the problem to derive the hydrodynamic behaviour for the AFP-process. To get it under control and to obtain valuable insights with respect to the non-product stationary measures, we treat a similar IPS in the next section.

4.2. Range-r Exclusion Process

As mentioned in the introduction to this chapter, we included this section to deal with a particle system similar to the one that describes the boundary's evolution away from poles. It nicely illustrates a part of the original problem and thereby lays the groundwork for the combinatorial derivation of the stationary measures. Also, it shows the relevance of the famous replacement lemma due to Guo, Papanicolaou and Varadhan.

In the following we deal with a type of symmetric exclusion process where particle jumps may occur to nearest-neighbour sites only, just as in the well known Simple Symmetric Exclusion Process (Section 2.2). The difference lies in the range of the exclusion-rule which in our case is enlarged and takes into account not only the occupancy of the jump-destination

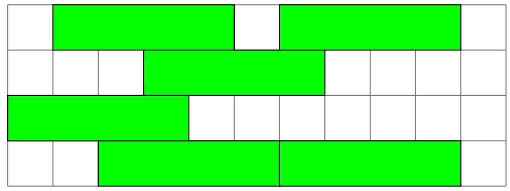

Figure 4.2.: Horizontally aligned cells of ratio 4:1 in a 2-dimensional lattice

site, but also the next site or more. In other words, we want the distance between two occupied sites (two particles) to be at least r, for fixed $r \geq 2$.

Similar models have been examined already. Early on we discovered that a particular interest for this kind of scheme seems to be present in traffic flow models and in Biology, describing cell-movements.

A simple application to traffic-flow models might be given by interpreting a particle as a moving vehicle, jumping (in one direction) from one site to the next in the microscopic (discrete) world. There is a natural exclusion rule as each part of the street, i.e. each site, should be occupied by one and only one particle/vehicle. The velocity in the macroscopic world corresponds to the jump rate for particles. In such a model, a natural problem is the accurate description of shocks or traffic jams. It is reasonable to assume that a vehicle, although only progressing site by site, takes into account the density of particles at sites even further away than just its next site in order to adapt his velocity (jump rate). Clearly, the higher density of particles in front, the higher the probability of a traffic jam, thus leading to a higher safety distance towards the next particle. In [2] there has been examined among others the bulk properties of the steady state for such a particular *asymmetric simple exclusion process* (ASEP) in a periodic system.

There are biological phenomena which are caused and can be described by discrete models of collective cell movements. Usually one cell corresponds to exactly one occupied site of the lattice in an interacting particle system. This approach however does not account for special forms of certain cells, whose longitudinal length might be considerably larger than their transverse length, thus giving them a rod-shaped form instead of a round one. An excerpt of a 2-dimensional lattice with 6 cells is presented in Figure (4.2). A cell can move one site at a time at a certain rate. However, there are no intersections allowed, thus giving a natural exclusion rule. These kind of models have been examined for example in [33].
It is worth noting that the evolution is actually very similar to the one from the example above: imagine the vehicle sits at the leftmost site of the occupied cell (or in the middle of the cell when the vehicle can move to the right and to the left), then the next three sites can be interpreted as the 'safety zone', i.e. the minimal possible distance between a vehicle and its successor (corresponding to a jump rate of 0 when the fourth-most site from the vehicle is occupied).

Schönherr and Schütz [32] have already succeeded in 2004 to derive the hydrodynamic equation for this model on heuristic grounds (cf. Remark 4.2.4). Their stochastic setup was motivated by protein synthesis, for which ribosomes (modelled as extended particles) undergo a local dynamics in order to generate the proteins [26]. As the ribosomes cannot occupy the same spot(s) at the same time and jump rates to the left differ from those to the right, the model is given by some hard-rod asymmetric exclusion process, which is referred to as l-ASEP, l being the horizontal length of the particles. They also mention another interpretation in form of *monomer chains*, for which a particle at one end of the individual chain jumps at a certain rate to the other end. However, considering that the RrEP was introduced in this thesis in order to get a better understanding of the AFP-process, it is certainly more natural to think of particles with a longer exclusion rule that occupy a single spot of the lattice. The reason being that we do not want to demand a rod-shaped particle (or monomer chain) to change its length from 2 to 3 (and vice versa) after every single jump.

A more direct heuristic approach was suggested to us by T. Kriecherbauer and is presented in the second to last Subsection 4.2.5. Unfortunately it does not quite give the correct hydrodynamic equation, but it is certainly possible to guess the correct one from there.

In this thesis, we use a more formal approach which is arduous at times, but also very fruitful in view of the upcoming tasks for the AFP-process in Section 4.3. In the next subsection we will introduce the formal Markov process, followed by stating the main theorem and analysing the hydrodynamic equation for the density evolution in 4.2.2. Subsection 4.2.3 introduces the stationary distribution as a grand-canonical measure and deals with the probability of certain events. Subsection 4.2.4 gives the proof of the main theorem and also, en passant, the classical martingale-approach heuristics for local equilibrium starting measures. We conclude this section with some generalisations and open questions to the RrEP.

4.2.1. Stochastic Model

We will now formalise the above stochastic process, in the following referred to as range-r-exclusion process (*RrEP*), $r \geq 1$. Write $\mathbb{T}_N$ for the torus $\mathbb{Z}/N\mathbb{Z}$ (whose elements we will simply address by $1, ..., N$), $\mathbb{T} = [0,1]$ for the unit interval and

$$\Omega_N := \Big\{ \eta \in \{0,1\}^{\mathbb{T}_N} \ : \ \eta(x) = 1 \ \Rightarrow \ \eta(x \pm k) = 0 \ \forall k \in \{1, \dots, r-1\}, \forall x \in \mathbb{T}_N \Big\}$$

for the space of configurations. As usual, for $\eta(x) = 1$, we write

$$\eta^{x,y}(z) = \begin{cases} \eta(z) & \text{if } z \neq x, y, \\ \eta(x) - 1 & \text{if } z = x, \\ \eta(y) + 1 & \text{if } z = y, \end{cases}$$

for a configuration obtained by a single particle jump from $x \in \mathbb{T}_N$ to $y \in \mathbb{T}_N$, whereas for $\eta(x) = 0$ we set $\eta^{x,y} = \eta$. Considering that only nearest-neighbour jumps may occur, the jump rate from a configuration η to ξ is zero, whenever $\xi \neq \eta^{x,x\pm 1}$ for some $x \in \mathbb{T}_N$.

Hence, the one-dimensional RrEP on the lattice $\mathbb{T}_N$ is a continuous-time Markov chain $(\eta_t)_{t\geq 0}$ with state space Ω_N, generator

$$(L_N f)(\eta) = \frac{1}{2} \sum_{\substack{x,y\in\mathbb{T}_N,\\ |x-y|=1}} \eta(x)\Big(\prod_{k=0}^{r-1}(1-\eta(y+(y-x)k))\Big)\big(f(\eta^{x,y})-f(\eta)\big) \tag{4.1}$$

and semigroup $S_N(t) = \exp(tL_N)$, when addition is taken modulo N. The generator assures that configurations containing particles being distant less than r sites are dynamically not accessible, so $L_N(\mathbb{1}_{\Omega_N}) = 0$. One can check that L_N is indeed a probability generator in the sense of Definition 3.12 of [25] and thus gives rise to a continuous-time Markov-chain with a collection $\{\mathbb{P}_\eta, \eta \in \Omega_N\}$ of probability measures on the set of càdlàg-functions from $\mathbb{R}_+$ to Ω_N such that

$$\mathbb{P}_\eta((\eta_t)_{t\geq 0} \ : \ \eta_0 = \eta) = 1.$$

For a probability measure μ on Ω_N we write $\mathbb{P}_\mu = \sum_{\eta\in\Omega_N} \mu(\{\eta\})\mathbb{P}_\eta$ for the $(\eta_t)_{t\geq 0}$-process started in the distribution μ.

We introduce the *empirical measure*

$$\pi_t^N(du) := \frac{1}{N}\sum_{x\in\mathbb{T}_N} \eta_t(x)\delta_{\frac{x}{N}}(\mathrm{d}u),$$

as a random variable into the space of finite positive measures on $\mathbb{T}$. Note that for a continuous function $G : \mathbb{T} \to \mathbb{R}$ we can write

$$\int_{\mathbb{T}} G(u)\pi_t^N(\mathrm{d}u) = \frac{1}{N}\sum_{x\in\mathbb{T}_N} G\left(\frac{x}{N}\right)\eta_t(x). \tag{4.2}$$

Mapping $\mathbb{T}_N$ to $\frac{\mathbb{T}_N}{N} \subset [0,1]$, in addition to accelerating the process by N^2, corresponds to the diffusive scaling. The hydrodynamic limit is obtained as $N \to \infty$. A stochastic process with generator $N^2 L_N$ will be referred to as N^2-*accelerated RrEP.*

4.2.2. Hydrodynamic Equation

Our goal is to show that, given a sequence of starting measures that are close to some initial density profile $\rho_0 : \mathbb{T} \to \mathbb{R}_+$, the family of accelerated empirical measures $(\pi^N_{N^2 t})_{t\in[0,T]}$ converges in distribution, as $N \to \infty$, towards the deterministic family of measures $(\rho(t,u)\,\mathrm{d}u)_{t\in[0,T]}$, where ρ is a weak solution (see Definition C.1.1 in the appendix) of the partial differential equation

$$\begin{cases} \partial_t \rho &= \frac{1}{2}\partial_x^2 \tilde{\Psi}(\rho), \\ \rho(0,\cdot) &= \rho_0(\cdot), \end{cases} \tag{4.3}$$

with

$$\tilde{\Psi}(\rho) := \frac{\rho}{1-(r-1)\rho},$$

i.e. Equation (4.3) is the hydrodynamic equation for the RrEP.

In order to state the main theorem below, we need some further definitions. At first we recall what was meant by a sequence of starting measures that are close to some function ρ_0.

Definition 4.2.1. *A sequence $(\mu^N)_{N\geq 1}$ of probability measures on $\{0,1\}^{\mathbb{T}_N}$ is* associated to a profile $\rho_0 : \mathbb{T} \to \mathbb{R}_+$ *if for every continuous function $G : \mathbb{T} \to \mathbb{R}$, and for every $\delta > 0$ we have*

$$\lim_{N\to\infty} \mu^N \Big(\Big| \int_{\mathbb{T}} G(u)\pi_0^N(\mathrm{d}u) - \int_{\mathbb{T}} G(u)\rho_0(u)\,\mathrm{d}u \Big| > \delta \Big) = 0. \tag{4.4}$$

We will also require the sequence μ^N to be comparable to a suitable sequence ν_ρ^N of grand-canonical measures (see Section 4.2.3 and in particular Equation (4.14) for their definition) in terms of relative entropies. Recall that for measures μ, ν, the relative entropy of μ given ν is

$$H(\mu|\nu) := \int \log \frac{\mathrm{d}\mu}{\mathrm{d}\nu}\,\mathrm{d}\mu.$$

In the following theorem, we will fix $r \geq 1$ and refer to these quantities with the suitable r.

Our main theorem states that, starting from associated measures which are close to some global stationary measure $\nu_{\alpha^*}^N$, the measures $S_N(N^2 t)\mu^N$ at a later time t remain close to some profile $\rho(t,u)$ which is determined by (4.3). The proof will be given in Section 4.2.4.

Theorem 4.2.1. *Let $\rho_0 : \mathbb{T} \to [0, \frac{1}{r}]$ be a Lebesgue-integrable function and $(\mu^N)_{N\geq 1}$ be a sequence of probability measures on Ω_N with the following properties:*

(a) $(\mu^N)_{N\geq 1}$ *is associated to* ρ_0*,*

(b) $\exists K \in \mathbb{R}$, $0 < \alpha^* < \frac{1}{r}$ *with*

$$H(\mu^N|\nu_{\alpha^*}^N) \leq K \cdot N.$$

Then , as $N \to \infty$,

$$\{\pi_{N^2 t}^N \;:\; t \in [0,T]\} \to \{\rho(t,u)\,\mathrm{d}u \;:\; t\in[0,T]\} \quad \textit{in distribution,}$$

where $\rho(t,u)$ is the unique weak solution in $L^2([0,T]\times\mathbb{T})$ of the nonlinear PDE (4.3)*.*

The set of measures $(\mu^N)_{N\geq 1}$ fulfilling properties (a) and (b) is not empty. For the Simple Symmetric Exclusion Process (or R1EP) it is well known ([14, Chapter 3]) that the sequence of *product measures with slowly varying parameter* $(\nu^N_{\rho_0(\cdot)})_{N\geq 1}$, which is characterised by its marginals

$$\nu^N_{\rho(\cdot)}(\{\eta \in \{0,1\}^{\mathbb{T}_N} \ : \ \eta(x)=k\}) = \nu^N_{\rho_0(\frac{x}{N})}(\{\eta \in \{0,1\}^{\mathbb{T}_N} \ : \ \eta(0)=k\}), \quad k \in \{0,1\},$$

is associated to the continuous density profile ρ_0. We can generate a similar example for the R2EP by a sequence of product-type (on $2\mathbb{N}\cap\mathbb{T}_N$) measures $(\tilde{\nu}^N_{\rho_0(\cdot)})_{N\geq 1}$ with

$$\tilde{\nu}^N_{\rho_0(\cdot)}(\{\eta \in \Omega_N \ : \ \eta(z)=1\}) = \begin{cases} 2\rho_0(\frac{z}{N}), & 2 \mid z \\ 0, & 2 \nmid z \end{cases}$$

and $\{\eta(x),\ x \in 2\mathbb{N}\cap\mathbb{T}_N\}$ independent w.r.t. $\tilde{\nu}^N_{\rho_0(\cdot)}$. At first we define $\tilde{\nu}_\alpha$ as the weak limit of $\tilde{\nu}^N_\alpha$ for $N\to\infty$.[1] This way there holds

$$\tau_{uN}\tilde{\nu}^N_{\rho_0(\cdot)} \to^w \tilde{\nu}_{\rho_0(u)}, \quad N \to \infty$$

for all continuity points $u \in \mathbb{T}$ of ρ_0. Next, we introduce another function

$$h_{N,l}(u) := \sum_{x\in\mathbb{T}_N} \mathbb{1}_{\{x\leq uN<x+1\}}(u)\cdot \mathbb{E}_{\tilde{\nu}^N_{\rho_0(\cdot)}}\left(\left|\frac{1}{2l+1}\sum_{\substack{y\in\mathbb{T}_N,\\ |y-x|\leq l}} \eta(y) - \rho_0\left(\frac{x}{N}\right)\right|\right),$$

which has the property that

$$h_{N,l}(u) \underset{N\to\infty}{\to} \mathbb{E}_{\tilde{\nu}_{\rho_0(u)}}\left(\left| \underbrace{\frac{1}{2l+1}\sum_{\substack{y\in\mathbb{T}_N,\\ |y|\leq l}} \eta(y)}_{\underset{l\to\infty}{\to}\frac{1}{2}\mathbb{E}_{\tilde{\nu}_{\rho_0(u)}}(\eta(0))=\rho_0(u)} -\rho_0(u)\right|\right) \underset{l\to\infty}{\to} 0$$

$\tilde{\nu}_{\rho_0(u)}$-a.s. by the strong law of large numbers. Consequently we know that

$$\begin{aligned}\int_{\mathbb{T}} h_{N,l}(u)\,\mathrm{d}u =& \mathbb{E}_{\tilde{\nu}^N_{\rho_0(\cdot)}}\left(\frac{1}{N}\sum_{x\in\mathbb{T}_N}\left|\frac{1}{2l+1}\sum_{\substack{y\in\mathbb{T}_N,\\ |y-x|\leq l}} \eta(y) - \rho_0\left(\frac{x}{N}\right)\right|\right) \\ =& \mathbb{E}_{\tilde{\nu}^N_{\rho_0(\cdot)}}\left(\left| \underbrace{\frac{1}{N}\sum_{x\in\mathbb{T}_N}\frac{1}{2l+1}\sum_{\substack{y\in\mathbb{T}_N,\\ |y-x|\leq l}} \eta(y)}_{=\sum_{x\in\mathbb{T}_N}\eta(x)} - \underbrace{\frac{1}{N}\sum_{x\in\mathbb{T}_N}\rho_0\left(\frac{x}{N}\right)}_{=\int_{\mathbb{T}}\rho_0(u)\,\mathrm{d}u+\mathcal{O}\left(\frac{1}{N}\right)}\right|\right)\end{aligned}$$

[1] As before, this is well-defined if one thinks of the measure $\tilde{\nu}^N_\alpha$ as being defined on $\{0,1\}^{\mathbb{Z}}$, but with positive weight only for configurations that have period N.

converges to zero by the dominated convergence theorem (of course the function $h_{N,l}$ is bounded everywhere), hence it follows for a continuous function $G : \mathbb{T} \to \mathbb{R}$

$$\tilde{\nu}^N_{\rho_0(\cdot)} \left(\Big| \frac{1}{N} \sum_{x \in \mathbb{T}_N} \eta(x) G\left(\frac{x}{N}\right) - \int_{\mathbb{T}} G(u)\rho_0(u)\,\mathrm{d}u \Big| > \delta \right) \underset{N\to\infty}{\to} 0.$$

The example above highlights one of the main difficulties for $r > 1$, as there is no stationary product-measure (w.r.t. the whole lattice, not only the even sites). To deal with it, we need to use the Replacement-Lemma due to Guo/Papanicolaou/Varadhan (stated in Section 4.2.4), which compares a sequence of measures associated to a profile with a suitable grand-canonical mixture of stationary distributions.

Remark 4.2.1. *Theorem 4.2.1 can be reformulated without too much additional work, as the measures $S_N(N^2t)\mu^N$ are also associated to the profile $\rho(t,u)$. This way, the theorem would state what is known as* conservation of associated measures*, which is neat and thus often stated in similar theorems.*

Properties of the Hydrodynamic Equation for the RrEP

At last we discuss some properties of the PDE (4.3) and its solution. Note that for $r \geq 2$ and

$$\Psi(\rho) := \frac{1}{1-(r-1)\rho}$$

we can simplify calculations due to $\partial_x^2 \tilde{\Psi} = \partial_x^2 \Psi$, obtaining the PDE

$$\begin{cases} \partial_t \rho &= \frac{1}{2(r-1)} \partial_x^2 \Psi(\rho), \\ \rho(0,\cdot) &= \rho_0(\cdot). \end{cases} \tag{4.5}$$

Existence and uniqueness statements will be given in the proof of the hydrodynamic evolution in Section 4.2.4. In general, the solution to this partial differential equation behaves as expected. Imposing periodic boundary conditions for ρ and $\partial_x \rho$, we obtain immediately the conservation law of mass

$$\partial_t \int_0^1 \rho(t,x)\,\mathrm{d}x = \frac{1}{2(r-1)} \int_0^1 \partial_x^2 \Psi(\rho(t,x))\,\mathrm{d}x = \frac{1}{2} \frac{\partial_x \rho(t,x)}{(1-(r-1)\rho(t,x))^2}\Big|_0^1 \equiv 0.$$

Additionally, the solution of (4.3) remains permanently in the interval $[0, \frac{1}{r}]$, given that $0 \leq \rho_0(\cdot) \leq \frac{1}{r}$, which can be proved by means of the maximum principle. The latter states that the maximum value of a solution ρ for PDE (4.3) in between $0 \leq x \leq 1$ and $0 \leq t \leq T$ must be obtained at a boundary point. Assuming further periodic boundary conditions up to order 2 in the space variable, we can even conclude that the maximum value must occur at time 0, where $0 \leq \rho_0(t,x) \leq \frac{1}{r}$. Thus

$$0 \leq \rho(t,x) \leq \frac{1}{r}$$

for all $x \in \mathbb{T}$ and $0 \leq t \leq T$.

Remark 4.2.2. *The general idea of showing the maximum principle for the RrEP directly is motivated by the following observation: assuming that the solution ρ attains a maximum value at an interior point $(t_0, x_0) \in (0,T)\times(0,1)$, implying both $(\partial_t \rho)(t_0, x_0) = 0$, $(\partial_x \rho)(t_0, x_0) = 0$ and $(\partial_x^2 \rho)(t_0, x_0) \leq 0$, one almost gets an immediate contradiction to the PDE (4.5), since the left-hand-side is 0 and the right-hand-side is greater or equal than 0 (due to possibilities like $f(y) = y^4$, having an obvious minimum at $y = 0$, although $\partial_y^2 f(0) = 0$).*

One way to avoid the unpleasant case is to define a new function

$$v(t,x) := \rho(t,x) + \varepsilon \cdot (T - t),$$

with $0 < \varepsilon < \frac{1}{r\cdot(r-1)\cdot T}$, where the second inequality guarantees that $1 - (r-1)v > 0$. The fact that $v(t,x)$ is not bounded above by $\frac{1}{r}$ does not play a role in the proof. It is straightforward to derive via contradiction that the maximum value of $v(t,x)$ must occur at $\{(t,x) \in [0,T] \times \mathbb{T} \ : \ t = 0\}$. Furthermore, one concludes that

$$\sup_{0\leq x\leq 1, 0\leq t\leq T} \rho(t,x) \leq \sup_{0\leq x\leq 1} \rho(0,x) + T\cdot\varepsilon.$$

As T is fixed in advance and this inequality is true for all sufficiently small $\varepsilon > 0$, we obtain

$$\sup_{0\leq x\leq 1, 0\leq t\leq T} \rho(t,x) = \sup_{0\leq x\leq 1} \rho(0,x) = \sup_{0\leq x\leq 1} \rho_0(x) \in [0, \frac{1}{r}].$$

This function has the properties

$$\partial_t v(t,x) = \partial_t \rho - \varepsilon$$

and

$$\begin{aligned}
\partial_x^2\left(\tfrac{1}{1-(r-1)v}\right) &= \partial_x^2\left(\tfrac{1}{1-(r-1)\rho+(r-1)\varepsilon(t-T)}\right)\\
&= \tfrac{(r-1)\partial_x^2\rho}{(1-(r-1)\rho+(r-1)\varepsilon(t-T))^2} + \tfrac{2((r-1)\partial_x\rho)^2}{(1-(r-1)\rho+(r-1)\varepsilon(t-T))^3}\\
&\overset{(r-1)\varepsilon(t-T)<0}{\geq} \partial_x^2\left(\tfrac{1}{1-(r-1)\rho}\right),
\end{aligned}$$

which leads to the strict inequality

$$\partial_t v - \frac{1}{2(r-1)}\partial_x^2\left(\frac{1}{1-(r-1)v}\right) \leq \partial_t\rho - \varepsilon - \frac{1}{2(r-1)}\partial_x^2\left(\frac{1}{1-(r-1)\rho}\right) \overset{(4.3)}{=} -\varepsilon < 0. \tag{4.6}$$

This way, we get a contradiction if we assume that $v(x,t)$ has a maximum value at an interior point (t_0,x_0), as

$$\partial_x^2\left(\frac{1}{1-(r-1)v}\right)(t_0,x_0)=\underbrace{\frac{(r-1)(\partial_x^2 v)(t_0,x_0)}{(1-(r-1)v)^2}}_{\leq 0}+\underbrace{\frac{2\left((r-1)(\partial_x v)(t_0,x_0)\right)^2}{(1-(r-1)v(t_0,x_0))^3}}_{=0}$$

$$\Rightarrow \underbrace{(\partial_t v)(t_0,x_0)}_{=0}-\frac{1}{2(r-1)}\partial_x^2\left(\frac{1}{1-(r-1)v}\right)(t_0,x_0)\geq 0.$$

By the same reasoning, one can also exclude $\{t=T\}$ to contain the maximum point, since

$$\begin{aligned}\partial_{t\_}v(T,x_0)&:=\lim_{h\downarrow 0}\frac{v(T,x_0)-v(T-h,x_0)}{h}\geq 0\\ \Rightarrow\quad \partial_{t\_}v(T,x_0)&-\frac{1}{2(r-1)}\partial_x^2\left(\frac{1}{1-(r-1)v(T,x_0)}\right)\geq 0,\end{aligned}$$

contradicting again (4.6). The above mentioned periodic boundary restrictions up to the second spatial derivative guarantee the possible further exclusion of $\{x=0\}$ and $\{x=1\}$, leaving only $\{t=0\}$ to contain the maximum point of $v(t,x)$ in $0\leq x\leq 1$, $0\leq t\leq T$. Thus both the equations

$$\begin{aligned}&\sup_{0\leq x\leq 1,0\leq t\leq T} v(t,x)=\sup_{0\leq x\leq 1} v(0,x)=\sup_{0\leq x\leq 1}\rho(0,x)+\varepsilon T \quad \textit{and}\\ &\sup_{0\leq x\leq 1,0\leq t\leq T}\rho(t,x)\leq \sup_{0\leq x\leq 1,0\leq t\leq T} v(t,x)\end{aligned}$$

hold, finally giving

$$\sup_{0\leq x\leq 1,0\leq t\leq T}\rho(t,x)\leq \sup_{0\leq x\leq 1}\rho(0,x)+T\cdot\varepsilon.$$

As T is fixed in advance and this inequality is true for all sufficiently small $\varepsilon>0$, we obtain

$$\sup_{0\leq x\leq 1,0\leq t\leq T}\rho(t,x)=\sup_{0\leq x\leq 1}\rho(0,x)=\sup_{0\leq x\leq 1}\rho_0(x)\in[0,\frac{1}{r}].$$

Applying the same proof to the function $-\rho$ shows that also the minimum value must be obtained at time $t=0$. Thus, once started in $[0,\frac{1}{r}]$, the solution to the PDE (4.3) never leaves this box.

Also, there is a notable connection between solutions to (4.3) for different $r\geq 2$. Indeed, it is enough to know the solution ρ^* for $r=2$, i.e. with

$$\begin{cases}\partial_t\rho^* &= \frac{1}{2}\partial_x^2\left(\frac{1}{1-\rho^*}\right)\\ \rho^*(0,\cdot) &= \rho_0(\cdot)\end{cases},\tag{4.7}$$

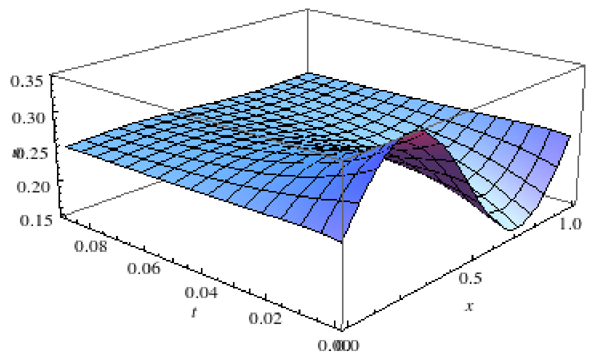

Figure 4.3.: solution $\rho(t,x)$ to (4.3) with initial boundary condition $\rho_0 := \frac{1}{4} + \frac{1}{10}\sin(2\pi x)$

in order to derive a solution $\rho := \frac{1}{r-1}\rho^*$ for the hydrodynamic equation of the RrEP, since

$$\partial_t \rho = \frac{1}{r-1}\partial_t \rho^* \overset{((4.7))}{=} \frac{1}{r-1}\frac{1}{2}\partial_x^2\left(\frac{1}{1-\rho^*}\right) = \frac{1}{2(r-1)}\partial_x^2\left(\frac{1}{1-(r-1)\rho}\right).$$

In Figure 4.3 the solution to (4.3) is plotted for $r = 2$ with initial condition

$$\rho_0 := \frac{1}{4} + \frac{1}{10}\sin(2\pi x).$$

In Section C.2 of the appendix, some more boundary conditions are illustrated.

4.2.3. Stationary Measures

In this section we will construct the stationary distribution of the RrEP and calculate the required probabilities used in Section 4.2.4. Despite its familiarity to the Simple Symmetric Exclusion Process with product-Bernoulli stationary measure, it is not clear from the start how the stationary measure for the RrEP generally looks like. It obviously cannot be product, since there is for example dependence between directly neighbouring sites; $\eta(x) = 1$ implies both $\eta(x-1) = 0$ and $\eta(x+1) = 0$ for $r > 1$. However, the product-form is still implicitly present, as was shown in [32]; compare Remark 4.2.3 after the next lemma.

We delve into the problem by considering the canonical setup first. As the number of particles is conserved under the time evolution, the state space Ω_N can be divided into non-communicating subspaces $\Omega_{N,K} := \{\eta \in \Omega_N \ : \ \sum_{x\in\mathbb{T}_N} \eta(x) = K\}$, each being irreducible due to the assumed periodic boundary conditions. Hence, one gets a unique stationary measure for each $K \in \{0, \dots, \lfloor \frac{N}{r} \rfloor\}$ and a whole family of stationary measures (called *canonical measures*) $(\mu_{N,K})_{K\in\{0,\dots,\lfloor \frac{N}{r}\rfloor\}}$ on Ω_N.

Lemma 4.2.1. *Let $K \in \{0, \dots, \lfloor \frac{N}{r} \rfloor\}$ and $\mu_{N,K}$ be the uniform distribution on $\Omega_{N,K}$. Then $\mu_{N,K}$ is invariant for the RrEP.*

Proof. According to Theorem 3.37 of [25] it is enough to show $\int L_N f \, \mathrm{d}\mu_{N,K} = 0$ for all $f : \Omega_{N,K} \to \mathbb{R}$. Given a configuration $\eta \in \Omega_{N,K}$ with $\eta(x) = 1$ it follows that $\eta^{x,x+1}(x+1) \cdot (1 - \eta^{x,x+1}(x-r+1)) = 1$ for all $x \in \mathbb{T}_N$, as well as similarly for jumps to the other side. Thus

$$
\begin{aligned}
&\int (L_N f)(\eta) \, \mathrm{d}\mu_{N,K}(\eta) \\
=&\sum_{\eta \in \Omega_{N,K}} \mu_{N,K}(\eta) \frac{1}{2} \sum_{\substack{x,y \in \mathbb{T}_N, \\ |x-y|=1}} \eta(x)(1 - \eta(y + (y-x)(r-1)))\,(f(\eta^{x,y}) - f(\eta)) \\
=&\frac{1}{2|\Omega_{N,K}|} \sum_{\substack{x,y \in \mathbb{T}_N, \\ |x-y|=1}} \sum_{\eta \in \Omega_{N,K}} \eta(x)(1 - \eta(y + (y-x)(r-1)))\,(f(\eta^{x,y}) - f(\eta)) = 0,
\end{aligned}
$$

due to

$$
\begin{aligned}
&\eta(x)(1 - \eta(y + (y-x)(r-1)))(f(\eta^{x,y}) - f(\eta)) \\
=& - \eta^{x,y}(y)(1 - \eta^{x,y}(x + (x-y)(r-1)))(f(\eta^{x,y}) - f(\eta))
\end{aligned}
$$

whenever $\eta, \eta^{x,y} \in \Omega_{N,K}$ for $|x-y| = 1$. □

Remark 4.2.3. *Schönherr and Schütz [32] gave quite an elegant reasoning for the previous lemma, owing to a one-to-one mapping of configurations in this model with configurations of a ZRP.*

At first, identify each extended particle (respectively in the RrEP-setting: each particle combined with its blocked neighbouring sites to his right) with a single site in the ZRP lattice. Then, identify the number of ZRP-particles on a site with the distance between the extended particle (that corresponds to this site) to the next extended particle. The rate at which a ZRP-particle leaves a site is 1 *(with equal probability to the left or right), independently of the total number of particles on this site. Given a periodic lattice of fixed size and a fixed number of particles, this means that it is possible to uniquely describe an RrEP dynamics by means of a ZRP dynamics.*[2] *From Subsection 2.3.2 we know the stationary measure of the latter, which happens to be of product form. Translated back to the extended particles, it follows that the uniform distribution on the lattice is stationary for them.*

In the following we will focus only on $K \geq 1$ and conveniently think of the measures $\mu_{N,K}$ as being defined on (the power set of) the whole Ω_N by putting $\mu_{N,K}(\Omega_{N,K}) = 1$. Before stating the next lemma regarding the probability of certain events under $\mu_{N,K}$, it is

[2] Of course the number of sites and the number of particles differ, but this is not important for the sake of this argument.

