ABSTRACT

The Structure of Norms and Legal Uncertainty:
A Framework for the Functional Analysis of Law as Transformed in Multi-Member
Decision Mechanisms
Gunnar Nordén 2016

Doctrinal studies of law describe *relationships* between conditioning legal facts and consequences, distinguishing between substantive and procedural norms. The latter constitute decision mechanisms that maintain the legal system's norms. These mechanisms generate binding decisions—ordered pairs of facts and consequences—that may obtain the status of *res judicata* and become part of the norm system in the *extensive* sense.

Functional analyses of law undertake to study agent equilibrium behavior under *given* norms, perceived as incentive structures.

Characteristically, norms that are maintained through adjudication (or arbitration) are not complete or unambiguous: In the ex ante sense, consequences are not uniquely implied by relevant conditioning facts. This indeterminacy has profound implications: first, in multi-member decision mechanisms norm structures are systematically *transformed*; second, these transformed norms or incentive structures guide agent (equilibrium) behavior. These observations challenge the approach that currently prevails in legal theory, namely of considering substantive norms as abstract entities independent of procedural mechanisms. They also suggest opportunities for widening the scope of functional or repercussion analyses of law.

This dissertation develops an analytical framework that seeks to enable the study of norm transformation in multi-member judicial decision mechanisms. The framework's relevance is demonstrated through numerous examples showing how equilibrium outcomes vary with mechanisms shaping the incentive structures. The framework is devel-

oped using formal *representation* of norms and by ex ante *identification* of judges (arbitrators) and norms. These representations combine functional relations with basic notions from probability theory that are simple enough to be operative in equilibrium analyses, and, at the same time, rich enough to embody detailed aspects of procedural law. The framework facilitates: (i) distinction between "meta-norms" (the doctrine of sources and judicial method) and "ordinary norms" (doctrine in the customary sense); (ii) depiction of (possible) indeterminacy at both levels; (iii) modeling of multi-member decision-making; and (iv) simultaneous consideration of epistemic uncertainty. The identification of norms and judges envisions judging as *commitment*, in contrast to a preference-based, rational choice account. The approach combines insights from logical aggregation, case-space, and evolutionary theories.

Under meta-level uncertainty, judges may base decisions on different substantive norms. Under ordinary-level uncertainty, they may reach different conclusions under the same substantive norm. In correspondence with standard conceptions of *legal* decision-making, judges *applying law* vote directly on outcomes or on substantive norm elements, not abstractly over substantive norms. The commitment notion assumes judges vote independently (non-strategically) as uncertainty is resolved. Both majority and super-majority aggregation rules are studied (the latter require default state specifications). Implications of sequential decision-making (bifurcation) and separation of (collective) decisions on law and facts are also analyzed.

The framework is applied to a detailed analysis of the model of precaution, which has a prevalent and unifying role in many areas of law. Equilibrium precautionary investment is derived under uncertain negligence standards, and under *mixed norms* (uncertainty between strict liability and negligence). Continuous comparative statics reflects the parametric representation of uncertainty on both the meta and ordinary level. Discrete comparative statics reflects decision mechanism size. A condensed analysis of final-offer arbitration demonstrates that the framework is applicable to strategic environments.

The norm representation enables one to distinguish between primary norms expressing obligations and power-conferring secondary norms, which express discretion or options. This distinction is reflected in suggested law-in-force notions, with discretion motivating a forward-looking, means-end approach (in fact, partly due to logical problems arising under aggregation). Examples used to illustrate the benefits from detailed attention to norms structures include entitlement-protection in exchange economies and legal commitment mechanisms in strategic environments.

Under indeterminacy, norm structures and induced equilibria vary systematically with decision mechanisms. This sensitivity, exacerbated by epistemic uncertainty, accentuates the question of evaluative criteria as discussed in legal and political philosophy. In reference to observed authority structures, the majority outcome in large panels is suggested as a benchmark, making possible a study of the link between (finite, real-world) mechanism choice and Type I and II error generation.

Some analytical results may be of independent interest. Judicial panels transform *marginal* dimensions of incentive structures. This is important when conditioning legal fact sets are choice or strategy spaces for optimizing agents (*level* transformations correspond to Condorcet-type theorems). Second-order stochastic dominance is used to describe panel size effects on a domain of substantive norms. First-order stochastic dominance is used to compare unitary and bifurcated trials.

Due to analytical challenges, some results are based on asymptotic theory. The equilibrium analyses are supported by simulations.

The Structure of Norms and Legal Uncertainty:

A Framework for the Functional Analysis of Law as Transformed in Multi-Member Decision Mechanisms

A Dissertation
Presented to the Faculty of the Law School
of
Yale University
In Candidacy for the Degree of
Doctor of the Science of Law

By
Gunnar Nordén

Dissertation Committee Supervisor: Richard R.W. Brooks
Readers: Ian Ayres and Daniel Markovits
June 2016

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

LIST OF EXAMPLES

AKNOWLEDGEMENTS

PART ONE

INTRODUCTION

> Surely, as in other cases of the progress from empiricism to science, the first step must be in the direction of mathematical or symbolic expression of the facts. The value of such a symbolisism is twofold: first, as an aid to precision of thought; and second, as a preliminary to generalisation.
>
> "An Application of Mathematics to Law"
> —*H. E. Potts, 1913*

> Statistical reasoning, like mathematical reasoning, legal reasoning, or any other form of reasoning, is essentially independent of its content.
> —*W. A. Wallis & H. V. Roberts, 1956*

1   Background

In this work, legal norms are understood as *relationships* between sets of conditioning legal facts X, and sets of (conditioned) legal consequences Y. Legal systems are understood as constituted by sets of such relationships. Characteristically, the systems include mechanisms (courts and tribunals) M that maintain and make authoritative decisions regarding a subset of the relationships, be they classified as substantive or procedural. Maintenance of procedural norms is related to mechanisms themselves and is made possible through hierarchical court organization.

The authoritative decisions, *judgments*, essentially take the form of *ordered pairs* of facts and conclusions, $\langle x, y \rangle$, $x \in X$ and $y \in Y$. The constituting elements can be complex bundles, given by direct enumeration or by propositions

describing fact patterns and conclusions (the sets X and Y are abstract and can be given general interpretations).

Let $\underline{d}$ be a vector of data and $\mathcal{M}$ the set of admissible decision mechanisms. A judgment j is more completely denoted $\langle x, y \rangle^{d_j}$, $\underline{d}_j$ describing additional information made available, including the mechanism $M \in \mathcal{M}$ rendering the decision, time and place, and (possibly) the distribution of votes in judicial panels and opinions (including concurring and dissenting ones).

A subset of the judgments obtains the status of *res judicata* and becomes enforceable. The set of enforceable judgments, $\left\{ \langle x, y \rangle^{d_j} \right\}_{j=1}^{J}$ say, represents the normative relationship in *extensive form*. The set of judgments forms a *binary relation* in the Cartesian product space of possible judgments: $\left\{ \langle x, y \rangle^{d_j} \right\}_{j=0}^{J} \subseteq X \times Y$. In a rudimentary sense, legal systems can be understood as composed of such relations (subsets or extensions).

Through legal systems' meta-norms (relationships constituting the doctrine of law sources), individual judgments $\langle x, y \rangle^{d_j}$ may influence legal relationships in other ways than in the direct extensive sense (keyword: precedent), if they are not *formally* recognized as a source class in some jurisdictions.[1] Furthermore, modern legal systems contain more potent mechanisms for dynamic evolution, distinguishing them sharply from other normative orders. Most importantly, they contain institutions for legislation and contract formation (includ-

---

[1] This work concerns the operational sources and norms. See Sacco [1991a,1991b] and Monateri & Sacco [1998] on *legal formants*. See Ross [1959] on the foundational role of the doctrine of sources and method to legal systems (esp. pp. 75–78).

ing international conventions).[2] These norms are (directly or indirectly) maintained in applications to new cases. Courts ensure that system specific criteria for norm generation are satisfied and may screen enacted or contracted norms for content (judicial review). However, the selection of cases for adjudication ("dockets") is only negatively controlled, if at all.[3]

In theory, a legislative body could produce *complete norms,* in the sense of mapping every relevant conditioning legal fact x on a domain X to a *unique* element in Y, $y = r(x)$, for example. In this case, $r(x)$ is called the *image* of x and the set of ordered pairs $\left\{ \langle x, y \rangle \in X \times Y \mid x \in X \text{ and } y = r(x) \right\}$ a *functional relation.* The rule connecting elements in X to elements in Y is denoted $r : X \to Y$. In this way, parties, in theory, could design *contracts* (including corporate forms) as complete mappings $c : X \to Y$, specifying a unique action $y = c(x)$ for every possible circumstance (contingency) $x \in X$.[4]

Ideals of completeness and determinacy have long traditions in law. The *Panel on Statistical Assessments as Evidence in the Courts*, which describes ju-

---

[2] Enacted formants. In addition, legal systems may make unilateral promises and self-legislation enforceable.

Historically, legislation occurs much later than judge made law (Ross [1958:78–84], Mackaay [1998]). As precedents, contracts are not universally (formally) recognized as a source of law (Ross [1958:84–91], Merryman & Pérez-Perdomo [2007:22–23]).

[3] As emphasized by Damaška [1986:88], judicial decision making concerns settlement of contested matters between parties by passive adjudicators. Rare exceptions of self referral concern constitutional courts (Ferejohn & Pasquino [2004:1682–83]).

The passing of retroactive legislation, reversing existing decisions $\langle x, y \rangle^{d_j}$ is generally blocked by maintained notions of court independence (Shetreet [1985:609]) or constitutional constraints.

[4] As will be discussed, there are various notions of completeness in modern contract theory. For the present purpose, X corresponds to a set of third party verifiable states and $c : X \to Y$ is *obligationally complete,* in the sense of Ayres & Gertner [1989:92].

The possibility of writing complete contracts and codes is heavily dependt on the development of measurement systems and monitoring costs (see, generally, Allen [2012]).

risprudence as evolving parallel to science, points out that earlier procedural forms were premised on certainty, as exemplified by divine intervention in trial by ordeal—a system that guarantees correct answers.[5] As classical physics developed between 1600 and 1900, leading scientists and mathematicians understood the universe to be regulated by strict determinism.[6] Probabilistic notions of uncertainty were tolerated as a useful tool when data were insufficient, but "in a sense, as a lesser discipline, because if our ignorance were only eliminated we wouldn't need probability […]".[7]

In emerging monolithic nation states on the European continent, Enlightenment ideas about supremacy of the legislature and separation of powers meant that extensive codes were promulgated and envisioned as complete, unambiguous, coherent and judge-proof.[8] Given any fact pattern, the legal order dictated the outcome and was *eine logische Geschlossenheit*[9]: law application merely re-

---

[5] The distinguished Panel, pooling information from a variety of academic fields, studies courts' factual assessments. The report is published in Fienberg [1989].

Elster [1989:104] suggests that "legal lotteries" used to select (impartial) judges, originally were interpreted as implementing intentional acts.

[6] "In many respects, the world of physics mirrors its surrounding cultural milieu, and, to some extent helps shape it. The classical and formal art, music, literature, and mathematics created during the Renaissance and Enlightenment periods, until the dawn of the 20th century, were complemented by the classical physics of Newton's dynamics, Maxwell's electromagnetism and Bolzman's thermodynamics. [---] Physics was precise, predictable and deterministic" (Reese [2000:1250]). As luminously explained in Ekeland [1988], with sufficient information about initial conditions, the future (and past) could be perfectly predicted.

[7] Isaac [1995:2] (see also Ekeland [1988:20]). As pointed out by Ekeland [1988:49] and Hacking [1975:1,148], the complex probability notion has the dual aspects of tendency and incomplete knowledge.

[8] See, generally, Merryman & Pérez-Perdomo [2007] and von Mehren & Gordley [1977] on the French and German legal system. See Ferris [2010] on the simultaneous rise of science and liberal democracies.

[9] von Mehren & Gordley [1977:1137]: if not even metaphysically predetermined (see Ross [1958:344-7] on the historical school).

quired "subsumption" of facts to self-applying norms.[10] The commitment to general rules, formal legality and separation of powers was famously formulated by Cesare Beccaria in 1764:

> Only the laws can determine the punishment of crimes; and the authority of making penal laws can reside only with the legislator, who represents the whole society united by the social compact.[11]

The revolutionary period's ideals of certainty and determinacy have affected *formal* sources of law doctrines, creating a rigid hierarchical system, that only accepts three classes: (i) legislation, dominating (ii) regulation, dominating (iii) custom. More fundamentally, these early ideals have had a lasting effect on legal decision mechanisms.

The characteristic sequential decision-making in adjudication, be it in or between mechanisms M, *compartmentalizes* uncertainty. Bifurcation (keywords: separation of questions of liability and consequences of liability) and sep-

---

[10] Under Frederick the Great, Prussia promulgated a code with more than 17,000 articles (in comparison, Code Napolén has 2,281), and judges caught interpreting the code were severely punished (Merryman & Pérez-Perdomo [2007:30,39]).

    The same attitude towards the third branch did not exist in revolutionary England and the U. S. (Merryman & Pérez-Perdomo [2007:16–19]). Diggins [1987] points out that ideas from classical mechanics (forces and counter-forces) helped secure an autonomous role for courts in the tri-partition of powers in the U.S. federal government and in the division between federal and state governments. (See La Porta et al. [2004] on the distinct development of judicial independence and constitutional review.)

[11] Beccaria's *Of Crimes and Punishment* is the most influential work on criminal law and procedure in Western history (claim and quote from Merryman & Pérez-Perdomo [2007:125]). A modern statement is provided in the European Court of Human Rights' plenary judgment *Sunday Times v. United Kingdom*: "[T]he law must be adequately accessible: the citizen must be able to have an indication that is adequate in the circumstances of the legal rules applicable to a given case. [A] norm cannot be regarded as a "law" unless it is formulated with sufficient precision to enable the citizen to regulate his conduct: he must be able […] to foresee, to a degree that is reasonable in the circumstances, the consequences which a given action may entail" (April 26, 1979, Sec. 49).

    Fuller [1978:373,380] stresses inter-subjectively available principles, and rational reasoning, as necessary for adjudication: "Without some standard of decision the requirement that the judge be impartial becomes meaningless. Similarly, without such a standard the litigants' participation through reasoned argument looses its meaning. [---] adjudication is a form of social ordering institutionally committed to 'rational' decision."

    Coleman & Leiter [1993] emphasize the link between determinacy, the possibility of democratic rule, and protection of agents' liberal autonomy (see also Allan [1998]).

aration of decisions on law and facts (according to various doctrines) are notorious examples.[12] While purely factual questions, as well as mixtures of law and facts (such as in the determination of liability), are often determined using supermajority thresholds in judicial panels (prioritizing default states, such as no liability), purely legal questions are resolved using symmetric majority rules. In hierarchical systems, peak courts typically are available only to settle pure questions of law (*de jure* in criminal cases and regularly in private law). These structures arguably assume the existence of accessible and independent answers to legal questions.[13]

The emphasis on (apparent) certainty in civil law traditions is also expressed in norms regulating decision announcements: In judicial panels, the distribution of votes are not noted, and concurring or dissenting opinions are not published.[14] Because judges, according to civil law traditions, are stipulated not to interpret law, pronouncements on law in higher echelons of judicial hierarchies formally do not bind lower courts.[15]

---

[12] Stein [1992:8] notes: "Western legal tradition, whether in its Roman or its common law form, has always required that a legal action produce a winner and a loser. Legal issues are seen in terms of black and white; either the defendant is liable or he is not liable. Courts have not been at liberty to strike an equitable balance."

See Landes [1998] on sequential decision making, and Coons [1964] and Kornhauser [1992a] on dichotomization in legal thought more generally (in Ferris [2010:288] described as "on-off swiches" characterizing moral precepts).

[13] See Damaška [1986] on legal authority structures. The suggested link between aggregation rules and (epistemic) access to independent truths are discussed at various points below (see generally Nitzan [2010]). On the distinction between epistemic and ontological questions with particular reference to the legal domain, see Coleman & Leiter [1993].

[14] Merryman & Pérez-Perdomo [2007:36–37] and von Mehren & Gordley [1977:1148–50]. This is in contrast to common law traditions (Merryman & Pérez-Perdomo [2007:48–55]; Kornhauser [1998]; see Ledebur [2009] on the U.S. Supreme Court).

[15] Damaška [1986:36–39]; Merryman & Pérez-Perdomo [2007:24–26].

Legislation and contracts are the results of *collective decisions*. In 1951, Kenneth J. Arrow opened the vast field of axiomatic social choice theory by noting that "[i]n a capitalist democracy there are essentially two methods by which social choices can be made: voting, typically used to make 'political' decisions, and the market mechanism, typically to make 'economic' decisions."[16] Choices are made from a set of possible states, S, and system members are represented by complete and transitive *preference orderings* on S.[17] A list of such orderings for all $i = 1,\ldots,I$ members, $\langle \succ^1,\ldots,\succ^I \rangle$, is called a preference profile. Defining a *constitution* as a *function* from the set of all possible preference profiles (a product space of orderings) to a set of possible societal preference orderings, and endowing the constitution with basic normative properties, Arrow famously proved that no such constitution (function) exists. Various versions of this *impossibility theorem* have devastating implications in many contexts, challenging the *meaning* of collective preferences (and a fortiori, of legislation and legislative intent[18]). Drawing on the link to game theory, Karl Borch noted:[19]

> [C]omplete knowledge about the rules of the game and the objectives of the players would in general only make it possible to specify a probability distribution over outcomes of the game. This should indicate that group decisions and group preferences can only be predicted in a probabilistic sense, <u>even if we have full knowledge about individual preferences</u>. [---] game theory has brought into economics an *uncertainty principle*, similar to the one brought into physics by the quantum theory. (Emphasis in original, underlining added).

---

[16] All references are to the third edition of *Social Choice and Individual Values*, here quoted from p. 1.

[17] A preference ordering $\succ^i$ is a set (binary relation) in $S \times S$.

[18] See Rodriguez [1998].

[19] Borch [1968:115,222]. See Maskin [2012] and Laffont & Martimort [2002] on the links between Arrow's theorem and game theory.

The uncertainty principle "revolutionized the foundations of physics [---] now saying that in principle you could not make exact predictions; the best you could do would be to make probability statements no matter how much data you collected."[20] From the more profane perspective of jurisprudence, in the 20th century, the mechanical and abstract approach to law has been replaced by more conjectural and complex visions.[21] In particular, the spread of functionally rigid constitutions, constitutional courts, judicial review, and super-national courts after the Second World War has revealed the impossibility of a sharp separation of legislative and judicial powers, implying a reconceptualization of sources of law doctrines.[22]

---

[20] Isaac [1995:2–3]. Modern physics began developing around 1900. "[P]hysicists discovered a new abstract formulation of the physical world as Bohr, de Broglie, Schrödinger, Heisenberg and Dirac elucidated the features of a totally new and unexpected (almost counterintuitive) type of mechanics: quantum mechanics. [---] Strict determinism was replaced by probability, uncertainty, and an unfamiliar new world of nature" (Reese [2000:1250]).

Remarkably, Leibniz, who concurrently with Newton discovered the calculus that made classical physics possible (Devlin [1994:87]), at the same time made fundamental contributions to probability theory. Developed in his work on "natural jurisprudence", Leibniz' essay *De conditionibus* of 1665 is a study of legal relations, *jus purum*, *jus nullum*, and *jus conditionale* meaning absolute, void and conditional rights, respectively (Hacking [1975:85-92]).

[21] See generally Merryman & Pérez-Perdomo [2007] and von Mehren & Gordley [1977:1138–39] on civil law systems; Langbein, Lerner & Smith [2009:982–91] on common law systems.

Parisi [1992] gives a comprehensive historical and comparative account of negligence liability in civil and common law jurisdictions (including the mixed Louisiana traditions), concluding that "[a] substantial level of unpredictability is innate in the negligence process. [---] The recognition of the crucial role played by judicial discretion in the negligence process is necessary for—what I am afraid is—a more skeptical understanding of the history of negligence, and for a more informed discussion over negligence rules and standards of liability" (443).

[22] The need to decide cases *erga omnes* in the case of judicial review (keyword: uniform development of law) has accentuated prior decisions as law sources (distinctions between cassation and revision echoes historical functions). In civil law jurisdictions, and in constitutional courts and the ECJ, the norm remains suppression of uncertainty in the decision process and announcement of unanimous decisions (Merryman & Pérez-Perdomo [2007:125]; Ferejohn & Pasquino [2004:1692–99]).

Understanding the degree to which *legal sources* determine outcomes can be used to classify jurisprudential positions. Broadly, a range from "legalism" to "legal realism" may be identified.[23] However, as argued by Kornhauser & Sager [1986], a mapping of positions onto this spectrum is still too simple and intimately linked to the *collective character* of legal decision-making, an aspect (they point out) that generally has been ignored in jurisprudence. Notoriously, apex legal decision-mechanisms $M*$ are collective. Table 1.1. illustrates that subsets of the relations defining law in the extensive sense,

$$\left\{\langle x, y\rangle^{d_j} \,\middle|\, \text{jugdment rendered by } M*\right\} \subseteq \left\{\langle x, y\rangle^{d_j}\right\}_{j=1}^{J} \subseteq X \times Y$$

are generated by judicial panels. All render decisions by majority rule.[24]

Table 1.1 Select apex mechanisms, $M*$

n	panel
3	European Court of Justice (ECJ), chamber
5	ECJ, chamber
7	European Court of Human Rights (ECtHR), chamber
8	German Const. Ct., senate
9	U.S. Sup. Ct.; Italian Sup. Ct.; French Const. Council
13	ECJ, grand chamber
15	International Court of Justice; Italian Const. Ct.
16	German Const. Ct., plenum
17	ECtHR, grand chamber
19	Norwegian Sup. Ct., plenary session
28(+)	ECJ, full court
501(+)	Athenian juries

---

[23] See Coleman & Leiter [1993] on notions of determinacy applied to the legal domain.

[24] Whitman [2008] criticizes jurisprudence for ignoring important institutional features described e.g. in comparative studies of law.
    See Cabrillo & Fitzpatrick [2008] on court structures in the U.S., England, France, Germany, and Spain. Mackenzie et al. [2010] provide information on a large number of international courts and tribunals (arbitration panels typically have three or five members). Working juries have not been included in the table, even though they decide a subset of issues with finality, typically under super-majority rules in criminal procedure. See French [1964:151] on Athenian juries.

If normative sources applied to facts do not determine outcomes uniquely, the difficulties identified in the impossibility theorems recur under collective *judicial* decision-making. Kornhauser & Sager's seminal [1986] paper persuaded economists and political scientists alike to pay closer attention to the complex structures of adjudication. The discovery (in fact, re-discovery) of a new kind of impossibility, arising in environments with multiple judges, now called a *doctrinal paradox*, originates in their paper. Two lines of literature have developed. The first, known as logical aggregation theory, extends the axiomatic Arrovian preference-based theory to the aggregation of (individual) judgments on interconnected propositions.[25] The second line, case-based theory, puts structure on the domain of judicial preferences, allowing a more explicit role for legal doctrine.[26]

Both strands of literature demonstrate that the confluence of *legal doctrine* and *structures of adjudication* lead to new classes of paradoxes (impossibility theorems) and ambiguity in multi-member panels. These challenges remain,

---

[25] Building on Kornhauser & Sager [1986], the axiomatic theory starts with List & Pettit [2002]. However, as emphasized in Mongin [2012], their version of the paradox (called the *discursive dilemma*) deviates from the initial, legal formulation by letting judges vote on connection rules. Mongin describes the theoretical development with particularly close attention to the initial formulation (as explained below, in courts, voting generally does not take place on doctrine or connection rules as such).

    As pointed out in Nordén [2012,2015], the doctrinal paradox has been well known in European jurisprudence for hundreds of years, and has been continuously struggled with in Anglo-American procedural contexts. A main source is Albert Gottlieb Heckscher's [1892] largely unknown work. Heckscher was a practicing lawyer in Copenhagen. His dissertation was considered too esoteric by the law faculty there, but was recognized by Thorvald N. Thiele (one of Niels Bohr's teachers) and awarded the Dr.phil. degree at the Faculty of Mathematics and Natural Sciences. It has recently been argued that his work belongs "in the pantheon of the theory of social choice" (Lagerspetz [2014:338]).

[26] The case-based approach can be interpreted as concerning the evolution of norms in an extensive sense, explicitly modelling multi-member decisions. The approach originates in Kornhauser [1992a,1992b]; see generally Landa & Lax [2009].

    A quantitatively dominating third line of literature is based on rational choice theory and judges having preferences over general policy spaces (keywords: constitutional economics, positive political theory, principal agent, and team theory). This literature has a more implicit role for legal doctrine (see Landa [2007] on the distinction between the "case space" and "policy space" approaches).

even if judges vote non-strategically (sincerely). In short, the investigations have led to remarkable insights about the possibility of coherence and determinacy that challenge the normative expectations expressed in jurisprudence and reflected in the quotes above.

2 Conceptual framework, approach and main themes

As suggested in Section 1, legal decision-making—the processes leading to formally binding decisions $\langle x, y \rangle^M$ based on substantive norms in mechanisms $M \in \mathcal{M}$—has more structure than decisions in legislatures and decisions regarding conventions or contract formation in political or market contexts.[27] Under legal decision-making, sources of indeterminacy can be traced to:

- the structure of meta-norms (the doctrine of sources and judicial method);

- the structure of ordinary norms I (static considerations), such as the effects of logical, strategic, and cost constraints on promulgated law or contracts;

- the structure of ordinary norms II (dynamic considerations), which reflects the fact that even if norms initially *were* complete, "conditions change, and, no matter how explicit the code may have been in the first place determining how society shall act in different circumstances, its meaning becomes ambiguous with the passage of time"[28]; and

---

[27] No clear demarcation needs to be made between political and market choices (see Lax [2007]). In the limiting case of contracts formed in so-called perfectly competitive markets, there is no direct interaction among agents. The question of completeness takes on a different perspective, in this context, and concerns the span of markets (see e.g. Laffont [1989]).

Basic notions of enforceable property rights and mechanisms for their transfer are necessary in all but the most rudimentary contexts, and presuppose existence of collective goods. Resources necessary to establish and maintain these goods are called transaction costs (see e.g. Allen [2012:19,230]). In this sense, collective action is therefore prior to market mechanisms (Coleman [1987]).

[28] Arrow [2012:1]. In dynamic settings, even single judge decision-making creates possibilities of inconsistency and path dependence (see e.g. Saari [2001:117–19] on priority systems and Gilmore [1961] on "legal cycles").

- epistemic uncertainty concerning conditioning legal facts, $x \in X$.

The *Panel on Statistical Assessments as Evidence in the Courts,* while documenting a preoccupation with certainty and even the appearance of certainty in law, emphasizes "a probabilistic notion of justice" as "realistic, sensible, natural and inevitable."[29] Probability theory provides "a model for situations in which like or similar causes can produce one of a number of unlike effects."[30]

In this dissertation, norms—conceived as relationships between conditioning legal facts X and (conditioned) legal consequences Y —are *represented* as functions from X to the set of (simple) probability distributions over Y , denoted $\mathbb{P}_Y$. A norm is denoted $g : X \rightarrow \mathbb{P}_Y$, and the set of possible norms $\mathbb{P}_Y^X$. In this way, the Section 1 extensional representation is replaced with an intentional formulation: the objects in $\mathbb{P}_Y^X$ are *probabilistically complete*, assigning a unique distribution to each conditioning fact. At $x \in X$, the norm is locally represented by the conditional distribution or prospect $g(\cdot|x) \in \mathbb{P}_Y$. This definition is flexible and can represent locally or globally determinative norms, by letting g map to degenerate probability distributions at relevant fact bundles (regions) in X .[31] By slightly varying definition elements, a distinction is made between relationships expressing obligations (keywords: conduct or primary norms) and relationships expressing protected options, and, at the same time, possible uncertainty about the range of options (keywords: discretion, power-conferring or secondary norms). Both norm classes are important in legal systems; the latter, arguably, is

---

[29] Fienberg [1989:139].

[30] Bickel & Doksum [2001:441] (see Feynman [1994:121–42] for a fascinating explication).

[31] In these regions, like causes produce like effects.

more rarely directly maintained by courts (having received less attention in functionally oriented analyses).

Meta-norms have the same logical structure as ordinary norms, mapping from a set of legal sources LS (a product space of classes, $LS = LS_1 \times \cdots \times LS_l$, say) to the set of (simple) distributions over the set of ordinary norms, $\eta : LS \to \mathbb{P}_{\mathbb{P}_Y^X}$.[32] If the meta-norm image at source constellation $ls \in LS$ is non-degenerate, the resulting probability-weighted combination of norms (weights given by the prospect $\eta(\cdot | ls) \in \mathbb{P}_{\mathbb{P}_Y^X}$) is a convex combination of elements in $\mathbb{P}_Y^X$ and called a *mixed norm.* Importantly, mixed norms have the same structure as "pure" norms ($\mathbb{P}_Y^X$ is a convex set).

Judges are *identified,* in the ex ante sense, with meta-level norms and assumed to vote non-strategically (sincerely) in and across decision mechanisms $M \in \mathcal{M}$ as uncertainty is resolved. The identification of judges and norms reflects normative expectations, as formulated in Section 1, and envisions judging as *commitment*, in contrast to (purely) preference based accounts.[33] This repre-

---

[32] The source classes must represent all formants of the system (Sacco [1991b:343]).

[33] The notion of judging as *commitment,* in contrast to preference based rational choice accounts, is motivated by Sen [2009] (see in particular pp. 174–93 and the discussion of Adam Smith's *The Theory of Moral Sentiments*). It may also be noted that Heckscher [1892] conceives judicial decisions as representing cognition (*qvod est eller fuit*), in contrast to political decisions expressing preferences (*qvod fiery debet*), linking the distinction to judicial and parliamentary decision mechanism design and to expectations of non-strategic behaviour in the former (see Nordén [2015]). Judgments "are not merely reports of a judge's preference" (Coleman & Leiter [1993:600]).—"[C]courts must declare the sense of the law; and if they should be disposed to excercise *will* instead of *judgment*, the consequence would equally be the substation of their pleasure to that of the legislative body" (Alexander Hamilton in *The Federalist No. 78*, quoted from Ferejohn & Pasquino [2004:1680]).
    In legal proceedings, the bearing role of parties' claims and a doctrine such as *res judicata* attest to judges duty to follow stipulations and the laws institutional and supra-individual character (see e.g. James, Hazard & Leubsdorf [2001] on claim and issue preclusion; many dimensions of evidentiary rules could be added under epistemic uncer-

sentation and identification makes it possible to account for uncertainty (and discretion) on both the meta-level and ordinary level.[34] The approach has similarities to judgment aggregation theory, the case-space approach, and evolutionary theories of law.[35] However, discretion accentuates preference-based accounts of judging (if in a more limited sense), complicating the analysis of multi-member decision-making as seen in the ensuing discussion of law-in-force notions, where positive and normative questions merge.

Represented norms $g \in \mathbb{P}_Y^X$ are called *abstract,* and *mixed abstract*, if there is explicit uncertainty on the level of meta-norms, reflecting the two-stage character of legal reasoning. The process of law application in courts is conceived of as implying direct voting under abstract (mixed) norms, or under abstract (mixed) norm elements. Voting on complete norms is not admissible: The

---

tainty). In bifurcated criminal trials, a judge or juror concluding no liability, may fully participate in the determination of penal reactions (Nordén [2015]).

[34] Judges are fully informed about legal sources and applicable norms (common knowledge) and subscribe to the norms of (propositional) logic. It corresponds to a "zero method" approach, as advocated by Popper [1964] (see esp. pp. 140–42 and 158). Arguably, consistency on the level of individual judges is a fundamental element in any rule of law notion.
    Technically, judges (in most situations) are modeled as independent drawings from intangible populations. However, it does not mean that probability theory is suggested as a "plausible account of the semantics of [legal] discourse" (Coleman & Leiter [1993:610]). Independence, given the normative sources and facts produced at trial, is partly motivated by the idea of commitment, corresponding to so-called substantive ("functional" or "decisional") independence, which includes "internal" relations to joint panel members, judicial colleagues and superior judges (Shetreet [1985:630–38].) See also Caminker [1999] for a normative discussion of voting in multi-member courts.
    Intra-panel decisional independence can also be motivated by epistemic arguments, see Part VII below.

[35] As emphazised by Kornhauser [1992a:1], economists have predominantly analyzed effects of given norms, rather than their creation. The inattention to judicial process is ascribed to the difficulty of imputing a proper objective function to judges (courts "are not representative institutions" and "[o]n most jurisprudential accounts […] not even political institutions"). Kornhauser suggests that the development of evolutionary theories law is a reflection of these difficulties. Recent formulations use probabilistic norm formulations in single-judge contexts. See Borchgrevink [2011] for a discussion of the literature.

binding decisions concern ordered pairs $\langle x, y \rangle^M \in X \times Y$, not $\langle x, g \rangle^M \in X \times \mathbb{P}_Y^X$

or $\langle g \rangle^M \in \mathbb{P}_Y^X$. This restriction on procedures reflects a fundamental distinction

between legal decision-making (law application) and legislation or contract for-

mation.[36] Since abstract (mixed) norms are elements in $\mathbb{P}_Y^X$, and judges vote un-

der ordinary (mixed) norms, or on ordinary (mixed) norm elements, decision-

mechanisms can be analyzed as *operators* $M : \mathbb{P}_Y^X \to \mathbb{P}_Y^X$, transforming abstract

(mixed) norms in the ex ante sense. Images, denoted $M(g)$ or g_M, are called

transformed norms. The mechanisms are varied exogenously in $\mathcal{M}$, variations

referring to size and aggregation rules (e.g. the number of votes required for re-

jection of a default proposition or default state: voting on premises or out-

comes).[37] Sequential decision-making, and in particular bifurcated trials, are con-

sidered in detail.[38] Implications of separation of collective decisions on law and

on facts are studied under *joint* legal and epistemic uncertainty.

---

[36] A court generating decisions $\langle x, g \rangle^M$ forms views on aspects that go beyond the imme-
diate case and, hence, treads into the realms of other branches of government (which may
undermine its independence, Shetreet [1985:636]). The voting regime restriction is re-
flected in the formulation of the doctrinal paradox and the discursive dilemma as formal-
ized in Mongin [2012]. (The finer nuances in judgment aggregation theory are, regretta-
bly, beyond the reach of the present work.)
    Some constitutional courts vote on norms in abstraction from individual cases
(facts) $x \in X$ generating decisions $\langle g \rangle^M \in \mathbb{P}_Y^X$. However, they are staffed by specifical-
ly elected judges (and are, in fact, not called courts by some authors), see Section III.1.

[37] Procedural norms define the mechanisms, whose maintenance is made possible by hi-
erarchical court systems. In Elster's [2013:21–23] terminology, lower level courts are
heteronomous, apex courts autonomous (in effect controlling their own procedures).

[38] Landes [1998] argues that the sequential nature of legal decision making largely has
been overlooked in functionally oriented literature. For example, the U.S. Federal Rules
of Civil Procedure 42(b) gives courts wide discretion to invoke sequential decision mak-
ing. Rules of the American Arbitration Association and the World Intellectual Property
Organization allow bifurcation, while rules of the International Chamber of Commerce
and UNCITRAL do not address the issue. In criminal procedural law, bifurcation is gen-
erally compulsory, but at the same time, further decomposition of issue determination
blocked (keywords: protection of general verdicts).

The fact that images $M(g) = g_M \in \mathbb{P}_Y{}^X$ depend systematically on $M \in \mathcal{M}$ challenges a prevailing approach in legal theory of considering substantive norms as abstract entities, independent of the procedural mechanisms. It also suggests a way of opening the scope for the functional analysis of law: Agent equilibrium behavior arguably is guided not by abstract norms but by transformed norms (or their shadows).[39]

In order to demonstrate that the analytical framework is fruitful and operative, it is applied to a condensed analysis of final-offer arbitration and to an extensive analysis of the model of precaution. The analyses establish an equilibrium correspondence from ordered pairs of abstract norms and legal decision mechanisms $\langle g, M \rangle$ to the set of legal facts $x^* : \mathbb{P}_Y^X \times \mathcal{M} \rightrightarrows X$. The study of arbitration illustrates the confluence of substantive and procedural norms, even in single-judge cases. Although using a strategic equilibrium notion, arbitration turns out to be a remarkably simple context in which to start discovering the links between norm structures, multi-member decision-making, and equilibrium outcomes.[40]

The model of precaution has a prevalent and unifying role in many areas of law, and a central role in the development of the law and economics disci-

---

[39] Following North [1994], norms and their enforcement characteristics are formal constraints defining incentive structures, constituting (jointly with informal constraints such as conventions and self-imposed norms of conduct) *institutions*. The institutions, defining game structures, are distinguished from the agents and organizations (players).

[40] The model is a reinterpretation of Farber [1980] and Gibbons [1992:22–26], transposed to a multi-member context.

From the point of view of formal legal theory, there is no reason to distinguish between *arbitration* and *adjudication*, see Ross [1947:275–86]. Elster [1989:94] points out that mechanisms similar to final-offer arbitration were used by Athenian juries. However, from a functional perspective, there are important differences with respect to the environment in which arbitrational panels and full fledged courts operate (see Posner & Yoo [2005] and Section VI.3.2 below).

pline.[41] Hence, uncertain negligence standards, and *mixes* of negligence standards with strict liability, are analysed in a simple unilateral model with stochastic accident technology.[42] A distinction is made between discrete and continuous comparative statics. The former refers to the equilibrium effects of making mechanism changes. The latter concerns the impact from (parameterized) norm-based uncertainty, referring both to meta-level and ordinary norms.

The framework is applied to an analysis of law-in-force notions. In this system-oriented context, the distinction between norms representing obligations and norms representing protected options (discretion), takes on particular importance. Under discretion, a forward looking, means-end perspective on normative relationships are accentuated and, in fact, partly motivated by logical difficulties arising under aggregation. From a control theoretic point of view, under *structural* (epistemic) uncertainty, norm-based and preference-based notions of judging converge. In strategic environments, analyses of credibility and *institutional design* aimed at implementing formally enacted targets benefit from detailed attention to norm structures.

In environments with *additive* epistemic uncertainty, judges observe not actual or "true" states $x \in X$, but verifiable legal facts $\mathbf{x} = x + \varepsilon \in X$, where ε is an error term. Judges are assumed to have homogenous epistemic competencies (the same error distributions) and to observe facts independently (conditioned on information that becomes available in proceedings). Under (pure) legal uncertainty, norm structures and induced equilibria vary systematically with the procedur-

---

[41] See Cooter [1985] and Polinsky [2003] (Appendix), respectively.

[42] The analysis of (pure) uncertain negligence standards builds on the (implicitly) single-judge analyses Craswell & Calfee [1986] and Shavell [1987]. The analysis of mixed liability regimes is tentatively suggested by Posner [1990:438–39].

al mechanism design. This sensitivity is further exacerbated by epistemic uncertainty, accentuating dimensions of indeterminacy as discussed in legal and political philosophy (notions of justice and ontological questions). In reference to observed authority structures, the outcome in large majority voting mechanisms (or population majorities in suitably defined pools of potential adjudicators) is suggested as a benchmark with respect to *legal* variables. The benchmark makes possible a study of the link between actual (finite, costly, real-world) decision mechanisms $M \in \mathcal{M}$ and Type I and II *errors*.

3   Organization and outline

The dissertation contains nine main parts, numbered I–IX, and three appendices, A.1–A.3. The main parts contain subsections, each with a series of numbered propositions, equations, examples, remarks, and figures. Objects referred to without roman numerals belong to the same main part. Conclusion of proofs, examples, and remarks are indicated by ◄, ■, and □, respectively.

The subject matter has required attention to analytical detail, but hopefully that does not detract from the methodological perspectives offered. Formulation of explicit propositions, consistent notation, and more than 40 figures should help make the arguments clear. A series of examples illustrate concepts or introduce notions that recur in later sections. Proposition proofs are included in the running text. (Some have a purely technical character and can be skipped with no loss of meaning.)

Part II develops the general norm representation (and includes a primer on analytical concepts). Rules and standards are carefully distinguished, reflecting local and global notions of determinacy. The representation of the two-stage

character of legal reasoning implies that mixed norms occur when meta-norms are (locally) indeterminate. Convexity of $\mathbb{P}_Y^X$ simplifies the ensuing analyses.

Part III studies the transformation of (pure and mixed) norms in multi-member mechanisms. Particular emphasis is on (general) dichotomous situations, due to their prevalence from the characteristic sequential structuring of legal decision-making. Both majority and super-majority mechanisms are discussed (including asymptotic properties). Polychotomous situations are analyzed under the *classical rule,* a fundamental *legal* aggregation method that assumes an ordering of alternatives along a single dimension. Premise-based and outcome-based voting regimes are studied in the context of two-dimensional norms. Ordinary-level doctrinal paradox probabilities are calculated globally over the set of legal facts. The calculations are directly relevant to the ensuing comparison of unitary procedures (direct voting on consequences) and bifurcated procedures (first-stage decisions on liability, second-stage decisions on consequences of liability). In addition, voting rules and mechanism size (mechanism combinations, in the case of bifurcated trials) are demonstrated to affect the final probability distributions over legal consequences, enabling a first-order stochastic dominance characterization, in the case of low-dimensional norms. A separate section studies decision procedures determining legal standard elements prior to fact application. The aggregation method is based on the classical rule and named "theoretical norm element determination." The section yields analytical results that are used in the ensuing analyses. Part III is closed with a detailed study of majority mechanisms: Under equilibrium analysis, conditioning sets X correspond to choice or strategy spaces for optimizing agents. In this case, the aggregation-function transformation of the *marginal* dimension of legal uncertainty or incentive structures is

essential (transformation of the *level* dimension is handled by reference to Condorcet-type theorems, if not under the traditional parametric interpretation of probabilities). Majority transformation of a large norm class is characterized by the notions of *mean-preserving reduction in risk* and *second-order stochastic dominance*.

Parts IV and V concern equilibrium analysis, establishing a link between abstract (mixed) norms, decision mechanisms and outcomes in systems endowed by agents who are fully informed about the legal regimes. Part IV is limited to pure norms. First, (single issue) final-offer arbitration is studied. Contract-based standards are represented by uniform and normal distributions. "Contract zones" in Farber's [1980] terminology are established as functions of parameters, describing the norms and the arbitration panels. Next, uncertain negligence standards are analyzed, in the context of the model of precaution: Craswell & Calfee [1986] and Shavell [1987] have demonstrated that uncertainty about abstract norm content creates opposing effects on compliance incentives: On the one hand, a "level effect" from existence of uncertainty weakens incentives to invest in precaution. On the other hand, a "marginal effect" on liability probability strengthens investment incentives. Defining compliance as efficient precaution (minimization of total expected real costs), Calfee-Craswell-Shavell were able to demonstrate that overcompliance is connected to narrow (concentrated) uncertainty and undercompliance to broad (dispersed) uncertainty.

Judicial panels affect both the level and marginal dimension of uncertainty. Analysis of an abstract standard class, with the median equal to efficient precaution, shows that sequences of equilibria, induced by increasing majority panels, eventually converge toward the efficient point from above. However, while equilibrium investments do not switch between over and under-compliance, the

convergence need not be monotonic. The general analysis is enhanced by a detailed study of uniform standards: In this case, unique (global) solutions are ensured under abstract norms, and (continuous) comparative static analysis conducted in a parametric measure of legal uncertainty for fixed panel sizes. The abstract norm equilibria corroborate the Calfee-Craswell-Shavell results and are used as input to the analysis of panels. While some of the difficulties in characterizing undercomplying solutions recur, the input solutions allow sharper statements about panel-size effects (discrete comparative static analysis). The analysis is supplemented with simulations. The study of bounded standards demonstrate that determinative ("bright line") rules must be defined with care in the context of judicial panels: Due to the transformation of marginal incentives at the upper support boundary, a class of abstract norms that locally function as determinate cast a shadow in panels.

Part V focuses on equilibrium analysis under mixed norms. The effects of uncertainty between strict liability and global negligence rules and between strict liability and negligence standards are investigated. Comparative statics refer to both meta-level and ordinary-level norm-based uncertainty. Due to the analytical difficulties faced in Part IV, the analysis of panels is limited to mixes of global rules.

Part VI syntheses the Parts III through V analyses, focusing on system aspects. If both meta- and ordinary-level norms can be understood as obligatory (primary), "deductive" probabilistic law-in-force notions are almost directly suggested by the framework. If admissible and relevant, equilibrium analysis enables a sharpening of propositions. In subsets of legal source constellations, legal systems function as (locally) determinative, a crucial ability that is reflected in the developed notions.

35

If norms are conceived as power-conferring, be it at the meta or ordinary level, conceptual challenges arise in multi-member mechanisms, even in the absence of norm-based uncertainty. Because power-conferring norms have been given limited attention in functionally oriented literature, their representation is thoroughly discussed. A tentative analysis suggests that norm-based and preference-based notions of judging converge (and includes discussion of a recently developed generalized doctrinal paradox). Discretion and logical difficulties arising under aggregation accentuate a forward-looking, means-end perspective on law and are investigated from a control-theoretic perspective. Under a realistic assumption of structural epistemic uncertainty, the distinction between normative positions regulated by obligatory and power-conferring norms seems to evaporate.

The use of Pareto-efficiency as a means in norm design is discussed in a general equilibrium context. In the absence of anchoring in formal sources, positive analyses of simple exchange economies suggest there is value in distinguishing between efficiency as a criterion in contract and tort law. Power-conferring norms are used to illuminate rule-bound and discretionary policy implementation in strategic environments. Positive analyses suggest the value (and possibly legitimacy) of system-based commitment mechanisms constraining legislatures in (and only in) a limited subset of situations. The nuts and bolts character of Section VI.3 illustrates the limitations on regularity that arise from norm-based protected choice.

Part VII is on joint legal and (additive) epistemic uncertainty. Based on the suggested benchmark for decisions on legal variables, Type I and II error generation in multi-member mechanisms is discussed under normality assumptions and homogeneous, epistemic competencies. The effects of separating col-

lective decisions on facts and law are investigated. Analytical challenges have necessitated the use of asymptotic theory and a reliance on approximations. However, a tentative insight (with parallels in reliability theory) concerns the costs of splitting decision makers into subgroups. The question is of considerable interest, because decisions on law and facts are often separated in legal decision mechanisms.

Part VIII suggests further applications and framework development. Part IX concludes. Simulation results used in Parts IV and V are reported in Appendix A.1. Appendices A.2 and A.3 give additional results on mixed and transformed power-conferring norms, respectively.

PART TWO

REPRESENTATION OF NORMS

1   Basic concepts and notation

Part I introduces analytical notions, such as relations and (descriptive) functions. This section provides precise definitions and introduces concepts and notation that will be used in the ensuing analyses.[1]

The abstract sets used to define legal relations in Part I, allow general interpretations. They will given more structure in the applications and defined by enumeration of elements (definition by "extension"), or by propositions characterizing the elements (definition by "intention"). In some applications, sets correspond to *sample spaces* in the sense of probability theory. Subsets that exhibit common features are called *events*.[2]

If V and V' are two sets, $V \setminus V'$ denotes *set difference*, that is, the set of elements in V but not in V'. Assume that V has a finite number of L elements, denoted $\#|V| = L$. The *power-set* of V, $\wp V$, is the set of all subsets of V, including the empty set $\varnothing$ and the set V itself:

---

[1] The following is largely abridged from Suppes [1972], Bartle [1976], and Carter [2001].

[2] Individual sample space elements are called *elementary outcomes*, and are mutually exclusive outcomes of actual or conceptual *experiments* (Bickel & Doksum [2001], Appendix A; Bartoszyński & Niewiadomska-Bugaj [1996], Ch. 1).

$$\wp V = \left\{ \varnothing, \left\{ v^1 \right\}, \dots, \left\{ v^L \right\}, \left\{ v^1, v^2 \right\}, \dots, \left\{ v^1, \dots, v^{L-1} \right\}, V \right\}.\,^3$$

An ordered pair (couple) is formed by two objects in a fixed order. $\langle u, v \rangle$ denotes the ordered pair whose first member is u and second member is v. Two ordered pairs $\langle u, v \rangle$ and $\langle u', v' \rangle$ are identical if and only if $u = u'$ and $v = v'$, The Cartesian product of two (non-empty, but not necessarily distinct) sets U and V, denoted $U \times V$, is the set of all ordered pairs which can be formed from the two sets in a fixed order: $U \times V = \left\{ \langle u, v \rangle : u \in U \text{ and } v \in V \right\}$.

Notions of relationship—that things stand in some given relation to each other—can be made precise with the notion of a (binary) relation: A *binary relation* R is a set of ordered pairs, $R \subseteq U \times V$. Equivalent notation for relation membership $\langle u, v \rangle \in R$, is uRv. The relation's *domain*, $D(R)$, is the set of all $u \in U$ such that, for some v, $\langle u, v \rangle \in R$. Its *counter-domain* (range), $C(R)$, is the set of $v \in V$ such that for some u, $\langle u, v \rangle \in R$.[4]

A relation $f \subseteq U \times V$ is a *function* if $\langle u, v \rangle \in f$ and $\langle u, v' \rangle \in f$ implies that $v = v'$: a function is a binary relation which relates a unique element in the counter-domain to each element in its domain. The element v is called the *image* of u under f, and denoted $f(u)$. Alternatively, the argument u is said to be mapped to $v = f(u)$ by f, or $f : u \mapsto v$. It is important to distinguish individual

---

[3] If V has $\#|V| = L$ members, the power-set has $\#|\wp V| = 2^L$ elements (sets), explaining the origin of the name (Suppes [1972:47]).

[4] Each $u \in U$ may be a list of v aspects (an ordered v-tuple) $u = \langle u_1, \dots, u_v \rangle$ and each $v \in V$ an ordered τ-tuple $v = \langle v_1, \dots, v_\tau \rangle$. In this case, U and V are the Cartesian products $U = U_1 \times \dots \times U_v$ and $V = V_1 \times \dots \times V_\tau$, respectively (themselves examples of relations).

elements of the function and the function itself. [5] The symbols f, $f(\cdot)$ and $f:U \to V$ are used to denote functional relations. In the latter case, it is understood that $U = D(f)$, but it is possible that $C(f)$ is a proper subset of V (V contains elements not in $C(f)$). The ordered pairs that constitute the function are called the function *graph*, conveniently used to visualize relations. The following definitions will be employed:

If $f:U \to V$ and $U' \subseteq U$, the function $f_1 : U' \to V$, defined by $f_1(u') = f(u')$ for $u' \in U'$ is called the *restriction* of f to U' (also written $f_1 = f \mid_{U'}$). If $U'' \supseteq U$ then any function f_2 with domain U'' such that $f_2(u) = f(u)$ for all $u \in U$ is called an *extension* of f to U''.

Let $f:U \to V$. f is called *injective* if $f(u_1) = f(u_2)$ implies $u_1 = u_2$ for every $u_1, u_2 \in D$. If $C(f) = V$, f is called *surjective*. A function which is both injective and surjective is called *bijective*.

Let $f:U \to V$ be bijective. Then there exists an *inverse* function $g:V \to U$, denoted $g = f^{-1}$, such that $g(f(u)) = u$ for all $u \in U$, and $f(g(v)) = v$ for all $v \in V$.

Let $f:U \to V$ and $U' \subseteq U$, $V' \subseteq V$. The *direct image of U' under f* is $f(U') = \{f(u) : u \in U'\}$. The *inverse image of V'* is $f^{-1}(V') = \{u : u \in U \text{ and } f(u) \in V'\}$. [6]

---

[5] If the function domain is a product set $D(f) = U_1 \times ... \times U_\nu$, the notation $f(\langle u_1,...,u_\tau \rangle)$ is conventionally simplified to $f(u_1,...,u_\tau)$.

[6] It is not necessary that f is bijective to define $f^{-1}(V') \subseteq U$.

Let U, V and W be sets and $f : U \to V$ and $g : V \to W$. The *composite function* $g \circ f : U \to W$ is defined by $g \circ f(u) = g\big(f(u)\big)$, for each $u \in U$.

Let $f : U \to V$. The *set* of functions from U to V is denoted V^U. Finally, a *correspondence* ϕ is a generalized (multi-valued) function from U to V, assigning a non-empty set $\phi(u) \subseteq V$ to each $u \in D(\varphi) = U$. It is denoted $\phi : U \rightrightarrows V$.[7]

2 Legal norms: definitions and examples

As discussed in Part I, the extensive perspective on legal norms corresponds to an ex post vision of law, and possibly a very incomplete one: If R denotes a legal relation in $X \times Y$, its domain $D(R)$ typically is a proper subset of the set of conditioning facts, X.[8] To study norms from a realistic ex ante perspective, extensive representations are replaced by intensions. The suggested notion assigns a consequence (or a set of consequences), to any fact bundle in $X \supseteq D(R)$, if only in a probabilistic sense: the norm is (probabilistically) complete.[9] This representation combines functional relations with basic notions from probability theory.[10]

---

[7] A correspondence can also be interpreted as a special type of relation on $U \times V$ (Carter [2001:178]).

[8] To exemplify: If a statute is promulgated according to system-specific norms (qualifying as an element in the legal source set), the norm is empty in the extensional sense ($R = \varnothing$) because no judgments have been rendered. Still, it is not satisfactory to claim that it does not exist. Similar remarks apply more generally to norms not directly maintained by courts.

[9] See generally Kaplow & Shavell [2002:436–44] on the importance of studying norms from an ex ante and global perspective (to consider all possible configurations of conditioning facts).

[10] The notions of experiment, sample space, events, and elementary outcomes and have been mentioned. A probabilistic description of an experiment has three elements: (i) the sample space, (ii) a set of "probabilizable" subsets in the sample space, and (iii) a proba-

The conditioning facts $x \in X$ are assumed non-stochastic and verifiable (assumptions relaxed in Part VII).

Let $\mathbb{P}_Y$ be the set of simple probability distributions over the set of legal consequences, *Y*. Let $g : X \to \mathbb{P}_Y$ be a function from $X = D(g)$ to $C(g) \subseteq \mathbb{P}_Y$.

Definition 2.1 A norm is a triple $\left(X, \mathbb{P}_Y, g : X \to \mathbb{P}_Y \right)$. The set of possible norms (functions g from *X* to $\mathbb{P}_Y$) is denoted $\mathbb{P}_Y{}^X$.

Remark 2.1 The general norm concept is formally equivalent to an object that occurs in axiomatic (subjective) expected utility theory.[11] In the context of decision mechanisms, the norm defined in 2.1 is called *abstract*, in contradiction to norms *transformed* in mechanisms $M \in \mathcal{M}$. Adding epistemic uncertainty, the objects become similar to so-called information structures with noise, used in mathematical communication theory and in information economics.[12] □

Remark 2.2 Functions (norms) define their own domains. Probabilities are a part of the *objective* norm description. The distributions $g(\cdot | \cdot) \in \mathbb{P}_Y$ can take any form, as long as they satisfy the axioms of probability. Probability is linked to (conceptual) random experiments. Their interpretation vary in different situations. It expresses notions of regularity (random does not mean arbitrary!). □

---

bility measure assigning a number between 0 and 1 to each of the subsets. See generally Bickel & Doksum [2001], Appendix A, and Bartoszyński & Niewiadomska-Bugaj [1996], Ch. 2.

[11] Technically, Definition 2.1 corresponds to so-called Savage-acts with an added element of probability distributions as introduced by Anscombe & Aumann [1963] (see, generally, Kreps [1988] and Gilboa [2009]).

[12] See Laffont [1989].

Definition 2.1 implies that *each* conditioning legal fact is mapped to a distribution over the set of legal consequences, $x \mapsto g(\cdot \mid x) \in \mathbb{P}_Y$: $g(\cdot \mid x)$ is the *image* of $x \in X$. The simple distribution $g(\cdot \mid x)$ at $x \in X$ assigns probability to a finite number of elements in Y, notably in its support, $\operatorname{supp} g(\cdot \mid x) = \{y^1, y^2, \ldots, y^L\} \subseteq Y$, say. Locally, for a given conditioning fact $x \in X$, the norm corresponds to the *prospect*[13]

$$g(\cdot \mid x) \equiv \left\langle g(y^1 \mid x), g(y^2 \mid x), \ldots, g(y^L \mid x) \right\rangle.$$

For later reference, let the number of elements in the support, conditioned on x, be denoted $\tau^g(x)$ ($\tau^g(x) = \#\left| \operatorname{supp} g(\cdot \mid x) \right|$). The prospect is a row vector with dimension $\tau^g(x)$ and may be depicted as the chance node in Figure 2.1:

$$x \mapsto \quad \begin{array}{l} g(y^1 \mid x) \quad y^1 \\ g(y^2 \mid x) \quad y^2 \\ \vdots \\ g(y^L \mid x) \quad y^L \end{array}$$
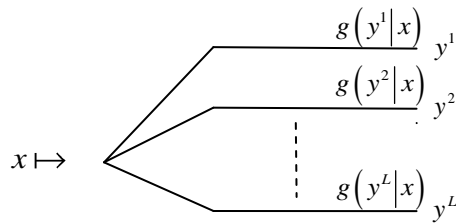
Figure 2.1 Local representation of g

For clarity, elements in $\operatorname{supp} g(\cdot \mid x)$ may be included in the prospect description:

$$\left\langle g(y^1 \mid x), g(y^2 \mid x), \ldots, g(y^L \mid x); y^1, y^2, \ldots, y^L \right\rangle.$$

The norm associates a probability distribution (possibly degenerate) to every element in its domain, $X = D(g)$. In this sense, g is *complete*. Figure 2.2 exemplifies a norm assigning a degenerate distribution $g(\cdot \mid \bar{x})$ to $\bar{x}$

---

[13] See Borch [1968:23].

($\operatorname{supp} g\left(\cdot\mid\overline{x}\right)=\left\{y^1\right\}$) and a non-degenerate distribution $g\left(\cdot\mid\overline{\overline{x}}\right)$ to $\overline{\overline{x}}$

($\operatorname{supp} g\left(\cdot\mid\overline{\overline{x}}\right)=\left\{y^{k'},y^k\right\}$).
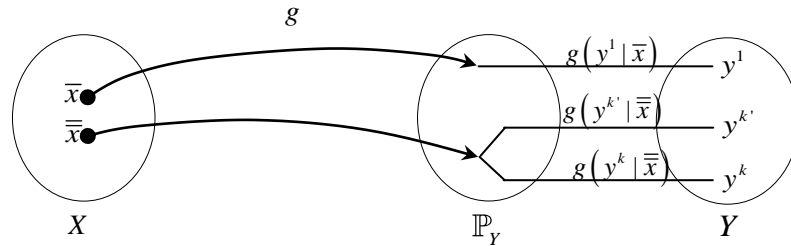
Figure 2.2 Global representation of g

Being a function, g assigns one and only one element in $\mathbb{P}_Y$ to each element in its domain X. It may assign the same element in $\mathbb{P}_Y$ to a subset (even all) elements in X (see Figure 3.1). If subsets in X are focused, $\underline{X}\subseteq X$ say, $g\big|_{\underline{X}}$ denotes the restriction of g to $\underline{X}$. The importance of keeping a sufficiently global perspective on norms will become evident when the set of legal facts is constituted by choice or strategy sets for optimizing agents.

In representing norms, system-aspects inform the choice of suitable sets X and Y (keyword: individuation). Functions may be composed (or decomposed!) to obtain adequate formulations (this will be demonstrated at several points).[14]

In Figure 2.2, the image of g at $\overline{x}$ is the prospect $g\left(\cdot\mid\overline{x}\right)=\left\langle g\left(y^1\mid\overline{x}\right)\right\rangle$ and at $\overline{\overline{x}}$, $g\left(\cdot\mid\overline{\overline{x}}\right)=\left\langle g\left(y^{k'}\mid\overline{\overline{x}}\right),g\left(y^k\mid\overline{\overline{x}}\right)\right\rangle$. With a view to simplifying the presentation, it will be convenient to represent norms as prospects of a uniform

---

[14] See generally Bartle [1976:15–18] and Carter [2001].

dimension, $\tau^g = \#\left|\bigcup_{x \in X} \operatorname{supp} g(\cdot \mid x)\right|$. For example, if $\bar{x}$ and $\bar{\bar{x}}$ are the only elements in X, $\tau^g = 3$, the representation is:

$$\begin{cases} g(\cdot \mid \bar{x}) = \left\langle 1, 0, 0; y^1, y^{k'}, y^k \right\rangle \\ g(\cdot \mid \bar{\bar{x}}) = \left\langle 0, g\left(y^{k'} \mid \bar{\bar{x}}\right), g\left(y^k \mid \bar{\bar{x}}\right); y^1, y^{k'}, y^k \right\rangle \end{cases}.$$

The following paradigmatic example demonstrates that the suggested norm definition is relevant and introduces notation and concepts that will be used repeatedly.

Example 2.1 (one-dimensional legal standards). Let a be a proposition such as 'the agent's driving is negligent', 'the agent's pollution is a negligent taking of entitlement', and let $a = 1$ and $a = 0$ denote that the proposition is found to hold and not hold, respectively (also denoted a, $\neg a$). Let $x_a \in X_a$ be an index of facts deemed relevant to the evaluation of a under applicable law, such as speed or amount of pollution type per time period. Assume that all other variables of interest are known and constant. The legal standard can be represented by a *stochastic variable* A on the set of legal facts $X_a = \mathbb{R}_+$ with *cumulative distribution function* (cdf) $F_A : \mathbb{R} \to [0,1]$. The cdf is assumed to be continuous and bounded with support $[\underline{x}_a, \bar{x}_a] \subset \mathbb{R}_{++}$, and sufficiently smooth to have a density f_A.[15] Liability occurs iff $A \le x_a$. The interpretation is that sufficiently high speeds or pollution levels $x_a \ge \bar{x}_a$, imply liability with certainty, and sufficiently low speeds or

---

[15] $\mathbb{R}$ and $\mathbb{R}_+$ denote the set of real and positive real numbers, respectively. On cdfs and densities (or probability mass functions), see Apostol [1969], Ch. 14, and Bartoszyński & Niewiadomska-Bugaj [1996], Ch. 6.

   The use of bounded legal standards is instructive with respect to classification of norms (Section 3), and will be seen to have interesting consequences in the context of judicial panels. For analytical convenience, unbounded (normal) standards are assumed in Example IV.2.2 and Part VII and appear as limit distributions in other contexts.

   The terminology regarding the (normative) propositions is chosen to avoid discussion about truth value (see, generally, Cohen & Nagel [1964], Ch. 18 and Sections III.1.2, and VI.3 below).

pollution levels $x_a \leq \underline{x}_a$ imply no liability with certainty. In interim situations, $x_a \in (\underline{x}_a, \overline{x}_a)$, the law is indeterminate from the ex ante perspective. [16] As a mnemonic device, liability and no liability are denoted $\otimes$ and $\oplus$, respectively. It follows that

$$P\{a=1|x_a\} = P\{\otimes|x_a\} = P\{A \leq x_a\} = F_A(x_a) = \int_{-\infty}^{x_a} f_A(\tau)d\tau .$$

The situation is illustrated in Figure 2.3.
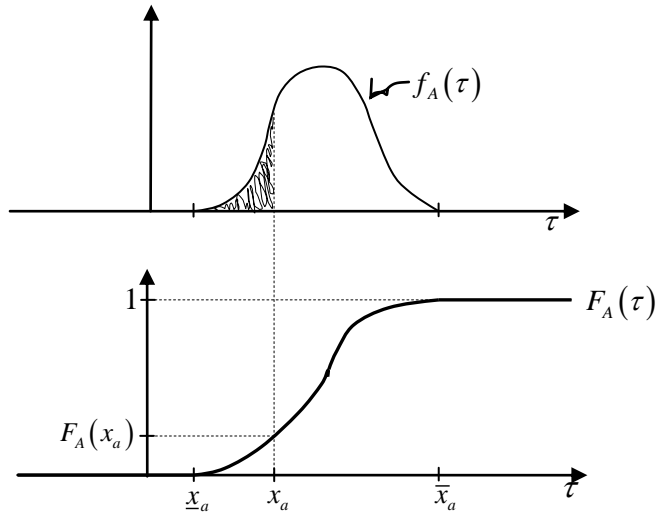
Figure 2.3 A legal standard density and cumulative distribution function

Each conditioning fact $x_a \in X_a$ is mapped to a probability distribution over the set of legal consequences, $Y = \{\oplus, \otimes\}$. Locally, $g(\cdot|x_a) = \langle g(\oplus|x_a), g(\otimes|x_a) \rangle$ $= \langle 1 - F_A(x_a), F_A(x_a) \rangle \in \mathbb{P}_{\{\oplus,\otimes\}}$ is depicted in Figure 2.4. Outside its support, the distribution degenerates and the norm locally functions as a rule (see Section 3 for precise definitions). Globally, the norm is the mapping $g : X_a \to \mathbb{P}_{\{\oplus,\otimes\}}$, given by:

$$\langle g(\oplus|x_a), g(\otimes|x_a) \rangle \big|_{x \in X_a} = \langle 1 - F_A(x_a), F_A(x_a) \rangle \big|_{x \in X_a} .$$

---

[16] The interval boundary descriptions are not important if the cdfs are is continuous. They are chosen with a view to analytical convenience in later parts, and the convention of defining cdf's as right continuous (Bartoszyński & Niewiadomska-Bugaj [1996:165]).

In compact notation, $g_{F_A} \in \mathbb{P}_{\{\oplus,\otimes\}}{}^{X_a}$. A perturbation of F_A (f_A) leads to a different norm

in $\mathbb{P}_{\{\oplus,\otimes\}}{}^{X_a}$. ∎

$$x_a \longmapsto \begin{cases} g(\oplus \mid x_a) = 1 - F_A(x_a) \quad \oplus \\[2em] g(\otimes \mid x_a) = F_A(x_a) \quad \otimes \end{cases}$$
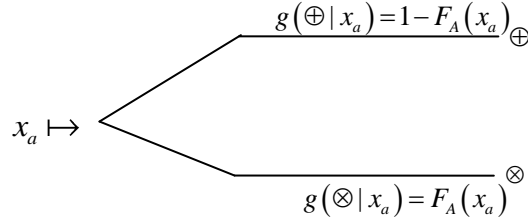
Figure 2.4 Chance node representation

<u>Remark 2.3.A</u> Taroni et al. [2006:37] date Leibniz' transformation of propositions into numerical values to 1669 (Section I.2). Modern probabilistic formulations of (abstract, one-dimensional) norms include Craswell & Calfee [1986] and Shavell [1987]. □

<u>Remark 2.3.B</u> Example 2.1 is intended to represent a ("primary") conduct norm, with a "core of certainty" [$x_a \notin \operatorname{supp} f_A$] and "a fringe of vagueness or 'open texture'" [$x \in \operatorname{supp} f_A$], to use a well-known formulation.[17] □

<u>Remark 2.4</u> Assume that a court M at time t is to decide a case defined by $_t x_a \in (\underline{x}_a, \overline{x}_a)$, based on $\langle 1 - F_A(_t x_a), F_A(_t x_a); \oplus, \otimes \rangle$. After the judgment is rendered and becomes final, $\langle _t x_a, \otimes \rangle^M$ or $\langle _t x_a, \oplus \rangle^M$ becomes a legal system element in the extensive sense "plot[ing] a point on the graph of tort" [18]. Meta-

---

[17] Hart [1961:119–20].

[18] G. W. Paton cited from Ross [1959:88].

norms (the doctrine of precedent) suggest that the standard should be updated to

cdf $F_A\left(\tau \,|\, \langle {}_t x_a, \otimes \rangle^M\right)$ or cdf $F_A\left(\tau \,|\, \langle {}_t x_a, \oplus \rangle^M\right)$.[19] $\square$

Example 2.2 (higher-dimensional standards). Assume that second norm dimension is described by a proposition b (e.g. referring to a driver's mental state, or to causation regarding pollution), and let $x_b \in X_b$ be an index of facts relevant to the proposition evaluation, with $P\{b=1|x_b\} = F_B(x_b)$. Let c denote the proposition that an agent is liable, and assume that doctrine dictates liability iff a and b :

$$a \wedge b \leftrightarrow c \tag{2.1}$$

Let $X = X_a \times X_b$, $\otimes = c$, and $\oplus = \neg c$. If the norm dimensions are stochastically independent, $g \in \mathbb{P}_{\{\oplus,\otimes\}}{}^{X_a \times X_b}$ is given by:

$$g\left(\cdot \,|\, x_a, x_b\right)\Big|_{\langle x_a, x_b\rangle \in X_a \times X_b} = \Big\langle 1 - F_A(x_a) F_B(x_b), F_A(x_a) F_B(x_b); \oplus, \otimes \Big\rangle \Big|_{\langle x_a, x_b\rangle \in X_a \times X_b}. \tag{2.2}$$

Two-dimensional, orthogonal and conjunctive norm formulations suffice for the applications in Part III.[20] $\blacksquare$

Remark 2.5 Propositional logic dominates legal literature on judgment aggregation, while more sophisticated formulations are used in logical aggregation theory.[21] Both literatures represent legal doctrine on a "second-stage level" (as illus-

---

[19] Under a doctrine of *stare decisis* an updated cdf $F_A\left(\tau \,|\, \langle {}_t x_a, \otimes \rangle^M\right)$ should have a new upper support boundary ${}_t \bar{x}_a = {}_t x_a < \bar{x}_a$ (or $F_A\left(\tau \,|\, \langle {}_t x_a, \oplus \rangle^M\right)$ a new lower support boundary ${}_t \underline{x}_a = {}_t x_a > \underline{x}_a$). Borchgrevink [2011] reviews and contributes to a literature on case selection, evolution of precedent, and dynamic norm notions in a single-judge context.

[20] Formulation (2.2) generalizes to higher dimensions. The disjunctive form $a \vee b \leftrightarrow c$ under independence gives $\langle 1 - \{F_A + F_B - F_A F_B\}, F_A + F_B - F_A F_B; \oplus, \otimes \rangle$. If the norm dimensions are stochastically dependent, the multiplicative forms are replaced by bi- or multivariate cdf's (see Nordén [2015]).

[21] See, generally, Mongin [2012].

trated by (2.1)), with no explicit role for the underlying conditioning legal facts

$\langle x_a, x_b \rangle$. A similar remark applies to the case-space approach. □

Remark 2.6 Inspired by procedural literature, the terminology *ultimate legal facts*

will be used about norm premises, such as a ("liability") and b ("causality").[22]

The terminology, however, is not stabilized.[23] □

Remark 2.7 Discussions of legal relations typically concern ultimate legal facts

(Ross' [1959], esp. Sec. 35, is a prominent example). Arguably, the analyses can

be enhanced by the use of notions from general relation theory. Recent contribu-

tions include an ex ante perspective in rich environments (see e.g. Cooter [1998]

and the references to option theory in Remark 5.1 below). □

Under the *classical* aggregation-rule assumptions discussed in Part III, the set of

legal consequences Y is, or can be, associated with a subset of the real numbers.

Let $\text{supp } g(\cdot \mid x) = \{t^1, t^2, ..., t^L\}$, with elements named such that $t^k < t^{k'}$ if $k < k'$.

---

[22] "When [the judge] tells the jury how to determine what [a and b] are, he is giving the legal definitions. Thus, the legal definitions are the norms which must be applied to the evidentiary facts in order to derive the ultimate facts [a and b] which, once a legal rule is applied [$a \wedge b \leftrightarrow c$], will provide a result." Dudnik [1965:493].

[23] Ultimate facts are called "legal facts" in Coleman & Leiter [1993]. Elements in the *case space* defined in Example 3.5 below, are called "legal findings" or "doctrinal factors", and "may be thought of as a particular mix of both purely objective facts and intermediate legal conclusions" Landa & Lax [2009:949]. Ross [1959:215–16] uses the term "operative facts" about ultimate and non-ultimate facts: "Some operative facts are purely factual (for example, birth, death, fire, collision at sea), others are legally conditioned, which means they are defined relative to the law. [---] this means that legal rule R_1 describes its operative facts, not directly, but by reference to the circumstances operative in relation to other legal rules R_2, R_3, and so on. [---] The same is also the case where a term has reference not to a formal legal rule, but to a legal standard. That a person has acted 'negligently,' for example, is not a purely factual statement, but has reference to a presupposed standard in respect of the course of action that can be expected by a reasonable man in the given situation." The quote points to system-aspects (individuation). As long as any chain component has standard character, from the ex ante perspective, a non-degenerate distribution over a penultimate set of consequences is implied.

In this class of situations, it is convenient to represent norms with a discrete conditional (right continuous) cdf $G(\cdot \mid x): \mathbb{R} \to [0,1]$:[24]

$$G(t \mid x) = \begin{cases} 0 \text{ if } t < \min\{\operatorname{supp} g(\cdot \mid x)\} = t^1 \\ g(t^1 \mid x) \text{ if } t \in [t^1, t^2) \\ g(t^1 \mid x) + g(t^2 \mid x) \text{ if } t \in [t^2, t^3) \\ \vdots \\ 1 \text{ if } t \geq \max\{\operatorname{supp} g(\cdot \mid x)\} = t^L \end{cases} \tag{2.3}$$

The cdf is illustrated in Figure 2.5.
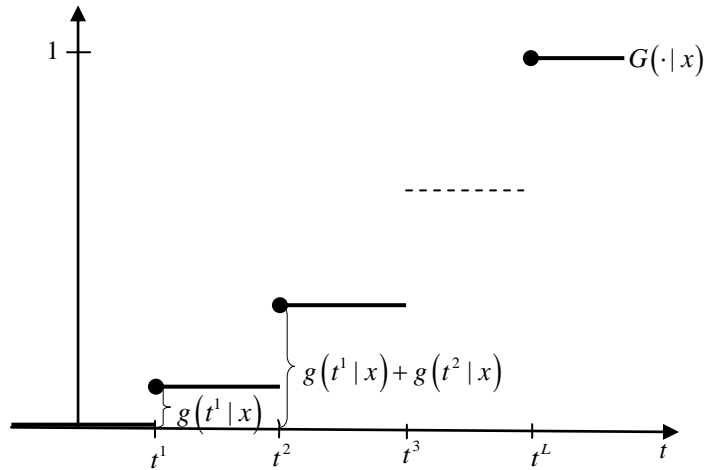
Figure 2.5 Cdf representation of $g \in \mathbb{P}_{\mathbb{R}}^{X}$

Example 2.3 (Ex. 2.1 continued). In the dichotomous case $Y = \{\oplus, \otimes\}$, no liability can be associated with $t^1 = 0$ and liability with $t^2 = 1$. Hence, at $x \in X$, $g \in \mathbb{P}_{\mathbb{R}}^{X}$ is given by $g(\cdot \mid x) = \langle g(0 \mid x), g(1 \mid x) \rangle = \langle 1 - F(x), F(x) \rangle$, with cdf:

---

[24] $G(\cdot \mid x)$ "jumps" at all points in $t^k \in \operatorname{supp} g(\cdot \mid x)$, with jumps equal to the point probabilities $g(t^k \mid x) = G(t^k \mid x) - \lim_{t \to t^{k-}} G(t \mid x)$, $k = 1,...,L$.

$$G(t|x) = \begin{cases} 0 \text{ if } t < 0 \\ 1 - F(x) \text{ if } t \in [0,1) \\ 1 \text{ if } t \geq 1 \end{cases}.$$

■

Definition 2.1, mapping legal facts directly to distributions over individual (conditioned) legal consequence bundles $y \in Y$, is well suited to model duty-imposing ("primary") norms (obligations). However, legal systems contain norms which endow agents with *options* or *discretion.* To model protected choices, and (possible) uncertainty about their range, a generalization of the representation is suggested:

> <u>Definition 2.1'</u> Let $\wp Y$ be the power-set of Y. A norm is a triple
> $\left(X, \mathbb{P}_{\wp Y}, g : X \to \mathbb{P}_{\wp Y} \right)$. The set of norms (functions g from X to $\mathbb{P}_{\wp Y}$)
> is denoted $\mathbb{P}_{\wp Y}{}^X$.

The definition is illustrated in Figure 2.6. For each $x \in X$, the support is a *family* of sets, $\operatorname{supp} g\left(\cdot | \bar{x} \right) = \left\{ \left\{ y^k, y^{k'} \right\} \right\}$; $\operatorname{supp} g\left(\cdot | \bar{\bar{x}} \right) = \left\{ \left\{ y^{k'}, y^{k''} \right\}, \left\{ y^{k'}, y^{k''}, y^{k'''} \right\} \right\}$). In situation $\bar{x}$, options y^k and $y^{k'}$ are certainly available. Likewise, in situation $\bar{\bar{x}}$, options $y^{k'}$ and $y^{k''}$ are certainly available, but option $y^{k'''}$ only with probability $g\left(\left\{ y^{k'}, y^{k''}, y^{k'''} \right\} | \bar{\bar{x}} \right)$.[25]

<u>Remark 2.8</u> Importantly, legal systems contain norms that constitute the ability to select new norms becoming system elements (legal formants), giving legal systems their characteristic dynamic character. To model "power-conferring" or

---

[25] Section VI.2 introduces notions that describe these positions more efficiently.

"secondary" norms (including delegation arrangements), Y is taken to be a set of ordinary norms $\mathbb{P}_Y^{\ X}$, and $\wp\mathbb{P}_Y^{\ X}$ is a family of norm sets, see Section VI.2.[26] □
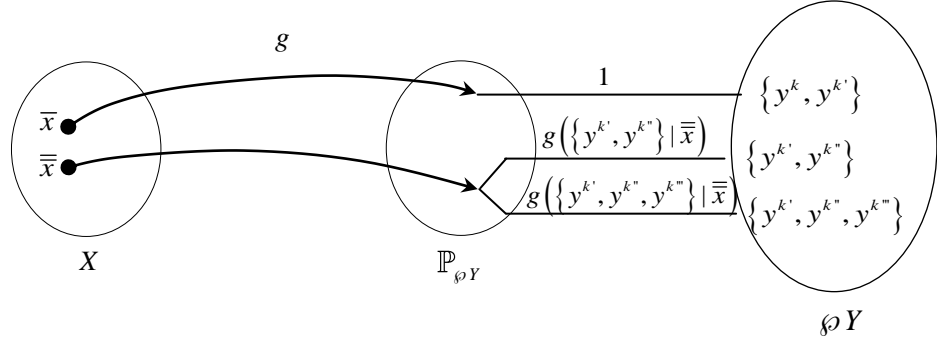
Figure 2.6 Global representation of g (the set of consequences a power set)

3   Rules and standards: local and global definitions

A degenerate probability distribution in $\mathbb{P}_Y$ assigns all probability mass to a single element in Y (such as $g(\cdot\,|\,\overline{x})$ in Figure 2.2 leading to y^1 with probability 1). Let the subset of degenerate probability distributions in $\mathbb{P}_Y$ be denoted $\delta\mathbb{P}_Y$. The *inverse image* of $\delta\mathbb{P}_Y$ under g, $g^{-1}(\delta\mathbb{P}_Y)\subseteq X$, is given by

$$g^{-1}(\delta\mathbb{P}_Y)=\left\{x\in X\,\middle|\,g(\cdot\,|\,x)\in\delta\mathbb{P}_Y\right\}.$$

> <u>Definition 3.1</u> Norm $g\in\mathbb{P}_Y^{\ X}$ locally has *rule character* at all points in $g^{-1}(\delta\mathbb{P}_Y)$, and locally *standard character* at all points in $X\setminus g^{-1}(\delta\mathbb{P}_Y)$.

> <u>Definition 3.2</u> Norm $g\in\mathbb{P}_Y^{\ X}$ is a *global rule* if $g^{-1}(\delta\mathbb{P}_Y)=X$ and a *global standard* if $g^{-1}(\delta\mathbb{P}_Y)=\varnothing$.

---

[26] See Ross [1959,1962], Hart [1961] and Tur [1998] on norm notions. Contestable concepts in norms (Waldron 1994) arguably also call for a Definition 2.1' type representation.

<u>Definition 3.3</u> Let $x \in g^{-1}(\delta\mathbb{P}_Y)$. The unique element to which g in effect maps is denoted $!y^{x,g}$.

The set of global rules or globally determinative norms, which map all $x \in X$ to $\delta\mathbb{P}_Y$, is denoted $\delta\mathbb{P}_Y^{\,X}$. Norms in $\mathbb{P}_Y^{\,X} \setminus \delta\mathbb{P}_Y^{\,X}$ will be called *standards*. Non-global standards are determinative on subsets of X (locally have rule character). Because global rules $g \in \delta\mathbb{P}_Y^{\,X}$, in effect, map conditioning facts $x \in X$ to unique elements $!y^{x,g} \in Y$, a more direct representation is as *functions* $r_g : X \to Y$. The set of such functions from X to Y is denoted Y^X. Figure 3.1 illustrates the set of such rules from a two-point set X to a two-point set Y.[27] The mode of representation is a matter of convenience. The seemingly more complex formulation $g \in \delta\mathbb{P}_Y^{\,X}$ will, in fact, in many situations, be easier to handle than the corresponding $r_g \in Y^X$.[28]
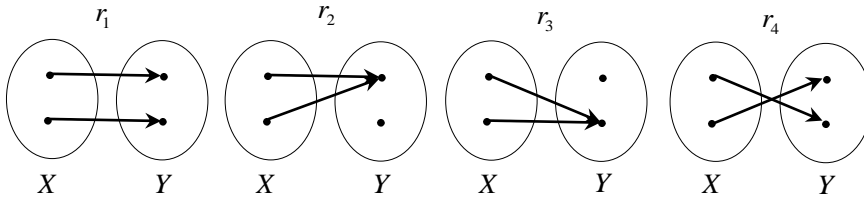
Figure 3.1 The universe of global rules from a two-point set X to a two-point set Y

---

[27] It is worth noting that that Y^X grows rapidly in the number of defining set elements: Assume that X and Y are finite with $\#|X| = m$ and $\#|Y| = L$ ($\#|\delta\mathbb{P}_Y| = L$), respectively. It follows that $\#|Y^X| = \#|\delta\mathbb{P}_Y^{\,X}| = (L)^m$ (Cameron [1999], Theorem 1.13); a remarkably large number even for "small" sets. (The introduction of probabilities means a continuum of possibilities even if $\#|X| = 1$ and $\#|Y| = 2$.)

[28] In contrast to Y^X, $\delta\mathbb{P}_Y^{\,X}$ is convex (see Section 4).

Part III models decision mechanisms M as binary operators, that is, as functions $M : \mathbb{P}_Y{}^X \rightarrow \mathbb{P}_Y{}^X$, mapping abstract norms $g \in \mathbb{P}_Y{}^X$ into *transformed norms* $M(g) \in \mathbb{P}_Y{}^X$. These mechanisms are reduced expressions of (procedural) global rules.

<u>Remark 3.1</u> Global rules are essential in any legal system, guiding endless actions and transactions. If all norms involved in a decision are (local) rules, the system works an algorithm, see Table VI.1. The propositional calculus in (2.1) is an example of a formal system and, hence, an algorithm.[29]

Kaplow [1992] and Fon & Parisi [2007] consider optimal ex ante information content in norms (the optimal degree of incompleteness) and their relation to promulgation and enforcement costs (including adjudication). For a general discussion of contractual form and organization of economic activity, see Hansmann [1996]. See Salzberger [1993] on strategic delegation to (independent) courts. Due to the logical difficulties inherent in case-by-case adjudication in multi-member courts, Landa & Lax [2009:961] suggest that courts may openly endorse "indeterminate legal standards" (see Section VI.3 below). ☐

<u>Example 3.1</u> (contract notions and representation). Let $\Omega = \{\theta_1, \ldots, \theta_n\}$ be a set of contingencies, and E_j an event (a subset of Ω). Let $\{E_j\}_{j=1}^m$ be a family of mutually exclusive events, and a_j an act selected from a set of possible acts, A. Shavell [2006] defines a (general) *contract term* as an ordered pair $\langle E_j, a_j \rangle$. The term is said to be *specific* if E_j contains a single element. A contract K is defined as the set:

$$K = \{\langle E_1, a_1 \rangle, \ldots, \langle E_m, a_m \rangle\}.$$

---

[29] See, generally, Berlinski [2000].

K is said to have *gaps* if there are unmapped contingencies, $\bigcup_{j=1}^{m} E_j \neq \Omega$. If $\bigcup_{j=1}^{m} E_j = \Omega$, K

is said to be *obligationally complete,* if possibly incomplete in the economic sense by

being "insufficiently state contingent" or "insensitive to relevant future contingencies".[30]

If $\bigcup_{j=1}^{m} E_j = \Omega$ and each term is specific, K is *fully detailed complete,* and is defined by:

$$\left\{ \left\langle \{\theta_1\}, a(\theta_1) \right\rangle, \left\langle \{\theta_2\}, a(\theta_2) \right\rangle, \ldots, \left\langle \{\theta_n\}, a(\theta_n) \right\rangle \right\}.$$

It follows that a contract K is a binary relation in $\wp\Omega \times A$. It has gaps if the relation

domain $D(K)$ is a proper subset of $\wp\Omega \setminus \varnothing$. An obligationally complete contract can be

written as $g_K : \Omega \to \delta\mathbb{P}_A$, such that for all $\theta_i \in E_j$, θ_i is mapped to a_j with probability

one (to $!a_j^{\theta_i, g_K|_{E_j}}$; $g_K|_{E_j}$ denoting the restriction of g_K to E_j), $j = 1, \ldots, m$. If K is fully

detailed complete, K is a functional relation, and $g_K : \Omega \to \delta\mathbb{P}_A$ such all $\theta_i \in \Omega$ gets

mapped to a uniquely determined act $a(\theta_i)$ with probability one (to $!a(\theta_i)^{\theta_i, g_K}$),

$i = 1, \ldots, n$.

There is no language ambiguity in Shavell's framework. Hermalin, Katz &

Craswell [2007:71] represent contracts as functions $C : D(C) \subseteq \Omega \to \wp A$. The contin-

gencies in $D(C)$ are addressed, but "whenever the contract mapped [a contingency] to a

set with more than one outcome" parties have "said too little or too much" (named lin-

guistic under- and over-determination, p. 71). In all contingencies $\theta_i \in D(C)$, the con-

tract determines exactly which subset of actions in $\wp A$ becomes available, hence the

contract corresponds to $C \in \left(\delta\mathbb{P}_{\wp A} \right)^{D(C)}$, in terms of Definition 2.1'.[31] ∎

---

[30] Ayres & Gertner [1989:92].

[31] Contingencies not addressed, gaps $\theta_i \in \Omega \setminus D(C)$, are called unmapped contingencies
(see pp. 69–73 for further nuances). In this case, the notion of *partial functions* can be
used in the contract definition (Kornhauser [1992:174] suggests their use in the context of
so-called extended rules).

Example 3.2 (Ex. 2.1 continued: global rules in a cdf framework). Let $c > 0$ be a parameter. A global negligence rule—partitioning the set of legal facts such that liability $\otimes$ results if and only if $x \geq c$—can handled in the cdf-framework, by introducing a right continuous function:

$$F_{\delta c}(x) = \begin{cases} 0 \text{ if } x < c \\ 1 \text{ if } x \geq c \end{cases},$$

where δc indicates the point of discontinuity of the distribution function. For any random variable S with cdf G and $t_0 \in \mathbb{R}$, $\lim_{t \to t_0^+} G(t) = G(t_0)$ and $\lim_{t \to t_0^-} G(t)$

$= G(t_0) - P\{S = t_0\}$: the "jump" at t_0 is equal to the probability mass concentrated at the point.[32] Hence, a global *negligence rule* in $\delta\mathbb{P}_{\{\oplus,\otimes\}}{}^X$ can be represented by

$g(\cdot \mid x) = \langle 1 - F_{\delta c}(x), F_{\delta c}(x); \oplus, \otimes \rangle \big|_{x \in X}$, the jump at $x = c$ equal to 1. The notation $g_{\delta c}$

will be used for $g_{F_{\delta c}} \in \delta\mathbb{P}_{\{\oplus,\otimes\}}{}^X$, with the understanding that the the mass at c is equal to one. A continuum of negligence rules is generated as c varies in $\mathbb{R}_{++}$. ∎

Remark 3.2.A Ehrlich & Posner [1974:268] suggest combining a rule-element with a standard-element ("unlawful to drive more than 60 miles per hour *or* to drive at any lower speed that is unreasonably fast in the particular circumstances"). Assume that, under the "particular circumstances", the 60 miles per hour corresponds to an interior point in the standard density support. The resulting

---

An alternative formulation, emphasizing the relation between $D(C)$ and A is as a correspondence (multi-valued function) from $D(C)$ to A, $C^*: D(C) \rightrightarrows A$, assigning a non-empty subset in A to every conditioning state (Carter [2001:177]). Definition 2.1' allows uncertainty over the sets representing linguistic under- and over-determination ($C \in \mathbb{P}_{\wp A}{}^{D(C)}$).

[32] Apostol [1969], Theorem 14.5.

norm can be represented by a cdf $G_{\delta 60}(t)$ continuously increasing on $[\underline{x}, 60)$, making a jump from $0 < \lim_{t \to 60^-} G_{\delta 60}(t) < 1$ to 1 at $x = 60$. $\square$

Remark 3.2.B A similar object results from dynamic law evolution (see Remark 2.4 with further references). $\square$

Example 3.3 (representation of liability norms). In the Parts IV–V analyses of the precaution model, care levels are measured positively in X, noncompliance ($\otimes$) occurrs if $x < S$ (too low precaution investment compared to the stochastic requirement, S). The event has probability $P\{S > x\} = 1 - P\{S \le x\} = 1 - F(x)$.

It will be convenient to represent liability norms as mapping directly to damages. Hence, negligence standards map noncompliance to losses L and compliance to 0 (no damage payment). The standards $g_F \in \mathbb{P}_{\{0,L\}}{}^X$ are represented as:

$$g_F(\cdot \mid x)\big|_{x \in X} = \langle F(x), 1 - F(x); 0, L \rangle\big|_{x \in X}.$$

The subclass of global negligence rules $\delta\mathbb{P}_{\{0,L\}}{}^X \subset \mathbb{P}_{\{0,L\}}{}^X$ are represented as ($c \in \mathbb{R}_{++}$)

$$g_{\delta c}(\cdot \mid x)\big|_{x \in X} = \langle F_{\delta c}(x), 1 - F_{\delta c}(x); 0, L \rangle\big|_{x \in X}, \text{ or:}$$

$$g_{\delta c}(\cdot \mid x) = \begin{cases} \langle 0,1;0,L \rangle\big|_{x<c} \\ \langle 1,0;0,L \rangle\big|_{x \ge c} \end{cases}.$$

Situation $c = 0$ corresponds to the global rule of *no liability*, $g_{\delta 0} \in \delta\mathbb{P}_{\{0,L\}}{}^X$. A global rule of *strict liability* is denoted (slightly abusing notation) $g_{\delta \infty} \in \delta\mathbb{P}_{\{0,L\}}{}^X$. The norms are represented by:

$$\begin{cases} g_{\delta 0}(\cdot \mid x)\big|_{x \in X} = \langle 1,0;0,L \rangle\big|_{x \in X} \\ g_{\delta \infty}(\cdot \mid x) = \langle 0,1;0,L \rangle\big|_{x \in X} \end{cases},$$

respectively. [33] As *functions* from X to $\{0, L\}$ (elements in $\{0, L\}^X$), the global rules may be defined as $r_{\delta 0}(x) \equiv 0$ and $r_{\delta \infty}(x) \equiv L$ for all $x \in X$, respectively. Global negligence rules are given by:

$$r_c(x) = \begin{cases} L \text{ if } x < c \\ 0 \text{ if } x \geq c \end{cases}, \ c > 0.$$

■

Example 3.4 (state contingent rules). Consider a set of global rules in $r \in Y^X$ and assume that the rules' domain can be *decomposed* into a Cartesian product space $X_1 \times \cdots \times X_\nu = X$. Relative to X, a distinction can be made between *completely state contingent* rules (CSC) and *partially state contingent* rules (PSC). A CSC is a function of the set of variables $\langle x_1, \ldots, x_\nu \rangle$. A PSC is a globally determinative norm $r_{PSC}(\cdot)$ defined over X, but restricted to depend on a subset of the variables $\langle x_1, \ldots, x_\nu \rangle$. ■

Example 3.5 (base rules). In the study of case disposition and *aggregation* of legal rules in multi-member courts, Landa & Lax [2009] introduce *base rules* that map from a *case set* C to a dichotomous outcome set, $br: C \rightarrow \{'no', 'yes'\}$. A case $c \in C$ has k *dimensions* or *factors* potentially relevant to a decision. Each case dimension is coded as a binary variable, taking value 1 if the factor is present and 0 if not. Accordingly, a case $c = \langle c_1, \ldots, c_k \rangle$ is an element in $C = \{0, 1\} \times \cdots \times \{0, 1\} = \{0, 1\}^k$.[34] For example, $c = \langle 1, \ldots, 1 \rangle$ has all factors present, $c = \langle 1, \ldots, 1, 0 \rangle$ all but the last, etc. Let $rd = \langle rd_1, \ldots, rd_k \rangle$ be a vector defining potentially relevant rule dimensions, $rd_d = 1$ if

---

[33] In the model of precaution, legal fact sets can be assumed closed and finite, and c an interior point. (Parts IV and V identify classes of negligence rules that function as strict liability.)

[34] Case dimensions are called "legal (sub)findings" or "doctrinal factors", and "may be thought of as a particular mix of both purely objective facts and intermediate legal conclusions" (p. 949).

59

dimension d is relevant, and $rd_d = 0$ otherwise. Let $\tau \in \{0,1,\dots,k\}$ be the so-called *rule threshold*. Base rules are given a simple *additive structure*: outcome '*yes*' is implied iff the inner product $c \cdot rd \geq \tau$. A base rule, therefore, is compactly defined as the ordered pair $\langle rd, \tau \rangle$, $rd \in \{0,1\}^k$, $\tau \in \{0,1,\dots,k\}$. Hence, $\langle \langle 1,\dots,1 \rangle, k \rangle$ corresponds to *the* conjunctive base rule (all factors are necessary for outcome '*yes*', threshold value $\tau = 1$ corresponds to disjunctive base rules (one factor is sufficient), and $1 < \tau < k$ to intermediate base rules. ∎

4 Convex combinations of norms (mixed norms)

Convex combinations of norms arise due to the system character of legal norms. This section discusses the notion independently of its applications. The following assumes a finite set of J (potential) norms $\{g_1, g_2, \dots, g_J\}$, $g_j \in \mathbb{P}_Y^X$. Norms may be depicted as prospects of the same dimension: [35]

$$g_j(\cdot \mid x)\big|_{x \in X} = \Big\langle g_j(y^1 \mid x), g_j(y^2 \mid x), \dots, g_j(y^L \mid x) \Big\rangle\Big|_{x \in X},$$

with $g_j(y^k \mid x) = 0$ if $y^k \notin \operatorname{supp} g_j(\cdot \mid x)$.

> <u>Definition 4.1</u> Let $\{g_1, g_2, \dots, g_J\}$ be a subset of norms in $\mathbb{P}_Y^X$, and $\vec{\lambda}$ a vector of weights $\vec{\lambda} = \langle \lambda_1, \lambda_2, \dots, \lambda_J \rangle$, such that $0 \leq \lambda_j \leq 1$ and $\sum_{j=1}^{J} \lambda_j = 1$.
>
> A *convex combination* of $\{g_1, g_2, \dots, g_J\}$, denoted $g_{\vec{\lambda}} = \lambda_1 g_1 \Lambda \lambda_2 g_2 \Lambda \cdots \Lambda \lambda_J g_J$, is defined point-wise for each $x \in X$ by:

---

[35] $L = \tau^{\{g_1, g_2 \dots, g_J\}} = \#\left|\bigcup_{j=1}^{J}\left\{\bigcup_{x \in X} \operatorname{supp} g_j(\cdot \mid x)\right\}\right|$, compare Section 2.

$$g_{\bar{\lambda}}(\cdot\mid x)=\lambda_1 g_1(\cdot\mid x)+\lambda_2 g_2(\cdot\mid x)+\cdots+\lambda_J g_J(\cdot\mid x)$$

$$=\left\langle \sum_{j=1}^{J}\lambda_j g_j\left(y^1\mid x\right),\sum_{j=1}^{J}\lambda_j g_j\left(y^2\mid x\right),\ldots,\sum_{j=1}^{J}\lambda_j g_j\left(y^L\mid x\right)\right\rangle.$$

The second definition equality is obtained by the linearity properties of probability and vector addition.[36] The convex combination of (simple) prospects is a so-called *compound* or *composite* prospect. The next example illustrates how a compound prospect is reduced to a simple prospect.[37]

Example 4.1 (simple and compound prospects) Let $J=2$ and

$$g_1(\cdot\mid x)=\left\langle 0,\ldots 0,\underset{k'\text{th comp.}}{1},0\ldots,0;y^1,\ldots,y^{k'},\ldots,y^k,\ldots,y^L\right\rangle$$

$$g_2(\cdot\mid x)=\left\langle 0,\ldots 0,\underset{k'\text{-th comp.}}{1-\alpha},0\ldots,\underset{k\text{-th comp.}}{\alpha},0\ldots,0;y^1,\ldots,y^{k'},\ldots,y^k,\ldots,y^L\right\rangle$$

Point-wise, the combination of $g_1(\cdot\mid x)$ and $g_2(\cdot\mid x)$ is illustrated in Figure 4.1:

$$g_{\langle\lambda_1,\lambda_2\rangle}(\cdot\mid x)=\left\langle 0,\ldots 0,\underbrace{\lambda_1+\lambda_2[1-\alpha]}_{k'\text{-th comp.}},0\ldots,\underset{k\text{-th comp.}}{\lambda_2\alpha},0\ldots,0;y^1,\ldots,y^{k'},\ldots,y^k,\ldots,y^L\right\rangle.\ \blacksquare$$
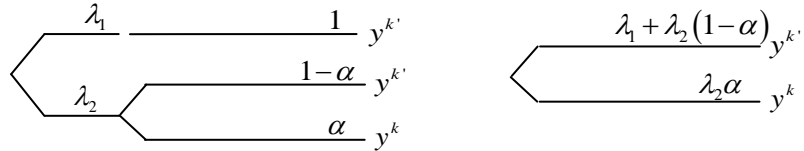
Figure 4.1 Reduction from compound to simple prospects

---

[36] For each $x\in X$, $g_{\bar{\lambda}}(\cdot\mid x)$ is given by:

$$\lambda_1\left\langle g_1\left(y^1\mid x\right),\ldots,g_1\left(y^L\mid x\right)\right\rangle+\lambda_2\left\langle g_2\left(y^1\mid x\right),\ldots,g_2\left(y^L\mid x\right)\right\rangle+\cdots+\lambda_J\left\langle g_J\left(y^1\mid x\right),\ldots,g_J\left(y^L\mid x\right)\right\rangle$$

$$=\left\langle\lambda_1 g_1\left(y^1\mid x\right),\ldots,\lambda_1 g_1\left(y^L\mid x\right)\right\rangle+\left\langle\lambda_2 g_2\left(y^1\mid x\right),\ldots,\lambda_2 g_2\left(y^L\mid x\right)\right\rangle+\cdots+\left\langle\lambda_J g_J\left(y^1\mid x\right),\ldots,\lambda_J g_J\left(y^L\mid x\right)\right\rangle$$

$$=\left\langle\lambda_1 g_1\left(y^1\mid x\right)+\cdots+\lambda_J g_J\left(y^1\mid x\right),\ldots,\lambda_1 g_1\left(y^L\mid x\right)+\cdots+\lambda_J g_J\left(y^L\mid x\right)\right\rangle=\left\langle g_{\bar{\lambda}}\left(y^1\mid x\right),\ldots,g_{\bar{\lambda}}\left(y^L\mid x\right)\right\rangle$$

[37] See Mas-Colell, Whinston & Green [1995:169-70], Kreps [1988:50], Wakker [2010:59–62], and Borch [1968:32–33].

Norms are defined *globally* on X. The next example and Figure 4.2, therefore, transplant Example 4.1 to a global context. Because the Part V applications consider mixes of two norms, the simpler notation $\vec{\lambda} = \langle \lambda, 1-\lambda \rangle$ and $g_\lambda = \lambda g_1 \Lambda (1-\lambda) g_2$ is used.

Example 4.2 (convex combinations of norms). Consider $g_1, g_2 \in \mathbb{P}_Y^{\ X}$, locally represented in Example 4.1. The convex norm combination $g_\lambda = \lambda g_1 \Lambda (1-\lambda) g_2$ can be illustrated as:
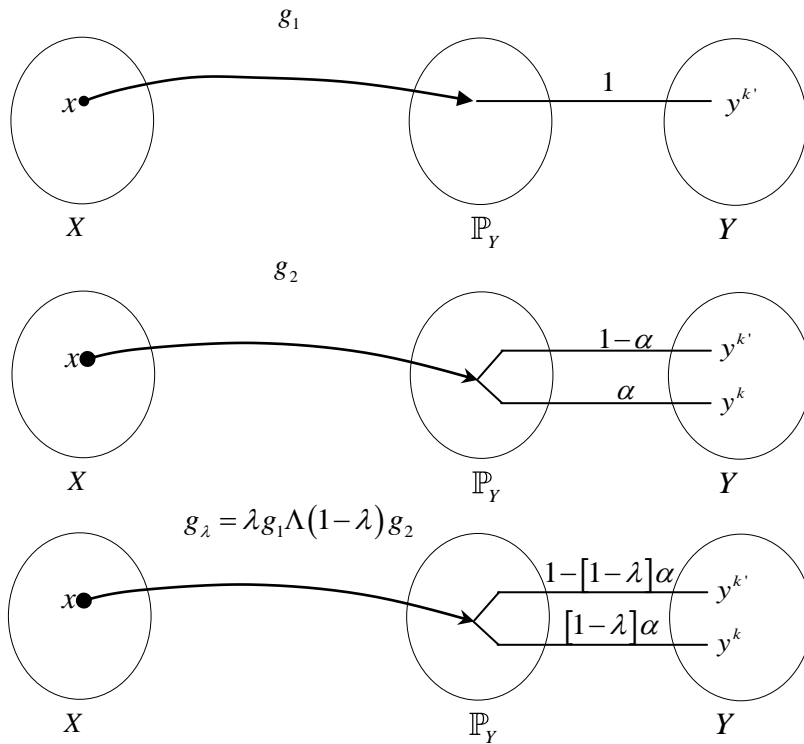
Figure 4.2 Convex combination of norms (global representation)

∎

It can be directly verified that for all $x \in X$ and all admissible $\vec{\lambda}$,

$$\sum_{y \in \text{supp } g_{\vec{\lambda}}(\cdot|x)} g_{\vec{\lambda}}(y \mid x) = 1.^{38}$$ It follows that $g_{\vec{\lambda}} \in \mathbb{P}_Y^X$: the function space $\mathbb{P}_Y^X$ is

convex and, in fact, a so-called mixture space. [39] This convexity property in part

has motivated the representation of norms (Definition 2.1). The term *mixed*

norms will be used about explicitly convex combinations of "pure" norms.

5 Meta-norms

This section introduces the notion of "norms over norms", arising from the char-

acteristic two- stage character of legal decision-making: Judges are identified

with the legal system meta-norms, constituted by the doctrine of sources and

method, expressing a notion of judging as commitment (Section I.2). This per-

spective distinguishes *legal* decision-making from agents an organizations mak-

ing preference-based decisions in legislatures or in markets (and is reflected in

legal aggregation mechanisms, see Part III).

Let LS be a set of legal sources (formants), a Cartesian product of clas-

ses $LS = LS_1 \times \cdots \times LS_l$, exemplified by statutes, regulations, prior judgments,

---

[38] $\sum_{k=1}^{L} \left\{ \sum_{j=1}^{J} \lambda_j g_j \left(y^k \mid x \right) \right\} = \sum_{j=1}^{J} \left\{ \lambda_j \left(\sum_{k=1}^{L} g_j \left(y^k \mid x \right) \right) \right\} = \sum_{j=1}^{J} \lambda_j = 1$ (using that

$g_j \in \mathbb{P}_Y^X$ for all j, hence for all $x \in X$, $\sum_{k=1}^{L} g_j \left(y^k \mid x \right) = 1$).

[39] A set S is convex if and only if it contains all convex combinations of each *pair* of
points in S (see Sydsæter [1981:152–54]). The convex combination (mixture) corre-
sponds to a *J*-nary *operation* on $\mathbb{P}_Y^X$ (a function from $\left(\mathbb{P}_Y^X \right)^J = \mathbb{P}_Y^X \times \cdots \times \mathbb{P}_Y^X$ to $\mathbb{P}_Y^X$).

Locally, if $Y = \mathbb{R}$, the mixed norm can be represented by the cdf
$G_{\vec{\lambda}} \left(t \mid x \right) = \sum_{j=1}^{J} \lambda_j G_j \left(t \mid x \right)$ (Mas-Colell, Whinston & Green [1995:183]). See
Kreps [1988:52–53] on mixture-spaces.

and contracts.[40] The set of potential (ordinary) norms is given by $\mathbb{P}_Y^X$. Norms functioning at the meta-level are represented by:

> <u>Definition 5.1</u> A meta-norm is a triple $\left(LS, \mathbb{P}_{\mathbb{P}_Y^X}, \eta : LS \to \mathbb{P}_{\mathbb{P}_Y^X} \right)$. The set of meta-norms is denoted $\mathbb{P}_{\mathbb{P}_Y^X}^{LS}$.

Meta-norms have the same logical structure as ordinary-level norms (and are elaborated in Part VI based on Definition 2.1'). Given source constellation $ls \in LS$ and $\eta \in \mathbb{P}_{\mathbb{P}_Y^X}^{LS}$, the correspondence $\mathrm{supp}\,\eta : LS \rightrightarrows \mathbb{P}_Y^X$ maps to a subset of ordinary norms, $\mathrm{supp}\,\eta(\cdot \,|\, ls) \subseteq \mathbb{P}_Y^X = \{ g_1, g_2, \ldots, g_J \}$, say. This is the set of norms assigned positive probability, given the normative sources, called *enforceable* in Part VI. However, the framework gives more than a set of enforceable norms: As explained in Section 4, conditioned on $ls \in LS$, the convex combination:

$$g_{\eta(\cdot|ls)} = \eta(g_1 \,|\, ls) g_1 \wedge \eta(g_2 \,|\, ls) g_2 \wedge \cdots \wedge \eta(g_J \,|\, ls) g_J, \qquad (5.1)$$

is a *mixed norm* $g_{\eta(\cdot|ls)} \in \mathbb{P}_Y^X$, given by:

$$g_{\eta(\cdot|ls)}(\cdot \,|\, x)\Big|_{x \in X} =$$
$$\left\langle \sum_{j=1}^J \eta(g_j \,|\, ls) g_j(y^1 \,|\, x), \sum_{j=1}^J \eta(g_j \,|\, ls) g_j(y^2 \,|\, x), \ldots, \sum_{j=1}^J \eta(g_j \,|\, ls) g_j(y^L \,|\, x) \right\rangle\Bigg|_{x \in X} \qquad (5.2)$$

---

[40] This is a rough description. Legislation arguably should be considered as ordered pair of enacted norms and preparatory works (if applicable). More complete representations of objects based on Definition II.2.1' are suggested below both with regard to legislation and contracts (see also Examples 3.1–3.5). The source-classes (formants) are the ones actually influencing decisions, not official ideology (Section I.1).

The mixing weights $0 \leq \eta(g_j \mid ls) \leq 1$ are provided by the meta-norm ($\sum_{j=1}^{J} \eta(g_j \mid ls) = 1$). The reduction and resulting mixed norm $g_{\eta(\cdot \mid ls)} : X \to \mathbb{P}_Y$, are illustrated in Figure 5.1.

$$g_{\eta(\cdot \mid ls)} = \eta(g_1 \mid ls) g_1 \wedge \eta(g_2 \mid ls) g_2 \wedge \cdots \wedge \eta(g_J \mid ls) g_J$$
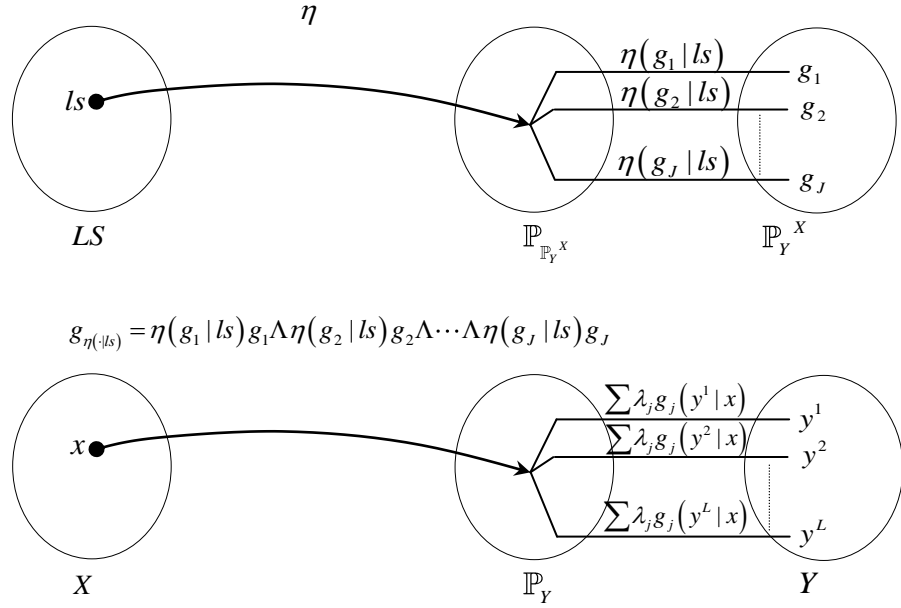
Figure 5.1 Mixed norm generated by meta-norm $\eta \in \mathbb{P}_{\mathbb{P}_Y^{X}}^{LS}$

As elements in $\mathbb{P}_Y^{X}$, mixed norms can be classified as discussed in Section 3. The next example introduces (abstract) mixed norms, which are used in Part V.

Example 5.1 (Ex. 3.3 continued: mixed liability regimes). A pollutee's entitlement might be protected by a negligence standard $g_F \in \mathbb{P}_{\{0,L\}}^{X}$, given by $g_F(\cdot \mid x) = \langle F(x), 1 - F(x); 0, L \rangle \big|_{x \in X}$, or by strict liability $g_{\delta\infty} \in \delta \mathbb{P}_{\{0,L\}}^{X} \subset \mathbb{P}_{\{0,L\}}^{X}$, given by $g(\cdot \mid x) \big|_{x \in X} = \langle 0, 1; 0, L \rangle \big|_{x \in X}$. Under a constellation of legal sources $ls \in LS$ and meta-norm $\eta \in \mathbb{P}_{\mathbb{P}_{\{0,L\}}^{X}}^{LS}$, it may be uncertain which liability regime prevails. Let

$\eta\left(g_{\delta\infty} \mid ls\right) = \lambda \in (0,1)$ (that   is, $ls \in LS \setminus \eta^{-1}\left(\delta\mathbb{P}_{\mathbb{P}_{\{0,L\}}}{}^{X}{}^{LS}\right)$).   The   mixed   norm

$\lambda g_{\delta\infty} \Lambda (1-\lambda) g_F \in \mathbb{P}_{\{0,L\}}{}^{X}$ is given by:

$$g_{\lambda g_{\delta\infty} \Lambda (1-\lambda) g_F}\left(\cdot \mid x\right)\Big|_{x \in X} = \lambda \left\langle 0,1;0,L\right\rangle\Big|_{x \in X} + (1-\lambda)\left\langle F(x), 1-F(x); 0, L\right\rangle\Big|_{x \in X}$$
$$= \left\langle [1-\lambda] F(x), 1-[1-\lambda] F(x); 0, L\right\rangle\Big|_{x \in X}.$$

Mixing   strict   liability   with   global   negligence   rules   corresponds   to

$\lambda g_{\delta\infty} \Lambda (1-\lambda) g_{\delta c} \in \delta\mathbb{P}_{\{0,L\}}{}^{X}$, defined by:

$$g_{\lambda g_{\delta\infty} \Lambda (1-\lambda) g_{\delta c}}\left(\cdot \mid x\right)\Big|_{x \in X} = \left\langle [1-\lambda] F_{\delta c}(x), 1-[1-\lambda] F_{\delta c}(x); 0, L\right\rangle\Big|_{x \in X}.$$

The (pure) global rule of *no liability* corresponds to $g_{\delta 0} \in \delta\mathbb{P}_{\{0,L\}}{}^{X}$.

Strict liability and (global) negligence rules may, alternatively, be represented

by functions $r_{\delta\infty}(\cdot), r_{\delta c}(\cdot) \in \{0,L\}^{X}$, convex combinations that correspond to ordinary

sums:

$$\lambda r_{\delta\infty}(x) + (1-\lambda) r_{\delta c}(x) = \begin{cases} L \text{ if } x < c \\ \lambda L \text{ if } x \geq c \end{cases}.$$

That the resulting functions are not elements in $\{0,L\}^{X}$ (the function space is not con-

vex). ∎

Remark 5.1 The Example 5.1 liability regimes allocate a so-called *call option* to

the polluter: the ability to choose $x \in X$, while paying an exercise price as de-

fined by the law (see Example VI.3.4 for a general equilibrium perspective;

probabilistic externalities are considered in Parts IV and V). However, a

pollutee's entitlement can be protected by a larger set of norms, such as being

assigned a *put option*. In this case, the pollutee decides on the level of x and can

exercise a right to be paid, according to a price defined by law. See, generally,

Ayres [2005], including rich extensions of the Calabresi & Melamed [1972]

property rule regimes. □

<u>Remark 5.2</u> Brooks & Schwartz [2005] observe that uncertainty about entitlement protection underlies the availability of preliminary injunctions. They discuss doctrinal design which enables efficient breach under contracts and, more generally, efficient "takings" (including areas such as intellectual property and constitutional law). In the present setting, it corresponds to meta-norm design. $\square$

<u>Example 5.2</u> (Ex. 3.1 continued: contract interpretation). Obligationally complete contracts K can be represented as elements in $\left(\delta\mathbb{P}_A\right)^{\Omega}$, and contracts with gaps as elements in $\left(\delta\mathbb{P}_A\right)^{\Omega\backslash\Omega_G}$, where Ω_G corresponds to unmapped contingencies. Define $\left(\delta\mathbb{P}_A\right)^{\Omega}\cup\left(\delta\mathbb{P}_A\right)^{\Omega\backslash\Omega_G}$. Shavell's [2006] definition of contract interpretation corresponds to the meta-norm

$$\eta^K:\left(\delta\mathbb{P}_A\right)^{\Omega}\cup\left(\delta\mathbb{P}_A\right)^{\Omega\backslash\Omega_G}\to\delta\mathbb{P}_{\left(\delta\mathbb{P}_A\right)^{\Omega}},$$

mapping from the source class "written contracts" $\left(\delta\mathbb{P}_A\right)^{\Omega}\cup\left(\delta\mathbb{P}_A\right)^{\Omega\backslash\Omega_G}$ (a set of possible contracts) to the set of (degenerate) distributions over the set of obligationally complete contracts $\left(\delta\mathbb{P}_A\right)^{\Omega}$ (the latter corresponds to an image set of interpreted contracts). See Remark IV.1.2 for additional comments on $\eta^K\left(\cdot\right)$. ∎

In some circumstances, it is relevant to base representation of meta-norms on Definition 2.1'. Under a notion of discretion as norm protected options (power-conferring norms), judges *select* different norms. This change in emphasis leads to conceptual challenges in multi-member court contexts. Meta-norms also uphold characteristic hierarchical norm structures. These, and similar system-oriented aspects, are considered in Sections VI.2–3.