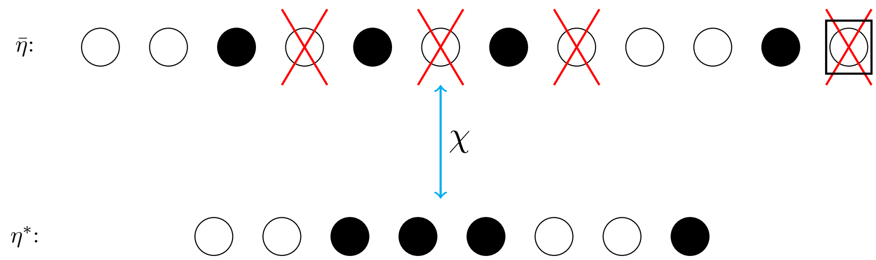

Figure 4.4.: example of two corresponding configurations η for the R2EP and η^* for the R1EP under the map χ which removes the rightward (vacant) neighbouring site to each occupied site; a black circle symbolises a particle at this site, a white circle stands for a vacant site; the rectangle around the rightmost site of $\bar{\eta}$ reflects the conditioning on $\{\eta \in \Omega_{12,4} \mid \eta(12) = 0\}$.

important to know how many configurations are possible in total. In order to find $|\Omega_{N,K}|$ we will identify the connection to the range-1-exclusion case. Due to the periodic lattice, we have to condition first on $r-1$ connected sites:

$$
\begin{aligned}
|\Omega_{N,K}| = & \,|\underbrace{\{\eta \in \Omega_{N,K} : \eta(x+k) = 0 \;\forall k \in \{0, \dots, r-2\}\}}_{=:\Omega^0_{N,K}}| \\
& + |\underbrace{\{\eta \in \Omega_{N,K} : \eta(x) = 1\}}_{=:\Omega^{1,x}_{N,K}}| + \dots + |\underbrace{\{\eta \in \Omega_{N,K} : \eta(x + (r-2)) = 1\}}_{=:\Omega^{1,x+(r-2)}_{N,K}}|
\end{aligned}
$$

for some $x \in \mathbb{T}_N$. Starting with the condition $\eta(x+k) = 0$ for all $k \in \{0, \dots, r-2\}$, one gets a bijection $\chi : \Omega^0_{N,K} \to \left\{\eta \in \{0,1\}^{N-(r-1)K} \;:\; \sum_{j=1}^{N-(r-1)K} \eta(j) = K\right\}$ by removing the first $r-1$ rightward neighbouring sites to each particle (see Figure 4.4 for an example with $r = 2$). Since there are K particles, the image space consists of $N - (r-1)K$ sites and particles might very well be neighbours, such that we obtained a suitable state space for the R1EP. Its cardinality is given by

$$
|\Omega^0_{N,K}| = \binom{N-(r-1)K}{K}. \tag{4.8}
$$

Regarding the other conditions, note at first that $|\Omega^{1,x}_{N,K}| = \dots = |\Omega^{x+(r-2)}_{N,K}|$ once again due to periodicity. Assuming that $\eta(x) = 1$, the matter is slightly different compared to the above case. There is still a bijection between the state space for the RrEP and an appropriate state space for the R1EP, but one has to take into account that we have to distribute $K-1$ particles instead of K (as one is already on site x), and that there is one less site at disposal. Hence,

$$
|\Omega^{1,x}_{N,K}| = \binom{N-(r-1)K-1}{K-1}, \tag{4.9}
$$

giving in total

$$
\begin{aligned}
|\Omega_{N,K}| &= |\Omega^0_{N,K}| + (r-1)|\Omega^{1,x}_{N,K}| \\
&= \binom{N-(r-1)K}{K} + (r-1)\binom{N-(r-1)K-1}{K-1}.
\end{aligned} \tag{4.10}
$$

Lemma 4.2.2. *For all $x \in \mathbb{T}_N$, we have*

$$\mu_{N,K}(\{\eta \in \Omega_{N,K} \ : \ \eta(x) = 1\}) = \frac{K}{N}, \tag{4.11}$$

$$\mu_{N,K}(\{\eta \in \Omega_{N,K} \ : \ \eta(x) = \eta(x+r) = 1\}) = \frac{K}{N} \cdot \frac{K-1}{N-(r-1)K-1}. \tag{4.12}$$

Proof. Making use of (4.9) and (4.10), we get

$$\mu_{N,K}(\{\eta \in \Omega_{N,K} \ : \ \eta(x) = 1\}) = \frac{\binom{N-(r-1)K-1}{K-1}}{(r-1)\binom{N-(r-1)K-1}{K-1} + \binom{N-(r-1)K}{K}} = \frac{K}{N}.$$

Analogously to the above, combinatorics give

$$|\{\eta \in \Omega_{N,K} \ : \ \eta(x) = \eta(x+r) = 1\}| = \binom{N-(r-1)K-2}{K-2},$$

and thus

$$
\begin{aligned}
\mu_{N,K}(\{\eta \in \Omega_{N,K} \ : \ \eta(x) = \eta(x+r) = 1\}) &= \frac{\binom{N-(r-1)K-2}{K-2}}{(r-1)\binom{N-(r-1)K-1}{K-1} + \binom{N-(r-1)K}{K}} \\
&= \frac{K}{N} \cdot \frac{K-1}{N-(r-1)K-1}.
\end{aligned} \tag{4.13}
$$

□

The grand-canonical measure ν_ρ^N is obtained by an appropriate convex combination of the canonical measures $\mu_{N,K}$. We set

$$\nu_\rho^N = \sum_{K=0}^{\lfloor \frac{N}{r} \rfloor} \alpha_{N,K}(\rho) \cdot \mu_{N,K}, \tag{4.14}$$

where the $(\alpha_{N,K}(\rho))_{K \in \{0,\dots,\lfloor \frac{N}{r} \rfloor\}}$ are the weights of a convex combination such that $\nu_\rho^N(\{\eta \in \Omega_N \ : \ \eta(x) = 1\}) = \rho$. In the R1EP-case, the weights are $\alpha_{N_0,K}(\rho) = \binom{N_0}{K}\rho^K(1-\rho)^{N_0-K}$, which leads to the Bernoulli-product measure. In the RrEP-case, the sum only goes up to $\lfloor \frac{N}{r} \rfloor$. In order to still get weights which are concentrated around $\rho \cdot N$, the generic choice is

$$\alpha_{N,K}(\rho) := \binom{\lfloor \frac{N}{r} \rfloor}{K} \left(\frac{N}{\lfloor \frac{N}{r} \rfloor} \rho \right)^K \left(1 - \frac{N}{\lfloor \frac{N}{r} \rfloor} \rho \right)^{\lfloor \frac{N}{r} \rfloor - K}. \tag{4.15}$$

Lemma 4.2.3. *Let $\alpha_{N,K}$ be defined as in* (4.15)*. Then there holds*

$$\sum_{K=0}^{\lfloor \frac{N}{r} \rfloor} \alpha_{N,K}(\rho) = 1,$$

and for all $x \in \mathbb{T}_N$

$$\nu_\rho^N(\{\eta \in \Omega_N \ : \ \eta(x) = 1\}) = \sum_{K=0}^{\lfloor \frac{N}{r} \rfloor} \alpha_{N,K}(\rho) \cdot \mu_{N,K}(\{\eta \in \Omega_N \ : \ \eta(x) = 1\}) = \rho.$$

Proof. The first statement is clear due to the binomial theorem, as

$$\sum_{K=0}^{\lfloor \frac{N}{r} \rfloor} \alpha_{N,K}(\rho) = (\frac{N}{\lfloor \frac{N}{r} \rfloor}\rho + (1 - \frac{N}{\lfloor \frac{N}{r} \rfloor}\rho))^{\lfloor \frac{N}{r} \rfloor}.$$

For the second statement we calculate

$$\begin{aligned}
&= \sum_{K=0}^{\lfloor \frac{N}{r} \rfloor} \binom{\lfloor \frac{N}{r} \rfloor}{K} \left(\frac{N}{\lfloor \frac{N}{r} \rfloor}\rho \right)^K \left(1 - \frac{N}{\lfloor \frac{N}{r} \rfloor}\rho \right)^{\lfloor \frac{N}{r} \rfloor - K} \cdot \frac{K}{N} \\
&= \frac{\lfloor \frac{N}{r} \rfloor}{N} \sum_{K=1}^{\lfloor \frac{N}{r} \rfloor} \binom{\lfloor \frac{N}{r} \rfloor - 1}{K-1} \left(\frac{N}{\lfloor \frac{N}{r} \rfloor}\rho \right)^K \left(1 - \frac{N}{\lfloor \frac{N}{r} \rfloor}\rho \right)^{\lfloor \frac{N}{r} \rfloor - K} \\
&= \frac{\lfloor \frac{N}{r} \rfloor}{N} \sum_{j=0}^{\lfloor \frac{N}{r} \rfloor - 1} \binom{\lfloor \frac{N}{r} \rfloor - 1}{j} \left(\frac{N}{\lfloor \frac{N}{r} \rfloor}\rho \right)^{j+1} \left(1 - \frac{N}{\lfloor \frac{N}{r} \rfloor}\rho \right)^{\lfloor \frac{N}{r} \rfloor - 1 - j} \\
&= \frac{\lfloor \frac{N}{r} \rfloor}{N} \cdot \left(\frac{N}{\lfloor \frac{N}{r} \rfloor}\rho \right) \underbrace{\sum_{j=0}^{\lfloor \frac{N}{r} \rfloor - 1} \binom{\lfloor \frac{N}{r} \rfloor - 1}{j} \left(\frac{N}{\lfloor \frac{N}{r} \rfloor}\rho \right)^{j} \left(1 - \frac{N}{\lfloor \frac{N}{r} \rfloor}\rho \right)^{\lfloor \frac{N}{r} \rfloor - 1 - j}}_{=(\frac{N}{\lfloor \frac{N}{r} \rfloor}\rho + (1 - \frac{N}{\lfloor \frac{N}{r} \rfloor}\rho))^{\lfloor \frac{N}{r} \rfloor - 1} = 1}.
\end{aligned}$$

□

Next, we state a technical lemma that is going to be used in the following proposition.

Lemma 4.2.4. *For all $x \in \mathbb{R}$,*

$$\sum_{t=0}^{q} \frac{(-1)^{q-t}}{t!(q-t)!} \cdot \prod_{j=0, j \neq t}^{q} (x - j) = 1. \tag{4.16}$$

Proof. Equation (4.16) is a polynomial of degree q in $x \in \mathbb{R}$, say $P(x)$. It is enough to show $P(0) = P(1) = \cdots = P(q) = 1$. But for $x \in \{0, 1, \ldots, q\}$ all but one term are 0, namely $x = t$. Thus

$$P(x) = \frac{(-1)^{q-x}}{x!(q-x)!} \underbrace{(x-0)\cdot(x-1)\cdot\dots\cdot(x-(x-1))}_{=x!} \cdot \underbrace{(x-(x+1))\cdot\dots\cdot(x-q)}_{=(-1)^{x-q}(q-x)!} = 1.$$

□

We will now address the problem of finding the limit of ν_ρ^N-measures of configurations having particles on two fixed sites with minimal distance r.

Proposition 4.2.1. *Consider the RrEP on the torus $\mathbb{T}_N$. Let ν_ρ^N be the grand-canonical measure according to* (4.14) *with weights given by* (4.15)*. Then, for all $x \in \mathbb{T}_N$,*

$$\nu_\rho^N(\{\eta \in \Omega_N \ : \ \eta(x) = \eta(x+r) = 1\}) \to \frac{\rho^2}{1-(r-1)\rho} \quad \textit{for } N \to \infty. \tag{4.17}$$

Proof. At first, bringing the demanded probability into a convenient form, we get

$$\begin{aligned}
&\nu_\rho^N(\{\eta \in \Omega_N \ : \ \eta(x) = \eta(x+r) = 1\}) \\
\overset{(4.13)}{=} \quad & \sum_{K=2}^{\lfloor \frac{N}{r} \rfloor} \tfrac{K}{N} \cdot \tfrac{K-1}{N-(r-1)K-1} \tbinom{\lfloor \frac{N}{r} \rfloor}{K} \left(\tfrac{N}{\lfloor \frac{N}{r} \rfloor} \rho \right)^K \left(1 - \tfrac{N}{\lfloor \frac{N}{r} \rfloor} \rho \right)^{\lfloor \frac{N}{r} \rfloor - K} \\
= \quad & \tfrac{\lfloor \frac{N}{r} \rfloor \cdot (\lfloor \frac{N}{r} \rfloor - 1)}{N} \sum_{K=2}^{\lfloor \frac{N}{r} \rfloor} \tbinom{\lfloor \frac{N}{r} \rfloor - 2}{K-2} \left(\tfrac{N}{\lfloor \frac{N}{r} \rfloor} \rho \right)^K \left(1 - \tfrac{N}{\lfloor \frac{N}{r} \rfloor} \rho \right)^{\lfloor \frac{N}{r} \rfloor - K} \tfrac{1}{N-(r-1)K-1} \\
= \quad & \tfrac{\lfloor \frac{N}{r} \rfloor (\lfloor \frac{N}{r} \rfloor - 1)}{N(N-(r-1)K-1)} \sum_{K=2}^{\lfloor \frac{N}{r} \rfloor} \tbinom{\lfloor \frac{N}{r} \rfloor - 2}{K-2} \sum_{s=0}^{\lfloor \frac{N}{r} \rfloor - K} \rho^{K+s} \cdot \left((\tfrac{N}{\lfloor \frac{N}{r} \rfloor})^{K+s} (-1)^s \tbinom{\lfloor \frac{N}{r} \rfloor - K}{s} \right) \\
\overset{m:=K+s}{=} \quad & \sum_{m=2}^{\lfloor \frac{N}{r} \rfloor} \tfrac{((r-1)\rho)^m}{(r-1)^2} \cdot a_N(m),
\end{aligned}$$

where

$$a_N(m) := \frac{1}{(r-1)^{m-2}} \left(\frac{N}{\lfloor \frac{N}{r} \rfloor} \right)^m \sum_{K=2}^{m} \binom{\lfloor \frac{N}{r} \rfloor - K}{m-K} \binom{\lfloor \frac{N}{r} \rfloor - 2}{K-2} \frac{(-1)^{m-K} \lfloor \frac{N}{r} \rfloor (\lfloor \frac{N}{r} \rfloor - 1)}{N(N-(r-1)K-1)}$$

Clearly, it holds

$$\sum_{m=2}^{\lfloor \frac{N}{r} \rfloor} \frac{((r-1)\rho)^m}{(r-1)^2} = \sum_{j=0}^{\lfloor \frac{N}{r} \rfloor - 2} \frac{((r-1)\rho)^{j+2}}{(r-1)^2} = \rho^2 \sum_{j=0}^{\lfloor \frac{N}{r} \rfloor - 2} ((r-1)\rho)^j \to \frac{\rho^2}{1-(r-1)\rho},$$

for $N \to \infty$ as $|(r-1)\rho| < 1$. Additionally, for fixed m, we have $\frac{((r-1)\rho)^m}{(r-1)^2} \cdot a_N(m) \geq 0$. Reformulating $a_N(m)$ further gives

$$\begin{aligned}
a_N(m) \quad &= \quad \frac{1}{(r-1)^{m-2}} \left(\frac{N}{\lfloor\frac{N}{r}\rfloor}\right)^m \frac{\lfloor\frac{N}{r}\rfloor!}{N(\lfloor\frac{N}{r}\rfloor-m)!} \sum_{K=2}^{m} \frac{(-1)^{m-K}}{(m-K)!(K-2)!(N-(r-1)K-1)} \\
&\overset{\substack{q:=m-2 \\ t:=K-2}}{=} \quad \frac{1}{(r-1)^q} \left(\frac{N}{\lfloor\frac{N}{r}\rfloor}\right)^{q+1} \frac{(\lfloor\frac{N}{r}\rfloor-1)!}{(\lfloor\frac{N}{r}-q-2)!} \sum_{t=0}^{q} \frac{(-1)^{q-t}}{(q-t)!t!(N-2r+1-(r-1)t)} \\
&= \quad \underbrace{\left(\frac{N}{\lfloor\frac{N}{r}\rfloor}\right)^{q+1} \frac{(\lfloor\frac{N}{r}\rfloor-1)!}{(\lfloor\frac{N}{r}-q-2)!\prod_{j=0}^{q}(N-2r+1-(r-1)j)}}_{=:E} \\
&\quad \cdot \underbrace{\sum_{t=0}^{q} \frac{(-1)^{q-t}}{t!(q-t)!}\frac{1}{(r-1)^q} \prod_{j=0,\ j\neq t}^{q} ((N-2r+1)-(r-1)j)}_{=:F}.
\end{aligned}$$

On the one hand, it holds that for fixed m (i.e. for fixed q) $E \to 1$ for $N \to \infty$, since both $\left(\lfloor\frac{N}{r}\rfloor\right)^{q+1} \prod_{j=0}^{q}(N-2r+1-(r-1)j)$ and $N^{q+1}\frac{(\lfloor\frac{N}{r}\rfloor-1)!}{(\lfloor\frac{N}{r}\rfloor-q-2)!}$ are polynomials in $N \cdot \lfloor\frac{N}{r}\rfloor$ of degree $q+1$ with leading coefficient 1. On the other hand, there actually holds equality for F, i.e. for $1 \leq q \leq \lfloor\frac{N}{r}\rfloor - 2$ we have $F = 1$, due to Lemma 4.2.4, putting $x = \frac{N-2r+1}{r-1}$. As E is monotonically increasing in N, so is $a_N(m)$.

Hence, we can apply the Monotone Convergence Theorem with $f(m) := \frac{((r-1)\rho)^m}{(r-1)^2}$ and $f_N(m) := \frac{((r-1)\rho)^m}{(r-1)^2} \cdot a_N(m) \cdot \mathbb{1}_{\{m \leq \lfloor\frac{N}{r}\rfloor\}}$, giving

$$\lim_{N\to\infty} \int\limits_{\mathbb{N}\setminus\{0,1\}} f_N(m)\,\mathrm{d}\mathcal{C} = \int\limits_{\mathbb{N}\setminus\{0,1\}} f(m)\,\mathrm{d}\mathcal{C} = \sum_{m=2}^{\infty} \frac{((r-1)\rho)^m}{(r-1)^2} = \frac{\rho^2}{1-(r-1)\rho},$$

where $\mathcal{C}$ is the counting measure on $\mathbb{N}\setminus\{0,1\}$. $\square$

Apart from knowing a stationary distribution on Ω_N, we are also interested in defining a stationary distribution ν_ρ on the whole lattice $\mathbb{Z}$, i.e. on the configuration space

$$\Omega_{\mathbb{Z}} := \left\{\eta \in \{0,1\}^{\mathbb{Z}} \ : \ \eta(y) = 1 \ \Rightarrow \ \eta(y \pm k) = 0 \ \forall k \in \{1,\dots,r-1\}, \forall y \in \mathbb{Z}\right\}. \tag{4.18}$$

We will do so by fixing a converging subsequence of ν_ρ^N. As $\mathbb{Z}$ is countable and $\{0,1\}^{\mathbb{Z}}$ is equipped with the product topology, the latter is compact. Thus the above configuration space, as a closed subset of $\{0,1\}^{\mathbb{Z}}$, is compact as well. This means that the sequence ν_ρ^N of measures on $\mathcal{B}(\Omega_{\mathbb{Z}})$ is tight, implying by Prokhorov's Theorem that a converging subsequence $\nu_\rho^{N_k}$ indeed exists. It follows from propositions I.1.8 and I.2.14 in [23] that the stationarity property is preserved for ν_ρ.

4.2.4. Proof of Theorem 4.2.1

The proof follows closely the standard method for reversible gradient systems as was illustrated for the zero range process in Chapter 5 of [14]. However the resemblance to the proof for the SSEP is obvious as well, since we permit at most one particle per site.

Throughout the proof we will use the following notation. We write $\mathcal{M}_+$ for the space of finite positive measures on the unit interval $\mathbb{T}$, endowed with the weak*-topology. It is possible to define a metric on $\mathcal{M}_+$ by means of a dense countable family of continuous functions $(f_k)_{k\geq 1}$ on $\mathbb{T}$ by

$$\delta(\mu,\nu) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{|\langle \mu, f_k\rangle - \langle \nu, f_k\rangle|}{1 + |\langle \mu, f_k\rangle - \langle \nu, f_k\rangle|}.$$

For fixed $T \in \mathbb{R}_+$, we set $D([0,T],\mathcal{M}_+)$ as the space of càdlàg-functions with values in $\mathcal{M}_+$ and $\pi^N := \{\pi^N_{N^2t} \ : \ t \in [0,T]\}$. Furthermore, we write $Q^N := (\mathbb{P}_{\mu^N})_{\pi^N}$ for the pushforward measure of $\mathbb{P}_{\mu^N}$ under π^N, i.e. $Q^N(\mathcal{Z}) = \mathbb{P}_{\mu^N}((\pi^N)^{-1}(\mathcal{Z}))$, where $\mathcal{Z}$ is an element of the σ-algebra on $D([0,T],\mathcal{M}_+)$ generated by π^N. Put differently, Q^N is the measure on $D([0,T],\mathcal{M}_+)$ corresponding to the N^2-accelerated Markov-process $\pi^N_{N^2t}$. Elements of the latter space are denoted by $(\pi_t)_{t\in[0,T]}$. It will be convenient later on to apply the translation map τ_x to configurations η, meaning that the latter is shifted by x units. The same holds for continuous functions f, where $\tau_x f(\eta) := f(\tau_x \eta)$ for all configurations η. Also, we write

$$\langle \pi, G\rangle := \int_{\mathbb{T}} G(u)\pi(\mathrm{d}u)$$

for positive measures π on $\mathbb{T}$ of finite total mass and remind the reader of Equation (4.2), which gives

$$\langle \pi^N_t, G\rangle = \frac{1}{N} \sum_{x\in\mathbb{T}_N} G\left(\frac{x}{N}\right) \eta_t(x)$$

for the empirical measure π^N_t.

We will show that $\pi^N_{N^2t}$ converges to a measure $\pi_t(\mathrm{d}u) := \rho(t,x)\,\mathrm{d}u$ which satisfies

$$\int_{\mathbb{T}} G(u)\pi_t(\mathrm{d}u) = \int_{\mathbb{T}} G(u)\pi_0(\mathrm{d}u) + \frac{1}{2}\int_0^t \int_{\mathbb{T}} \partial_x^2 G(u) \cdot \frac{\rho(s,u)}{1-(r-1)\rho(s,u)}\,\mathrm{d}u\,\mathrm{d}s, \tag{4.19}$$

for smooth functions $G : \mathbb{R} \to \mathbb{R}$ with period 1 and $t \in [0,T]$.

The proof can be divided up into 3 main parts. We will show that Q^N converges to a measure concentrated on the deterministic path $\{\rho(t,u)\,\mathrm{d}u,\ 0 \leq t \leq T\}$ by

1. proving that $(Q^N)_{N\geq 1}$ is relatively compact,

2. showing convergence of subsequences $(Q^{N_k})_{k\geq 1}$ towards a unique measure, which turns out to be the Dirac-measure on the solution of (4.19),
3. confirming uniqueness of weak solutions to (4.3).

Prior to proceeding with the first step, we execute some necessary calculations. We fix $N \in \mathbb{N}$, a function $G : \mathbb{R} \to \mathbb{R}$ with period 1 and start with the following process $(M_t^{G,N})_{t\geq 0}$, which is given at time $N^2 t$ by

$$\begin{aligned} M_{N^2t}^{G,N} &= \langle \pi_{N^2t}^N, G\rangle - \langle \pi_0^N, G\rangle - \int_0^{N^2t} L_N \langle \pi_s^N, G\rangle \,\mathrm{d}s \\ &= \langle \pi_{N^2t}^N, G\rangle - \langle \pi_0^N, G\rangle - \int_0^{t} N^2 L_N \langle \pi_{N^2s}^N, G\rangle \,\mathrm{d}s. \end{aligned} \tag{4.20}$$

According to Lemma A.1.1 and Remark A.1.1 in the appendix, putting

$$F(s, \eta_s) := \langle \pi_s^N, G\rangle,$$

the process $(M_t^{G,N})_{t\geq 0}$ is a martingale. Introducing the discrete Laplacian

$$\Delta_N G(\frac{x}{N}) := N^2 \left(G\left(\frac{x+1}{N}\right) + G\left(\frac{x-1}{N}\right) - 2G\left(\frac{x}{N}\right)\right), \tag{4.21}$$

the integrand in the last line of Equation (4.20) gives

$$\begin{aligned} &N^2 L_N \langle \pi_{N^2s}^N, G\rangle \\ =&N \sum_{x\in\mathbb{T}_N} G\left(\frac{x}{N}\right) \Big(-\frac{1}{2}\eta_{N^2s}(x)(1-\eta_{N^2s}(x-r)) - \frac{1}{2}\eta_{N^2s}(x)(1-\eta_{N^2s}(x+r)) \\ &\quad + \frac{1}{2}\eta_{N^2s}(x-1)(1-\eta_{N^2s}(x+r-1)) + \frac{1}{2}\eta_{N^2s}(x+1)(1-\eta_{N^2s}(x-r+1))\Big) \\ =&\frac{N}{2} \sum_{x\in\mathbb{T}_N} G\left(\frac{x}{N}\right) \Big(-\eta_{N^2s}(x) + \eta_{N^2s}(x)\eta_{N^2s}(x-r) - \eta_{N^2s}(x) + \eta_{N^2s}(x)\eta_{N^2s}(x+r) \\ &+ \eta_{N^2s}(x-1) - \eta_{N^2s}(x-1)\eta_{N^2s}(x+r-1) + \eta_{N^2s}(x+1) - \eta_{N^2s}(x+1)\eta_{N^2s}(x-r+1)\Big) \\ =&\frac{1}{2N}\langle \pi_{N^2s}^N, \Delta_N G\rangle \\ &+ \frac{N}{2}\sum_{x\in\mathbb{T}_N} \eta_{N^2s}(x)\eta_{N^2s}(x+r)\cdot \Big(G(\frac{x+r}{N}) + G(\frac{x}{N}) - G\left(\frac{x+1}{N}\right) - G\left(\frac{x+r-1}{N}\right)\Big), \end{aligned} \tag{4.22}$$

taking advantage of the fact that products of the form $\eta_{N^2s}(x) \cdot \eta_{N^2s}(x+k)$ are 0 for $|k| < r$. Note that for the SSEP (R1EP), the second summand is just 0. Continuing with (4.22) by applying summation by parts, we get

$$\begin{aligned} &N^2 L_N \langle \pi^N_{N^2s}, G\rangle \\ =&\frac{1}{2N}\langle \pi^N_{N^2s}, \Delta_N G\rangle + \frac{1}{2N}\sum_{x\in\mathbb{T}_N} \eta_{N^2s}(x)\eta_{N^2s}(x+r)\cdot\left(\sum_{j=x+1}^{x+r-1}\Delta_N G\left(\frac{j}{N}\right)\right) \\ =&\frac{1}{2N}\sum_{x\in\mathbb{T}_N}\Delta_N G\left(\frac{x}{N}\right)\Big(\eta_{N^2s}(x) + \eta_{N^2s}(x-r+1)\eta_{N^2s}(x+1) + \dots \\ &\qquad\qquad\qquad \dots + \eta_{N^2s}(x-1)\eta_{N^2s}(x-1+r)\Big). \end{aligned} \tag{4.23}$$

Remark 4.2.4 (Heuristics based on a local equilibrium ansatz)**.** *Already at this point, one is able to predict the hydrodynamic equation under some stronger assumptions. First, let the N^2-accelerated empirical measures converge to an absolutely continuous measure with density $\rho(t,u)$. Assume further that for all local, bounded functions $F : \{0,1\}^{\mathbb{Z}} \to \mathbb{R}$ and ν_ρ as on page 46, we have a sequence $(\mu^N)_{N\geq 1}$ of local equilibrium measures for the starting profile $\rho_0 : \mathbb{T} \to \mathbb{R}_+$, i.e. recalling from Subsection 2.2.3*

$$\lim_{N\to\infty} \mathbb{E}_{\tau_{\lfloor uN\rfloor}\mu^N}(F(\eta)) = \mathbb{E}_{\nu_{\rho_0(u)}}(F(\eta)) \tag{4.24}$$

for all continuity points u of ρ_0. Having this assumption on initial measures is quite a restrictive condition, as we demand weak convergence in every continuity point of the initial density profile ρ_0. Certainly for continuous functions $G : \mathbb{T} \to \mathbb{R}$ a local equilibrium sequence implies the convergence in Definition 4.2.1, where we only demanded the spatial means to converge. On the other hand this assumption facilitates calculations for the purpose of this remark. Another property that we shall assume for the moment is that local equilibrium is conserved for the function $\rho : \mathbb{R}_+ \times \mathbb{T} \to \mathbb{R}_+$ in the sense that

$$S_N(N^2t)\tau_{\lfloor uN\rfloor}\mu^N \to^w \nu_{\rho(t,u)}, \quad N \to \infty \tag{4.25}$$

for all $t \geq 0$ and all continuity points u of $\rho(t,\cdot)$.

Later on it will turn out that, in the limit $N \to \infty$, the measures Q^N are concentrated on a deterministic path in $D([0,T],\mathcal{M}_+)$. Thus, it is a valid heuristic approach to take the expectation in Equation (4.20). Since $M_0^{G,N} = 0$, we have

$$\mathbb{E}_{\mathbb{P}_{\mu^N}}(M^{G,N}_{N^2t}) = 0 \quad \forall t \geq 0.$$

Combined with Equation (4.23) *and Equation* (4.2)*, we get*

$$\begin{aligned}&\frac{1}{N}\sum_{x\in\mathbb{T}_N} G\left(\frac{x}{N}\right)\mathbb{E}_{\mathbb{P}_{\tau_x\mu^N}}(\eta_{N^2t}(0)) - \frac{1}{N}\sum_{x\in\mathbb{T}_N} G\left(\frac{x}{N}\right)\mathbb{E}_{\mathbb{P}_{\tau_x\mu^N}}(\eta_0(0))\\ =&\frac{1}{2N}\int_0^t \sum_{x\in\mathbb{T}_N}\Delta_N G\left(\frac{x}{N}\right)\cdot\left(\mathbb{E}_{\mathbb{P}_{\tau_x\mu^N}}\left(\eta_{N^2s}(0)\right) + (r-1)\cdot\mathbb{E}_{\mathbb{P}_{\tau_x\mu^N}}\left(\eta_{N^2s}(-r)\eta_{N^2s}(0)\right)\right)\mathrm{d}s.\end{aligned} \tag{4.26}$$

Taking the limit $N\to\infty$, we obtain with (4.25)

$$\begin{aligned}\lim_{N\to\infty}\left(\frac{1}{N}\sum_{x\in\mathbb{T}_N} G\left(\frac{x}{N}\right)\mathbb{E}_{\mathbb{P}_{\tau_x\mu^N}}(\eta_{N^2t}(0))\right) &= \lim_{N\to\infty}\left(\frac{1}{N}\sum_{x\in\mathbb{T}_N} G\left(\frac{x}{N}\right)\underbrace{\nu_{\rho(t,\frac{x}{N})}(\{\eta(0)=1\})}_{=\rho(t,\frac{x}{N})}\right)\\ &=\int_{\mathbb{T}} G(u)\rho(t,u)\,\mathrm{d}u\end{aligned}$$

and analogously for the second term in the first line of (4.26) *thanks to* (4.24)*. Also, there holds (after applying Taylor's formula twice in $\frac{x}{N}$)*

$$\begin{aligned}&\Delta_N G\left(\frac{x}{N}\right) = NG'\left(\frac{x}{N}\right) + \frac{1}{2}G''\left(\frac{x}{N}\right) + \mathcal{O}\left(\frac{1}{N}\right) - NG'\left(\frac{x}{N}\right) + \frac{1}{2}G''\left(\frac{x}{N}\right) + \mathcal{O}\left(\frac{1}{N}\right)\\ =&G''\left(\frac{x}{N}\right) + \mathcal{O}\left(\frac{1}{N}\right)\end{aligned} \tag{4.27}$$

and furthermore, not only $S_N(N^2t)\tau_{\lfloor uN\rfloor}\mu^N$ converges weakly to $\nu_{\rho(t,u)}$ after assuming a conservation of local equilibrium according to (4.25)*, but also $\nu^N_{\rho(t,u)} \to^w \nu_{\rho(t,u)}$. Finally we obtain from* (4.26)

$$\begin{aligned}&\int_{\mathbb{T}} G(u)\rho(t,u)\,\mathrm{d}u - \int_{\mathbb{T}} G(u)\rho_0(u)\,\mathrm{d}u\\ =&\lim_{N\to\infty}\left(\frac{1}{2N}\int_0^t\sum_{x\in\mathbb{T}_N}\partial_x^2 G\left(\frac{x}{N}\right)\cdot\left(\mathbb{E}_{\nu^N_{\rho(s,\frac{x}{N})}}(\eta_{N^2s}(0)) + (r-1)\cdot\mathbb{E}_{\nu^N_{\rho(s,\frac{x}{N})}}(\eta_{N^2s}(-r)\eta_s(0))\right)\mathrm{d}s\right).\end{aligned}$$

At this point we benefit from Proposition 4.2.1 and get

$$\int_{\mathbb{T}} G(u)\rho(t,u)\,\mathrm{d}u - \int_{\mathbb{T}} G(u)\rho_0(u)\,\mathrm{d}u = \frac{1}{2}\int_0^t\int_{\mathbb{T}}\partial_x^2 G(u)\cdot\underbrace{\frac{\rho(s,u)}{1-(r-1)\cdot\rho(s,u)}}_{=:\Phi(\rho)}\,\mathrm{d}u\,\mathrm{d}s. \tag{4.28}$$

Thus, the only candidate for the hydrodynamic equation of the RrEP is

$$\begin{cases} \partial_t \rho & = \frac{1}{2}\partial_x^2 \Phi(\rho) \\ \rho(0,\cdot) & = \rho_0(\cdot) \end{cases}. \tag{4.29}$$