<u>Remark 5.3</u> Landa & Lax [2009] consider the *construction* of global rules based on individual judgments generated by judge selected base-rules (Example 3.5).

From the present perspective, it concerns the possibility of a rational construction of legal doctrine—a problem on the level of meta-norms—based on extensive formulations (see Example VI.3.2). □

PART THREE

TRANSFORMATION OF NORMS IN MULTI-MEMBER DECISION

MECHANISMS

1 Introduction

This part studies how abstracts norms $g \in \mathbb{P}_Y{}^X$, from the ex ante perspective, are transformed in decision mechanisms $M \in \mathcal{M}$. The mechanisms are constituted by global rules, regulating the mapping of judgment profiles to binding decisions $\langle x, y \rangle^M \in X \times Y$, which become law in the extensive sense.[1]

Throughout, it is assumed that voting takes place directly, on consequences under ordinary norms (outcome-based voting) or on ordinary norm elements (premise-based voting). Fundamentally, judges—identified with the doctrine of sources of law—do not vote on ordinary norms as such, but *apply* the law to individual cases $x \in X$, generating ordered pairs $\langle x, y \rangle^M \in X \times Y$.[2]

---

[1] See Urfalino [2010] on the notion of binding decisions. To become legal system elements in the extensive sense, the judgments should be final according to system internal criteria (Section I.1).

[2] Generation of $\langle x, g \rangle^M \in X \times \mathbb{P}_Y^X$ essentially corresponds to declaratory judgments and generation of $\langle g \rangle^M \in \mathbb{P}_Y^X$ to abstract decision-making or legislation. These phenomena occur in some constitutional and super-national courts, generally not considered a part of the judiciary (Ferejohn & Pasquino [2004]), and in fact not called "courts" by some purists (Merryman & Péres-Perdomo [2007:90]). Such decisions are not studied here (but see Example VI.1). (Eng [2000] points out that arguments on the level of meta-norms typically are not discussed by courts. It arguably reflects the voting methods.)

This part is organized as follows: The present gives an overview of legal aggregation methods. General dichotomous situations, of wide applicability due to the characteristic sequential structure of legal decision-making, are analyzed in Section 2. The results are relevant to the analysis of polychotomous environments in Section 3, which studies the classical aggregation rule. Higher-dimensional norms, and outcome-based and premise-based voting protocols, are considered in Section 4. The analysis is conducted with a view to the study of bifurcated trials, which follows in Section 5. Collective decisions on legal standard elements S prior to their application to facts $x \in X$ —named theoretical norm element determination—are considered in Section 6. The section provides analytical results which are used in Parts IV and VII. Section 7 prepares for the equilibrium analyses, depicting majority aggregation-function curvature properties.

1.1   *Legal aggregation methods*

In collectives of n judges, v or more votes are required for *rejection* of a *default* proposition or default state (such as no liability, $\oplus$).[3] All judges are required to vote (abstentions are not allowed).[4] Such mechanisms are denoted $M_{[v;n]}$. By de-

---

[3] The terminology is inspired by scientific use of a null-hypothesis being exposed to rejection under criteria defined in testing regimes (see Larsen & Marx [1986:286–304]). By rejecting a null, the alternative hypothesis is not claimed to be true. However, in law, the doctrines of *res judicata* and similar inspires terminology such as the defendant *is* liable or similar.

    Interestingly, in Scottish law, the form "not proven" is used when defendants are acquitted in criminal cases (Schauer & Zeckhauser [1996]).

[4] In this sense, legal norm systems are ex post complete: action for denial of justice can be brought against a judge who refuses to give judgment on the ground of incomplete norms (von Mehren & Gordley [1977:1135]; for similar observations regarding international law, Ross [1947:278–79]).

    To the extent that a dispute arises about the meaning or scope of a judgment $\langle x, y \rangle^M \in X \times Y$, parties, generally, can obtain a new judgment interpreting the former judgment. Such mechanisms must be distinguished sharply from revision and appeal and

fining appropriate defaults, attention can be limited to majority and super-majority mechanisms.[5] A *super-majority* requirement is:

$$n \geq v \geq \begin{cases} \frac{n}{2} + 1 \text{ if } n \text{ is even} \\ \frac{n+1}{2} + 1 \text{ if } n \text{ is odd} \end{cases}.$$

A *majority* requirement is:

$$v = \begin{cases} \frac{n}{2} \text{ if } n \text{ is even} \\ \frac{n+1}{2} \text{ if } n \text{ is odd} \end{cases}.$$

In the case of majority rule, for analytical convenience, an odd number of judges will be assumed throughout, $n = 2m+1$, $m \in \{0,1,2,...\}$. Majority mechanisms are denoted M_{2m+1}. Due a fundamental symmetry property (*neutrality* among outcomes), they do not require default state specification.

Dichotomous situations are of particular importance in law. Substantive norms may map directly over binary spaces. In legal proceedings, multi-dimensional questions are often mapped into (sequences of) dichotomous questions. Bifurcated proceedings provide a prominent example: Consider a situation with possible damage levels $Y = \{0, l^1, ..., l^L\}$, 0 corresponding to no liability, and $l^j < l^{j+1}$ to findings of liability; $l^j \in \mathbb{R}_{++}$, $j \in \{1, 2, ..., L-1\}$. Instead of voting directly on a large set of consequences in unitary trials, Y may be partitioned into:

$$\left\{ \underbrace{0}_{\oplus} \mid \underbrace{l^1, ..., l^L}_{\otimes} \right\},$$

are exercised with care not to underscore judgment properties of *res judicata* (see generally Mackenzie *et al.* [2010]).

[5] As documented by Vermeule [2005], sub-majority rules have legal relevance and include the U.S. Supreme Court's "Rule of Four", permitting four out of nine judges to grant a writ of certiorari.

and the question of liability determined at a first stage (voting on $Y^I = \{\oplus, \otimes\}$),

prior to a conditional second stage determination of damages (given liability vot-

ing on $Y^{II} = \{l^1, \ldots, l^L\}$).

In non-dichotomous situations, the set of possible legal consequences is

assumed to be linearly ordered and associated with a subset of the real numbers.

The modeling reflects assumptions of the *classical rule* (extended to super-

majority settings). This fundamental legal aggregation method proceeds in two

steps. Let $y = \langle y_1, y_2, \ldots, y_n \rangle$ be a vector (profile) of individual judgment conclu-

sions, $y_i \in Y$, and:

I.   rank the conclusions in non-decreasing order:

$$r(y) = \langle y_{1:n}, y_{2:n}, \ldots, y_{n:n} \rangle;^6$$

II. select the conclusion backed by a (super-)majority of v or more

votes as the collective judgment, either:

- counting votes *from above* in the mechanism denoted $M_{[v;n]^\Downarrow}$;

(prioritizing as default states at the lower end of Y). It corre-

sponds to selection of $y_{n-v+1:n}$

- counting votes *from below* in the mechanism denoted $M_{[v;n]^\Uparrow}$ (pri-

oritizing as default states at the higher end of Y). It corresponds

to selection of $y_{v:n}$; or

---

[6] Formally, the procedural rule component $r(\cdot)$ is a function $r : \mathbb{R}^n \to \mathbb{R}^n$ defined recur-

sively as (dropping possible ties in its statements):

$$y_{1:n} = \min_i y_i, \ y_{2:n} = \min_{y_{2:n} \neq y_{1:n}} y_i, \ldots, \ y_{n:n} = \max_i y_i,$$

see Carter [2001:429].

- counting votes from above or below in (neutral) majority mechanisms M_{2m+1}. It corresponds to selection of the middlemost vote, $y_{m+1:2m+1}$.

These methods produce a collective decision—an ordered pair $\langle x, y \rangle^M \in X \times Y$—which has as concluding element y, identical to the individual judgment conclusion of at least one judge, $y \in \{y_i\}_{i=1}^n$.

Remark 1.1 The *classical rule* is extensively discussed in Heckscher [1892]: The method goes back hundreds of years and, apparently, is so natural in judicial contexts, that its legislation is often considered unnecessary.[7] Although considering judging as a form of cognition, Heckscher emphasizes that the rule leads to a binding decision corresponding to the complete judgment $\langle x, y_i \rangle$ of at least one of the judges (Section I.2 above). An aggregator, such as the mean, is, therefore, rejected, even if it would be a better tracker of an external truth ("det Sande").[8] Supermajority rules do not concern the collective's meaning ("Mening"), but prioritize certain default states.[9] □

---

[7] P. 154. The rule fits jurisdictions in the civil law tradition particularly well, where claims, in general, must be collected by obtaining money judgments and contract-based promises that cannot be converted into money, are not seen as creating legal obligations (Merryman & Pérez-Perdomo [2007:123]).

[8] Pp. 161–62 and "Efterskrift"; see Remarks VII.5.1–2 below.

[9] Pp. 17 and 106–8. For further remarks on Heckscher and implementation of the classical rule in more complex environments, see Nordén [2015].
    The classical rule is, for example, formulated in the Norwegian Civil Procedure Act (with a parallel provision in the Criminal Procedure Act): "Each ruling shall be made by majority vote unless otherwise provided by statute. [---] If there is no majority for any outcome when a sum of money or other quantity is to be determined, the votes in favor of higher amounts or quantities shall be added to the votes in favour of the closest amounts or quantities until a majority has been reached" (Sec. 19–3(4)).

1.2   *Modeling the decision process*

The conception of judging as one of *commitment,* and ex ante identification of judges and norms, assume that judges vote independently—in and across stages in sequential trials—as uncertainty is resolved (intra-court decisional independence is a part of fair trial requirements). Because judges *apply* the law, voting fundamentally concerns consequences under substantive norms in outcome-based regimes or substantive norm elements in premise-based regimes. It does not take place on norms in $\operatorname{supp}\eta(\cdot\,|\,ls)\subseteq\mathbb{P}_Y^{\,X}$.

In most applications, the ex ante identification of norms and judges implies that judges are modeled as random drawings (i.i.d.'s) from intangible populations, defined by norm structures $g_{\eta(\cdot|ls)}\in\mathbb{P}_Y^{\,X}$, and conditioned by sources and facts $\langle ls,x\rangle$. A theory *about* norms and legal phenomena is constructed.[10] Probability theory is not suggested as a "plausible account of the semantics of [legal] discourse" (Coleman & Leiter [1993:610]): judges are not perceived as performing lotteries.[11]

---

[10] See e.g. Wedberg [1951] and Ross [1958:6–11]. As emphazised by Gilboa [2009:55–64], an advantage of formal representation is that questions about modality can be relegated to the level of interpretation (see Section VI.3).

[11] On the level of individual judges, Coleman & Leiter [1993:561] formulate the process that is modeled in the following manner: "The set of legal reasons is a function of two elements: (i) the set of valid or binding legal sources; and (ii) the set of *interpretive* operations that can be legitimately performed on those sources (to generate rules and principles of law) and the set of *rational* operations that can be performed on law and facts (to generate outcomes)."

In some circumstances, it *is* relevant to directly identify judges and (weighted) lotteries. Lotteries are used to break ties in judicial panels, and procedural rules define lotteries over potential judges (corresponding to random sampling), representing notions of impartiality and independence (Sections I.2 and VII.3). Elster [1989] describes important examples of substantive law-prescribed lotteries (such as to allocation of scarce resources). Optimal contracts may be designed with explicitly stochastic elements (Bolton & Dewatripont [2005:81]). Proof standards in both civil and criminal procedure are candidates for being stated in probabilistic terms (Fienberg [1989], Appendix A).

The constellation of sources and facts that condition ordinary norms, $\langle ls, x \rangle \in LS \times X$, become common knowledge in formal proceedings. From the ex ante perspective, the decision process may be described as follows: Judges independently:

- *observe* constellation $\langle ls, x \rangle \in LS \times X$;

- *apply* meta norm $\eta \in \mathbb{P}_{\mathbb{P}_Y^X}{}^{LS}$ to $ls \in LS$, leading to the abstract

    mixed norm $\eta(g_1 \mid ls) g_1 \wedge \eta(g_2 \mid ls) g_2 \wedge \cdots \wedge \eta(g_J \mid ls) g_J$
    $\equiv g_{\eta(\cdot \mid ls)} \in \mathbb{P}_Y^X$, and

- *vote* in M conditioned on $x \in X$ (or abstractly in a two-stage procedure in theoretical norm element determination), generating a binding decision $\langle x, y \rangle^M$.

Due to convexity of $\mathbb{P}_Y^X$, it is not necessary to explicitly distinguish the levels of legal argumentation in the study of norm transformations. It will be demonstrated that mechanisms correspond to *operators*

$$M : \mathbb{P}_Y^X \to \mathbb{P}_Y^X,$$

mapping *abstract* (mixed) norms $g \in \mathbb{P}_Y^X$ to *transformed* (mixed) norms $M(g) \equiv g_M \in \mathbb{P}_Y^X$ ($M\left(g_{\eta(\cdot \mid ls)}\right) \equiv g_{\eta(\cdot \mid ls), M} \in \mathbb{P}_Y^X$). In the case of a single judge (M_1), the transformation is given by the identity mapping $id_{\mathbb{P}_Y^X}(g) = g$. Due to the Section II.2 convention of writing prospects as vectors of a uniform dimension τ^g ($g(\cdot \mid x)|_{x \in X} \in \mathbb{R}^{\tau^g}$), $M : \mathbb{P}_Y^X \to \mathbb{P}_Y^X$ can be depicted as a function with the product space $[0,1] \times \cdots \times [0,1] = [0,1]^{\tau^g}$ as counter-domain: Decomposed to $M = \left\langle M^1, \ldots, M^{\tau^g} \right\rangle$, the τ^g components are given by $M^i : \mathbb{P}_Y^X \to [0,1]$. The im-

age at $g \in \mathbb{P}_Y^X$ is $M(g) = \left\langle M^1(g), \ldots, M^{\tau^g}(g) \right\rangle$.[12] In the binary case ($\tau^g = 2$), it

suffices to focus on one component by the laws of probability ($\sum_{i=1}^{\tau^g} M^i(g) = 1$).

2   Dichotomous environments: basic legal geometries

To guide intuition and exemplify the approach, this section starts with an analysis

of the abstract legal standard $g_F \in \mathbb{P}_{\{\oplus,\otimes\}}^X$ defined in Example II.2.1, applied in

three-member panels, $M_{[v;3]}$, $v \in \{2,3\}$. The norm is given by

$g_F(\cdot \mid x)\big|_{x \in X} = \left\langle 1 - F(x), F(x); \oplus, \otimes \right\rangle\big|_{x \in X}$, with F the cdf of the stochastic varia-

ble S, liability $\otimes$ occurring iff $S \leq x$.

Example 2.1 (Ex. II.2.1 continued: three-member panels). Judges vote directly on

$Y = \{\oplus, \otimes\}$ conditioned on $x \in X$ in $M_{[v;3]}$. Due to independence, the sequence in

which judges vote does not matter. In the Figure 4.1 probability tree, at the first stage,

Judge 1 concludes liability ($\otimes$) with probability $F(x)$ ("up") and no liability ($\oplus$) with

probability $1 - F(x)$ ("down"). At the second stage, the two nodes which directly follow

Judge 1's possible decisions, Judge 2 concludes liability and no liability, with probabili-

ties $F(x)$ and $1 - F(x)$, respectively. The same applies to the four nodes at the third

stage. The 2^3 possible sequences of votes (elementary sample space outcomes) corre-

spond to tree branches. Each sample space point has probability equal to the product of

the assigned branch segment probabilities:[13]

---

[12] Carter [2001:173].

[13] Formally, the segment probabilities are conditional, the condition being that the rele-
vant node is reached. Branch probabilities are calculated according to the chain rule
(Bartoszyński & Niewiadomska-Bugaj [1996:95–97]), independence implying that mar-
ginal and conditional probabilities are equal.

sample   elementary out-
space   come-probabilities

Judge 3

Judge 2

$F(x)$ $\otimes\otimes\otimes$ $[F(x)]^3$

Judge 1 $F(x)$

$F(x)$ $1-F(x)$ $\otimes\otimes\oplus$ $[F(x)]^2[1-F(x)]$

$F(x)$ $\otimes\oplus\otimes$ $[F(x)]^2[1-F(x)]$

$1-F(x)$

$1-F(x)$ $\otimes\oplus\oplus$ $[1-F(x)]^2F(x)$

$F(x)$ $\oplus\otimes\otimes$ $[F(x)]^2[1-F(x)]$

$F(x)$

$1-F(x)$ $1-F(x)$ $\oplus\otimes\oplus$ $[1-F(x)]^2F(x)$

$F(x)$ $\oplus\oplus\otimes$ $[1-F(x)]^2F(x)$

$1-F(x)$

$1-F(x)$ $\oplus\oplus\oplus$ $[1-F(x)]^3$
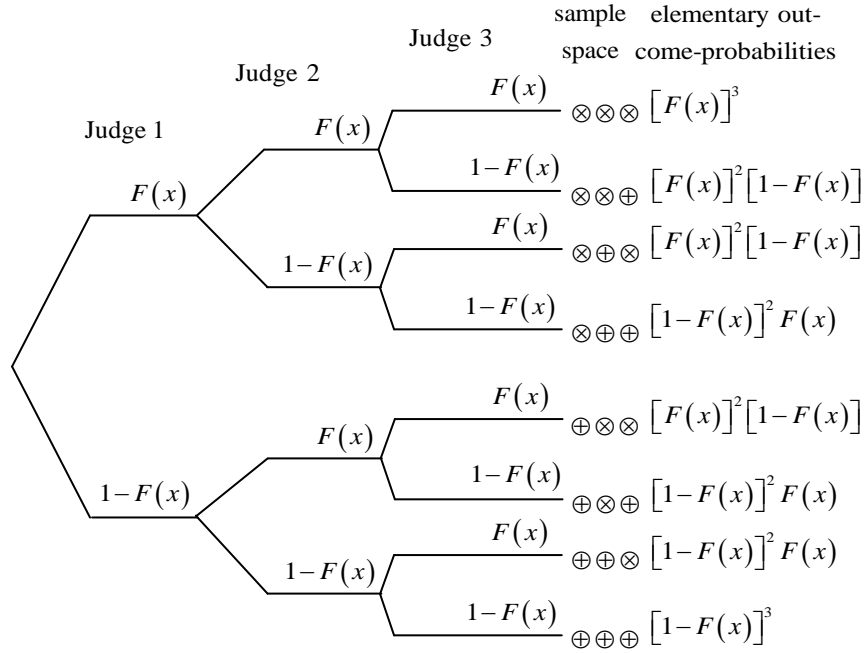
Figure 2.1 Probability tree and elementary outcomes

An unanimity requirement ($v=3$) for rejection of default state $\oplus$, corresponds to the event $\{\otimes\otimes\otimes\}$. It has probability $\left[F(x)\right]^3$. A majority requirement ($v=2$) corresponds to $\{\otimes\otimes\otimes,\otimes\otimes\oplus,\otimes\oplus\otimes,\oplus\otimes\otimes\}$, with probability $\left[F(x)\right]^3+3\left[F(x)\right]^2\left[1-F(x)\right]$. Hence, norms generated by $M_{[3;3]}$ and $M_{[2;3]}$ are given by:

$$\left\langle 1-\left[F(x)\right]^3,\left[F(x)\right]^3;\oplus,\otimes\right\rangle\Big|_{x\in X}, \text{ and}$$

$$\left\langle 1-\left[F(x)\right]^3+3\left[F(x)\right]^2\left[1-F(x)\right],\left[F(x)\right]^3+3\left[F(x)\right]^2\left[1-F(x)\right];\oplus,\otimes\right\rangle\Big|_{x\in X}, \quad (2.1)$$

respectively. Limiting attention to the liability component and abstracting from the underlying legal fact, it is seen that $M_{[3,3]}$ implies a strictly increasing and strictly convex transformation (F^3) globally on $[0,1]$, whereas M_3 implies a transformation strictly increasing and strictly convex on $\left[0,\frac{1}{2}\right]$ and strictly increasing and strictly concave on $\left[\frac{1}{2},1\right]$. Both transformations are bijections, see Figure 2.2 and 7.1.

The liability component, as a (composite) mapping from X to $[0,1]$, cannot be so easily described. This will be a recurrent theme.[14] ∎

Abstract norms $g \in \mathbb{P}_{\{\oplus,\otimes\}}{}^X$ are transformed $M_{[v;n]}$ with default $\oplus$. In transformed norms $M_{[v;n]}(g) \equiv g_{[v;n]} \in \mathbb{P}_{\{\oplus,\otimes\}}{}^X$, also denoted $g_{[v;n]}(\cdot \,|\, x)\big|_{x \in X}$

$= \left\langle g_{[v;n]}(\oplus \,|\, x), g_{[v;n]}(\otimes \,|\, x) \right\rangle\big|_{x \in X}$,. liability components $g_{[v;n]}(\otimes \,|\, x)$ are focused, without loss of generality ($g_{[v;n]}(\oplus \,|\, x) = 1 - g_{[v;n]}(\otimes \,|\, x)$ for all $x \in X$).

Proposition 2.1 If each judge rejects $\oplus$ with probability $g(\otimes \,|\, x)$, in $M_{[v;n]}$ with default $\otimes$, for all $x \in X$:

$$g_{[v;n]}(\otimes \,|\, x) = \sum_{i=v}^{n} \binom{n}{i} \big[g(\otimes \,|\, x) \big]^i \big[1 - g(\otimes \,|\, x) \big]^{n-i} \equiv H_{[v;n]}\big(g(\otimes \,|\, x) \big) \ (2.2)$$

Proof. The proposition gives the tail probability of a binomial distribution, starting from v with success probability $g(\otimes \,|\, x)$ and n trials. It follows directly from the fact that Bernoulli Trial conditions are satisfied if judges observe the same non-stochastic conditioning fact $x \in X$ and vote independently (see e.g. Bartoszyński & Niewiadomska-Bugaj [1996:334-6]). ◄

In (2.2), a distinction can be made between the "outer" aggregation $H_{[v;n]} : [0,1] \to [0,1]$ and the abstract norm component $g(\otimes \,|\, \cdot) : X \to [0,1]$. The *transformed* norm component $g_{[v;n]}(\otimes \,|\, \cdot) : X \to [0,1]$, as a function of legal fact

---

[14] Both transformed norm liability-components are increasing on $[\underline{x}, \overline{x}] \subseteq X$, but the classification of convex and concave regions is in general not preserved. (Outside the support, the probability distributions are degenerate, mapping to $\oplus$ with certainty if $x < \underline{x}$ and to $\otimes$ with certainty if $x > \overline{x}$.)

$x \in X$, is given by the *composition* $H_{[v;n]} \circ g(\otimes | \cdot): X \rightarrow [0,1]$, defined by

$H_{[v;n]}\big(g(\otimes | x)\big)\Big|_{x \in X}$. This section studies the effects of the outer mechanism, sep-

arately from the underlying abstract norm. It gives simple results that reappear in

more complex settings. Hence, in the next propositions, $g(\otimes | x) = F \in [0,1]$ is

taken as parametrically given ($x \in X$ is fixed). In *majority-mechanisms,*

$h_{2m+1}(\cdot) \equiv H_{[m+1;2m+1]}(\cdot)$ denote aggregation functions.[15]

Remark 2.1 Outer aggregation function values can be obtained from binomial

tables. Let $b(\cdot;n,F)$ denote the binomial probability mass function and

$B(\cdot;n,F)$ the corresponding cumulative distribution function:

$$H_{[v;n]}(F) = \sum_{i=v}^{n} b(i;n,F) = 1 - \sum_{i=0}^{v-1} b(i;n,F) = 1 - B(v-1;n,F). \qquad (2.3)$$

$\square$

Proposition 2.2 If $n \geq 3$ the bijection $H_{[v;n]}(F)$ is (i) strictly increasing

on $[0,1]$ with $H_{[v;n]}(0) = 0$, $H_{[v;n]}(1) = 1$, (ii) strictly convex on

$\big[0, \breve{F}(v,n)\big]$ and strictly concave on $\big[\breve{F}(v,n),1\big]$, with *inflection point*

$\breve{F}(v,n) = (v-1)/(n-1) > \frac{1}{2}$, and $H_{[v;n]}{}'(0) = H_{[v;n]}{}'(1) = 0$. In majority

mechanisms, the inflection point is invariant for all $m \geq 1$,

$\breve{F}(m+1,2m+1) = \frac{1}{2}$, with $h_{2m+1}(\breve{F}) = \frac{1}{2}$.[16]

---

[15] The majority case notation is borrowed from Boland [1989].

[16] If $n > 1$, $H_{[v;n]}{}'(\cdot)$ increases monotonically from 0 on $\big[0,\frac{v-1}{n-1}\big]$, and thereafter falls to 0 from above on $\big[\frac{v-1}{n-1},1\big]$: $H_{[v;n]}{}'(F)$ has a strict global maximum at $\breve{F}$ (see Sydsæter [1981], Definition 5.13 and Theorem 5.6 regarding the point of inflection). Functions that are strictly convex (concave) on the open interval (a,b) are strictly con-

<u>Proof.</u>   The   identity $H_{[v;n]}(F) = \sum_{i=v}^{n}\binom{n}{i}[F]^i[1-F]^i = C(v,n)\int_0^F \tau^{v-1}(1-\tau)^{n-v}\,d\tau$,

$C(v,n) = n!/(n-v)!(v-1)!$,   allows   differentiation.   From   Leibniz's   formula

$H_{[v;n]}'(F) = C(v,n)[F]^{v-1}[1-F]^{n-v}$,   which   is   strictly   positive   if $F \neq 0,1$.[17]   From

$H_{[v;n]}(0) = 0$ and $H_{[v;n]}(1) = 1$, the continuous function is a bijection. Furthermore,

$$H_{[v;n]}''(F) = C(v,n)\left\{(v-1)[F]^{v-2}(1-F)^{n-v} - (n-v)[F]^{v-1}[1-F]^{n-v-1}\right\}$$
$$= C(v,n)[F]^{v-2}[1-F]^{n-v-1}\{v-1-(n-1)F\}$$

Hence, $H_{[v;n]}''(F) \geq 0 (<0)$ iff $F \leq (>)\{[v-1]/[n-1]\} = \breve{F}(v,n)$.   In   super-majority

mechanisms with an even number of judges, $v \geq (n/2)+1$ implies $\breve{F} \geq n/2(n-1)$, and in

super-majority mechanisms with an odd number of judges, $v \geq ((n+1)/2)+1$, implies

$\breve{F} \geq (n+1)/2(n-1)$. In   majority   mechanisms   with   and   odd   number   of   judges

$v = (n+1)/2$, gives $\breve{F} = \frac{1}{2}$ and $h_{2m+1}\left(\frac{1}{2}\right) = \frac{1}{2}$. ◄

The next proposition is illustrated in Figures 2.2 (part A) and 7.1 (part B).

   <u>Proposition 2.3</u> If $F \in (0,1)$:

   A) In all super-majority mechanisms $M_{[v;n]}$, $1 - H_{[v;n]}(F) > H_{[v;n]}(1-F)$.

   B) In majority mechanisms M_{2m+1}, $1 - h_{2m+1}(F) = h_{2m+1}(1-F)$.

---

vex (concave) on the closed interval $[a,b]$ (see Sydsæter [1981:247–54] for precise defi-
nitions and characterizations of concavity and convexity). $h_1(\cdot)$ is the identity mapping
(hence concave and convex).

[17] See Arnold, Balakrishnan & Nagaraja [2008], equation 2.2.14, regarding the identity.
See Bartle [1976], Theorem 31.8, for Lebniz' formula (at the end points in $[0,1]$ the de-
rivatives are right and left hand, respectively).

Proof. A: From Remark 2.1, $H_{[v;n]}(F) = \sum_{i=v}^{n} b(i;n,F) = 1 - \sum_{i=0}^{v-1} b(i;n,F)$. The general

relation $b(i;n,F) = b(n-i;n,1-F)$ gives $H_{[v;n]}(1-F) = \sum_{i=v}^{n} b(i;n,1-F)$

$= \sum_{i=v}^{n} b(n-i;n,F) = \sum_{i=0}^{n-v} b(i;n,F)$.[18] It follows that $1 - H_{[v;n]}(F) - H_{[v;n]}(1-F)$

$= \sum_{i=0}^{v-1} b(i;n,F) - \sum_{i=0}^{n-v} b(i;n,F) = B(v-1;n,F) - B(n-v;n,F)$. It remains to ensure

that $v-1-(n-v) = 2v-n-1$ is positive ($B(\cdot;n,F)$ is increasing). With an even number

of judges, super-majority implies $v \geq (n/2)+1$. It follows that

$2v-n-1 \geq 2((n/2)+1)-n-1 = 1$. With an odd number of judges, super-majority implies

$v \geq ((n+1)/2)+1$. It follows that $2v-n-1 \geq 2(([n+1]/2)+1)-n-1 = 2$. B: In the case

of majority rule and an odd number of judges, $v = m+1$ and

$2v-n-1 = 2(m+1)-(2m+1)-1 = 0$, for all m (cf. Boland [1989:181]). ◄
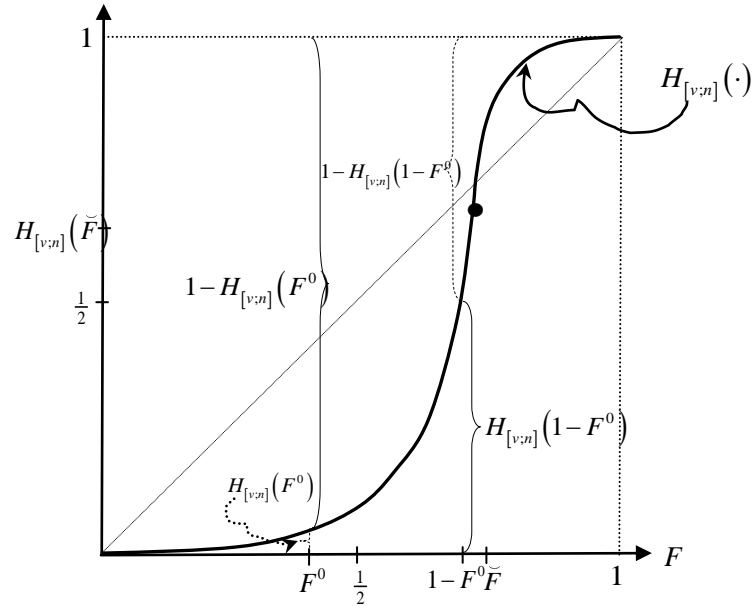
Figure 2.2 Liability probability in $M_{[v;n]}$ (with inflection point ●)

---

[18] See Bartoszyński & Niewiadomska-Bugaj [1996], equation 3.14, on the general relation.

The impact of decision mechanism size-increases are most easily described in the context of *majority rules*. In this case, the inflextion point is fixed ($\breve{F} \equiv \frac{1}{2}$), and the *S*-shaped curves $h_{2m+1}(\cdot)$ monotonically become more pronounced, see Figure 7.1. For a given input $F \in (0,1)$, the following are finite and asymptotic aspects, respectively, of the famous:

Condorcet theorem[19]

$$
\begin{array}{ll}
\text{C1} & \begin{cases} \text{if } \frac{1}{2} < F < 1, & h_{2m+1}(F) > F \text{ for all } m \geq 1 \\ \text{if } 0 < F < \frac{1}{2}, & h_{2m+1}(F) < F \text{ for all } m \geq 1 \end{cases} \\[2em]
\text{C2} & \begin{cases} \text{if } \frac{1}{2} < F < 1, & h_{2m+1}(F) \uparrow 1 \text{ as } m \to \infty \\ \text{if } 0 < F < \frac{1}{2}, & h_{2m+1}(F) \downarrow 0 \text{ as } m \to \infty \end{cases}
\end{array}
\tag{2.4}
$$

Section 7 gives additional results on majority transformations.

For a fixed *n*, increasing *v* shifts the *S*-shaped function graph to the south-east in Figure 2.2 (fewer terms are added in (2.2).[20]

In small super-majority mechanisms, increasing size affects inflection points and curvature properties, making precise statements complicated (for interim increases in n, there are no monotonicity properties corresponding to Proposition C2). Table 2.1 illustrates that increasing v and n, such that $vn^{-1} \equiv \frac{2}{3}$, tends to make the *S*-shaped curves more pronounced, monotonically

---

[19] Boland [1989]. The symbol $\uparrow$ ($\downarrow$) means that for any $F \in (\frac{1}{2},1)$ (any $F \in (0,\frac{1}{2})$) and any m, h_{2m+1} gets larger (smaller) as m increases: the bounded sequence $\left(h_{2m+1} : m \in \{0,1,2...\}\right)$ is increasing (decreasing). $h_{2m+1}(\frac{1}{2}) = \frac{1}{2}$ for all m.

[20] A qualitative change happens if unanimity results: $H_{[n;n]}(F) = [F]^n$, is strictly convex on $[0,1]$.

reducing $H_{[v;n]}(F)$ for sufficiently small F, and monotonically increasing $H_{[v;n]}(F)$ for sufficiently large F.[21]

Table 2.1: Size-effects under super-majority rules

F	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$H_{[2;3]}(F)$	0.028	0.104	0.216	0.352	0.500	0.648	0.784	0.896	0.972
$H_{[4;6]}(F)$	0.001	0.017	0.070	0.179	0.344	0.544	0.774	0.901	0.984
$H_{[6;9]}(F)$	0.000	0.003	0.025	0.099	0.254	0.483	0.730	0.914	0.992

In sufficiently large mechanisms, monotonicity is obtained for all F, relative to a suitably defined point. In the limit, the aggregation function is turned into a step function, as illustrated in Figure 2.3. If $q \in \left(\frac{1}{2}, 1\right)$ and $\lceil qn \rceil$ denotes the smallest integer larger than or equal to qn, a "q-rule mechanism" is defined as $M_{[\lceil qn \rceil;n]}$ with $H_{[\lceil qn \rceil;n]}(F) = P\left\{\sum_{i=1}^{n} J_i \geq \lceil qn \rceil \mid F\right\}$. It follows:

q-rule theorem (Fey [2003])

$$i) \begin{cases} \text{If } F > q, \text{ there exists an integer } \bar{n} \text{ such that for all } n \geq \bar{n} \\ \quad H_{[\lceil qn \rceil;n]}(F) > F \text{ and } \lim_{n \to \infty} H_{[\lceil qn \rceil;n]}(F) = 1^- \end{cases}$$

$$ii) \begin{cases} \text{ii) If } F \leq q, \text{ there exists an integer } \bar{\bar{n}} \text{ such that for all } n \geq \bar{\bar{n}} \\ \quad H_{[\lceil qn \rceil;n]}(F) < F \text{ and } \lim_{n \to \infty} H_{[\lceil qn \rceil;n]}(F) = 0^+ \end{cases}$$

More powerful asymptotic results are discussed in Section 6.

---

[21] Nordén [2015] compares $M_{[5;7]}$ and $M_{[7;10]}$ in detail.
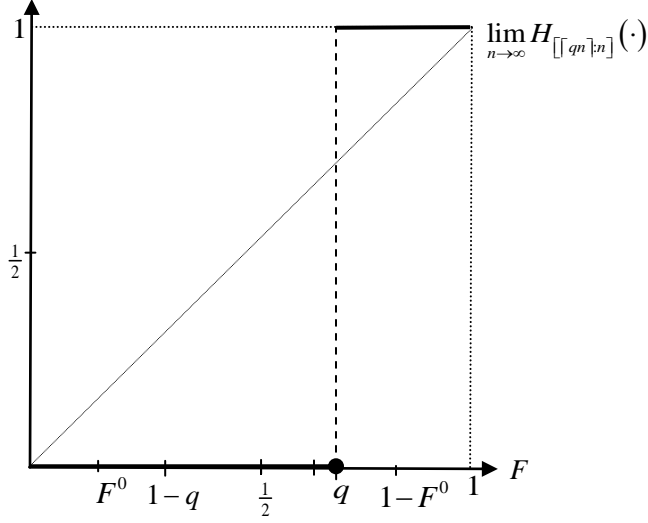
Figure 2.3 Liability probability under a q-rule ($n \to \infty$)

3   Polychotomous environments and the classical rule

This section studies transformation of abstract norms $g \in \mathbb{P}_Y^{\ X}$ with $Y = \mathbb{R}$. Let

the $\tau^g = L$ elements in $\underset{x \in X}{\cup} \mathrm{supp}\, g\left(\cdot \mid x\right) = \left\{ t^1, t^2, \ldots, t^L \right\}$ be denoted such that

$t^{k'} < t^k$ if $k' < k$. With $g\left(\cdot \mid x\right) \in \left[0,1\right]^{\tau^g}$ for all $x \in X$, the the transformed norms

$g_{[v;n]^{\Downarrow}}, g_{[v;n]^{\Uparrow}}, g_{2m+1} \in \mathbb{P}_{\mathbb{R}}^{\ X}$ generated in $M_{[v;n]^{\Downarrow}}$, $M_{[v;n]^{\Uparrow}}$, and M_{2m+1} are depicted

as:

$$g_{[v;n]^{\Downarrow}}\left(\cdot \mid x\right) = \left\langle g_{[v;n]^{\Downarrow}}\left(t^1 \mid x\right), g_{[v;n]^{\Downarrow}}\left(t^2 \mid x\right), \ldots, g_{[v;n]^{\Downarrow}}\left(t^L \mid x\right) \right\rangle;$$

$$g_{[v;n]^{\Uparrow}}\left(\cdot \mid x\right) = \left\langle g_{[v;n]^{\Uparrow}}\left(t^1 \mid x\right), g_{[v;n]^{\Uparrow}}\left(t^2 \mid x\right), \ldots, g_{[v;n]^{\Uparrow}}\left(t^L \mid x\right) \right\rangle; \text{ and },$$

$$g_{2m+1}\left(\cdot \mid x\right) = \left\langle g_{2m+1}\left(t^1 \mid x\right), g_{2m+1}\left(t^2 \mid x\right), \ldots, g_{2m+1}\left(t^L \mid x\right) \right\rangle,$$

respectively. From independence, the order in which judges vote does not matter.

The vector of individual conclusions $y = \left\langle y_1, y_2, \ldots, y_n \right\rangle \in \mathbb{R}^n$ corresponds to a

random sample (i.i.d.'s) of size n, from a population defined by the *discrete*

84

conditional cdf $G(t|x)\big|_{x \in X}$, see Figure II.2.5. Ranking of the judgment conclu-

sions in non-decreasing order $y_{1:n} \le y_{2:n} \le \cdots \le y_{n:n}$, as dictated by the classical

rule (Section 1.1), generates a *variational sequence*, the stochastic variable $y_{k:n}$

corresponding to the *k*-th (single) *order statistic*.[22] The aggregator (transfor-

mation) involves the entire set of variables, $\{y_i\}_{i=1}^{n}$.[23] It follows from the classical

rule variants that $M_{[v;n]^{\Downarrow}}$, $M_{[v;n]^{\Uparrow}}$, and M_{2m+1} correspond to the $n-v+1$ 'th order

statistic $y_{n-v+1:n}$, the v 'th order statistic $y_{v:n}$, and the *median* $y_{m+1:2m+1}$, respective-

ly. The following result will be needed:

> <u>Proposition 3.1</u> If the individual judgment conclusions $\{y_i\}_{i=1}^{n}$ are i.i.d.'s
>
> from a discrete cdf $G(t|x)$, for all $x \in X$, the following cdfs are gener-
>
> ated:
>
> (i) in $M_{[v;n]^{\Downarrow}}$
>
> $$P\{y_{n-v+1:n} \le t|x\}\big|_{M_{[v;n]^{\Downarrow}}} \equiv G_{n-v+1:n}(t|x)$$
>
> $$= \sum_{i=n-v+1}^{n} \binom{n}{i}\left[G(t|x)\right]^{i}\left[1-G(t|x)\right]^{n-i}$$
>
> $$= H_{[n-v+1;n]}\big(G(t|x)\big) = 1 - H_{[v;n]}\big(1-G(t|x)\big), \ -\infty < t < \infty$$
>
> (ii) in $M_{[v;n]^{\Uparrow}}$
>
> $$P\{y_{v:n} \le t|x\}\big|_{M_{[v;n]^{\Uparrow}}} \equiv G_{v:n}(t|x) = H_{[v;n]}\big(G(t|x)\big), \ -\infty < t < \infty$$

---

[22] Arnold, Balakrishnan & Nagaraja [2008:1].

[23] In the case of a continuous population cdf, the probability of variational sequence equalities is zero, simplifying derivations (see the rank order function defined in Section 1.1 and Section 6).

(iii) in M_{2m+1}

$$P\left\{y_{m+1:2m+1} \le t \big| x\right\}\Big|_{M_{2m+1}} \equiv G_{m+1:2m+1}\left(t \big| x\right)$$
$$= h_{2m+1}\left(G\left(t \big| x\right)\right), \quad -\infty < t < \infty$$

Proof. The notation stresses the fact that cdfs are obtained (and, hence, defined for all $t \in \mathbb{R}$). (i): The first *equality* follows from Arnold, Balakrishnan & Nagaraja [2008], equation 2.2.13 and pp. 41–42. The second equality follows from Proposition 2.1 above. The third equality follows from Proposition 3.2 below. (ii) and (iii) follow by the same lines of argument. ◄

Proposition 3.2 The following identity holds for all $F \in [0,1]$:

$$H_{[n-v+1;n]}\left(F\right) \equiv 1 - H_{[v;n]}\left(1-F\right).$$

Proof. From Remark 2.1, $H_{[n-v+1;n]}\left(F\right) = \sum_{i=n-v+1}^{n} b\left(i;n,F\right) = 1 - \sum_{i=0}^{n-v} b\left(i;n,F\right)$. Due to

the general relation $b\left(i;n,F\right) = b\left(n-i;n,1-F\right){}^{24}$, $\quad 1 - \sum_{i=0}^{n-v} b\left(i;n,F\right) =$

$1 - \sum_{i=0}^{n-v} b\left(n-i;n,1-F\right) \qquad = 1 - \left\{b\left(n;n,1-F\right) + b\left(n-1;n,1-F\right) + \ldots + b\left(v;n,1-F\right)\right\}$

$= 1 - \sum_{i=v}^{n} b\left(i;n,1-F\right) = 1 - H_{[v;n]}\left(1-F\right)$. ◄

Transformed norm components can be obtained from $G_{n-v+1:n}\left(t \big| x\right)\Big|_{x \in X}$,

$G_{v:n}\left(t \big| x\right)\Big|_{x \in X}$, and $G_{m+1:2m+1}\left(t \big| x\right)\Big|_{x \in X}$:

Proposition 3.3 Let $t^0 \in \bigcup_{x \in X} \text{supp } g\left(\cdot \big| x\right) = \left\{t^1, t^2, \ldots, t^L\right\}$. For each $x \in X$,

prospect component $g\left(t^0 \big| x\right)$ in the abstract norm $g \in \mathbb{P}_{\mathbb{R}}{}^{X}$ is trans-

formed to:

---

[24] Bartoszyński & Niewiadomska-Bugaj [1996], equation 3.14.

(i) in $M_{[v;n]^{\Downarrow}}$,

if $t^0 \in \operatorname{supp} g(\cdot \mid x)$,

$$g_{[v;n]^{\Downarrow}}\left(t^0 \mid x\right) = H_{[v;n]}\left(1 - \lim_{t \to t^{0-}} G(t \mid x)\right) - H_{[v;n]}\left(1 - G\left(t^0 \mid x\right)\right).$$

if $t^0 \notin \operatorname{supp} g(\cdot \mid x)$, $\quad g_{[v;n]^{\Downarrow}}\left(t^0 \mid x\right) = 0$

(ii) in $M_{[v;n]^{\Uparrow}}$

if $t^0 \in \operatorname{supp} g(\cdot \mid x)$,

$$g_{[v;n]^{\Uparrow}}\left(t^0 \mid x\right) = H_{[v;n]}\left(G\left(t^0 \mid x\right)\right) - H_{[v;n]}\left(\lim_{t \to t^{0-}} G(t \mid x)\right).$$

if $t^0 \notin \operatorname{supp} g(\cdot \mid x)$, $\quad g_{[v;n]^{\Uparrow}}\left(t^0 \mid x\right) = 0$

(iii) in M_{2m+1}

if $t^0 \in \operatorname{supp} g(\cdot \mid x)$,

$$g_{2m+1}\left(t^0 \mid x\right) = h_{2m+1}\left(G\left(t^0 \mid x\right)\right) - h_{2m+1}\left(\lim_{t \to t^{0-}} G(t \mid x)\right).$$

if $t^0 \notin \operatorname{supp} g(\cdot \mid x)$, $\quad g_{2m+1}\left(t^0 \mid x\right) = 0$

Proof. (i) By right continuity of the Proposition 3.1.i cdf, at each point t^0 with positive

probability (each *possible* point under the discrete cdf $G(t \mid x)\big|_{x \in X}$), $g_{[v;n]^{\Downarrow}}\left(t^0 \mid x\right) =$

$$= G_{n-v+1:n}\left(t^0 \mid x\right) - \lim_{t \to t^{0-}} G_{n-v+1:n}(t \mid x) = 1 - H_{[v;n]}\left(1 - G\left(t^0 \mid x\right)\right) - \lim_{t \to t^{0-}}\left\{1 - H_{[v;n]}\left(1 - G(t \mid x)\right)\right\}.$$

By continuity of $H_{[v;n]}(\cdot)$, $\lim_{t \to t^{0-}}\left\{1 - H_{[v;n]}\left(1 - G(t \mid x)\right)\right\} = 1 - H_{[v;n]}\left(1 - \lim_{t \to t^{0-}} G(t \mid x)\right)$, giving

(i). (ii) and (iii) follow by the same argument. ◄

Example 3.1 (Ex. II.2.3 continued: Proposition II.2.1 derived with a cdf). To demonstrate

the approach outlined in this section, let $\oplus = 0$ and $\otimes = 1$ and define cdf $G(t \mid x)$, as in

Example II.2.3:

$$G(t \mid x) = \begin{cases} 0 \text{ if } t < 0 \\ g(0 \mid x) \text{ if } t \in [0,1) \\ \underbrace{g(0 \mid x) + g(1 \mid x)}_{1} \text{ if } t \geq 1 \end{cases}.$$

From Propositions 3.3 and 2.2:

$$g_{[v;n]^{\psi}}(0 \mid x) = H_{[v;n]}\left(1 - \lim_{t \to 0^-} G(t \mid x)\right) - H_{[v;n]}\left(1 - G(0 \mid x)\right)$$

$$= H_{[v;n]}(1-0) - H_{[v;n]}\left(1 - g(0 \mid x)\right)$$

$$= \begin{cases} 1 - H_{[v;n]}(1-1) = 1 \text{ if } x < \underline{x} \\ 1 - H_{[v;n]}\left(1 - \{1 - F(x)\}\right) = 1 - H_{[v;n]}\left(F(x)\right) \text{ if } x \in [\underline{x}, \overline{x}] \\ 1 - H_{[v;n]}(1-0) = 0 \text{ if } x > \overline{x} \end{cases}$$

It follows that:

$$g_{[v;n]^{\psi}}(1 \mid x) = 1 - g_{[v;n]^{\psi}}(0 \mid x) = \begin{cases} 0 \text{ if } x < \underline{x} \\ H_{[v;n]}\left(F(x)\right) \text{ if } x \in [\underline{x}, \overline{x}] \\ 1 \text{ if } x > \overline{x} \end{cases}.$$

Hence, Proposition 3.3 gives the "direct" Proposition 2.1 (binary) result

$$\left\langle g_{[v;n]}(\oplus \mid x), g_{[v;n]}(\otimes \mid x)\right\rangle\Big|_{x \in X} = \left\langle 1 - H_{[v;n]}\left(F(x)\right), H_{[v;n]}\left(F(x)\right)\right\rangle\Big|_{x \in X}. \blacksquare$$

4   Higher-dimensional norms and voting protocols

Under the conjunctive "second stage" norm formulation $c \leftrightarrow a \wedge b$ in Example II.2.2 and stochastically independent norm dimensions, the abstract standard $g \in \mathbb{P}_{\{\neg c,c\}}^{X_a \times X_b}$ is given by

$$g(\cdot \mid x_a, x_b)\Big|_{\langle x_a, x_b \rangle \in X_a \times X_b} =$$
$$\left\langle 1 - F_A(x_a) F_B(x_b), F_A(x_a) F_B(x_b); \neg c, c \right\rangle\Big|_{\langle x_a, x_b \rangle \in X_a \times X_b}. \tag{4.1}$$

Under an outcome-based voting regime (OBV), judges vote directly on consequences, the set $Y = \{c, \neg c\}$, to reject or not reject the (default) proposition of 'no liability' ($\neg c$). Under a premise-based regime (PBV), judges vote separately

on proposition a and on proposition b. The connection rule $a \wedge b \leftrightarrow c$, is subsequently used to generate the collective decisions on liability.

Judges observe facts $\langle x_a, x_b \rangle \in X_a \times X_b$ prior to voting (Section 6 considers voting on conditioning norm elements in abstraction from facts). The same aggregation rule is used for decisions on all propositions (composite or simple). Under super-majority, proposition $\neg c$ is taken as a default in outcome-based regimes, and $\neg a$ and $\neg b$ as defaults in the premise-based regimes: v or more votes out of n are required for proposition rejections.

Let $\neg c \equiv \oplus$ and $c \equiv \otimes$. From independence (orthogonality), conjunction, and Proposition 2.1, under OBV the transformed norm $g_{[v;n]}^{OBV} \in \mathbb{P}_{\{\oplus,\otimes\}}^{X_a \times X_b}$ is given by:

$$g_{[v;n]}^{OBV} \left(\cdot \mid x_a, x_b \right) =$$
$$\left\langle 1 - H_{[v;n]}\left(F_A\left(x_a\right) F_B\left(x_b\right) \right), H_{[v;n]}\left(F_A\left(x_a\right) F_B\left(x_b\right) \right); \oplus, \otimes \right\rangle \tag{4.2}$$

for all $\langle x_a, x_b \rangle \in X_a \times X_b$. Under PBV transformed norm $g_{[v;n]}^{PBV} \in \mathbb{P}_{\{\oplus,\otimes\}}^{X_a \times X_b}$ is given by:

$$g_{[v;n]}^{PBV} \left(\cdot \mid x_a, x_b \right) =$$
$$\left\langle 1 - H_{[v;n]}\left(F_A\left(x_a\right) \right) \times H_{[v;n]}\left(F_B\left(x_b\right) \right), H_{[v;n]}\left(F_A\left(x_a\right) \right) \times H_{[v;n]}\left(F_B\left(x_b\right) \right); \oplus, \otimes \right\rangle \tag{4.3}$$

for all $\langle x_a, x_b \rangle \in X_a \times X_b$.

The next example demonstrates that, for some constellations of premise findings ("ultimate legal facts", see Remark II.2.6), $M_{[v;n]}^{OBV}$ lead to collective decision $\oplus$, whereas $M_{[v;n]}^{PBV}$ lead to $\otimes$.

<u>Example 4.1</u> (a doctrinal paradox). Consider a case $\langle x_a, x_b \rangle \in X_a \times X_b$ decided under

$c \leftrightarrow a \wedge b$ in M_3. Let $\langle a,b \rangle^i \in \{0,1\} \times \{0,1\}$ denote Judge i's findings with respect to the

premises and assume the profile of findings $\langle \langle 1,1 \rangle^1, \langle 1,0 \rangle^2, \langle 0,1 \rangle^3 \rangle$. Each judge applies

the doctrine consistently. In M_3^{OBV}, two judges conclude no liability, and the judgment

$\langle \langle x_a, x_b \rangle, \oplus \rangle^{M_3^{OBV}}$ is rendered. In M_3^{PBV}, two judges conclude that $a = 1$ and that $b = 1$.

The doctrine implies liability and the judgment $\langle \langle x_a, x_b \rangle, \otimes \rangle^{M_3^{PBV}}$ is rendered.[25] ∎

Example 4.1 illustrates the *doctrinal paradox*: the judges (or, more generally, a

group of agents) who decide under OBV and the constraint (or theory) $c \leftrightarrow a \wedge b$

cannot reason their decision (OBV is a legislated procedure in many situations).

It illustrates deep logical problems arising under aggregation of judgments of

interconnected propositions. The phenomenon potentially affects any collective

making decisions under a doctrine or theory.[26] The present discussion is limited

to aspects needed in the analysis of bifurcated trials in Section 5. For this purpose

the simple two-dimensional, conjunctive formulation of doctrine is sufficient (a

recent generalized version of the paradox is considered in Example VI.2).

Let the probability of a paradox occurring in $M_{[v;n]}$ be denoted $\rho_{[v;n]}$. It

depends on:

- the abstract norm structure, $g \in \mathbb{P}_{\{\oplus, \otimes\}}^{X_a \times X_b}$;

---

[25] Independence means that the sequence in which judges vote is irrelevant. There are
$4^3 = 64$ possible profiles of findings. In addition to the Example 4.1 profile, five other
profiles generate a paradox: $\langle \langle 1,1 \rangle, \langle 0,1 \rangle, \langle 1,0 \rangle \rangle$, $\langle \langle 1,0 \rangle, \langle 1,1 \rangle, \langle 0,1 \rangle \rangle$, $\langle \langle 0,1 \rangle, \langle 1,1 \rangle, \langle 1,0 \rangle \rangle$,
$\langle \langle 1,0 \rangle, \langle 0,1 \rangle, \langle 1,1 \rangle \rangle$, and $\langle \langle 0,1 \rangle, \langle 1,0 \rangle, \langle 1,1 \rangle \rangle$ See List [2005] for a systematic combinatori-
al approach.

[26] See Part I and Mongin [2012].

- fact location, $\langle x_a, x_b \rangle \in X_a \times X_b$, and

- mechanism size n and aggregation rule threshold v.

<u>Proposition 4.1</u> Let $g \in \mathbb{P}_{\{\oplus,\otimes\}}{}^{X_a \times X_b}$ be defined by

$\langle 1 - F_A(x_a) F_B(x_b), F_A(x_a) F_B(x_b); \oplus, \otimes \rangle \big|_{\langle x_a, x_b \rangle \in X_a \times X_b}$ in (4.1). The doctri-

nal paradox probability in $M_{[v;n]}$ under the conjunctive connection rule

and orthogonality, $\rho_{[v;n]} : X_a \times X_b \to [0,1]$, is given by:

$$\rho_{[v;n]}(x_a, x_b) = g_{[v;n]}^{PBV}(\otimes \mid x_a, x_b) - g_{[v;n]}^{OBV}(\otimes \mid x_a, x_b)$$
$$= H_{[v;n]}(F_A(x_a)) H_{[v;n]}(F_B(x_b)) - H_{[v;n]}(F_A(x_a) F_B(x_b))^{\cdot}$$

$\rho_{[v;n]}$ is strictly positive in the interior of $[\underline{x}_a, \bar{x}_a] \times [\underline{x}_b, \bar{x}_b] \subseteq X_a \times X_b$,

and zero elsewhere.

<u>Proof.</u> See Nordén [2015]. ◄

<u>Remark 4.1.A</u> The proposition generalizes directly to higher-dimensional norms

(Nordén [2015]). The function $\rho_{[v;n]}(\cdot)$ is complex, even under Proposition 4.1

assumptions. □

<u>Remark 4.1.B</u> In Landa & Lax's [2009] framework (see Example II.3.5), Exam-

ple 4.1 can be illustrated as follows: Let the case-space $C = \{0,1\} \times \{0,1\}$, the first

and second dimension representing proposition a and b, respectively, coded such

that $\langle 1,1 \rangle$ corresponds to $a = 1$ and $b = 1$, $\langle 1,0 \rangle$ to $a = 1$ and $b = 0$, etc. The

judges share the (conjunctive) base rule $\langle rd, \tau \rangle = \langle \langle 1,1 \rangle, 2 \rangle$, corresponding to

$c \leftrightarrow a \wedge b$, but disagree about legal findings (case dimensions). □

Due to the central role of majority mechanisms, paradox probabilities $\rho_{[m+1;2m+1]} \equiv \rho_{2m+1}$ are particularly interesting. They can be calculated globally on $X_a \times X_b$ based on Proposition 4.1.[27] The next result concerns *maximal* paradox probabilities on $X_a \times X_b$:

> Conjecture 4.2 The constellation of facts solving $\underset{x_a, x_b}{\arg\max}\, \rho_{2m+1}(x_a, x_b)$ is
>
> given by $\left\langle x_a^{\rho,2m+1}, x_b^{\rho,2m+1} \right\rangle = \left\langle F_A^{-1}(q_{2m+1}*), F_B^{-1}(q_{2m+1}*) \right\rangle$, where for all
>
> $m \geq 1$ $q_{2m+1}* \in \left(\frac{1}{2}, 1\right)$ is the unique (implicit) solution to:
>
> $$h_{2m+1}(q_{2m+1}*) = [q_{2m+1}*]^{m+1}[1 + q_{2m+1}*]^m, \qquad (4.4)$$
>
> with maximal values $\rho_{2m+1}* = \left[h_{2m+1}(q_{2m+1}*)\right]^2 - h_{2m+1}\left([q_{2m+1}*]^2\right)$.

Solutions to (4.4) and function values $\rho_{2m+1}*$ for $m \in \{1,\dots,9\}$ are given in Table 4.1.