For $r = 1$ this is just the well-known heat equation for SSEP. Excluding this case, taking $r \geq 2$, the PDE can be simplified to

$$\begin{cases} \partial_t \rho & = \frac{1}{2(r-1)}\partial_x^2 \Psi(\rho) \\ \rho(0,\cdot) & = \rho_0(\cdot) \end{cases}, \tag{4.30}$$

with

$$\Psi(\rho) := \frac{1}{1-(r-1)\rho}.$$

In order to carry out step 1 and show relative compactness of $(Q^N)_{N\geq 1}$ on $D([0,T],\mathcal{M}_+)$, we will make use of Proposition B.2.2 in the appendix. Given a suitable family of test-functions $\{g_k;\ k \geq 1\}$, it allows to treat the same problem for measures $(Q^{N,G})_{N\geq 1}$ on $D([0,T],\mathbb{R})$ instead, where those measures are defined by

$$Q^{N,g_k}(A) = Q^N(\{(\pi_t)_{t\in[0,T]} \in D([0,T],\mathcal{M}_+)\ :\ (\langle \pi_t, g_k\rangle)_{t\in[0,T]} \in A\})$$

for measurable sets $A \subset D([0,T],\mathbb{R})$. The dense subfamily of $C(\mathbb{T})$ mentioned in the proposition will be in our case the space of twice continuously differentiable functions $G : \mathbb{R} \to \mathbb{R}$ with period 1, denoted by $C^2(\mathbb{T})$. Of course, the constant 1-function on $\mathbb{T}$ is in $C^2(\mathbb{T})$ and $C^2(\mathbb{T})$ is dense in $C(\mathbb{T})$ for the uniform topology. Thus, we only have to check whether $(Q^{N,G})_{N\geq 1}$ is relatively compact in $D([0,T],\mathbb{R})$ for every test-function $G \in C^2(\mathbb{T})$, since $(\langle \pi^N_{N^2 t}, G\rangle)_{t\in[0,T]}$ is a real-valued process.
At this point, we can check a version of Prohorov's criterions for relative compactness (cf. Theorem B.2.1 in the appendix). Therefor, for a function $f : [0,T] \to \mathbb{R}$, we introduce a *modified uniform modulus of continuity* by

$$w'_f(\gamma) := \inf_{\{t_i\}_{0\leq i\leq \bar{r}}} \max_{0\leq i<\bar{r}} \sup_{t_i\leq s'<t<t_{i+1}} |f_t - f_{s'}|, \tag{4.31}$$

where the infimum is taken over all partition points $\{t_i, 0 \leq i \leq \bar{r}\}$ of $[0,T]$ such that $0 = t_0 < t_1 < ... < t_{\bar{r}-1} < t_{\bar{r}} = T$ and $t_i - t_{i-1} > \gamma$ for all $i = 1, ..., \bar{r}$. In our current context Prohorov's Theorem states that, for a function $G \in C(\mathbb{T})$, a sequence $(Q^{N,G})_{N\geq 1}$ of probability measures on $D([0,T],\mathbb{R})$ is relatively compact if and only if

1. for every $t \in [0,T]$ and every $\varepsilon > 0$, there is a compact set $K(t,\varepsilon) \subset \mathbb{R}$ such that

$$\sup_{N\geq 1} Q^{N,G}(f\ :\ f_t \notin K(t,\varepsilon)) \leq \varepsilon, \tag{4.32}$$

2. for every $\varepsilon > 0$,

$$\lim_{\gamma \to 0} \limsup_{N \to \infty} Q^{N,G}(f \ : \ w'_f(\gamma) > \varepsilon) = 0. \tag{4.33}$$

Condition (4.32) demands compactness of the marginals at every fixed point in time. It is satisfied directly since $\pi_t^N(\mathbb{T}) \leq 1$ for all $N \geq 1$ and our test function $G \in C^2(\mathbb{T})$ is in particular continuous, such that the image $G(\mathbb{T})$ is compact as well, hence $|\langle \pi_t^N, G \rangle|$ is bounded. Instead of checking condition (4.33), we might check the sufficient condition

$$\lim_{\gamma \to 0} \limsup_{N \to \infty} \sup_{\tau \in \mathcal{T}_T, \theta \leq \gamma} Q^{N,G}(f \ : \ |f_{\tau+\theta} - f_\tau| > \varepsilon) = 0 \qquad \forall \varepsilon > 0, \tag{4.34}$$

given in [1] (see also Section B.2 in the appendix), where $\mathcal{T}_T$ is the family of stopping times bounded by T. With Equation (4.20), we obtain[3]

$$\begin{aligned} &Q^{N,G}\left(\left| \langle \pi^N_{N^2(\tau+\theta)}, G \rangle - \langle \pi^N_{N^2\tau}, G \rangle \right| > \varepsilon \right) \\ \leq &Q^{N,G}\left(\left| M^{G,N}_{N^2(\tau+\theta)} - M^{G,N}_{N^2\tau} \right| > \frac{\varepsilon}{2} \right) + Q^{N,G}\left(\left| \int\limits_\tau^{\tau+\theta} N^2 L_N \langle \pi^N_{N^2 s}, G \rangle \, \mathrm{d}s \right| > \frac{\varepsilon}{2} \right), \end{aligned} \tag{4.35}$$

such that it is sufficient to show convergence towards 0 for each summand in the last line separately. Utilising Equation (4.23) and writing abbreviatory

$$h(\eta_{N^2 s}) := \eta_{N^2 s}(0) + \eta_{N^2 s}(1-r)\eta_{N^2 s}(1) + \cdots + \eta_{N^2 s}(-1)\eta_{N^2 s}(r-1), \tag{4.36}$$

as well as $D := \sup\limits_{N \geq 1} ||\Delta_N G||_\infty < \infty$, we obtain

$$\begin{aligned} &\mathbb{P}_{\mu^N}\left(\left| \frac{1}{2N} \int\limits_\tau^{\tau+\theta} \sum_{x \in \mathbb{T}_N} \Delta_N G\left(\frac{x}{N}\right) \cdot \tau_x h(\eta_{N^2 s}) \, \mathrm{d}s \right| > \frac{\varepsilon}{2} \right) \\ \leq &\mathbb{P}_{\mu^N}\left(\frac{1}{2} \theta D > \frac{\varepsilon}{2} \right) \underset{\theta \leq \gamma}{\leq} \mathbb{P}_{\mu^N}\left(\gamma D > \varepsilon \right) \underset{\gamma \to 0}{\to} 0, \end{aligned} \tag{4.37}$$

which implies by the definition of $Q^{N,G}$ that the second term of the right hand side in inequality (4.35) converges to 0, too, as $\gamma \to 0$ (and thus $\theta \to 0$). Concerning the first term of the right hand side in inequality (4.35), condition (4.34) does not follow immediately only by the martingale property. Showing that $\mathbb{E}_{Q^{N,G}}[(M^{G,N}_{N^2(\tau+\theta)} - M^{G,N}_{N^2\tau})^2] \to 0$ would

[3] For two random variables X, Y there holds

$$\mathbb{P}(|X+Y| > \varepsilon) \leq \mathbb{P}(|X| + |Y| > \varepsilon) \leq \mathbb{P}(\{|X| > \frac{\varepsilon}{2}\} \cup \{|Y| > \frac{\varepsilon}{2}\}) \leq \mathbb{P}(|X| > \frac{\varepsilon}{2}) + \mathbb{P}(|Y| > \frac{\varepsilon}{2}),$$

as $\{|X| + |Y| > \varepsilon\} \subset \left\{ \{|X| > \frac{\varepsilon}{2}\} \cup \{|Y| > \frac{\varepsilon}{2}\} \right\}$.

do the trick though due to Chebyshev's inequality. The connection to the quadratic term is achieved by Lemma 5.1 in Appendix 1 of [14], which is also stated in the appendix as Lemma A.1.1. In our case, we conclude that

$$\begin{aligned} N^G_{N^2t} &:= \left(M^{G,N}_{N^2t}\right)^2 - \int_0^{N^2t} \left(L_N\langle\pi^N_s, G\rangle^2 - 2\langle\pi^N_s, G\rangle L_N\langle\pi^N_s, G\rangle\right) \mathrm{d}s \\ &= (M^{G,N}_{N^2t})^2 - \int_0^t \big(\underbrace{N^2 L_N\langle\pi^N_{N^2s}, G\rangle^2 - 2N^2\langle\pi^N_{N^2s}, G\rangle L_N\langle\pi^N_{N^2s}, G\rangle}_{:=B^G_{N^2s}}\big) \mathrm{d}s \end{aligned} \tag{4.38}$$

is a new martingale. In order to derive $B^G_{N^2s}$, we first note that for sites $x, y \in \mathbb{T}_n,\ |x-y| = 1$, with $\eta_{N^2s}(x) = 1$ and such that a jump to $y \in \mathbb{T}_N$ is allowed under the dynamics, we have with the (third) binomial formula

$$\begin{aligned} &\left(\sum_{z\in\mathbb{T}_N} \eta^{x,y}_{N^2s}(z)G\left(\frac{z}{N}\right)\right)^2 - \left(\sum_{z\in\mathbb{T}_N} \eta_{N^2s}(z)G\left(\frac{z}{N}\right)\right)^2 \\ =&\left(\sum_{z\in\mathbb{T}_N} (\eta^{x,y}_{N^2s}(z) + \eta_{N^2s}(z))G\left(\frac{z}{N}\right)\right) \cdot \left(\sum_{z\in\mathbb{T}_N} (\eta^{x,y}_{N^2s}(z) - \eta_{N^2s}(z))G\left(\frac{z}{N}\right)\right) \\ =&\left(\left(2\sum_{z\in\mathbb{T}_N} \eta_{N^2s}(z)G\left(\frac{z}{N}\right)\right) + G\left(\frac{y}{N}\right) - G\left(\frac{x}{N}\right)\right) \cdot \left(G\left(\frac{y}{N}\right) - G\left(\frac{x}{N}\right)\right). \end{aligned} \tag{4.39}$$

We can use this equation to derive

$$\begin{aligned} &N^2 L_N\left(\langle\pi^N_{N^2s}, G\rangle^2\right) \\ =&\frac{1}{2}\sum_{\substack{x,y\in\mathbb{T}_N,\\ |x-y|=1}} \eta_{N^2s}(x)\left(\prod_{k=0}^{r-1}(1-\eta_{N^2s}(y+(y-x)k))\right)\cdot \\ &\qquad\cdot\left(\left(\sum_{z\in\mathbb{T}_N} \eta^{x,y}_{N^2s}(z)G\left(\frac{z}{N}\right)\right)^2 - \left(\sum_{z\in\mathbb{T}_N} \eta_{N^2s}(z)G\left(\frac{z}{N}\right)\right)^2\right) \\ =&\frac{1}{2}\sum_{\substack{x,y\in\mathbb{T}_N,\\ |x-y|=1}} \eta_{N^2s}(x)\left(\prod_{k=0}^{r-1}(1-\eta_{N^2s}(y+(y-x)k))\right)\left(G\left(\frac{y}{N}\right) - G\left(\frac{x}{N}\right)\right)^2 \\ &+ 2N^2\langle\pi^N_{N^2s}, G\rangle L_N\langle\pi^N_{N^2s}, G\rangle, \end{aligned} \tag{4.40}$$

from which we can conclude

$$B^G_{N^2s} = \frac{1}{2} \sum_{\substack{x,y\in\mathbb{T}_N,\\ |x-y|=1}} \eta_{N^2s}(x) \left(\prod_{k=0}^{r-1} (1-\eta_{N^2s}(y+(y-x)k)) \right) \left(G\left(\frac{y}{N}\right) - G\left(\frac{x}{N}\right)\right)^2 . \quad (4.41)$$

Also, it holds

$$\mathbb{E}_{Q^{N,G}}[(M^{G,N}_{N^2(\tau+\theta)} - M^{G,N}_{N^2\tau})^2] = \mathbb{E}_{Q^{N,G}}[(M^{G,N}_{N^2(\tau+\theta)})^2 - (M^{G,N}_{N^2\tau})^2],$$

due to the martingale properties of $(M^{N,G}_t)$, which can be seen by conditioning properly. This way

$$\begin{aligned} &\mathbb{E}_{Q^{N,G}}[(M^{G,N}_{N^2(\tau+\theta)} - M^{G,N}_{N^2\tau})^2] \\ =& \underbrace{\mathbb{E}_{Q^{N,G}}[N^G_{N^2(\tau+\theta)} - N^G_{N^2\tau}]}_{=0,\text{ since } N^G_t \text{ is a martingale}} + \int_\tau^{\tau+\theta} \mathbb{E}_{Q^{N,G}}[B^G_{N^2s}]\,\mathrm{d}s. \end{aligned} \quad (4.42)$$

and thus

$$\lim_{N\to\infty} \mathbb{E}_{Q^{N,G}}[(M^{G,N}_{N^2(\tau+\theta)} - M^{G,N}_{N^2\tau})^2] \leq \lim_{N\to\infty} \frac{||(\partial_x G)^2||_\infty \cdot \theta}{2N} = 0. \quad (4.43)$$

Finally, for fixed $\varepsilon > 0$,

$$\begin{aligned} &\lim_{\gamma\to 0} \limsup_{N\to\infty} \sup_{\tau\in\mathcal{T}_T,\theta\leq\gamma} Q^{N,G}(|M_{N^2(\tau+\theta)} - M_{N^2\tau}| > \frac{\varepsilon}{2}) \\ \leq& \lim_{\gamma\to 0} \limsup_{N\to\infty} \sup_{\tau\in\mathcal{T}_T,\theta\leq\gamma} \frac{4\mathbb{E}_{Q^{N,G}}[(M^{G,N}_{N^2(\tau+\theta)} - M^{G,N}_{N^2\tau})^2]}{\varepsilon^2} \\ \leq& \lim_{\gamma\to 0} \limsup_{N\to\infty} \frac{4||\partial_x G||^2_\infty \cdot \gamma}{\varepsilon^2 \cdot N} = 0. \end{aligned} \quad (4.44)$$

Thus condition (4.34) is satisfied with (4.37), (4.44), (4.35) and the fact that by construction, $Q^{N,G}$ lays full measure on paths of the form $t \mapsto \langle \pi^N_{N^2t}, G\rangle$. All combined, the relative compactness has been shown.

The next step is to show uniqueness of the limit along subsequences of $(Q^N)_{N\geq 1}$. Let therefore Q^{N_k} be a subsequence converging to a limit point Q^*.
At first we show that the pushforward measure $Q^*_{p_t}$ under the projection-map p_t onto time $t \in [0,T]$ is concentrated on absolutely continuous measures with respect to the Lebesgue

measure. Similarly to SSEP, there is at most one particle per site, such that for trajectories $(\pi_t)_{t\in[0,T]} \in D([0,T],\mathcal{M}_+)$, there holds

$$\sup_{t\in[0,T]} |\langle \pi_t, G\rangle| \le \frac{1}{N_k}\sum_{x\in\mathbb{T}_{N_k}} |G(\frac{x}{N_k})| \quad Q^{N_k} - \text{a.s.}. \tag{4.45}$$

Furthermore the function

$$(\pi_t)_{t\in[0,T]} \mapsto \sup_{t\in[0,T]} |\langle \pi_t, G\rangle|$$

is continuous with respect to the Skorohod-topology, which implies that the set

$$\{(\pi_t)_{t\in[0,T]} \in D([0,T],\mathcal{M}_+) \ : \ \sup_{t\in[0,T]} |\langle \pi_t, G\rangle| \le \int_{\mathbb{T}} |G(u)|\,\mathrm{d}u\}$$

is closed. Thus, it is possible to apply the Portmanteau-Theorem for closed sets, obtaining

$$\begin{aligned}
1 &\overset{(4.45)}{=} \limsup_{k\to\infty} Q^{N_k}(\{(\pi_t)_{t\in[0,T]} \in D([0,T],\mathcal{M}_+) \ : \ \sup_{t\in[0,T]} |\langle \pi_t, G\rangle| \le \int_{\mathbb{T}} |G(u)|\,\mathrm{d}u\}) \\
&\le Q^*(\{(\pi_t)_{t\in[0,T]} \in D([0,T],\mathcal{M}_+) \ : \ \sup_{t\in[0,T]} |\langle \pi_t, G\rangle| \le \int_{\mathbb{T}} |G(u)|\,\mathrm{d}u\}).
\end{aligned}$$

Thus Q^* is concentrated on trajectories $(\pi_t)_{t\in[0,T]} \in D([0,T],\mathcal{M}_+)$ such that

$$\sup_{t\in[0,T]} |\langle \pi_t, G\rangle| \le \int_{\mathbb{T}} |G(u)|\,\mathrm{d}u,$$

and we conclude by the monotone-class theorem that Q^* is concentrated on paths which are absolutely continuous with respect to the Lebesgue-measure at every instant in time. We will write $\rho(s,u)$ for the associated density at time s.

Naturally, the assumption of associated starting measures $(\mu^N)_{N\ge 1}$ guarantees that Q^* is also concentrated on paths that at time 0 have density ρ_0:

$$\begin{aligned}
&Q^*\Big(\Big\{(\pi_t)_{t\in[0,T]} \in D([0,T],\mathcal{M}_+) \ : \ \Big|\int_{\mathbb{T}} G(u)\pi_0(\mathrm{d}u) - \int G(u)\rho_0(u)\,\mathrm{d}u\Big| > \varepsilon\Big\}\Big) \\
&\le \liminf_{k\to\infty} Q^{N_k}\Big(\Big|\int_{\mathbb{T}} G(u)\pi_0(\mathrm{d}u) - \int G(u)\rho_0(u)\,\mathrm{d}u\Big| > \varepsilon\Big) \\
&= \liminf_{k\to\infty} Q^{N_k}\Big(\Big|\frac{1}{N_k}\sum_{x\in\mathbb{T}_{N_k}} G\left(\frac{x}{N_k}\right)\eta_0(x) - \int G(u)\rho_0(u)\,\mathrm{d}u\Big| > \varepsilon\Big) \\
&= \lim_{k\to\infty} \mu^{N_k}\Big(\Big|\frac{1}{N_k}\sum_{x\in\mathbb{T}_{N_k}} G\left(\frac{x}{N_k}\right)\eta(x) - \int G(u)\rho_0(u)\,\mathrm{d}u\Big| > \varepsilon\Big) = 0.
\end{aligned}$$

Consider now smooth functions $G : [0,T] \times \mathbb{T} \to \mathbb{R}$ fulfilling $G(t,0) = G(t,1)$ as well as $\partial_x G(t,0) = \partial_x G(t,1)$ for all $t \in [0,T]$. Once again we refer to Lemma A.1.1, which tells us that the process M^F from Equation (A.2) is another martingale. Applied to the function

$$F(s,\eta_s) := \frac{1}{N_k} \sum_{x \in \mathbb{T}_{N_k}} G\left(\frac{s}{N_k^2}, \frac{x}{N}\right) \eta_s(x)$$

at time $N_k^2 t$, in our case it reads[4]

$$\begin{aligned} M^{G,N_k}_{N_k^2 t} &= \langle \pi^{N_k}_{N_k^2 t}, G(t,\cdot)\rangle - \langle \pi^{N_k}_0, G(0,\cdot)\rangle - \int_0^{N_k^2 t} (\partial_s + L_{N_k}) \langle \pi^{N_k}_s, G\left(\frac{s}{N_k^2}, \cdot\right)\rangle \,\mathrm{d}s \\ &= \langle \pi^{N_k}_{N_k^2 t}, G(t,\cdot)\rangle - \langle \pi^{N_k}_0, G(0,\cdot)\rangle - \int_0^{t} (\partial_s + N_k^2 L_{N_k}) \langle \pi^{N_k}_{N_k^2 s}, G(s,\cdot)\rangle \,\mathrm{d}s, \end{aligned} \tag{4.46}$$

which will show that Q^* is concentrated on trajectories such that

$$\begin{aligned} \langle \pi_t, G\rangle &= \langle \pi_0, G\rangle + \int_0^t \int_{\mathbb{T}} \left(\rho \cdot \partial_s G + \frac{1}{2}\Phi(\rho) \cdot \partial_x^2 G\right) \mathrm{d}u \,\mathrm{d}s, \\ \langle \pi_0, G\rangle &= \int_{\mathbb{T}} G(0,u) \cdot \rho_0(u) \,\mathrm{d}u, \end{aligned} \tag{4.47}$$

where we recall

$$\Phi(\alpha) := \lim_{N\to\infty} \mathbb{E}_{\nu_\alpha^N}[h] \overset{h \in C_b}{=} \mathbb{E}_{\nu_\alpha}[h] = \frac{\alpha}{1-(r-1)\alpha}$$

for $0 \le \alpha \le \frac{1}{r}$ as in Remark 4.2.4.
One gets for every $\delta > 0$

$$\begin{aligned} Q^{N_k}\big(\sup_{0\le t\le T} |M^{G,N_k}_{N_k^2 t}| > \delta\big) \le & \frac{\mathbb{E}_{Q^{N_k}}\left(\left(\sup_{0\le t\le T} |M^{G,N_k}_{N_k^2 t}|\right)^2\right)}{\delta^2} \\ \le & \frac{4}{\delta^2} \mathbb{E}_{Q^{N_k}}\left(\big(M^{G,N_k}_{N_k^2 T}\big)^2\right) \to 0 \end{aligned}$$

for $k \to \infty$, by applying Chebyshev's inequality, followed by Doob's L^2-inequality. Unlike the approach for SSEP, it is not straightforward to derive an expression for the martingale

[4] Contrary to prior applications of Lemma A.1.1, this time the operator ∂_s cannot be neglected, since the function G depends on $s \in \mathbb{R}_+$. In particular, this means that we have to check condition (A.1), which holds true however due to our smoothness assumption on $G : [0,T] \times \mathbb{T} \to \mathbb{R}$ above.

in (4.46) that is supposed to be a function of the empirical measure. This is due to the fact that product terms of the form $\eta_{N^2t}(x-r+1)\cdot\eta_{N^2t}(x+1)$ occur. More precisely, we obtain

$$\begin{aligned}\lim_{k\to\infty} P_{\mu^{N_k}}\Big(\sup_{t\in[0,T]}\Big|\langle\pi^{N_k}_{N_k^2t},G\rangle-\langle\pi^{N_k}_0,G\rangle-\int_0^t\langle\pi^{N_k}_{N_k^2s},\partial_sG\rangle\,\mathrm{d}s\\ -\frac{1}{2}\int_0^t\frac{1}{N_k}\sum_{x\in\mathbb{T}_{N_k}}\Delta_{N_k}G(s,\frac{x}{N_k})\tau_xh(\eta_{N_k^2s})\,\mathrm{d}s\Big|>\delta\Big)=0.\end{aligned}\tag{4.48}$$

Similarly to the heuristic derivation of the hydrodynamic equation in Remark 4.2.4, once again the means to get a closed version of (4.48) in terms of the empirical measure will be a connection to the expectation of h with respect to the stationary measure. This can be obtained by the replacement lemma due to Guo, Papanicolaou and Varadhan (Lemma 3.3 in [20]). First, we introduce the local empirical density of range $2l+1$ centred at x

$$\eta^l(x)=\frac{1}{2l+1}\sum_{|y-x|\le l}\eta(y).$$

Then the replacement lemma for a fixed sequence μ^N of probability measures on $\{0,1\}^{\mathbb{T}_N}$ with $H(\mu^N|\nu^N_\alpha)\le C_0N$ for some finite C_0, states that for every $\delta>0$ and every local function h,

$$\limsup_{\varepsilon\to0}\limsup_{N\to\infty}\mathbb{P}_{\mu^N}\Big(\int_0^t\frac{1}{N}\sum_{x\in\mathbb{T}_N}\tau_xV_{\varepsilon N}(\eta_s)\ ds\ge\delta\Big)=0,\tag{4.49}$$

where

$$V_l(\eta)=\Big|\Big(\frac{1}{2l+1}\sum_{|y|\le l}\tau_yh(\eta)\Big)-\Phi(\eta^l(0))\Big|.$$

Once again we can consider

$$\frac{1}{2N_k}\sum_{x\in\mathbb{T}_{N_k}}\partial_x^2G(s,\frac{x}{N_k})\tau_xh(\eta_{N_k^2s})\tag{4.50}$$

instead of the version with the discrete Laplacian Δ_{N_k} in (4.48). We obtain

$$\begin{aligned}&\frac{1}{2N_k}\sum_{x\in\mathbb{T}_{N_k}}\Delta_{N_k}G(s,\frac{x}{N_k})\tau_xh(\eta_{N_k^2s})\\ =&\frac{1}{2N_k}\sum_{x\in\mathbb{T}_{N_k}}\frac{1}{2\varepsilon N_k+1}\sum_{|y-x|\le\varepsilon N_k}\partial_x^2G(s,\frac{y}{N_k})\tau_yh(\eta_{N_k^2s})\ +\ \mathcal{O}(\frac{1}{N}),\end{aligned}$$

and a first-order Taylor-expansion of $\partial_x^2 G(s, \frac{y}{N_k})$ in $\frac{x}{N_k}$ gives[5]

$$\frac{1}{2N_k} \sum_{x \in \mathbb{T}_{N_k}} \partial_x^2 G(s, \frac{x}{N_k}) \frac{1}{2\varepsilon N_k + 1} \sum_{|y-x| \leq \varepsilon N_k} \tau_y h(\eta_{N_k^2 s}) \ + \mathcal{O}(\varepsilon) + \mathcal{O}(\frac{1}{N}). \tag{4.51}$$

Intuitively, there must be a simple connection between the local empirical density and the empirical measure and indeed, defining

$$\iota_\varepsilon(u) := \frac{1}{2\varepsilon} \mathbb{1}_{\{|u| \leq \varepsilon\}},$$

one gets approximately (for large N)

$$\begin{aligned} \eta_{N^2 s}^{\varepsilon N}(0) &= \frac{1}{2\varepsilon N + 1} \sum_{|y| \leq \varepsilon N} \eta_{N^2 s}(y) \\ &\sim \frac{1}{2\varepsilon N} \sum_{|y| \leq \varepsilon N} \eta_{N^2 s}(y) = \frac{1}{N} \sum_{x \in \mathbb{T}_N} \eta_{N^2 s}(x) \cdot \mathbb{1}_{\{|\frac{x}{N}| \leq \varepsilon\}} \cdot \frac{1}{2\varepsilon} = \langle \pi_{N^2 s}^N, \iota_\varepsilon \rangle. \end{aligned}$$

Hence, applying the replacement lemma, replacing the local empirical density by a function of the empirical measure and lightening the notation by putting $a = \langle \pi_{N_k^2 s}^{N_k}, \iota_\varepsilon \rangle$ for the parameter of the grand-canonical measure, expression (4.51) is equal to

$$\frac{1}{2N_k} \sum_{x \in \mathbb{T}_{N_k}} \partial_x^2 G(s, \frac{x}{N_k}) \tau_x \Phi(a) \ + \ R_{N_k, \varepsilon},$$

where $R_{N_k,\varepsilon}$ is an expression which vanishes in probability as $N_k \to \infty$ and $\varepsilon \to 0$ afterwards. Furthermore, replacing the sum by an integral, we only make an error of order $\mathcal{O}\left(\frac{1}{N}\right)$. Hence, there holds

$$\begin{aligned} \limsup_{\varepsilon \to 0} \limsup_{k \to \infty} Q^{N_k} \Big(\Big| \langle \pi_t, G \rangle - \langle \pi_0, G \rangle - \int_0^t \langle \pi_s, \partial_s G \rangle \, \mathrm{d}s \\ - \frac{1}{2} \int_0^t \int_{\mathbb{T}} \partial_x^2 G(s, u) \Phi\Big(\int_{\mathbb{T}} \iota_\varepsilon(v - u) \pi_s(dv) \Big) \, \mathrm{d}u \, \mathrm{d}s \Big| \geq \delta \Big) = 0. \end{aligned} \tag{4.52}$$

Observe that for each $\varepsilon > 0$ the map

$$\begin{aligned} (\pi_s)_{s \in [0,T]} \mapsto & \langle \pi_t, G \rangle - \langle \pi_0, G \rangle - \int_0^t \langle \pi_s, \partial_s G \rangle \, \mathrm{d}s \\ & - \frac{1}{2} \int_0^t \frac{1}{N_k} \sum_{x \in \mathbb{T}_{N_k}} \partial_x^2 G(s, \frac{x}{N_k}) \Phi\Big(\int_{\mathbb{T}} \iota_\varepsilon(v - u) \pi_s(dv) \Big) \, \mathrm{d}s \end{aligned}$$

[5] For simplicity we treat εN_k as an integer.

is continuous. Thus the Portmanteau-Theorem is applicable, giving

$$\limsup_{\varepsilon\to 0} Q^*\Big(\Big|\langle \pi_t, G\rangle - \langle \pi_0, G\rangle - \int_0^t \langle \pi_s, \partial_s G\rangle \,\mathrm{d}s \\ - \frac{1}{2}\int_0^t \int_{\mathbb{T}} \partial_x^2 G(s,u)\Phi(\int_{\mathbb{T}} \iota_\varepsilon(v-u)\pi_s(dv))\,\mathrm{d}u\,\mathrm{d}s\Big| \geq \delta\Big) = 0. \tag{4.53}$$

The final goal in this step consists in getting rid of the ε, obtaining a function of the density. Pointwise, by Lebesgue's differentiation theorem, we see that

$$\langle \pi^{N_k}_{N_k^2 s}, \iota_\varepsilon\rangle = \frac{1}{2\varepsilon}\int_{\mathbb{T}} \mathbb{1}_{[-\varepsilon,\varepsilon]}(u)\ \pi^{N_k}_{N_k^2 s}(du) \underset{N_k\to\infty}{\to} \frac{1}{2\varepsilon}\int_{\mathbb{T}} \mathbb{1}_{[-\varepsilon,\varepsilon]}(u)\cdot\rho(s,u)\,\mathrm{d}u \underset{\varepsilon\to 0}{\to} \rho(s,0), \tag{4.54}$$

since we know that Q^* is concentrated on absolutely continuous measures. Formally we use that Φ is continuous, $\rho(s,\cdot)\in[0,\frac{1}{r}]$ and $|\partial_x^2 G| < \infty$, such that by dominated convergence for $\varepsilon \downarrow 0$

$$\int_0^t \int_{\mathbb{T}} \partial_x^2 G(s,u)\Phi(\int_{\mathbb{T}} \iota_\varepsilon(v-u)\pi_s(\mathrm{d}v))\,\mathrm{d}u\,\mathrm{d}s \to \int_0^t \int_{\mathbb{T}} \partial_x^2 G(s,u)\Phi(\rho(s,u))\,\mathrm{d}u\,\mathrm{d}s \quad Q^* - \text{a.s.}.$$

We conclude that

$$\limsup_{\varepsilon\to 0} Q^*\Big(\Big|\int_0^t \int_{\mathbb{T}} \partial_x^2 G(s,u)(\Phi(\int_{\mathbb{T}} \iota_\varepsilon(v-u)\ \pi_s(dv)) - \Phi(\rho(s,u)))\,\mathrm{d}u\,\mathrm{d}s\Big| > \delta\Big) = 0$$

for all $\delta > 0$ and follow up by letting $\delta \to 0$.

We will close the proof by showing the uniqueness of weak solutions of (4.3) in $L^2([0,T]\times\mathbb{T})$, i.e. the third step of the proof. We benefit from the discussion in Appendix A.2 of [14] and in particular one theorem (cited here in Theorem C.1.1), which states that for the quasi-linear parabolic equation

$$\begin{cases} \partial_t \rho &= \sigma \partial_x^2 \Psi(\rho), \\ \rho(0,\cdot) &= \rho_0(\cdot), \end{cases}$$

where Ψ is a smooth, strictly increasing function with $||\Psi'||_\infty \leq g^* < \infty$, σ is a constant and ρ_0 a bounded profile, there exists a unique weak solution in $L^2([0,T]\times\mathbb{T})$.

By definition of the RrEP, the starting profile ρ_0 must be bounded by $\frac{1}{r}$ and $\frac{1}{2(r-1)} > 0$. The restriction of the time line to $[0,T]$, the whole $\mathbb{R}_+$ in the Theorem, does not cause any problems neither, as a unique solution in the bigger space $\mathbb{R}_+ \times \mathbb{T}$ obviously implies uniqueness in $[0,T] \times \mathbb{T}$. Finally, the function

$$\Psi : [0, \frac{1}{r}] \to [1, r],\ \rho \mapsto \frac{1}{1-(r-1)\rho}$$

is smooth and strictly increasing with

$$||\Psi^{'}||_\infty \leq r^2(r-1) < \infty,$$

fulfilling all prerequisites for Theorem C.1.1 in the appendix.

4.2.5. Heuristic Approach based on SSEP

In this section our interest lies in deriving the hydrodynamic equation by a more direct heuristic method. The approach was suggested to us by T. Kriecherbauer, and it turns out that it almost works. It relies on the direct connection to the SSEP, i.e. the range-1 case, and our knowledge of the heat equation being its hydrodynamic equation. To formalise this connection, we define the 1-to-1 function χ which eliminates the $(r-1)$ empty sites to the right of each particle in a configuration $\xi \in \Omega_{N,K}$, where $\Omega_{N,K} := \{\eta \in \Omega_N \mid \sum_{x\in\mathbb{T}_N} \eta(x) = K\}$, obtaining an element of $\{\eta \in \{0,1\}^{N-K} \mid \sum_{j=1}^{N-K} \eta(j) = K\}$. Figure 4.4 in Section 4.2.3 shows a simple example for $r = 2$. Neglecting the periodic boundary conditions for once, the RrEP-dynamics corresponds exactly to the SSEP-dynamics of $\eta = \chi(\xi)$. Thus it is reasonable to expect the heat equation to govern the RrEP-density in some kind of way.