Remark 4.2 In the case of $m \in \{1,2\}$, solutions to (4.4) are easily found:

$h_3(q_3*) = [q_3*]^2[1 - q_3*] \leftrightarrow q_3* = \frac{2}{3}$. $\qquad h_5(q_5*) \qquad = [q_5*]^3[1 - q_5*]^2 \qquad \leftrightarrow$

$5[q_5*]^2 - 17[q_5*] + 9 = 0$, with positive root $q_5* \approx 0.656$. Solutions $q_{2m+1}*$ for

---

[27] Nordén [2015] reports function values ρ_3 and $\rho_{[5;7]}$ over the *probability space* (that is, alternative combinations $(F_A, F_B) \in (0,1) \times (0,1)$, corresponding to fact variations $\langle x_a, x_b \rangle$ in supp $f_A \times$ supp $f_B \subseteq X_a \times X_b$ (the paper focuses variation in decision mechanisms, rather than the abstract norm structure). List [2005] calculates doctrinal paradox probabilities in majority mechanisms in what may be characterized as a sampling framework with exogenous parameters describing population parameters (see Remark VII.3.2). List also calculates maximal paradox probabilities. As explained in Nordén [2015], the substantive norm formulation (4.1) reduces the degrees of freedom compared to List's constraint $c \leftrightarrow a \wedge b$ and therefore must imply lower maximal paradox probabilities.

$m \geq 3$ are approximate and based on simulations. $\rho_{2m+1}*$ can be obtained using Remark 2.1:

$$\left[h_{2m+1}\left(q_{2m+1}* \right) \right]^2 - h_{2m+1}\left(\left[q_{2m+1}* \right]^2 \right)$$

$$= \left[1 - B\left(m; 2m+1, q_{2m+1}* \right) \right]^2 - \left\{ 1 - B\left(m; 2m+1, \left[q_{2m+1}* \right]^2 \right) \right\} \qquad ,$$

$$= B\left(m; 2m+1, q_{2m+1}* \right)\left\{ B\left(m; 2m+1, q_{2m+1}* \right) - 2 \right\} + B\left(m; 2m+1, \left[q_{2m+1}* \right]^2 \right)$$

$B(\cdot)$ is the binomial cdf (with parameters $2m+1$ trials and success probability $q_{2m+1}*$ or $\left[q_{2m+1}* \right]^2$). □

Proof (outline). Due to due to continuity of h_{2m+1}, given continuous cdfs F_A, F_B, $\rho_{2m+1}(\cdot)$ defined in Proposition 4.1 is continuous, hence obtains a maximum on the closed set $[\underline{x}_a, \overline{x}_a] \times [\underline{x}_b, \overline{x}_b]$. The maximum lies in the interior ($\rho_{2m+1} \equiv 0$ on the boundary). Assuming sufficiently smooth cdfs, a maximum is necessarily described by the first order conditions:

$$\partial \rho_{2m+1} / \partial x_a =$$
$$h_{2m+1}\left(F_B\left(x_b \right) \right) h_{2m+1}'\left(F_A\left(x_a \right) \right) F_A'\left(x_a \right) - h_{2m+1}'\left(F_A\left(x_a \right) F_B\left(x_b \right) \right) F_B\left(x_b \right) F_A'\left(x_a \right) = 0$$
$$\partial \rho_{2m+1} / \partial x_b = \qquad\qquad (4.5)$$
$$h_{2m+1}\left(F_A\left(x_a \right) \right) h_{2m+1}'\left(F_B\left(x_b \right) \right) F_B'\left(x_b \right) - h_{2m+1}'\left(F_A\left(x_a \right) F_B\left(x_b \right) \right) F_A\left(x_a \right) F_B'\left(x_b \right) = 0$$

On $\text{int}[\underline{x}_a, \overline{x}_a] \times [\underline{x}_b, \overline{x}_b]$, the derivatives F_A', F_B' cancel. Assuming a symmetric equilibrium, $F_A = F_B = q$, $q \in (0,1)$, (4.5) reduces to $h_{2m+1}'(q) h_{2m+1}(q) - h_{2m+1}'(q^2) = 0$. Substituting for h_{2m+1}' from (7.3) in Section 7 below gives (4.4).

Uniqueness of solutions to (4.4) can be established as follows: Define the strictly convex function $\varphi_{2m+1} : [0,1] \to \mathbb{R}_+$ by $\varphi_{2m+1}(q) = [q]^{m+1}[1+q]^m$. $\varphi_{2m+1}(0) = 0$ and $\varphi_{2m+1}(1) = 2^m$, with $\varphi_{2m+1}'(0) = 0$, $\varphi_{2m+1}'(q), \varphi_{2m+1}''(q) > 0$ if $q \in (0,1)$ and $m \geq 1$. Consider $\Phi_{2m+1}(q) = h_{2m+1}(q) - \varphi_{2m+1}(q)$. Solutions to (4.4) corresponds to points $q_{2m+1}*$ such that $\Phi_{2m+1}(q_{2m+1}*) = 0$. $\Phi_{2m+1}\left(\tfrac{1}{2} \right) = h_{2m+1}\left(\tfrac{1}{2} \right) - \varphi_{2m+1}\left(\tfrac{1}{2} \right) = \tfrac{1}{2} - 3^m / 2^{2m+1} > 0$

93

for all $m \geq 1$ and $\Phi_{2m+1}\left(\frac{1}{2}\right) \uparrow \frac{1}{2}$ if $m \uparrow$. $\Phi_{2m+1}(1) = h_{2m+1}(1) - \varphi_{2m+1}(1) = 1 - 2^m < 0$ for all

$m \geq 1$ and $\Phi_{2m+1}(1) \downarrow -\infty$ if $m \uparrow$. From Proposition 2.2, $h_{2m+1}(\cdot)$ is strictly concave on

$\left[\frac{1}{2},1\right]$. Because $\varphi_{2m+1}(\cdot)$ is strictly convex, $-\varphi_{2m+1}(\cdot)$ is strictly concave. It follows that

$\Phi_{2m+1}(\cdot)$, being a sum of concave functions, is (strictly) concave on $\left[\frac{1}{2},1\right]$.[28] Hence,

$\Phi_{2m+1}(q) = 0$ has a unique solution $q_{2m+1}*$ in $\left(\frac{1}{2},1\right)$ for all $m \geq 1$. ◄

Table 4.1 Maximal doctrinal paradox probabilities $\rho_{2m+1}*$

m	1	2	3	4	5	6	7	8	9
$q_{2m+1}*$	.667	.656	.650	.645	.642	.640	.638	.636	.635
$\rho_{2m+1}*$	.132	.228	.306	.371	.427	.478	.520	.559	.594

<u>Remark 4.3</u> The proof of Conjecture 4.2 is incomplete because the possibility of non-symmetric solutions (in the probability space), has not been ruled out for $m \geq 2$.[29] In large mechanisms ($\lim_{m \to \infty} M_{2m+1}$), , $X_a \times X_b$ is partitioned into areas where the probability of paradoxes is 0 and 1, respectively (Nordén [2015]). □

5   Unitary versus bifurcated trials

In Section 3, voting takes place directly on the set of legal consequences Y. This section considers the effect of partitioning Y, such that a decision is first made between dichotomous alternatives, followed by a decision on a subset of alternatives conditioned by the first stage outcome. Such bifurcation of proceedings is typically *jus cogens* with respect to criminal liability, and representative of a

---

[28] Sydsæter [1981], Theorem 5.14.i.

[29] Nordén [2015] proves a symmetric solution if $m = 1$ (and for $\rho_{[5;7]}$ using a theorem on global univalence).

wider class of sequential decision-making regimes, frequently observed in legal procedures (Section I.1).

This section considers a negligence standard protecting legal entitlements, as introduced in Example II.2.1, combined with uncertainty regarding the scope of liability (damages). The negligence standard is given by the stochastic variable S with cdf F, liability occurring iff $S \leq x_s$. The standard $g \in \mathbb{P}_{\{0,1\}}^{X_s}$ is represented by $\langle 1 - F(x_s), F(x_s); 0, 1 \rangle\big|_{x_s \in X_s}$, where 0 and 1 denotes no liability ($\oplus$) and liability ($\otimes$), respectively.

Let $x_l \in X_l$ be an index of (non-stochastic) facts relevant to the scope of liability. The norm defining damages $g_p \in \mathbb{P}_{\{l^1, l^2, \ldots, l^L\}}^{X_l}$ is given by:

$$\langle p_1(x_l), p_2(x_l), \ldots, p_L(x_l); l^1, l^2, \cdots, l^L \rangle\big|_{x_l \in X_l},$$

with $0 < l^1 < l^2 < \cdots < l^L$.[30] Substantive law is assumed to treat liability and damages as independent (a fact attested to by the various mechanisms that separate the decisions!).[31] The substantive norm can be represented as in Figure 5.1. Using independence and the linear structure of probabilities, a reduced form (final distribution over outcomes) is obtained:

$$g_r(\cdot \mid x_s, x_l) = \langle 1 - F(x_s), F(x_s) p_1(x_l), \ldots, F(x_s) p_L(x_l); 0, l^1, l^2, \cdots, l^L \rangle,$$

$\langle x_s, x_l \rangle \in X_s \times X_l$. The (abstract) reduced norm is compactly denoted $g_r \in \mathbb{P}_{\{0, l^1, l^2, \ldots, l^L\}}^{X_s \times X_l}$.

---

[30] Because damages are independent of x_s, a simplified (discrete) version of Shavell's [1987] loss function is defined (see Section IV.5.1 below).

[31] See Wissler, Rector & Saks [2001] on the possible fusion (lack of independence) between decisions on liability and damages in actual decision-making.
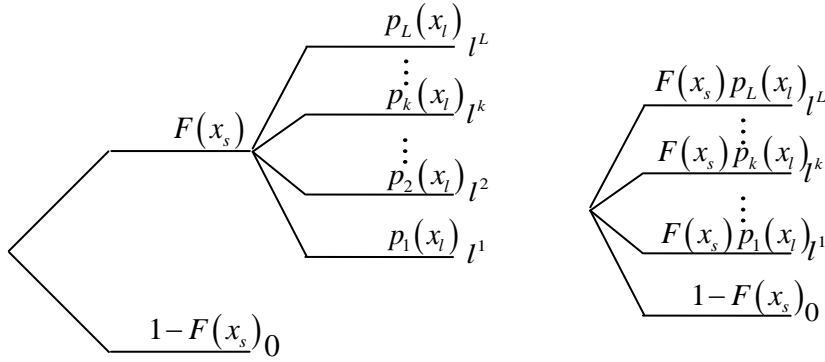
Figure 5.1 Abstract and abstract reduced norm (local representations)

In unitary trials (U), judges vote directly on consequences $\{0, l^1, l^2, ..., l^L\}$ under g_r. In two-stage bifurcated trials (B), judges vote on liability ($\{0,1\}$) under g_F at stage one, and over awards ($\{l^1, l^2, ..., l^L\}$) under g_p at stage two, given liability at stage one.

In dichotomous settings, with a super-majority rule, no liability is assumed to be the default state. In polychotomous environments, the *classical rule*, aggregating "from above" ($\Downarrow$), protects default 0, or states corresponding to lower damage awards (Section 1.1). Unitary trial mechanisms are denoted $M^U_{[v;n]^{\Downarrow}}$. Under bifurcation, panel size and aggregation rule may vary at the two stages. The (combined) mechanisms are denoted $M^B_{[v^I;n^I] \wedge [v^{II};n^{II}]^{\Downarrow}}$. Varying first stage (I) and second stage (II) parameters, $\langle v^I, n^I \rangle$ and $\langle v^{II}, n^{II} \rangle$, respectively, a large set of mechanisms in $\mathcal{M}$ result.

Judges vote conditionally on non-stochastic legal facts $\langle x_s, x_l \rangle \in X_s \times X_l$. Commitment and independence are interpreted to imply that judges vote independently, across stages, in bifurcated trials.

The next propositions compare transformation of the abstract reduced norm $g_r \in \mathbb{P}_{\{0,l^1,...,l^L\}}^{X_s \times X_l}$ in $M^U_{[v;n]^{\Downarrow}}$ to the transformed norm generated in $M^B_{[v^I;n^I] \wedge [v^{II};n^{II}]^{\Downarrow}}$ leading to $g^U_{r,[v;n]^{\Downarrow}}, g^B_{[v^I;n^I] \wedge [v^{II},n^{II}]^{\Downarrow}} \in \mathbb{P}_{\{0,l^1,l^2...,l^L\}}^{X_s \times X_l}$, respectively. Particular attention will be paid to the case of uniformly sized mechanism at all "decision points" ($v = v^I = v^{II}$ and $n = n^I = n^{II}$) and two-dimensional damage functions ($L = 2$), e.g. interpreted as uncertainty regarding availability of *punitive damages*. These situations give sharp results. First, however, general transformations are established (if somewhat tersely). Let

$$\tau^{g_r} = \# \left| \bigcup_{\langle x_s, x_l \rangle \in X_s \times X_l} \mathrm{supp}\, g_r(\cdot \mid x_s, x_l) \right| = L+1:$$

Proposition 5.1 (Unitary Trials). Direct voting on $\{0,l^1,l^2...,l^L\}$ under $g^U_{r,[v;n]^{\Downarrow}} \in \mathbb{P}_{\{0,l^1,l^2,...,l^L\}}^{X_s \times X_l}$ in $M^U_{[v;n]^{\Downarrow}}$ results in $g^U_{r,[v;n]^{\Downarrow}} \in \mathbb{P}_{\{0,l^1,l^2...,l^L\}}^{X_s \times X_l}$ with $\tau^{g_r} = (L+1)$ prospect components $g^U_{r,[v;n]^{\Downarrow}}(t^0 \mid x_s, x_l)$ given by

($F = F(x_s)$ and $p_k = p_k(x_s)$, $k = 1,2,...,L-1$):

$$\begin{cases}
g^U_{r,[v;n]^{\Downarrow}}(l^L \mid x_s, x_l) = H_{v.n}(F[1 - p_1 - \cdots - p_{L-1}]) \\
g^U_{r,[v;n]^{\Downarrow}}(l^{L-1} \mid x_s, x_l) = H_{v.n}(F[1 - p_1 - \cdots - p_{L-2}]) - H_{v.n}(F[1 - p_1 - \cdots - p_{L-1}]) \\
\quad\vdots \\
g^U_{r,[v;n]^{\Downarrow}}(l^k \mid x_s, x_l) = H_{v.n}(F[1 - p_1 - \cdots - p_{k-1}]) - H_{v.n}(F[1 - p_1 - \cdots - p_k]) \\
\quad\vdots \\
g^U_{r,[v;n]^{\Downarrow}}(l^2 \mid x_s, x_l) = H_{v.n}(F[1 - p_1]) - H_{v.n}(F[1 - p_1 - p_2]) \\
g^U_{r,[v;n]^{\Downarrow}}(l^1 \mid x_s, x_l) = H_{v.n}(F) - H_{v.n}(F[1 - p_1]) \\
g^U_{r,[v;n]^{\Downarrow}}(0 \mid x_s, x_l) = 1 - H_{v.n}(F)
\end{cases}$$

Proof. The (reduced) abstract norm has cdf:

$$G(t \mid x_s, x_l) = \begin{cases} 0 \text{ if } t \in (-\infty, 0) \\ 1 - F(x_s) \text{ if } t \in [0, l^1) \\ 1 - F(x_s) + F(x_s) p_1(x_l) \text{ if } t \in [l^1, l^2) \\ \vdots \\ 1 - F(x_s) + F(x_s) [p_1(x_l) + p_2(x_l) + \cdots + p_k(x_l)] \text{ if } t \in [l^k, l^{k+1}) \\ \vdots \\ 1 - F(x_s) + F(x_s) [p_1(x_l) + p_2(x_l) + \cdots + p_{L-1}(x_l)] \text{ if } t \in [l^{L-1}, l^L) \\ 1 \text{ if } t \in [l^L, \infty) \end{cases}$$

From Proposition 3.3.i $g^U_{r,[v:n]^{\Downarrow}}(t^0 \mid x) = H_{[v:n]}\left(1 - \lim_{t \to t^{0-}} G(t \mid x_s, x_l)\right) - H_{[v:n]}\left(1 - G(t^0 \mid x_s, x_l)\right)$,

$t^0 \in \{0, l^1, ..., l^L\}$. Then $g^U_{r,[v:n]^{\Downarrow}}(0 \mid x) = H_{[v:n]}\left(1 - \lim_{t \to 0^-} G(t \mid x_s, x_l)\right) - H_{[v:n]}\left(1 - G(0 \mid x_s, x_l)\right)$.

From right continuity of $G(t \mid x_s, x_l)$ and Proposition 2.2, it equals

$H_{v:n}(1-0) - H_{v:n}\left(1 - [1 - F(x_s)]\right) = 1 - H_{v:n}(F(x_s))$.   Similarly, $(F = F(x_s)$

$p_k = p_k(x_l), \ k = 1, ..., L-1)$

$g^U_{r,[v:n]^{\Downarrow}}(l^1 \mid x) = H_{[v:n]}\left(1 - [1 - F]\right) - H_{[v:n]}\left(1 - [1 - F + Fp_1]\right) = H_{[v:n]}(F) - H_{[v:n]}(F[1 - p_1])$

$g^U_{r,[v:n]^{\Downarrow}}(l^2 \mid x) = H_{[v:n]}\left(1 - [1 - F + Fp_1]\right) - H_{[v:n]}\left(1 - [1 - F + Fp_1 + Fp_2]\right)$
$= H_{[v:n]}(F[1 - p_1]) - H_{[v:n]}\left(F[1 - (p_1 + p_2)]\right)$

$g^U_{r,[v:n]^{\Downarrow}}(l^k \mid x) = H_{[v:n]}\left(1 - [1 - F + Fp_1 + \cdots + Fp_{k-1}]\right) - H_{v:n}\left(1 - [1 - F + Fp_1 + \cdots + Fp_k]\right)$
$= H_{[v:n]}\left(F[1 - (p_1 + \cdots + p_{k-1})]\right) - H_{[v:n]}\left(F[1 - (p_1 + \cdots + p_k)]\right)$

$g^U_{r,[v:n]^{\Downarrow}}(l^{L-1} \mid x) = H_{[v:n]}\left(1 - [1 - F + Fp_1 + \cdots + Fp_{L-2}]\right) - H_{[v:n]}\left(1 - [1 - F + Fp_1 + \cdots + Fp_{L-1}]\right)$
$= H_{[v:n]}\left(F[1 - (p_1 + \cdots + p_{L-2})]\right) - H_{[v:n]}\left(F[1 - (p_1 + \cdots + p_{L-1})]\right)$

$g^U_{r,[v:n]^{\Downarrow}}(l^L \mid x) = H_{[v:n]}\left(1 - [1 - F + Fp_1 + \cdots + Fp_{L-1}]\right) - H_{[v:n]}\left(1 - [1 - F + Fp_1 + \cdots + Fp_L]\right)$
$= H_{[v:n]}\left(F[1 - (p_1 + \cdots + p_{L-1})]\right) - H_{[v:n]}(0) = H_{[v:n]}\left(F[1 - (p_1 + \cdots + p_{L-1})]\right)$

◄

Proposition 5.2 (Bifurcated Trials). A bifurcated trial (first-stage voting on $\{0,1\}$ under $g \in \mathbb{P}_{\{0,1\}}{}^{X_s}$ in $M_{\left[v^I;n^I\right]}$, and second-stage voting on $\{l^1, l^2, ..., l^L\}$ under $g_p \in \mathbb{P}_{\{l^1,l^2,...,l^L\}}{}^{X_l}$ in $M_{\left[v^{II};n^{II}\right]^{\Downarrow}}$ results in $g^B_{\left[v^I;n^I\right]\wedge\left[v^{II};n^{II}\right]^{\Downarrow}} \in \mathbb{P}_{\{0,l^1,l^2,...,l^L\}}{}^{X_s \times X_l}$ with $\tau = (L+1)$ prospect components $g^B_{\left[v^I;n^I\right]\wedge\left[v^{II};n^{II}\right]^{\Downarrow}}\left(t^0 \mid x_s, x_l\right)$ given by given by ($F = F(x_s)$ and $p_k = p_k(x_l)$, $k = 1, 2, ..., L-1$):

$$
\begin{cases}
g^B_{\left[v^I;n^I\right]\wedge\left[v^{II};n^{II}\right]^{\Downarrow}}\left(l^L \mid x_s, x_l\right) \\[4pt]
= H_{\left[v^I;n^I\right]}(F) H_{\left[v^{II};n^{II}\right]}\left(1 - p_1 - \cdots - p_{L-1}\right) \\[4pt]
g^B_{\left[v^I;n^I\right]\wedge\left[v^{II};n^{II}\right]^{\Downarrow}}\left(l^{L-1} \mid x_s, x_l\right) \\[4pt]
= H_{\left[v^I;n^I\right]}(F)\left\{H_{\left[v^{II};n^{II}\right]}\left(1 - p_1 - \cdots - p_{L-2}\right) - H_{\left[v^{II};n^{II}\right]}\left(1 - p_1 - \cdots - p_{L-1}\right)\right\} \\[4pt]
\quad\vdots \\[4pt]
g^B_{\left[v^I;n^I\right]\wedge\left[v^{II};n^{II}\right]^{\Downarrow}}\left(l^k \mid x_s, x_l\right) \\[4pt]
= H_{\left[v^I;n^I\right]}(F)\left\{H_{\left[v^{II};n^{II}\right]}\left(1 - p_1 - \cdots - p_{k-1}\right) - H_{\left[v^{II};n^{II}\right]}\left(1 - p_1 - \cdots - p_k\right)\right\} \\[4pt]
\quad\vdots \\[4pt]
g^B_{\left[v^I;n^I\right]\wedge\left[v^{II};n^{II}\right]^{\Downarrow}}\left(l^2 \mid x_s, x_l\right) \\[4pt]
= H_{\left[v^I;n^I\right]}(F)\left\{H_{\left[v^{II};n^{II}\right]}\left(1 - p_1\right) - H_{\left[v^{II};n^{II}\right]}\left(1 - p_1 - p_2\right)\right\} \\[4pt]
g^B_{\left[v^I;n^I\right]\wedge\left[v^{II};n^{II}\right]^{\Downarrow}}\left(l^1 \mid x_s, x_l\right) H_{\left[v^I;n^I\right]}(F)\left\{1 - H_{\left[v^{II};n^{II}\right]}\left(1 - p_1\right)\right\} \\[4pt]
g^B_{\left[v^I;n^I\right]\wedge\left[v^{II};n^{II}\right]^{\Downarrow}}\left(0 \mid x_s, x_l\right) = 1 - H_{\left[v^1;n^1\right]}(F)
\end{cases}
$$

Proof. It follows directly from Proposition 2.1, that the first stage transformed norm is $\left\langle 1 - H_{\left[v^I,n^I\right]}\left(F(x_s)\right), H_{\left[v^I,n^I\right]}\left(F(x_s)\right); 0, 1\right\rangle$. The second stage transformed norm follows from Proposition 5.1 with $F \equiv 1$, $n = n^{II}$ and $v = v^{II}$ (in the discrete setting, $F = 1$ im-

plies that outcome 0 is impossible; $H_{\left[v''; n''\right]}(1)=1$). The reduced distribution over the set

of consequences is, therefore, given by the product of the probabilities assigned to con-

secutive segments of the corresponding probability tree branch. ◄

Comparison of $g^{U}_{r,[v;n]^{U}}$ and $g^{B}_{[v;n]\wedge[v;n]^{U}}$ (a uniform number of judges and the same

super-majority rule at all "decision points"), corresponds to a court voting direct-

ly on outcomes, or splitting the proceedings, respectively.

<u>Proposition 5.3</u> The (component-wise) difference between $g^{B}_{[v;n]\wedge[v;n]^{U}}$ and

$g^{U}_{r,[v;n]^{U}}$ is ($F=F(x_{s})$; $p_{k}=p_{k}(x_{l})$, $k=1,2,...,L$; $\rho_{[v;n]}(\cdot,\cdot)$ denotes the

function $\rho_{[v;n]}:[0,1]\times[0,1]\to[0,1]$ defined in Proposition 4.1,

$\rho_{[v;n]}(\alpha,\beta)=H_{[v;n]}(\alpha)H_{[v;n]}(\beta)-H_{[v;n]}(\alpha\beta),$

$\langle\alpha(x_{s}),\beta(x_{l})\rangle\in[0,1]\times[0,1]$):

$$
\begin{cases}
g^{B}_{[v;n]\wedge[v;n]^{U}}\left(l^{L}\mid x_{s},x_{l}\right)-g^{U}_{r,[v;n]^{U}}\left(l^{L}\mid x_{s},x_{l}\right)\\
=\rho_{[v;n]}(F,p_{L})>0 \text{ if } x\in(\underline{x},\overline{x}), p_{L}\in(0,1)\\
g^{B}_{[v;n]\wedge[v;n]^{U}}\left(l^{L-1}\mid x_{s},x_{l}\right)-g^{U}_{r,[v;n]^{U}}\left(l^{L-1}\mid x_{s},x_{l}\right)\\
=\rho_{[v;n]}(F,1-p_{1}-\cdots-p_{L-2})-\rho_{[v;n]}(F,1-p_{1}-\cdots-p_{L-1})\\
\quad\vdots\\
g^{B}_{[v;n]\wedge[v;n]^{U}}\left(l^{k}\mid x_{s},x_{l}\right)-g^{U}_{r,[v;n]^{U}}\left(l^{k}\mid x_{s},x_{l}\right)\\
=\rho_{[v;n]}(F,1-p_{1}-\cdots-p_{k-1})-\rho_{[v;n]}(F,1-p_{1}-\cdots-p_{k})\\
\quad\vdots\\
g^{B}_{[v;n]\wedge[v;n]^{U}}\left(l^{2}\mid x_{s},x_{l}\right)-g^{U}_{r,[v;n]^{U}}\left(l^{2}\mid x_{s},x_{l}\right)\\
=\rho_{[v;n]}(F,1-p_{1})-\rho_{[v;n]}(F,1-p_{1}-p_{2})\\
g^{B}_{[v;n]\wedge[v;n]^{U}}\left(l^{1}\mid x_{s},x_{l}\right)-g^{U}_{r,[v;n]^{U}}\left(l^{1}\mid x_{s},x_{l}\right)\\
=-\rho_{[v;n]}(F,1-p_{1})<0 \text{ if } x\in(\underline{x},\overline{x}), p_{1}\in(0,1)\\
g^{B}_{[v;n]\wedge[v;n]^{U}}\left(0\mid x_{s},x_{l}\right)-g^{U}_{r,[v;n]^{U}}\left(0\mid x_{s},x_{l}\right)=0
\end{cases}
$$

<u>Proof.</u> From Proposition 5.2 and 5.1 with $v = v^I = v^{II}$ and $n = n^I = n^{II}$, the k-th component difference $g^B_{[v;n] \wedge [v;n]^{\Downarrow}} \left(l^k \mid x_a, x_b \right) - g^U_{r,[v;n]} \left(l^k \mid x_a, x_b \right)$ is given by:

$$H_{[v;n]}(F) \left\{ H_{[v;n]}\left(1 - p_1 - \cdots - p_{k-1} \right) - H_{[v;n]}\left(1 - p_1 - \cdots - p_k \right) \right\}$$
$$- \left\{ H_{[v;n]}\left(F\left[1 - (p_1 + \cdots + p_{k-1}) \right] \right) - H_{[v;n]}\left(F\left[1 - (p_1 + \cdots + p_k) \right] \right) \right\}$$
$$= H_{[v;n]}(F) H_{[v;n]}\left(1 - p_1 - \cdots - p_{k-1} \right) - H_{[v;n]}\left(F\left[1 - (p_1 + \cdots + p_{k-1}) \right] \right)$$
$$- \left\{ H_{[v;n]}(F) H_{v.n}\left(1 - p_1 - \cdots - p_k \right) - H_{[v;n]}\left(F\left[1 - (p_1 + \cdots + p_k) \right] \right) \right\}$$
$$= \rho_{[v;n]}\left(F, 1 - p_1 - \cdots - p_{k-1} \right) - \rho_{[v;n]}\left(F, 1 - p_1 - \cdots - p_k \right)$$

The last equality follows from Proposition 4.1. If $k = 1$, $\rho_{[v;n]}(F,1) - \rho_{[v;n]}(F,1-p_1)$

$= -\rho_{[v;n]}(F, 1 - p_1)$.[32] If $k = L$, $\rho_{[v;n]}\left(F, 1 - p_1 - \cdots - p_k \right) = \rho_{[v;n]}(F,0) = 0$, hence

$$g^B_{[v;n] \wedge [v;n]^{\Downarrow}} \left(l^L \mid \cdot \right) - g^U_{[v;n]^{\Downarrow}} \left(l^L \mid \cdot \right) = \rho_{[v;n]}\left(F, 1 - p_1 - \cdots - p_{L-1} \right) = \rho_{[v;n]}(F, p_L). \blacktriangleleft$$

Proposition 5.3 confirms that the mechanisms are equivalent under determinacy,

$(\alpha, \beta) \in \{(1,1), (1,0), (0,1), (0,0)\}$ and that they are equivalent for all values of β

given $\alpha \equiv 0$ (if $x \leq \underline{x}$, no liability is the only possible outcome under the sub-

stantive norm). In higher dimensions ($L \geq 3$), if $2 \leq k \leq L - 1$, the component

differences ($F = F(x_s)$ and $p_k = p_k(x_l)$, $k = 1, 2, ..., L$)

$$g^B_{[v;n] \wedge [v;n]^{\Downarrow}} \left(l^k \mid x_s, x_l \right) - g^U_{r,[v;n]^{\Downarrow}} \left(l^k \mid x_s, x_l \right)$$
$$= \rho_{[v;n]}\left(F, 1 - p_1 - \cdots - p_{k-1} \right) - \rho_{[v;n]}\left(F, 1 - p_1 - \cdots - p_k \right)$$

may be positive or negative depending on the location $\langle x_s, x_l \rangle \in X_s \times X_l$. Few

general statements seem possible, except regarding l^1 and l^L, due to the com-

plex structure of $\rho_{[v;n]}(\cdot, \cdot)$.[33]

---

[32] The paradox probabilities collapse at boundaries (certainty with respect to at least one norm component).

[33] See Nordén [2015] for some curvature properties.

In the binary case ($L=2$), sharper characterizations can be given. Let $l^2 = \bar{l}$ and $l^1 = \underline{l}$, $p_2 = p(x_l)$ and $p_1 = 1 - p(x_l)$. Figure 5.2 illustrates $g^B_{[v;n]\wedge[v;n]^{\Downarrow}}(\cdot \mid x_s, x_l)$ and $g^U_{r,[v;n]^{\Downarrow}}(\cdot \mid x_s, x_l)$.

$$
\begin{array}{cc}
\underbrace{\hspace{2.5cm}}\;H_{[v;n]}(F)H_{[v;n]}(p)\,\overline{l} & \underbrace{\hspace{2.5cm}}\;H_{[v;n]}(Fp)\,\overline{l} \\
H_{[v;n]}(F)\{1-H_{[v;n]}(p)\}\,\underline{l} & H_{[v;n]}(F)-H_{[v;n]}(Fp)\,\underline{l} \\
1-H_{[v;n]}(F)\,0 & 1-H_{[v;n]}(F)\,0
\end{array}
\;,
$$
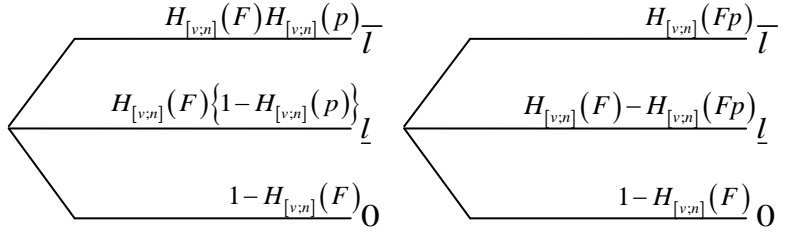
Figure 5.2 Transformed reduced norms from bifurcated trials (left node) and unitary tri-als (right node), $L=2$, $F = F(x_s)$, $p = p(x_l)$

Component-wise probability differences are given by:

$$
\begin{aligned}
g^B_{[v;n]\wedge[v;n]^{\Downarrow}}(\bar{l} \mid x) - g^U_{r,[v;n]^{\Downarrow}}(\bar{l} \mid x_s, x_l) &= \rho_{[v;n]}\left(F(x_s), p(x_l)\right) \\
g^B_{[v;n]\wedge[v;n]^{\Downarrow}}(\underline{l} \mid x) - g^U_{r,[v;n]^{\Downarrow}}(\underline{l} \mid x_s, x_l) &= -\rho_{[v;n]}\left(F(x_s), p(x_l)\right). \\
g^B_{[v;n]\wedge[v;n]^{\Downarrow}}(0 \mid x) - g^U_{r,[v;n]^{\Downarrow}}(0 \mid x_s, x_l) &= 0
\end{aligned}
\qquad (5.1)
$$

The corresponding cdf's, $G^B_{[v;n]\wedge[v;n]^{\Downarrow}}(t \mid x_s, x_l)$ and $G^U_{r,[v;n]^{\Downarrow}}(t \mid x_s, x_l)$ are illustrated in Figure 5.3:

$G^B_{[v;n]\wedge[v;n]^{\Downarrow}}(t \mid x_s, x_l)$ ——

$G^U_{r,[v;n]^{\Downarrow}}(t \mid x_s, x_l)$ - - -

$\rho_{[v;n]}\left(F(x_s), p(x_l)\right)$
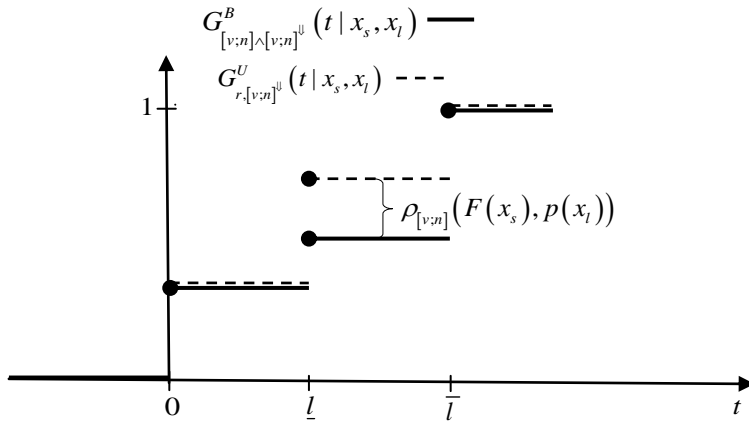
Figure 5.3 Transformed norm cdfs in bifurcated and unitary trials: first-order stochastic dominance

102

Assume that an agent (plaintiff) is an expected utility maximizer, with (an increasing) Bernoulli utility function $u(\cdot)$. Cdf $G^*(t)$ *first-order stochastically dominates* cdf $\tilde{G}(t)$ if, for every non-decreasing function $u:\mathbb{R}\to\mathbb{R}$,

$$\int u(t)dG^*(t) \geq \int u(t)d\tilde{G}(t).^{34}$$

<u>Proposition 5.4</u>   (First-Order   Stochastic   Dominance).   If $L=2$,

$g^B_{[v;n]\wedge[v;n]^{\Downarrow}} \in \mathbb{P}^{X_s\times X}_{\{0,l,\bar{l}\}}$ ($G^B_{[v;n]\wedge[v;n]^{\Downarrow}}\left(t\mid x_s,x_l\right)\Big|_{\langle x_s,x_l\rangle\in X_s\times X_l}$) first-order stochasti-

cally dominates $g^U_{r,[v;n]\wedge[v;n]^{\Downarrow}} \in \mathbb{P}^{X_s\times X}_{\{0,l,\bar{l}\}}$ ($G^U_{r,[v;n]\wedge[v;n]^{\Downarrow}}\left(t\mid x_s,x_l\right)\Big|_{\langle x_s,x_l\rangle\in X_s\times X_l}$) for

all $\langle x_s,x_l\rangle\in X_s\times X_l$.

<u>Proof.</u> The bounded cdf $G^*(t)$ first-order stochastically dominates $\tilde{G}(t)$ if and only if

$G^*(t)\leq\tilde{G}(t)$ for every t (Mas-Colell, Whinston & Green [1995], Proposition 6.D.1).

Therefore, Proposition 5.4 is a direct consequence of (5.1), as illustrated in Figure 5.3. ◄

The degree to which $g^B_{[v;n]\wedge[v;n]^{\Downarrow}}$ ($G^B_{[v;n]\wedge[v;n]^{\Downarrow}}\left(t\mid x_s,x_l\right)\Big|_{\langle x_s,x_l\rangle\in X_s\times X_l}$) dominates

$g^U_{r,[v;n]\wedge[v;n]^{\Downarrow}}$ ($G^U_{r,[v;n]\wedge[v;n]^{\Downarrow}}\left(t\mid x_s,x_l\right)\Big|_{\langle x_s,x_l\rangle\in X_s\times X_l}$) depends on:

- the structure of the underlying abstract norms (as represented by norm components $F(\cdot)$ and $p(\cdot)$;

- the location of $\langle x_s,x_l\rangle\in X_s\times X_l$; and

- the specifics of the mechanism used ($M_{[v;n]}$ shaping $\rho_{[v;n]}(\cdot)$).

---

[34] Mas-Colell, Whinston & Green [1995], Definition 6.D.1.

The majority case comparison of g_{2m+1}^B ($G_{[m+1;2m+1]\wedge[m+1;2m+1]}^B(t\mid x,p)\big|_{\langle x_s,x_l\rangle\in X_s\times X_l}$)

and $g_{r,2m+1}^U$ ($G_{r,[m+1;2m+1]\wedge[m+1;2m+1]}^U(t\mid x,p)\big|_{\langle x_s,x_l\rangle\in X_s\times X_l}$) is of particular relevance to

civil law. In this case, the maximum difference between the cdfs (maximum cdf

jumps in Figure 5.3) is obtained at $\langle x_s,x_l\rangle=\langle F^{-1}(q_{2m+1}*),p^{-1}(q_{2m+1}*)\rangle$, with

maximum values given in Table 4.1.[35]

From a rule of law perspective, it seems questionable to assign courts

discretion as to sequencing of trials.[36]

6   Theoretical norm element determination

In Sections 2–5, judges vote on (ordinary-level) norm conclusions, or possibly

over norm premises, *conditional on* legal facts $x\in X$ or $\langle x_a,x_b\rangle\in X_a\times X_b$. This

section considers effects of judicial panels determining (ordinary-level) legal

standard *elements*, prior to the application to facts. The assumption of non-

stochastic facts is maintained (Section VII.5 studies separation of decisions on

law and facts under joint legal and epistemic uncertainty).

Consider the standard $g_F\in\mathbb{P}_{\{\oplus,\otimes\}}^X$ introduced in Example II.2.1, liability

($\otimes$) occurring iff $S\le x$, S with cdf $F(t)$. In super-majority mechanisms, no

liability ($\oplus$) is the default state. In panel decisions on S, in abstraction from or

prior to the application to facts, the *classical rule* corresponds to "aggregation

from below ($\Uparrow$)" and selection of $S_{v:n}$ (realizations of S towards the upper sup-

---

[35] In the case of $X_s=X_l$, a maximum is obtained under the restriction $x_s=x_l$ and there-
fore reduced (in general, the restriction will be binding in the optimization problem).

[36] Under indeterminacy, the splitting of trial into two stages, can be seen as involving a
structural commitment to "polarized justice" as criticized by Coons [1964].

port boundary protects the defendant, Section 1.1). Applied to fact $x \in X$, liability concluded iff $S_{v:n} \leq x$. The mechanisms are denoted $M_{[v;n]^{\Uparrow,T}}$ and the transformed standards denoted $g_{F,[v;n]^{\Uparrow,T}} \in \mathbb{P}_{\{\oplus,\otimes\}}{}^X$. Parallel results for $M_{[v;n]^{\Downarrow,T}}$ are stated and will be used regarding factual aggregation in Section VII.5. Majority mechanisms, denoted M^T_{2m+1}, are neutral and do not require specification of defaults.[37] Transformed norms are denoted $g^T_{F,2m+1} \in \mathbb{P}_Y{}^X$.

> Proposition 6.1 If $g_F \in \mathbb{P}^X_{\{\oplus,\otimes\}}$ is a standard with stochastic variable S
>
> with cdf $F(t)$, theoretical panel decisions on S according to the classical rule, lead to:
>
> (i) in $M_{[v;n]^{\Uparrow,T}}$ (state $\oplus$ prioritized, liability iff $S_{v:n} \leq x$),
>
> $g_{F,[v;n]^{\Uparrow,T}} \in \mathbb{P}_{\{\oplus,\otimes\}}{}^X$, given by:
>
> $$g_{F,[v;n]^{\Uparrow,T}}\left(\cdot \mid x\right)\Big|_{x \in X} = \left\langle 1 - H_{[v;n]}\left(1 - F(x)\right), H_{[v;n]}\left(F(x)\right); \oplus, \otimes \right\rangle\Big|_{x \in X}$$
>
> (ii) in $M_{[v;n]^{\Downarrow,T}}$ (state $\otimes$ prioritized, liability iff $S_{n-v+1:n} \leq x$),
>
> $g_{F,[v;n]^{\Downarrow,T}} \in \mathbb{P}_{\{\oplus,\otimes\}}{}^X$, given by:
>
> $$g_{F,[v;n]^{\Downarrow,T}}\left(\cdot \mid x\right)\Big|_{x \in X} = \left\langle H_{[v;n]}\left(1 - F(x)\right), 1 - H_{[v;n]}\left(1 - F(x)\right); \oplus, \otimes \right\rangle\Big|_{x \in X}$$
>
> (iii) in M^T_{2m+1} (no default, liability iff $S_{m+1:2m+1} \leq x$), $g^T_{F,2m+1} \in \mathbb{P}_{\{\oplus,\otimes\}}{}^X$,
>
> given by:
>
> $$g^T_{F,2m+1}\left(\cdot \mid x\right)\Big|_{x \in X} = \left\langle 1 - h_{2m+1}\left(F(x)\right), h_{2m+1}\left(F(x)\right); \oplus, \otimes \right\rangle\Big|_{x \in X}$$

---

[37] $M_{[m+1;2m+1]^{\Uparrow,T}} = M_{[m+1;2m+1]^{\Downarrow,T}}$.

<u>Proof.</u> Judges are i.i.d.'s $\{S_i\}_{i=1}^{n}$ from the *continuous* cdf $F(t)$ (no conditioning on legal facts!). The resulting variational sequence is $S_{1:n} < S_{2:n} < \cdots < S_{n:n}$ (the probability of equalities is zero due to continuity). $M_{[v;n]^{\Uparrow,T}}$ corresponds to the single order statistic $S_{v:n}$ (Section 1.1). From Arnold, Balakrishnan & Nagaraja [2008], equation 2.2.13, it has cdf:

$$P\{S_{v:n} \leq t\} = F_{v:n}(t) = \sum_{i=v}^{n} \binom{n}{i} \left[F(t)\right]^{i} \left[1-F(t)\right]^{n-i}, \quad -\infty < t < \infty. \tag{6.1}$$

It follows that given $x \in X$, $P\{S_{v:n} \leq x\} = F_{v:n}(x)$. From Proposition 2.1, $F_{v:n}(x) = H_{[v;n]}(F(x))$. This proves (i).

$M_{[v;n]^{\Downarrow,T}}$ corresponds to order statistic $S_{n-v+1:n}$ (Section 1.1). By the same argument, $P\{S_{n-v+1:n} \leq x\} = H_{[n-v+1;n]}(F(x))$. From the Proposition 3.2 identity, $H_{[n-v+1;n]}(F(x)) = 1 - H_{[v;n]}(1-F(x))$. Hence (ii).

Majority mechanisms correspond to $S_{m+1:2m+1}$ (Section 1.1). Hence, $P\{S_{m+1:2m+1} \leq x\} = F_{m+1:2m+1}(x) = h_{2m+1}(F(x))$, confirming (iii). ◄

<u>Remark 6.1.A</u> In super-majority mechanisms, $g_{F,[v;n]^{\Uparrow,T}}(\otimes \mid x) \leq g_{F,[v;n]^{\Downarrow,T}}(\otimes \mid x)$ for all $x \in X$ with strict inequality on the support $[\underline{x}, \overline{x}] \subseteq X$. The component difference $g_{F,[v;n]^{\Downarrow,T}}(\otimes \mid x) - g_{F,[v;n]^{\Uparrow,T}}(\otimes \mid x)$ is equal to $B(v-1;n,F(x)) - B(n-v;n,F(x))$, see Proposition 2.3.A. □

<u>Remark 6.1.B</u> In majority mechanisms, for all $x \in X$ Proposition 2.3.B implies $g_{F,2m+1}^{T}(\otimes \mid x) = h_{2m+1}(F(x)) \equiv 1 - h_{2m+1}(1-F(x))$. □

<u>Remark 6.1.C</u> From Propositions 6.1.i and 2.1 (with $g(\cdot \mid x) = \langle 1-F(x), F(x); \oplus, \otimes \rangle \equiv g_F(\cdot \mid x)$, $g_{F,[v;n]^{\Uparrow,T}}$ generated in $M_{[v;n]^{\Uparrow,T}}$ is equal to

$g_{F,[v;n]}$ generated in $M_{[v;n]}$ (in majority mechanisms, $g^{T}_{F,2m+1}$ generated in M^{T}_{2m+1}

is equal to $g_{F,2m+1}$ generated in M_{2m+1}). $\square$

Remark 6.1.C implies that that norm transformations in $M_{[v;n]}$ (M_{2m+1}) can be

analyzed *as if* taking place in $M_{[v;n]^{\Uparrow,T}}$ (M^{T}_{2m+1}). This fact has analytical interest

and will be utilized repeatedly. For example, if the abstract standard $g_{F} \in \mathbb{P}^{X}_{\{\oplus,\otimes\}}$

has a sufficiently smooth cdf F, it has *density* $f(t)$, see Example II.2.1. Let

$f_{v:n}(t)$ be the *transformed* density corresponding to $F_{v:n}(t)$ arising from $M_{[v;n]^{\Uparrow,T}}$

(see the proof of Proposition 6.1.i). From Remark 6.1.C, (6.1) and Bartoszyński

& Niewiadomska-Bugaj [1996], equation 6.32, the transformed density generated

by $M_{[v;n]^{\Uparrow,T}}$ and $M_{[v;n]}$ is given by:

$$f_{v:n}(t) = F_{v:n}'(t) = H_{[v;n]}'\big(F(t)\big)F'(t) = H_{[v;n]}'\big(F(t)\big)f(t), \quad -\infty < t < \infty, \quad (6.2)$$

and the majority mechanism density generated by M^{T}_{2m+1} and M_{2m+1} is given by:

$$f_{m+1:2m+1}(t) = F_{m+1:2m+1}'(t) = h_{2m+1}'\big(F(t)\big)f(t), \quad -\infty < t < \infty. \quad (6.3)$$

The "outer" results in Propositions 2.2 and 2.3 give insight into these complex

transformations (majority mechanisms are considered in detail in Section 7).

Example 6.1 (uniform densities). Parts IV and V use a class of uniform distributions

$S \sim U\big(\ln L - \underline{x}, \ln L + \overline{x}\big)$. The abstract norm has density:

$$f(t;\overline{x},L) = \begin{cases} 0 & \text{if } t < \ln L - \overline{x} \\ \dfrac{1}{2\overline{x}} & \text{if } t \in \big[\ln L - \overline{x}, \ln L + \overline{x}\big]. \\ 0 & \text{if } t > \ln L + \overline{x} \end{cases}$$

Closed form transformed densities may be obtained. For example:

$$f_{m+1:2m+1}(t;\underline{x},L) = \begin{cases} 0 \text{ if } t < \ln L - \overline{x} \\ \dfrac{k_m}{(2\overline{x})^{2m+1}}\left(t - (\ln L - \underline{x})\right)^m \left(\ln L + \overline{x} - x\right)^m \text{ if } t \in [\ln L - \overline{x}, \ln L + \overline{x}]. \\ 0 \text{ if } t > \ln L + \overline{x} \end{cases}$$

∎

In large courts, the following asymptotic result provides approximations for analyses in later sections. Let $q \in (0,1)$ and $\lceil nq \rceil$ denote the smallest integer, larger than or equal to nq:

<u>Proposition 6.2</u> (Legal standard asymptotic distribution under the classical rule in $M_{[\lceil qn \rceil ;n]^T}$). If the standard cdf $F(t)$ is sufficiently smooth with pdf $f(t)$, $f(F^{-1}(q)) > 0$ and $f(t)$ continuous at $F^{-1}(q)$, and $n \to \infty$

$$\sqrt{n} f\left(F^{-1}(q)\right) \frac{\left(S_{\lceil nq \rceil :n} - F^{-1}(q)\right)}{\sqrt{q(1-q)}} \xrightarrow{\;D\;} N(0,1).^{38}$$

<u>Proof.</u> The proposition follows directly from Arnold, Balakrishnan & Nagaraja [2008], Theorem 8.5.1 on central order statistics. ◄

<u>Remark 6.2</u> $q = \frac{1}{2}$ corresponds to majority mechanisms. $q > \frac{1}{2}$ corresponds to $M_{[v:n]^{\uparrow,T}}$ and $q < \frac{1}{2}$ to $M_{[v:n]^{\downarrow,T}}$, respectively. For sufficiently large, but finite n,

---

[38] The symbol $\xrightarrow{\;D\;}$ denotes convergence in distribution: Let $\xi_0, \xi_1, \xi_2, \ldots$ be a sequence of random variables and $P\{\xi_k \leq t\} = F_k(t)$, $k = 0,1,2,\ldots$ be their cdf's. The sequence $\{\xi_k\}$ converges in distribution if $\lim_{k \to \infty} F_k(t) = F_0(t)$ at every t at which $F_0(t)$ is continuous. See Bartoszyński & Niewiadomska-Bugaj [1996], in and at Definition 10.2.3, for concept motivation and discussion.

$S_{\lceil nq \rceil:n}$ is approximately $N\left(F^{-1}(q), \frac{q(1-q)}{n\left[f\left(F^{-1}(q)\right)\right]^2}\right)$.[39] If the abstract norm distribu-

tion is symmetric, $E\left\{S_{\lceil nq \rceil:n}\right\} \equiv \mu_{\lceil nq \rceil:n} = F^{-1}(q)$, otherwise $\mu_{\lceil nq \rceil:n} \approx F^{-1}(q)$

($Var\left\{S_{\lceil nq \rceil,n}\right\} \equiv \sigma^2_{\lceil nq \rceil:n} \approx \frac{q(1-q)}{n\left[f\left(F^{-1}(q)\right)\right]^2}$). $\square$

<u>Example 6.2</u> (Ex. 6.1 continued: approximated density). In the majority case, $q = \frac{1}{2}$. Be-

cause $f\left(t; \underline{\overline{x}}, L\right)$ is symmetric, $F^{-1}\left(\frac{1}{2}; \underline{\overline{x}}, L\right) = \ln L$ and $f\left(\ln L; \underline{\overline{x}}, L\right) = \frac{1}{2\overline{x}} > 0$ and con-

tinuous at that point. From Proposition 6.2 and Remark 6.2, for sufficiently large m,

$S_{m+1:2m+1}$ is approximately from $N\left(\ln L, \frac{\overline{x}^2}{2m+1}\right)$. $\blacksquare$

7   Majority mechanisms

Due to the central role of majority mechanisms in civil procedure and in deci-

sions on (pure) law, this section studies their properties in further detail. Curva-

ture aspects are highlighted, reflecting their impact in equilibrium analysis (see,

especially, Parts IV and V). Without loss of generality, liability components of

transformed standards $g_F \in \mathbb{P}_{\{\oplus,\otimes\}}{}^X$ are considered. Due to the functional equiva-

lence of M^T_{2m+1} and M_{2m+1}, $g^T_{F,2m+1}\left(\otimes \mid x\right) = g_{F,2m+1}\left(\otimes \mid x\right) = h_{2m+1}\left(F(x)\right)$ are not

distinguished (Remark 6.1.C).

    Propositions 2.1 and 2.2 imply that the "outer" aggregation functions

$h_{2m+1}: [0,1] \to [0,1]$ are strictly increasing bijections. If $m \geq 1$, they are strictly

---

convex on $[0, \frac{1}{2}]$, and strictly concave on $[0,1]$, with inflection point $\breve{F} = \frac{1}{2}$ (h_1 is $id_{[0,1]}$). In convenient notation:

$$h_{2m+1}(F) = k_m \int_0^F \tau^m (1-\tau)^m \, d\tau \text{ , with} \tag{7.1}$$

$$k_m \equiv (2m+1)!/[m!]^2 = (2m+1)\{(2m)!/[m!]^2\} = (2m+1)\binom{2m}{m} \geq 1. \tag{7.2}$$

$$\begin{cases} h_{2m+1}'(F) = k_m [F(1-F)]^m \\ h_{2m+1}''(F) = k_m m [F(1-F)]^{m-1} (1-2F) \end{cases} . \tag{7.3}$$

Fundamental symmetry properties around $\breve{F} = \frac{1}{2}$—corresponding to the *median of underlying abstract norm distributions*—are expressed by $h_{2m+1}\left(\frac{1}{2}\right) = \frac{1}{2}$ and $h_{2m+1}(F) = 1 - h_{2m+1}(1-F)$, for all m and all $F \in [0,1]$, see Proposition 2.3.B and Figure 7.1. The Condorcet theorem monotonicity property (C2, Section 6) is also illustrated. The $45°$ line represents a single judge, $h_1(F) = F$.
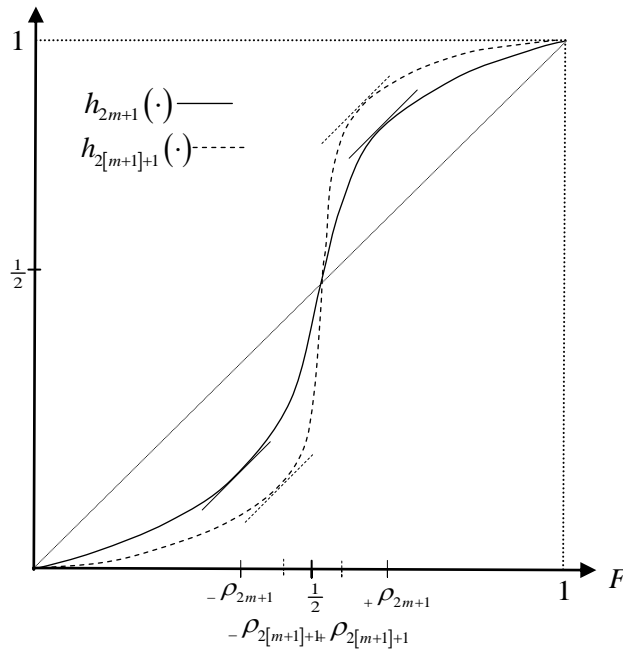
Figure 7.1 The graph of $h_1(\cdot), h_{2m+1}(\cdot)$, and $h_{2[m+1]+1}(\cdot)$

110

At $\breve{F} = \frac{1}{2}$, the slope of the aggregation function is given by:

$$h_{2m+1}{}'\left(\tfrac{1}{2}\right) = k_m / 2^{2m} = (2m+1)(2m)! / 2^{2m}[m!]^2 . \qquad (7.4)$$

These monotonically increasing derivatives will play a key role in the equilibrium analysis. Some function values are given in Table 7.1 For large m, $(2m)![m!]^{-2}$ may be approximated by $(\pi m)^{-\frac{1}{2}} 2^{2m}$ (see the Proposition 7.1–2 proof). Accordingly:

$$h_{2m+1}{}'\left(\tfrac{1}{2}\right) \approx (2m+1) / \sqrt{\pi m} . \qquad (7.5)$$

In the limit, the derivative $h_{2m+1}{}'$ does not exist at $\breve{F} = \frac{1}{2}$.[40]

Table 7.1 Derivatives at the point of inflection

m	0	1	2	3	4	5	6	7	8	9
$h_{2m+1}{}'\left(\tfrac{1}{2}\right)$	0	1.5	1.875	2.188	2.461	2.707	2.933	3.142	3.339	3.542

An immediate consequence of convergence property C2 in (2.4) is that for any (fixed) $F \neq \frac{1}{2}$, $\lim\limits_{m \to \infty} h_{2m+1}{}'(F) = 0^+$.[41] However, there is no parallel to level-monotonicity property C1, with respect to marginals $h_{2m+1}{}'$. As seen in Figure 7.1, if input values F are sufficiently close to $\frac{1}{2}$, derivatives $h_{2m+1}{}'$ *grow* in interim increases of m. These facts motivate considering the following experiments: For $m \geq 1$, find input levels ρ_{2m+1} such that:

$$h_{2m+1}{}'\left(\rho_{2m+1}\right) = h_1{}'(F) \equiv 1. \qquad (7.6)$$

---

[40] By L'Hospital's Rule (Bartle & Sherbert [1982], Theorem 5.3.4) $\lim\limits_{m \to \infty}\left\{(2m+1) / \sqrt{\pi m}\right\} = \lim\limits_{m \to \infty}\left\{\left(4/\sqrt{\pi}\right)\sqrt{m}\right\} = \infty .$

[41] Because the limit function is constant on the half open intervals $\left[0, \tfrac{1}{2}\right)$ and $\left(\tfrac{1}{2}, 1\right]$, in the limit h_{2m+1} must have a vanishing derivative.

The equation roots ρ_{2m+1} partition the input set $[0,1]$ into intervals where marginal incentives are boosted (slope of the aggregation functions increased), constant (slope invariant) and weakened (slope decreased) relative to the single judge mechanism (45^0-line). The roots also identify the points at which the *level-differences* in probability between majority voting panels (M_{2m+1}^T, M_{2m+1}) and the single-judge mechanisms (M_1^T and M_1) are maximized and minimized, respectively.[42]

Proposition 7.1 Equation (7.6) has roots:

$$_{+,-}\rho_{2m+1} = \frac{1}{2}\left(1 \pm \sqrt{1 - \frac{4}{\sqrt[m]{k_m}}}\right), \quad k_m = \frac{(2m+1)}{[m!]^2}.$$

Figure 7.1 and Table 7.2 suggest that $_{+,-}\rho_{2m+1}$ converge monotonically toward $\frac{1}{2}$ from above and below, respectively:

Proposition 7.2 For large *m*, the equations in (7.6) have roots:

$$_{+,-}\rho_{2m+1} \sim \frac{1}{2}\left(1 \pm \sqrt{1 - 4\left[\frac{(2m+1)2^{2m}}{\sqrt{\pi m}}\right]^{-\frac{1}{m}}}\right),$$

with $_{+}\rho_{2m+1} \downarrow \frac{1}{2}$ and $_{-}\rho_{2m+1} \uparrow \frac{1}{2}$ as $m \to \infty$.

---

[42] For example, $_{+,-}\rho_3 = \frac{1}{2}\left(1 \pm \frac{1}{\sqrt{3}}\right)$. The $m=1$ solutions corresponds to maximizing or minimizing the difference $d(g) = h_3(g) - h_1(g)$ on $[0,1]$. $d(g) = g(1-g)(2g-1)$, with $d(0) = d(1) = 0$. Critical points are roots of $d'(g) = 0$, equaling $_{+,-}\rho_3$ and $\frac{1}{2}$. Because $_{+}\rho_3 > \frac{1}{2}$, $d(_{+}\rho_3) > 0$, $_{-}\rho_3 < \frac{1}{2}$, $d(_{-}\rho_3) < 0$, and $d(\frac{1}{2}) = 0$, $_{+}\rho_3$ and $_{-}\rho_3$ are strict global maximum and minimum points for $d(\cdot)$ in $[0,1]$, respectively. (Because $d(\cdot)$ is continuously differentiable, no other candidates exist, Bartle & Sherbert [1982], Theorem 5.2.1.)

Table 7.2 Partitioning input-values in majority mechanisms (M_{2m+1} , M_{2m+1}^{T})

$n=2m+1$	m	$_{-}\rho_{2m+1}$	$_{+}\rho_{2m+1}$
3	1	.2113	$\frac{1}{2}(1+0.577)=.7887$
5	2	.2403	$\frac{1}{2}(1+0.519)=.7597$
7	3	.2604	$\frac{1}{2}(1+0.479)=.7396$
9	4	.2755	$\frac{1}{2}(1+0.449)=.7245$
11	5	.2875	$\frac{1}{2}(1+0.425)=.7125$
13	6	.2974	$\frac{1}{2}(1+0.405)=.7026$
15	7	.3058	$\frac{1}{2}(1+0.388)=.6942$
17	8	.3130	$\frac{1}{2}(1+0.374)=.6870$
19	9	.3193	$\frac{1}{2}(1+0.361)=.6807$

<u>Proof (Propositions 7.1 and 7.2).</u> The requirements $h_{2m+1}{}'(F)=1$ are equivalent to

$k_m F^m (1-F)^m = 1$ or $\left(F(1-F)\right)^m = (k_m)^{-1}$, with k_m given in (7.2). Taking the m-th root

and arranging terms, $F^2 - F + \left(1/\sqrt[m]{k_m}\right) = 0$. The cubic equations admit two real roots:

$$_{+,-}\rho_{2m+1} = \frac{1}{2}\left(1 \pm \sqrt{1 - \frac{4}{\sqrt[m]{k_m}}}\right),$$

if and only if $1 - \frac{4}{\sigma_m} > 0$, or $\sigma_m > 4$, $\sigma_m \equiv \sqrt[m]{k_m}$, $m \geq 1$. Table 7.3 gives values for a sub-

set of values of σ_m . For large m, the binomial coefficient in (7.2) may be approximated

with *Stirling's formula*[43]:

$$\binom{2m}{m} = \frac{(2m)!}{[m!]^2} \approx \sqrt{2\pi}(2m)^{2m+\frac{1}{2}}\exp[-2m]\left/\left[\sqrt{2\pi}m^{m+\frac{1}{2}}e^{-m}\right]^2\right. = \frac{1}{\sqrt{\pi m}}2^{2m}.$$

It follows that: $\sqrt[m]{k_m} \approx \left[(2m+1)2^{2m}/\sqrt{\pi m}\right]^{\frac{1}{m}}$. Because:

$$\left[(2m+1)2^{2m}/\sqrt{\pi m}\right]^{\frac{1}{m}} = 4\left\{(2m+1)^{\frac{1}{m}}/\sqrt{\pi m}^{\frac{1}{m}}\right\},$$

$$= 4\left\{\exp\left[\ln(2m+1)/m\right]/\exp\left[\ln\left(\sqrt{\pi m}\right)/m\right]\right\}$$

---

[43] Stirling's formula: $n! \approx \sqrt{2\pi}n^{n+\frac{1}{2}}e^{-n}$. It is accurate even for small m (Feller [1968:52–54]) σ_{50} is approximated as 4.1703. Direct calculation gives 4.1702.

by *L'Hospital's Rule*:[44] $\lim_{m\to\infty}\left\{\ln(2m+1)/m\right\}=\lim_{m\to\infty}\left\{2/(2m+1)\right\}=0$, and

$\lim_{m\to\infty}\left\{\ln\sqrt{\pi m}/m\right\}=\lim_{m\to\infty}\left\{1/2m\right\}=0$, implying $\lim_{m\to\infty}\sqrt[m]{k_m}=4$.

Table 7.3 Parameter values

m	1	2	3	4	5	6	7	8	9
σ_m	6	5.4772	5.1925	5.010	4.8816	4.7856	4.7108	4.6505	4.6009
b_m	5	4.667	4.5	4.4	4.333	4.286	4.25	4.222	4.20

◄

Because sequences of roots $_+\rho_{2m+1}$ are falling in m, and $h_{2m+1}(\cdot)$ are strictly increasing and strictly concave on $\left[\tfrac{1}{2},1\right]$ for all $m\geq1$, $h_{2[m+1]+1}{}'(F)<h_{2m+1}{}'(F)$ if $F\in\left[{}_+\rho_{2[m+1]+1},{}_+\rho_{2m+1}\right]$, see Figure 7.1. The inequalities extend to $\left[{}_+\rho_{2[m+1]+1},1\right)$.[45] This gives structure to the derivatives $h_{2m+1}{}'(\cdot)$ that will be used in comparative statics:

<u>Proposition 7.3</u>

i) For all $m\geq0$ and all $F\in\left[{}_{+,}\rho_{2[m+1]+1},1\right)$, $h_{2[m+1]+1}{}'(F)\leq h_{2m+1}{}'(F)$ with equality if and only if $m=0$ and $F={}_+\rho_3$.

ii) For all $m\geq0$ and all $F\in\left(0,{}_-\rho_{2[m+1]+1}\right]$, $h_{2[m+1]+1}{}'(F)\leq h_{2m+1}{}'(F)$ with equality if and only if $m=0$ and $F={}_-\rho_3$.

<u>Proof.</u> (i) Let $m\geq1$ and consider the difference:

---

[44] Bartle & Sherbert [1982], Theorem 5.3.4.

[45] At $F=1$, $h_{2m+1}{}'(F)=0$ for all $m\geq1$.

$$h_{2[m+1]+1}'(F) - h_{2m+1}'(F) = k_{m+1}F^{m+1}(1-F)^{m+1} - k_m F^m (1-F)^m$$
$$= \left[F(1-F)\right]^m \left\{k_{m+1}F(1-F) - k_m\right\},$$

with

$$k_{m+1} = (2m+3)\binom{2m+2}{m+1} = (2m+3)\left\{(2m+2)!/(m+1)!(m+1)!\right\}$$
$$= (2m+3)\left\{(2m+2)(2m+1)(2m)!/(m+1)m!(m+1)m!\right\} \quad .$$
$$= (2m+3)\left\{2(2m+1)(2m)!/(m+1)[m!]^2\right\}.$$

It follows that $k_{m+1} = 2(2m+3)(m+1)^{-1}k_m$, and:

$$h_{2[m+1]+1}'(F) - h_{2m+1}'(F) = \left[F(1-F)\right]^m k_m \left\{b_m F(1-F) - 1\right\},$$

$b_m = 2(2m+3)/(m+1)$. The sign of the difference is determined by the sign of

$b_m F(1-F) - 1$. The coefficient b_m falls monotonically in m with $b_1 = 5$, and $\lim\limits_{m\to\infty} = 4^+$.

The function $\Psi_m(F) = -b_m F^2 + b_m F - 1$ is strictly concave and has a global maximum at

$F = \frac{1}{2}$ for all m, with maximum value $(b_m - 4)/4 > 0$, and $\Psi_m(1) = -1$. Accordingly for

each m, a unique (non-empty) interval where $\Psi_m(F) \leq 0$ ($\Psi_m(F) < 0$) is determined

by $F \geq r_m$ ($F > r_m$), with r_m defined as the positive root of $\Psi_m(F) = 0$, or:

$r_m = (1/2)\left\{1 + \left(1 - (4/b_m)\right)^{\frac{1}{2}}\right\}$. Because $b_m < \sigma_m = \sqrt[m]{k_m}$, for all values of $m \geq 1$,

$r_m < {}_+\rho_{2m+1}$ (see the Proposition 7.1–2 proof). Hence, it can be concluded that

$h_{2[m+1]+1}'(F) < h_{2m+1}'(F)$, not only in $\left[{}_+\rho_{2[m+1]+1}, {}_+\rho_{2m+1}\right]$, but in $\left[{}_+\rho_{2m+1}, 1\right)$.

If $m = 0$, $h_3' - h_1' < 0$ for all $F \in \left({}_+\rho_3, 1\right]$ by the definition of ρ_3 and linearity

of h_1.

Proposition 7.3.ii follows by symmetry: $h_{2m+1}(1-F) = 1 - h_{2m+1}(F)$. ◄

Remark 6.1.C and $F_{m+1:2m+1}(t) = h_{2m+1}(F(t))$ imply that richer characterizations

of majority mechanism standard transformations are possible. Consider the class

of abstract standards, with mean equal to the median, $\mu_F \equiv \int t\,dF(t) = F^{-1}\left(\frac{1}{2}\right)$

(the class includes all symmetric ones[46]). As t increases in supp f, F increases

from 0 to 1. Majority transformations of distributions F with $F^{-1}\left(\frac{1}{2}\right) = \mu_F$ are

mean-preserving because $h_{2m+1}\left(\frac{1}{2}\right) = \frac{1}{2}$ for all m. From monotonicity property C2

in (2.4) and Figure 7.1, it follows that $F_{m'+1:2m'+1}(t) = h_{2m'+1}(F(t))$ is a *mean-*

preserving reduction in risk compared to $F_{m+1:2m+1}(t) = h_{2m+1}(F(t))$ if $m' > m$,

displacing weight from the tails of the distributions to the center, while keeping

the mean constant.[47] If supp f is interpreted as an outcome space for expected

utility maximizing agents with (increasing) Bernoulli utility function $u(\cdot)$, as

defined in and at Proposition 5.4, an even sharper characterization can be given:

Let $G^*(t)$ and $\tilde{G}(t)$ be (any) cdfs with the same mean. $G^*(t)$ *second-order*

stochastically dominates $\tilde{G}(t)$ if for every non-decreasing function $u : \mathbb{R} \to \mathbb{R}$:

$$\int u(t)\,dG^*(t) \geq \int u(t)\,d\tilde{G}(t).^{48}$$

---

[46] A standard is symmetric around a constant (mirror line) κ if the relation $F(\kappa + t) = 1 - F(\kappa - t)$ ($f(\kappa + t) = f(\kappa - t)$) holds identically in t. Symmetry implies that the median is equal to the expectation (Larsen & Marx [1986:155,119]).

[47] Mean-preserving reduction in risk is also called a reduction in the Rothschild-Stigliz sense. See Laffont [1989:24–30] and Mas-Colell, Whinston & Green [1995:194–99] for alternative characterizations.

       Substantive norm symmetry implies that transformed norms are symmetric: From $M_{2m+1} = M_{2m+1}^T$, $F_{m+1:2m+1}{}'(\kappa + t) = f_{m+1:2m+1}(\kappa + t)$. From (7.3), the transformed *density* equals $k_m\left\{F(\kappa + t)\left[1 - F(\kappa + t)\right]\right\}^m = k_m\left\{(1 - F(\kappa - t))\left[1 - \{1 - F(\kappa - t)\}\right]\right\}^m = k_m\left\{F(\kappa - t)\left[1 - F(\kappa - t)\right]\right\}^m$. Hence, $f_{m+1:2m+1}(\kappa + t) = f_{m+1:2m+1}(\kappa - t)$.

[48] Mas-Colell, Whinston & Green [1995], Definition 6.D.2.

Proposition 7.4 If $m' > m$, the transformed norm liability component

$$F_{m'+1:2m'+1}(t) = h_{2m'+1}(F(t)) \quad \text{second order stochastically dominates}$$

$$F_{m+1:2m+1}(t) = h_{2m+1}(F(t)).$$

Proof. See Laffont [1989:32–33]. ◄

Remark 7.1 In large courts, the property of mean-preserving reduction of risk is applies to *any* underlying substantive norm distribution F (as long as it is suffi-ciently continuous around the median $F^{-1}\left(\frac{1}{2}\right)$ and $f\left(F^{-1}\left(\frac{1}{2}\right)\right) > 0$). By Proposi-tion 6.2 ($q = \frac{1}{2}$ and $n = 2m+1$):

$$S_{m+1:2m+1} \approx N\left(F^{-1}\left(\tfrac{1}{2}\right), \left\{ 4(2m+1)\left[f\left(F^{-1}\left(\tfrac{1}{2}\right)\right)\right]^{2}\right\}^{-1}\right). \square$$

PART FOUR

EQUILIBRIUM UNDER TRANSFORMED (PURE) NORMS

1 Equilibrium analysis

This part allows the set of legal facts to be X constituted by choice or strategy spaces for agents with complete information about legal source constellations, meta-norms, and the decision mechanism in place. It corresponds to knowing the triple $\langle ls, \eta, M \rangle$, with $ls \in LS$, $\eta \in \mathbb{P}_{\mathbb{P}_Y^X}^{LS}$, and $M \in \mathcal{M}$. A positive theory capable of, in effect, "contracting" the set of conditioning legal facts enhances comparison of the legal norms, objects in $\mathbb{P}_Y^X$, defined globally on X. Such a theory may also allow a sharpening of law-in-force statements and even inform the analysis of meta-norms (Part VI).

The legal source data and meta-norms generate abstract mixed norms $g_{\eta(\cdot|ls)} \in \mathbb{P}_Y^X$ (Section II.5). Part III explains how such abstract (mixed) norms are transformed in a subset of mechanisms $M \in \mathcal{M}$, leading to transformed incentive structures (transformed mixed norms) $g_{\eta(\cdot|ls),M} \in \mathbb{P}_Y^X$. A relevant theory ($T$) for studying agent behavior under these structures, may be represented by a set of equations:

$$T\left(x, g_{n(\cdot|ls),M}\left(\cdot \mid x\right)\right)\Big|_{x \in X} = 0. \tag{1.1}$$

A solution correspondence:

$$x^T*: \mathbb{P}_Y^X \times \mathcal{M} \rightrightarrows X \,, \tag{1.2}$$

is sought, assigning to each $\left\langle g_{\eta(\cdot|ls)}, M \right\rangle \in \mathbb{P}_Y^X \times \mathcal{M}$ system determined variables

$x*\left(g_{\eta(\cdot|ls),M}\right) \subseteq X$ such that $T\left(x*\left(g_{\eta(\cdot|ls),M}\right), g_{n(\cdot|ls)}\left(\cdot \mid x*\left(g_{\eta(\cdot|ls),M}\right)\right)\right) \equiv 0 .^1$

While some general system-theoretic concepts are discussed in Section VI.3, economic theory is a particularly relevant and powerful source for theory relations. Parametric equilibrium notions are used in the precaution model analysis in Parts IV and V and in the Section VI.3.1 applications to exchange economies (the latter include the core solution from cooperative game theory). Strategic non-cooperative solutions appear in Sections 2 and VI.3.2.[2] The use of a solution correspondence formulation reflects possible non-uniqueness of equilibria (and core allocations). Continuous and discrete comparative statics is conducted, corresponding to changes in parameters which describe legal norms and legal decision mechanisms, respectively. This part considers pure norms (abstracts from uncertainty on the level of meta-norms). Part V considers (explicitly) mixed norms, permitting comparative static analyses referring both to ordinary and meta-level norms. Throughout Parts IV and V absence of epistemic uncertainty is assumed, meaning that abstract norms $g \in \mathbb{P}_Y^X$ and norms transformed in single-judge mechanisms, $g_1 = M_1(g)$ need not be distinguished (compare Section VII.2).

The confluence of legal uncertainty and uncertainty from the precaution-model accident technology creates analytical challenges. Therefore, links be-

---

[1] The equations in (1.1) correspond to propositional functions. Solutions (1.2) reduce the equations to identities.

[2] See, generally, Arrow [1968] and Hahn [1974] on equilibrium notions and the methodological approach.

tween decision mechanisms and equilibrium outcomes are first investigated in the context of final offer arbitration in Section 2. Section 3 introduces the model of precaution. Equilibrium analysis is conducted under general assumptions about the abstract norm and the accident technology in Section 4. Section 5 analyzes uniform distributions. Section 6 uses results from Section III.6 to motivate the direct relevance of the Crasswell & Calfee [1986] analysis to large but finite mechanisms. The analysis is supported by simulations reported in Appendix A.1.

<u>Remark 1.1</u> Particularly sophisticated analyses of abstract norms are found in modern theories on incentives and contracts (see Laffont & Martimort [2002] and Bolton & Dewatripont [2005]). The theories concern "the choice of constraints as opposed to the choice within constraints".[3] □

<u>Remark 1.2</u> In Shavell's [2006] framework (Example II.5.2), contract parties have complete information about a fixed meta-norm $\eta^K \in \left(\delta\mathbb{P}_{(\delta\mathbb{P}_A)^\Omega} \right)^{(\delta\mathbb{P}_A)^\Omega \cup (\delta\mathbb{P}_A)^{\Omega \backslash \Omega_G}}$.

Contract interpretation corresponds to use of this global rule to produce an obligationally complete contract. Given η^K, parties chose *equilibrium contracts* $g_K(\eta^K)$, in a model with all variables contractible, and writing costs which increase in number of terms.[4] An optimal rule $\eta^M *$ is derived., and gives an important example of choice of meta-norms under equilibrium analysis. □

The Parts IV and V equilibrium analyses concern choice *within constraints*, varied exogenously via parametric change which represent decision mechanisms and

---

[3] James Buchanan (on constitutional economics) quoted from Stremitzer [2005:82].

[4] Risk neutral parties maximize expected value (no renegotiation). In applying η^K, courts only have information about the distribution of contract parties (types).

norm-based uncertainty on the ordinary and meta-level. Sections VI.3 and VII give examples of *constraint choice* at the ordinary and meta-level, respectively (VI.3.2 under equilibrium agent behavior).

2   Introductory example: final-offer arbitration

Final-offer arbitration (FOA) is used in important contract disputes. This section reinterprets and extends a simple version of Farber's [1980] model of single issue final offer arbitration to situations with arbitration panels. Rather than letting agents be uncertain about a single arbitrator's *preference* (fair settlement notion), the panel's decision basis will be a contract-based standard. Panel members are identified, in the ex ante sense, with the abstract standard. Following Gibbons' [1992:23-6] variant of the underlying model, a firm (f) and union (u) are unable to reach a settlement on wages. Under FOA, each party's *strategy* is given by an offer $w_f \in W_f$ and $w_u \in W_u$, respectively.[5] The defining global procedural rule requires arbitrators to choose the offer closest to the wage defined by the contract-based standard. The contract norm price element is represented by a stochastic variable S with cdf F. Arbitrators are i.i.d.'s from F and assumed to observe the strategy profile $\langle w_f, w_u \rangle$ before voting in majority mechanism M_{2m+1} (or in M_{2m+1}^T). Both the contract-based norm and the procedural rules defining the game structure are is common knowledge.

The set of legal facts is taken to be $X = W_f \times W_u$. The FAO procedure means:

---

[5] The strategy sets are assumed to be convex and compact in $\mathbb{R}_+$.

$$\langle w_f, w_u \rangle \mapsto w_f \text{ if } |w_f - s| \le |w_u - s|$$
$$\langle w_f, w_u \rangle \mapsto w_u \text{ if } |w_f - s| > |w_u - s|$$
. $\quad$ (2.1)

The equations in (2.1) partition X. Let $W^+ = \left\{ \langle w_f, w_u \rangle \in W_f \times W_u \,\middle|\, w_u \ge w_f \right\}$ and

$W^- = \left\{ \langle w_f, w_u \rangle \in W_f \times W_u \,\middle|\, w_u < w_f \right\}$. The norm resulting from the contract and

procedural norm confluence can be represented as $g_F \in \mathbb{P}_{\mathbb{R}_+ \times \mathbb{R}_+}{}^{W_f \times W_u}$, given by:

$$g_F \left(\cdot \,|\, w_f, w_u \right)\Big|_{W^+} = \left\langle F\left(\tfrac{w_f + w_u}{2}\right), 1 - F\left(\tfrac{w_f + w_u}{2}\right); w_f, w_u \right\rangle$$
$$g_F \left(\cdot \,|\, w_f, w_u \right)\Big|_{W^-} = \left\langle 1 - F\left(\tfrac{w_f + w_u}{2}\right), F\left(\tfrac{w_f + w_u}{2}\right); w_f, w_u \right\rangle$$
, $\quad$ (2.2)

$g_F\big|_{W^+}$ and $g_F\big|_{W^-}$ denotes the restriction to W^+ and W^-, respectively.[6]

---

[6] Ad $g_F\big|_{W^+}$: If $w_u > w_f$ and $s \in (w_f, w_u)$, first line inequality in (2.1) is equivalent to $s \le (w_f + w_u)/2$. The event has probability $F\left(\tfrac{w_f + w_u}{2}\right) - F(w_f)$. Clearly, $\langle w_f, w_u \rangle$ is also mapped to w_f if $s \le w_f$. The event has probability $F(w_f)$ The union of the two disjoint events has probability $F\left(\tfrac{w_f + w_u}{2}\right)$. If $w_f = w_u = \tilde{w}$, say, $P\{S = \tilde{w}\} = 0$. It follows that each $\langle w_f, w_u \rangle \in W^+$ is mapped to w_f with probability $F\left(\tfrac{w_f + w_u}{2}\right)$ (and hence to w_u with probability $1 - F\left(\tfrac{w_f + w_u}{2}\right)$). Similarly, $g_F\big|_{W^-}$ follows from the first line in (2.1) by noting that if $s \in (w_u, w_f)$, the inequality is equivalent to $s \ge (w_f + w_u)/2$, an event with probability $F(w_f) - F\left(\tfrac{w_f + w_u}{2}\right)$. Clearly, $\langle w_f, w_u \rangle$ is also mapped to w_f if $s \ge w_f$, an event with probability $1 - F(w_f)$. It follows that $\langle w_f, w_u \rangle$ is mapped to w_f with probability $1 - F\left(\tfrac{w_f + w_u}{2}\right)$ (disjoint events), and finally $\langle w_f, w_u \rangle$ to w_u with probability $F\left(\tfrac{w_f + w_u}{2}\right)$.
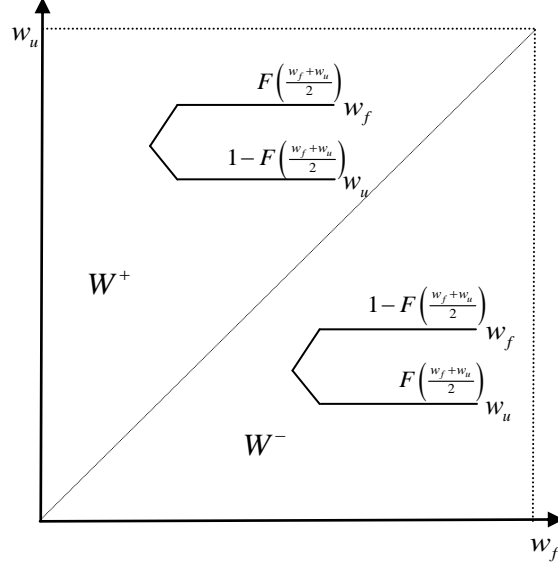
Figure 2.1 Abstract norm defined by final-offer arbitration

It follows directly from Propositions III.2.1 and III.2.3.B that in majority panels, g_F is transformed to:

$$g_{F,2m+1}\left(\cdot\mid w_f,w_u\right)\big|_{W^+} = \left\langle h_{2m+1}\left(F\left(\tfrac{w_f+w_u}{2}\right)\right),1-h_{2m+1}\left(F\left(\tfrac{w_f+w_u}{2}\right)\right);w_f,w_u\right\rangle$$

$$g_{F,2m+1}\left(\cdot\mid w_f,w_u\right)\big|_{W^-} = \left\langle 1-h_{2m+1}\left(F\left(\tfrac{w_f+w_u}{2}\right)\right),h_{2m+1}\left(F\left(\tfrac{w_f+w_u}{2}\right)\right);w_f,w_u\right\rangle \qquad (2.3)$$

Assuming risk-neutral agents, minimizing or maximizing the expected wage settlement, respectively, the payoff functions induced by (2.3) are:[7]

$$\pi_{f,2m+1}\left(w_f,w_u\right) = -\left\{w_f h_{2m+1}\left(F\left(\tfrac{w_f+w_u}{2}\right)\right) + w_u\left[1-h_{2m+1}\left(F\left(\tfrac{w_f+w_u}{2}\right)\right)\right]\right\}$$

$$\pi_{u,2m+1}\left(w_u,w_f\right) = w_f h_{2m+1}\left(F\left(\tfrac{w_f+w_u}{2}\right)\right) + w_u\left[1-h_{2m+1}\left(F\left(\tfrac{w_f+w_u}{2}\right)\right)\right] \qquad (2.4)$$

---

[7] The payoff formulation has been simplified by noting that a Nash equilibrium profile $\langle w_f*,w_u*\rangle$ must lie in W^+: Assume that an equilibrium profile is in W^- and that w_f* is chosen by the panel. It follows (by continuity) that it could have been reduced, imply-ing a direct increase in π_f as well an indirect increase in π_f (by increasing the probabil-ity that w_f is chosen). Similarly, had w_u* been chosen by the panel, by the same argu-ments it's increase would have resulted in a higher π_u. Hence, a Nash equilibrium cannot exist in W^-.

A Nash equilibrium $\left\langle w_f{}^*, w_u{}^* \right\rangle$ is given by simultaneous optimization:[8]

$$w_f{}^*\left(2m+1\right) = \arg\max_{w_f \in W_f} \pi_{f,2m+1}\left(w_f, w_u{}^*\right)$$

$$w_u{}^*\left(2m+1\right) = \arg\max_{w_u \in W_u} \pi_{u,2m+1}\left(w_u, w_f{}^*\right)$$

Define:

$$\partial \pi_{f,2m+1}\left(w_f, w_u{}^*\right)\big/\partial w_f =$$
$$-\left\{ h_{2m+1}\left(F\left(\tfrac{w_f + w_u{}^*}{2}\right)\right) - \left[w_u{}^* - w_f\right]\tfrac{1}{2}h_{2m+1}{}'\left(F\left(\tfrac{w_f + w_u{}^*}{2}\right)\right)F'\left(\tfrac{w_f + w_u{}^*}{2}\right) \right\} \tag{2.5}$$

$$\partial^2 \pi_{f,2m+1}\left(w_f, w_u{}^*\right)\big/\partial w_f{}^2 = -\left\{ h_{2m+1}{}'\left(F\left(\tfrac{w_f + w_u{}^*}{2}\right)\right)F'\left(\tfrac{w_f + w_u{}^*}{2}\right) \right.$$
$$\left. +\tfrac{1}{2}w_f\left\{ h_{2m+1}{}''\left(F\left(\tfrac{w_f + w_u{}^*}{2}\right)\right)\left[F'\left(\tfrac{w_f + w_u{}^*}{2}\right)\right]^2\tfrac{1}{2} + h_{2m+1}{}'\left(F\left(\tfrac{w_f + w_u{}^*}{2}\right)\right)F''\left(\tfrac{w_f + w_u{}^*}{2}\right)\tfrac{1}{2} \right\} \right\} \tag{2.6}$$

$$\partial \pi_{u,2m+1}\left(w_u, w_f{}^*\right)\big/\partial w_u =$$
$$1 - h_{2m+1}\left(F\left(\tfrac{w_f{}^* + w_u}{2}\right)\right) - \left[w_u - w_f{}^*\right]\tfrac{1}{2}h_{2m+1}{}'\left(F\left(\tfrac{w_f{}^* + w_u}{2}\right)\right)F'\left(\tfrac{w_f{}^* + w_u}{2}\right) \tag{2.7}$$

$$\partial^2 \pi_{u,2m+1}\left(w_u, w_f{}^*\right)\big/\partial w_u{}^2 = -h_{2m+1}{}'\left(F\left(\tfrac{w_f{}^* + w_u}{2}\right)\right)F'\left(\tfrac{w_f{}^* + w_u}{2}\right)$$
$$-\tfrac{1}{2}w_u\left\{ h_{2m+1}{}''\left(F\left(\tfrac{w_f{}^* + w_u}{2}\right)\right)\left[F'\left(\tfrac{w_f{}^* + w_u}{2}\right)\right]^2\tfrac{1}{2} + h_{2m+1}{}'\left(F\left(\tfrac{w_f{}^* + w_u}{2}\right)\right)F''\left(\tfrac{w_f{}^* + w_u}{2}\right)\tfrac{1}{2} \right\} \tag{2.8}$$

From the first order conditions:

$$\begin{cases} \dfrac{\partial \pi_{f,2m+1}\left(w_f, w_u{}^*\right)}{\partial w_f} = 0 \\[2mm] \dfrac{\partial \pi_{u,2m+1}\left(w_u, w_f{}^*\right)}{\partial w_u} = 0 \end{cases}, \tag{2.9}$$

---

[8] It is maintained that this and Section VI.3.2 use, in principle, sophisticated equilibrium notions: "In the one-person context, we are usually led to a well-defined optimization problem […]. While this problem might be difficult to solve in practice, it involves no conceptual issue. [---] But in the interactive multi-person context, the very meaning of "optimal decision" is unclear, since in general, no one player completely controls the final outcome. One must address the conceptual issue of *defining* the problem before one can start *solving* it. Game Theory is concerned with both matters: defining "solution concepts", and then investigating their properties, in general as well as in specific models coming from the various areas of application" (Aumann & Hart [1992], preface).

it follows that $h_{2m+1}\left(F\left(\frac{w_f*+w_u*}{2}\right)\right)=\frac{1}{2}$ in equilibrium. Because $h_{2m+1}:[0,1]\to[0,1]$

is strictly increasing, bijective, and $h_{2m+1}\left(\frac{1}{2}\right)=\frac{1}{2}$ for all $m\in\{0,1,2,...\}$ (Proposi-

tion III.2.2), $F\left(\frac{w_f*+w_u*}{2}\right)=\frac{1}{2}$, hence (†) $\frac{w_f*+w_u*}{2}=F^{-1}\left(\frac{1}{2}\right)$: the average of the equi-

librium offers equals the legal standard median. Substituting back into the first

order conditions, (††) $w_u*-w_f*=\left\{h_{2m+1}'\left(\frac{1}{2}\right)F'\left(F^{-1}\left(\frac{1}{2}\right)\right)\right\}^{-1}$, relating the offer

gap to the inverse of the derivative of the aggregation function and standard den-

sity property at the median ($F'\left(F^{-1}\left(\frac{1}{2}\right)\right)=f\left(F^{-1}\left(\frac{1}{2}\right)\right)$) given sufficient smooth-

ness). From (†) and (††),

$$w_f*(2m+1)=F^{-1}\left(\tfrac{1}{2}\right)-\frac{1}{h_{2m+1}'\left(\tfrac{1}{2}\right)2F'\left(F^{-1}\left(\tfrac{1}{2}\right)\right)},$$
$$w_u*(2m+1)=F^{-1}\left(\tfrac{1}{2}\right)-\frac{1}{h_{2m+1}'\left(\tfrac{1}{2}\right)2F'\left(F^{-1}\left(\tfrac{1}{2}\right)\right)},\qquad(2.10)$$

with $h_{2m+1}'\left(\frac{1}{2}\right)=(2m+1)\{(2m)!\}/2^{2m}[m!]^2$, see equation (III.7.4). The solutions

define a what Farber calls "contract zone[s]" $\left[w_f*(2m+1),w_u*(2m+1)\right]$, a

range of settlements that the agents prefer to the FOA-induced outcome in

M_{2m+1}.[9] The solutions (2.10) illustrate that the effects of legal uncertainty ("law

shadows") are eliminated in large courts because the derivatives $h_{2m+1}'\left(\frac{1}{2}\right)$ grow

monotonically towards infinity. The solutions converge on the standard median

irrespective of the amount of uncertainty in the underlying abstract norm.

---

[9] See Farber [1980:694–98] on M_1, importantly including risk-averse agents.

To ensure that the first order conditions define solutions, saddle-point conditions must be checked. Sufficient conditions for strict local equilibria are demonstrated for two families of distributions:[10]

<u>Example 2.1</u> (uniform distributions $S \sim U(\underline{w}, \overline{w})$). The cdf is:

$$F(t; \underline{w}, \overline{w}) = \begin{cases} 0 \text{ if } t < \underline{w} \\ \frac{t-\underline{w}}{\overline{w}-\underline{w}} \text{ if } t \in [\underline{w}, \overline{w}) \\ 1 \text{ if } t > \overline{w} \end{cases}.$$

$F'(t; \underline{w}, \overline{w}) \equiv \frac{1}{\overline{w}-\underline{w}}$ and $F''(t; \underline{w}, \overline{w}) \equiv 0$. $F^{-1}\left(\frac{1}{2}; \underline{w}, \overline{w}\right) = \frac{\underline{w}+\overline{w}}{2}$. Hence, from (2.6) and

(2.8) (as $h_{2m+1}''\left(\frac{1}{2}\right) = 0$), $\partial^2 \pi_{f,2m+1}\left(w_f*, w_u*; \underline{w}, \overline{w}\right) / \partial w_f^2$

$= \partial^2 \pi_{u,2m+1}\left(w_u*, w_f*; \underline{w}, \overline{w}\right) / \partial w_u^2 = -h_{2m+1}'\left(\frac{1}{2}\right) / (\overline{w}-\underline{w}) < 0$, It follows that locally

unique interior equilibria are given by:

$$w_f*\left(2m+1; \underline{w}, \overline{w}\right) = \frac{\underline{w}+\overline{w}}{2}\left\{1 - \frac{1}{h_{2m+1}'\left(\frac{1}{2}\right)}\right\}$$

$$w_u*\left(2m+1; \underline{w}, \overline{w}\right) = \frac{\underline{w}+\overline{w}}{2}\left\{1 + \frac{1}{h_{2m+1}'\left(\frac{1}{2}\right)}\right\}$$.

■

<u>Example 2.2</u> (normal distributions $S \sim N(\mu, \sigma^2)$). The cdf is $F(t; \mu, \sigma^2) =$

$\left(2\sigma^2\pi\right)^{-\frac{1}{2}} \int_{-\infty}^{t} \exp\left[-\frac{1}{2\sigma^2}(\tau-\mu)^2\right] d\tau$, with derivative (density) $F'(t; \mu, \sigma^2) =$

$\left(2\sigma^2\pi\right)^{-\frac{1}{2}} \exp\left[-\frac{1}{2\sigma^2}(\tau-\mu)^2\right]$. The second order derivative is $F''(t; \mu, \sigma^2) =$

$-(t-\mu)\sigma^{-3}(2\pi)^{-\frac{1}{2}} \exp\left[-\frac{1}{2\sigma^2}(\tau-\mu)^2\right]$. By symmetry $F^{-1}\left(\frac{1}{2}; \mu, \sigma^2\right) = \mu$

---

[10] Because the cdf's are continuous, existence of equilibria follows from appropriate fixed-point theorems, see in and at Proposition 8.D.3 in Mas-Colell, Whinston & Green [1995]. (The possibility of parameter constellations giving rise to boundary solutions in the uniform case, or unbounded solutions in the normal case, has not been investigated.)

$$F'\left(F^{-1}\left(\tfrac{1}{2};\mu,\sigma^2\right);\mu,\sigma^2\right) = \left(2\sigma^2\pi\right)^{-\tfrac{1}{2}},$$ and $F''\left(F^{-1}\left(\tfrac{1}{2};\mu,\sigma^2\right);\mu,\sigma^2\right) = 0$ (μ is the

distribution expectation, median and mode). $h_{2m+1}''\left(\tfrac{1}{2}\right) = 0$, (2.6) and (2.8) give:

$$\partial^2\pi_{f,2m+1}\left(w_f{*},w_u{*};\mu,\sigma^2\right)\big/\partial w_f{}^2 = \partial^2\pi_{u,2m+1}\left(w_u{*},w_f{*};\mu,\sigma^2\right)\big/\partial w_u{}^2,$$ which is

equal to $-h_{2m+1}'\left(\tfrac{1}{2}\right)F'\left(F^{-1}\left(\tfrac{1}{2};\mu,\sigma^2\right);\mu,\sigma^2\right) < 0$. Hence, locally unique (interior)

Nash-equilibria are given by:

$$w_f{*}\left(2m+1;\mu,\sigma^2\right) = \mu - \frac{\sqrt{\sigma^2\pi}}{h_{2m+1}'\left(\tfrac{1}{2}\right)\sqrt{2}}$$
$$w_u{*}\left(2m+1;\mu,\sigma^2\right) = \mu - \frac{\sqrt{\sigma^2\pi}}{h_{2m+1}'\left(\tfrac{1}{2}\right)\sqrt{2}}$$

∎

3 The model of precaution

The analysis is limited to variations over the paradigmatic model of unilateral "accidents" with fixed activity levels and identical, risk-neutral (potential) injurers. Losses are non-stochastic and there is no contact (bargaining) between (potential) injurers and victims.[11] Precaution investments are measured by costs $x \in \mathbb{R}_+$, which reduce the probability of an adverse outcome p at a decreasing rate as given by the function $p : X \to [0,1)$, $p' < 0$ and $p'' > 0$. If an adverse outcome occurs, the loss $L > 0$ is incurred. The maintained assumption $-p'(0)L > 1$ ensures that it is efficient to commit resources to precaution. Total expected (real) costs are:

---

[11] Autarky in Ayres' [2005] terminology. See Kraakman *et al.* [2004:76–77] regarding insulation of shareholders from damages and law that controls externalities. On the notion of externalities, see Arrow [1970] and Example VI.3.5 below (allowing contracting around background liability law in a general equilibrium context).

$$k(x) = x + p(x)L. \tag{3.1}$$

$k(0) = p(0)L \leq L$. $k(L) = L + p(L)L > L$, hence $k(L) > k(0)$ and $k(x) > L$

for $x > L$. $k(\cdot)$ is continuous and strictly convex. $k'(x)\big|_{x=0} = 1 + p'(0)L < 0$.

Hence, $k(\cdot)$ has a unique strict minimum $x^{PE} = \arg\min_x \{k(x)\}$ in $(0, L)$, satisfy-

ing:

$$k'(x^{PE}) = 1 - p'(x^{PE})L = 0. \tag{3.2}$$

The abstract liability standard and induced incentive structures

The set of legal facts is identified with the set of precaution investments,

$X = \mathbb{R}_+$. An *uncertain negligence standard* is defined by a bounded stochastic

variable S on X with a continuous cdf F. Given an accident and care level x,

liability is incurred if and only if $S > x$ (noncompliance $\otimes$), an event with prob-

ability $P\{S > x\} = 1 - F(x)$. Liability implies payment of damages equal to the

loss, L. Hence, the abstract negligence standard $g_F \in \mathbb{P}_{\{0,L\}}{}^X$ is represented as:

$$g_F(\cdot \mid x)\big|_{x \in X} = \langle F(x), 1 - F(x); 0, L \rangle\big|_{x \in X}, \tag{3.3}$$

see Example II.3.3.[12] The cdf $F(t)$ is assumed to be bounded with support

$\operatorname{supp} F = [\underline{x}, \overline{x}]$, and to be weakly uni-modal around the median, $F^{-1}\left(\frac{1}{2}\right) = x^{PE}$.[13]

---

[12] Liability is incurred only if an accident occurs according to a realization of $p(\cdot)$, embedding the notion of causality (see Shavell [1987:118–26] on uncertainty over causation).

    The norm is defined over losses $\{0, L\}$ rather than liability $\{\oplus, \otimes\}$. This convenient norm individuation is inconsequential because losses (damages) are non-stochastic (compare Section III.5).

[13] Weak uni-modality means that there exists a point x^0 such that the density function f is weakly increasing for $x \leq x^0$ and weakly decreasing for $x \geq x^0$. (The property is not required in Proposition 4.1.)

$F(\cdot)$ is supposed to be twice continuously differentiable everywhere except possibly at the support boundaries: $F'(x)=0$ if $x<\underline{x}$ and $x>\overline{x}$, $F'(x)=f(x)$ if $x\neq\underline{x},\overline{x}$. At the upper boundary, left and right hand derivatives are given by (separating only if the density function jumps at $\overline{x}$),

$$\begin{cases}F'\left(\overline{x}^{-}\right)=\lim_{h\to 0^{-}}\dfrac{F\left(\overline{x}+h\right)-F\left(\overline{x}\right)}{h}=f\left(\overline{x}^{-}\right)\\[2mm]F'\left(\overline{x}^{+}\right)=\lim_{h\to 0^{+}}\dfrac{F\left(\overline{x}+h\right)-F\left(\overline{x}\right)}{h}=0\end{cases}\text{.[14]}\qquad(3.4)$$

In M_{2m+1}, g_F is transformed to $g_{F,\,2m+1}\in\mathbb{P}_{\{0,L\}}{}^{X}$, given by (Proposition III.2.2)

$\left\langle 1-h_{2m+1}\left(1-F\left(x\right)\right),h_{2m+1}\left(1-F\left(x\right)\right);0,L\right\rangle\Big|_{x\in X}$, from Proposition III.2.3B equal to

$\left\langle 1-h_{2m+1}\left(1-F\left(x\right)\right),1-h_{2m+1}\left(F\left(x\right)\right);0,L\right\rangle\Big|_{x\in X}$. Hence, minimands:

$$K_{2m+1}\left(x\right)=x+\left[1-h_{2m+1}\left(F\left(x\right)\right)\right]p\left(x\right)L,\qquad(3.5)$$

define agents' decision problems. Equivalent formulations are:

$$K_{2m+1}\left(x\right)=x+\left[1-F_{m+1:2m+1}\left(x\right)\right]p\left(x\right)L.$$

$F_{m+1:2m+1}\left(x\right)$ denotes the cdf from theoretical norm element determination in Section III.6.

4   Equilibrium analysis I: the structure of solutions

Let $x*(2m+1)$ denote the level of care that minimizes expected cost defined by the minimands in (3.5). In case of non-uniqueness, $x*(2m+1)$ denotes a solution set. The term compliance will be used about efficient equilibria, and under-

---

[14] Continuity of F means that density function jumps are finite ($0<f\left(\overline{x}^{-}\right)<\infty$). It will be demonstrated that solutions cannot occur at $\underline{x}$.

and overcompliance about $x*(2m+1) < x^{PE}$ and $x*(2m+1) > x^{PE}$, respectively.[15]

Equilibrium under legal certainty

Solutions under legal certainty are considered as benchmarks. In this case, the abstract negligence norms are *global rules* $g_{\delta c} \in \delta\mathbb{P}_{\{0,L\}}{}^X$, described by $\langle F_{\delta c}(x), 1 - F_{\delta c}(x); 0, L \rangle_{x \in X} \big|$, $c \in \mathbb{R}_{++}$, mapping care levels $x < c$ to liability if an accident occurs, and to no liability in all other situations (Example II.3.3).

Define care level $\tilde{x}$ such that $\tilde{x} = k(x^{PE})$, see Figure 4.1. If $m = 0$, the following solution correspondence follows (Shavell [1987], Proposition 4.5):[16]

$$x*(1;c) = \begin{cases} \{c\} & \text{if } 0 < c < \tilde{x} \\ \{x^{PE}, \tilde{x}\} & \text{if } c = \tilde{x} .^{17} \\ \{x^{PE}\} & \text{if } c > \tilde{x} \end{cases} \tag{4.1}$$

Under legal certainty, the same solution is reproduced by all collectives: the equilibrium (set) equality $x*(2m+1;c) = x*(1;c)$ holds for all m, because all judges reach the unique outcome dictated by the doctrine.[18]

---

[15] Asymptotic outcomes may be used to motivate the terminology (Sections 4.1 and 6).

[16] Set notation is used because $c = \tilde{x}$ admits two equilibria.

[17] Regarding $c = \tilde{x}$, $x*(1; \tilde{x}) = \tilde{x}$, while implying a low accident probability, leaves victims uninsured, whereas $x*(1; \tilde{x}) = x^{PE}$ implies a higher accident probability but fully insures the victim. If $c > \tilde{x}$, the negligence norm functions as strict liability ($g_{\delta\infty}$).

[18] If $x < c$, $F(x) = 0$ and $h_{2m+1}(F) = 0$ for all m. If $x \geq c$, $F(x) = 1$ and $h_{2m+1}(F) = 1$ for all m. From (3.5),
$$K_{2m+1}(x) = \begin{cases} x + p(x)L & \text{if } x < c \\ x & \text{if } x \geq c \end{cases}.$$

Interior solutions to the minimization problem defined in (3.5) *exists* in $(0, L)$ for all m.[19] Candidates in $(0, L)$ are either at points where the derivative $K_{2m+1}{}'(x)$ does not exist (that is, possibly at support boundaries $\underline{x}$ and $\bar{x}$) or where $K_{2m+1}{}'(x) = 0$ (that is, at critical points).[20]

From the Condorcet theorem (Section III.2, monotonicity property C2), the following relations hold for all (finite) $n = 2m + 1$:[21]

$$\begin{cases} k(x) = K_n(x) = \cdots = K_3(x) = K_1(x) \text{ on } [0, \underline{x}]; \\ k(x) > K_n(x) > \cdots > K_3(x) > K_1(x) \text{ on } (\underline{x}, x^{PE}); \\ K_n(x^{PE}) = \cdots = K_3(x^{PE}) = K_1(x^{PE}) = x^{PE} + \frac{1}{2}p(x^{PE})L \equiv \nabla; \\ k(x) > K_1(x) > K_3(x) > \cdots > K_n(x) \text{ on } (x^{PE}, \bar{x}); \text{ and} \\ k(x) > K_1(x) = K_3(x) = \cdots = K_n(x) = x \text{ on } [\bar{x}, L]. \end{cases} \qquad (4.2)$$

Figure 4.1 summarizes this situation (anticipating some results derived below). Because $k(x)$ is strictly falling on $[0, x^{PE}]$, by the first two lines in (4.2),

---

[19] By continuity of $K_{2m+1}(\cdot)$, an absolute minimum exists in $[0, L]$ (Bartle & Sherbert [1982], Theorem 4.6.4). At x^{PE}, $F(x^{PE}) = \frac{1}{2}$ and $K_{2m+1}(x^{PE}) = x^{PE} + \frac{1}{2}p(x^{PE})L < k(x^{PE})$. From $k'(x)\big|_{x=0} < 0$, and $k(0) < k(L)$, $k(x^{PE}) < k(0) < k(L)$, hence the solution lies in $(0, L)$ (because $K_{2m+1}(x) \geq x$ if $x \geq L \geq k(0)$, there are no candidates $x > L$).

[20] Sydæter [1981], Theorem 5.3.

[21] On $[0, \underline{x}]$ $F = 0$ and $K_{2m+1} = k$ since $h_{2m+1}(0) = 0$. This establishes the first line. Regarding the second line, on $(\underline{x}, \frac{1}{2})$, $F \in (0, \frac{1}{2})$. From C2, $h_{2m+1}(F) < h_{2m^0+1}(F)$ if $m > m^0$. Hence $[1 - h_{2m+1}(F)] > [1 - h_{2m^0+1}(F)]$, implying $K_{2m+1} > K_{2m^0+1}$. The third line follows from $F(x^{PE}) = \frac{1}{2}$ and $h_{2m+1}(\frac{1}{2}) = \frac{1}{2}$ for all m. The fourth line follows from $F \in (\frac{1}{2}, 1)$ on $(x^{PE}, \bar{x})$, hence $h_{2m+1}(F) > h_{2m^0+1}(F)$ for $m > m^0$ (C2). Therefore $[1 - h_{2m+1}(F)] < [1 - h_{2m^0+1}(F)]$ and $K_{2m+1} < K_{2m^0+1}$. The fifth line follows from $F = 1$ on $[\bar{x}, L]$ and $h_{2m+1}(1) = 1$ for all m.

equilibria are not possible at the lower support boundary for any m; $\underline{x} < x^*(2m+1)$. Because $K_{2m+1}(x) = x$ for all m if $x \geq \bar{x}$ (fifth line), $x^*(2m+1) \leq \bar{x}$. Because $K_{2m+1}(x^{PE}) = \nabla$ for all m (third line), $x^*(2m+1) \leq \min\{\nabla, \bar{x}\}$ for all m. Section 4.2 demonstrates that $x^*(2m+1) < \min\{\bar{x}, \nabla\}$, with the exception of a possible boundary solution $x^*(2m+1) = \bar{x}$ only if $m=0$ and $\bar{x} < \nabla$ (necessary, not sufficient, conditions).
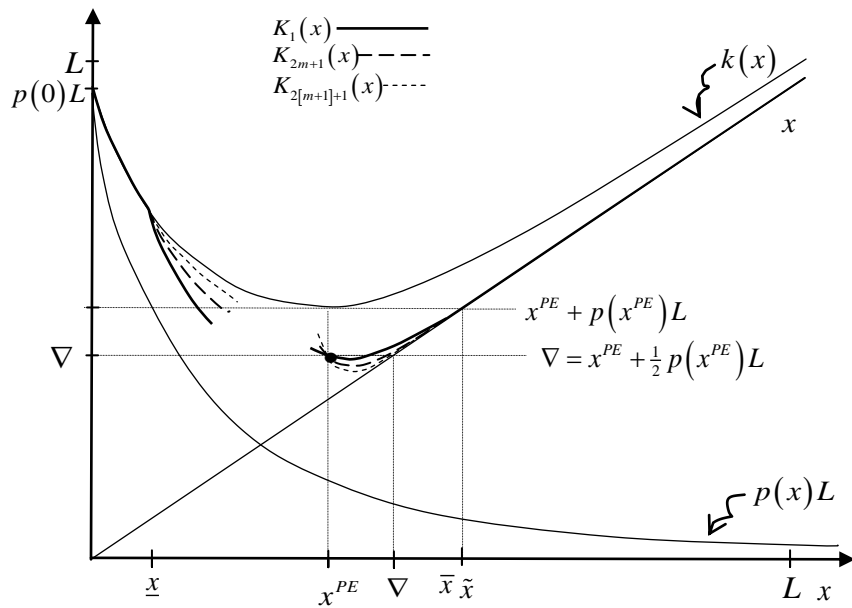
Figure 4.1 Incentive structures (illustrated with $\bar{x} > \nabla$ and $f(\underline{x}) > 0$ and indicating the area where curvature properties are "open")

4.1 Large panels

In large panels, the following results:

Proposition 4.1 The sequence of functions $(K_{2m+1} : m \in N)$ converges point-wise to the limit function K on $[0, L]$ with:

133

$$K(x) = \lim_{m \to \infty} K_{2m+1}(x) = \begin{cases} k(x) & \text{if } 0 \le x < x^{PE} \\ x^{PE} + \frac{1}{2} p\left(x^{PE}\right) L = \nabla & \text{if } x = x^{PE} \\ x & \text{if } x > x^{PE} \end{cases}.$$

The proposition follows directly from the relations in (4.2) and the Condorcet theorem limit property (C2, see Section II.2).[22] A large majority-voting panel transforms the minimization problem under uncertainty into a situation approximating an efficient *rule* under certainty: As m gets (sufficiently) large, the transformed norm induces the injurer to take (from above, arbitrarily close to) efficient care.[23]

4.2   Finite panels

In finite panels, detailed aspects of the aggregation function $h_{2m+1}(\cdot)$ and the abstract norm $g^F \in \mathbb{P}_{\{0,L\}}{}^X$ influence outcomes. Denote strictly undercomplying solution candidates in $\left(\underline{x}, x^{PE}\right)$ by $x^{*<}(2m+1)$ and overcomplying candidates in $\left[x^{PE}, \min\{\nabla, \overline{x}\}\right]$ by $x^{*\ge}(2m+1)$. Incentives to invest in precaution are determined by:

---

[22] See Bartle & Sherbert [1982] Sec. 7.1 on point-wise convergence of sequences of functions (smoothness and weak unimodality of F is not required for the proposition). While each of the minimands $K_{2m+1}(x)$ is continuous, the limit function is not: On $\left(\underline{x}, x^{PE}\right)$, $F \in \left(0, \frac{1}{2}\right)$ and $h_{2m+1}(F) \to 0^+$ if $m \to \infty$ (C2). Hence $\left[1 - h_{2m+1}(F)\right] \to 1^-$ and $K_{2m+1} \to k^-$. At x^{PE}, $F = \frac{1}{2}$ and $h_{2m+1} \equiv \frac{1}{2}$. Hence, $K_{2m+1} = \nabla$ for all m. On $\left(x^{PE}, \overline{x}\right)$, $F \in \left(\frac{1}{2}, 1\right)$ and $h_{2m+1}(F) \to 1^-$ if $m \to \infty$ (C2). Hence $\left[1 - h_{2m+1}(F)\right] \to 0^+$ and $K_{2m+1} \to x^+$.

[23] The discontinuity point of the limit function is determined by the median of F. Hence, more generally, a convergence result holds with $F^{-1}\left(\frac{1}{2}\right)$ replacing x^{PE} in Proposition 4.1: A large panel transforms the minimization problem under uncertainty to a problem parallel to a certain legal standard with $c = F^{-1}\left(\frac{1}{2}\right)$. (Section 5.2.2 gives a more detailed argument regarding large, but finite, panels.)

$$K_{2m+1}{}'(x) = 1 + \left[1 - h_{2m+1}(F(x))\right] p'(x)L - h_{2m+1}{}'(F(x))F'(x)p(x)L, \quad (4.3)$$

left and right derivatives possibly separating at support boundaries. *Critical points* in $(\underline{x}, \overline{x})$ are defined by:

$$K_{2m+1}{}'(x) = 0. \tag{4.4}$$

Legal uncertainty affects precaution investment incentives through two terms:

- negatively, by discounting the marginal real value of precaution, $|p'(x)L|$, through the liability probability $\left[1 - h_{2m+1}(F(x))\right]$. The weakened incentives through $h_{2m+1}(F(x))$—or $F_{m+1:2m+1}(x)$—will be called a *level dimension* of legal uncertainty
- positively, by increasing the probability of a compliance decision $h_{2m+1}{}'(F(x))F'(x)$, valued at $p(x)L$. The strengthened incentives through $h_{2m+1}{}'(F(x))F'(x)$—or $f_{m+1:2m+1}(x) = F_{m+1:2m+1}{}'(x)$—will be called a *marginal dimension* of legal uncertainty

Equation (4.3) shows that aggregation affects the level and marginal dimensions of legal uncertainty and thereby (interior) equilibria through (4.4). Due to the offsetting effects of level and marginal dimensions, and their interplay with the accident technology, it is difficult to characterize equilibria even in a single-judge context: Potential injurers may under or overcomply depending on specific aspects of the standard and the accident technology. However, Calfee-Craswell-Shavell were able to demonstrate that overcompliance is connected to concentrated distributions ("narrow uncertainty") and undercompliance to dispersed distributions ("broad uncertainty"), see Section 5.1.3.

Unfortunately, it is not possible to sign the second-order derivative of the minimand K_{2m+1} uniformly on $(\underline{x}, \overline{x})$:

$$K_{2m+1}"(x) = L\left\{ \underbrace{\left[1 - h_{2m+1}\big(F(x)\big)\right]p"(x)}_{\circ} \underbrace{-2h_{2m+1}'\big(F(x)\big)f(x)p'(x)}_{\circ\circ} \right.$$

$$\left. \underbrace{-h_{2m+1}"\big(F(x)\big)f(x)^2 p(x)}_{\circ\circ\circ} \underbrace{-h_{2m+1}'\big(F(x)\big)f'(x)p(x)}_{\circ\circ\circ\circ} \right\}$$

(4.5)

However, it can be demonstrated that $K_{2m+1}(\cdot)$ is *strictly convex* for all m on an interval $(a_{2m+1}, \bar{x})$, with $a_{2m+1} < x^{PE}$.[24] Strict convexity implies local uniqueness of solution candidates $x^{*\geq}(2m+1)$ and endows the derivative of K_{2m+1} at x^{PE} with important information regarding the structure of solutions. At x^{PE}, $F'(x^{PE}) = f(x^{PE})$ and $-p'(x^{PE})L = 1$, implying:

$$K_{2m+1}'(x^{PE}) = \tfrac{1}{2} - \underbrace{h_{2m+1}'(\tfrac{1}{2})f(x^{PE})}_{F_{m+1:2m+1}'(x^{PE}) = f_{m+1:2m+1}(x^{PE})} p(x^{PE})L.$$

(4.6)

At the efficient point, panel size only affects marginal dimensions of legal uncertainty ($F(x^{PE}) = \tfrac{1}{2}$, and $h_{2m+1}(\tfrac{1}{2}) = \tfrac{1}{2}$ for all m, eliminating the level effect). It follows that the incentive to invest in precaution increases monotonically in m at x^{PE}, see equation (III.7.4) and Table III.7.1.[25]

4.2.1 Overcomplying solutions

---

[24] On $\left[x^{PE}, \bar{x}\right)$, $F \in \left[\tfrac{1}{2}, 1\right)$, hence $h_{2m+1} \in \left[\tfrac{1}{2}, 1\right)$, $h_{2m+1}' > 0$ and $h_{2m+1}" \leq 0$ (Propostion III.2.2). (If $m \geq 1$ $h_{2m+1}" = 0$ only if $x = x^{PE}$. $h_1" \equiv 0$.). It follows that on $\left[x^{PE}, \bar{x}\right)$ $\circ > 0$, $\circ\circ > 0$ and $\circ\circ\circ \geq 0$. Due to weak unimodality, $f' \leq 0$ on $\left[x^{PE}, \bar{x}\right)$, hence $\circ\circ\circ\circ \geq 0$. It follows that $K_{2m+1}" > 0$ on $\left[x^{PE}, \bar{x}\right)$. By continuity of, there is an interval $\left(a_{2m+1}, x^{PE}\right]$ to the left of x^{PE} where $K_{2m+1}" > 0$ ($K_{2m+1}"(x^{PE}) > 0$).