It is important to note that under the function χ, a tagged site of a configuration ξ differs from the tagged site of $\eta = \chi(\xi)$ according to the particle distribution. That is, a site $u \in \mathbb{T}_N$ for the configuration $\xi \in \Omega_{N,K}$ corresponds to the site $u - (r-1)\sum_{j=1}^{u-1} \xi(j)$ in $\mathbb{T}_{N-K}$, since for every particle to the left of u, $r-1$ sites are eliminated.

Let us assume that for a sequence $(\xi_N)_{N\geq 1}$ with $\xi_N \in \Omega_N$ we know that

$$\rho^{RrEP}(y) := \lim_{\varepsilon\to 0} \lim_{N\to\infty} \frac{1}{2\varepsilon N} \sum_{j=-\lfloor \varepsilon N\rfloor}^{\lfloor \varepsilon N\rfloor} \xi_N(\lfloor yN\rfloor + j) \tag{4.55}$$

exists for all $y \in [0,1]$ and that ρ^{RrEP} is continuous. Let $\eta_N \in \Omega_{N-\sum_{x\in\mathbb{T}_N}\xi_N(x)}$ be the image of ξ_N under χ. Then

$$\rho^{R1EP}(x) := \lim_{\delta\to 0} \lim_{N\to\infty} \frac{1}{2\delta N} \sum_{j=-\lfloor \delta N\rfloor}^{\lfloor \delta N\rfloor} \eta_N(\lfloor xN\rfloor + j) \tag{4.56}$$

exists for all $x \in [0, 1-(r-1)\int_0^1 \rho^{RrEP}(u)\,\mathrm{d}u]$. To see this, and to relate ρ^{R1EP} with ρ^{RrEP}, first note that since we assume ρ^{RrEP} to be continuous, we can replace the two-sided average in (4.55) by a one-sided one, i.e.

$$\rho^{RrEP}(y) = \lim_{\varepsilon\to 0} \lim_{N\to\infty} \frac{1}{\varepsilon N} \sum_{j=1}^{\lfloor \varepsilon N \rfloor} \xi_N(\lfloor yN \rfloor + j)$$

and the same for ρ^{R1EP}. Next, define the function

$$G(y) := \int_0^y (1-(r-1)\rho^{RrEP}(u))\,\mathrm{d}u$$

and note that $G : [0,1] \to [0, 1-(r-1)\int_0^1 \rho^{RrEP}(u)\,\mathrm{d}u]$ is a 1-to-1 map. For $\delta > 0$ we define $\varepsilon = \varepsilon(y)$ by the equation

$$\delta := \varepsilon - (r-1) \int_y^{y+\varepsilon} \rho^{RrEP}(u)\,\mathrm{d}u.$$

Since $\rho^{RrEP} \le \frac{1}{r}$, we have $\varepsilon \to 0$ whenever $\delta \to 0$. With $x := G(y)$, we thus obtain

$$\begin{aligned}
\tilde{\rho}^{R1EP}_\delta(x) &:= \lim_{N\to\infty} \frac{1}{\delta N} \sum_{j=1}^{\lfloor \delta N \rfloor} \eta_N(\lfloor xN \rfloor + j) \\
&= \lim_{N\to\infty} \frac{\sum_{j=1}^{\lfloor \varepsilon N - (r-1)N \int_y^{y+\varepsilon} \rho^{RrEP}(u)\,\mathrm{d}u \rfloor} \eta_N(\lfloor xN \rfloor + j)}{\varepsilon N - (r-1)N \int_y^{y+\varepsilon} \rho^{RrEP}(u)\,\mathrm{d}u} \\
&= \lim_{N\to\infty} \frac{\frac{1}{\lfloor \varepsilon N \rfloor} \sum_{j=1}^{\lfloor \varepsilon N \rfloor} \xi_N(\lfloor yN \rfloor + j)}{1-(r-1)\frac{1}{\varepsilon} \int_y^{y+\varepsilon} \rho^{RrEP}(u)\,\mathrm{d}u}.
\end{aligned}$$

Taking δ (and thus ε) to zero and using the continuity of ρ^{RrEP}, we get

$$\tilde{\rho}^{R1EP}(x) = \lim_{\delta\to 0} \tilde{\rho}^{R1EP}_\delta(x) = \frac{\rho^{RrEP}(y)}{1-(r-1)\rho^{RrEP}(y)} = \frac{\rho^{RrEP}(G^{-1}(x))}{1-(r-1)\rho^{RrEP}(G^{-1}(x))}.$$

For the same reason and since $(r-1)\rho^{RrEP} < 1$ it follows that $\tilde{\rho}^{R1EP}$ is continuous. Hence, we get

$$\rho^{R1EP}(x) = \tilde{\rho}^{R1EP}(x) = \frac{1}{r-1}\left(\frac{1}{1-(r-1)\rho^{RrEP}(G^{-1}(x))} - 1\right)$$

for all $x \in [0, 1-(r-1)\int_0^1 \rho^{RrEP}(u)\,\mathrm{d}u]$. Using the function

$$F(x) := \int\limits_0^x (1+(r-1)\rho^{R1EP}(u))\,\mathrm{d}u$$

we can vice versa express ρ^{RrEP} by ρ^{R1EP} in view of

$$\begin{aligned}\rho^{RrEP}(y) &= \frac{\rho^{R1EP}(F^{-1}(y))}{1+(r-1)\rho^{R1EP}(F^{-1}(y))} \\ &= \frac{1}{r-1}\left(1-\frac{1}{1+(r-1)\rho^{R1EP}(F^{-1}(y))}\right)\end{aligned} \tag{4.57}$$

for all $y \in [0,1]$ since $G^{-1}(F^{-1}(y)) = y$.

Now we can take the time derivative of (4.57) and use that ρ^{R1EP} solves the heat equation. Note however that the functions F^{-1} and G^{-1} also depend on time.
Applying the time derivative to the equation $F^{-1}(t, F(t,x)) = x$ we obtain

$$\begin{aligned}\partial_t F^{-1}(t, F(t,x)) &= -\partial_y F^{-1}(t, F(t,x)) \cdot \partial_t F(t,x) \\ &= -\frac{1}{1+(r-1)\rho^{R1EP}(t,x)} \cdot \frac{1}{2}(r-1)\left(\partial_x \rho^{R1EP}(t,x) - \partial_x \rho^{R1EP}(t,0)\right),\end{aligned} \tag{4.58}$$

where we use the knowledge of ρ^{R1EP} solving the heat equation. Furthermore

$$\begin{aligned}\partial_x \rho^{R1EP}(t,x) &= \frac{\partial_x \rho^{RrEP}(t, G^{-1}(t,x)) \cdot \partial_x G^{-1}(t,x)}{\left(1-(r-1)\rho^{RrEP}(t, G^{-1}(t,x))\right)^2} \\ &= \frac{\partial_x \rho^{RrEP}(t, G^{-1}(t,x))}{(1-(r-1)\rho^{RrEP}(t, G^{-1}(t,x)))^3}\end{aligned} \tag{4.59}$$

and

$$\begin{aligned}\partial_x^2 \rho^{R1EP}(t,x) = &\left(1-(r-1)\rho^{RrEP}(t, G^{-1}(t,x))\right)^{-5} \\ &\cdot \Big((1-(r-1)\rho^{RrEP}(t, G^{-1}(t,x)))\partial_{yy}\rho^{RrEP}(t, G^{-1}(t,x)) \\ &\quad + 3(r-1)\left(\partial_y \rho^{RrEP}(t, G^{-1}(t,x))\right)^2\Big)\end{aligned} \tag{4.60}$$

Hence,

$$\begin{aligned}\partial_t \rho^{RrEP}(t,y) &= \frac{\partial_t \rho^{R1EP}(t, F^{-1}(t,y)) + \partial_x \rho^{R1EP}(t, F^{-1}(t,y)) \cdot \partial_t F^{-1}(t,y)}{\left(1+(r-1)\rho^{R1EP}(t, F^{-1}(t,y))\right)^2} \\ &= \frac{\frac{1}{2}\partial_x x \rho^{R1EP}(t, F^{-1}(t,y)) + \partial_x \rho^{R1EP}(t, F^{-1}(t,y)) \cdot \partial_t F^{-1}(t,y)}{\left(1-(r-1)\rho^{RrEP}(t,y)\right)^{-2}}.\end{aligned}$$

Inserting (4.58), (4.59) and (4.60), this leads to the partial differential equation

$$\begin{cases} \partial_t \rho^{RrEP}(t,y) & = \frac{1}{2(r-1)} \partial_y^2 \left(\Psi(\rho(t,y)) \right) + \frac{1}{2(r-1)} \cdot \partial_y \rho^{RrEP}(t,y) \cdot \frac{\partial_y \rho^{RrEP}(t,0)}{(1-(r-1)\rho^{RrEP}(t,0))^3}, \\ \rho(0,\cdot) & = \rho_0(\cdot), \end{cases} \tag{4.61}$$

with

$$\Psi(\rho) := \frac{1}{1-(r-1)\rho}.$$

Curiously enough, this PDE differs from the one in (4.3) by a term which depends on the density's space derivative at $x = 0$, which is due to the fact that the transformation function χ had to start at some fixed point. It turns out that there is a difference between taking the hydrodynamic limit directly for the RrEP on the one hand, and on the other hand applying χ for discrete configurations, evolving those configurations in time according to the SSEP-dynamics, applying the hydrodynamic limit and transforming back.
This problem can be avoided by studying the RrEP and the PDE on the whole real line, with zero boundary conditions, in which case the evaluation point of $\partial_y \rho$ in the last term of (4.61) can be pushed to $-\infty$ and the extra term disappears. However, the fact that the term shows up in the periodic setting shows that hydrodynamic limits are a delicate matter and that heuristic derivations have to be treated with care.

4.2.6. Conclusion

The RrEP offers a broad range of interpretations and modelling possibilities. It comes as no surprise that there are plenty of applications to real-world phenomena, as mentioned in the beginning. There naturally come along a couple of related generalisations and open questions. We will now mention a few of them.

Due to our model of spatial random permutations on the lattice, we were only interested in the symmetric case. However, an obvious generalisation is to account for a directional bias which leads to systems out of equilibrium with a non-vanishing macroscopic current [16, 32].

In order to derive the hydrodynamic equation, we mainly used the stationary distribution of configurations with two particles being as close together as possible. However, we gave no explicit formula for other events. It is noteworthy that the number of possible configurations having particles at sites x and y say, does not only depend on whether/how their exclusion-ranges overlap, but also whether the influenced sites are connected or not.

Furthermore, the treatment here has been conducted entirely in 1 dimension. Considering higher dimensions, on a lattice $\Lambda \subset \mathbb{Z}^d$, different scenarios are possible. An example in 2 dimensions in which horizontal and vertical exclusion-ranges differ has already been mentioned in the introduction to this section, see Figure 4.2.

At last, recalling Equation (4.3), one realises that this PDE is well-posed not only for integers r, but also for arbitrary $r \in [1, \infty)$. Given such a general r, an interesting question is if there is a reasonable stochastic system that has this hydrodynamic equation for its particle density evolution.

4.3. AFP Model

After the excursion to the range-r exclusion process, we are now ready for the interacting particle system that describes the zero-temperature permutations away from the poles. We will refer to this particle system in the following as *AFP*-process, due to its origin *away from poles*. We are going to derive the hydrodynamic equation of this process, following the martingale-approach as in the previous section. Unfortunately, the procedure runs not as smoothly as for the range-r exclusion process, owing to the additional difficulties that the system is not translation-invariant and, more importantly, of non-gradient type.

4.3.1. Stochastic Model

An informal description of the process has already been given in Section 4.1, such that we can dive directly into the technical definition as a Markov/Feller process. As before, we write $\mathbb{T}_N$ for the torus $\mathbb{Z}/N\mathbb{Z}$ and $\mathbb{T} := [0,1]$ for the unit interval. This time, however, we want $N \in \mathbb{N}$ to be even, since the model contains two different kinds of sites. Of course this restriction is no real loss of generality, as we will take the hydrodynamic limit eventually. Define the space of configurations by

$$\Omega_N := \left\{ \eta \in \{0,1\}^{\mathbb{T}_N} \ : \ \eta(x) = 1 \ \Rightarrow \begin{cases} \eta(x \pm 1) = 0 & \forall x \in \mathbb{T}_N \text{ even}, \\ \eta(x \pm 1) = \eta(x \pm 2) = 0 & \forall x \in \mathbb{T}_N \text{ odd}. \end{cases} \right\}$$

In other words, we only allow configurations for which a particle on an even site has a distance of at least 2 to the next particle. On an odd site a particle must have a distance of at least 3 to his next neighbour. Given a configuration $\eta \in \Omega_N$ with a particle present at site $x \in \mathbb{T}_N$, we define the new configuration obtained by a jump from site x to $y \in \mathbb{T}_N$ (with $x \neq y$) by

$$\eta^{x,y}(z) := \begin{cases} \eta(z) & \text{if } z \neq x, y, \\ \eta(x) - 1 & \text{if } z = x, \\ \eta(y) + 1 & \text{if } z = y. \end{cases}$$

For $\eta(x) = 0$, i.e. when no particle is present at $x \in \mathbb{T}_N$, we simply set $\eta^{x,y} = \eta$. Now consider the linear operator L_N on $C(\Omega_N)$ given by

$$\begin{aligned}(L_N f)(\eta) =& \frac{1}{2} \sum_{\substack{y,z \in \mathbb{T}_N, 2|y, \\ |y-z|=1}} \eta(y) \left(\prod_{k=0}^{2} (1 - \eta(z + k(z-y))) \right) (f(\eta^{y,z}) - f(\eta)) \\ &+ \frac{1}{2} \sum_{\substack{x,z \in \mathbb{T}_N, 2\nmid x, \\ |x-z|=1}} \eta(x)(f(\eta^{x,z}) - f(\eta)),\end{aligned} \tag{4.62}$$

where addition is taken modulo N. This defines a continuous-time Markov process on the finite space Ω_N with off-diagonal elements of the q-matrix[6] given by

$$q(\eta, \eta^{x,y}) = \begin{cases} \frac{1}{2}\eta(x) \prod_{k=0}^{2} (1 - \eta(y + k(y - x))) & \text{if } |y - x| = 1 \text{ and } 2 \mid x, \\ \frac{1}{2}\eta(x) & \text{if } |y - x| = 1 \text{ and } 2 \nmid x, \\ 0 & \text{otherwise.} \end{cases} \tag{4.63}$$

But with the tools of the introductory Section 2.1 at hand, we might as well state the stronger next proposition.

Proposition 4.3.1. *The linear operator L_N in Equation* (4.62) *is a probability generator in the sense of Definition 2.1.3. Furthermore, it gives rise to a Feller process $(\eta_t)_{t\in\mathbb{R}_+}$ with probability semigroup $S_N(t) = \exp(tL_N)$.*

Proof. Since the configuration space contains only elements that have at most one particle per site on the finite torus $\mathbb{T}_N$, we have $|\Omega_N| < \infty$ and one gets a compact (metric) space with respect to the discrete topology. Clearly,

$$||L_N f|| < \infty, \quad \forall f \in C(\Omega_N),$$

such that $\mathcal{D}(L_N) = C(\Omega_N)$. The (finite) jump rates easily meet the conditions of Theorem I.3.9. in [23], which gives a probability semigroup $S_N(t) = \exp(tL_N)$. Theorem A.2.2 then guarantees the existence of an appropriate (quasi-left continuous) Feller process $(\eta_t)_{t\in\mathbb{R}_+}$. $\square$

The path measure with starting configuration η will be denoted by $\mathbb{P}_\eta$ and we write

$$\mathbb{P}_\mu = \sum_{\eta\in\Omega_N} \mu(\{\eta\})\mathbb{P}_\eta \tag{4.64}$$

for the AFP path measure starting from the distribution μ on Ω_N. The dynamics and the state space are well-defined in the sense that

$$L_N(\mathbb{1}_{\Omega_N}) = 0,$$

which implies that once the process starts with a configuration $\eta^0 \in \Omega_N$, it remains there at all later times, i.e. $\eta_t \in \Omega_N, \ \forall t \in \mathbb{R}_+$.

Another feature worth mentioning is that the number of particles is conserved in this model, i.e. there is no sudden „birth“ or „death“ of particles and the number of particles remains constant in time. However, this does not mean that we could simply consider the system for a fixed particle number right away, since the approach of taking the hydrodynamic limit requires the description of the *local* particle evolution, whose density can vary whether or not the global particle ratio is fixed.

[6] We prefer the following notation of $q(\eta, \xi)$ for the rate to go from state η to state ξ, in comparison to the notation $q(x, y, \eta)$ specifically suited for IPS in Chapter 2 to signalise a particle jump from lattice site x to y.

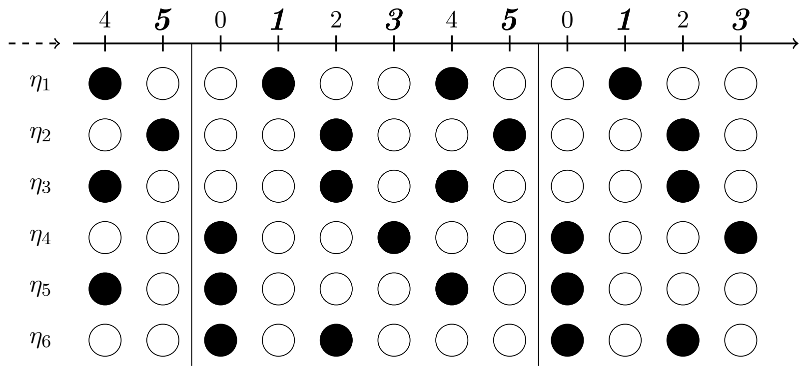

Figure 4.5.: Possible configurations for the AFP in $\Omega_{6,2}$ on the periodic lattice $\mathbb{Z}/6\mathbb{Z}$.

4.3.2. Stationary Measures

The martingale approach relies heavily on knowing the family of stationary measures, which will be derived in this subsection. After that, we will be able to state and proof the hydrodynamic equation.

In contrast to the particle systems that occurred for the curve shortening flow in the Ising model [19], the AFP-process does not possess a stationary measure of product form. At first glance one might think that this characteristic is due to the different behaviour for even and odd sites, but this is not the case (as becomes clear, for example, in [37]). Instead, the particular form of a longer-range exclusion condition is responsible, as can be seen immediately by the identity

$$\mathbb{P}(\{\eta \in \Omega_N \ : \ \eta(x) = 1 \ \wedge \ \eta(x+1) = 1\}) = \mathbb{P}(\emptyset) = 0,$$

for all $x \in \mathbb{T}_N$ and all measures $\mathbb{P}$ on the power set of Ω_N. Consequently an explicit formula for the stationary measure is hard to come by. Instead, we proceed similarly to the Range-r exclusion process.

At first we deal with the case of a fixed number of particles $K \leq \frac{N}{2}$ and fixed $N \in \mathbb{N}$ with $2 \mid N$. Note that the highest particle density can only be achieved when every even site is occupied and every odd site is vacated. Define

$$\Omega_{N,K} := \{\eta \in \Omega_N \ : \ \sum_{x \in \mathbb{T}_N} \eta(x) = K\}$$

as a new state space. As an example consider Figure 4.5, which lists all elements of $\Omega_{6,2}$. The AFP-process on $\Omega_{N,K}$ is implicitly defined by (4.62), with the defining family of probability measures for the Feller process simply being the probability measures on Ω_N conditioned on the subset $\Omega_{N,K}$.

Lemma 4.3.1. *Let $\mu_{N,K}$ be the uniform distribution on $\Omega_{N,K}$ for $0 \leq K \leq \frac{N}{2}$. Then $\mu_{N,K}$ is the unique stationary distribution for the AFP-process and*

$$\mu_{N,K}(\{\eta\}) = \left(\sum_{i=0}^{K \wedge \lfloor \frac{N}{4} \rfloor} \binom{\frac{N}{2} - i}{K} \binom{K}{i} \frac{N}{N-2i} \right)^{-1} \quad \forall \eta \in \Omega_{N,K}. \tag{4.65}$$

Proof. Following the pattern in the proof of Lemma 4.2.1, it is easy to show that

$$\sum_{\eta \in \Omega_{N,K}} \mu_{N,K}(\eta)(L_N f)(\eta) = 0 \quad \forall f : \Omega_{N,K} \to \mathbb{R}$$

due to the reversibility of the jump rates in the sense that a jump $\eta \mapsto \eta^{x,y}$ is allowed if and only if $\eta^{x,y} \mapsto \eta$ is and and those jumps have the same rate. Also, the stochastic process on $\Omega_{N,K}$ is both recurrent and irreducible. Thus, according to Proposition 2.1.1, there is a unique stationary measure up to constant multiples. By construction of $\mu_{N,K}$, we trivially have $\mu_{N,K}(\Omega_{N,K}) = 1$, which makes $\mu_{N,K}$ a unique stationary distribution.

In order to obtain the explicit formula (4.65), we need to know the cardinality of the state space $|\Omega_{N,K}|$. The number of configurations that have $i \leq (\lfloor \frac{N}{4} \rfloor \wedge K)$ particles on odd sites $|\Omega^i_{N,K}| := |\{\eta \in \Omega_{N,K} \ : \ \sum_{x \in \mathbb{T}_{N}, 2 \nmid x} \eta(x) = i\}|$ is the product of possible arrangements of i particles on $\frac{N}{2}$ sites for a R2EP-configuration (see red ellipse with north-east line pattern in Figure 4.6) times the number of possible arrangements of the remaining particles with range 1 (SSEP) on $\frac{N}{2} - 2i$ sites (blue ellipse with crosshatch dots).[7] We get, similarly to Equation (4.10),

$$\begin{aligned} |\Omega^i_{N,K}| &= \left(\binom{\frac{N}{2} - i}{i} + \binom{\frac{N}{2} - i - 1}{i - 1} \right) \cdot \binom{\frac{N}{2} - 2i}{K - i} \\ &= \frac{(\frac{N}{2} - i - 1)! \cdot \frac{N}{2}}{i!(K-i)!(\frac{N}{2} - K - i)!} = \binom{\frac{N}{2} - i}{K} \cdot \binom{K}{i} \cdot \frac{N}{N-2i}. \end{aligned} \tag{4.66}$$

The number of possible configurations with K particles can now be calculated by summing over $0 \leq i \leq (\lfloor \frac{N}{4} \rfloor \wedge K)$, which gives (4.65). $\square$

Applying the formula to our earlier example we see that

$$\mu_{6,2}(\{\eta\}) = \left(\sum_{i=0}^{2 \wedge 1} \binom{3-i}{2} \binom{2}{i} \frac{6}{6-2i} \right)^{-1} = (3+3)^{-1} = \frac{1}{6} \quad \forall \eta \in \Omega_{N,K},$$

[7] There is no need for a case-by-case analysis after placing i particles at odd sites. It always leaves $\frac{N}{2} - 2i$ even sites available for the remaining particles. This number is independent of the exact placement of the odd-site particles since a double exclusion (on sites that are blocked both by a particle present on the left and by a particle on the right) affects only odd sites.

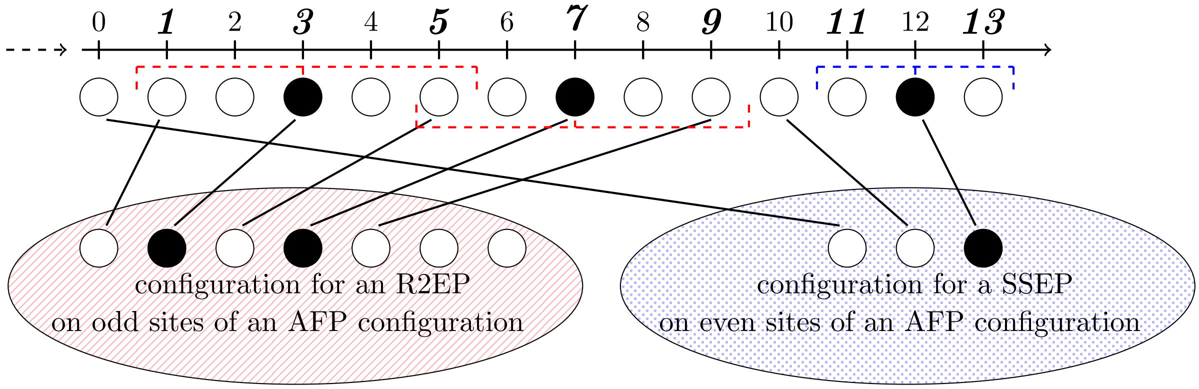

Figure 4.6.: Combinatorial derivation of Equation (4.65). The sites within the red, dashed area cannot be occupied due to a particle's presence at an odd site (3 and 7). The sites within the blue, dashed area cannot be occupied due to the particle at even site 12.

as expected from Figure 4.5, where 3 configurations have no particles on odd sites (corresponding to the index $i = 0$) and 3 configurations have one particle on an odd site ($i = 1$ respectively).

Considering $K \in \{0, 1\}$ particles in the previous lemma leads to obvious degeneracies due to the non-existing exclusion rule. For example, $\mu_{N,1}(\eta) = \frac{1}{N}$ $\forall \eta \in \Omega_{N,1}$, when the single particle has a free choice over the N sites, no matter if even or not. Astonishingly enough, formula (4.65) still remains true.

Not talking about the evolution yet, instead simply asking for the particle density, is an interesting question on its own. Obviously the answer depends on whether we consider even or odd sites. Averaging over two neighboured sites, the particle density should be $\frac{K}{N}$. For the rest of this section, we will use the symbol „$\approx$“ whenever the limits on the left-hand side and right-hand side coincide for $N \to \infty$ with $\frac{K}{N}$ converging to a value within the open interval $(0, \frac{1}{2})$[8]. As usual, the symbol „$\sim$“ represents asymptotic behaviour; e.g. in this context $f(N, K_N) \sim g(N, K_N)$ if and only if $\lim\limits_{N\to\infty} \frac{f(N,K_N)}{g(N,K_N)} = 1$ for sequences (K_N) between 0 and $\frac{N}{4}$.

Lemma 4.3.2. *Let $x \in \mathbb{T}_N$ with $2 \nmid x$ and $K \leq \frac{N}{2}$. Then for the stationary measure $\mu_{N,K}$*

[8]In particular $K \in \Theta(N)$, i.e. there holds both $K \in \mathcal{O}(N)$ and $N \in \mathcal{O}(K)$.

of the AFP-process on $\Omega_{N,K}$

$$\mu_{N,K}(\eta(x)=1) \approx \frac{2\frac{K}{N}(1-2\frac{K}{N})}{1+\sqrt{1-4\frac{K}{N}+8\left(\frac{K}{N}\right)^2}}, \tag{4.67}$$

$$\mu_{N,K}(\eta(x+1)=1) \approx \frac{2\frac{K}{N}\left(2\frac{K}{N}+\sqrt{1-4\frac{K}{N}+8(\frac{K}{N})^2}\right)}{1+\sqrt{1-4\frac{K}{N}+8(\frac{K}{N})^2}}. \tag{4.68}$$

Moreover, the average particle density is given by

$$\frac{1}{2}\Big(\mu_{N,K}(\eta(x)=1)+\mu_{N,K}(\eta(x+1)=1)\Big) \approx \frac{K}{N}. \tag{4.69}$$

Proof. Since the lemma is based on configurations that already have a particle sitting on $x \in \mathbb{T}_N$ ($x+1 \in \mathbb{T}_N$ respectively), we need $K \geq 3$ to be able to state some of the following closed combinatorial formulas. Of course this restriction means no loss of generality for the resulting lemma.

We can derive the number of configurations that have a particle sitting at $x \in \mathbb{T}_N$ in a similar way to $|\Omega_{N,K}|$ in the proof of Lemma 4.3.1, except that we have to count the number of configurations with $K-1$ particles distributed on the remaining $\frac{N}{2}-3$ odd and $\frac{N}{2}-2$ even sites, since the neighbourhood of x (i.e. $x-2, x-1, x, x+1, x+2$) is forced to be vacant. Forcing a number of $0 \leq i \leq ((K-1) \wedge \lfloor \frac{N}{4}-1 \rfloor)$ particles to sit on odd sites outside of the blocked area around $x \in \mathbb{T}_N$, they can be arranged like configurations of a R2EP (except for the restriction to odd sites only). Thus we can apply Equation (4.9) with $i = K-1$ and $\frac{N}{2}$ instead of N, giving

$$\binom{\frac{N}{2}-2-i}{i}$$

possibilities. Then, as every single one of the i particles blocks 2 even sites, there are

$$\binom{\frac{N}{2}-2-2i}{K-i-1}$$

possibilities to arrange the remaining $K-i-1$ particles. Hence,

$$|\{\eta \in \Omega_{N,K} \ : \ \eta(x)=1\}| = \sum_{i=0}^{(K-1)\wedge\lfloor\frac{N}{4}-1\rfloor} \binom{\frac{N}{2}-2-i}{i}\binom{\frac{N}{2}-2-2i}{K-i-1}. \tag{4.70}$$

It is convenient to rewrite the summands such that they include the right-hand side of

(4.66) as a factor. This gives

$$\begin{aligned}
&\sum_{i=0}^{K\wedge\lfloor\frac{N}{4}\rfloor}\binom{\frac{N}{2}-i}{K}\binom{K}{i}\frac{N}{N-2i}\cdot\frac{(1-2\frac{i}{N})(\frac{K}{N}-\frac{i}{N})(\frac{1}{2}-\frac{K}{N}-\frac{i}{N})}{(\frac{1}{2}-\frac{i}{N})(\frac{1}{2}-\frac{i}{N}-\frac{1}{N})}\\
=&\sum_{i=0}^{K\wedge\lfloor\frac{N}{4}\rfloor}|\Omega_{N,K}^{i}|\cdot\frac{(1-2\frac{i}{N})(\frac{K}{N}-\frac{i}{N})(\frac{1}{2}-\frac{K}{N}-\frac{i}{N})}{(\frac{1}{2}-\frac{i}{N})(\frac{1}{2}-\frac{i}{N}-\frac{1}{N})}.
\end{aligned}\tag{4.71}$$

Note that we also adapted the upper bound of the sum's index to fit the bounds for (4.66) with respect to the index i. If $i=\lfloor\frac{N}{4}\rfloor$, the first binomial coefficient in (4.70) is 0 and if $i=K$, the second binomial coefficient is 0, such that we did no harm in adding another summand that is 0. By Stirling's formula $N!=\sqrt{2\pi N}\left(\frac{N}{e}\right)^N\cdot\left(1+\mathcal{O}\left(\frac{1}{N}\right)\right)$ applied to all occurring factorials we get[9] for $K,i\in\mathcal{O}(N)$

$$\begin{aligned}
|\Omega_{N,K}^{i}|=&\frac{(\frac{N}{2}-i)!}{(\frac{N}{2}-i-K)!i!(K-i)!}\cdot\frac{N}{N-2i}\\
=&\frac{1}{2\pi}\sqrt{\frac{\frac{N}{2}-i}{(\frac{N}{2}-i-K)i(K-i)}}\frac{N}{N-2i}\\
&\cdot e^{(\frac{N}{2}-i)\log(\frac{N}{2}-i)-(\frac{N}{2}-i-K)\log(\frac{N}{2}-i-K)-i\log(i)-(K-i)\log(K-i)}\cdot\frac{1+\mathcal{O}\left(\frac{1}{N}\right)}{\left(1+\mathcal{O}\left(\frac{1}{N}\right)\right)^3}\\
=&\frac{1}{2\pi}\frac{1}{N}f\left(\frac{i}{N}\right)e^{-Ng(\frac{i}{N})}\cdot\left(1+\mathcal{O}\left(\frac{1}{N}\right)\right),
\end{aligned}\tag{4.72}$$

where

$$f(y):=\frac{1}{2}\left(\left(\frac{1}{2}-y-\frac{K}{N}\right)\left(\frac{K}{N}-y\right)\left(\frac{1}{2}-y\right)\right)^{-\frac{1}{2}},\tag{4.73}$$

and

$$\begin{aligned}
g(y):=&-\left(\frac{1}{2}-y\right)\log\left(\frac{1}{2}-y\right)+\left(\frac{1}{2}-y-\frac{K}{N}\right)\log\left(\frac{1}{2}-y-\frac{K}{N}\right)\\
&+y\log(y)+\left(\frac{K}{N}-y\right)\log\left(\frac{K}{N}-y\right).
\end{aligned}\tag{4.74}$$

Both functions f and g do not depend on neither N nor K alone, but only on the ratio $\frac{K}{N}$, which converges for $N\to\infty$. Also, $g(y)\le 0$ for $y\in(0,\frac{1}{4}]$ with $g(\frac{1}{4})=0$. Summing over i

[9] Note that while the lemma gives a statement for $K\in\mathcal{O}(N)$, we do not necessarily know at this point that also $i\in\mathcal{O}(N)$ (it will follow from this very lemma). However, as $i\in o(N)$ implies $i\in\mathcal{O}(N)$, the following equality remains true either way.

in Equation (4.72) and using a Riemann-type approximation owing to [27][10] gives

$$|\Omega_{N,K}| = \sum_{i=0}^{K\wedge\lfloor\frac{N}{4}\rfloor} |\Omega^i_{N,K}| \sim \sum_{i=0}^{K\wedge\lfloor\frac{N}{4}\rfloor} \frac{1}{2\pi}\frac{1}{N} f\left(\frac{i}{N}\right) e^{-Ng\left(\frac{i}{N}\right)} \tag{4.75}$$

$$\sim \frac{1}{2\pi} \int_0^{\frac{K}{N}\wedge(\frac{1}{2}-\frac{K}{N})\wedge\frac{1}{4}} f(y)e^{-Ng(y)}\,\mathrm{d}y. \tag{4.76}$$

Technically, we had to add the condition $y \leq (\frac{1}{2} - \frac{K}{N})$ for the integral in (4.75), which was hidden previously in the discrete case, for example in the binomial coefficient $\binom{\frac{N}{2}-i}{K}$. Next, we want to apply *Laplace's method* (cf. Section B.1), which is possible here since $f, g \in \mathcal{O}(1)$. At first we observe for the function $g : (0, \frac{K}{N} \wedge (\frac{1}{2} - \frac{K}{N}) \wedge \frac{1}{4}) \to \mathbb{R}$ that

$$\begin{aligned} & g'(y) = \log\left(\frac{1}{2} - y\right) - \log\left(\frac{1}{2} - y - \frac{K}{N}\right) + \log(y) - \log\left(\frac{K}{N} - y\right) \overset{!}{=} 0 \\ \Leftrightarrow \quad & y^2 - \frac{1}{2}y + \left(\frac{1}{4}\frac{K}{N} - \frac{y^2}{2}\right) = 0, \end{aligned}$$

which gives an extremal value at

$$y_{\min} := \frac{1}{4} - \sqrt{\frac{1}{16} - \frac{1}{4}\frac{K}{N} + \frac{K^2}{2N^2}}, \tag{4.77}$$

with $0 < y_{\min} < \frac{1}{4} \wedge \left(\frac{1}{2} - \frac{K}{N}\right) \wedge \frac{K}{N}$. The inequality

$$g''(y) = \frac{1}{\frac{1}{2} - y - \frac{K}{N}} + \frac{1}{y} + \frac{1}{\frac{K}{N} - y} - \frac{1}{\frac{1}{2} - y} > 0$$

is fulfilled for $0 < y < \frac{1}{4} \wedge (\frac{1}{2} - \frac{N}{K}) \wedge \frac{K}{N}$, so g attains the unique minimum in $y_{\min}$. To continue, it is straightforward to check the prerequisites of Theorem 1 in [27]. In particular, the pair (f, g) is admissible (see Definition 1 therein) by Lemma 1 and the differentiability conditions are easily met.[11] Hence, with Theorem 1 we conclude

$$|\Omega_{N,K}| = \frac{e^{-Ng(y_{\min})}}{\sqrt{2\pi N g''(y_{\min})}} \cdot \left(f(y_{\min}) + \mathcal{O}\left(\frac{1}{N^\beta}\right)\right)\left(1 + \mathcal{O}\left(\frac{1}{N}\right)\right) \tag{4.78}$$

for $\beta \in (0, \frac{1}{2})$.

[10] [27] even considers a more general step size of order $\frac{1}{N^\alpha}$ and shows that the asymptotic behaviour holds whenever the partition is fine enough, i.e. whenever $\alpha > \frac{1}{2}$, which is obviously the case for $\alpha = 1$ in this work.

[11] One reason for the necessity of error estimates for Proposition 4.3.2 is the differentiability condition on the function f.

The same procedure can be applied to the number of configurations that have a particle at $x \in \mathbb{T}_N$. Comparing Equation (4.71) with (4.78), we find that the difference lies in the factor

$$\frac{(1-2\frac{i}{N})(\frac{K}{N}-\frac{i}{N})(\frac{1}{2}-\frac{K}{N}-\frac{i}{N})}{(\frac{1}{2}-\frac{i}{N})(\frac{1}{2}-\frac{i}{N}-\frac{1}{N})} \approx \frac{2(\frac{K}{N}-\frac{i}{N})(\frac{1}{2}-\frac{K}{N}-\frac{i}{N})}{(\frac{1}{2}-\frac{i}{N})} \in \mathcal{O}(1),$$

which enters Laplace's method by defining the new function

$$f^x(y) := f(y) \cdot \frac{2(\frac{K}{N}-y)(\frac{1}{2}-\frac{K}{N}-y)}{(\frac{1}{2}-y)}.$$

The function g in the exponential (and consequently its extreme value) remains the same, though, because g was derived entirely through $|\Omega^i_{N,K}|$ in (4.72). It follows that

$$|\{\eta \in \Omega_{N,K} \ : \ \eta(x)=1\}| = \frac{e^{-Ng(y_{\min})}}{\sqrt{2\pi N g''(y_{\min})}} \cdot \left(f^x(y_{\min}) + \mathcal{O}\left(\frac{1}{N^\beta}\right)\right) \cdot \left(1+\mathcal{O}\left(\frac{1}{N}\right)\right).$$

This means

$$\begin{aligned}
\mu_{N,K}(\eta(x)=1) =& \frac{|\{\eta \in \Omega_{N,K} \ : \ \eta(x)=1\}|}{|\Omega_{N,K}|} = \frac{f^x(y_{\min}) + \mathcal{O}\left(\frac{1}{N^\beta}\right)}{f(y_{\min}) + \mathcal{O}\left(\frac{1}{N^\beta}\right)} \cdot \frac{1+\mathcal{O}\left(\frac{1}{N}\right)}{1+\mathcal{O}\left(\frac{1}{N}\right)} \\
=& \left(\frac{2\frac{K}{N}(1-2\frac{K}{N})}{1+\sqrt{1-4\frac{K}{N}+8\left(\frac{K}{N}\right)^2}} + \mathcal{O}\left(\frac{1}{N^\beta}\right)\right) \cdot \left(1+\mathcal{O}\left(\frac{1}{N}\right)\right) \qquad (4.79) \\
=& \frac{2\frac{K}{N}(1-2\frac{K}{N})}{1+\sqrt{1-4\frac{K}{N}+8\left(\frac{K}{N}\right)^2}} + \mathcal{O}\left(\frac{1}{N^\beta}\right).
\end{aligned}$$

Putting a particle at $x+1 \in \mathbb{T}_N$ (with $2 \mid (x+1)$), both nearest neighbours must stay vacant. The remaining $K-1$ particles can be distributed among $\frac{N}{2}-2$ odd sites and $\frac{N}{2}-1$ even sites, as long as the obtained configuration is in $\Omega_{N,K}$. Similarly to the above for site $x \in \mathbb{T}_N$, we get

$$|\{\eta \in \Omega_{N,K} \ : \ \eta(x+1)=1\}| = \sum_{i=0}^{(K-1)\wedge\lfloor\frac{N}{4}-\frac{1}{2}\rfloor} \binom{\frac{N}{2}-1-i}{i}\binom{\frac{N}{2}-1-2i}{K-i-1}, \qquad (4.80)$$

which can be transformed into the convenient form

$$|\{\eta \in \Omega_{N,K} \ : \ \eta(x+1)=1\}| = \sum_{i=0}^{K\wedge\lfloor\frac{N}{4}\rfloor} |\Omega^i_{N,K}| \cdot \frac{\frac{K}{N}-\frac{i}{N}(2\frac{K}{N}+1)+2\left(\frac{i}{N}\right)^2}{\frac{1}{2}-\frac{i}{N}}. \qquad (4.81)$$

Putting

$$f^{x+1}(y) := f(y) \cdot \frac{\frac{K}{N}-y(2\frac{K}{N}+1)+2y^2}{\frac{1}{2}-y},$$

we have that Equation (4.81) asymptotically behaves like

$$\frac{1}{2\pi} \int_0^{\frac{K}{N} \wedge (\frac{1}{2} - \frac{K}{N}) \wedge \frac{1}{4}} f^{x+1}(y) e^{-Ng(y)} \, \mathrm{d}y. \tag{4.82}$$

The same procedure as above applied to (4.82) and (4.75) leads to

$$\begin{aligned} \mu_{N,K}(\eta(x+1)=1) =& \frac{|\{\eta \in \Omega_{N,K} \ : \ \eta(x)=1\}|}{|\Omega_{N,K}|} \approx \frac{\frac{K}{N} - y_{\min}(2\frac{K}{N}+1) + 2y_{\min}^2}{\frac{1}{2} - y_{\min}} \\ \approx& \frac{\frac{K}{N}\left(4\frac{K}{N} + 2\sqrt{1-4\frac{K}{N}+8(\frac{K}{N})^2}\right)}{1+\sqrt{1-4\frac{K}{N}+8(\frac{K}{N})^2}}. \end{aligned} \tag{4.83}$$

From equations (4.79) and (4.83) we have

$$\begin{aligned} \frac{1}{2}\Big(\mu_{N,K}(\eta(x)=1) + \mu_{N,K}(\eta(x+1)=1)\Big) \approx& \frac{2\frac{K}{N}(1-2\frac{K}{N}) + \frac{K}{N}\left(4\frac{K}{N} + 2\sqrt{1-4\frac{K}{N}+8(\frac{K}{N})^2}\right)}{2\cdot(1+\sqrt{1-4\frac{K}{N}+8(\frac{K}{N})^2})} \\ =& \frac{2\cdot\frac{K}{N}\left(1+\sqrt{1-4\frac{K}{N}+8(\frac{K}{N})^2}\right)}{2\cdot\left(1+\sqrt{1-4\frac{K}{N}+8(\frac{K}{N})^2}\right)} = \frac{K}{N}, \end{aligned} \tag{4.84}$$

as desired.

□

In Figure 4.7 the marginal distributions of the previous lemma are illustrated for $\rho \in (0, \frac{1}{2})$, which corresponds to the particle density $\frac{K}{N} \in (0, \frac{1}{2})$. It becomes obvious that in general particles prefer to sit on even sites. However, for small densities close to 0, there are only few interactions and particles can move rather freely on the lattice. This explains why under the stationary measure, it is just as likely to find a particle on an even site, as it is to find one on an odd site. The opposite is the case for ρ close to the maximum particle density $\frac{1}{2}$. Particles are forced to sit on even sites, as the odd sites have a larger exclusion rule and thus permit fewer particles on the lattice overall. The maximum value on odd sites is achieved for $\rho = \frac{1}{4}$ by $\mu_{N,\frac{N}{4}}(\eta(x)=1) = \frac{1}{4+\sqrt{8}} \approx 0.146$ and there holds the symmetry

$$\mu_{N,\rho N}(\eta(x)=1) = \mu_{N,(0.5-\rho)N}(\eta(x)=1).$$

In order to prove the following lemma, it is convenient to state a combinatorial formula first, which deals with the case of non-periodicity. In principle, this corollary is just a generalisation of previous thoughts in the proof on page 70, that relied on knowledge for the R2EP in form of $|\Omega_{N,K}^{1,x}|$.

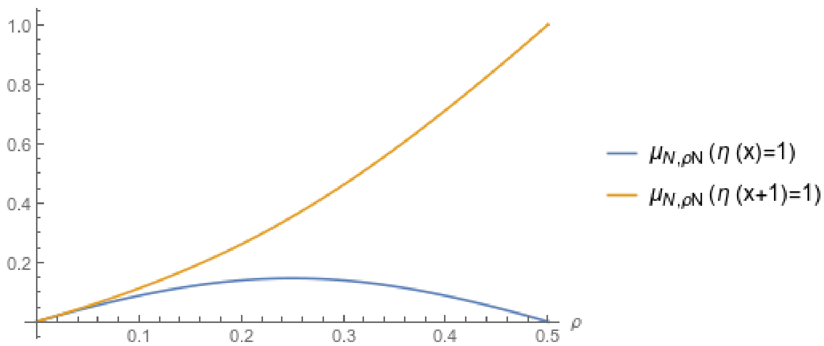

Figure 4.7.: Marginal distributions of the stationary measure $\mu_{N,\rho N}$ for a particle density $\rho \in (0, \frac{1}{2})$ on an odd site (blue, concave function) and an even site (orange, convex function); see Lemma 4.3.2.

Corollary 4.3.1. *Let $\tilde{\mathbb{T}}_{\tilde{N}} := \{0, \dots, \tilde{N}-1\}$ with $2 \nmid \tilde{N}$ be a (non-periodic) lattice for AFP-configurations with $2 \le \tilde{K} \le \frac{\tilde{N}+1}{2}$ particles. Then the number of AFP-configurations is given by*

$$\sum_{j=0}^{\tilde{K} \wedge \frac{\tilde{N}-1}{2}} \binom{\frac{\tilde{N}-1}{2} - j + 1}{j} \binom{\frac{\tilde{N}+1}{2} - 2j}{\tilde{K} - j}. \tag{4.85}$$

Proof. Given $j \le (\tilde{K} \wedge \frac{\tilde{N}-1}{2})$ particles that sit on odd sites of $\tilde{\mathbb{T}}_N$, there are $\binom{\frac{\tilde{N}-1}{2}-j+1}{j}$ possible ways to arrange them. This follows from Equation (4.9) by putting $\frac{\tilde{N}-1}{2} = N - 3$ (since the particle present at $x \in \mathbb{T}_N$ blocks 3 sites in the R2EP setting) and $j = K - 1$. The remaining $\tilde{K} - j$ particles can be distributed in $\binom{\frac{\tilde{N}+1}{2}-2j}{\tilde{K}-j}$ different ways on $\frac{\tilde{N}+1}{2} - 2j$ even sites since a particle at an odd site blocks the 2 neighbouring even sites. □

Contrary to the derivation of the hydrodynamic limit for standard particle systems like the SSEP, it is not enough to know the marginal distributions. We will need additional knowledge of the expectations of the following random variables:

$$\begin{aligned}
\tau_x C_1 :=& \eta(x-4) \cdot \eta(x-1) \cdot \eta(x+1), \\
\tau_x C_2 :=& \eta(x-3) \cdot \eta(x-1) \cdot \eta(x+1), \\
\tau_x C_3 :=& \eta(x-4) \cdot \eta(x-1) \cdot \eta(x+2), \\
\tau_x C_4 :=& \eta(x-3) \cdot \eta(x-1) \cdot \eta(x+2), \\
\tau_x C_5 :=& \eta(x-3) \cdot \eta(x+1), \\
\tau_x C_6 :=& \eta(x-1) \cdot \eta(x+1), \\
\tau_{x+1} D :=& \eta(x-1) \cdot \eta(x+2), \\
\tau_x D :=& \eta(x-2) \cdot \eta(x+1).
\end{aligned} \tag{4.86}$$

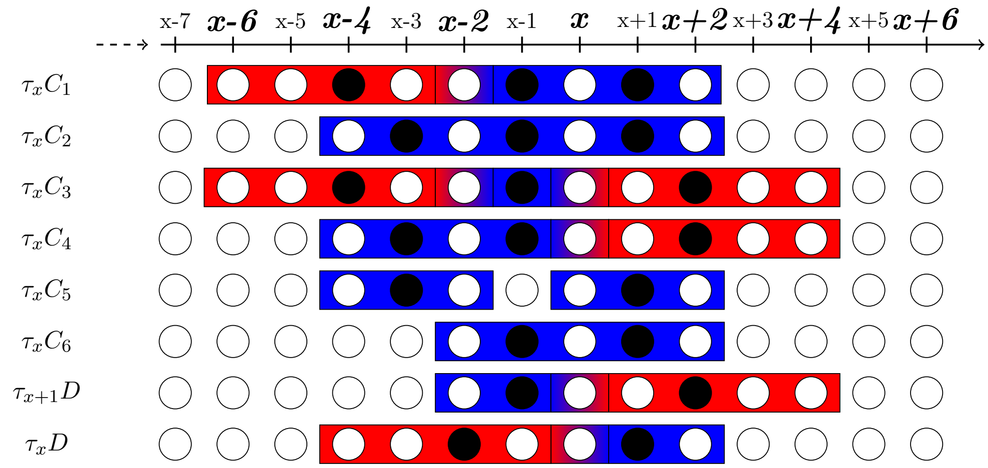

Figure 4.8.: Blocked sites for the crucial events $\tau_x C_1, \dots, \tau_x C_6, \tau_{x+1} D, \tau_x D$ with $2 \nmid x$. The red regions are influenced by the exclusion rule for odd sites, the blue regions are influenced by even sites. Odd site numbers are emphasised in italics and a taller font size.

Even though it is not necessary at this stage, we mention in passing that we will use brackets to indicate the time component, e.g.

$$C_1(t) := \eta_t(-4) \cdot \eta_t(-1) \cdot \eta_t(1). \tag{4.87}$$

The random variables in (4.86) will appear naturally by means of the martingale approach in the following sections. In Figure 4.8 the local configurations around $x \in \mathbb{T}_N$ for $\tau_x C_i = 1$ ($i = 1, \dots, 6$), $\tau_x D = 1$ and $\tau_{x+1} D = 1$ are visualised. It is interesting to note that these configurations are the only necessary cases. For example, we do not need to know about configurations that have 4 particles locally arranged on the lattice.

Unlike Lemma 4.3.2, the measures in the upcoming lemma are stated with $y_{\min}$ from (4.77), since the insertion does not always simplify the formulas.

Lemma 4.3.3. *Let $x \in \mathbb{T}_N$ with $2 \nmid x$ and $K \leq \frac{N}{2}$. Then*

$$\begin{aligned}
\mu_{N,K}(\tau_x C_1 = 1) = \mu_{N,K}(\tau_x C_4 = 1) &\approx \frac{2(\frac{K}{N} - y_{min})^3(\frac{1}{2} - \frac{K}{N} - y_{min})}{(\frac{1}{2} - y_{min})^3}, \\
\mu_{N,K}(\tau_x C_2 = 1) &\approx \frac{2(\frac{K}{N} - y_{min})^3}{(\frac{1}{2} - y_{min})^2}, \\
\mu_{N,K}(\tau_x C_3 = 1) &\approx \frac{2(\frac{K}{N} - y_{min})^3(\frac{1}{2} - \frac{K}{N} - y_{min})^2}{(\frac{1}{2} - y_{min})^4}, \\
\mu_{N,K}(\tau_x C_5 = 1) &\approx \frac{2(\frac{K}{N} - y_{min})^2(\frac{1}{2} - 2y_{min})}{(\frac{1}{2} - y_{min})^2}, \\
\mu_{N,K}(\tau_x C_6 = 1) &\approx \frac{2(\frac{K}{N} - y_{min})^2}{\frac{1}{2} - y_{min}}, \\
\mu_{N,K}(\tau_x D = 1) = \mu_{N,K}(\tau_{x+1} D = 1) &\approx \frac{2(\frac{K}{N} - y_{min})^2(\frac{1}{2} - \frac{K}{N} - y_{min})}{(\frac{1}{2} - y_{min})^2}.
\end{aligned} \tag{4.88}$$

Proof. We put $K \geq 5$ without loss of generality. Both $\tau_x C_1$, $\tau_x C_4$ and $\tau_x D$, $\tau_{x+1} D$ are mirrored versions of one another. The dynamics of the AFP is symmetric, i.e. there is no bias of a particle jump towards either side, so these two pairs must each attain the same value under $\mathbb{E}_{\mu_{N,K}}$.

We can read from the blocked sites in Figure 4.8 that the cardinality

$$|\{\eta \in \Omega_{N,K} \ : \ \tau_x C_1 = 1\}|$$

can be calculated by finding the possible arrangements for $K-3$ particles on the remaining $\frac{N}{2} - 5$ odd and $\frac{N}{2} - 4$ even sites. The particle presences according to $\tau_x C_1 = 1$ undermine the periodicity of the lattice (w.r.t. the combinatorial derivation), such that we can use Corollary 4.3.1 with $i = j$, $N - 9 = \tilde{N}$ (since there are 9 blocked sites for $\tau_x C_1 = 1$) and $K - 3 = \tilde{K}$ (since there are 3 particles in between sites $x - 6$ to $x + 2$) in order to get

$$|\{\eta \in \Omega_{N,K} \ : \ \tau_x C_1 = 1 \ \wedge \sum_{z \in \mathbb{T}_N, \ 2 \nmid z} \eta(z) = i + 1\}| = \binom{\frac{N}{2} - 4 - i}{i}\binom{\frac{N}{2} - 4 - 2i}{K - 3 - i}.$$

The last expression can be rewritten as a product

$$\begin{aligned}
&|\Omega^i_{N,K}| \cdot \frac{(1 - 2\frac{i}{N})(\frac{K}{N} - \frac{i}{N})(\frac{K}{N} - \frac{i}{N} - \frac{1}{N})(\frac{K}{N} - \frac{i}{N} - \frac{2}{N})(\frac{1}{2} - \frac{K}{N} - \frac{i}{N})}{(\frac{1}{2} - \frac{i}{N})(\frac{1}{2} - \frac{i}{N} - \frac{1}{N})(\frac{1}{2} - \frac{i}{N} - \frac{2}{N})(\frac{1}{2} - \frac{i}{N} - \frac{3}{N})} \\
\sim &|\Omega^i_{N,K}| \cdot \underbrace{\frac{2(\frac{K}{N} - \frac{i}{N})^3(\frac{1}{2} - \frac{K}{N} - \frac{i}{N})}{(\frac{1}{2} - \frac{i}{N})^3}}_{=:\bar{f}_{C_1}(\frac{i}{N})},
\end{aligned} \tag{4.89}$$

containing $|\Omega^i_{N,K}|$, of which we have studied the asymptotic behaviour (w.r.t. $\frac{i}{N}$) before, and a function $\bar{f}_{C_1}\left(\frac{i}{N}\right) \in \mathcal{O}(1)$. With Stirling's formula we get similar to the calculations[12] in (4.72)

$$|\{\eta \in \Omega_{N,K} \ : \ \tau_x C_1 = 1 \ \wedge \sum_{z \in \mathbb{T}_N,\ 2 \nmid z} \eta(z) = i+1\}| \sim \frac{1}{2\pi}\frac{1}{N} f_{C_1}\left(\frac{i}{N}\right) e^{-Ng\left(\frac{i}{N}\right)}, \tag{4.90}$$

where

$$f_{C_1}\left(\frac{i}{N}\right) := f\left(\frac{i}{N}\right) \cdot \bar{f}_{C_1}\left(\frac{i}{N}\right)$$

and f, g are the same functions as in (4.73),(4.74). Outside of the sites $x-6$ to $x+2$, there are $\frac{N-10}{2}$ odd sites. Since periodicity plays no role in this case, there can be at most $\lfloor \frac{\frac{N-10}{2}+1}{2} \rfloor = \lfloor \frac{N}{4} - 2 \rfloor$ particles on odd sites outside of the C_1-blocked area. Hence with (4.89) and (4.90)

$$\begin{aligned}
|\{\eta \in \Omega_{N,K} \ : \ \tau_x C_1 = 1\}| &\sim \sum_{i=0}^{(K-3)\wedge \lfloor \frac{N}{4} - 2 \rfloor} |\Omega^i_{N,K}| \cdot \bar{f}_{C_1}\left(\frac{i}{N}\right) \\
&= \sum_{i=0}^{K \wedge \lfloor \frac{N}{4} \rfloor} |\Omega^i_{N,K}| \cdot \bar{f}_{C_1}\left(\frac{i}{N}\right) \\
&\sim \frac{1}{2\pi} \int_0^{\frac{K}{N} \wedge (\frac{1}{2} - \frac{K}{N}) \wedge \frac{1}{4}} f_{C_1}(y) \cdot e^{-Ng(y)} \,\mathrm{d}y,
\end{aligned} \tag{4.91}$$

using Riemann-type approximation [27]. The equality on the second line as well as the additional bound $\frac{1}{2} - \frac{K}{N}$ for the integral's variable follow the arguments on pages 71, 72. Laplace's method shows that the last equation is asymptotically close to

$$\frac{f_{C_1}(y_{\min})}{\sqrt{2\pi N g''(y_{\min})}} e^{-Ng(y_{\min})},$$

which, using Equation (4.78), gives us

$$\mu_{N,K}(\tau_x C_1 = 1) = \frac{|\{\eta \in \Omega_{N,K} \ : \ \tau_x C_1 = 1\}|}{|\Omega_{N,K}|} \approx \bar{f}_{C_1}(y_{\min}) = \frac{2(\frac{K}{N} - y_{\min})^3(\frac{1}{2} - \frac{K}{N} - y_{\min})}{(\frac{1}{2} - y_{\min})^3},$$

as desired.

The cardinality

$$|\{\eta \in \Omega_{N,K} \ : \ \tau_x C_2 = 1\}|$$

[12] For simplicity we skip the error terms here; their behaviour will be just as before.

can be calculated very similarly. We read from Figure 4.8 that $K-3$ particles can be distributed onto $\frac{N}{2}-4$ odd and $\frac{N}{2}-3$ even sites with the appropriate (new) boundary condition of the lattice. The number of those configurations that have i particles on odd sites outside from $x-4$ to $x+2$ is given[13] by

$$|\{\eta \in \Omega_{N,K} \ : \ \tau_x C_2 = 1 \ \wedge \sum_{z \in \mathbb{T}_N,\ 2 \nmid z} \eta(z) = i\}| = \binom{\frac{N}{2}-3-i}{i}\binom{\frac{N}{2}-3-2i}{K-3-i}.$$

The last term can be written as

$$\begin{aligned} &|\Omega^i_{N,K}| \cdot \frac{(N-2i)(K-i)(K-i-1)(K-i-2)}{N(\frac{N}{2}-i)(\frac{N}{2}-i-1)(\frac{N}{2}-i-2)} \\ \sim &|\Omega^i_{N,K}| \cdot \underbrace{\frac{2(\frac{K}{N}-\frac{i}{N})^3}{(\frac{1}{2}-\frac{i}{N})^2}}_{=:\bar{f}_{C_2}(\frac{i}{N})} \sim \frac{1}{2\pi}\frac{1}{N} f_{C_2}\left(\frac{i}{N}\right) e^{-Ng(\frac{i}{N})}, \end{aligned}$$

where $\bar{f}_{C_2}\left(\frac{i}{N}\right) \in \mathcal{O}(1)$ and $f_{C_2}\left(\frac{i}{N}\right) := f\left(\frac{i}{N}\right) \cdot \bar{f}_{C_2}\left(\frac{i}{N}\right)$. Summing over the number of particles on odd sites outside of the area of blocked sites according to $\{\tau_x C_2 = 1\}$, we obtain

$$\begin{aligned} |\{\eta \in \Omega_{N,K} \ : \ \tau_x C_2 = 1\}| \sim &\sum_{i=0}^{(K-3) \wedge \lfloor \frac{N}{4} - \frac{3}{2} \rfloor} |\Omega^i_{N,K}| \cdot \bar{f}_{C_2}\left(\frac{i}{N}\right) \\ \sim &\frac{1}{2\pi} \int_0^{\frac{K}{N} \wedge (\frac{1}{2} - \frac{K}{N}) \wedge \frac{1}{4}} f_{C_2}(y) \cdot e^{-Ng(y)} \,\mathrm{d}y. \end{aligned}$$

At last, Laplace's method leads to

$$\mu_{N,K}(\tau_x C_2 = 1) = \frac{|\{\eta \in \Omega_{N,K} \ : \ \tau_x C_2 = 1\}|}{|\Omega_{N,K}|} \approx \bar{f}_{C_2}(y_{\min}) = \frac{2(\frac{K}{N} - y_{\min})^3}{(\frac{1}{2} - y_{\min})^2}.$$

The same scheme applies to the formulas for $\mu_{N,K}(\tau_x C_3 = 1)$, $\mu_{N,K}(\tau_x C_6 = 1)$ and $\mu_{N,K}(\tau_x D = 1)$, so this part of the proof boils down to

$$\begin{aligned} |\{\eta \in \Omega_{N,K} \ : \tau_x C_3 = 1 \ \wedge &\sum_{z \in \mathbb{T}_N,\ 2 \nmid z} \eta(z) = i+2\}| = \binom{\frac{N}{2}-5-i}{i}\binom{\frac{N}{2}-5-2i}{K-3-i} \\ =&|\Omega^i_{N,K}| \cdot \frac{(N-2i)(K-i)(K-i-1)(K-i-2)(\frac{N}{2}-K-i)(\frac{N}{2}-K-i-1)}{N(\frac{N}{2}-i)(\frac{N}{2}-i-1)(\frac{N}{2}-i-3)(\frac{N}{2}-i-4)} \\ \sim&|\Omega^i_{N,K}| \cdot \underbrace{\frac{2(\frac{K}{N}-\frac{i}{N})^3(\frac{1}{2}-\frac{K}{N}-\frac{i}{N})^2}{(\frac{1}{2}-\frac{i}{N})^4}}_{=:\bar{f}_{C_3}(\frac{i}{N})}, \end{aligned}$$

[13] In the case of $\tau_x C_2$, this actually equals an overall particle number of i on odd sites

for the number of configurations that have particles present at sites $x-4, x-1$ and $x+2$, as well as

$$\begin{aligned}|\{\eta \in \Omega_{N,K} \ : \tau_x C_6 = 1\}| &= \binom{\frac{N}{2}-2-i}{i}\binom{\frac{N}{2}-2-2i}{K-2-i} \\ &= |\Omega^i_{N,K}| \cdot \frac{(N-2i)(K-i)(K-i-1)}{N(\frac{N}{2}-i)(\frac{N}{2}-i-1)} \sim |\Omega^i_{N,K}| \cdot \underbrace{\frac{2(\frac{K}{N}-\frac{i}{N})^2}{(\frac{1}{2}-\frac{i}{N})}}_{=:\bar{f}_{C_6}(\frac{i}{n})},\end{aligned}$$

and

$$\begin{aligned}|\{\eta \in \Omega_{N,K} \ : \tau_x D = 1 \ \wedge \sum_{z\in\mathbb{T}_N,\ 2\nmid z} \eta(z) = i+1\}| &= \binom{\frac{N}{2}-3-i}{i}\binom{\frac{N}{2}-3-2i}{K-2-i} \\ &= |\Omega^i_{N,K}| \cdot \frac{(N-2i)(K-i)(K-i-1)(\frac{N}{2}-K-i)}{N(\frac{N}{2}-i)(\frac{N}{2}-i-1)(\frac{N}{2}-i-2)} \\ &\sim |\Omega^i_{N,K}| \cdot \underbrace{\frac{2(\frac{K}{N}-\frac{i}{N})^2(\frac{1}{2}-\frac{K}{N}-\frac{i}{N})}{(\frac{1}{2}-\frac{i}{N})^2}}_{=:\bar{f}_D(\frac{i}{N})}.\end{aligned}$$

A particular case occurs for

$$|\{\eta \in \Omega_{N,K} \ : \ \tau_x C_5 = 1\}|,$$

as there might be a particle present at the even site $x-1 \in \mathbb{T}_N$, in between blocked sites. Fortunately this issue can be divided into two problems that have been dealt with already. Once again we refer to Figure 4.8 for an illustration. Note that if there is a particle at $x-1$ (together with $\{\tau_x C_5 = 1\}$), we have exactly $\{\tau_x C_2 = 1\}$. If there is no particle at $x-1 \in \mathbb{T}_N$, we have not considered such configurations yet. However, this consideration is not necessary, as it suffices to compare $\{\tau_x C_5 = 1 \ \wedge \eta(x-1) = 0\}$ with $\{\tau_x D = 1\}$. The number of configurations must be the same in both cases, since the blocked area is the same and there are 2 particles within it. Consequently

$$\mu_{N,K}(\tau_x C_5 = 1) = \mu_{N,K}(\tau_x D = 1) + \mu_{N,K}(\tau_x C_2 = 1) \approx \frac{2(\frac{K}{N}-y_{\min})^2(\frac{1}{2}-2y_{\min})}{(\frac{1}{2}-y_{\min})^2}.$$

□

Figure 4.9 shows the values of $\mu_{N,\rho N}$ for the above events. If an event contains a particle on an odd site, e.g. $\{\tau_x C_1 = 1\}$, the probability tends to 0 as the particle density ρ approaches $\frac{1}{2}$, which can be seen in the left graph. The orange function ($\mu_{N,\rho N}(\tau_x C_3 = 1)$) is significantly smaller than all others, owing to the fact that the event $\{\tau_x C_3 = 1\}$ is the

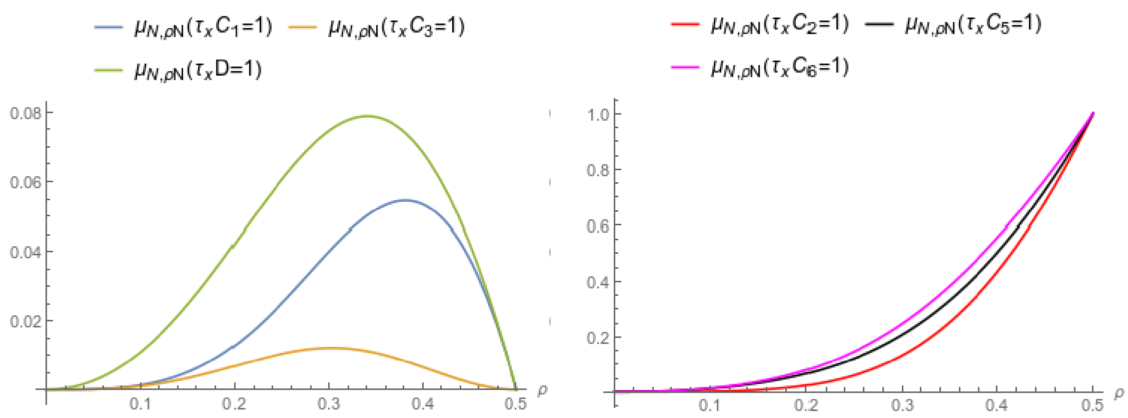

Figure 4.9.: Crucial values of the stationary measure $\mu_{N,\rho N}$ for the AFP; see Lemma 4.3.3.

only one that has 2 particles on odd sites. In the right graph, all particles of the events are on even sites, thus all the measures tend to 1 as $\rho \to \frac{1}{2}$.