[25] In fact, for all $m \geq 1$, $h_{2m+1}'(\cdot)$ is maximized at the point of inflection $\tfrac{1}{2} = F(x^{PE})$ (Proposition III.2.2).

By convexity of the minimand $K_{2m+1}(\cdot)$ in $(a_{2m+1}, \bar{x})$, a *necessary* condition for $x^*(2m+1) > x^{PE}$ is $K_{2m+1}'(x^{PE}) < 0$, or equivalently:

$$h_{2m+1}'\left(\tfrac{1}{2}\right) f\left(x^{PE}\right) p\left(x^{PE}\right) L = f_{m+1:2m+1}\left(x^{PE}\right) p\left(x^{PE}\right) L > \tfrac{1}{2},$$

marginal incentives for precaution from legal uncertainty, as transformed by the collective, must be sufficiently strong at x^{PE}. This condition is satisfied irrespective of the standard (density) and accident technology for sufficiently large m ($f_{m+1:2m+1}(x)$ peaks).

However, the condition is not sufficient for a global overcomplying solution (K_{2m+1} is not necessarily convex on $[\underline{x}, \bar{x}]$). For *interim* values of m, solutions $x^{*<}(2m+1)$ are possible. For sufficiently large m, however, overcompliance must result (see (4.2) and Section 4.3 below).

It has been established that $x^{*\geq}(2m+1) \leq \min\{\bar{x}, \nabla\}$, see in and at (4.2). If $\bar{x} = \nabla$, $K_{2m+1}(\bar{x}) = K_{2m+1}(x^{PE}) = \nabla$. Due to strict convexity on $[a_{2m+1}, \bar{x}]$ and necessity of $K_{2m+1}'(x^{PE}) < 0$, if solutions are strictly overcomplying, $K_{2m+1}(x^{*\geq}(2m+1)) < \nabla$. Hence, $x^{*\geq}(2m+1) < \nabla$. It follows that a solution at the upper support boundary $\bar{x}$ is possible only if $\bar{x} < \nabla$. Hence, $x^{*\geq}(2m+1) < \nabla$, with $x^{*\geq}(2m+1) = \bar{x}$ only if $\bar{x} < \nabla$ (necessary, not sufficient conditions).

The right hand derivative of K_{2m+1} at $\bar{x}$ is constant and positive for all m; $K_{2m+1}'(\bar{x}^+) \equiv 1$. The left derivative of K_{2m+1} at $\bar{x}$ is given by (from (4.3), (3.4), continuity of F, and Proposition III.2.2):

$$K_{2m+1}{}'\left(\overline{x}^{-}\right)=1+\left[1-\underbrace{h_{2m+1}\left(F\left(\overline{x}\right)\right)}_{1}\right]p{}'\left(\overline{x}\right)L$$

$$-\underbrace{h_{2m+1}{}'\left(F\left(\overline{x}\right)\right)}_{\substack{0\text{ if }m\geq1\\1\text{ if }m=0}}\underbrace{F{}'\left(\overline{x}^{-}\right)}_{f\left(\overline{x}^{-}\right)}p\left(\overline{x}\right)L=\begin{cases}1\text{ if }m\geq1\\1-f\left(\overline{x}^{-}\right)p\left(\overline{x}\right)L\text{ if }m=0\end{cases}\qquad(4.7)$$

Hence, for all $m\geq1$, K_{2m+1} is smooth and the derivative strictly positive at $\overline{x}$.

The same applies if $m=0$ and $f\left(\overline{x}^{-}\right)=0$. Because $K_{2m+1}{}'\left(\overline{x}^{-}\right)=K_{2m+1}{}'\left(\overline{x}^{+}\right)=1>0$, $x^{PE}<x^{*\geq}\left(2m+1\right)<\min\left\{\overline{x},\nabla\right\}$ are unique interior solutions given by first-order condition (4.4), see Figure 4.2.a ($\overline{x}\geq\nabla$) and 4.2.b ($\overline{x}<\nabla$).

It remains to investigate the possibility of boundary solutions $x^{*\geq}\left(1\right)=\overline{x}$ if $K_{1}{}'\left(x^{PE}\right)<0$, $\overline{x}<\nabla$, and $f\left(\overline{x}^{-}\right)>0$; the continuous $K_{1}\left(\cdot\right)$ has a kink at $\overline{x}<\nabla$ (necessary, not sufficient conditions).

If $K_{1}{}'\left(\overline{x}^{-}\right)>0$, the solution candidate $x^{*\geq}\left(1\right)$ is interior with $x^{PE}<x^{*\geq}\left(1\right)<\overline{x}<\nabla$ uniquely described by first-order condition (4.4), see Figure 4.2.c. However, if $K_{1}{}'\left(\overline{x}^{-}\right)\leq0$, the boundary solution candidate $x^{*\geq}\left(1\right)=\overline{x}<\nabla$ results (Figure 4.2.d). From (4.7) this condition is equivalent to $f\left(\overline{x}^{-}\right)p\left(\overline{x}\right)L=f_{m+1:2m+1}\left(\overline{x}^{-}\right)p\left(\overline{x}\right)L\geq1$: sufficiently high investment incentives at the boundary.[26] (Section 5.1 identifies families of uniform distributions which implement global solutions $x*\left(1\right)=\overline{x}$.)

---

[26] The condition is not strong enough to rule out global solution candidates in $\left(\underline{x},x^{PE}\right)$.
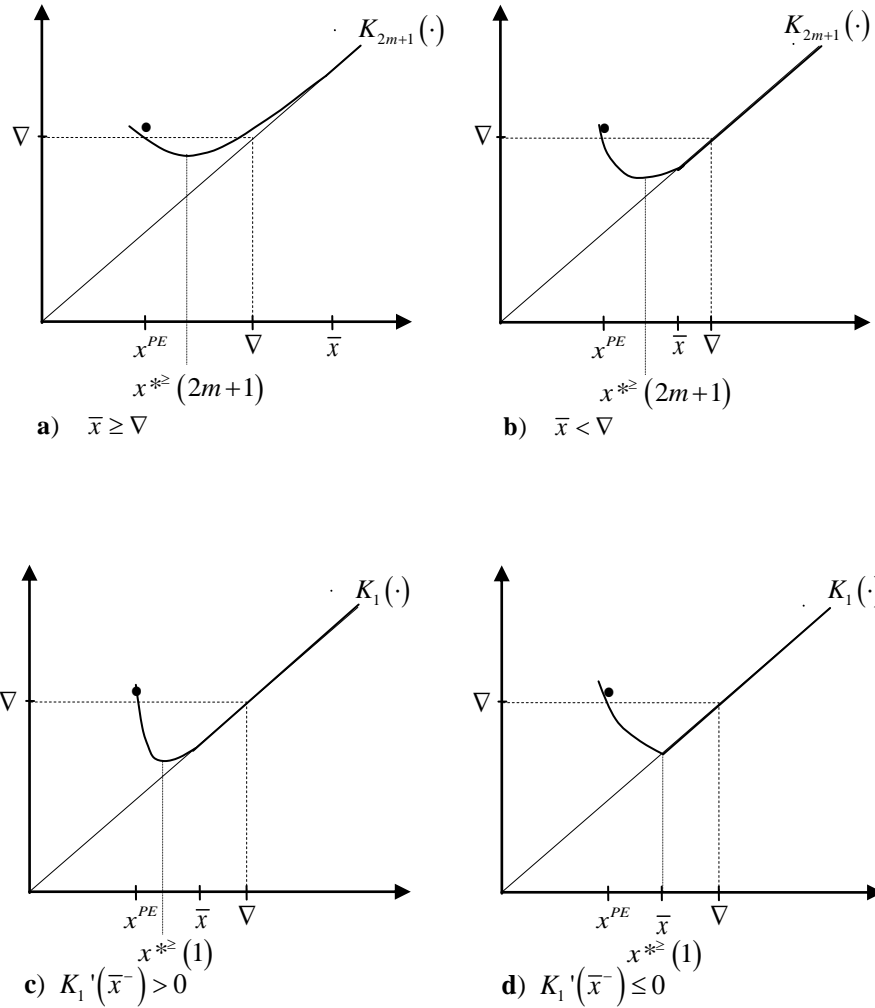
Figure 4.2 Overcomplying solution candidates ($K_{2m+1}{'}\left(x^{PE}\right)<0$). a) and b):

$m \geq 1$, or $m = 0$ and $f\left(\overline{x}^-\right) = 0$. c) and d): $m = 0$, $f\left(\overline{x}^-\right) > 0$ and $\overline{x} < \nabla$.

4.2.2 Efficient solutions

The "hairline" situation:

$$h_{2m+1}{'}\left(\tfrac{1}{2}\right)f\left(x^{PE}\right)p\left(x^{PE}\right)L = f_{m+1:2m+1}\left(x^{PE}\right)p\left(x^{PE}\right)L = \tfrac{1}{2},$$

by strict convexity of the criterion in $\left(a_{2m+1}, \overline{x}\right)$, implies that $x^{*\geq}\left(2m+1\right) = x^{PE}$

is a *local* strict minimum. Solution candidates $x^{*<}\left(2m+1\right) \leq a_{2m+1}$ cannot be

ruled out.

Should an efficient equilibrium prevail, it would be shifted by a change in panel size or a perturbation of the distribution function F (or the accident technology).

For large values of m, there might not exist admissible distributions implementing efficient solutions. (Section 5.2 derives a unique sequence of uniform distributions ensuring efficient solutions for small and interim values of m.)

4.2.3 Undercomplying solutions

If $K_{2m+1}'(x^{PE}) > 0$, strict convexity of K_{2m+1} on $(a_{2m+1}, \overline{x})$ and the increasing $K_{2m+1} = x$ on $[\overline{x}, \infty)$ implies that a *global* solution must be undercomplying (sufficiency). Furthermore, solutions are found among critical points $\underline{x} < x^{*<}(2m+1) < x^{PE}$, defined by (4.4).[27]

The sufficient condition for undercompliance is equivalent to:

$$h_{2m+1}'\left(\tfrac{1}{2}\right) f\left(x^{PE}\right) p\left(x^{PE}\right) L = f_{m+1:2m+1}\left(x^{PE}\right) p\left(x^{PE}\right) L < \tfrac{1}{2},$$

marginal incentives for precaution from legal uncertainty, as transformed by the collective, must not be too strong at x^{PE}. Low expected (real) accident costs at the efficient level of care facilitates undercompliance, while increasing panels

---

[27] Regarding exclusion of the lower boundary solution candidates (in terms of derivatives),

$$K_{2m+1}'\left(\underline{x}^+\right) = 1 + p'(\underline{x})L - h_{2m+1}'(0) f(\underline{x}) p(\underline{x})L = k'(\underline{x}) - h_{2m+1}'(0) f(\underline{x}) p(\underline{x})L.$$

If $m \geq 1$, $h_{2m+1}'(0) = 0$ (equation (III.7.3)). Hence $K_{2m+1}'\left(\underline{x}^+\right) = k'(\underline{x})$ and K_{2m+1} smooth at $\underline{x}$ and its strictly negative derivative independent of m. If $m = 0$, $K_1'\left(\underline{x}^+\right) = k'(\underline{x}) - f(\underline{x}) p(\underline{x})L$. It follows that $K_1'\left(\underline{x}^+\right) \leq K_{2m+1}'\left(\underline{x}^+\right) = k'(\underline{x}) < 0$, the first inequality strict if and only if $f(\underline{x}) > 0$ (the density function is discontinuous at $\underline{x}$). This situation is illustrated in Figure 4.1. Since all derivatives are strictly negative at $\underline{x}$, it is confirmed that $x^{*<}(2m+1) > \underline{x}$ for all m.

and a large abstract norm distribution mode tend to undermine the condition (see Table 2.1).

Due to the possible lack of convexity in $\left(\underline{x}, x^{PE}\right)$, uniqueness of solutions to (4.4) is not guaranteed. In order to ensure that a solution candidate is not (locally) maximizing expected cost, it is necessary that $K_{2m+1}"\left(x^{*<}\left(2m+1\right)\right) \geq 0$. If $K_{2m+1}"\left(x^{*<}\left(2m+1\right)\right) > 0$, minima are locally unique.[28]

Because $K_{2m+1}'\left(x^{PE}\right) > 0$ is only sufficient, not necessary, for undercomplying solutions, breach of the inequality does not mean that a global minimum is overcomplying. But for large m, solutions must be overcomplying, see Section 4.3. (Section 5.2 identifies classes of uniform distributions that implement undercomplying equilibria, for small and interim values of m.)

4.2.4   Summary

The structure of solutions has been characterized for finite panels and weakly uni-modal standards, with median at the efficient level of precaution, x^{PE}. Only the case of $K_{2m+1}'\left(x^{PE}\right) > 0$ allows definitive conclusions: under-investment $\underline{x} < x^*\left(2m+1\right) < x^{PE}$ is implied (if not uniquely). If $K_{2m+1}'\left(x^{PE}\right) \leq 0$ characterizations of (unique) candidates $x^{PE} \leq x^{*\geq}\left(2m+1\right) \leq \min\{\overline{x}, \nabla\}$ are local: the possibility of global minima in $\left(\underline{x}, x^{PE}\right)$ cannot be ruled out, for small or interim values of m. However, as will be demonstrated in Section 4.3, for sufficiently large m, (unique) global solutions lie in $\left(x^{PE}, \min\{\overline{x}, \nabla\}\right)$.

---

[28] $K_{2m+1}" \geq 0$ is necessary for a critical point to be a local minimum. A strict inequality is sufficient (Sydsæter [1982], Proposition 5.9 and Theorem 5.5.ii). A property of $K_{2m+1}" \neq 0$ at critical points ensures isolated equilibria. It is a weak assumption of *regularity* and allows use of the Implicit function theorem (see Section 5).

4.3 Comparative statics: panel size

By the second line in (4.2), the sequence $K^{*<}(2m+1) = \inf\left\{K_{2m+1}(x) : x \in \left(\underline{x}, x^{PE}\right)\right\}$ is monotonically increasing in m toward a number strictly larger than ∇. By the fourth line in (4.2), the sequence $K^{*>}(2m+1) = \inf\left\{K_{2m+1}(x) : x \in \left(x^{PE}, \overline{x}\right)\right\}$ is monotonically decreasing in m to a number strictly smaller than ∇. These monotonicity properties—deriving from transformation of the legal uncertainty *level* dimension $h_{2m+1}(F(x)) = F_{m+1:2m+1}(x)$—imply that in sufficiently large panels, solution candidates are overcomplying: From strict convexity on $\left(a_{2m+1}, \overline{x}\right)$ unique global solutions $x*(2m+1)$ in $\left(x^{PE}, \min\{\overline{x}, \nabla\}\right)$ defined by first-order condition (4.4) eventually result.[29] By proposition 4.1, solutions are arbitrarily close to x^{PE} from above for sufficiently large m.

Monotonicity implies that solutions $x*(2m+1)$ do not switch back and forth between under- and overcompliance as m increases. But a path from undercomplying equilibria needs not be unique (due to possible non-convexity of $K_{2m+1}(\cdot)$ in $\left(\underline{x}, a_{2m+1}\right)$) nor monotonic, and (unique) overcomplying solutions $x^{*\geq}(2m+1)$ need not contract monotonically towards x^{PE}. These difficulties in describing equilibrium effects of "interim" increases in m are due to the joint transformation of the *marginal* dimension of legal uncertainty:

$$h_{2m+1}{}'(F(x))F'(x) = f_{m+1:2m+1}(x).$$

---

[29] It is possible (if unlikely) that a single m^0 implies $x*(2m^0+1) = x^{PE}$ (see Sections 4.2.2 and 5.2.1). Because $K_{2m+1}{}'$ is strictly falling in m, $x*(2m+1) > x^{PE}$ for $m > m^0$.

To illustrate, consider an abstract norm with cdf F and an accident technology such that an overcomplying global minimum $F^{-1}\left(\frac{1}{2}\right) = x^{PE} < x*(1) < \overline{x}$ results, corresponding to an input value $\frac{1}{2} < F\left(x*(1)\right) < 1$ in Figure III.7.1. From the Craswell-Calfee-Shavell results, this is the case if the distribution is not too dispersed. Let the solid curve represent the tree-member panel, h_3. For all such $F(x)$, $h_3\left(F(x)\right) > h_1\left(F(x)\right)$: the *level* dimension of legal uncertainty—, ceteris paribus, pushing towards reduced precaution—is strengthened by the panel pushing towards reduced precaution.[30]

Globally on $F \in \left(\frac{1}{2}, 1\right)$ ($x \in \left(x^{PE}, \overline{x}\right)$), there is not a corresponding unidirectional impact from the panel, via the *marginal* dimension of legal uncertainty: If $\frac{1}{2} < F <_{+} \rho_3$ ($x^{PE} < x < F^{-1}\left(_{+}\rho_3\right)$), the marginal dimension is strengthened by the panel $h_3{}'\left(F(x)\right)F'(x) > h_1{}'\left(F(x)\right)F'(x) > F'(x)$, pushing towards increased precaution. However, the condition $_{+}\rho_3 < F$, that is $F^{-1}\left(_{+}\rho_3\right) < x*(1) < \overline{x}$, from *strict concavity* of $h_3(\cdot)$ on $\left[\frac{1}{2}, 1\right]$ is sufficient for the two dimensions of legal uncertainty to pull towards a contracted equilibrium $x*(3) < x*(1)$. Because $K_3(\cdot)$ is strictly convex on $(a_3, \overline{x})$, $x*(3)$ is a unique global optimum in $\left(x^{PE}, x*(1)\right)$.[31]

---

[30] x^{PE} ($F = \frac{1}{2}$) and $x = \overline{x}$ ($F = 1$) are "level fixed-points".

[31] The condition is "too sufficient" in the sense that $h_3{}'$ changes continuously around $_{+}\rho_3$, whereas there is a discrete jump between h_1 and h_3 at $_{+}\rho_3$, implying that the conclusion $x*(3) < x*(1)$ can be extended to an interval larger than $\left[_{+}\rho_3, 1\right)$ (larger than $\left[F^{-1}\left(_{+}\rho_3\right), \overline{x}\right)$ in X).

The argument generalizes to any increase in m from an input $x^{PE} < F^{-1}\left(_+\rho_{2m+1}\right) \le x^*(1)$, using the monotonicity property of $h_{m+1}(\cdot)\uparrow$ on $\left(\frac{1}{2},1\right)$ regarding the level dimension (the Condorcet theorem), and strict concavity of $h_{2m+1}(\cdot)$ on $\left(\frac{1}{2},1\right)$. But it also follows that, if $F^{-1}\left(_+\rho_{m+1}\right) \le x^*(m+1)$ $\left(_+\rho_{m+1} \le F\right)$, then $x^*\left([m+1]+1\right) < x^*(m+1)$ (see Figure III.7.1). For this result, the Proposition 2.3.i conclusion that $h_{2[m+1]+1}(F) < h_{2m+1}(F)$ if $F \in \left[_+\rho_{2[m+1]},1\right)$ if $F \in \left[_+\rho_{2[m+1]},1\right)$ is invoked.[32] Using, then, the fact that the sequene of roots $_+\rho_{2m+1}$ is falling and strict convexity of $K_{2m+1}(\cdot)$ on $\left(a_{2m+1},\bar{x}\right)$, $a_{2m+1} < x^{PE}$, it follows:

> Proposition 4.2 (sufficient conditions). Assume that the abstract norm cdf and accident technology is such that $x^*(1)$ is a global solution in $\left(x^{PE},\bar{x}\right]$. Define the unique and falling sequence $x^{+\langle 2m+1\rangle} = F^{-1}\left(_+\rho_{2m+1}\right)$ in $\left(x^{PE},\bar{x}\right)$. For all $m \ge 1$, global equilibria $x^{PE} < x^*(2m+1) < \bar{x}$ are unique and the following relations hold (ad inf) conditionally on the abstract norm equilibrum $x^*(1)$:

---

[32] The conditions clearly are sufficient, not necessary.

$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$
$m=3$		$x*(7)<\begin{cases}x*(5)\\x*(3)\\x*(1)\end{cases}$	$x*(7)<x*(5)<\begin{cases}x*(3)\\x*(1)\end{cases}$	$x*(7)<\cdots<x*(1)$
$m=2$			$x*(5)<\begin{cases}x*(3)\\x*(1)\end{cases}$	$x*(5)<x*(3)<x*(1)$
$m=1$				$x*(3)<x*(1)$

| $x*(1)$ | $\downarrow\cdots$ | $x*(1)\in\left[x^{+\langle7\rangle},x^{+\langle5\rangle}\right)$ | $x*(1)\in\left[x^{+\langle5\rangle},x^{+\langle3\rangle}\right)$ | $x*(1)\in\left[x^{+\langle3\rangle},\bar{x}\right]$ |

<u>Remark 4.1.A</u> The "stacked" variables cannot be ranked internally. Derivatives $h_{2m+1}{}'\big(F(x)\big)F'(x)$ change continuously *around* a point $x^{+\langle2m+1\rangle}$ in X. By continuity, inequalities established in $\left[x^{+\langle2[m+1]+1\rangle},x^{+\langle2m+1\rangle}\right)$ will also hold in the larger intrerval $\left(l_{+\langle2[m+1]+1\rangle},x^{+\langle2m+1\rangle}\right)$, where $l_{+\langle2[m+1]+1\rangle}<x^{+\langle2[m+1]+1\rangle}$ is some number sufficiently close to $x^{+\langle2[m+1]+1\rangle}$. $\square$

<u>Remark 4.1.B</u> Indeterminacy for interim increases in *m* depends on the location of "input value" $x*(1)$ and the absolute size of panel variations considered. For $x*(1)$ sufficiently close to x^{PE}, and *m* sufficiently small, an increase in precaution, as a response to panel size growth, is a *possibility* for several size increases. This is illustrated in Figure 5.4. $\square$

<u>Remark 4.2.</u> In case of a corner solution, the abstract norm maps legal fact $x*(1)=\bar{x}$ to no liability, with probability one ($F(\bar{x})=1$). Hence, locally, the

abstract norm $g_F\left(\cdot\,|\,x*(1)\right)=\left\langle F\left(\bar{x}\right),1-F\left(\bar{x}\right);0,L\right\rangle=\left\langle 1,0;0,L\right\rangle$ is determinate (a

"bright line"). However, it casts a shadow in the institutional context, due to the

transformation of marginal incentives at $\bar{x}$:[33] Transformed norms

$g_{F,2m+1}\left(\cdot\,|\,x*(1)\right)=\left\langle h_{2m+1}\left(F\left(x*(2m+1)\right)\right),1-h_{2m+1}\left(F\left(x*(2m+1)\right)\right);0,L\right\rangle$ have

non-collapsing nodes implementing equilibria $x*(1)=\bar{x}>x*(3)>x*(5)>\ldots$[34]

Section 5.1.1 derives a family of uniform standards, which implements

$x*(1)=\bar{x}$. See Figure 5.4. $\square$

Section 5.2.1 defines explicit families of distributions that identify input intervals

assumed in Proposition 4.2.

> Proposition 4.3 Assume that the abstract norm distribution cdf and the
>
> accident technology is such that $x*(1)=x^{PE}$ is a global solution. For
>
> each $m\geq 1$, $x^{PE}<x*(2m+1)<\bar{x}$ is uniquely determined, eventually
>
> falling in m.

The proposition is a direct implication of the monotonicity properties of

$K^{*<}(2m+1)$ and $K^{*>}(2m+1)$, from relations (4.2) and derivative (4.6) at x^{PE}:

Overcompliance is activated through transformation of the marginal dimension

---

[33] A necessary condition for the corner solution $x*(1)=\bar{x}$ is $K_1{}'\left(\bar{x}^-\right)\leq 0$. At $\bar{x}$, because $F\left(\bar{x}\right)=1$, $h_{2m+1}\left(F\left(\bar{x}\right)\right)\equiv 1$ for all m, including $m=0$. Hence, at $\bar{x}$ the level effect is invariant in panel size. However, $h_1{}'\left(F\left(\bar{x}\right)\right)=1$ ($K_1{}'\left(\bar{x}^-\right)\leq 0$), but $h_{2m+1}{}'\left(F\left(\bar{x}\right)\right)=0$ ($K_{2m+1}{}'\left(\bar{x}^-\right)=1$) if $m\geq 1$ implying a fall in precaution investment (see (4.7) and Figure 4.2.d).

[34] The weaker conclusion $x*(2m+1)<x*(1)=\bar{x}$ for all $m\geq 1$ follows from the proof of interior solutions in Section 4.2.1.

of legal uncertainty (the level dimension is invariant at x^{PE}, but $h_{2m+1}'\left(\frac{1}{2}\right)$ is strictly increasing in m). For sufficiently large m, $x*(2m+1)$ falls monotonically in m (compare Remark 4.1.B).

Assume that the abstract legal standard distribution cdf F and the accident technology is such that a global minimum $\underline{x} < x*(1) < x^{PE} = F^{-1}\left(\frac{1}{2}\right)$ exists, corresponding to an input value $0 < F < \frac{1}{2}$ in Figure 2.2. From the Craswell-Calfee-Shavell-results, this correspondins to a sufficiently dispersed standard. For all $F \in \left(0, \frac{1}{2}\right)$, $h_{2m+1}(F) \downarrow 0$ as $m \uparrow$. Ceteris paribus, the *level* effect of legal uncertainty is dampened by increased panel size that pushes towards higher precaution levels.

As described above, a similar monotonicity property does not apply to marginals $h_{2m+1}'(\cdot)$. Let the solid line in Figure 2.2 represent $h_3(\cdot)$. If $x*(1) \in \left[F^{-1}\left(\underline{\ }\rho_3\right), x^{PE}\right)$ corresponding to $\left[\underline{\ }\rho_3, \frac{1}{2}\right)$, from *strict convexity* of $h_3(\cdot)$ on $\left[0, \frac{1}{2}\right]$ (Proposition III.2.2), $h_3'(\cdot) > h_1'(\cdot) \equiv 1$ on $\left(\underline{\ }\rho_3, \frac{1}{2}\right]$, with equality at $\underline{\ }\rho_3$: Marginal precaution incentives are (weakly) strengthened, ceteris paribus, pushing towards increased precaution investments. Accordingly $x*(1) \in \left[F^{-1}\left(\underline{\ }\rho_3\right), x^{PE}\right)$ is *sufficient* for $x*(3) > x*(1)$.[35] If, however, $x*(1) \in \left(\underline{x}, F^{-1}\left(\underline{\ }\rho_3\right)\right)$, the level and marginal effects from legal uncertainty transformation work in opposite directions, and the net effect is uncertain.

---

[35] Because the derivative h_3' changes continuously around $\underline{\ }\rho_3$, but the level jumps from $h_1\left(\underline{\ }\rho_3\right) = \underline{\ }\rho_3$ to $h_3\left(\underline{\ }\rho_3\right) < \underline{\ }\rho_3$, the condition is "too sufficient": it holds for an interval larger than $\left(\underline{\ }\rho_3, \frac{1}{2}\right)$ in $\left(0, \frac{1}{2}\right)$ (larger than $\left(F\left(\underline{\ }\rho_3\right)^{-1}, x^{PE}\right)$ in X).

From the fact that the level mononicity property holds for all *m,* and us-

ing the increasing sequence of input levels $F^{-1}\left(_{-}\rho_{2m+1}\right) = x^{-\langle 2m+1\rangle}$ from Proposi-

tion 2.1-2 and the property of the strictly convex marginals $h_{2m+1}{}'(\cdot)$ on $\left[0,\tfrac{1}{2}\right]$

noted in Proposition III.7.3.ii, it follows:

Proposition 4.4 (sufficient conditions). Assume that the abstract norm

distribution cdf and the accident technology is such that $x^*(1)$ is a glob-

al solution candidate in $\left(\underline{x}, x^{PE}\right)$ and define the increasing sequence,

$x^{-\langle 2m+1\rangle} = F^{-1}\left(_{-}\rho_{2m+1}\right)$. If $x^*(1) \geq x^{-\langle 2m+1\rangle}$, then $x^*(2m+1) > x^*(1)$ for

all $m \geq 1$.

$$\vdots \qquad\qquad\qquad\qquad \vdots \qquad \cdots$$

$m = 3$ $\qquad\qquad\qquad\qquad \left.\begin{array}{l} x^*(7) \\ x^*(5) \\ x^*(3) \end{array}\right\} > x^*(1) \quad \cdots$

$m = 2$ $\qquad \left.\begin{array}{l} x^*(5) \\ x^*(3) \end{array}\right\} > x^*(1) \quad \left.\begin{array}{l} x^*(5) \\ x^*(3) \end{array}\right\} > x^*(1) \quad \cdots$

$m = 1$ $\quad x^*(3) > x^*(1) \qquad x^*(3) > x^*(1) \qquad x^*(3) > x^*(1) \quad \cdots$

$x^*(1)$ $\Big|$ $x^*(1) \in \left[x^{-\langle 3\rangle}, x^{-\langle 5\rangle}\right) \quad x^*(1) \in \left[x^{-\langle 5\rangle}, x^{-\langle 7\rangle}\right) \quad x^*(1) \in \left[x^{-\langle 7\rangle}, x^{-\langle 9\rangle}\right) \quad \cdots \downarrow$

Remark 4.3 From the inequalities in (4.2), $x*(2m+1) > x^{PE}$ for sufficiently large m from any initial $x*(1) \in \left(\underline{x}, x^{PE} \right]$. The lack of structure in the table (weak predictions) reflects that the (eventual) convergence to efficiency, in general, is not monotonic.[36] Remark 4.1.A about local extension of the comparative statics from continuity applies. $\square$

Section 4.2.3 establishes sufficient (not necessary) conditions for global undercomplying solutions $x*(2m+1)$. In terms of size m:

> Proposition 4.5. If there exists an $\bar{m} \in N$ such that $h_{2\bar{m}+1}{}'\left(\frac{1}{2}\right) f\left(x^{PE}\right) p\left(x^{PE}\right) L < \frac{1}{2}$, there exists global undercomplying solutions $\underline{x} < x*(2m+1) < x^{PE}$ for all $m \leq \bar{m}$, $m, \bar{m} \in \{0,1,2,...\}$.

Remark 4.4 Relative to a class of abstract norms and accident technologies, there might not exist an $\bar{m}$ such that the proposition inequality holds. Function values $h_{2m+1}{}'\left(\frac{1}{2}\right)$ are reported in Table III.7.1. (Sufficient conditions for undercomplying equilibria under uniform distributions are given in Proposition 5.7.) $\square$

5　Equilibrium analysis II: uniform distributions

This section specializes on a class of abstract norms with distributions symmetric and uniformly distributed around the efficient point, $S \sim U\left(x^{PE} - \underline{\overline{x}}, x^{PE} + \underline{\overline{x}}\right)$ and $p(x) = e^{-x}$. Because $-p'(0)L > 1$, an interior efficient point $x^{PE} = \ln L$ is given by (3.2). The family of cdfs is given by:

---

[36] All abstract equilibria in Figure 5.4 share this property.

$$F(x;\underline{x})\big|_{\overline{x}\in(0,\ln L]} = \begin{cases} 0 \text{ if } x < \ln L - \overline{x} \\ \dfrac{x - (\ln L - \overline{x})}{2\overline{x}} \text{ if } \ln L - \overline{x} \le x \le \ln L + \overline{x} \\ 0 \text{ if } x > \ln L + \overline{x} \end{cases}.$$

The parameter $\overline{x} \in (0, \ln L]$ measures legal uncertainty and allows (continuous) comparative statics. An increase in $\overline{x}$ corresponds to a *mean-preserving increase in risk*, see Section III.7 and Figure 5.2. The equilibrium analysis is supported by simulations with $L = e^e \cong 15.154$ and $\overline{x} \in (0, e]$, solutions reported in Appendix A.1.[37]

The cdfs are continuous with $F(\underline{x}; \overline{x}) = 0$, $F(\ln L; \overline{x}) = \tfrac{1}{2}$, and $F(\overline{x}; \overline{x}) = 1$. $F(\cdot; \overline{x})$ is continuously differentiable everywhere, except at the support boundaries. Define $\partial F(x; \overline{x})/\partial x \equiv F'$. $F' = 0$ if $x < \underline{x}$ or $x > \overline{x}$, and $F'(x) = f(x; \overline{x}) \equiv \tfrac{1}{2\overline{x}}$ on $(\underline{x}, \overline{x})$. At $\overline{x}$, derivatives are given by:

$$\begin{cases} F'(\overline{x}^-) = f(\overline{x}; \underline{x}) = \dfrac{1}{2\overline{x}} \\ F'(\overline{x}^+) = 0 \end{cases}.$$

The accident technology allows *separation* of real and institutional factors. Critical points are given by:

$$\partial K_{2m+1}(x; \underline{x})/\partial x = K_{2m+1}'(x; \underline{x})$$

$$= \underbrace{1 - e^{-x}L}_{\text{real factors}} \underbrace{\left\{ 1 - \underbrace{h_{2m+1}(F(x;\underline{x}))}_{\text{level dimension}} + \underbrace{h_{2m+1}'(F(x;\underline{x}))F'(x;\underline{x})}_{\text{marginal dimension}} \right\}}_{\text{institutional factors}} = 0. \qquad (5.1)$$

---

[37] The qualitative illustrations below reflect that $L > e$, implying that $e^{-x}L$ and x intersect to the left of the efficient point, $\ln L$.

Let $\alpha_{2m+1} = \{1 - h_{2m+1} + h_{2m+1}{}'F'\}$. $\alpha_{2m+1} > 0$ on $(\underline{x}, \overline{x})$. Equation (5.1) is equivalent to:

$$x = \ln L + \ln \alpha_{2m+1}. \tag{5.2}$$

As is demonstrated in Section 4.2, solutions $x^* = x^*(2m+1)$ are characterized by $x^* > \underline{x}$ for all $m \geq 0$, and for all $m \geq 1$, $x^* < \min\{\overline{x}, \nabla\}$ $= \min\{\ln L + \overline{x}, \ln L + \tfrac{1}{2}\}$.[38] A boundary solution $x^* = \overline{x}$ is possible only if $m = 0$ and $\overline{x} < \nabla$, that is, $\overline{x} < \tfrac{1}{2}$ (necessary, not sufficient). Interior solutions can be characterized as follows:

- undercompliance iff $\alpha_{2m+1} < 1$, that is, the level effect dominates the marginal effect of legal uncertainty as transformed in panels:

  $h_{2m+1}(F(x^*)) > h_{2m+1}{}'(F(x^*))F'(x^*)$;

- compliance iff $\alpha_{2m+1} = 1 \leftrightarrow h_{2m+1}(F(x^*)) = h_{2m+1}{}'(F(x^*))F'(x^*)$; and

- overcompliance iff $\alpha_{2m+1} > 1$, that is, the marginal effect dominates the level effect, $h_{2m+1}(F(x^*)) < h_{2m+1}{}'(F(x^*))F'(x^*)$.

Detailed aspects of the solutions in the single-judge case ($m = 0$) are utilized for characterizing solutions with $m \geq 1$ (compare the role of $x^*(1)$ in Propositions 4.2–5). Abstract norm equilibria are studied in Section 5.1 and equilibria under transformed norms in Section 5.2. Frequent reference will be made to the minimand derivative at the efficient point (compare (4.6)):

$$K_{2m+1}{}'\left(x^{PE}; \overline{x}\right)\Big|_{x^{PE} = \ln L} = \frac{1}{2\underline{x}}\left\{\overline{x} - h_{2m+1}{}'\left(\tfrac{1}{2}\right)\right\}. \tag{5.3}$$

---

[38] $\nabla = x^{PE} + \tfrac{1}{2}p\left(x^{PE}\right)L = \ln L + \tfrac{1}{2}e^{-\ln L}L = \ln L + \tfrac{1}{2}$.

5.1   *Abstract norms*

$F(x;\overline{x})$ is linear in x on $[\underline{x},\overline{x}]$ and $h_1(\cdot)$ the identity function. It follows that the minimand $K_1(x;\overline{x}) = x + [1 - F(x;\overline{x})]e^{-x}L$ is *strictly convex* on $[\underline{x},\overline{x}]$.[39] Hence, for each $\overline{x} \in (0, \ln L]$, a *unique global* minimum $x^*(1)$ exists. The solution is a function of $\overline{x}$ and is denoted $\chi(1;\overline{x})$.

5.1.1   Solutions

<u>Proposition 5.1</u> If $S \sim U(\ln L - \overline{x}, \ln L + \overline{x})$ and $p(x) = e^{-x}$, $x^{PE} = \ln L$

and $\ln L + \underline{x} < \chi(1;\overline{x}) < \nabla = \ln L + \frac{1}{2}$ is uniquely given for all $\overline{x} \in (0, \ln L]$

with:

$$\chi(1;\overline{x}) = \ln L + \overline{x} \ \text{ if } \ 0 < \overline{x} \leq \hat{\overline{x}};$$
$$\ln L < \chi(1;\overline{x}) < \ln L + \hat{\overline{x}} \ \text{ if } \ \hat{\overline{x}} < \overline{x} < 1;$$
$$\chi(1;\overline{x}) = \ln L \ \text{ if } \ \overline{x} = 1; \text{ and}$$
$$\ln L - \overline{x} < \chi(1;\overline{x}) < \ln L \ \text{ if } \ \overline{x} > 1.$$

The interior equilibria, obtained for $\overline{x} > \hat{\overline{x}} \cong 0.352$, are given by the (implicit) solution to (5.2) with $m = 0$.

Solutions are illustrated in Figures 5.1 and 5.4.

<u>Proof.</u> From (5.3), $K_1'(\ln L;\overline{x}) = \frac{1}{2\overline{x}}\{\overline{x} - 1\}$. By strict convexity of $K_1(\cdot;\overline{x})$ on $[\underline{x},\overline{x}]$, a unique solution for each $\overline{x} \in (0, \ln L]$ follows:

$$\chi(1;\overline{x}) < \ln L = x^{PE} \text{ iff } \overline{x} > 1 (K_1'(\ln L;\overline{x}) > 0)$$
$$\chi(1;\overline{x}) = \ln L = x^{PE} \text{ iff } \overline{x} = 1 (K_1'(\ln L;\overline{x}) = 0)$$
$$\chi(1;\overline{x}) > \ln L = x^{PE} \text{ iff } \overline{x} < 1 (K_1'(\ln L;\overline{x}) < 0)$$

---

[39] Strict convexity follows from (4.5) because $h_1'' \equiv f' \equiv 0$, eliminating terms $\circ\circ\circ$ and $\circ\circ\circ\circ$, leaving $K_{2m+1}''(\cdot) > 0$ on $(\underline{x},\overline{x})$.

From Section 4.2: $\chi(1;\overline{x}) > \underline{x} = \ln L - \overline{x}$. From Section 4.2.1, $\chi(1;\overline{x}) < \nabla = \ln L + \frac{1}{2}$.

Only standards with $\overline{x} < \nabla$ ($\underline{x} < \frac{1}{2}$) need to be investigated regarding possible solutions

at $\overline{x}$. Interior solutions are the (implicit) solutions to (5.2) with $m = 0$. Boundary solu-

tions $\chi(1;\overline{x}) = \overline{x} = \ln L + \overline{x}$ obtain if $F(\overline{x})p(\overline{x})L = f(\overline{x}^-)p(\overline{x})L \geq 1$, that is,

$(1/2\overline{x})\exp\left[-(\ln L + \overline{x})\right]L = \left\{2\overline{x}\exp[\overline{x}]\right\}^{-1} \geq 1$, or $\psi(\overline{x}) \equiv \overline{x} + \ln\overline{x} + \ln 2 \leq 0$. $\psi(\cdot)$ is

strictly increasing and strictly concave with $\psi \to -\infty$ if $\overline{x} \to 0^+$ and $\psi\left(\frac{1}{2}\right) = \frac{1}{2}$. The

inequality is satisfied for all $0 < \overline{x} \leq \hat{\overline{x}}$, with $\hat{\overline{x}}$ approximately equal to 0.352.

From Proposition 5.2 below, $\partial\chi(1;\overline{x})/\partial\overline{x} < 0$ if $\overline{x} > \hat{\overline{x}}$. Hence, $\chi(1;\overline{x}) < \hat{\overline{x}}$ for

$\overline{x} > \hat{\overline{x}}$. The only way to make $\chi(1;\overline{x}) = \ln L$ is to have $\alpha_1 = 1$, or, equivalently,

$\ln L + \overline{x} + 1 - \ln L/2\overline{x} = 1$. It follows that $\chi(1;1) = \ln L$. ◄
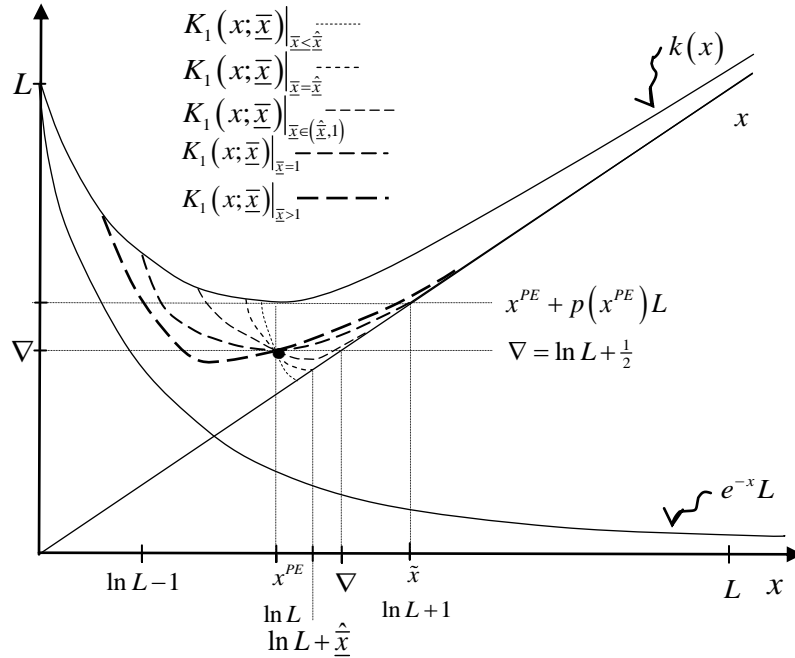
Figure 5.1. Equilibria under uniform abstract norms ($\overline{x} \in (0, \ln L]$).

153

5.1.2 Comparative statics: legal uncertainty

This section considers the structure of abstract equilibria in further detail, in particular their response to legal uncertainty, as measured by $\underline{x} \in (0, \ln L]$. The proof of Proposition 5.1 establishes results that will be used in Section 5.2. Let $\partial \chi(1; \underline{x})/\partial \underline{x} \equiv \chi_{\underline{x}}(1; \underline{x})$.

> **Proposition 5.2** Let $S \sim U(\ln L - \underline{x}, \ln L + \underline{x})$, $p(x) = e^{-x}$ and legal uncertainty be measured by $\underline{x} \in (0, \ln L]$.
>
> i) Precaution investment increases linearly in legal uncertainty if:
>
> $\underline{x} \in (0, \hat{\underline{x}})$: $\chi(1; \underline{x}) > \ln L$ and $\chi_{\underline{x}}(1; \underline{x}) = 1$.
>
> ii) Precaution investment decreases in legal uncertainty if $\underline{x} > \hat{\underline{x}}$:
>
> if $\underline{x} \in (\hat{\underline{x}}, 1)$, $\chi_{\underline{x}}(1; \underline{x}) < 0$, with $\chi_{\underline{x}}(1; \underline{x})\big|_{\underline{x}=1} = -\frac{1}{3}$ at the efficient point.

Proof. If $\underline{x} > \hat{\underline{x}}$, solutions are interior from Proposition 5.1. First-order condition (5.1) is an identity in $\underline{x}$. By the Implicit function theorem:[40]

$$\chi_{\underline{x}}(1; \underline{x}) = -\frac{\partial K_1{}'(\chi; \underline{x})}{\partial \underline{x}}\bigg|_{\chi = \chi(1; \underline{x}) \text{ const.}} \bigg/ K_1{}''(\chi(1; \underline{x}); \underline{x}). \tag{5.4}$$

By strict convexity of the minimand at interior points, $\left\{ \partial K_1{}'(\chi; \underline{x})/\partial \underline{x} \right\}\big|_{\chi = \chi(1; \underline{x}) \text{ const}} > 0$ implies $\chi_{\underline{x}}(1; \underline{x}) < 0$, and $\left\{ \partial K_1{}'(\chi; \underline{x})/\partial \underline{x} \right\}\big|_{\chi = \chi(1; \underline{x}) \text{ const}} < 0$ implies $\chi_{\underline{x}}(1; \underline{x}) > 0$.

Holding $\chi(1; \underline{x}) = \chi$ constant under differentiation in (5.1) ($m = 0$, $h_1(F) = F$ and $h_1{}' \equiv 1$), gives (after rearranging terms):

---

[40] The parameter $\underline{x}$ enters directly in the $K_1{}'(\cdot; \underline{x})$-function and through the equilibrium $\chi(1; \underline{x})$. $K_1{}''(\chi(1; \underline{x}); \underline{x}) > 0$ by strict convexity on $(x, \bar{x})$, and is sufficient for isolated equilibria and for C^1-functions $\chi(1; \underline{x})$ to exist locally (Sydsæter [1981], Theorem 3.3).

$$\frac{\partial K_1{}'(\chi;\overline{x})}{\partial\overline{x}} = e^{-\chi}L\left\{\underbrace{\frac{\partial F(\chi;\overline{x})}{\partial\overline{x}}}_{a} - \underbrace{\frac{\partial F{}'(\chi;\overline{x})}{\partial\overline{x}}}_{b}\right\}. \tag{5.5}$$

Component *a* measures the *direct* (χ fixed) impact of increased legal uncertainty on its *level dimension*. Component *b* measures the *direct* (χ fixed) impact of increased legal uncertainty on its *marginal dimension*. Because $\chi = \chi(1,\overline{x}) \in (\underline{x},\overline{x})$, $F(\chi;\overline{x}) = \{\chi - (\ln L - \overline{x})\}/2\overline{x}$ and $F{}'(\chi;\overline{x}) = f(\chi;\overline{x}) \equiv 1/2\overline{x}$. It follows:

$$\begin{cases} a & \dfrac{\partial F(\chi;\overline{x})}{\partial\overline{x}} = \dfrac{\ln L - \chi}{2\overline{x}^2} \\[2mm] b & \dfrac{\partial F{}'(\chi;\overline{x})}{\partial\overline{x}} = \dfrac{\partial f(\chi;\overline{x})}{\partial\overline{x}} \equiv -\dfrac{1}{2\overline{x}^2} \end{cases}. \tag{5.6}$$

Figure 5.2.b illustrates that component *b* is negative at all equilibrium levels $\chi \in (\underline{x},\overline{x})$, pulling towards a positive derivative $\partial K_1{}'(\chi;\overline{x})/\partial\overline{x}$ in (5.5) at the initial equilibrium (critical point) and hence, ceteris paribus, towards lower precaution investments.[41] Figure 5.2.a illustrates that component *a* is positive, at undercomplying equilibria, thereby also pulling towards reduced precaution investments. At overcomplying equilibria, component *a* is negative, and the direct level and direct marginal effect work in opposite directions. However, due to the upper bound on equilibria $\chi(1;\overline{x}) < \nabla = \ln L + \frac{1}{2}$, the direct marginal effect dominates:

$$a - b = \{\ln L + 1 - \chi\}/2\overline{x}^2 > 0, \tag{*}$$

implying $\partial K_1{}'(\chi;\overline{x})/\partial\overline{x} > 0$. It follows that $\chi_{\overline{x}}(1;\overline{x}) < 0$ for all $\overline{x} > \hat{\overline{x}}$.

From (4.5), $K_1{}''(\chi(1;\overline{x});\overline{x}) = \exp[-\chi(1;\overline{x})]\{1 - F(\chi(1;\overline{x});\overline{x}) + \frac{1}{\overline{x}}\}$. Accordingly $\chi_{\overline{x}}(1;\overline{x}) = [\chi(1;\overline{x}) - (\ln L + 1)]/(2\overline{x}^2\{1 - F(\chi(1;\overline{x});\overline{x}) + \frac{1}{\overline{x}}\})$, and $\chi_{\overline{x}}(1;1) = -\frac{1}{3}$.

---

[41] This "global" property is a feature of the uniform distribution. For example, if $S \sim N(\mu,\sigma^2)$, uncertainty measured by σ^2, precaution incentives from the marginal dimension would be strengthened at points sufficiently far from μ.

If $0 < \underline{\bar{x}} < \hat{\underline{\bar{x}}}$, from Proposition 5.1 $\chi(1;\underline{\bar{x}}) = \ln L + \underline{\bar{x}}$. Hence, $\chi_{\bar{x}}(1;\underline{\bar{x}}) = 1$.[42] ◄
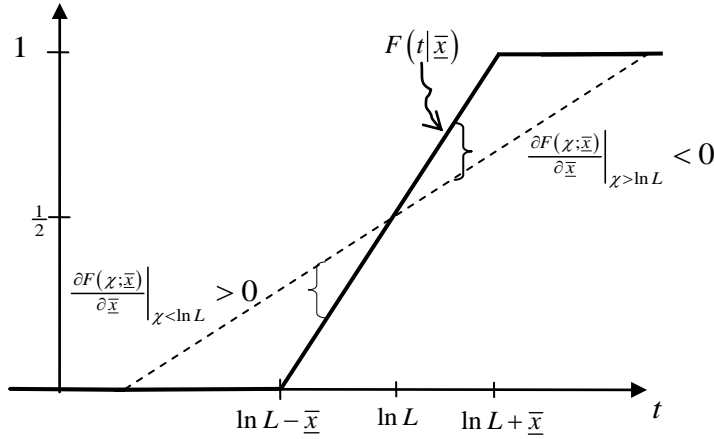
a) level dimension (interior solution)
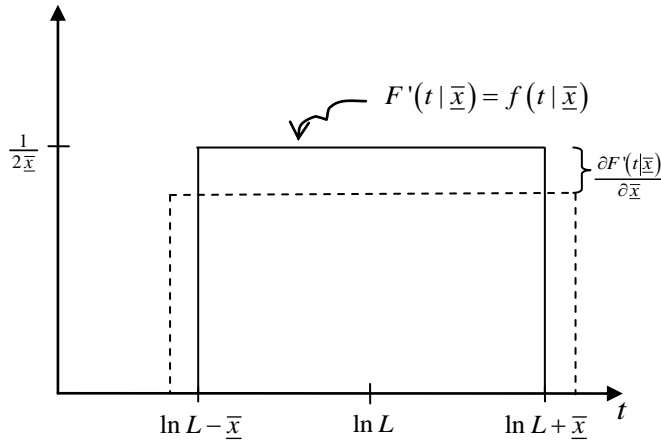
b) marginal dimension

Figure 5.2 Mean-preserving increase in abstract norm uncertainty ($\bar{x} \uparrow$; *direct*

impact on the a) level dimension and b) marginal dimension)

The value function:

$$\mathbf{K}_1(\underline{\bar{x}}) = K_1\big(\chi(1;\underline{\bar{x}});\underline{\bar{x}}\big) = \chi(1;\underline{\bar{x}}) + \Big[1 - F\big(\chi(1;\underline{\bar{x}});\underline{\bar{x}}\big)\Big]e^{-\chi(1;\underline{\bar{x}})}L,$$

also characterizes solutions (see Figure 5.1):

---

[42] The corner solution condition, requiring a sufficiently large marginal incentive $F'(\bar{x}^-;\underline{\bar{x}})p(\bar{x})L = f(\bar{x}^-;\underline{\bar{x}})p(\bar{x})L \geq 1$, even though the direct impact b is negative, remains satisfied unless initially satisfied with equality (corresponding to $\chi(1;\underline{\bar{x}}) = \ln L + \hat{\underline{\bar{x}}}$).

<u>Proposition 5.3</u>: Let $S \sim U\left(\ln L - \overline{\underline{x}}, \ln L + \overline{\underline{x}}\right)$, $p(x) = e^{-x}$ and legal uncertainty be measured by $\overline{\underline{x}} \in (0, \ln L]$.

    i) From overcomplying solutions—from given levels of uncertainty $\overline{\underline{x}} \in (0,1)$ —expected cost increases in legal uncertainty, $\mathbf{K}_1{'}(\overline{x}) > 0$.

    ii) From undercomplying solutions—from given levels of uncertainty $\overline{\underline{x}} > 1$ —expected cost falls in legal uncertainty, $\mathbf{K}_1{'}(\overline{x}) < 0$.

<u>Proof.</u>   If $\overline{\underline{x}} < \hat{\overline{\underline{x}}}$, $\chi(1;\overline{\underline{x}}) = \ln L + \overline{\underline{x}}$ and $F \equiv 1$ (Proposition 5.1).   Hence,

$\mathbf{K}_1{'}(\overline{x}) = \partial\{\ln L + \overline{\underline{x}}\}/\partial\overline{\underline{x}} = 1$. If $\overline{\underline{x}} \in \left(\hat{\overline{\underline{x}}}, \ln L\right)$, solutions are interior. By the Envelope

theorem $\mathbf{K}_1{'}(\overline{x})$ is given by:[43]

$$\left.\frac{\partial K_1(\chi;\overline{\underline{x}})}{\partial\overline{\underline{x}}}\right|_{\chi=\chi(1;\overline{\underline{x}})\text{ const.}} = -\exp\left[-\chi(1;\overline{\underline{x}})\right]L\left.\frac{\partial F(\chi;\overline{\underline{x}})}{\partial\overline{\underline{x}}}\right|_{\chi=\chi(1;\overline{\underline{x}})\text{ const.}} = \left\{\chi(1;\overline{\underline{x}}) - \ln L\right\}/2\overline{\underline{x}}^2 e^{\overline{\underline{x}}}.$$

Hence, i) and ii) follow from Proposition 5.1. ◄

5.1.3   Discussion

For the class of uniform standards symmetric around the efficient solution, equilibrium behavior implies overcompliance when the standard is concentrated, and undercompliance when it is dispersed. Hence, compared to a determinative negligence rule $g_{\delta c} \in \delta\mathbb{P}_{\{0,L\}}{}^X$ with care level $c = x^{PE} = \ln L$ (or the class of rules with $c > \tilde{x} = \ln L + 1$ functionally equivalent to strict liability), uncertain standards $g_F \in \mathbb{P}_{\{0,L\}}{}^X \setminus \delta\mathbb{P}_{\{0,L\}}{}^X$ impose real costs on the jurisdiction for all $\overline{\underline{x}} \neq 1$.[44]

---

[43] The set $[0,L]$ is compact and the derivative $\left\{\partial K_1\left(\chi(1;\overline{\underline{x}});\overline{\underline{x}}\right)/\partial\overline{\underline{x}}\right\}\bigg|_{\chi(1;\overline{\underline{x}})\text{ const}}$ continuous when $\overline{\underline{x}} > \hat{\overline{\underline{x}}}$ (interior solution). Hence, the conditions of the Envelope theorem are satisfied (Carter [2001], Theorem 6.2).

[44] The situation $c = \tilde{x}$ is indeterminate, see (4.1).

The findings corroborate Shavell's proof that overcompliance results if the legal standard distribution is not too dispersed and undercomplying solutions with dispersed standards.[45] Craswell & Calfee [1986] measure legal uncertainty with the standard deviation in normal distributions $N(\mu,\sigma^2)$, finding in simulations that, in the case of abstract norms with $\mu = x^{PE}$ (equal to the median by symmetry), that (i) if σ is large, undercompliance results (compare $\bar{x} > 1$), and (ii), if σ is small, overcompliance results (compare $\bar{x} < 1$).[46] They also find that a reduction of legal uncertainty for sufficiently low initial values of σ, will not necessarily improve compliance decisions, and, in fact, aggravates the overcompliance problem (compare $\bar{x} \downarrow$ in $\left(\hat{\bar{x}},1\right)$).

Due to corner solutions in the uniform case for $\bar{x} \in \left(0,\hat{\bar{x}}\right)$, a reduction in uncertainty from sufficiently low initial levels, reduces the overcompliance problem linarly.

---

[45] Shavell [1987], Remark 3 to Proposition 4. 4. His proof is valid for a general expected loss function $l(x) = p(x)\int g(l;x)dl$ ($g(\cdot;x)$ is the density of losses given precaution investment x and occurrence of an accident; Sec. 6A.1). The distribution of the legal standard is allowed to take any form as long as the probability of liability is strictly positive and it gets sufficiently concentrated around a level $\hat{x} < \tilde{x}' = \arg\min_{x}\{x + l(x)\}$ (Sections 4A.3.3 and 4A.3.1). In the present setting, with $\tilde{x}$ defined in (4.1) corresponding to $\tilde{x}'$ and $\ln L$ to $\hat{x}$, Shavell's requirements translates to i) $1 - F(\ln L;\bar{x}) \geq \gamma > 0$ and ii) $F(\cdot;\bar{x})$ becomes concentrated at $\ln L$ as $\bar{x} \to 0^+$ (see Property 1 and 2, p. 97, respectively). Both are satisfied because $1 - F(\ln L;\bar{x}) = \frac{1}{2}$ for all $\bar{x}$ and $F(\cdot;\bar{x})$ has a jump discontinuity at $\ln L$ in the limit (with the amount of probability mass concentrated at $\ln L$ equal to 1; Apostol [1969], Theorem 14.5).