It is clear that the procedure in the proof of Lemma 4.3.3 can be generalised to local events whose blocked area is connected. Events like $\{\tau_x C_5 = 1\}$ on the other hand are harder to state. For $x \in \mathbb{T}_N$ define

$$b(x) := \begin{cases} 1 & \text{for } 2 \mid x, \\ 2 & \text{for } 2 \nmid x \end{cases}.$$

Corollary 4.3.2. *Let $C \subset \Omega_{N,K}$ be a subset of the form $\{\eta \in \Omega_{N,K} \ : \ \prod_{i=1}^{n} \eta(x_i) = 1\}$ for some $n \leq K$, $x_i \in \mathbb{T}_N$, and with the property that the blocked sites are connected, i.e. the ordered sites $x_{(1)}, \dots, x_{(n)}$ fulfil*

$$\begin{aligned} &\{x_{(1)} - b(x_{(1)}), \dots, x_{(1)} + b(x_{(1)})\} \cup \dots \cup \{x_{(n)} - b(x_{(n)}), \dots, x_{(n)} + b(x_{(n)})\} \\ =&\{x_{(1)} - b(x_{(1)}), \dots, x_{(n)} + b(x_{(n)})\}. \end{aligned}$$

Then

$$\mu_{N,K}(C) \approx \frac{\left(\frac{K}{N} - y_{min}\right)^n \left(\frac{1}{2} - \frac{K}{N} - y_{min}\right)^{\lfloor \frac{v}{2} \rfloor - n}}{\left(\frac{1}{2} - y_{min}\right)^{\lfloor \frac{v}{2} \rfloor - 1}},$$

where $v := x_{(n)} + b(x_{(n)}) - (x_{(1)} - b(x_{(1)}))$.

Proof. The proof follows the methods from the proof of Lemma 4.3.3. □

As we do not want to be limited to the configuration space $\Omega_{N,K}$ with a fixed particle number K, we define a grand-canonical measure ν_ρ^N on Ω_N by means of

$$\nu_\rho^N := \sum_{K=0}^{\frac{N}{2}} \alpha_{N,K}(\rho) \cdot \mu_{N,K}, \tag{4.92}$$

where the convex combination $\alpha_{N,K}$ is defined by

$$\alpha_{N,K}(\rho) := \binom{\frac{N}{2}}{K} (2\rho)^K (1-2\rho)^{\frac{N}{2}-K}, \tag{4.93}$$

just as for the toy-model RrEP. This definition of $\alpha_{N,K}$ is reasonable in the following sense.

Proposition 4.3.2. *Let $N \in \mathbb{N}$ with $2 \mid N$, $\rho \in (0, \frac{1}{2})$ and $\alpha_{N,K}(\rho)$ defined as in (4.93). Then it holds true that*

$$\sum_{K=0}^{\frac{N}{2}} \alpha_{N,K}(\rho) = 1 \tag{4.94}$$

and

$$\frac{1}{2}\left(\nu_\rho^N((\eta(x)=1) + \nu_\rho^N((\eta(x+1)=1)\right) \to \rho, \quad N \to \infty. \tag{4.95}$$

Proof. The first statement is trivial due to

$$\sum_{K=0}^{\frac{N}{2}} \alpha_{N,K}(\rho) = (2\rho + (1-2\rho))^{\frac{N}{2}}.$$

For the second statement, we need to control the magnitude of the error for $\mu_{N,K}(\eta(x)=1)$ in Lemma 4.3.2. We recall from Equation (4.79)

$$\mu_{N,K}(\eta(x)=1) = \frac{2\frac{K}{N}(1-2\frac{K}{N})}{1+\sqrt{1-4\frac{K}{N}+8\left(\frac{K}{N}\right)^2}} + \mathcal{O}\left(\frac{1}{N^\beta}\right),$$

where $\beta \in (0, \frac{1}{2})$. Thus

$$\begin{aligned}
\nu_\rho^N(\eta(x)=1) &= \sum_{K=0}^{\frac{N}{2}} \alpha_{N,K}(\rho) \cdot \mu_{N,K}(\eta(x)=1) \\
&= \sum_{K=0}^{\frac{N}{2}} \alpha_{N,K}(\rho) \cdot \left(\frac{(2-4\frac{K}{N})\frac{K}{N}}{1+\sqrt{1-4\frac{K}{N}+8\left(\frac{K}{N}\right)^2}} + \mathcal{O}\left(\frac{1}{N^\beta}\right)\right) \\
&\sim \sqrt{\frac{1}{N}} \sum_{K=0}^{\frac{N}{2}} e^{-N\cdot g_\nu\left(\frac{K}{N}\right)} \cdot f_\nu\left(\frac{K}{N}\right),
\end{aligned}$$

with Stirling's approximation[14], where

$$f_\nu\left(\frac{K}{N}\right) := \frac{2\frac{K}{N}(1-2\frac{K}{N})}{\left(1+\sqrt{1-4\frac{K}{N}+8\left(\frac{K}{N}\right)^2}\right)\sqrt{2\pi\frac{K}{N}\left(1-\frac{2K}{N}\right)}},$$

and[15]

$$g_\nu\left(\frac{K}{N}\right) := \frac{K}{N}\log\left(\frac{2K}{N}\right)+\left(\frac{1}{2}-\frac{K}{N}\right)\log\left(1-\frac{2K}{N}\right)-\frac{K}{N}\log(2\rho)-\left(\frac{1}{2}-\frac{K}{N}\right)\log(1-2\rho).$$

The latter fulfils $g_\nu'(\rho)=0$, $g_\nu(\rho)=0$ and $g_\nu''(\rho)=\frac{1}{\rho(1-2\rho)}>0$. Now, without the expression $\mu_{N,K}(\eta(x)=1)$, it is obvious that the function f_ν is differentiable in a neighbourhood of ρ, which is one of the conditions to apply Theorem 1 in [27]. Together with Laplace's method it follows that

$$\begin{aligned}
\nu_\rho^N(\eta(x)=1) \sim& \sqrt{\frac{1}{N}}\cdot N\cdot\int_0^{\frac{1}{2}} e^{-Ng_\nu(y)}\cdot f_\nu(y)\ \mathrm{d}y \sim \sqrt{\frac{1}{N}}e^{-Ng_\nu(\rho)}\cdot f_\nu(\rho)\cdot\sqrt{\frac{2\pi N}{g_\nu''(\rho)}}\\
=& e^{-N\cdot 0}\frac{2(1-2\rho)\rho}{\left(1+\sqrt{1-4\rho+8\rho^2}\right)\cdot\sqrt{2\pi\rho(1-2\rho)}}\cdot\sqrt{2\pi\rho(1-2\rho)} \qquad (4.96)\\
=& \frac{2(1-2\rho)\rho}{1+\sqrt{1-4\rho+8\rho^2}}.
\end{aligned}$$

The same argument works for the even site $x+1\in\mathbb{T}_N$, which gives

$$\nu_\rho^N(\eta(x+1)=1)\sim\frac{2\rho\left(2\rho+\sqrt{1-4\rho+8\rho^2}\right)}{1+\sqrt{1-4\rho+8\rho^2}}.$$

Combined, we get

$$\frac{1}{2}\Big(\nu_\rho^N(\eta(x)=1)+\nu_\rho^N(\eta(x+1)=1)\Big)\to\rho,\quad N\to\infty,$$

basically as in (4.84). □

Finally we are ready to calculate the grand-canonical measure of the crucial events for the AFP. Comparing the upcoming corollary with Lemma 4.3.3, we observe an equivalence of ensembles, which is of course no surprise given our definition of ν_ρ^N.

[14]Note that due to the first statement in the proposition, the term $\mathcal{O}\left(\frac{1}{N^\beta}\right)\cdot\sum_{K=0}^{\frac{N}{2}}\alpha_{N,K}(\rho)$ can be neglected asymptotically

[15]Contrary to the function g in (4.74) that reflected the increasing behaviour (as $N\to\infty$) of $|\Omega_{N,K}^i|$, here there holds $g_\nu(y)\geq 0$ for all $y\in(0,\frac{1}{2})$, i.e. $e^{-Ng_\nu(y)}$ is decreasing because it reflects the convex combination $\alpha_{N,K}(\rho)$.

Corollary 4.3.3. *Let $x \in \mathbb{T}_N$ with $2 \nmid x$, $\rho \in (0, \frac{1}{2})$ and define*

$$\bar{y}_{min} := \frac{1}{4} - \sqrt{\frac{1}{16} - \frac{1}{4}\rho + \frac{1}{2}\rho^2}. \tag{4.97}$$

Then there holds for $N \to \infty$

$$\begin{aligned}
\nu_\rho^N(\tau_x C_1 = 1) = \nu_\rho^N(\tau_x C_4 = 1) &\to \frac{2(\rho - \bar{y}_{min})^3(\frac{1}{2} - \rho - \bar{y}_{min})}{(\frac{1}{2} - \bar{y}_{min})^3}, \\
\nu_\rho^N(\tau_x C_2 = 1) &\to \frac{2(\rho - \bar{y}_{min})^3}{(\frac{1}{2} - \bar{y}_{min})^2}, \\
\nu_\rho^N(\tau_x C_3 = 1) &\to \frac{2(\rho - \bar{y}_{min})^3(\frac{1}{2} - \rho - \bar{y}_{min})^2}{(\frac{1}{2} - \bar{y}_{min})^4}, \\
\nu_\rho^N(\tau_x C_5 = 1) &\to \frac{2(\rho - \bar{y}_{min})^2(\frac{1}{2} - 2\bar{y}_{min})}{(\frac{1}{2} - \bar{y}_{min})^2}, \\
\nu_\rho^N(\tau_x C_6 = 1) &\to \frac{2(\rho - \bar{y}_{min})^2}{\frac{1}{2} - \bar{y}_{min}}, \\
\nu_\rho^N(\tau_x D = 1) = \nu_\rho^N(\tau_{x+1} D = 1) &\to \frac{2(\rho - \bar{y}_{min})^2(\frac{1}{2} - \rho - \bar{y}_{min})}{(\frac{1}{2} - \bar{y}_{min})^2}.
\end{aligned} \tag{4.98}$$

Proof. This follows directly from Lemma 4.3.3 and the proof of Proposition 4.3.2. □

4.3.3. Hydrodynamic Equation

In this subsection we state the main result concerning the AFP, namely its behaviour in the hydrodynamic limit. The corresponding theorem including the hydrodynamic equation can be formulated in various ways, depending for example on the class of permitted starting measures or the type of convergence when the lattice spacing decreases. Having already defined a grand-canonical measure in the previous subsection, we can allow for a class of initial measures that are not too far away from ν_ρ^N (w.r.t. the entropy).

Theorem 4.3.1. *Let $\rho_0 : \mathbb{T} \to [0, \frac{1}{2}]$ be a Lebesgue-integrable function and $(\mu^N)_{N \geq 1}$ be a sequence of probability measures on Ω_N with the following properties:*

1. *$(\mu^N)_{N \geq 1}$ is associated to ρ_0,*
2. *$\exists K \in \mathbb{R},\ 0 < \alpha^* < \frac{1}{2}$ with*
$$H(\mu^N \mid \nu_{\alpha^*}^N) \leq K \cdot N.$$

Then, as $N \to \infty$,

$$\{\pi_{N^2 t}^N \ :\ t \in [0, T]\} \to \{\rho(t, u)\, \mathrm{d}u \ :\ t \in [0, T]\} \quad \text{in distribution,}$$

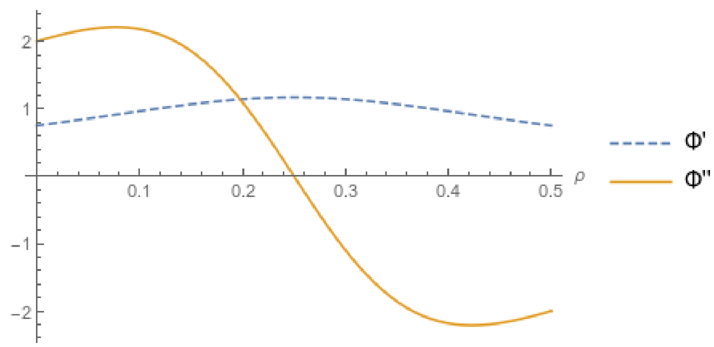

Figure 4.10.: Illustrations of (4.102) for Φ' and Φ'' in dependence of the particle density ρ.

where $\rho(t,u)$ is a solution of the PDE

$$\begin{cases} \partial_t \rho &= \frac{2}{3}\partial_x^2(\Phi(\rho)), \\ \rho(0,u) &= \rho_0(u), \end{cases} \tag{4.99}$$

with

$$\Phi(\rho) := \frac{2\rho\Big(3 - 8\rho + 3c(\rho) - 2\rho \cdot c(\rho) + 16\rho^2 + 8\rho^2 \cdot c(\rho) + 8\rho^3\Big)}{(1+c(\rho))^4}, \tag{4.100}$$

and

$$c(\rho) := \sqrt{1 - 4\rho + 8\rho^2}. \tag{4.101}$$

Properties of the Hydrodynamic Equation for the AFP-process

Compared to the heat equation for SSEP (which naturally appeared for the interface motion in the Ising-model) or the hydrodynamic equation for the RrEP, this PDE looks rather unusual. Still, it possesses all properties that one would expect from our model.

Lemma 4.3.4. *The function Φ has the symmetric property*

$$\partial_x^2 \Phi(\rho) = \partial_x^2 \Phi(\frac{1}{2} - \rho).$$

Proof. There holds

$$\partial_x^2 \Phi(\rho) = \Phi''(\rho) \cdot (\partial_x \rho)^2 + \Phi'(\rho) \cdot \partial_x^2 \rho$$

and

$$\partial_x^2 \Phi(\frac{1}{2} - \rho) = \Phi''(\frac{1}{2} - \rho) \cdot (\partial_x \rho)^2 - \Phi'(\frac{1}{2} - \rho) \cdot \partial_x^2 \rho,$$

such that there is to show (cf. Figure 4.10)

$$\Phi'(\rho) = \Phi'(\frac{1}{2} - \rho) \quad \text{and} \quad \Phi''(\rho) = -\Phi''(\frac{1}{2} - \rho). \tag{4.102}$$

We can write

$$\Phi'(\rho) = 4 \cdot \frac{-32\rho^3 + 32\rho^4 + 3(1 + c(\rho)) - \rho(13 + 7c(\rho)) + 2\rho^2(17 + 7c(\rho))}{c(\rho)(1 + c(\rho))^5},$$

which has the advantage that both numerator and denominator are symmetric around $\frac{1}{4}$. Next, we calculate

$$\Phi''(\rho) = -\frac{16(4\rho - 1)}{(c(\rho))^3(1 + c(\rho))^6} \cdot \Big(4(1 + c(\rho)) + 38\rho^2(2 + c(\rho)) - 8\rho^3(15 + 4c(\rho)) \\ + 8\rho^4(15 + 4c(\rho)) - \rho(23 + 15c(\rho))\Big).$$

This time one can check that the denominator is again axially symmetric around $\frac{1}{4}$, whereas the numerator is point-symmetric around $(\frac{1}{4}, 0)$. We have shown (4.102) and thus proved the lemma. $\square$

Lemma 4.3.4 is in accordance with the interface motion model, as the time evolution of the interface away from the poles should be symmetric around the diagonal from pole to neighbouring pole.

Imposing periodic boundary conditions for ρ and $\partial_x \rho$, we obtain immediately the conservation law of mass

$$\begin{aligned}\partial_t \int_0^1 \rho \,\mathrm{d}u =& \frac{2}{3} \int_0^1 \partial_x^2(\Phi(\rho)) \,\mathrm{d}x \\ =& \frac{8}{3} \partial_x \rho \frac{-32\rho^3 + 32\rho^4 + 3(1 + c(\rho)) - \rho(13 + 7c(\rho)) + 2\rho^2(17 + 7c(\rho))}{c(\rho)(1 + c(\rho))^5} \Bigg|_0^1 \equiv 0,\end{aligned}$$

which means that there is no increase or decrease of the total particle density over time.

Furthermore, there holds a maximum principle according to

$$\inf_{x \in \mathbb{T}} \rho_0(u) \leq \inf_{(t,x) \in \mathbb{R}_+ \times \mathbb{T}} \rho(t, x) \leq \sup_{(t,x) \in \mathbb{R}_+ \times \mathbb{T}} \rho(t, x) \leq \sup_{x \in \mathbb{T}} \rho_0(x), \tag{4.103}$$

such that the initial particle density diffuses over time and the overall minimum/maximum is attained at time 0. In particular this will guarantee uniqueness of weak solutions to the hydrodynamic equation in the proof later on. Owing to the somewhat impractical partial differential equation given by (4.99) - (4.101), we will not show (4.103) by hand, but instead refer to [17] for a general result for quasilinear parabolic equations of second order. An illustration of a solution to the hydrodynamic equation (4.100) with initial condition $\rho_0(x) := \frac{1}{4} + \frac{1}{10} \sin(2\pi x)$ is given in Figure 4.11.

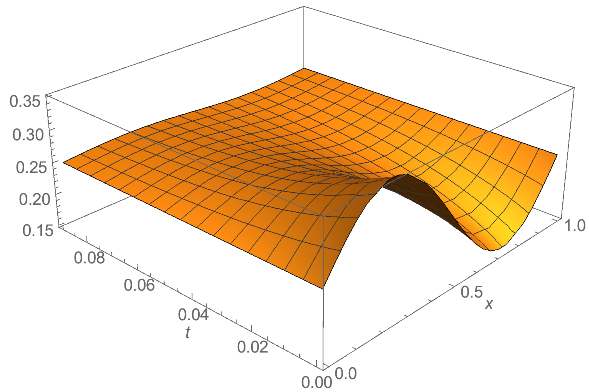

Figure 4.11.: Solution to the hydrodynamic equation (4.99) for the initial density profile $\rho_0(x) := \frac{1}{4} + \frac{1}{10}\sin(2\pi x)$.

4.3.4. Martingale Approach

In this subsection we lay the groundwork for the proof of Theorem 4.3.1. Similarly to the RrEP, we can derive the hydrodynamic equation heuristically by a *local equilibrium ansatz* (cf. Remark 4.2.4) starting from a *martingale approach* for the stochastic process at hand. Along the way, we are confronted with the characteristic difficulty for non-gradient systems, which makes it impossible to directly close the main equation in terms of the empirical measure by applying Varadhan's replacement lemma. However, the problem can be dealt with by means of another martingale equation applied to a particularly chosen function. Mathematical details of this procedure will be delayed to Subsection 4.3.5.

For fixed $z \in \mathbb{T}_N$ we define the function

$$f_z : \Omega_N \to \{0,1\},\ \eta \mapsto \eta(z),$$

that evaluates a configuration in the point z. With the generator L_N from (4.62) we get for $2 \nmid x$

$$\begin{aligned}(L_N f_x)(\eta) = \frac{1}{2}\big(&\eta(x-1) - 2\eta(x) + \eta(x+1) - 2\eta(x-1)\eta(x+1) + \underbrace{\eta(x-1)\eta(x+1)\eta(x+2)}_{=0\ \forall \eta \in \Omega_N} \\ &+ \underbrace{\eta(x+1)\eta(x-1)\eta(x-2)}_{=0\ \forall \eta \in \Omega_N} - \eta(x-1)\eta(x+2) - \eta(x+1)\eta(x-2)\big),\end{aligned} \tag{4.104}$$

and for $2 \mid y$ we have

$$(L_N f_y)(\eta) = \frac{1}{2}\big(\eta(y-1) - 2\eta(y) + \eta(y+1) + \eta(y)\eta(y-2) + \eta(y)\eta(y+2) + \eta(y)\eta(y-3) \\ + \eta(y)\eta(y+3) - \underbrace{\eta(y)\eta(y-2)\eta(y-3)}_{=0\ \forall \eta\in\Omega_N} - \underbrace{\eta(y)\eta(y+2)\eta(y+3)}_{=0\ \forall \eta\in\Omega_N}\big). \tag{4.105}$$

Starting from the martingale (cf. Lemma A.1.1 with $(X_t)_{t\geq 0} := (\eta_t)_{t\geq 0}$ and $F(t, X_t) := \langle \pi_t^N, G\rangle$, as well as Remark A.1.1)

$$\begin{aligned} M_{N^2 t}^{G,N} &= \langle \pi_{N^2 t}^N, G\rangle - \langle \pi_0^N, G\rangle - \int_0^{N^2 t} L_N \langle \pi_s^N, G\rangle \,\mathrm{d}s \\ &= \langle \pi_{N^2 t}^N, G\rangle - \langle \pi_0^N, G\rangle - \int_0^{t} N^2 L_N \langle \pi_{N^2 s}^N, G\rangle \,\mathrm{d}s \end{aligned} \tag{4.106}$$

for a smooth function $G : \mathbb{T} \to \mathbb{R}$, the integrand

$$N^2 L_N \langle \pi_{N^2 s}^N, G\rangle = N^2 L_N \left(\frac{1}{N} \sum_{z\in\mathbb{T}_N} G\left(\frac{z}{N}\right) \eta_{N^2 s}(z) \right)$$

is the focus of attention. Suppressing the time index for the moment, it can be calculated via (4.104) and (4.105) to give

$$\begin{aligned} &N \sum_{x\in T_N,\ 2\nmid x} G\left(\frac{x}{N}\right)(L_N f_x)(\eta) + N \sum_{y\in\mathbb{T}_N,\ 2\mid y} G\left(\frac{y}{N}\right)(L_N f_y)(\eta) \\ =&\frac{N}{2} \sum_{x\in\mathbb{T}_N, 2\nmid x} G\left(\frac{x}{N}\right)\Big(\eta(x-1) - 2\eta(x) + \eta(x+1) - 2\eta(x-1)\eta(x+1) \\ &\qquad - \eta(x-1)\eta(x+2) - \eta(x+1)\eta(x-2)\Big) \\ &+ \frac{N}{2} \sum_{x\in\mathbb{T}_N, 2\nmid x} G\left(\frac{x+1}{N}\right)\Big(\eta(x+2) + \eta(x) - 2\eta(x+1) + \eta(x+1)\eta(x-1) \\ &\qquad + \eta(x+1)\eta(x+3) + \eta(x+1)\eta(x-2) + \eta(x+1)\eta(x+4) \\ &\qquad - \eta(x+1)\eta(x-1)\eta(x-2) - \eta(x+1)\eta(x+3)\eta(x+4)\Big). \end{aligned}$$

This expression can be partially expressed by means of the discrete Laplacian (cf. (4.21)) and we obtain

$$\begin{aligned}
&\frac{1}{2N}\sum_{z\in\mathbb{T}_N}\Delta_N G(\frac{z}{N})\eta(z)+\frac{1}{2N}\sum_{x\in\mathbb{T}_N,2\nmid x}\eta(x-1)\eta(x+1)\Delta_N G\left(\frac{x}{N}\right)\\
&+\underbrace{\frac{N}{2}\sum_{x\in\mathbb{T}_N,2\nmid x}\eta(x-1)\eta(x+2)\big(G\left(\frac{x-1}{N}\right)-G\left(\frac{x}{N}\right)\big)}_{=\sum_{x\in\mathbb{T}_N,2\nmid x}\eta(x-1)\eta(x+2)\left(-G'\left(\frac{x}{N}\right)\frac{1}{N}+G''\left(\frac{x}{N}\right)\frac{1}{2N^2}+\mathcal{O}(\frac{1}{N^3})\right)}\\
&+\underbrace{\frac{N}{2}\sum_{x\in\mathbb{T}_N,2\nmid x}\eta(x-2)\eta(x+1)\big(G\left(\frac{x+1}{N}\right)-G\left(\frac{x}{N}\right)\big)}_{=\sum_{x\in\mathbb{T}_N,2\nmid x}\eta(x-2)\eta(x+1)\left(G'\left(\frac{x}{N}\right)\frac{1}{N}+G''\left(\frac{x}{N}\right)\frac{1}{2N^2}+\mathcal{O}(\frac{1}{N^3})\right)}.
\end{aligned}$$

Thus, combined with (4.27), $N^2L_N\langle\pi^N_{N^2s},G\rangle$ equals

$$\begin{aligned}
&\frac{1}{2N}\sum_{x\in\mathbb{T}_N}G''\left(\frac{x}{N}\right)\eta_{N^2s}(x)+\frac{1}{2N}\sum_{x\in\mathbb{T}_N,2\nmid x}G''\left(\frac{x}{N}\right)\Big(\eta_{N^2s}(x-1)\eta_{N^2s}(x+1)\\
&\qquad\qquad+\frac{1}{2}\eta_{N^2s}(x-1)\eta_{N^2s}(x+2)+\frac{1}{2}\eta_{N^2s}(x-2)\eta_{N^2s}(x+1)\Big)\\
&\underbrace{-\frac{1}{2}\sum_{x\in\mathbb{T}_N,2\nmid x}G'\left(\frac{x}{N}\right)\Big(\eta_{N^2s}(x-1)\eta_{N^2s}(x+2)-\eta_{N^2s}(x-2)\eta_{N^2s}(x+1)\Big)}_{=:H^{G'}_{\eta_{N^2s}}}+\mathcal{O}\left(\frac{1}{N}\right).
\end{aligned}\tag{4.107}$$

In comparison to the SSEP or RrEP, at this point there is an additional difficulty due to the term $H^{G'}_{\eta_{N^2s}}$. The index of the sum runs through the odd integers, i.e. makes steps of size 2, whereas with the local function $D(\eta):=\eta(-2)\eta(1)$ every summand in $H^{G'}_{\eta_{N^2s}}$ is only a difference of a „1-step“ translation of local functions, i.e.

$$H^{G'}_{\eta_{N^2s}}=\frac{1}{2}\sum_{x\in\mathbb{T}_N,2\nmid x}G'\left(\frac{x}{N}\right)(\tau_{x+1}D(\eta_{N^2s})-\tau_xD(\eta_{N^2s})).$$

This means that another summation by parts (in order to get $G''(\frac{x}{N})$) is not directly applicable here and it seems as if $H^{G'}_{\eta_{N^2s}}\in\mathcal{O}(N)$, which is precisely the difficulty for non-gradient systems. This means that the next step needs some further preparation.
We will follow a method which was applied in [37], where the author was confronted with a similar difficulty. The idea is to replace the term $\tau_xD(\eta)-\tau_{x+1}D(\eta)=\eta(x-2)\eta(x+1)-\eta(x-1)\eta(x+2)$ by a generator of a certain function and some „step-2“ differences.

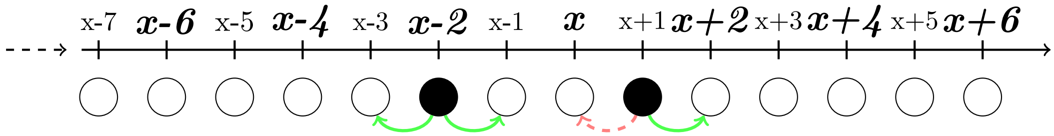

Figure 4.12.: Particle jumps for a configuration $\eta \in \Omega_N$ with $\eta(x-2) = \eta(x+1) = 1$. The jump depicted by the red (dotted) arrow is not allowed under the dynamics and therefor does not lead to a change of the function $\bar{f}_x$. The jumps from $x-2$ to $x-1/x+1$ are always allowed, whereas the jump from $x+1$ to $x+2$ depends on $\eta(x+3)$ and $\eta(x+4)$ according to the exclusion rule.

Therefore we define the function

$$\bar{f}_x(\eta) := \begin{cases} 1 & \text{for } \eta(x-2) = \eta(x+1) = 1, \\ -1 & \text{for } \eta(x-1) = \eta(x+2) = 1, \\ 0 & \text{else} \end{cases}$$

for $x \in \mathbb{T}_N$ odd. Loosely speaking, there are (seemingly) 16 possibilities of how $\bar{f}_x$ can change its value in an infinitesimal time slot. For example, given a configuration $\eta \in \Omega_N$ with $\eta(x-2) = \eta(x+1) = 1$ (implying $\bar{f}_x(\eta) = 1$), the function $\bar{f}_x$ would decrease by 1 after a jump of either the particle at site $x-2$ or $x+1$ to the left/right neighbouring site. However in this case, since the jump from site $x+1$ to x is not allowed under the dynamics, there remain only 3 (instead of 4) possible jumps that lead to a new configuration $\eta^{\text{new}} \in \Omega_N$ (cf. Figure 4.12). Thus, with a rate of $q(\eta, \eta^{\text{new}})$, we have $(\bar{f}_x(\eta^{\text{new}}) - \bar{f}_x(\eta)) = -1$. On the other hand, in the case that only one of the sites $x-2, x+1$ is occupied, there are 4 (in fact only 3, for the same reason as above) possible jumps that might lead to $\eta(x-2) = \eta(x+1) = 1$ and thus $(\bar{f}_x(\eta^{\text{new}}) - \bar{f}_x(\eta)) = 1$ with appropriate jump rates. The same method could be applied for the sites $x-1$ and $x+2$, leading to (at most) 8 further changes of $\bar{f}_x$ in an infinitesimal time slot, but of course it is much more convenient to use the symmetry at this point. With that in mind, we apply the generator L_N from (4.62) and find that $(L_N \bar{f}_x)(\eta)$ equals

$$\begin{aligned}
&\frac{1}{2}\eta(x+1)\eta(x-1)(1-\eta(x-3))(1-\eta(x-4)) + \frac{1}{2}\eta(x+1)\eta(x-3)(1-\eta(x-1)) \\
&+\frac{1}{2}\eta(x-2)\eta(x+2) - \eta(x+1)\eta(x-2) - \frac{1}{2}\eta(x-2)\eta(x+1)(1-\eta(x+3))(1-\eta(x+4)) \\
&-\Big(\frac{1}{2}\eta(x+2)\eta(x-2) + \frac{1}{2}\eta(x-1)\eta(x+1)(1-\eta(x+3))(1-\eta(x+4)) \\
&+\frac{1}{2}\eta(x-1)\eta(x+3)(1-\eta(x+1)) - \frac{1}{2}\eta(x+2)\eta(x-1)(1-\eta(x-3))(1-\eta(x-4)) \\
&-\eta(x-1)\eta(x+2)\Big),
\end{aligned}$$

which can be written as

$$\frac{1}{2}\Big((\tau_{x+2}C_1-\tau_xC_1)+2(\tau_{x+2}C_2-\tau_xC_2)+(\tau_{x+2}C_3-\tau_xC_3)+(\tau_{x+2}C_4-\tau_xC_4)$$
$$-(\tau_{x+2}C_5-\tau_xC_5)+3\underbrace{(\tau_{x+1}D(\eta)-\tau_xD(\eta))}_{=-\bar f_x(\eta)}\Big) \tag{4.108}$$

with the definitions from (4.86). Note that $L_N\bar{f}_x(\eta)$ contains $\bar{f}_x(\eta)$ and we can write

$$\bar f_x(\eta_{N^2s})=-\frac{1}{3}\big(2L_N(\bar f_x\eta_{N^2s})-(\tau_{x+2}C_1-\tau_xC_1)-2(\tau_{x+2}C_2-\tau_xC_2)-(\tau_{x+2}C_3-\tau_xC_3)$$
$$-(\tau_{x+2}C_4-\tau_xC_4)+(\tau_{x+2}C_5-\tau_xC_5)\big).$$

We proceed by replacing the term $\tau_{x+1}D(\eta)-\tau_xD(\eta)$ appearing in $H^{G'}_{\eta_{N^2s}}$ by means of Equation (4.108). Defining

$$F(s,\eta_s):=\frac{1}{N^2}\sum_{x\in\mathbb{T}_N,2\nmid x}G'\left(\frac{x}{N}\right)\bar f_x(\eta_s) \tag{4.109}$$

in (A.2), we recognise that this would be indeed an improvement in view of Lemma A.1.1, as

$$M^F_{N^2t}:=\frac{1}{N^2}\sum_{x\in\mathbb{T}_N,2\nmid x}G'\left(\frac{x}{N}\right)\bar f_x(\eta_{N^2t})-\frac{1}{N^2}\sum_{x\in\mathbb{T}_N,2\nmid x}G'\left(\frac{x}{N}\right)\bar f_x(\eta_0)$$
$$-\underbrace{\int_0^{N^2t}L_N\Big(\frac{1}{N^2}\sum_{x\in\mathbb{T}_N,2\nmid x}G'\left(\frac{x}{N}\right)\bar f_x(\eta_s)\Big)\,\mathrm{d}s}_{=\int_0^t\sum_{x\in\mathbb{T}_N,2\nmid x}G'\left(\frac{x}{N}\right)L_N\left(\bar f_x(\eta_{N^2s})\right)\mathrm{d}s} \tag{4.110}$$

is a martingale and taking the expectation with respect to the path measure $\mathbb{P}_{\mu^N}$ (recall our notation (4.64)) in (4.110) gives

$$\int_0^t\sum_{x\in\mathbb{T}_N,2\nmid x}G'(\frac{x}{N})\mathbb{E}_{\mathbb{P}_{\mu^N}}\big((L_N\bar f_x)(\eta_{N^2s})\big)\,\mathrm{d}s$$
$$=\mathbb{E}_{\mathbb{P}_{\mu^N}}\left(\frac{1}{N^2}\sum_{x\in\mathbb{T}_N,2\nmid x}G'\left(\frac{x}{N}\right)\bar f_x(\eta_{N^2t})-\frac{1}{N^2}\sum_{x\in\mathbb{T}_N,2\nmid x}G'\left(\frac{x}{N}\right)\bar f_x(\eta_0)\right) \tag{4.111}$$
$$=\frac{1}{N}\Big(\frac{1}{N}\sum_{x\in\mathbb{T}_N,2\nmid x}G'\left(\frac{x}{N}\right)\mathbb{E}_{\mathbb{P}_{\mu^N}}(\bar f_x(\eta_0)-\bar f_x(\eta_{N^2t}))\Big)\in\mathcal{O}\left(\frac{1}{N}\right),$$

where we used that $M_0^F = 0$ and thus

$$\mathbb{E}_{\mathbb{P}_{\mu^N}}(M_t^F) = 0 \quad \forall t \in \mathbb{R}_+$$

by the martingale properties. In other words, the expectation of the generator term disappears for $N \to \infty$. Hence, in spirit of a local equilibrium ansatz as for the RrEP, we take the expectation in (4.106)

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}_{\mu^N}} \langle \pi^N_{N^2 t}, G\rangle &= \mathbb{E}_{\mathbb{P}_{\mu^N}} \langle \pi^N_0, G\rangle + \int_0^t \frac{1}{2N} \sum_{x\in\mathbb{T}_N} G''\left(\frac{x}{N}\right) \mathbb{E}_{\mathbb{P}_{\mu^N}}(\eta_{N^2 s}(x))\,\mathrm{d}s \\
&+ \int_0^t \frac{1}{2N} \sum_{x\in\mathbb{T}_N, 2\nmid x} G''\left(\frac{x}{N}\right) \mathbb{E}_{\mathbb{P}_{\mu^N}}\Big(\eta_{N^2 s}(x-1)\eta_{N^2 s}(x+1) + \frac{1}{2}\eta_{N^2 s}(x-1)\eta_{N^2 s}(x+2) \\
&\qquad + \frac{1}{2}\eta_{N^2 s}(x-2)\eta_{N^2 s}(x+1)\Big)\,\mathrm{d}s \\
&+ \int_0^t \frac{1}{6} \sum_{x\in\mathbb{T}_N, 2\nmid x} G'\left(\frac{x}{N}\right) \mathbb{E}_{\mathbb{P}_{\mu^N}}\Big((\tau_{x+2}C_1(N^2 s) - \tau_x C_1(N^2 s)) + 2(\tau_{x+2}C_2(N^2 s) \\
&\qquad - \tau_x C_2(N^2 s)) + (\tau_{x+2}C_3(N^2 s) - \tau_x C_3(N^2 s)) + (\tau_{x+2}C_4(N^2 s) \\
&\qquad - \tau_x C_4(N^2 s)) - (\tau_{x+2}C_5(N^2 s) - \tau_x C_5(N^2 s))\Big)\,\mathrm{d}s + \mathcal{O}\left(\frac{1}{N}\right)
\end{aligned}$$

and after another summation by parts we get

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}_{\mu^N}} \langle \pi^N_{N^2 t}, G\rangle &= \mathbb{E}_{\mathbb{P}_{\mu^N}} \langle \pi^N_0, G\rangle + \int_0^t \frac{1}{2N}\Bigg(\sum_{x\in\mathbb{T}_N} G''(\frac{x}{N})\mathbb{E}_{\mathbb{P}_{\mu^N}}(\eta_{N^2 s}(x)) \\
&+ \sum_{x\in\mathbb{T}_N, 2\nmid x} G''(\frac{x}{N})\mathbb{E}_{\mathbb{P}_{\mu^N}}\Big(-\frac{2}{3}\tau_x C_1(N^2 s) - \frac{4}{3}\tau_x C_2(N^2 s) - \frac{2}{3}\tau_x C_3(N^2 s) \\
&- \frac{2}{3}\tau_x C_4(N^2 s) + \frac{2}{3}\tau_x C_5(N^2 s) + \underbrace{\eta_s(x-1)\eta_s(x+1)}_{=:\tau_x C_6(N^2 s)} + \frac{1}{2}\underbrace{\eta_s(x-1)\eta_s(x+2)}_{=\tau_{x+1}D(N^2 s)} \\
&+ \frac{1}{2}\underbrace{\eta_s(x-2)\eta_s(x+1)}_{=\tau_x D(N^2 s)}\Big)\Bigg)\,\mathrm{d}s + \mathcal{O}\left(\frac{1}{N}\right).
\end{aligned} \tag{4.112}$$

Note that an additional factor of 2 came in, owing to the step size of $\frac{2}{N}$

$$\frac{N}{2}(G'(\frac{x+2}{N}) - G'(\frac{x}{N})) = G''(\frac{x}{N}) + \mathcal{O}\left(\frac{1}{N}\right).$$

With this approach, we are able to predict the hydrodynamic equation. Assume for the purpose of the remaining subsection that the N^2-accelerated empirical measures converge to an absolutely continuous measure with density $\rho(t,u)$. Also, for all local, bounded functions $F:\{0,1\}^{\mathbb{Z}} \to \mathbb{R}$ and the grand-canonical measure ν_ρ^N as given in (4.92), we assume for the family of starting measures $(\mu^N)_{N\geq 1}$

$$\lim_{N\to\infty} \mathbb{E}_{\tau_{\lfloor uN\rfloor}\mu^N}(F(\eta)) = \lim_{N\to\infty} \mathbb{E}_{\nu^N_{\rho_0(u)}}(F(\eta)) \tag{4.113}$$

for all continuity points u of an initial density profile $\rho_0 : \mathbb{T} \to \mathbb{R}_+$. Furthermore, this property shall be conserved for the function $\rho : \mathbb{R}_+ \times \mathbb{T} \to \mathbb{R}_+$ in the sense that

$$\lim_{N\to\infty} \mathbb{E}_{S_N(N^2t)\tau_{\lfloor uN\rfloor}\mu^N}(F(\eta)) = \lim_{N\to\infty} \mathbb{E}_{\nu^N_{\rho(t,u)}}(F(\eta)) \tag{4.114}$$

for all $t \geq 0$ and all continuity points u of $\rho(t,\cdot)$. We have

$$\mathbb{E}_{\mathbb{P}_{\mu^N}}(\eta_0(x)) = \mathbb{E}_{\mathbb{P}_{\tau_x\mu^N}}(\eta_0(0)) = \mathbb{E}_{\tau_x\mu^N}(\eta(0)),$$

such that, given the local equilibrium property (4.113), there holds

$$\begin{aligned}
&\lim_{N\to\infty}\left(\mathbb{E}_{\mathbb{P}_{\mu^N}}\langle \pi_0^N, G\rangle\right) = \lim_{N\to\infty}\left(\frac{1}{N}\sum_{x\in\mathbb{T}_N} G\left(\frac{x}{N}\right)\mathbb{E}_{\mathbb{P}_{\mu^N}}(\eta_0(x))\right)\\
&= \lim_{N\to\infty}\left(\frac{1}{N}\sum_{x\in\mathbb{T}_N, 2\nmid x} G\left(\frac{x}{N}\right)\left(\mathbb{E}_{\tau_x\mu^N}(\eta(0)) + \mathbb{E}_{\tau_x\mu^N}(\eta(1))\right) + \mathcal{O}\left(\frac{1}{N}\right)\right)\\
&= \lim_{N\to\infty}\left(\frac{1}{N}\sum_{x\in\mathbb{T}_N, 2\nmid x} G\left(\frac{x}{N}\right)\underbrace{\left(\nu^N_{\rho_0\left(\frac{x}{N}\right)}(\{\eta(0)=1\}) + \nu^N_{\rho_0\left(\frac{x}{N}\right)}(\{\eta(1)=1\})\right)}_{\to 2\rho_0\left(\frac{x}{N}\right) \text{ by (4.95) in Proposition 4.3.2}} + \mathcal{O}\left(\frac{1}{N}\right)\right)\\
&= \int_{\mathbb{T}} G(u)\rho_0(u)\,\mathrm{d}u.
\end{aligned} \tag{4.115}$$

Analogously, as

$$\mathbb{E}_{\mathbb{P}_{\mu^N}}(\eta_{N^2t}(x)) = \mathbb{E}_{S_N(N^2t)\tau_x\mu^N}(\eta(0)),$$

we proceed for $\mathbb{E}_{\mathbb{P}_{\mu^N}}\langle \pi^N_{N^2t}, G\rangle$, getting

$$\lim_{N\to\infty} \mathbb{E}_{\mathbb{P}_{\mu^N}}\langle \pi^N_{N^2t}, G\rangle = \int_{\mathbb{T}} G(u)\rho(t,u)\,\mathrm{d}u, \tag{4.116}$$

due to the conservation assumption in (4.114). Generally, the same procedure works for the other terms like $-\frac{2}{3}\mathbb{E}_{\mathbb{P}_{\mu^N}}(\tau_x C_1(N^2 s))$ in Equation (4.112), e.g.

$$\begin{aligned}\lim_{N\to\infty} \mathbb{E}_{\mathbb{P}_{\mu^N}}(\tau_x C_1(N^2 s)) = \lim_{N\to\infty} \mathbb{E}_{S_N(N^2 s)\tau_x\mu^N}(C_1) &\overset{(4.114)}{=} \lim_{N\to\infty} \mathbb{E}_{\nu^N_{\rho\left(s,\frac{x}{N}\right)}}(C_1)\\ &= \lim_{N\to\infty} \nu^N_{\rho\left(s,\frac{x}{N}\right)}(C_1 = 1),\end{aligned}$$

with the additional step of applying the results from Corollary 4.3.3 at this point. Altogether, we can take the limit $N \to \infty$ on both sides of Equation (4.112) and we obtain that the particle density function ρ should fulfil the PDE

$$\partial_t \rho = \frac{1}{2}\partial_x^2(\bar{\Phi}(\rho)), \tag{4.117}$$

with

$$\begin{aligned}\bar{\Phi}(\rho) := \rho + \frac{1}{2}\Bigg(&- \frac{8}{3}\frac{(\rho-\bar{y}_{\min})^3(\frac{1}{2}-\rho-\bar{y}_{\min})}{(\frac{1}{2}-\bar{y}_{\min})^3} - \frac{4}{3}\frac{(\rho-\bar{y}_{\min})^3(\frac{1}{2}-\rho-\bar{y}_{\min})^2}{(\frac{1}{2}-\bar{y}_{\min})^4}\\ &+ \frac{4}{3}\Big(\frac{(\rho-\bar{y}_{\min})^2(\frac{1}{2}-\rho-\bar{y}_{\min})}{(\frac{1}{2}-\bar{y}_{\min})^2} + \frac{(\rho-\bar{y}_{\min})^3}{(\frac{1}{2}-\bar{y}_{\min})^2}\Big) + \frac{2(\rho-\bar{y}_{\min})^2}{\frac{1}{2}-\bar{y}_{\min}}\\ &- \frac{8}{3}\frac{(\rho-\bar{y}_{\min})^3}{(\frac{1}{2}-\bar{y}_{\min})^2} + \frac{2(\rho-\bar{y}_{\min})^2(\frac{1}{2}-\rho-\bar{y}_{\min})}{(\frac{1}{2}-\bar{y}_{\min})^2}\Bigg).\end{aligned} \tag{4.118}$$

and $\bar{y}_{\min}$ as in (4.97). The factor $\frac{1}{2}$ in front of the big bracket in (4.118) comes from the fact, that the Riemann-sum is taken over points with distance $\frac{2}{N}$. However, Equation (4.117) (and Equation (4.118) respectively) can be simplified to the form which is also stated in Theorem 4.3.1, namely

$$\partial_t \rho = \frac{2}{3}\partial_x^2(\Phi(\rho)),$$

where

$$\Phi(\rho) = \frac{2\rho\Big(3 - 8\rho + 3c(\rho) - 2\rho\cdot c(\rho) + 16\rho^2 + 8\rho^2\cdot c(\rho) + 8\rho^3\Big)}{(1+c(\rho))^4},$$

and

$$c(\rho) = \sqrt{1 - 4\rho + 8\rho^2}.$$

The last function is an axially symmetric function attaining its minimum value $\sqrt{\frac{1}{2}}$ for the particle density $\rho = \frac{1}{4}$.

4.3.5. Proof of Theorem 4.3.1

The general outline of the proof follows the methods from [14], which have been applied also in Subsection 4.2.4. The emphasis lies on dealing with the terms that are of non-gradient nature.

For simplicity, we mainly adopt the notation from the RrEP. As mentioned earlier, $\mathbb{P}$ now denotes the path measure for the AFP-process. Q^N will be the pushforward measure of $\mathbb{P}_{\mu^N}$ (the path measure starting from the distribution μ^N on Ω_N) under $\pi^N := \{\pi^N_{N^2t} \ : \ t \in [0,T]\}$, i.e. it is the measure on $D([0,T],\mathcal{M}_+)$ corresponding to the N^2-accelerated process $\pi^N_{N^2t}$. Based on Q^N, we define another measure for a function $G : \mathbb{T} \to \mathbb{R}$ by

$$Q^{N,G}(A) = Q^N(\{(\pi_t)_{t\in[0,T]} \in D([0,T],\mathcal{M}_+) \ : \ (\langle \pi_t, G\rangle)_{t\in[0,T]} \in A\})$$

for measurable sets $A \subset D([0,T],\mathbb{R})$.

We will show that the sequence $(Q^N)_{N\geq 1}$ converges to the Dirac-measure concentrated on the deterministic path $\{\rho(t,u)\,\mathrm{d}u,\ 0 \leq t \leq T\}$, where $\rho(t,u)$ satisfies

$$\int_{\mathbb{T}} G(u)\rho(t,u)\,\mathrm{d}u = \int_{\mathbb{T}} G(u)\rho_0(u)\,\mathrm{d}u + \frac{2}{3}\int_0^t \int_{\mathbb{T}} \partial_x^2 G(u)\cdot\Phi(\rho(s,u))\,\mathrm{d}u\,\mathrm{d}s, \tag{4.119}$$

for all $t \in [0,T]$ and with Φ defined as in (4.100). This implies Equation (C.2) by partial integration thanks to the boundary conditions, which means that ρ is a weak solution of the hydrodynamic equation. We will prove that

1. $(Q^N)_{N\geq 1}$ is relatively compact,
2. subsequences $(Q^{N_k})_{k\geq 1}$ converge towards a unique measure concentrated on the solution of (4.119) and
3. there is a unique weak solution to (4.99).

The first step, i.e. showing relative compactness of $(Q^N)_{N\geq 1}$ on $D([0,T],\mathcal{M}_+)$, can be simplified thanks to Proposition B.2.2. Therefor, we consider the class of twice continuously differentiable functions $G : \mathbb{R} \to \mathbb{R}$ with period 1, which will be denoted by $C^2(\mathbb{T})$, as before. Considering the real-valued process $(\langle \pi^N_{N^2t}, G\rangle)_{t\in[0,T]}$, we just have to prove for every $G \in C^2(\mathbb{T})$ that $(Q^{N,G})_{N\geq 1}$ is relatively compact in $D([0,T],\mathbb{R})$.
We take advantage of Prohorov's Theorem and verify Condition (B.3) therein directly, since $|\langle \pi^N_t, G\rangle|$ is bounded (cf. page 51). Regarding the second condition (B.4) in Prohorov's Theorem, we will show Aldous' condition regarding the oscillations in (B.5) instead, as it is easier to handle. Our intermediate goal thus becomes to control

$$\begin{aligned} &Q^{N,G}\left(\left|\langle \pi^N_{N^2(\tau+\theta)}, G\rangle - \langle \pi^N_{N^2\tau}, G\rangle\right| > \varepsilon\right) \\ \leq &Q^{N,G}\left(\left|M^{G,N}_{N^2(\tau+\theta)} - M^{G,N}_{N^2\tau}\right| > \frac{\varepsilon}{2}\right) + Q^{N,G}\left(\left|\int_\tau^{\tau+\theta} N^2 L_N\langle \pi^N_{N^2s}, G\rangle\,\mathrm{d}s\right| > \frac{\varepsilon}{2}\right), \end{aligned} \tag{4.120}$$

where we recall that τ is a bounded stopping time and $\theta \leq \gamma$ with $\gamma \to 0$ eventually. This goal can be achieved by controlling the first and second summand in the last line of (4.120) separately. Starting with the former, it is enough to show convergence of the second moment of $M^{G,N}_{N^2 t}$ due to Chebyshev's inequality. During the proof of the hydrodynamic equation for the RrEP, we have spent a big amount of time on showing

$$\mathbb{E}_{Q^{N,G}}[(M^{G,N}_{N^2(\tau+\theta)} - M^{G,N}_{N^2\tau})^2] \to 0, \quad \text{for } N \to \infty,$$

whereas the treatment of the second summand in (4.120) went rather smoothly. This time, the opposite is the case, as the second summand becomes much harder to handle due to the non-gradient nature of the AFP-process, and for the first summand, we benefit from our previous work. Taking the expectation in Equation (A.3), we know with Lemma A.1.1, as N^F_t is a martingale, that there holds

$$\mathbb{E}_{Q^{N,G}}\left(\left(M^{G,N}_{N^2 t}\right)^2\right) = \int_0^t N^2 \mathbb{E}_{Q^{N,G}}\left(L_N\left(\langle \pi^N_{N^2 s}, G\rangle^2\right) - 2\langle \pi^N_{N^2 s}, G\rangle L_N\left(\langle \pi^N_{N^2 s}, G\rangle\right)\right) \mathrm{d}s. \tag{4.121}$$

We proceed similarly to page 53 and get with rates $q(\eta, \eta^{x,y})$ as in (4.63) that $N^2 L_N\left(\langle \pi^N_{N^2 s}, G\rangle^2\right)$ equals

$$\sum_{\substack{x,y\in\mathbb{T}_N,\\ |x-y|=1}} q(\eta,\eta^{x,y})\left(G\left(\frac{y}{N}\right) - G\left(\frac{x}{N}\right)\right)^2 + 2N^2\langle \pi^N_{N^2 s}, G\rangle L_N\left(\langle \pi^N_{N^2 s}, G\rangle\right).$$

Since

$$N^2\left(G\left(\frac{y}{N}\right) - G\left(\frac{x}{N}\right)\right)^2 = \left(G'\left(\frac{x}{n}\right)\right)^2 + \mathcal{O}\left(\frac{1}{N}\right)$$

we conclude that (4.121) vanishes for $N \to \infty$, which implies for every $\varepsilon > 0$

$$\lim_{\gamma\to 0}\limsup_{N\to\infty}\sup_{\tau\in\mathcal{T}_T,\theta\leq\gamma} Q^{N,G}\left(\left|M^{G,N}_{N^2(\tau+\theta)} - M^{G,N}_{N^2\tau}\right| > \frac{\varepsilon}{2}\right) = 0.$$

Next, concerning the second summand in the bottom line of (4.120), we need to control the troublesome term $\sum_{x\in\mathbb{T}_N, 2\nmid x} G'\left(\frac{x}{N}\right) L_N \bar{f}_x(\eta_{N^2 s})$ that appeared in the martingale approach as a summand (times constant) of $H^{G'}_{\eta_{N^2 s}}$, which itself appeared in the term $N^2 L_N\langle \pi^N_{N^2 s}, G\rangle$. For

$$F(s,\eta_s) := \frac{1}{N^2}\sum_{x\in\mathbb{T}_N, 2\nmid x} G'\left(\frac{x}{N}\right)\bar{f}_x(\eta_s) \tag{4.122}$$

we obtain the new martingale

$$
\begin{aligned}
\bar{M}^{G',N}_{N^2t} :=& F(N^2t,\eta_{N^2t}) - F(0,\eta_0) - \int_0^{N^2t} L_N F(s,\eta_s)\,\mathrm{d}s + \mathcal{O}\left(\frac{1}{N}\right) \\
=& \underbrace{F(N^2t,\eta_{N^2t}) - F(0,\eta_0)}_{\in\mathcal{O}\left(\frac{1}{N}\right)} - \int_0^t \sum_{x\in\mathbb{T}_N, 2\nmid x} G'\left(\frac{x}{N}\right) L_N \bar{f}_x(\eta_{N^2s})\,\mathrm{d}s + \mathcal{O}\left(\frac{1}{N}\right),
\end{aligned}
\tag{4.123}
$$

with $\mathbb{E}_{\mathbb{P}_{\mu^N}}\left(\left(\bar{M}^{G',N}_{N^2t}\right)^2\right)$ given by

$$
\int_0^t \mathbb{E}_{\mathbb{P}_{\mu^N}}\left(N^2 L_N\left(\left(F(N^2s,\eta_{N^2s})\right)^2\right) - 2N^2 F(N^2s,\eta_{N^2s}) L_N\left(F(N^2s,\eta_{N^2s})\right)\right)\mathrm{d}s. \tag{4.124}
$$

For the first summand in the integral of (4.124) there holds

$$
N^2 L_N\left(\left(F(N^2s,\eta_{N^2s})\right)^2\right) = \frac{1}{N^2} L_N\left(\left(\sum_{z\in\mathbb{T}_N, 2\nmid z} G'\left(\frac{z}{N}\right)\bar{f}_z(\eta_{N^2s})\right)^2\right) \in \mathcal{O}\left(\frac{1}{N}\right),
$$

by a straightforward calculation similar to (4.40), using

$$
\begin{aligned}
& L_N\left(\left(\sum_{z\in\mathbb{T}_N, 2\nmid z} G'\left(\frac{z}{N}\right)\bar{f}_z(\eta_{N^2s})\right)^2\right) \\
&= \sum_{\substack{x,y\in\mathbb{T}_N,\\ |x-y|=1}} q(\eta,\eta^{x,y})\left(\left(\sum_{z\in\mathbb{T}_N, 2\nmid z} G'\left(\frac{z}{N}\right)\bar{f}_z(\eta^{x,y})\right)^2 - \left(\sum_{z\in\mathbb{T}_N, 2\nmid z} G'\left(\frac{z}{N}\right)\bar{f}_z(\eta)\right)^2\right) \\
&= \sum_{\substack{x,y\in\mathbb{T}_N,\\ |x-y|=1}} q(\eta,\eta^{x,y})\left(\sum_{z\in\mathbb{T}_N, 2\nmid z} G'\left(\frac{z}{N}\right)\left(\bar{f}_z(\eta^{x,y}) + \bar{f}_z(\eta)\right)\right)\left(\sum_{z\in\mathbb{T}_N, 2\nmid z} G'\left(\frac{z}{N}\right)\left(\bar{f}_z(\eta^{x,y}) - \bar{f}_z(\eta)\right)\right)
\end{aligned}
$$

where the last sum (in large brackets) of the last line simplifies to a term in $\mathcal{O}(1)$. For the second summand in (4.124) we get

$$
\begin{aligned}
&\int_0^t \mathbb{E}_{\mathbb{P}_{\mu^N}}\left[-2N^2 F(N^2 s,\eta_{N^2 s})\cdot L_N\left(F(N^2 s,\eta_{N^2 s})\right)\right]\mathrm{d}s\\
=&\frac{-2}{N^2}\int_0^t \mathbb{E}_{\mathbb{P}_{\mu^N}}\left[\left(\sum_{x\in\mathbb{T}_N,2\nmid x} G'\left(\frac{x}{N}\right)\bar{f}_x(\eta_{N^2 s})\right)\left(\sum_{x\in\mathbb{T}_N,2\nmid x} G'\left(\frac{x}{N}\right)L_N\left(\bar{f}_x(\eta_{N^2 s})\right)\right)\right]\mathrm{d}s\\
=&\frac{-2}{N^2}\int_0^t \mathbb{E}_{\mathbb{P}_{\mu^N}}\Bigg[\left(\sum_{x\in\mathbb{T}_N,2\nmid x} G'\left(\frac{x}{N}\right)\bar{f}_x(\eta_{N^2 s})\right)\Bigg(\frac{1}{2}\sum_{x\in\mathbb{T}_N,2\nmid x} G'\left(\frac{x}{N}\right)\Big((\tau_{x+2}C_1(N^2 s)\\
&\quad-\tau_x C_1(N^2 s))+2(\tau_{x+2}C_2(N^2 s)-\tau_x C_2(N^2 s))+(\tau_{x+2}C_3(N^2 s)-\tau_x C_3(N^2 s))\\
&\quad+(\tau_{x+2}C_4(N^2 s)-\tau_x C_4(N^2 s))-(\tau_{x+2}C_5(N^2 s)-\tau_x C_5(N^2 s))\Big)\\
&\quad-\frac{3}{2}\sum_{x\in\mathbb{T}_N,2\nmid x} G'\left(\frac{x}{N}\right)\bar{f}_x(\eta_{N^2 s})\Bigg)\Bigg]\mathrm{d}s.
\end{aligned}
$$

After a summation by parts for the sum with gradients over two lattice sites, the last equation becomes

$$
\begin{aligned}
&\frac{-2}{N^2}\int_0^t \mathbb{E}_{\mathbb{P}_{\mu^N}}\Bigg[-\frac{3}{2}\left(\sum_{x\in\mathbb{T}_N,2\nmid x} G'\left(\frac{x}{N}\right)\bar{f}_x(\eta_{N^2 s})\right)^2+\left(\sum_{x\in\mathbb{T}_N,2\nmid x} G'\left(\frac{x}{N}\right)\bar{f}_x(\eta_{N^2 s})\right)\\
&\quad\cdot\left(\frac{1}{N}\sum_{x\in\mathbb{T}_N,2\nmid x} G''\left(\frac{x}{N}\right)(-C_1(N^2 s)-2C_2(N^2 s)-C_3(N^2 s)-C_4(N^2 s)+C_5(N^2 s))\right)\\
&\quad+\left(\sum_{x\in\mathbb{T}_N,2\nmid x} G'\left(\frac{x}{N}\right)\bar{f}_x(\eta_{N^2 s})\right)\cdot\mathcal{O}\left(\frac{1}{N}\right)\Bigg]\mathrm{d}s.
\end{aligned}
$$

Here, the second line is of order $\mathcal{O}(1)$, thus the whole summand (the one which contains the term in the second line) disappears for $N\to\infty$, given the prefactor $\frac{-2}{N^2}$. The same holds of course for the third summand. The only clarification is necessary for the term

$$
3\int_0^t \mathbb{E}_{\mathbb{P}_{\mu^N}}\left[\left(\frac{1}{N}\sum_{x\in\mathbb{T}_N,2\nmid x} G'\left(\frac{x}{N}\right)\bar{f}_x(\eta_{N^2 s})\right)^2\right]\mathrm{d}s. \tag{4.125}
$$

Fortunately the replacement lemma due to Guo, Papanicolaou and Varadhan [13] helps us out here. At first, note that there holds (assuming for simplicity $\varepsilon N \in \mathbb{N}$)

$$\frac{1}{N} \sum_{x \in \mathbb{T}_N, 2 \nmid x} G'\left(\frac{x}{N}\right) \bar{f}_x(\eta_{N^2 s}) = \frac{1}{N} \sum_{x \in \mathbb{T}_N, 2 \nmid x} \frac{1}{\varepsilon N + 1} \sum_{\substack{y \in \mathbb{T}_N, 2 \nmid y \\ |y-x| \le \varepsilon N}} G'\left(\frac{y}{N}\right) \bar{f}_y(\eta_{N^2 s})$$

and that a first order Taylor expansion of G' in $\frac{x}{N}$ further gives

$$\frac{1}{N} \sum_{x \in \mathbb{T}_N, 2 \nmid x} G'\left(\frac{x}{N}\right) \frac{1}{\varepsilon N + 1} \sum_{\substack{y \in \mathbb{T}_N, 2 \nmid y \\ |y-x| \le \varepsilon N}} \bar{f}_y(\eta_{N^2 s}) + \mathcal{O}(\varepsilon). \tag{4.126}$$

The replacement lemma[16] now allows to replace the local averages of $\bar{f}_y(\eta_{N^2 s})$ by an expected value with respect to the stationary measure with parameter given by the local density. More precisely we have from (4.126) (cf. also equation (4.8) in [37]) and with the same notation as for the RrEP on page 58 that

$$3 \int_0^t \mathbb{E}_{\mathbb{P}_{\mu^N}} \left[\left(\frac{1}{N} \sum_{x \in \mathbb{T}_N, 2 \nmid x} G'\left(\frac{x}{N}\right) \bar{f}_x(\eta_{N^2 s}) - \frac{1}{N} \sum_{x \in \mathbb{T}_N, 2 \nmid x} G'\left(\frac{x}{N}\right) \mathbb{E}_{\nu_{\langle \pi^N_{N^2 s}, \iota_\varepsilon \rangle}} (\bar{f}_x(\eta_{N^2 s})) \right)^2 \right] \mathrm{d}s$$

converges to 0 for $N \to \infty$ and then $\varepsilon \to 0$. However, independently of whatever the actual value of $\langle \pi^N_{N^2 s}, \iota_\varepsilon \rangle$ is, we have

$$\mathbb{E}_{\nu_{\langle \pi^N_{N^2 s}, \iota_\varepsilon \rangle}} (\bar{f}_x(\eta_{N^2 s})) = 0$$

owing to the symmetry of $\tau_x D$ and $\tau_{x+1} D$ already mentioned at the beginning of the proof to Lemma 4.3.3. Hence the somewhat troublesome expression (4.125) vanishes for increasing N, and so does

$$\mathbb{E}_{\mathbb{P}_{\mu^N}} \left(\left(\bar{M}^{G',N}_{N^2 t} \right)^2 \right) \to 0, \quad N \to \infty.$$

Bearing this in mind, we go back to the initial martingale representation in (4.106). The analysis of the drift term $N^2 L_N \langle \pi^N_{N^2 s}, G \rangle$ led to sums including G'' plus the term $H^{G'}_{\eta_{N^2 s}}$, which can be represented now with the help of $\bar{M}^{G',N}_{N^2 t}$. All things combined, we have

$$\begin{aligned} \langle \pi^N_{N^2 t}, G \rangle - \langle \pi^N_0, G \rangle = \int_0^t & \left(\frac{1}{2N} \sum_{x \in \mathbb{T}_N} G''\left(\frac{x}{N}\right) \eta_{N^2 s}(x) + \frac{1}{2N} \sum_{x \in \mathbb{T}_N, 2 \nmid x} G''\left(\frac{x}{N}\right) \bar{h}_x(\eta_{N^2 s}) \right) \mathrm{d}s \\ & + M^{G,N}_{N^2 t} - \bar{M}^{G',N}_{N^2 s} + \mathcal{O}\left(\frac{1}{N}\right) \end{aligned}$$

[16] Compared to the statement (4.49) we just need a weaker version (containing the expected value), since the term we are considering originated from $\mathbb{E}_{\mathbb{P}_{\mu^N}} \left(\left(\bar{M}^{G',N}_{N^2 t} \right)^2 \right)$.

with

$$\bar{h}_x := \tau_x C_6 + \frac{1}{2}\tau_{x+1} D + \frac{1}{2}\tau_x D - \frac{2}{3}\tau_x C_1 - \frac{4}{3}\tau_x C_2 - \frac{2}{3}\tau_x C_3 - \frac{2}{3}\tau_x C_4 + \frac{2}{3}\tau_x C_5,$$

and we are finally able to control the second summand in the bottom line of (4.120), i.e. we have for fixed $G \in C^2(\mathbb{T})$

$$\begin{aligned}
&\mathbb{P}_{\mu^N}\left(\left|\int_\tau^{\tau+\theta} N^2 L_N \langle \pi^N_{N^2 s}, G\rangle \,\mathrm{d}s\right| > \frac{\varepsilon}{2}\right) \\
\leq &\mathbb{P}_{\mu^N}\left(\left|\int_\tau^{\tau+\theta}\left(\frac{1}{2N}\sum_{x\in\mathbb{T}_N} G''\left(\frac{x}{N}\right)\eta_{N^2 s}(x) + \frac{1}{2N}\sum_{x\in\mathbb{T}_N, 2\nmid x} G''\left(\frac{x}{N}\right)\bar{h}_x(\eta_{N^2 s})\right)\right| > \frac{\varepsilon}{4}\right) \\
&+ \mathbb{P}_{\mu^N}\left(\left|\bar{M}^{G',N}_{N^2(\tau+\theta)} - \bar{M}^{G',N}_{N^2\tau}\right| > \frac{\varepsilon}{4}\right).
\end{aligned}$$