[46] See column four in Table 1. (Second order conditions are not explicitly considered. The paper also discusses distributions with mean different from the efficient level.)

5.2   *Transformed norms*

If $m \geq 1$, *interior* equilibria $x*(2m+1) \in (\ln L - \overline{x}, \min\{\ln L + \overline{x}, \ln L + \frac{1}{2}\})$ are

defined for all $\overline{x} \in (0, \ln L]$ by (5.2).[47] Convexity is no longer ensured in

$(\underline{x}, a_{2m+1})$, implying that the difficulties encountered in Section 4 recur. However,

the unique, global solutions $\chi(1; \overline{x})$ derived in Section 5.1 allow sharper state-

ments about panel size effects.

5.2.1   Comparative statics: panel size

Concentrated distributions $0 < \overline{x} < 1$

If $\overline{x} \in (0,1)$, unique *global* solutions $\chi(1; \overline{x}) > \ln L$ follow from Proposition 5.1.

Hence, Proposition 4.2 implies that for all $m \geq 1$, there are unique overcomplying

global equilibria, $x* = \chi(2m+1; \overline{x}) > \ln L$, that eventually converge to

$x^{PE} = \ln L$ (Proposition 4.1). However, due to the mix of marginal and level di-

mensions of legal uncertainty, the convergence is not (in general) monotonic:

Proposition 4.2 can be sharpened to statements about families of legal

standards. To this end, for all $\overline{x} \in (0, \ln L]$ define the function:

$$\mathbf{F}(\overline{x}) = F(\chi(1; \overline{x}); \overline{x}). \tag{5.7}$$

As illustrated in Figure 5.3, $\mathbf{F}(\overline{x}) = 1$ if $\overline{x} \in (0, \hat{\overline{x}}]$ and $\mathbf{F}(1) = F(\chi(1;1);1) = \frac{1}{2}$.

$\mathbf{F}$ is continuous for all $\overline{x} > 0$ and continuously differentiable for all $\overline{x} \neq \hat{\overline{x}}$. $\mathbf{F}$ is

strictly falling on $[\hat{\overline{x}}, 1]$: If $\overline{x} \in (\hat{\overline{x}}, 1 + \varepsilon)$ and $\varepsilon > 0$ is sufficiently small:

---

[47] Section 4.2.1.

$$\mathbf{F}'(\underline{\overline{x}}) = F'\left(\chi(1;\underline{\overline{x}});\underline{\overline{x}}\right)\chi_{\underline{\overline{x}}}(1;\underline{\overline{x}}) + \frac{\partial F\left(\chi(1;\underline{\overline{x}});\underline{\overline{x}}\right)}{\partial \underline{\overline{x}}} < 0.^{48} \qquad (5.8)$$

This implies that to any element in the (falling) sequence of roots $_{+}\rho_{2m+1}$ in

$\left[\frac{1}{2},1\right)$, $m=1,2,\dots$ (Proposition 2.2), a unique (increasing) sequence of distribu-

tion parameters $\underline{\overline{x}}^{+\langle 2m+1\rangle}$ may be defined in $\left(\hat{\underline{\overline{x}}},1\right]$ such that

$_{+}\rho_{2m+1} = F\left(\chi\left(1;\underline{\overline{x}}^{+\langle 2m+1\rangle}\right);\underline{\overline{x}}^{+\langle 2m+1\rangle}\right)$, or:

$$\underline{\overline{x}}^{+\langle 2m+1\rangle} = \mathbf{F}^{-1}\left(_{+}\rho_{2m+1}\right). \qquad (5.9)$$
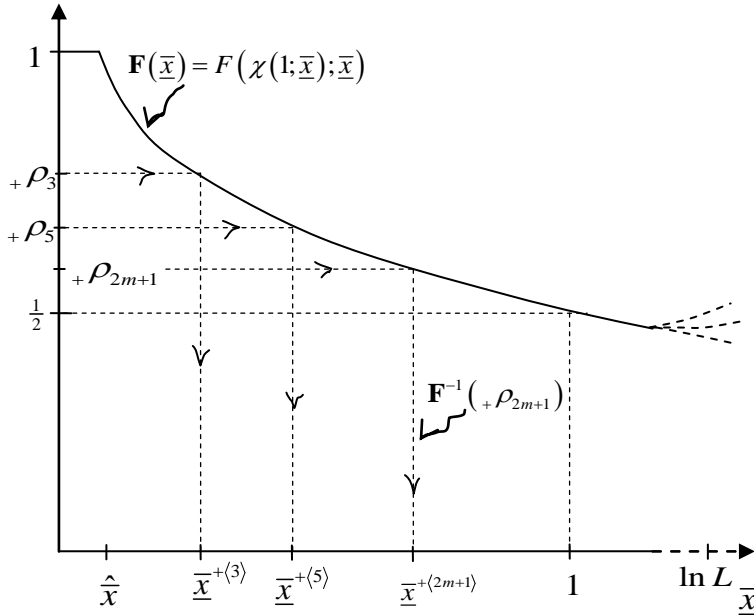

Figure 5.3 A "spanning property"

---

[48]   If $\quad \underline{\overline{x}} \in \left(\hat{\underline{\overline{x}}},1\right), \quad \ln L \le \chi(1;\underline{\overline{x}}) < \ln L + \hat{\underline{\overline{x}}} \quad$ (Proposition 5.1), implying

$F'\left(\chi(1;\underline{\overline{x}});\underline{\overline{x}}\right) = 1/2\underline{\overline{x}}$ and $\partial F\left(\chi(1;\underline{\overline{x}});\underline{\overline{x}}\right)/\partial\underline{\overline{x}} < 0$ (equation (5.6)). $\chi_{\underline{\overline{x}}}(1;\underline{\overline{x}}) < 0$ from

Proposition 5.2.ii. By the same argument, at $\underline{\overline{x}} = 1$, $\chi(1,1) = \ln L$, $F'\left(\chi(1,1);1\right) = \frac{1}{2}$,

$\partial F\left(\chi(1;1);1\right)/\partial\underline{\overline{x}} = 0$, and $\chi_{\underline{\overline{x}}}(1,1) = -\frac{1}{3}$ (Proposition 5.2.iii). Accordingly $\mathbf{F}'(1) = -\frac{1}{6}$.

By continuity of $\mathbf{F}'(\cdot)$, there exists an interval to the right of $\underline{\overline{x}} = 1$ such that $\mathbf{F}'(\underline{\overline{x}}) < 0$

(ε may be a small number).

Proposition 5.5 (sufficient conditions). For each $\underline{x} \in (0,1)$ and each $m \geq 1$, a unique overcomplying global solution $\ln L < \chi(2m+1; \underline{x}) < \ln L + \hat{\hat{x}}$, is defined by (5.2) and the following relations hold (ad inf) with $\overline{\underline{x}}^{+\langle 2m+1 \rangle} = \mathbf{F}^{-1}(_{+}\rho_{2m+1})$:

$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$
$m=3$	$\chi(7;\underline{x}) < \cdots < \chi(1;\underline{x})$	$\chi(7;\underline{x}) < \chi(5;\underline{x}) < \begin{cases} \chi(3;\underline{x}) \\ \chi(1;\underline{x}) \end{cases}$	$\chi(7;\underline{x}) < \begin{cases} \chi(5;\underline{x}) \\ \chi(3;\underline{x}) \\ \chi(1;\underline{x}) \end{cases}$	
$m=2$	$\chi(5;\underline{x}) < \chi(3;\underline{x}) < \chi(1;\underline{x})$	$\chi(5;\underline{x}) < \begin{cases} \chi(3;\underline{x}) \\ \chi(1;\underline{x}) \end{cases}$		
$m=1$	$\chi(3;\underline{x}) < \chi(1;\underline{x})$			

| $\underline{x}$ | $\underline{x} \in \left(0, \overline{\underline{x}}^{+\langle 3 \rangle}\right]$ | $\underline{x} \in \left(\overline{\underline{x}}^{+\langle 3 \rangle}, \overline{\underline{x}}^{+\langle 5 \rangle}\right]$ | $\underline{x} \in \left(\overline{\underline{x}}^{+\langle 5 \rangle}, \overline{\underline{x}}^{+\langle 7 \rangle}\right]$ | $\cdots \uparrow$ |

Remark 5.1 If $\underline{x} \in \left(0, \hat{\hat{x}}\right]$ ($\hat{\hat{x}} < \underline{x}^{+\langle 3 \rangle}$), the boundary solution $\chi(1;\underline{x}) = \ln L + \underline{x}$ obtains with $F \equiv 1$. Remark 4.2 applies: While the abstract norm is locally determinate ($\langle F(\chi(1;\underline{x}); \underline{x}), 1 - F(\chi(1;\underline{x}); \underline{x}); 0, L \rangle = \langle 1, 0; 0, L \rangle$), the transformation of marginal incentives at $\underline{x}$ implies that the corner equilibrium is perturbed by any panel size increase. $\square$

Table 5.1 gives values of $\overline{\underline{x}}^{+\langle 2m+1 \rangle}$, based on simulations with $L = e^e$. The equilibria reported in Appendix A.1 corroborate Propositions 4.2 and 5.5, see Figure 5.4. Remark 4.1.A on sufficiency is also reflected: If $\underline{x} \leq 0.47$, all

overcomplying equilibria $\chi(2m+1;\overline{x})$ fall in m towards $\ln L = x^{PE}$. For more dispersed distributions, increasing m initially increases overcompliance. The simulations demonstrate (giving more structure to the stacked variables in Propsitions 4.2 and 5.2) that if:

$\overline{x} \in [0.48, 0.59]$, $\chi(2m+1;\overline{x})$ increases in m from $m=0$ to $m=1$, then falls;

$\overline{x} \in [0.60, 0.69]$, $\chi(2m+1;\overline{x})$ increases in m from $m=0$ to $m=2$, then falls;

$\overline{x} \in [0.70, 0.78]$, $\chi(2m+1;\overline{x})$ increases in m from $m=0$ to $m=3$, then falls;

$\overline{x} \in [0.79, 0.86]$, $\chi(2m+1;\overline{x})$ increases in m from $m=0$ to $m=4$, then falls;

$\overline{x} \in [0.87, 0.93]$, $\chi(2m+1;\overline{x})$ increases in m from $m=0$ to $m=5$, then falls:

$\overline{x} \in [0.94, 0.99]$, $\chi(2m+1;\overline{x})$ increases in m from $m=0$ to $m=6$, then falls.

Table 5.1 Sequence of $\overline{x}^{+\langle 2m+1 \rangle}$-values (calculations based on $L = e^e$)

m	1	2	3	4	5	6	7	8	9
$\overline{x}^{+\langle 2m+1 \rangle}$	0.458	0.480	0.496	0.509	0.521	0.531	0.539	0.547	0.554

Next, consider the "hairline" case of $\overline{x} = 1$ (compare to Propostition 4.3):

Proposition 5.6

If $\overline{x} = 1$, for each $m \geq 1$, a unique *global* solution $\ln L < \chi(2m+1;\overline{x}) < \ln L + \hat{\overline{x}}$ is given by equation (5.2).

Remark 5.2 Because $F(\chi(1;1),1) = \frac{1}{2}$ and $h_{2m+1}(\frac{1}{2}) = \frac{1}{2}$ for all m, the level effect of legal uncertainty is invariant at x^{PE}. The panel effect is activated by the mono- tonically increasing $h_{2m+1}'(\frac{1}{2})$ through the marginal dimension of legal uncertain- ty (see Remark 4.3). □

As illustrated in Figure 5.4, solutions $\chi(2m+1;1)$ increase in m up to and in-

cluding 13-member panels, and then fall.

Dispersed distributions

If $\overline{x} \in (1, \ln L]$, $K_1'(\ln L; \overline{x}) > 0$, and by Propositions 5.1, 5.2, and 5.3, a unique

$\underline{x} < \chi(1; \overline{x}) < \ln L$ results with $\mathbf{K}_1(\overline{x}) < \nabla$, $\chi_{\underline{x}}(1; \overline{x}) < 0$ and $\mathbf{K}_1'(\overline{x}) < 0$.

Due to a possible lack of convexity in $\left(\underline{x}, x^{PE}\right)$, if $m \geq 1$, solution candi-

dates $\ln L + \overline{x} < \chi^{*<}(2m+1; \overline{x}) < \ln L$ might not be unique. To ensure that candi-

dates defined by first-order condition (5.2) do not (locally) maximize cost, it

must be ensured that $K_{2m+1}''\left(\chi^{*<}(2m+1; \overline{x}); \overline{x}\right) \geq 0$. It can be assumed that

$K_{2m+1}''\left(\chi^{*<}(2m+1; \overline{x}); \overline{x}\right) > 0$, ensuring local uniqueness (regularity). Equilibria

are denoted $\chi(2m+1; \overline{x})$.[49]

For fixed $\overline{x}$, the relations in (4.2) imply that the sequence of value func-

tions $\mathbf{K}_{2m+1}(\overline{x})$ increases in m, as long as solutions are undercomplying, and

that solutions eventually switch to overcompliance. However, sufficiently dis-

persed distributions retain under-compliant equilibria for interim increases in

panel size.

From Section 4.2.3, Proposition 4.5, and equation (5.3), convexity of

$K_{2m+1}(\cdot; \underline{x})$ in $\left(a_{2m+1}, \ln L + \overline{x}\right)$ implies undercomplying equilibria

---

[49] The probability of a non-regular equilibrium is ignored due to the multitude of impacts
of $\overline{x}$ on K_{2m+1}'', given by

$$
\begin{aligned}
K_{2m+1}''(\chi; \overline{x})\big|_{\chi = \chi^{*<}(2m+1; \overline{x})} &= Le^{-x}\Big\{1 - h_{2m+1}\left(F(\chi; \overline{x})\right) \\
&+ \tfrac{1}{\underline{x}} h_{2m+1}'\left(F(\chi; \overline{x})\right) - \tfrac{1}{4\underline{x}^2} h_{2m+1}''\left(F(\chi; \overline{x})\right)\Big\}\Big|_{\chi = \chi^{*<}(2m+1; \overline{x})} .
\end{aligned}
$$

(It is conjectured that $\partial\left\{K_{2m+1}''(\chi; \overline{x})\big|_{\chi = \chi^{*<}(2m+1; \overline{x})}\right\}\big/\partial \overline{x}$ is non-zero.)

$\ln L + \underline{x} < \chi(2m+1;\overline{x}) < \ln L$ if $K_{2m+1}{}'\!\left(\ln L;\overline{x}\right) = \frac{1}{2\overline{x}}\{\overline{x} - h_{2m+1}{}'\!\left(\frac{1}{2}\right)\} > 0$. Because

$h_{2m+1}{}'\!\left(\frac{1}{2}\right)$ increases in m (see equation (III.7.4)), it follows:

> Proposition 5.7 (Sufficient conditions for undercompliance). Define:

$$\underline{\overline{x}}^{2\hat{m}+1} = h_{2\hat{m}+1}{}'\!\left(\tfrac{1}{2}\right) = \frac{(2\hat{m}+1)(2\hat{m})!}{2^{2\hat{m}}[m!]^2}.$$

> If $\underline{x} \in \left(\overline{x}^{2\hat{m}+1}, \ln L + \underline{x}\right)$, then $\chi(2m+1;\overline{x}) < \ln L$ for all $m \le \hat{m}$,
>
> $m,\hat{m} \in \{0,1,2,3,4,5\}$.

Table 5.2 and Figure 5.4 report values for $\underline{\overline{x}}^{2\hat{m}+1}$ for a subset of panel sizes (the case $\hat{m}=0$ corresponds to $\chi(1;\underline{x}) \le \ln L$ if $\overline{x} \ge 1$ in Proposition 5.1). For panels larger than or equal to $m=6$ ($n \ge 13$), the interval collapses (the mechanisms only admits overcomplying solutions).

Table 5.2 Parameter values $\underline{\overline{x}}^{2\hat{m}+1}$ implying undercompliance

$\hat{m}$	0	1	2	3	4	5	6	7	8	9
$\underline{\overline{x}}^{2\hat{m}+1}$	1	1.5	1.875	2.188	2.461	2.707	–	–	–	–

Based on the sequence of *increasing* roots $_{-}\rho_{2m+1}$ in $\left(0,\frac{1}{2}\right]$, Proposition 4.4 can, in principle, be used to develop results for families of distributions, analogously to Proposition 5.5. It means defining a sequence of solutions $\overline{x}^{-\langle 2m+1\rangle}$ to:

$$_{-}\rho_{2m+1} = F\!\left(\chi\!\left(1;\underline{\overline{x}}^{-\langle 2m+1\rangle}\right);\underline{\overline{x}}^{-\langle 2m+1\rangle}\right). \tag{5.10}$$

However, as indicated in Figure 5.3, for small and interim values of m, i) solutions may not exist and ii) even if solutions exist, the monotonicity property may be absent. This is due to the fact that for small values of m, $_{-}\rho_{2m+1}$ might not be

164

in the range of $\mathbf{F}$ and that the inverse function $\mathbf{F}^{-1}$ might not exist ($\mathbf{F}'(\overline{x})$ defined in (5.8) is not necessarily negative when $\underline{x}$ is sufficiently larger than 1).[50] The following conditional statement of can be made:

> Proposition 5.8 Assume that $\tilde{m}$ is sufficiently high that a $\overline{x}^{-\langle 2\hat{m}+1\rangle}$ is a solution to (5.10) such that $\mathbf{F}(\overline{x})$ is strictly falling on $\left[1, \overline{x}^{-\langle 2\tilde{m}+1\rangle}\right]$. If
>
> $\overline{x} \in \left(1, \overline{x}^{-\langle 2\tilde{m}+1\rangle}\right]$ $\chi(2\tilde{m}+1;\overline{x}) > \chi(1;\overline{x})$ [51] and $\chi(2\tilde{\tilde{m}}+1;\overline{x}) > \chi(1;\overline{x})$ for
>
> all $\tilde{\tilde{m}} > \tilde{m}$.

From the simulations with $L = e^e$, $\min\left\{\mathbf{F}(\overline{x})\middle| \overline{x} \in [1,e]\right\} = 0.4440$, which is strictly larger than $-\rho_{19} = 0.3193$ (outside the range). In addition, $\mathbf{F}(e) = 0.4445 > 0.4440$, illustrating that the inverse does not exist on the whole of $(1,e]$. The simulations show that by choosing $\underline{x}$ sufficiently close to $\overline{x} = 1$ from below, for m sufficiently large, further increases in m lead solutions to fall off:

$\overline{x} \in [1.01, 1.06]$, $\chi(2m+1;\overline{x})$ increases in m from $m=0$ to $m=7$, then falls;

$\overline{x} \in [1.07, 1.11]$, $\chi(2m+1;\overline{x})$ increases in m from $m=0$ to $m=8$, then falls;

$\overline{x} \in [1.12, e]$, $\chi(2m+1;\overline{x})$ increases in m from $m=0$ to $m=9$.

5.2.2 Comparative statics: legal uncertainty in fixed panels

---

[50] Define $\mathbf{F}_L = F\left(\chi(1;\ln L);\ln L\right) = \mathbf{F}(\ln L) = \frac{\chi(1,\ln L)}{2\ln L} < \frac{1}{2}$ for undercomplying solutions. By continuity $\mathbf{F}$ takes all values between $\frac{1}{2}$ and $\mathbf{F}_L$ on $[1,\ln L]$.

[51] Since $\mathbf{F}'(1) < 0$ and $-\rho_{2m+1}$ converges to $\frac{1}{2}$, a solution $\overline{x}^{-\langle 2\tilde{m}+1\rangle}$ exists.

The increasing sequence of parameter values $\underline{\overline{x}}^{2\hat{m}+1}$ defined in Proposition 5.7 corresponds to finding the substantive norm (distribution) that implements the efficient solution for a given panel size, $\chi\left(2\hat{m}+1;\underline{\overline{x}}^{2\hat{m}+1}\right)=\ln L$. That is,

$$K_{2\hat{m}+1}{}'\left(\ln L;\underline{\overline{x}}^{2\hat{m}+1}\right)=0. \tag{5.11}$$

The left hand side of the equation is given by (5.3). By strict convexity of $K_{2m+1}(\cdot)$ on $\left(a_{2m+1},\overline{x}\right)$, solutions are uniquely given (but may not exist for admissible parameter values for large $\hat{m}$). At the efficient point:

$$\frac{\partial K_{2\hat{m}+1}{}'\left(\ln L;\underline{\overline{x}}^{2\hat{m}+1}\right)}{\partial\underline{\overline{x}}}=\frac{h_{2\hat{m}+1}{}'\left(\frac{1}{2}\right)}{2\left(\underline{\overline{x}}^{2\hat{m}+1}\right)^{2}}>0. \tag{5.12}$$

Hence, from strict convexity of K_{2m+1} on $\left(a_{2m+1},\overline{x}\right)$, for any (admissible) $\hat{m}$,

$\underline{\overline{x}}>\underline{\overline{x}}^{2\hat{m}+1}$ is sufficient for $\chi\left(2\hat{m}+1;\underline{\overline{x}}\right)<\ln L$: $\chi_{\underline{\overline{x}}}\left(2\hat{m}+1;\underline{\overline{x}}^{2\hat{m}+1}\right)<0$. The property is confirmed in the simulations.

The next proposition gives a more general description of equilibrium responses to increased legal uncertainty, as measured by $\underline{\overline{x}}$ (giving efficient point impacts as a special case).

Let $x*(2m+1)=\chi\left(2m+1;\underline{\overline{x}}\right)$ be an isolated solution (possibly not unique if undercomplying).

<u>Proposition 5.9</u> Let $m\geq1$, $S\sim U\left(\ln L-\underline{\overline{x}},\ln L+\underline{\overline{x}}\right)$, $p(x)=e^{-x}$ and legal uncertainty be measured by $\underline{\overline{x}}\in(0,\ln L]$. The sign of the derivative $\chi_{\underline{\overline{x}}}\left(2m+1;\underline{\overline{x}}\right)$ is determined by the sign of:

$$(\S) \quad \left\{ \underbrace{h_{2m+1}{}'\big(F\left(\chi;\overline{\underline{x}}\right)\big)\big(\chi-(\ln L+1)\big)}_{c<0}+\underbrace{\frac{1}{2\overline{\underline{x}}}h_{2m+1}{}''\big(F\left(\chi;\overline{\underline{x}}\right)\big)(\ln L-\chi)}_{d\geq 0}\right\}\Bigg|_{\chi=\chi(2m+1;\overline{\underline{x}})} \quad ,$$

- *strictly positive* for sufficiently concentrated distributions, $\overline{\underline{x}}\in\left(0,\overline{\underline{x}}_{2m+1}\right)$ for all m (the set of admissible parameter values smaller the smaller m); and

- *strictly negative* around efficient equilibria.

<u>Proof.</u> Solutions are interior for all $m\geq 1$ (Section 4). First-order condition (5.1) is an identity in $\overline{\underline{x}}$. By the Implicit function theorem[52]

$$\chi_{\overline{\underline{x}}}\left(2m+1;\overline{\underline{x}}\right)=-\frac{\left.\dfrac{\partial K_{2m+1}{}'\left(\chi;\overline{\underline{x}}\right)}{\partial\overline{\underline{x}}}\right|_{\chi=\chi(2m+1;\overline{\underline{x}})\,\text{const.}}}{K_{2m+1}{}''\left(\chi\left(2m+1;\overline{\underline{x}}\right);\overline{\underline{x}}\right)}\ . \tag{5.13}$$

$\chi\left(2m+1;\overline{\underline{x}}\right)=\chi$ is constant under differentiation. By the necessary convexity of the minimand at critical points, $\left\{\partial K_{2m+1}{}'\left(\chi;\overline{\underline{x}}\right)/\partial\overline{\underline{x}}\right\}\big|_{\chi=\chi(2m+1;\overline{\underline{x}})\,\text{const}}>0$ implies $\chi_{\overline{\underline{x}}}\left(2m+1;\overline{\underline{x}}\right)<0$, and $\left\{\partial K_{2m+1}{}'\left(\chi;\overline{\underline{x}}\right)/\partial\overline{\underline{x}}\right\}\big|_{\chi=\chi(2m+1;\overline{\underline{x}})\,\text{const}}<0$ implies $\chi_{\overline{\underline{x}}}\left(2m+1;\overline{\underline{x}}\right)>0$.

Holding $\chi\left(2m+1;\overline{\underline{x}}\right)$ constant under differentiation of (5.1) and rearranging:

$$\frac{\partial K_{2m+1}{}'\left(\chi;\overline{\underline{x}}\right)}{\partial\overline{\underline{x}}}=e^{-\chi}L\left\{\underbrace{h_{2m+1}{}'\big(F\left(\chi;\overline{\underline{x}}\right)\big)\frac{\partial F\left(\chi;\overline{\underline{x}}\right)}{\partial\overline{\underline{x}}}}_{A}\right.$$
$$\left.-\left\langle\underbrace{h_{2m+1}{}''\big(F\left(\chi;\overline{\underline{x}}\right)\big)\frac{\partial F\left(\chi;\overline{\underline{x}}\right)}{\partial\overline{\underline{x}}}F'\left(\chi;\overline{\underline{x}}\right)+h_{2m+1}{}'\big(F\left(\chi;\overline{\underline{x}}\right)\big)\frac{\partial F'\left(\chi;\overline{\underline{x}}\right)}{\partial\overline{\underline{x}}}}_{B}\right\rangle\right\} \tag{5.14}$$

---

[52] The parameter $\overline{\underline{x}}$ enters directly in the $K_{2m+1}{}'\left(\cdot;\overline{\underline{x}}\right)$-function and through the equilibria $\chi\left(2m+1;\overline{\underline{x}}\right)$. $K_{2m+1}{}''\left(\chi\left(2m+1;\overline{\underline{x}}\right);\overline{\underline{x}}\right)>0$ is sufficient for isolated equilibria and for C^{1}-functions $\chi\left(2m+1;\overline{\underline{x}}\right)$ to exist locally (Sydsæter [1981], Theorem 3.3).

Component *A* measures the *direct* (χ is fixed) impact of increased legal uncertainty on its *level dimension*, as transformed by the collective. Component *B* measures the *direct* impact of increased legal uncertainty on its *marginal dimension*, as transformed by the collective. A positive *B* would imply a negative effect on the sign in (5.14) contribution to a positive derivative $\chi_{\bar{x}}$ (the marginal dimension stimulating investment), whereas a positive *A* would contribute to a positive sign in (5.14), contributing to a negative derivative $\chi_{\bar{x}}$ (the level dimension dampening investment incentives).

Because solutions are interior, $h_{2m+1}' > 0$. From (5.6) the substantive norm direct effect $\dfrac{\partial F(\chi;\bar{x})}{\partial \bar{x}} = \dfrac{\ln L - \chi}{2\bar{x}^2}$, implying that *A* is positive at overcomplying equilibria and negavtive at undercomplying equilibria.

Substitutions from (5.6) and $F'(\chi;\bar{x}) = \frac{1}{2\bar{x}}$ give $B = h_{2m+1}''(\cdot)\frac{\ln L - \chi}{2\bar{x}^2}\frac{1}{2\bar{x}} - h_{2m+1}'(\cdot)\frac{1}{2\bar{x}^2}$. Due to convexity of h_{2m+1} at undercomplying equilibria and concavity at overcomplying equilibria, *B* is indeterminate. However, $A - B$ has the same sign as $h_{2m+1}'\{[\ln L + 1] - \chi\} - \frac{1}{2\bar{x}}h_{2m+1}''\{\ln L - \chi\}$. The sign of $\chi_{\bar{x}}(2m+1;\bar{x})$ is, therefore, given by:

$$(\S) \quad \underbrace{h_{2m+1}'\{\chi - [\ln L + 1]\}}_{c<0} + \underbrace{\tfrac{1}{2\bar{x}}h_{2m+1}''\{\ln L - \chi\}}_{d\geq 0} .$$

Because $\chi(2m+1;\bar{x}) < \nabla = \ln L + \frac{1}{2}$ (Section 4.2 and Proposition 5.1 proof), $c < 0$. $d > 0$ at all points different from the efficient one follows from $h_{2m+1}(\cdot)$ strictly convex at undercomplying equilibria and strictly concave at overcomplying equilibria.

To prove that ($\S$) becomes positive if $\bar{x}$ is sufficiently small, from equation (III.7.3), if $\chi \in (\underline{x}, \bar{x})$:

$$h_{2m+1}"\left(F\left(\chi;\overline{x}\right)\right) = m\left\{k_m\left[F\left(\chi;\overline{x}\right)\left(1-F\left(\chi;\overline{x}\right)\right)\right]^m\right\}\frac{1-2F\left(\chi;\overline{x}\right)}{F\left(\chi;\overline{x}\right)\left(1-F\left(\chi;\overline{x}\right)\right)}$$

$$= mh_{2m+1}'\left(F\left(\chi;\overline{x}\right)\right)\frac{1-2F\left(\chi;\overline{x}\right)}{F\left(\chi;\overline{x}\right)\left(1-F\left(\chi;\overline{x}\right)\right)}. \qquad (5.15)$$

Let $\beta\left(\chi;\overline{x}\right) = \dfrac{1-2F\left(\chi;\overline{x}\right)}{F\left(\chi;\overline{x}\right)\left(1-F\left(\chi;\overline{x}\right)\right)}$. Because $h_{2m+1}' > 0$ on $\left(\underline{x},\overline{x}\right)$, it follows that (§) is

positive if:

$$(\S') \quad \underbrace{\left(\chi-\left[\ln L+1\right]\right)}_{c'} + m\underbrace{\frac{\beta\left(\chi;\overline{x}\right)}{2\overline{x}}\left(\ln L - \chi\right)}_{d"} > 0 ,$$

$\chi = \chi\left(2m+1;\overline{x}\right)$. Because χ is fixed in the Implicit function theorem formula (5.4), the

sign of the sum in (§') can be determined if $\overline{x}$ becomes small: From Proposition 5.5

$\chi\left(2m+1;\overline{x}\right) > \ln L$ if $\overline{x} < 1$. In $d"$, $2\overline{x} \to 0^+$. For a fixed $\chi^0 > \ln L$, $\lim\limits_{\overline{x}\to 0^+} F\left(\chi^0,\overline{x}\right) = 1^-$

(see Figure 5.2.a). If $F \to 1^-$, $\beta\left(\chi^0;\overline{x}\right) \to -\infty$ ($\beta(\cdot)$ has a vertical asymptote at 1).

Accordingly, $d"$ must become strictly larger than c', if the distribution becomes suffi-

ciently concentrated. Also, an increase in m directly boosts $d"$, relaxing the required

constraint on $\overline{x}$. ◄

Remark 5.3 From Proposition 5.5, equilibria are overcomplying for all m if $\overline{x} < 1$.

Proposition 5.9 parallels the sharper $m = 0$ result in Proposition 5.2.i. on linear

growth in precaution investments as a function of $\overline{x}$ in sufficiently concentrated

distributions ($\overline{x} \in \left(0,\hat{\underline{x}}\right]$). The simulations have solutions with a uni-modal char-

acter; growing monotonically, then falling monotonically, with:

$x\left(3;\overline{x}\right)$ increasing in $\overline{x}$ up to and including 0,48;

$x\left(5;\overline{x}\right)$ increasing in $\overline{x}$ up to and including 0,59;

$x\left(7;\overline{x}\right)$ increasing in $\overline{x}$ up to and including 0,70;

$x(9;\overline{x})$ increasing in $\underline{x}$ up to and including 0,78;

$x(11;\overline{x})$ increasing in $\underline{x}$ up to and including 0,85;

$x(13;\overline{x})$ increasing in $\underline{x}$ up to and including 0,92;*

$x(15;\overline{x})$ increasing in $\underline{x}$ up to and including 1,00*;

$x(17;\overline{x})$ increasing in $\underline{x}$ up to and including 1.05*; and

$x(19;\overline{x})$ increasing in $\underline{x}$ up to and including 1,12*.

In the *marked situations, efficient (and undercomplying) solutions are not obtained, although solutions eventually fall in $\underline{x}$ (see in and at Table 5.2). □

Remark 5.4 The impact of increased legal uncertainty is negative around efficient equilibrium points ($d=0$). The increasing sequence of distributions $\overline{x}^{2\hat{m}+1}$ identify such solutions. However, for sufficiently large panels, there might not exist admissible distributions admitting efficient points. These simulations illustrate that this is the case, for $m \geq 6$ (Table 5.2). Because an efficient solution is sufficient, not necessary, for $\chi_{\underline{x}}(2m+1;\underline{x})<0$, there might still—for interim values of m—exist classes of distributions that imply falling precaution investments, in the face of increasing legal uncertainty (in equilibrium, there's a complex interaction between the components in (§')). In the simulations, all panels admit distributions characterized by $\chi_{\underline{x}}(2m+1;\overline{x})<0$, even though there are no admissible distributions (no $\underline{x}^{2\hat{m}+1} \in (0,e]$) implementing the efficient solution, if $m \geq 6$. □

Consider the effect of increased legal uncertainty for the potential injurer. In spite of the complexities in signing the impact of legal uncertainty on precaution lev-

els, the derivation of the next proposition is immediate. Define the value function:

$$\mathbf{K}_{2m+1}\left(\overline{\underline{x}}\right) = K_{2m+1}\left(\chi\left(2m+1;\overline{\underline{x}}\right);\overline{\underline{x}}\right)$$

$$= \chi\left(2m+1;\overline{\underline{x}}\right) + \left[1 - h_{2m+1}\left(F\left(\chi\left(2m+1;\overline{\underline{x}}\right);\overline{\underline{x}}\right)\right)\right]e^{-\chi(2m+1;\overline{\underline{x}})}L^{\cdot}$$

<u>Proposition 5.10</u> Let $S \sim U\left(\ln L - \overline{x}, \ln L + \overline{x}\right)$, $p(x) = e^{-x}$ and legal uncertainty be measured by $\overline{x} \in \left(0, \ln L\right]$.

i) From overcomplying solutions potential injurers' expected cost increases in legal uncertainty, $\mathbf{K}_{2m+1}{}'\left(\overline{\underline{x}}\right)\big|_{\chi(2m+1;\overline{\underline{x}})>\ln L} > 0$.

ii) From undercomplying solutions potential injurers' expected cost falls in legal uncertainty, $\mathbf{K}_{2m+1}{}'\left(\overline{\underline{x}}\right)\big|_{\chi(2m+1;\overline{\underline{x}})<\ln L} < 0$.

<u>Remark 5.5</u> For sufficiently concentrated distributions $\overline{x} \in \left(0,1\right)$, overcompliance results for all m (Proposition 5.5), and increases in legal uncertainty is costly for potential injurers.

Sufficiency conditions on parameter values for $\mathbf{K}_{2m+1}{}'\left(\overline{\underline{x}}\right) < 0$ are given in Proposition 5.7. (The case of $m = 0$ in Proposition 5.3 allows sharper result due to the one-to-one relation between compliance and parameter values.) □

<u>Remark 5.6</u> The efficient point $\ln L$ is a fixed point for minimands $K_{2m+1}\left(\ln L; \overline{\underline{x}}\right) = \nabla$ for all m and $\overline{\underline{x}}$. It follows that $K_{2\hat{m}+1}\left(\chi\left(2\hat{m}+1;\underline{\overline{x}}^{2\hat{m}+1}\right);\underline{\overline{x}}^{2\hat{m}+1}\right) \equiv \nabla$ and that $\mathbf{K}_{2\hat{m}+1}{}'\left(\overline{\underline{x}}^{2\hat{m}+1}\right) = 0$. □

<u>Proof.</u> By the Envelope theorem:[53]

---

[53] Carter [2001], Theorem 6.2.

$$\mathbf{K}_{2m+1}'(\overline{x}) = \frac{\partial K_{2m+1}(\chi;\overline{x})}{\partial \overline{x}}\Bigg|_{\chi = \chi(2m+1;\overline{x}) \text{ const}}$$

$$= -h_{2m+1}'\left(F\left(\chi(2m+1;\overline{x});\overline{x}\right)\right)\frac{\partial F(\chi;\overline{x})}{\partial \overline{x}}\Bigg|_{\chi = \chi(2m+1;\overline{x}) \text{ const}} e^{-\chi(2m+1;\overline{x})}L$$

Because $\chi(2m+1,\overline{x}) \in (\underline{x},\overline{x})$ if $m \geq 1$,

$$\frac{\partial F(\chi;\overline{x})}{\partial \overline{x}}\Bigg|_{\chi = (2m+1;\overline{x}) \text{ const.}} = \frac{\ln L - \chi(2m+1;\overline{x})}{2\overline{x}^2},$$

see the proof of Proposition 5.2. Accordingly,

$$\begin{cases} \mathbf{K}_{2m+1}'(\overline{x}) > 0 \text{ if } \chi(2m+1;\overline{x}) > \ln L \\ \mathbf{K}_{2m+1}'(\overline{x}) = 0 \text{ if } \chi(2m+1;\overline{x}) = \ln L. \blacktriangleleft \\ \mathbf{K}_{2m+1}'(\overline{x}) < 0 \text{ if } \chi(2m+1;\overline{x}) < \ln L \end{cases}$$

5.2.3 Discussion

The analyses show that in panels of fixed size, sufficiently concentrated distributions induce overcomplying solutions, whereas dispersed distributions induce undercomplying solutions, if the panel is not too large (smaller than $n = 11$ in the simulations). As in the single judge case, overcompliance increases in legal uncertainty (if not linearly) from sufficiently concentrated initial distributions, but eventually falls in growing uncertainty, to and below the efficient point (if the panel size m is not too large).

Figure 5.4 summarizes the effects of panel size variations qualitatively, reflecting the impact of level and marginal aspects of legal uncertainty and its transformation in multi-member panels (and the interaction with the accident technology). Consistent with the general observations in Section 4.3, for sufficiently concentrated distributions, the level and marginal effects push towards an efficient solution. However, as the abstract norm equilibrium lies closer to the efficient point from above, equilibria increase in interim panel size increases (reflecting a dominating marginal dimension effect), but eventually the level

172

effect dominates (in the simulations, for all $\underline{x} < 1$, equilibria fall in m for panels larger than $m = 6$). All undercomplying equilibria grow for initial increases in panel size. All abstract equilibria become overcomplying for $m \geq 6$, and equilibria induced by more concentrated distribution by smaller panels. The latter equilibriua, while growing beyond the efficient point as panels become large, start converging to the efficient point for sizes above $m = 6$. The most dispersed distributions induce equilibria that grow monotonically for all $m \leq 9$, meaning that the marginal dimension of legal uncertainty dominates.
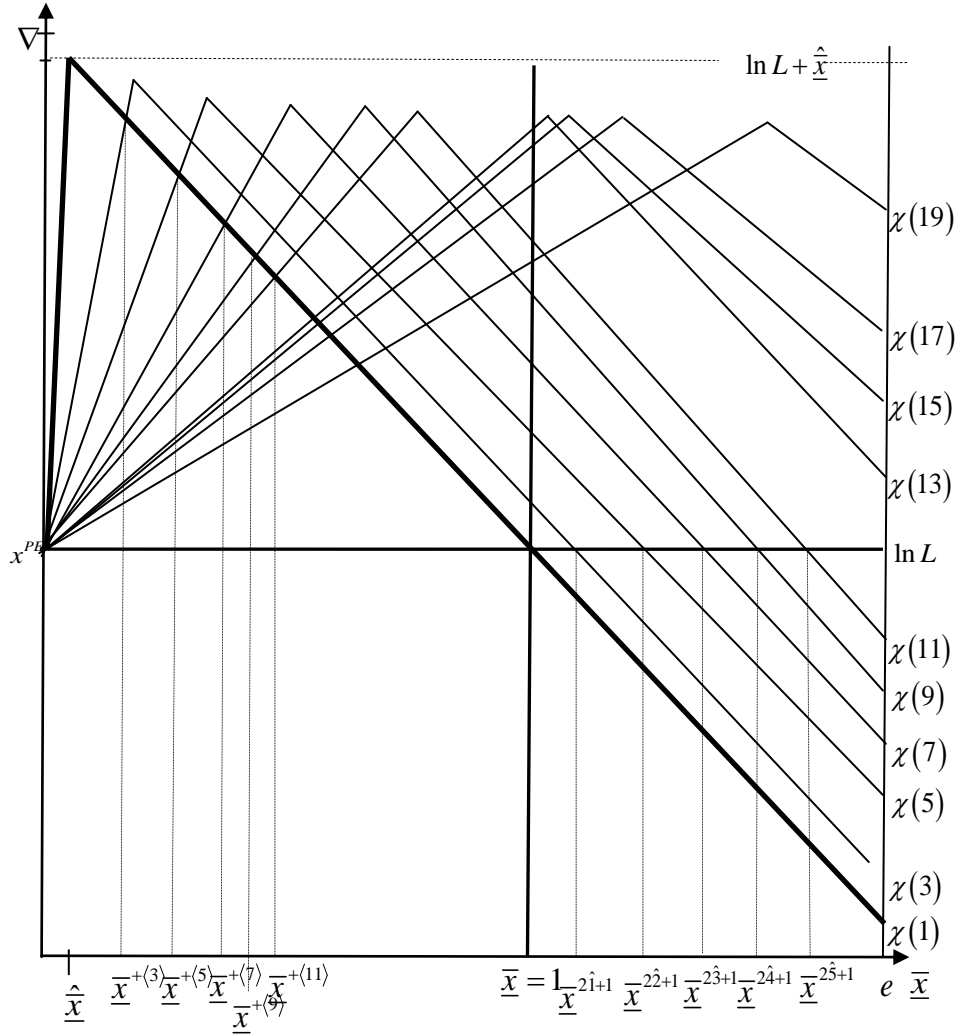
Figure 5.4 Equilibrium structure under uniform distributions, $m \leq 9$

6   Asymptotic approximation

Section 4.1 demonstrates convergence of equilibria from above in large panels, effectively eliminating the uncertainty in the legal standard $g_{F,2m+1} \in \mathbb{P}_{\{0,L\}}{}^{X}$. Sections 4 and 5 analyze equilibrium effects in small, finite panels. Considering large, but finite panels, by Proposition III.6.2 (requiring only that the legal standard cdf $F(t)$ continuous around the median $F^{-1}\left(\frac{1}{2}\right) = x^{PE}$ and $f\left(x^{PE}\right) > 0$) the transformed norm cdf in M_{2m+1}, $F_{m+1:2m+1}(t)$ is approximately from:

$$N\left(x^{PE}, \left\{ 4(2m+1)\left[f\left(x^{PE}\right)\right]^2 \right\}^{-1} \right).$$

In case of a uniform distribution $U\left(\ln L - \bar{x}, \ln L + \bar{x}\right)$, $F_{m+1:2m+1}(t; \bar{x})$ is approximately from (see Example III.6.2):

$$N\left(\ln L, \frac{\bar{x}^2}{2m+1} \right).$$

It follows that the single judge simulation results in Crasswell & Calfee [1986], with $\mu = x^{PE}$, can be applied directly to these larger mechanisms.

EQUILIBRUM UNDER (TRANSFORMED) MIXED NORMS

1   Introduction

Sections IV.2 through IV.6 implicitly assume (locally) determinate meta-norm ($ls \in \eta^{-1}\left(\delta\mathbb{P}_{\mathbb{P}_Y{}^X}\right)$): judges are identified with a unique ordinary norm in a single-element set, $\operatorname{supp}\eta(\cdot\,|\,ls) = \{g_F\}$. From a *functional* perspective, reference to meta-norms is superfluous. This part assumes a fixed meta-norm $\eta \in \mathbb{P}_{\mathbb{P}_{\{0,L\}}{}^X}{}^{LS}$ and

$ls \in LS \setminus \eta^{-1}\left(\delta\mathbb{P}_{\mathbb{P}_{\{0,L\}}{}^X}\right)$, that is, a non-degenerate distribution over ordinary liability norms. Hence, (explicitly) *mixed norms* are analyzed.

In the case of (probabilistic) externalities, entitlements can be protected by a variety of regimes (Remark II.5.1). This part considers mixes of strict liability with negligence standards, $g_{\delta\infty}, g_F \in \mathbb{P}_{\{0,L\}}{}^X$ as introduced in Example II.3.3. Mixing weights are given by $\eta(g_{\delta\infty}\,|\,ls) = \lambda$ and $\eta(g_F\,|\,ls) = 1 - \lambda$, $\lambda \in (0,1)$. The resulting abstract mixed norms $\lambda g_{\delta\infty} \Lambda (1-\lambda) g_F \in \mathbb{P}_{\{0,L\}}{}^X$ are denoted $g_{\lambda g_{\delta\infty} \Lambda (1-\lambda) g_F}$ and are given by:

$$g_{\lambda g_{\delta\infty} \Lambda (1-\lambda) g_F}\left(\cdot\,|\,x\right)\Big|_{x \in X} = \left\langle [1-\lambda]F(x), 1 - [1-\lambda]F(x); 0, L \right\rangle\Big|_{x \in X}, \qquad (1.1)$$

see Example II.5.1. Transformations of $g_{\lambda g_{\delta\infty} \Lambda (1-\lambda) g_F}$ in M_{2m+1} lead to:

$$g_{\lambda g_{\delta\infty}\wedge(1-\lambda)g_F,2m+1}\left(\cdot\mid x\right)\Big|_{x\in X} = \left\langle h_{2m+1}\left(\left[1-\lambda\right]F\left(x\right)\right),h_{2m+1}\left(1-\left[1-\lambda\right]F\left(x\right)\right);0,L\right\rangle\Big|_{x\in X},$$

see Proposition III.2.1. Using Proposition III.2.3B identities, the norms are conveniently represented by:

$$g_{\lambda g_{\delta\infty}\wedge(1-\lambda)g_F,2m+1}\left(\cdot\mid x\right)\Big|_{x\in X} =$$
$$\left\langle h_{2m+1}\left(\left[1-\lambda\right]F\left(x\right)\right),1-h_{2m+1}\left(\left[1-\lambda\right]F\left(x\right)\right);0,L\right\rangle\Big|_{x\in X}. \tag{1.2}$$

Use of mechanism M_1 (in absence of factual uncertainty) corresponds to the identity function $id_{\mathbb{P}_{\{0,L\}}{}^X}\left(g\right)=g$ for all $g\in\mathbb{P}_{\{0,L\}}{}^X$. Hence, abstract mixed norms $g_{\lambda g_{\delta\infty}\wedge(1-\lambda)g_F}$ and transformed mixed norms $M_1\left(g_{\lambda g_{\delta\infty}\wedge(1-\lambda)g_F}\right)$ need not be distinguished.

This part is organized as follows: Section 2 analyzes global rules. Strict liability $g_{\delta\infty}\in\delta\mathbb{P}_{\{0,L\}}{}^X$ is mixed with global negligence rules, $g_{\delta c}\in\delta\mathbb{P}_{\{0,L\}}{}^X$, $c\in\mathbb{R}_{++}$. The resulting class of global rules, $g_{\lambda g_s\wedge(1-\lambda)g_{\delta c}}\in\delta\mathbb{P}_{\{0,L\}}{}^X$, exemplifies situations were all *legal* uncertainty derives from the meta-level. In Section 3, strict liability is mixed with negligence standards $g_F\in\mathbb{P}_{\{0,L\}}{}^X\setminus\delta\mathbb{P}_{\{0,L\}}{}^X$. To utilize the Section IV.5 analysis, that section's accident technology and legal standard assumptions are maintained. In this case, legal uncertainty derives from both meta-level norms and ordinary-level norms, allowing richer comparative statics. Section 4 considers mixed norms transformed in multi-member courts. Due to the difficulties encountered in Part IV, the analysis is limited global rules (under general assumptions about the accident technology). Therefore, all legal uncertainty derives from the level of methodology.

2   Mixed norms transformed in M_1: strict liability and negligence rules

Under the mixed norm $\lambda g_{\delta\infty} \Lambda (1-\lambda) g_{\delta c} \in \mathbb{P}_{\{0,L\}}{}^X$, the minimand is given by:

$$x + \left\{ 1 - [1-\lambda] F_{\delta c}(x) \right\} p(x) L, \qquad (2.1)$$

with $F_{\delta c}(x) = 0$ if $x < c$ and $F_{\delta c}(x) = 1$ if $x \geq c$. Potential injurers choose:

$$x^*\left(\lambda g_{\delta\infty} \Lambda (1-\lambda) g_{\delta c}\right) = \underset{x \in X}{\arg\min} \begin{cases} x + p(x)L & \text{if } x < c \\ x + \lambda p(x)L & \text{if } x \geq c \end{cases} \qquad (2.2)$$

To describe the equilibrium, it is convenient to define a family of functions $\phi_\lambda : X \to R_{++}$, $\lambda \in (0,1)$, by:

$$\phi_\lambda(x) = x + \lambda p(x) L. \qquad (2.3)$$

For all $x \in X$ and $\lambda \in (0,1)$:

- $x = \phi_0(x) < \phi_\lambda(x) < \phi_1(x) = k(x)$;

- ϕ_λ is strictly convex[1];

- $\bar{\lambda} > \underline{\lambda} \implies \phi_{\bar{\lambda}}(x) > \phi_{\underline{\lambda}}(x)$, $\phi_{\bar{\lambda}}'(x) < \phi_{\underline{\lambda}}'(x)$, $\phi_1'(x) < \phi_\lambda'(x) < \phi_0'(x) = 1$.

The functions are illustrated in Figures 2.1 and 2.2. Program (2.2) may be formulated as:

$$x^*\left(\lambda g_{\delta\infty} \Lambda (1-\lambda) g_{\delta c}\right) = \underset{x \in X}{\arg\min} \begin{cases} \phi_1(x) & \text{if } x < c \\ \phi_\lambda(x) & \text{if } x \geq c \end{cases}. \qquad (2.4)$$

As a benchmark, under a pure strict liability regime ($\lambda = 1$), the minimand in (2.4) is continuous and strictly convex, $\phi_1(x) = k(x)$, implementing the efficient solution, x^{PE}. Under the maintained assumption $k'(0) = \phi_1'(0) < 0$, that is,

---

[1] Ad convexity: $\phi_\lambda''(x) = \lambda p''(x) L > 0$.

$-p'(0)L>1$, the efficient point is interior with $\phi_1'\left(x^{PE}\right)=0$, see Figures IV.4.1

and V.2.2. The value of the minimand at x^{PE}, is denoted $\tilde{x}=k\left(x^{PE}\right)$.

Under a pure negligence regime ($\lambda=0$), the minimand is discontinuous

at $x=c$, shifting from ϕ_1 to ϕ_0, and the equilibrium given by the solution corre-

spondence:

$$x*(1;c)=\begin{cases}\{c\} & \text{if } 0<c<\tilde{x}\\ \{x^{PE},\tilde{x}\} & \text{if } c=\tilde{x}, \\ \{x^{PE}\} & \text{if } c>\tilde{x}\end{cases}$$

see Figure 2.1 and Section IV.4.

2.1 The structure of solutions

Define:

$$\tilde{x}^{\lambda} : \phi_{\lambda}\left(\tilde{x}^{\lambda}\right)=k\left(x^{PE}\right). \tag{2.5}$$

A solution $\tilde{x}^{\lambda}\in\left(x^{PE},\tilde{x}\right)$ to (2.5) exists for all $\lambda\in(0,1)$, satisfying $\tilde{x}^{\lambda}\downarrow x^{PE}$ as

$\lambda\uparrow 1$ and $\tilde{x}^{\lambda}\uparrow\tilde{x}$ as $\lambda\downarrow 0$, see Figures 2.1 and 2.2.[2]

"Low" probability of strict liability

If $\phi_{\lambda}'(0)\geq 0$, by strict convexity, $\phi_{\lambda}'(x)>0$ for all $x>0$. It implies that for all for

all $c\in\left(0,\tilde{x}^{\lambda}\right)$, $x*=c$, see Figure 2.1: Agents comply with the (quite probable)

---

[2] More precisely, fix $\lambda^{0}\in(0,1)$. By the the Implicit function theorem, around $\tilde{x}^{\lambda^{0}}$ equation (2.5) is reduced to an identity in λ and $\tilde{x}^{\lambda}{}'(\lambda^{0})=-p\left(\tilde{x}^{\lambda^{0}}\right)L\left\{1+\lambda^{0}p'\left(\tilde{x}^{\lambda^{0}}\right)\right\}^{-1}$
$=-p\left(\tilde{x}^{\lambda^{0}}\right)L\left\{g_{\lambda^{0}}{}'\left(\tilde{x}^{\lambda^{0}}\right)\right\}^{-1}<0$ (Sydsæter [1981], Theorem 3.3). (The derivative is strictly negative because $\phi_{\lambda}'(\tilde{x}^{\lambda})>0$, following from $\phi_1'(x^{PE})=0$, $\phi_{\lambda}'(x^{PE})>\phi_1'(x^{PE})$, $\tilde{x}^{\lambda}>x^{PE}$ and strict convexity of ϕ_{λ}.)

negligence rule. Due to the low value of λ, it does not pay off to insure against strict liability by further precaution. When $c \in (0, \tilde{x}^\lambda)$, precaution investments rise one-to-one in the required care level. In the "hairline" case of $c = \tilde{x}^\lambda > x^{PE}$, agents are indifferent between $x^* = x^{PE}$ and $x^* = \tilde{x}^\lambda$. If $c > \tilde{x}^\lambda$, meeting the negligence requirement is so costly, that it is better for agents not to satisfy the standard and be liable for adverse outcomes. The care level jumps down and the accident probability jumps up the efficient, but higher level, the potential victims, however, now fully insured. The solution tracks pure negligence regimes, except for $\tilde{x}^\lambda < \tilde{x}$ and partial insurance when $c \in (0, \tilde{x}^\lambda)$.

The condition $\phi_\lambda'(0) \geq 0$ is equivalent to:

$$\lambda \leq \frac{1}{-p'(0)L} < 1, \tag{2.6}$$

the strict inequality by the condition which ensures that it is efficient to commit resources to precaution.
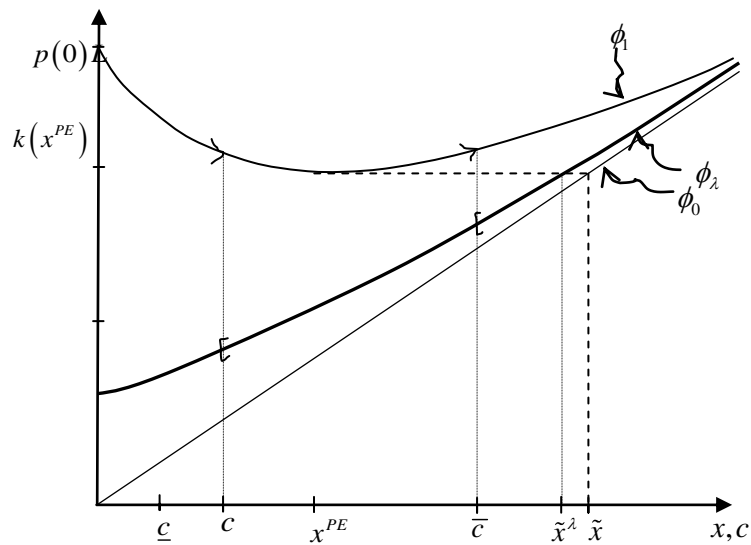
Figure 2.1 Incentive structures, low probability of strict liability

"High" probability of strict liability

If $\lambda > 1/-p'(0)L$, $\phi_\lambda '(0) < 0$, see Figure 2.2. Define the critical point:

$$x_{\min}^\lambda : \ \phi_\lambda '\left(x_{\min}^\lambda\right) = 0. \tag{2.7}$$

Because $\phi_1 '(x^{PE}) = 0$, $\phi_\lambda ' > \phi_1 '$ for all $x \in X$, $\phi_\lambda '(x^{PE}) > 0$. Because ϕ_λ is strict-

ly convex and $\phi_\lambda '(0) < 0$, an interior solution $0 < x_{\min}^\lambda < x^{PE}$ to (2.7) exists for all

$\lambda \in \left(\frac{1}{-p'(0)L}, 1\right)$, and it is a unique global minimum point of $\phi_\lambda (x)$.[3] Hence, if

$c \in \left(0, x_{\min}^\lambda\right]$, $x^* = x_{\min}^\lambda$ (as exemplified by $\underline{c}$ in Figure 2.2): due to a "high" risk

of strict liability, agents overcomply relative to the pure negligence requirement.

The required λ is less stringent the higher the cost of an adverse outcome, L. If

$c \in \left(x_{\min}^\lambda, \tilde{x}^\lambda\right)$, because $\phi_\lambda (x)$ is strictly convex and $\phi_\lambda '(c) > 0$, $x^* = c$ (as ex-

emplified by $\bar{c}$): At costly negligence levels, agents satisfy the requirement, but

do not insure against the risk of strict liability. The solution is as discussed

above: In the "hairline" case $c = \tilde{x}^\lambda$, by (2.5), agents are indifferent between

$x^* = x^{PE}$ and $x^* = \tilde{x}^\lambda$. If $c > \tilde{x}^\lambda$, they choose not to meet the negligence re-

quirement and implement the solution corresponding to strict liability regime,

$x^* = x^{PE}$.