With Chebyshev's inequality there holds for every $\varepsilon > 0$

$$\lim_{\gamma\to 0}\limsup_{N\to\infty}\sup_{\tau\in\mathcal{T}_T,\theta\leq\gamma}\mathbb{P}_{\mu^N}\left(\left|\bar{M}^{G',N}_{N^2(\tau+\theta)} - \bar{M}^{G',N}_{N^2\tau}\right| > \frac{\varepsilon}{4}\right) = 0,$$

as well as with $D := ||G''||_\infty$

$$\begin{aligned}
&\mathbb{P}_{\mu^N}\left(\left|\int_\tau^{\tau+\theta}\left(\frac{1}{2N}\sum_{x\in\mathbb{T}_N} G''\left(\frac{x}{N}\right)\eta_{N^2 s}(x) + \frac{1}{2N}\sum_{x\in\mathbb{T}_N, 2\nmid x} G''\left(\frac{x}{N}\right)\bar{h}_x(\eta_{N^2 s})\right)\right| > \frac{\varepsilon}{4}\right) \\
\leq &\mathbb{P}_{\mu^N}\left(\frac{1}{2}\theta D > \frac{\varepsilon}{4}\right) \underset{\theta\leq\gamma}{=} \mathbb{P}_{\mu^N}\left(\gamma D > \frac{\varepsilon}{2}\right) \to 0, \quad \text{for } \gamma \to 0,
\end{aligned}$$

since the function $\bar{h}_x$ is bounded for all $\eta \in \Omega_N$. By definition of $Q^{N,G}$, this implies Aldous' sufficient condition for the second condition in Prohorov's Theorem about relative compactness. We have shown

$$\lim_{\gamma\to 0}\limsup_{N\to\infty}\sup_{\tau\in\mathcal{T}_T,\theta\leq\gamma} Q^{N,G}\left(\left|\langle \pi^N_{N^2(\tau+\theta)}, G\rangle - \langle \pi^N_{N^2\tau}, G\rangle\right| > \varepsilon\right) = 0.$$

Concerning the second step of the proof, i.e. uniqueness of converging subsequences $(Q^{N_k})_{k\geq 1}$, a major part has already been done either during this proof or for the RrEP on pages 55 onwards. Let Q^{N_k} be a subsequence converging to a limit point Q^*.
We recall that $p_t : D([0,T],\mathcal{M}_+) \to \mathcal{M}_+$ denotes the projection-map onto time $t \in [0,T]$ and $Q^*_{p_t}$ then is the pushforward measure of Q^* with respect to p_t. At first we will show that $Q^*_{p_t}$ is

concentrated on absolutely continuous measures. For trajectories $(\pi_t)_{t\in[0,T]} \in D([0,T],\mathcal{M}_+)$, there holds

$$\sup_{t\in[0,T]} |\langle \pi_t, G\rangle| \leq \frac{1}{N_k}\sum_{x\in\mathbb{T}_{N_k}} |G(\frac{x}{N_k})| \quad Q^{N_k} - \text{a.s.}. \tag{4.127}$$

Furthermore, we obtain with the Portmanteau-Theorem for closed sets

$$\begin{aligned} 1 &\overset{(4.45)}{=} \limsup_{k\to\infty} Q^{N_k}(\{(\pi_t)_{t\in[0,T]} \in D([0,T],\mathcal{M}_+) \ : \ \sup_{t\in[0,T]} |\langle \pi_t, G\rangle| \leq \int_{\mathbb{T}} |G(u)|\,\mathrm{d}u\}) \\ &\leq Q^*(\{(\pi_t)_{t\in[0,T]} \in D([0,T],\mathcal{M}_+) \ : \ \sup_{t\in[0,T]} |\langle \pi_t, G\rangle| \leq \int_{\mathbb{T}} |G(u)|\,\mathrm{d}u\}). \end{aligned}$$

Thus Q^* is concentrated on trajectories $(\pi_t)_{t\in[0,T]} \in D([0,T],\mathcal{M}_+)$ such that

$$\sup_{t\in[0,T]} |\langle \pi_t, G\rangle| \leq \int_{\mathbb{T}} |G(u)|\,\mathrm{d}u,$$

which implies by the monotone-class theorem that, for every $t \in [0,T]$, Q^* is concentrated on paths which are absolutely continuous with respect to the Lebesgue-measure. We will write $\rho(s,u)$ for the associated density at time s.
The first assumption in Theorem 4.3.1, regarding the starting measures $(\mu^N)_{N\geq 1}$, guarantees that Q^* is concentrated on paths that at time 0 have density ρ_0. This can be seen by

$$\begin{aligned} &Q^*\Big(\Big\{(\pi_t)_{t\in[0,T]} \in D([0,T],\mathcal{M}_+) \ : \ \Big|\int_{\mathbb{T}} G(u)\pi_0(\mathrm{d}u) - \int G(u)\rho_0(u)\,\mathrm{d}u\Big| > \varepsilon\Big\}\Big) \\ &\leq \liminf_{k\to\infty} Q^{N_k}\Big(\Big|\int_{\mathbb{T}} G(u)\pi_0(\mathrm{d}u) - \int G(u)\rho_0(u)\,\mathrm{d}u\Big| > \varepsilon\Big) \\ &= \liminf_{k\to\infty} Q^{N_k}\Big(\Big|\frac{1}{N_k}\sum_{x\in\mathbb{T}_{N_k}} G\left(\frac{x}{N_k}\right)\eta_0(x) - \int G(u)\rho_0(u)\,\mathrm{d}u\Big| > \varepsilon\Big) \\ &= \lim_{k\to\infty} \mu^{N_k}\Big(\Big|\frac{1}{N_k}\sum_{x\in\mathbb{T}_{N_k}} G\left(\frac{x}{N_k}\right)\eta(x) - \int G(u)\rho_0(u)\,\mathrm{d}u\Big| > \varepsilon\Big) = 0. \end{aligned}$$

Now, fix a smooth function $G : [0,T]\times\mathbb{T} \to \mathbb{R}$ fulfilling $G(t,0) = G(t,1)$ as well as $\partial_x G(t,0) = \partial_x G(t,1)$ for all $t \in [0,T]$. We will show that Q^* is concentrated on trajectories such that

$$\begin{aligned} \langle \pi_t, G\rangle &= \langle \pi_0, G\rangle + \int_0^t \int_{\mathbb{T}} \left(\rho\cdot\partial_s G + \frac{1}{2}\bar{\Phi}(\rho)\cdot\partial_x^2 G\right)\mathrm{d}u\,\mathrm{d}s, \\ \langle \pi_0, G\rangle &= \int_{\mathbb{T}} G(0,u)\cdot\rho_0(u)\,\mathrm{d}u, \end{aligned} \tag{4.128}$$

with $\bar{\Phi}(\rho)$ as in (4.118). By Lemma A.1.1 applied to

$$F(s,\eta_s) := \frac{1}{N_k} \sum_{x\in\mathbb{T}_{N_k}} G\left(\frac{s}{N_k^2}, \frac{x}{N_k}\right) \eta_s(x),$$

we know that

$$M^{G,N_k}_{N_k^2 t} = \langle \pi^{N_k}_{N_k^2 t}, G(t,\cdot)\rangle - \langle \pi^{N_k}_0, G(0,\cdot)\rangle - \int_0^t (\partial_s + N_k^2 L_{N_k}) \langle \pi^{N_k}_{N_k^2 s}, G(s,\cdot)\rangle \,\mathrm{d}s, \tag{4.129}$$

is another martingale. For the second moment of this martingale the additional time dependency (in comparison to the prior treatment (4.121) in step 1 of this proof) does not play any role, as it does not enter the martingale N^F from (A.3). Thus, just as before on page 96, we have

$$\lim_{k\to\infty} \mathbb{E}_{Q^{N_k}} \left(\left(M^{G,N_k}_{N_k^2 t}\right)^2 \right) = 0$$

for all $t \in [0,T]$ and in particular for $t = T$. We get for every $\delta > 0$

$$\begin{aligned} Q^{N_k}\Big(\sup_{0\le t\le T} |M^{G,N_k}_{N_k^2 t}| > \delta\Big) \le & \frac{\mathbb{E}_{Q^{N_k}}\left(\left(\sup_{0\le t\le T} |M^{G,N_k}_{N_k^2 t}| \right)^2 \right)}{\delta^2} \\ \le & \frac{4}{\delta^2} \mathbb{E}_{Q^{N_k}} \left(\left(M^{G,N_k}_{N_k^2 T}\right)^2 \right) \to 0 \end{aligned}$$

for $k \to \infty$, by applying Chebyshev's inequality, followed by Doob's L^2-inequality. Consequently, we know that

$$\lim_{k\to\infty} P_{\mu^{N_k}} \Bigg(\sup_{t\in[0,T]} \Bigg| \langle \pi^{N_k}_{N_k^2 t}, G\rangle - \langle \pi^{N_k}_0, G\rangle - \int_0^t \langle \pi^{N_k}_{N_k^2 s}, \partial_s G\rangle \,\mathrm{d}s \tag{4.130}$$

$$- \int_0^t \left(\frac{1}{2N_k} \sum_{x\in\mathbb{T}_{N_k}} G''\left(\frac{x}{N_k}\right) \eta_{N_k^2 s}(x) + \frac{1}{2N_k} \sum_{x\in\mathbb{T}_{N_k}, 2\nmid x} G''\left(\frac{x}{N_k}\right) \bar{h}_x(\eta_{N_k^2 s}) \right) \mathrm{d}s \Bigg| > \delta \Bigg) = 0 \tag{4.131}$$

for all $\delta > 0$. The rest is familiar by now. Once more we have to apply the replacement lemma for the second summand of the integral in the bottom line of Equation (4.130). This time, all remaining summands will contribute to the equation - contrary to the term (4.125) - and the approach follows precisely the approach for gradient systems from here on forward (see pages 57 to 59, only with a different function h). As explained before, the replacement lemma allows to bring the expectation with respect to the grand-canonical measure ν^N_ρ into

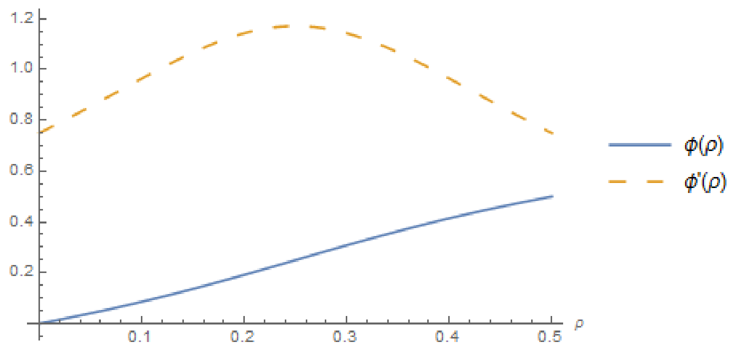

Figure 4.13.: Plots for Φ from (4.100) and its first derivative in dependence of the particle density. The function Φ is very close to the identity map $\rho \mapsto \rho$ and has symmetric derivatives around $\rho = \frac{1}{4}$.

the equation, which enables us to close the equation with respect to the empirical measure. Then, with Corollary 4.3.3, we can show the concentration of the measures $(Q^N)_{N\in\mathbb{N}}$ on the dirac-measure on absolutely continuous measures whose density solves (4.119).

The third and last step of the proof consists in showing uniqueness of weak solutions to the hydrodynamic equation. At first, we remind the reader that a maximum principle holds according to (4.103), such that

$$\rho_0(x) \in \left[0, \frac{1}{2}\right] \; \forall x \in \mathbb{T} \quad \Rightarrow \quad \rho(t,x) \in \left[0, \frac{1}{2}\right] \; \forall t \in \mathbb{R}_+, x \in \mathbb{T}.$$

Then, we note that the function Φ from (4.100) is smooth and strictly increasing for $\rho \in [0, \frac{1}{2}]$, which is illustrated in Figure 4.13. Furthermore, due to the symmetric properties of the derivatives (cf. (4.102)), we know that $||\Phi'||_\infty = \Phi'(\frac{1}{4}) < \infty$. Hence, we have shown that all requirements of Theorem C.1.1 are fulfilled and we conclude that there exists a unique weak solution in $L^2([0,T] \times \mathbb{T})$.[17]

[17]For the constraint of the time line we refer to the same discussion on page 59.

5. Conclusions

5.1. Dynamics at poles

Following the method applied by Spohn, Lacoin, Simenhaus and Toninelli, we have to ask what the evolution of the droplet's boundary γ will be like *around* the poles for a zero-temperature dynamics of planar random permutations. Our starting point is again an interface between two coexisting phases which can be approximated by a long permutation cycle with nearest-neighbour permutation jumps, just as in the scenario away from poles.

Creating a corresponding interacting particle system (like the AFP-process away from the poles) is initially a very creative and joyful exercise. Many systems have been tried already, but few of them seemed decently easy to handle. Basically with every system, there occurred some difficulty in deriving the hydrodynamic behaviour, such that standard techniques were not applicable. Of course, given the arduous work that went into the scaling limits for the AFP-process, this should not come as a surprise.

Still, we would like to present an IPS which is based on the ZRP with two species of particles which eliminate each other. Just as in our model away from the poles, we introduce a 1-dimensional lattice (such as $\mathbb{Z}$ or the torus $\mathbb{Z}/N\mathbb{Z}$ for $N \in \mathbb{N}$) with two different (alternating) types of sites, say *even* and *odd*. On even sites, we basically have a configuration space just like for the above mentioned ZRP with A and B particles, with A particles modelling positive gradients of the interface's height function (compare with the second subsubsection in Subsection 3.1.3) and B particles modelling negative gradients. In particular, many particles of the same type can occupy a single lattice site, and the dynamics is such that the leaving rate is independent from the amount of particles at a given even site. On the other hand, on odd sites, there can be only one A or one B particle at a time. Such a particle corresponds to a diagonal jump, which, around the poles, can take two possible forms: either from the bottom left to the top right, or from the bottom right to the top left (assuming w.l.o.g. a model for the north-pole, so to speak). Elimination of two particles (one each) can happen at every site, whenever an A particle jumps to a site that is occupied by (a) B particle(s) or vice versa.

For the normal two-species ZRP with instantaneous annihilation it is known that in the scaling limit there eventually appear macroscopic regions where one particle type dominates the other. In those regions, the density evolution should be slower but similar to the standard ZRP. The difficulty lies of course in combining those regional evolutions.

5.2. Future Work

The original goal of this Ph.D. project was the derivation of mean-curvature type droplet shrinking for spatial random permutations at zero temperature. By now, we have only treated the hydrodynamic equation away from poles. Even though the arising partial differential equation looks rather scary at first sight, it still seems to be very similar to the heat equation, which - of course - would be very desirable for mean-curvature motion (cf. Section 3.1). Thus, a further step could consist in deriving a result such as Corollary 3.5 in [18], in which the rescaled microscopic boundary could be compared to solutions of the heat equation.

In the previous section it became clear that the treatment of the situation at the poles is probably just as hard as away from the poles.

However, being optimistic, one still has to "glue together" the results for the auxiliary dynamics and translate it into a single partial differential equation that describes the curve shortening flow.

Furthermore, another obvious generalisation is the consideration of general α for the Gibbs measures, i.e. not only dealing with the zero-temperature case.

At last, one could consider less rigorous assumptions on the initial domain $\mathcal{D}$, which has been achieved for example within 2 years from [18] to [19] in the zero-temperature Ising model.

A. Stochastic Processes: Markov, Feller, Particle Systems

A.1. Markov Processes

For Markov processes, there are several natural martingales that play an important role, particularly in this thesis. From [14, Appendix 1.5] (cf. [25, Theorem 3.32]) we take the following well known results.

Let $(X_t)_{t\geq 0}$ be a continuous-time Markov process on the countable state space E. Consider a bounded function $F : \mathbb{R}_+ \times E \to \mathbb{R}$ smooth in the first coordinate, uniformly over the second: for each x in E, $F(\cdot, x)$ is twice continuously differentiable and there exists a finite constant C such that

$$\sup_{(s,x)} \left| (\partial_s^j F)(s,x) \right| \leq C \tag{A.1}$$

for $j = 1, 2$. To each function F satisfying (A.1), define M_t^F and N_t^F by

$$M_t^F := F(t, X_t) - F(0, X_0) - \int_0^t \mathrm{d}s (\partial_s + L) F(s, X_s) \tag{A.2}$$

and

$$N_t^F := \left(M_t^F\right)^2 - \int_0^t \mathrm{d}s \Big\{ LF(s, X_s)^2 - 2F(s, X_s) LF(s, X_s) \Big\}, \tag{A.3}$$

where L denotes the generator of the Markov process $(X_t)_{t\geq 0}$.

Lemma A.1.1 (Kipnis/Landim)**.** *Denote by $\{\mathcal{F}_t,\ t \geq 0\}$ the filtration induced by the Markov process: $\mathcal{F}_t = \sigma(X_s,\ s \leq t)$. The processes M_t^F and N_t^F are $\mathcal{F}_t$-martingales.*

Proof. See Lemma A1.5.1 in [14]. □

Remark A.1.1. *In most of the cases in previous chapters, our choice of the function $F : \mathbb{R}_+ \times E \to \mathbb{R}$ does not depend on the first variable, i.e. on the time component per se. This means that $\partial_s F = 0$ and the right-hand side of* (A.2) *simplifies to*

$$F(t, X_t) - F(0, X_0) - \int_0^t LF(s, X_s) \, \mathrm{d}s.$$

A.2. Feller Processes

The following results are taken out of the works by Thomas M. Liggett on Markov processes, Feller processes and interacting particle systems in [23, 24, 25]. They are either explicitly used in the preceding chapters or give additional insight to the basic introduction to the matter in Section 2.1. Also, the notation is taken from the latter.

At first we note that a semigroup $\{T(t) \ : \ t \in \mathbb{R}_+\}$ is a contraction operator at any point in time

$$||T(t)f|| \leq ||f|| \quad \text{for all } f \in C(X), \tag{A.4}$$

which is a direct consequence of properties 4. and 5. in Definition 2.1.2

Definition A.2.1. *Given a semigroup $T(t)$, its Laplace transform*

$$U(\alpha)f = \int_0^\infty e^{-\alpha t} T(t) f \,\mathrm{d}t, \quad \alpha > 0,$$

is called the resolvent of the semigroup.

The resolvent is a linear operator on $C(X)$. It is well defined due to the continuity of the function $t \mapsto e^{-\alpha t}T(t)f$ for all $\alpha > 0$ and due to

$$||e^{\alpha t}T(t)f|| \leq e^{-\alpha t}||f||,$$

based on equation (A.4). We are now prepared to state the theorem that enables us to go from a probability semigroup to a generator.

Theorem A.2.1 (Liggett). *Suppose that $T(t)$ is a probability semigroup, and define L by*

$$Lf := \lim_{t \downarrow 0} \frac{T(t)f - f}{t} \tag{A.5}$$

on

$$\mathcal{D}(L) = \{f \in C(X) \ : \ \textit{the (strong) limit in } \text{(A.5)} \textit{ exists}\}.$$

Then L is a probability generator. Furthermore, the following statements hold.

1. *For any $g \in C(X)$ and $\alpha > 0$,*

$$f = \alpha U(\alpha) g \ \textit{iff} \ f \in \mathcal{D}(L) \textit{ and satisfies } f - \alpha^{-1} L f = g.$$

2. *If $f \in \mathcal{D}(L)$, then $T(t)f \in \mathcal{D}(L)$ for all $t \in \mathbb{R}_+$, is a continuously differentiable function of t, and satisfies*

$$\frac{d}{dt} T(t) f = T(t) L f = L T(t) f.$$

3. *For $f \in C(X)$ and $t > 0$,*

$$\lim_{n\to\infty} \left(I - \frac{t}{n}L\right)^{-n} f = T(t)f.$$

Proof. See Theorem 3.16 in [25]. □

For the next theorem, we introduce quasi-left continuity.

Definition A.2.2. *The process $(\eta_t)_{t\in\mathbb{R}_+}$ is said to be* quasi-left continuous *if whenever a sequence of stopping times τ_n increases to τ, it follows that*

$$\eta_{\tau_n} \to \eta_\tau \quad \textit{a.s. on the event } \{\tau < \infty\}.$$

Starting from a probability generator, we have seen in Section 2.1 how to obtain a probability semigroup $\{T(t) \,:\, t \in \mathbb{R}_+\}$. The operator L_ε in (2.5) is well defined by condition 3 of Definition 2.1.3, which can be seen by the equivalence $f - \varepsilon Lf = g \;\Leftrightarrow\; f = (I-\varepsilon L)^{-1}g$. The latter also implies the inequality

$$||L_\varepsilon g|| = ||Lf|| \leq \frac{||f|| + ||g||}{\varepsilon} \leq \frac{2}{\varepsilon}||g||.$$

It follows that L_ε is a bounded linear operator and that the approximation $T_\varepsilon(t)$ in (2.6) is indeed well defined. For the last step of this direction, i.e. getting a Feller process, we referred to the following statement.

Theorem A.2.2 (Liggett). *If $T(t)$ is a probability semigroup, then there is a Feller process $(\eta_t)_{t\in\mathbb{R}_+}$ satisfying*

$$\mathbb{E}^{\eta^0} f(\eta_t) = T(t)f(\eta^0)$$

for $\eta^0 \in X,\; t \in \mathbb{R}_+$, and $f \in C(X)$. Furthermore, $(\eta_t)_{t\in\mathbb{R}_+}$ is quasi-left continuous.

Proof. See Theorem 3.26 in [25]. □

B. Miscellaneous

B.1. Laplace's Method

Laplace's method plays a crucial role in deriving the stationary measures in this thesis. Even though we frequently referred to [27], in this section we will give an introduction to the topic based on [10]. Of course, the original technique is due to Laplace (1820).

Laplace's method is used to estimate integrals of the type

$$I(\lambda) := \int_a^b e^{-\lambda p(t)} q(t)\,\mathrm{d}t,$$

for values a, b that might or might not be finite. The idea is based on the observation that the peak value of the function $e^{-\lambda p(t)}$ occurs at the point $t = t_0$ where $p(t)$ is a minimum. For large λ the peak is concentrated in a neighborhood of $t - t_0$.

Suppose now that $t_0 = a$ and $p'(a) > 0,\ q(a) \neq 0$. If we replace $p(t), q(t)$ in $I(\lambda)$ by a Taylor expansion near $t = t_0$, we get

$$I(\lambda) \sim \int_a^b e^{-\lambda(p(a)+p'(a)(t-t_0))} q(a)\,\mathrm{d}t.$$

Putting $b = \infty$, we obtain

$$I(\lambda) \sim q(a) e^{-\lambda p(a)} \int_a^\infty e^{-\lambda(t-a)p'(a)}\,\mathrm{d}t$$

and hence

$$I(\lambda) \sim q(a) \frac{e^{-\lambda p(a)}}{\lambda p'(a)}.$$

Otherwise, when $t = t_0$ is an interior point and $p''(t_0) > 0$, then

$$I(\lambda) = \int_a^b e^{-\lambda p(t)} q(t)\,\mathrm{d}t \sim \int_a^b e^{-\lambda(p(t_0)+\frac{1}{2}p''(t_0)(t-t_0)^2)} q(t_0)\,\mathrm{d}t.$$

At this point, we may replace $a = -\infty$ and $b = \infty$ with negligible error. As

$$\int_{-\infty}^{\infty} e^{-at^2}\,\mathrm{d}t = \sqrt{\frac{\pi}{a}}$$

for $a > 0$, we get

$$I(\lambda) \sim e^{-\lambda p(t_0)} q(t_0) \int_{-\infty}^{\infty} e^{-\lambda \frac{(t-t_0)^2}{2} p''(t_0)}\,\mathrm{d}t = e^{-\lambda p(t_0)} q(t_0) \sqrt{\frac{2\pi}{\lambda p''(t_0)}}. \tag{B.1}$$

This derivation of the asymptotics has been very informal, of course. For a proper statement and proof we refer once again to [10].

B.2. Compactness results

The method used to prove the hydrodynamic evolution of particle systems in this thesis relies on showing convergence of sequences (more precisely: sequences of distributions on a space of measure-valued càdlàg functions). This can be achieved by showing that the sequence is relatively compact and that all subsequences converge to the same limit. The proofs of Theorem 4.2.1 and 4.3.1 refer to results from [8] and a standard reference for probability researchers, Billingsley's [6], where an entire chapter is devoted to measures on càdlàg spaces.

For the next Theorem, we recall some notation from page 51 first. For a function $f : [0, T] \to \mathbb{R}$ a modified uniform modulus of continuity is defined by

$$w'_f(\gamma) := \inf_{\{t_i\}_{0\le i\le \bar r}} \max_{0\le i<\bar r} \sup_{t_i\le s'<t<t_{i+1}} |f_t - f_{s'}|, \tag{B.2}$$

where the infimum is taken over all partition points $\{t_i, 0 \le i \le \bar r\}$ of $[0, T]$ such that $0 = t_0 < t_1 < ... < t_{\bar r-1} < t_{\bar r} = T$ and $t_i - t_{i-1} > \gamma$ for all $i = 1, ..., \bar r$. The modified modulus of continuity allows to characterise compact sets in $D([0, T], \mathbb{R})$ by the Ascoli Theorem (cf. [14, Proposition 4.1.2]) and gives the following statement of Prohorov's Theorem.

Theorem B.2.1 (Ascoli, Prohorov)**.** *Let $(P^N)_{N\ge1}$ be a sequence of probability measures on $D([0, T], \mathbb{R})$. The sequence is relatively compact if and only if*

1. *for every $t \in [0, T]$ and every $\varepsilon > 0$, there is a compact set $K(t, \varepsilon) \subset \mathbb{R}$ such that*

$$\sup_{N\ge1} P^N(f \ : \ f_t \notin K(t, \varepsilon)) \le \varepsilon, \tag{B.3}$$

2. *for every $\varepsilon > 0$,*

$$\lim_{\gamma\to0} \limsup_{N\to\infty} P^N(f \ : \ w'_f(\gamma) > \varepsilon) = 0. \tag{B.4}$$

Proof. See Theorem 1.3 in Chapter 4 of [14]. □

Since the second condition in Theorem B.2.1 is hard to verify in our setting, we will use the following proposition instead.

Proposition B.2.1. *Let $\mathcal{T}_T$ be the family of stopping times bounded by T. A sequence of probability measures $(P^N)_{N\geq 1}$ on $D([0,T],\mathbb{R})$ satisfies Equation* (B.4) *for every $\varepsilon > 0$ provided*

$$\lim_{\gamma\to 0}\limsup_{N\to\infty}\sup_{\tau\in\mathcal{T}_T,\theta\leq\gamma} P^N(f \ : \ |f_{\tau+\theta} - f_\tau| > \varepsilon) = 0 \tag{B.5}$$

for every $\varepsilon > 0$.

Proof. See Proposition 1.6 in Chapter 4 of [14]. □

Fortunately, the problem of proving relative compactness for distributions on $D([0,T],\mathcal{M}_+)$ (which is the problem that we are dealing with in this thesis) can be simplified significantly by means of the following proposition.

Proposition B.2.2 (Kipnis/Landim)**.** *Let $\{g_k;\ k\geq 1\}$ be a dense subfamily of $C(\mathbb{T})$ containing the constant* 1*-function. A family of probability measures $(Q^N)_{N\geq 1}$ on $D([0,T],\mathcal{M}_+)$ is relatively compact if for every positive integer k the family $(Q^{N,g_k})_{N\geq 1}$ of probabilities on $D([0,T],\mathbb{R})$ has this property, where the latter family is defined for a measurable set $A\subset D([0,T],\mathbb{R})$ by*

$$Q^{N,g_k}(A) = Q^N(\{(\pi_t)_{t\in[0,T]}\in D([0,T],\mathcal{M}_+) \ : \ (\langle\pi_t, g_k\rangle)_{t\in[0,T]}\in A\}).$$

Proof. See Proposition 1.7 in Chapter 4 of [14]. □

Thus, the preceding results from this Section, namely Theorem B.2.1 and Proposition B.2.1, can be applied for every test-function $G : \mathbb{T}\to\mathbb{R}$ and its associated measure $P^N := Q^{N,G}$.

C. Analysis of Partial Differential Equations

C.1. Quasilinear Parabolic Partial Differential Equations

Both the hydrodynamic equation for the RrEP and the hydrodynamic equation for the AFP-process are *quasilinear parabolic equations of second order*. In order to guarantee existence and uniqueness results for weak solutions of the partial differential equations, we cite from Appendix 2.4 in [14], but restrict ourselves to a setting in 1 space dimension.

In the following we will refer to the general *Cauchy problem*

$$\begin{cases} \partial_t \rho &= \sigma \partial_x^2 \Phi(\rho), \\ \rho(0,\cdot) &= \rho_0(\cdot), \end{cases} \tag{C.1}$$

where Φ is a smooth, strictly increasing function such that $||\Phi'||_\infty \leq g^* < \infty$ and $\sigma > 0$.

Definition C.1.1. *Fix a bounded initial profile $\rho_0 : \mathbb{T} \to \mathbb{R}$. A measurable function $\rho : \mathbb{R}_+ \times \mathbb{T} \to \mathbb{R}$ is a* weak solution *of the Cauchy problem* (C.1) *if for every function $G : \mathbb{R}_+ \times \mathbb{T} \to \mathbb{R}$ of class $C^{1,2}(\mathbb{R}_+ \times \mathbb{T})$ with compact support*

$$\int_0^\infty \mathrm{d}t \int_{\mathbb{T}} \mathrm{d}u \Big\{ \rho(t,u)\partial_t G + \Phi(\rho(t,u))\sigma\partial_u^2 G \Big\} + \int_{\mathbb{T}} \mathrm{d}u G(0,u)\rho_0(u) = 0. \tag{C.2}$$

It has been shown [29] that bounded weak solutions are uniformly Hölder continuous on each compact subset of $(0,\infty) \times \mathbb{T}$. This is an important tool to prove that there exists a bounded weak solution of (C.2) for bounded initial profiles ρ_0. It is possible to derive the following even stronger result, which is used both in Subsection 4.2.4 and 4.3.5.

Theorem C.1.1 (Kipnis/Landim)**.** *Fix a bounded profile ρ_0. There exists a unique weak solution of the quasi-linear parabolic equation* (C.2) *that belongs to $L^2([0,T] \times \mathbb{T})$.*

C.2. Illustrations of the Particle Density Evolution

In Figures C.1 to C.3 we give some more illustrations of solutions to hydrodynamic equation for the RrEP (4.3). Note the similarity to solutions of the heat equation.

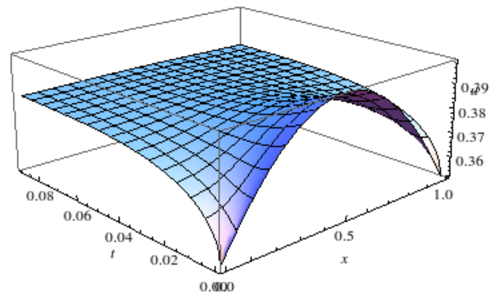

Figure C.1.: Solution to (4.3) with $\rho_0(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(x-\frac{1}{2}\right)^2}$

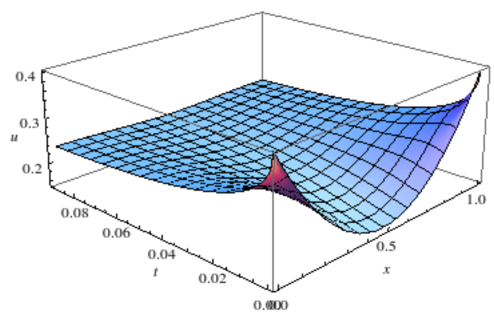

Figure C.2.: Solution to (4.3) with $\rho_0(x) = \frac{2}{5} - x(x-1)$

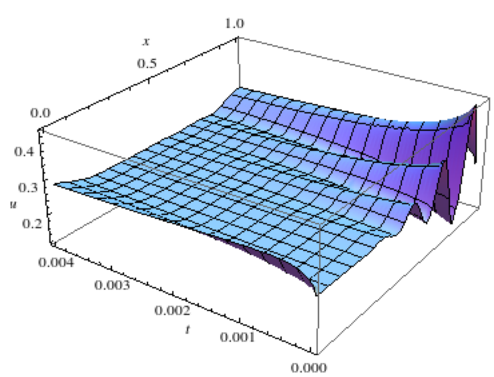

Figure C.3.: Solution to (4.3) with $\rho_0(x) = \frac{1}{4} + \frac{1}{4000}\sin(10\pi x)\left(4x^7 - 1000x^3 + 81x^2 + 20x\right)$

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