By the Implicit function theorem,[4] (2.7) is an identity in λ around

$\lambda \in \left(\frac{1}{-p'(0)L}, 1\right)$, with:

$$x_{\min}^\lambda '(\lambda) = -p'\left(x_{\min}^\lambda (\lambda)\right) \Big/ \lambda p''\left(x_{\min}^\lambda (\lambda)\right) > 0. \tag{2.8}$$

---

[3] Darboux's intermediate value theorem (Bartle & Sherbert [1982], Theorem 5.2.12).

[4] Sydsæter [1981], Theorem 3.3.

It follows that with a higher risk of strict liability, the set of negligence rules relative to which the agent will overinvest in precaution ($c \in \left(0, x_{\min}^{\lambda}\right]$) increases, as do precaution investments ($x* = x_{\min}^{\lambda}$).
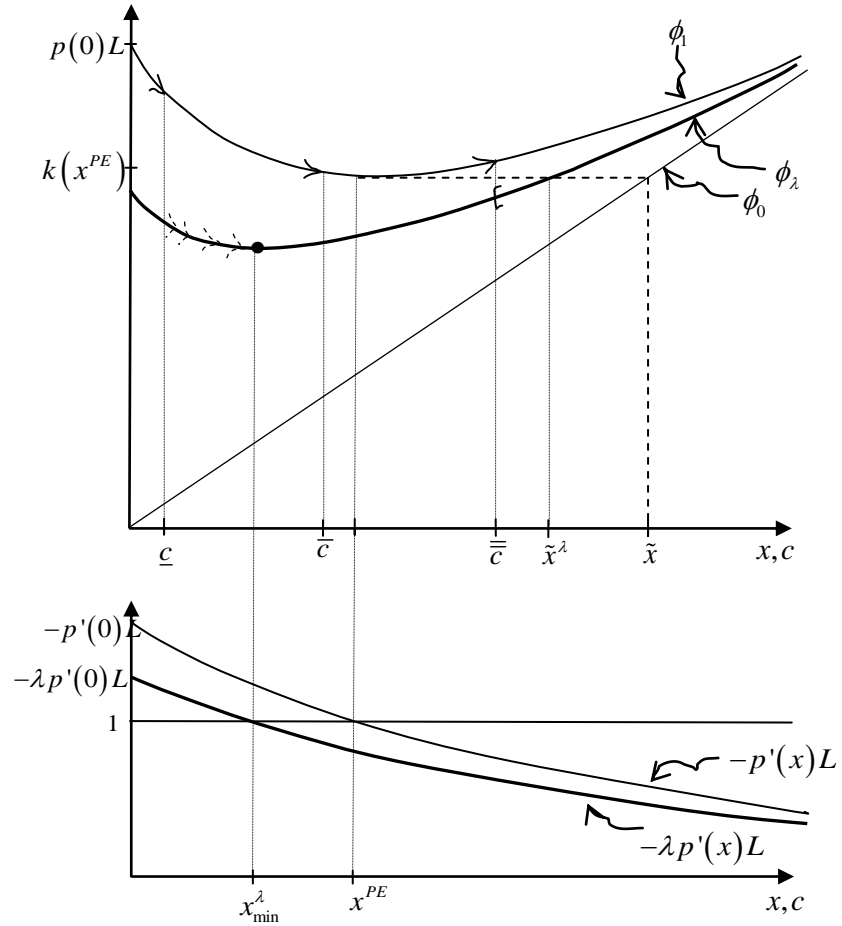

Figure 2.2 Incentive structures, high probability of strict liability

The solutions are summarized in the following proposition:

Proposition 2.1 Under the mixed norm $\lambda g_{\delta\infty} \Lambda (1-\lambda) g_{\delta c} \in \mathbb{P}_{\{0,L\}}^{X}$, $p : X \to \mathbb{R}_{++}$, $p' < 0$, $p'' > 0$ and $-p'(0)L > 1$:

$$(A) \quad x*\left(\lambda g_{\delta\infty}\Lambda(1-\lambda)g_{\delta c}\right)\Big|_{\lambda\le-\frac{1}{p'(0)L}} = \begin{cases} \{c\} & \text{if } c\in\left(0,\tilde{x}^{\lambda}\right) \\ \left\{x^{PE},\tilde{x}^{\lambda}\right\} & \text{if } c=\tilde{x}^{\lambda} \\ \left\{x^{PE}\right\} & \text{if } c>\tilde{x}^{\lambda} \end{cases} \qquad (2.9)$$

$$(B) \quad x*\left(\lambda g_{\delta\infty}\Lambda(1-\lambda)g_{\delta c}\right)\Big|_{\lambda>-\frac{1}{p'(0)L}} = \begin{cases} \left\{x_{\min}^{\lambda}\right\} & \text{if } c\in\left(0,x_{\min}^{\lambda}\right] \\ \{c\} & \text{if } c\in\left(x_{\min}^{\lambda},\tilde{x}^{\lambda}\right) \\ \left\{x^{PE},\tilde{x}^{\lambda}\right\} & \text{if } c=\tilde{x}^{\lambda} \\ \left\{x^{PE}\right\} & \text{if } c>\tilde{x}^{\lambda} \end{cases} \qquad (2.10)$$

with $\tilde{x}^{\lambda}$ defined in (2.5) and falling in λ, and $x_{\min}^{\lambda}$ defined in (2.7) and

increasing in λ, $0<x_{\min}^{\lambda}<x^{PE}<\tilde{x}^{\lambda}<\tilde{x}$.

2.2   *Comparative statics: the probability of strict liability*

Assume an initial "low" probability of strict liability, $\lambda\le\frac{1}{-p'(0)L}$. The only effect

on equilibria of an increased probability of strict liability, is through a fall in the

switching point to strict liability solutions ($\tilde{x}^{\lambda}$ falls in λ).

From an initial "high" probability of strict liability, $\lambda>\frac{1}{-p'(0)L}$, an further

increase mixes with the class $g_{\delta c}\in\delta\mathbb{P}_{\{0,L\}}{}^{X}$, $c\in\left(0,x_{\min}^{\lambda}\right]$ induce a higher level of

precaution relative to the negligence requirement ($x_{\min}^{\lambda}$ increases in λ). In mixes

with negligence rules defined by $c>x_{\min}^{\lambda}$, effects are as described above.

The expected cost for potential injurers is independent of λ if strict lia-

bility is mixed with the class of global negligence rules $g_{\delta c}\in\mathbb{P}_{\{0,L\}}{}^{X}$ defined by

$c\ge\tilde{x}^{\lambda}$, and increases if $c<\tilde{x}^{\lambda}$.[5]

---

[5] If $\lambda>1/-p'(0)L$ and $c\in\left(0,x_{\min}^{\lambda}\right]$, the value function is $x_{\min}^{\lambda}(\lambda)+\lambda p\left(x_{\min}^{\lambda}(\lambda)\right)L$. The
equilibrium response to an increase in λ cancels due to $\phi_{\lambda}{}'\left(x_{\min}^{\lambda}\right)=0$ ("envelope proper-

3   Mixed norms transformed in M_1: strict liability and negligence standards

The mixed norm $g_{\lambda g_s, \Lambda(1-\lambda)g_F} \in \mathbb{P}_{\{0,L\}}{}^X$ is given by ($\lambda \in (0,1)$):

$$\left\langle [1-\lambda]F(x;\underline{x}), 1-[1-\lambda]F(x;\underline{x}); 0, L \right\rangle\Big|_{x \in X},$$

the family of uniform negligence standard cdfs $F(t;\overline{x})$ defined in Section IV.5 and $p(x) = e^{-x}$, giving minimand:

$$K_1^\lambda(x;\underline{x}) = x + \left[1 - [1-\lambda]F(x;\underline{x})\right]e^{-x}L. \tag{3.1}$$

3.1   The structure of solutions

The ordinary-norm legal uncertainty, defined by $F(t;\underline{x})$ in the negligence standard, which, ceteris paribus, pushes towards undercompliance, is dampened by the strict liability probability given by meta-norm uncertainty $\lambda = h(g_s \mid ls)$. The next propositions demonstrate how mixing strict liability with negligence standards stabilizes equilibria around the efficient point.

Note that for all x:

$$K_1^\lambda(x;\underline{x}) = \lambda k(x) + [1-\lambda]K_1(x;\underline{x}). \tag{3.2}$$

As demonstrated in Section IV.5.1, $k(x)$ is strictly convex on X and $K_1(x;\overline{x})$ strictly convex on $[\underline{x},\overline{x}]$. $K_1^\lambda(x;\overline{x})$ is a convex combination of $k(x)$ and, hence,

---

ty"), giving an increase in λ at the rate $p(x_{\min}^\lambda)L$. Otherwise, the value function is $c + \lambda p'(c)L$, giving an increase in λ at the rate $p(c)L$.

$K_1(x;\underline{\overline{x}})$ strictly convex on $[\underline{x},\overline{x}]$.[6] The minimand is smooth at all x (except at

the support boundaries $\overline{x}, \underline{x} = \ln L \pm \underline{\overline{x}}$), with:

$$K_1^{\lambda}{}'(x;\underline{\overline{x}}) = \lambda k'(x) + [1-\lambda] K_1'(x;\underline{\overline{x}}).\qquad(3.3)$$

In $[0,\underline{x}]$, $K_1^{\lambda}(x;\underline{\overline{x}}) = k(x)$. Because $K_1^{\lambda}{}'(x;\underline{\overline{x}}) = k'(x)$ on $[0,\underline{x})$, and $k'(x) < 0$

on $[0,\ln L)$, a solution must be larger than or equal to $\underline{x}$. $K_1^{\lambda}(\underline{x};\underline{\overline{x}}) = k(\underline{x})$. At

the efficient point, $x^{PE} = \ln L$,

$$K_1^{\lambda}\left(\ln L;\underline{\overline{x}}\right) = \ln L + \tfrac{1}{2} + \tfrac{\lambda}{2} = \nabla + \tfrac{\lambda}{2} \equiv \nabla_1^{\lambda}.\qquad(3.4)$$

Because $\min_x\{k(x)\} = \ln L + 1 < \nabla_1^{\lambda}$, solutions cannot lie at the lower support

boundary $\underline{x}$. Hence, the solution lies to the right of $\underline{x}$.

Consider, then, the upper support boundary, $\overline{x}$. If $x \geq \overline{x}$, the function

$K_1^{\lambda}(x;\underline{\overline{x}}) = \lambda k(x) + (1-\lambda)x = x + \lambda e^{-x} L$ is strictly increasing and strictly con-

vex. The probability of liability under the "pure" negligence standard g_F is re-

duced to zero, but agents still risk strict liability and hence face (expected) costs

that lie above the linear precaution cost function, x. Hence, it is never cost-

efficient (even for $\lambda \uparrow 1$) to invest more in precaution than at that level which

eliminates the possibility of negligence liability. Solutions are bounded by $\overline{x}$,

where $K_1^{\lambda}(\overline{x};\underline{\overline{x}}) = \ln L + \overline{x} + \tfrac{\lambda}{e^{\overline{x}}}$.

Hence, for all $\lambda \in (0,1)$ and all $\underline{\overline{x}} \in (0, \ln L]$ solutions lie in $(\underline{x},\overline{x})$. Due

to strict convexity of $K_1^{\lambda}(\cdot;\underline{\overline{x}})$ in $[\underline{x},\overline{x}]$, unique solutions are ensured for all

$\langle \lambda, \underline{\overline{x}} \rangle \in (0,1) \times (0, \ln L]$. They are denoted $\chi^{\lambda}(1;\underline{\overline{x}})$ and illustrated in Figure 3.1.

---

[6] Sydsæter [1981], Theorem 5.14.i, Note 1.
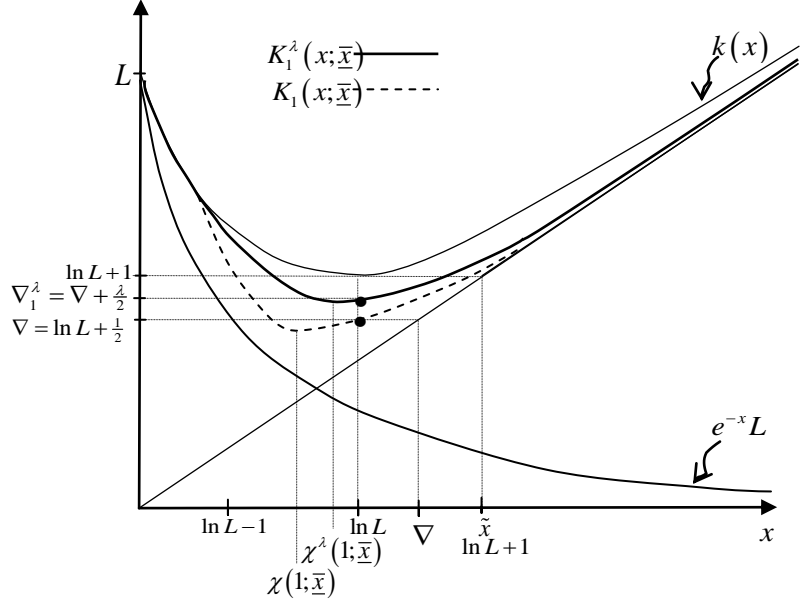
Figure 3.1 Equilibria under mixed norms (illustration based on $\overline{x} > 1$ and $L > e$)

At $\overline{x}$, the right hand derivative is:

$$K_1^\lambda{}'\left(\overline{x}^+;\underline{x}\right) = \lambda k'\left(\overline{x}\right) + \left[1-\lambda\right]K_1{}'\left(\overline{x}^+;\underline{x}\right) = 1 - \lambda e^{-\overline{x}}L = 1 - \tfrac{\lambda}{e^{\overline{x}}} > 0. \qquad (3.5)$$

As illustrated in Figures 3.2 and 3.3, the expected *marginal* total cost increases at the upper support boundary at a rate lower than under a pure negligence regime, $K_1{}'\left(\overline{x}^+;\underline{x}\right) = 1$, reflecting the value of reducing $p(x)$ as liability occurs with probability λp in adverse situations.

Let the parameter $\hat{\underline{x}}$ and "pure" equilbria $\chi\left(1;\overline{x}\right)$ be defined in Proposition IV.5.1. It follows:

Proposition 3.1 Under the mixed norm $g_{\lambda g_s \wedge (1-\lambda) g_F} \in \mathbb{P}_{\{0,L\}}{}^X$, cdf F from

$U\left(\ln L - \overline{x}, \ln L + \overline{x}\right)$ and $p(x) = e^{-x}$, a unique solution

$\ln L - \overline{x} < \chi^\lambda\left(1;\overline{x}\right) < \nabla_1^\lambda = \ln L + \tfrac{1+\lambda}{2}$ exists for all $\langle \lambda, \overline{x} \rangle \in (0,1) \times (0, \ln L]$

with $\hat{\underline{x}}^\lambda < \hat{\underline{x}}$, the parameter $\hat{\underline{x}}^\lambda$ falling in λ and:

185

$$\chi^{\lambda}\left(1;\overline{x}\right)=\chi\left(1;\overline{x}\right)=\ln L+\underline{x}\ \text{if}\ \langle\lambda,\overline{x}\rangle\in(0,1)\times\left(0,\hat{\tilde{x}}^{\lambda}\right];$$

$$\chi^{\lambda}\left(1;\overline{x}\right)\in\left(\ln L,\hat{\tilde{x}}^{\lambda}\right)\ \text{and}\ \chi^{\lambda}\left(1;\overline{x}\right)<\chi\left(1;\overline{x}\right)\ \text{if}\ \langle\lambda,\overline{x}\rangle\in(0,1)\times\left(\hat{\tilde{x}}^{\lambda},1\right);$$

$$\chi^{\lambda}\left(1;1\right)=\chi\left(1;1\right)=\ln L\ \text{if}\ \lambda\in(0,1);\ \text{and}$$

$$\chi^{\lambda}\left(1;\overline{x}\right)\in\left(\ln L-\underline{x},\ln L\right)\ \text{and}\ \chi^{\lambda}\left(1;\overline{x}\right)>\chi\left(1;\overline{x}\right)\ \text{if}\ \langle\lambda,\overline{x}\rangle\in(0,1)\times(1,\ln L].$$

Proof. At the efficient point, $k'(\ln L)=0$ and:

$$K_{1}^{\lambda}{}'\left(x;\overline{x}\right)\big|_{x=\ln L}=[1-\lambda](1-\underline{x})/2\underline{x}.\tag{3.6}$$

From (3.6) and strict convexity of $K_{1}^{\lambda}\left(\cdot;\overline{x}\right)$ on $[\underline{x},\overline{x}]$, all $\lambda\in(0,1)$ solutions are (i) undercomplying if $\underline{x}>1$ (ii) efficient if $\underline{x}=1$ and (iii) overcomplying if $\underline{x}<1$.

(i) Dispersed standards, $\underline{x}>1$. At undercomplying solutions $\chi^{\lambda}\left(1;\overline{x}\right)<\ln L$, the corresponding pure negligence equilibria ($\lambda=0$) are also undercomplying, $\chi\left(1;\overline{x}\right)<\ln L$ (Proposition IV.5.1), implying $k'\left(\chi\left(1;\overline{x}\right)\right)<0$. By definition $K_{1}{}'\left(\chi\left(1;\overline{x}\right);\overline{x}\right)=0$.

Hence, (3.3) gives $K_{1}^{\lambda}{}'\left(\chi\left(1;\overline{x}\right);\overline{x}\right)=\lambda k'\left(\chi\left(1;\overline{x}\right)\right)<0$. Therefore, from strict convexity of $K_{1}^{\lambda}\left(\cdot;\overline{x}\right)$ on $[\underline{x},\overline{x}]$, $\ln L-\underline{x}<\chi\left(1;\overline{x}\right)<\chi^{\lambda}\left(1;\overline{x}\right)<\ln L$ for all $\lambda\in(0,1)$ and all $\underline{x}\in(1,\ln L]$. The situation is illustrated in Figure 3.1.

(ii) Efficient standard, $\underline{x}=1$. In this "hairline" case, $\chi^{\lambda}\left(1;\overline{x}\right)\big|_{\underline{x}=1}=\chi\left(1;\overline{x}\right)\big|_{\underline{x}=1}=\ln L$, solutions are independent of λ: increasing the probability of strict liability only affects expected cost levels $K_{1}^{\lambda}\left(\chi^{\lambda}\left(1;\overline{x}\right);\overline{x}\right)\big|_{\underline{x}=1}=\nabla_{1}^{\lambda}$.

(iii) Concentrated standards, $\underline{x}<1$. Overcomplying equilibria are interior in $(\ln L,\overline{x})$ or at the support boundary, $\overline{x}=\ln L+\underline{x}$. Because $K_{1}^{\lambda}{}'\left(\ln L;\overline{x}\right)<0$ and $K_{1}^{\lambda}\left(\cdot;\overline{x}\right)$ is strictly convex on $[\underline{x},\overline{x}]$, a boundary solution $\chi^{\lambda}=\overline{x}$ requires $K_{1}^{\lambda}{}'\left(\overline{x}^{-};\overline{x}\right)\leq0$. Because $k'(\overline{x})>0$, $K_{1}{}'\left(\overline{x}^{-},\overline{x}\right)<0$:

$$K_{1}^{\lambda}{}'\left(\overline{x}^{-};\overline{x}\right)=\lambda k'(\overline{x})+[1-\lambda]K_{1}{}'\left(\overline{x}^{-};\overline{x}\right).\tag{3.7}$$

186

The situation is illustrated in Figure 3.2.
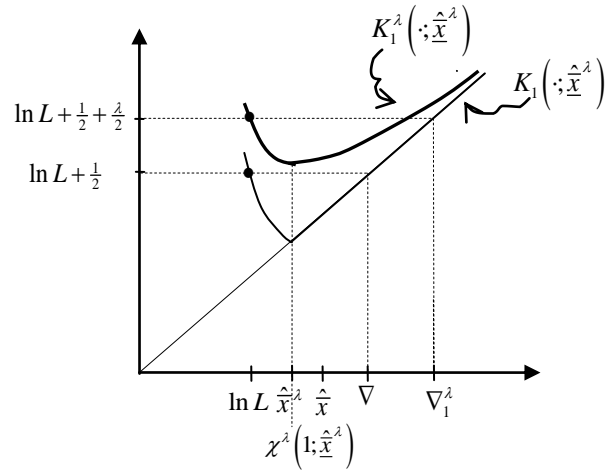
Figure 3.2 Mixed norm boundary solution at $\hat{\bar{x}}^{\lambda} = \ln L + \hat{\underline{x}}^{\lambda}$

At $\hat{\underline{x}}$, defined in Proposition IV.5.1 and corresponding to $\hat{\bar{x}} = \ln L + \hat{\underline{x}}$ in Figure 3.3, $K_1'\left(\hat{\bar{x}}^{-};\hat{\underline{x}}\right) = 0$. Therefore, $K_1^{\lambda}{}'\left(\hat{\bar{x}}^{-};\hat{\underline{x}}\right) = \lambda k'\left(\hat{\underline{x}}\right) > 0$. Hence, from strict convexity of $K_1^{\lambda}\left(\cdot;\hat{\underline{x}}\right)$, an interior solution $\chi^{\lambda}\left(1;\hat{\underline{x}}\right) < \chi\left(1;\hat{\underline{x}}\right) = \ln L + \hat{\underline{x}}$ is implied. It follows that more concentrated distributions $\bar{x} \ll \hat{\underline{x}}$ are required for support boundary solutions in the mixed-norm case.
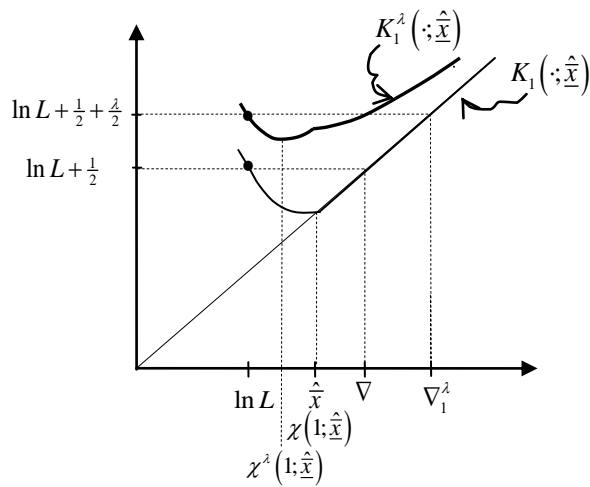
Figure 3.3 Mixed norm interior solution (boundary solution under a pure negligence re-

gime)

Define $\hat{\bar{x}}^{\lambda} = \ln L + \hat{\underline{x}}^{\lambda}$ such that $K_1^{\lambda}{}'\left(\hat{\bar{x}}^{\lambda-};\hat{\underline{x}}^{\lambda}\right) = 0$. Such an $\hat{\underline{x}}^{\lambda} < \hat{\underline{x}}$ exists for all values

of $\lambda \in (0,1)$ and is falling in λ: $K_1^{\lambda}{}'\left(\bar{x}^-;\underline{x}\right) = 1 - \frac{1}{2\bar{x}}e^{-\bar{x}}\left\{1 + \lambda\left(2\underline{x} - 1\right)\right\}$.[7] Hence.

$K_1^{\lambda}{}'\left(\bar{x}^-;\underline{x}\right) \leq 0$ is equivalent to (*) $\underline{x} + \ln\underline{x} + \ln 2 \equiv \psi\left(\underline{x}\right) \leq \ln\left\{1 + \lambda\left(2\underline{x} - 1\right)\right\} \equiv \Psi^{\lambda}\left(\underline{x}\right)$.

$\psi(\cdot)$ is the strictly increasing and strictly concave function defined in the proof of Prop-

osition IV.5.1 ($\lim_{\underline{x} \to 0^+} \psi\left(\underline{x}\right) = -\infty$). The function $\Psi^{\lambda}(\cdot)$ is strictly increasing and is strictly

concave on $[0,1]$ with $\Psi^{\lambda}(0) = \ln\{1 - \lambda\}$, $\Psi^{\lambda}(1) = \ln\{1 + \lambda\}$, and $\Psi^{\lambda}\left(\frac{1}{2}\right) = 0$ and inde-

pendent of λ. As illustrated in Figure 3.4, the uniquely determined $\hat{\underline{x}}^{\lambda}$ falls in λ.
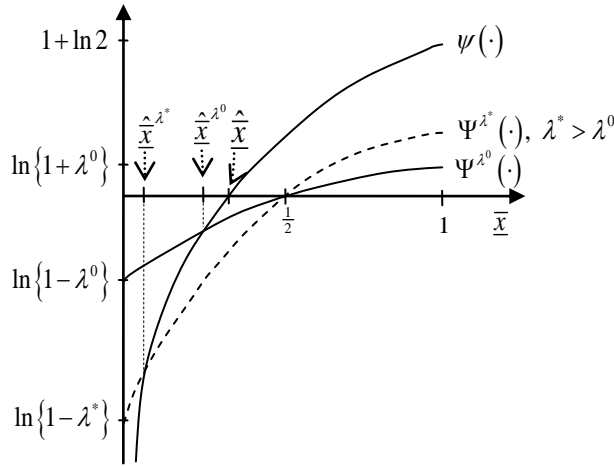
Figure 3.4 Parameter values $\left(0, \hat{\underline{x}}^{\lambda}\right]$ giving mixed norm support boundary solutions

$$\chi^{\lambda}\left(1;\underline{x}\right) = \bar{x}$$

It follows that if $\underline{x} \leq \hat{\underline{x}}^{\lambda}$ $\chi^{\lambda}\left(1;\underline{x}\right) = \chi\left(1;\underline{x}\right) = \ln L + \underline{x}$. If $\underline{x} \in \left(\hat{\underline{x}}^{\lambda},1\right)$ solutions

$\ln L < \chi^{\lambda}\left(1;\underline{x}\right) < \ln L + \underline{x}$ are interior.

If $\underline{x} \in \left(\hat{\underline{x}}^{\lambda},\hat{\underline{x}}\right)$, the pure negligence equilibrium $\chi\left(1;\underline{x}\right)$ continues the linear in-

crease in $\underline{x}$, hence $\chi^{\lambda}\left(1;\underline{x}\right) < \chi\left(1;\underline{x}\right)$ in this interval.

---

[7] $K_1{}'\left(\bar{x}^-;\underline{x}\right) = 1 - \underbrace{\left[1 - F\left(\bar{x};\underline{x}\right)\right]}_{0}e^{-\bar{x}}L - \underbrace{F'\left(\bar{x}^-;\underline{x}\right)}_{\frac{1}{2\bar{x}}}e^{-\bar{x}}L = 1 - \frac{1}{2\bar{x}}e^{-\bar{x}}L$ and $k'\left(\bar{x}\right) = 1 - e^{-\bar{x}}L$.

188

If $\underline{\overline{x}} \in \left(\hat{\overline{x}}, 1 \right)$, both mixed and pure equilibria are overcomplying and interior.

From (3.2), $K_1^{\lambda}{}'\left(\chi(1;\underline{\overline{x}}); \underline{\overline{x}} \right) = \lambda k'\left(\chi(1;\underline{\overline{x}}) \right) > 0$ because $K_1{}'\left(\chi(1;\underline{\overline{x}}); \underline{\overline{x}} \right) = 0$ (Proposition IV.5.1). From strict convexity of $K_1^{\lambda}\left(\cdot; \underline{\overline{x}} \right)$ on $[\ln L, \overline{x}]$, $\chi^{\lambda}\left(1;\underline{\overline{x}}\right) < \chi\left(1;\underline{\overline{x}}\right)$ for all $\lambda \in (0,1)$ and $\underline{\overline{x}} \in \left(\hat{\overline{x}}, 1 \right)$. ◀

3.2   *Comparative statics: uncertainty on the level of ordinary norms*

Let the parameter $\hat{\underline{\overline{x}}}^{\lambda}$ and the unique equilibrium $\chi^{\lambda}\left(1;\underline{\overline{x}}\right)$ be defined in Proposition 3.1. It follows:

Proposition 3.2 For all $\lambda \in (0,1)$:

$$\frac{\partial \chi^{\lambda}\left(1;\underline{\overline{x}}\right)}{\partial \underline{\overline{x}}} = 1 \text{ if } \underline{\overline{x}} \in \left(0, \hat{\underline{\overline{x}}}^{\lambda} \right), \text{ and}$$

$$\frac{\partial \chi^{\lambda}\left(1;\underline{\overline{x}}\right)}{\partial \underline{\overline{x}}} < 0 \text{ if } \underline{\overline{x}} \in \left(\hat{\underline{\overline{x}}}^{\lambda}, \ln L \right).$$

Proof. If $\underline{\overline{x}} \in \left(0, \hat{\underline{\overline{x}}}^{\lambda} \right)$, by Proposition 3.2, $\chi^{\lambda}\left(1;\underline{\overline{x}}\right) = \ln L + \underline{\overline{x}}$. Hence, $\partial \chi^{\lambda}\left(1;\underline{\overline{x}}\right)/\partial \underline{\overline{x}} = 1$.

By the Implicit function theorem, at all interior solutions ($\underline{\overline{x}} \in \left(\hat{\underline{\overline{x}}}^{\lambda}, \ln L \right)$):

$$\frac{\partial \chi^{\lambda}\left(1;\underline{\overline{x}}\right)}{\partial \underline{\overline{x}}} = - \left. \frac{\partial K_1^{\lambda}{}'\left(x;\underline{\overline{x}}\right)}{\partial \underline{\overline{x}}} \right|_{x=\chi^{\lambda}(1;\underline{\overline{x}}) \text{ const.}} \Bigg/ K_1^{\lambda}{}''\left(\chi^{\lambda}\left(1;\underline{\overline{x}}\right); \underline{\overline{x}} \right), \qquad (3.8)$$

that is, by strict convexity of $K_1^{\lambda}\left(\cdot; \underline{\overline{x}} \right)$ on $[\underline{x}, \overline{x}]$ $\chi^{\lambda}\left(1;\underline{\overline{x}}\right)$ falls in $\underline{\overline{x}}$ if $\partial K^{\lambda}\left(1;\underline{\overline{x}}\right)/\partial \underline{\overline{x}} > 0$ and increases if $\partial K^{\lambda}\left(1;\underline{\overline{x}}\right)/\partial \underline{\overline{x}} < 0$. From (3.3):

$$\left. \frac{\partial K_1^{\lambda}{}'\left(x;\underline{\overline{x}}\right)}{\partial \underline{\overline{x}}} \right|_{x=\chi^{\lambda}(1;\underline{\overline{x}}) \text{ const.}} = (1-\lambda) \left. \frac{\partial K_1{}'\left(x;\underline{\overline{x}}\right)}{\partial \underline{\overline{x}}} \right|_{x=\chi^{\lambda}(1;\underline{\overline{x}}) \text{ const.}}.$$

It follows from the proof of Proposition IV.5.2 that the sign of $\partial K_1{}'\left(x;\underline{\overline{x}}\right)/\partial \underline{\overline{x}} \big|_{x=\chi^{\lambda}(1;\underline{\overline{x}}) \text{ const.}}$

is determined by the sign of $\left\{ \ln L + 1 - \chi^{\lambda}\left(1;\underline{\overline{x}}\right) \right\}$. The latter is strictly positive due to

189

$\chi^\lambda(1;\overline{x}) < \nabla_1^\lambda < \ln L + 1$.[8] Hence, for all $\lambda \in (0,1)$ and $\overline{x} \in \left(\hat{\overline{x}}^\lambda, \ln L\right)$, $\partial\chi^\lambda(1;\overline{x})/\partial\overline{x} < 0$.

◄

Regarding the effects of $\overline{x}$ on expected costs under the mixed norm, consider the value function:

$$\mathbf{K}_1^\lambda(\overline{x}) = K_1^\lambda\left(\chi^\lambda(1;\overline{x});\overline{x}\right) = \chi^\lambda(1;\overline{x}) + \left\{1 - [1-\lambda]F\left(\chi^\lambda(1;\overline{x});\overline{x}\right)\right\}e^{-\chi^\lambda(1;\overline{x})}L \quad (3.9)$$

It follows:

<u>Proposition 3.3</u> For all $\lambda \in (0,1)$:

$$\frac{\partial\mathbf{K}_1^\lambda(\overline{x})}{\partial\overline{x}} = \begin{cases} > 0 \text{ if } \overline{x} \in (0,1) \\ < 0 \text{ if } \overline{x} \in (1,\ln L) \end{cases}$$

<u>Proof.</u> If $\overline{x} \in \left(0,\hat{\overline{x}}^\lambda\right)$, boundary solutions give the explicit value function

$\mathbf{K}_1^\lambda(\overline{x}) = \ln L + \overline{x} + \lambda e^{-\overline{x}}$. Direct differentiation gives $\partial\mathbf{K}_1^\lambda(\overline{x})/\partial\overline{x} = 1 - \lambda e^{-\overline{x}} > 0$. If

$\overline{x} >\in \hat{\overline{x}}^\lambda$, solutions are interior. Hence, by the Envelope theorem:[9]

$$\frac{\partial\mathbf{K}_1^\lambda(\overline{x})}{\partial\overline{x}} = \frac{\partial K_1^\lambda(x;\overline{x})}{\partial\overline{x}}\bigg|_{x=\chi^\lambda(1;\overline{x})\text{ const}}$$

$$= -\langle 1-\lambda\rangle\frac{\partial F(x;\overline{x})}{\partial\overline{x}}\bigg|_{x=\chi^\lambda(1;\overline{x})\text{ const}} e^{-\chi^\lambda(1;\overline{x})}L = -[1-\lambda]\left\{\ln L - \chi^\lambda(1;\overline{x})\right\}e^{-\chi^\lambda(1;\overline{x})}L,$$

---

[8] $K_1^\lambda(\ln L;\overline{x}) = \nabla_1^\lambda$. At $K_1^\lambda(\nabla_1^\lambda;\overline{x}) > \nabla_1^\lambda$ because $K_1^\lambda(x;\overline{x}) > x$ for all x. Hence, $\chi^\lambda \geq \nabla_1^\lambda$ cannot be an equilibrium for any parameter constellation $(\lambda,\overline{x})$.

[9] Interior solutions $\underline{x} < \chi^\lambda(1;\overline{x}) < \overline{x}$ if $\overline{x} \in \left(\hat{\overline{x}}^\lambda, \ln L\right]$ and strict convexity of $K_1^\lambda(\cdot;\overline{x})$ on $[\underline{x},\overline{x}]$ ensures that the assumptions of the theorem are satisfied (Carter [2001], Theorem 6.2).

implying that expected cost increases in $\underline{x}$ at all overcomplying mixed equilibria (for all $\underline{x} \in \left(\hat{\underline{x}}^{\lambda}, 1 \right)$ and $\lambda \in (0,1)$), and falls at all undercomplying mixed equilibria (all $\underline{x} \in (1, \ln L)$ and $\lambda \in (0,1)$). ◄

<u>Remark 3.1</u> At the efficient mixed equilibrium corresponding to $\underline{x} = 1$, expected cost is invariant with respect to a small change in $\underline{x}$: $K_1^{\lambda} \left(\chi^{\lambda} (1; \underline{x}); \underline{x} \right) \Big|_{\underline{x}=1} = \nabla^{\lambda}$

$= \ln L + \frac{1}{2} + \frac{\lambda}{2}$. □

3.3 Comparative statics: uncertainty on the level of meta-norms

This subsection considers the equilibrium dependence on meta-level uncertainty (mixing weight λ) and characterizes its influence on expected cost.

<u>Proposition 3.4</u> For all $\lambda \in (0,1)$:

$$\frac{\partial \chi^{\lambda} (1; \underline{x})}{\partial \lambda} = 0 \text{ if } \underline{x} \in \left(0, \hat{\underline{x}}^{\lambda} \right);$$
$$\frac{\partial \chi^{\lambda} (1; \underline{x})}{\partial \lambda} < 0 \text{ if } \underline{x} \in \left(\hat{\underline{x}}^{\lambda}, 1 \right);$$
$$\frac{\partial \chi^{\lambda} (1; 1)}{\partial \lambda} = 0 \text{ ; and}$$
$$\frac{\partial \chi^{\lambda} (1; \underline{x})}{\partial \lambda} > 0 \text{ if } \underline{x} \in (1, \ln L).$$

<u>Remark 3.2</u> If $\underline{x} \in \left(0, \hat{\underline{x}}^{\lambda} \right)$, $\chi^{\lambda} (1; \underline{x}) = \ln L + \underline{x}$, and the mixed norm equilibria are independent of λ. A necessary, but not sufficient, condition for this situation is an upper support boundary solution in the corresponding pure negligence regime, eliminating risk of liability under $g_{F(x; \underline{x})}$. The more likely strict liability, the more concentrated the negligence distribution needs to be to ensure this situation (sufficiency). □

<u>Proof.</u> By the Implicit function theorem, at interior solutions ($\bar{x} \in \left(\hat{\bar{x}}^{\lambda}, \ln L \right)$):

$$\frac{\partial \chi^{\lambda}\left(1;\bar{x}\right)}{\partial \lambda} = -\left.\frac{\partial K_1^{\lambda}{}'\left(x;\bar{x}\right)}{\partial \lambda}\right|_{x=\chi^{\lambda}(1;\bar{x})\text{ const.}} \Bigg/ K_1^{\lambda}{}''\left(\chi^{\lambda}\left(1;\bar{x}\right);\bar{x}\right). \qquad (3.10)$$

From (3.3):

$$\left.\frac{\partial K_1^{\lambda}{}'\left(x;\bar{x}\right)}{\partial \lambda}\right|_{x=\chi^{\lambda}(1;\bar{x})\text{ const.}} = \left\{k'\left(\chi^{\lambda}\left(1;\bar{x}\right)\right) - K_1{}'\left(\chi^{\lambda}\left(1;\bar{x}\right);\bar{x}\right)\right\} \qquad (3.11)$$

At all interior overcomplying equilibria, corresponding to $\bar{x} \in \left(\hat{\bar{x}}^{\lambda}, 1 \right)$, relation $\chi^{\lambda}\left(1;\bar{x}\right) < \chi\left(1;\bar{x}\right)$ hold for all $\lambda \in (0,1)$ (Proposition 3.1). Because $K_1\left(\cdot;\bar{x}\right)$ is strictly convex on $\left[\ln L, \bar{x}\right]$, $K_1{}'\left(\chi^{\lambda}\left(1;\bar{x}\right);\bar{x}\right) < 0$. Clearly, $k'\left(\chi^{\lambda}\left(1;\bar{x}\right)\right) > 0$. Hence $\left.\partial K_1^{\lambda}{}'\left(x;\bar{x}\right)\big/\partial \lambda\right|_{x=\chi^{\lambda}(1;\bar{x})\text{ const.}} > 0$ and $\partial \chi^{\lambda}\left(1;\bar{x}\right)\big/\partial \lambda < 0$.

At undercomplying equilibria, corresponding to $\bar{x} \in (1, \ln L)$, the relation $\chi^{\lambda}\left(1;\bar{x}\right) > \chi\left(1;\bar{x}\right)$ holds for all $\lambda \in (0,1)$ (Proposition 3.1). Because $K_1\left(\cdot;\bar{x}\right)$ is strictly convex on $\left[\underline{x}, \ln L\right]$, $K_1{}'\left(\chi^{\lambda}\left(1;\bar{x}\right);\bar{x}\right) > 0$. Clearly, $k'\left(\chi^{\lambda}\left(1;\bar{x}\right)\right) < 0$. Hence $\left.\partial K_1^{\lambda}{}'\left(x;\bar{x}\right)\big/\partial \lambda\right|_{x=\chi^{\lambda}(1;\bar{x})\text{ const.}} < 0$ and $\partial \chi^{\lambda}\left(1;\bar{x}\right)\big/\partial \lambda > 0$.

If $\bar{x} = 1$, implementing the efficient solution in all regimes changing the mixture weights has no equilibrium effects ($k' = K_1{}' = 0$); see also in and at (3.6). ◄

Let the value function $\mathbf{K}_1^{\lambda}\left(\bar{x}\right)$ be defined in (3.9). It follows:

<u>Proposition 3.5</u> For all $\langle \lambda, \bar{x} \rangle \in (0,1) \times (0, \ln L]$:

$$\frac{\partial \mathbf{K}_1^{\lambda}\left(\bar{x}\right)}{\partial \lambda} > 0.$$

192

<u>Proof.</u> If $\overline{x} \in \left(0, \hat{\overline{x}}^{\lambda}\right)$, $\mathbf{K}_1^{\lambda}\left(\overline{x}\right) = \ln L + \overline{x} + \lambda e^{-\overline{x}}$ and direct differentiation gives

$\partial \mathbf{K}_1^{\lambda}\left(\overline{x}\right) / \partial \lambda = e^{-\overline{x}} > 0$. Precaution is never above $\overline{x}$. In situations (initial equilibria) where there is no risk of negligence liability, increasing the probability of strict liability does not affect the optimal precaution level, but directly increases the cost of being held liable for damages $e^{-\overline{x}} L = e^{-\overline{x}}$.

If $\overline{x} > \hat{\overline{x}}^{\lambda}$, solutions are interior. By the Envelope theorem:[10]

$$\frac{\partial \mathbf{K}_1^{\lambda}\left(\overline{x}\right)}{\partial \lambda} = \left.\frac{\partial K_1^{\lambda}\left(x;\overline{x}\right)}{\partial \lambda}\right|_{x=\chi^{\lambda}(1;\overline{x}) \text{ const}} = F\left(\chi^{\lambda}\left(1;\overline{x}\right);\overline{x}\right) e^{-\chi^{\lambda}(1;\overline{x})} L > 0. \blacktriangleleft$$

<u>Remark 3.3</u> Because $\mathbf{K}_1^{\lambda}\left(1\right) = \nabla^{\lambda}$, the derivative has a particularly simple form at the efficient point; $\partial \nabla^{\lambda} / \partial \lambda = \frac{1}{2}$. $\square$

4   Mixed norms transformed in M_{2m+1}: strict liability and negligence rules

From (1.2), under the transformed mixed norm $g_{\lambda g_{\delta c} \Lambda (1-\lambda) g_{\delta c}, 2m+1} \in \delta \mathbb{P}_{\{0,L\}}^X$, the minimand is given by:

$$x + \left\{1 - h_{2m+1}\left(\left[1-\lambda\right] F_{\delta c}\left(x\right)\right)\right\} p\left(x\right) L. \tag{4.1}$$

Because $F_{\delta c}\left(x\right) = 0$ if $x < c$, $F_{\delta c}\left(x\right) = 1$ if $x \geq c$, $h_{2m+1}\left(0\right) = 0$ for all m (Proposition III.2.2) and $h_{2m+1}\left(1-\lambda\right) = 1 - h_{2m+1}\left(\lambda\right)$ for all m and all $\lambda \in \left(0,1\right)$, agents choose:

$$x^*\left(g_{\lambda g_{\delta c} \Lambda (1-\lambda) g_{\delta c}, 2m+1}\right) = \arg\min_{x \in X} \begin{cases} x + p\left(x\right) L \text{ if } x < c \\ x + h_{2m+1}\left(\lambda\right) p\left(x\right) L \text{ if } x \geq c \end{cases}. \tag{4.2}$$

---

[10] Carter [2001], Theorem 6.2.

With $\phi_\lambda(\cdot)$ defined in (2.3), the incentive structure induces the program:

$$\arg\min_{x \in X} \begin{cases} \phi_1(x) \text{ if } x < c \\ \phi_{h_{2m+1}(\lambda)}(x) = x + h_{2m+1}(\lambda) p(x) L \text{ if } x \geq c \end{cases}$$

Because $h_{2m+1}(\lambda) \in (0,1)$ for all $\lambda \in (0,1)$, it follows that $\phi_{h_{2m+1}(\lambda)}(x)$ inherits all

the properties of $\phi_\lambda(x) = \phi_{h_{2m+1}(\lambda)}(x)\big|_{m=0}$ in the Section 2 bullet points. By the

Condorcet theorem monotonicity properties (Section III.2), $h_{2m+1}(\lambda) \downarrow 0$ if

$\lambda < \frac{1}{2}$, $h_{2m+1}(\lambda) \uparrow 1$ if $\lambda > \frac{1}{2}$, and $h_{2m+1}(\frac{1}{2}) = \frac{1}{2}$ for all m. It, therefore, follows

from Proposition 2.1 that the structure of equilibria can be summarized as

$(x^{PE} = \arg\min_x \{k(x) = x + p(x)L\}$, $k(x^{PE}) = \tilde{x}$, and the functions $h_{2m+1}(\lambda)$ giv-

en in equation (III.7.1)):

> Proposition 4.1 Under the transformed mixed norm
>
> $g_{\lambda g_{\delta\infty} \wedge (1-\lambda) g_{\delta c}, 2m+1} \in \mathbb{P}_{\{0,L\}}{}^X$, $p: X \to \mathbb{R}_{++}$, $p' < 0$, $p'' > 0$ and $-p'(0)L > 1$
>
> solution correspondences $x* = x*\left(g_{\lambda g_{\delta\infty} \wedge (1-\lambda) g_{\delta c}, 2m+1}\right)$ are given by:

	$\lambda < \dfrac{1}{2}$	$\lambda = \dfrac{1}{2}$	$\lambda > \dfrac{1}{2}$
$\dfrac{1}{-p'(0)L} \geq \dfrac{1}{2}$	$x*$ as in P.2.1A, λ decreasing acc. to $h_{2m+1}(\lambda)$, converging to a pure $g_{\delta c}$-regime	$x*$ constant for all m, def. in P.2.1A	for small/interm. m, $x*$ possibly as in P.2.1A, eventually as $\frac{1}{-p'(0)L} < \frac{1}{2}$ case
$\dfrac{1}{-p'(0)L} < \dfrac{1}{2}$	for small/interm. m, $x*$ possibly as in P.2.1B, eventually as $\frac{1}{-p'(0)L} \geq \frac{1}{2}$ case	$x*$ constant for all m, def. in P.2.1B	$x*$ as in P.2.1B, λ incr. acc. to $h_{2m+1}(\lambda)$, converging to a pure $g_{\delta\infty}$-regime; $x_{\min}^{\lambda} \uparrow x^{PE},\ \tilde{x}^{\lambda} \downarrow x^{PE}$

<u>Remark 4.1</u> Transformed mixed standards $g_{\lambda g_{\delta\infty} \Lambda(1-\lambda)g_F, 2m+1} \in \mathbb{P}_{\{0,L\}}{}^X$, leading to

minimands $K_{2m+1}^{\lambda}(x;\underline{\overline{x}}) = x + \left\{1 - h_{2m+1}\left(\left[1-\lambda\right]F\left(x;\underline{\overline{x}}\right)\right)\right\}e^{-x}L$, may also be con-

sidered, combining the Section IV.5.2 results with the Section 3 approach. □

PART SIX

LAW-IN-FORCE NOTIONS AND SYSTEM-BASED REPERCUSSION

ANALYSES

This part considers decision-making in judicial panels, with emphasis on the two-stage character of legal reasoning deriving from the distinction between meta-norms and ordinary norms. Law-in-force notions based on Definition II.2.1, in which norms express obligations, are discussed in Section 1. Section 2 gives a general definition of power-conferring norms, and develops some of their properties. Expressing the notion of protected options (discretion), power-conferring norms are used as elements in more complex law in-force-notions, in Section 3. Discretion, and logical difficulties which arise under judgment aggregation, are used to motivate the relevance of forward-looking, means-end analyses of law. Illustrating examples include normative use of Pareto-efficiency in the design of contract and tort law in a general equilibrium context in Section 3.1. More general aspects of means-end analyses are also considered, and translated to strategic environments in Section 3.2.

1   Configurations of norm-based uncertainty and law-in-force notions I: (probabilistically) deductive systems

Let the meta-norm $\eta \in \mathbb{P}_{\mathbb{P}_Y^X}{}^{LS}$ represent the doctrine of sources and method, that is, a mapping from the set of legal sources LS to the set of (simple) probability

distributions over the set of ordinary norms (Definition II.5.1). Legal decision-making, from the ex ante perspective, is conceived as follows: At point in time t, judges $j \in \{1, 2, \ldots, n\}$ independently:

I. *observe* the constellation of data $\langle {}_t ls, {}_t x \rangle \in LS \times X$;

II. *apply* meta-norm $\eta \in \mathbb{P}_{\mathbb{P}_Y^X}^{LS}$ to ${}_t ls$, leading to the (abstract) mixed

norm $g_{\eta(\cdot \mid {}_t ls)} \in \mathbb{P}_Y^X$ given by[1]

$$\langle g_{\eta(\cdot \mid {}_t ls)}\left(y^1 \mid x\right), g_{\eta(\cdot \mid {}_t ls)}\left(y^2 \mid x\right), \ldots, g_{\eta(\cdot \mid {}_t ls)}\left(y^L \mid x\right) \rangle \Big|_{x \in X};$$

III. *apply* mixed norm $g_{\eta(\cdot \mid {}_t ls)}$ to fact ${}_t x$ in mechanism $M \in \mathcal{M}$, leading

to $g_{\eta(\cdot \mid {}_t ls)}\left(\cdot \mid x\right)\Big|_{x \in X} = \langle g_{\eta(\cdot \mid {}_t ls)}\left(y^1 \mid x\right), g_{\eta(\cdot \mid {}_t ls)}\left(y^2 \mid x\right), \ldots, g_{\eta(\cdot \mid {}_t ls)}\left(y^L \mid x\right) \rangle \Big|_{x \in X},$

that is, a final distribution over the set of consequences. The process is illustrated in Figure 1.1. The following definition is suggested:

Definition 1.1 At time t, conditional on data $\langle {}_t ls, {}_t x \rangle \in LS \times X$,

$\eta \in \mathbb{P}_{\mathbb{P}_Y^X}^{LS}$, and $M \in \mathcal{M}$, the set of *enforceable norms* is

supp $\eta(\cdot \mid {}_t ls) \subseteq \mathbb{P}_Y^X$, and the set of *formally enforceable judgments* the

set of ordered pairs $\left\{ \langle {}_t x, y \rangle^M \in X \times Y \mid y \in \text{supp } g_{\eta(\cdot \mid {}_t ls), M}\left(\cdot \mid {}_t x\right) \right\} \subseteq X \times Y$,

with elements (judgments) realized with probability $g_{\eta(\cdot \mid {}_t ls), M}\left(\cdot \mid {}_t x\right)$.

---

[1] See Section II.5 regarding the reduction ($L = \# \left| \bigcup_{j=1}^{J} \left\{ \bigcup_{{}_t x \in X} \text{supp } g_j\left(\cdot \mid {}_t x\right) \right\} \right|$, see Section II.4).
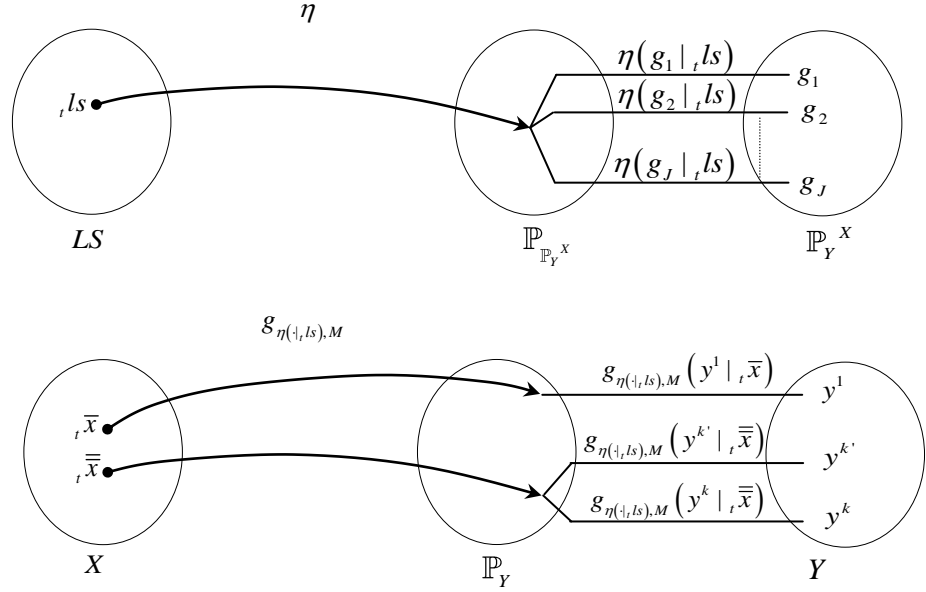
Figure 1.1 The process generating formally enforceable norms and judgments (node $g_{\eta(\cdot|_t ls),M}\left(y^1 |_t \overline{x}\right)$ is degenerate)

As a function of the legal source constellation $_t ls \in LS$, meta-norm η, and mechanism M, Definition 1.1 at time t delineates a set of conceivable (probabilistically complete) ordinary-level norms: the correspondence $\operatorname{supp}\eta : LS \rightrightarrows \mathbb{P}_Y^X$ assigns positive probability to the norms in $\operatorname{supp}\eta(\cdot |_t ls) \subseteq \mathbb{P}_Y^X$. The set of formally enforceable judgments are elements in a binary relation (compare the extensive norm notions in Section I.1).

However, the definition involves more than delineation of a range, it provides explicit distributions over the ordered pairs, reflecting:

- the meta-norm and ordinary norm structures,
- detailed aspects of the decision mechanism, and
- location of input data in $LS \times X$.

To exemplify, if the transformed mixed norm is locally determinate, as illustrated by $_t \overline{x} \in g_{\eta(\cdot|_t ls),M}^{-1}\left(\delta\mathbb{P}_Y\right)$ in Figure 1.1, the set of formally enforceable judgments

is the single element set $\left\{ \left\langle {}_t\bar{x}, !\, y^{\,\prime\,\bar{x}, g_{\eta(\cdot|_t ls), M}} \right\rangle^M \right\}$. If ${}_t\bar{\bar{x}} \in X \setminus g_{\eta(\cdot|ls)}^{-1}(\delta\mathbb{P}_Y)$, the set of

formally enforceable judgments equals $\left\{ \left\langle {}_t\bar{\bar{x}}, y^{k'} \right\rangle^M, \left\langle {}_t\bar{\bar{x}}, y^k \right\rangle^M \right\}$, generated with

probability $g_{\eta(\cdot|_t ls), M}\left(y^{k'} \mid {}_t\bar{\bar{x}} \right)$ and $g_{\eta(\cdot|_t ls), M}\left(y^k \mid {}_t\bar{\bar{x}} \right)$, respectively.

       Equilibrium analysis may reduce the set of formally enforceable judg-

ments to a subset named enforceable judgments:

> <u>Definition 1.2</u> At time t, conditional on data ${}_t ls \in LS$, $\eta \in \mathbb{P}_{\mathbb{P}_Y X}^{\,LS}$, and
>
> $M \in \mathcal{M}$, the set of *enforceable judgments* is the set ordered pairs
>
> $$\left\{ \left\langle x^*\!\left(g_{\eta(\cdot|_t ls), M} \right), y \right\rangle^M \in X \times Y \,\middle|\, y \in \operatorname{supp} g_{\eta(\cdot|_t ls), M}\left(\cdot \mid x^*\!\left(g_{\eta(\cdot|_t ls), M} \right) \right) \right\} \subseteq X \times Y$$
>
> , with elements (judgments) realized with probability
>
> $g_{\eta(\cdot|_t ls), M}\left(\cdot \mid x^*\!\left(g_{\eta(\cdot|_t ls), M} \right) \right)$.

Parts IV and V give a series of propositions characterizing solutions ${}_t x^*$ and

distributions $g_{\eta(\cdot|_t ls), M}\left(\cdot \mid x^*\!\left(g_{\eta(\cdot|_t ls), M} \right) \right)$, as functions of parameters describing

mechanisms $M \in \mathcal{M}$ and ordinary and meta-level norms.

Table 1.1 classifies alternative configurations of legal uncertainty. Because legal

source data and meta-norms are exogenously given, it suffices to distinguish situ-

ations where the latter *locally* function as rules or standards. Mechanisms

$M \in \mathcal{M}$ only affect norm structures and equilibrium outcomes in †-marked sit-

uations in the table: If abstract norms are determinative, they remain so in multi-

member mechanisms, generating *functional relations*. (Local) indeterminacy is necessary for legal decision mechanisms to have effect.[2]

Table 1.1 Configurations of norm-based uncertainty (relevant section references are indicated)

	meta-norm (local) rule	meta-norm (local) standard
ord. norm(s) global rule(s)	$ls \in \eta^{-1}\left(\delta\mathbb{P}_{\mathbb{P}_Y^X}\right); !g_j^{\,ls,\eta} \in \delta\mathbb{P}_Y^{\,X}$ deductive system (IV.4)	$ls \in \eta^{-1}\left(\mathbb{P}_{\mathbb{P}_Y^X} \setminus \delta\mathbb{P}_{\mathbb{P}_Y^X}\right); \mathrm{supp}\,\eta(\cdot \mid ls) \subseteq \delta\mathbb{P}_Y^{\,X}$ meta-level uncertainty (V.2)†
ord. norm(s) local rule(s)	$ls \in \eta^{-1}\left(\delta\mathbb{P}_{\mathbb{P}_Y^X}\right); x \in !g_j^{\,ls,\eta\,-1}\left(\delta\mathbb{P}_Y\right)$ locally deductive system (IV.4-IV.5)	$ls \in \eta^{-1}\left(\mathbb{P}_{\mathbb{P}_Y^X} \setminus \delta\mathbb{P}_{\mathbb{P}_Y^X}\right); x \in g_{\eta(\cdot\mid ls)}^{-1}\left(\delta\mathbb{P}_Y\right)$ meta-level uncertainty (V.3)†
ord. norm(s) local std(s)	$ls \in \eta^{-1}\left(\delta\mathbb{P}_{\mathbb{P}_Y^X}\right); x \in !g_j^{\,ls,\eta\,-1}\left(\mathbb{P}_Y \setminus \delta\mathbb{P}_Y\right)$ ord. lev. uncertainty (IV.4-IV.5)†	$ls \in \eta^{-1}\left(\mathbb{P}_{\mathbb{P}_Y^X} \setminus \delta\mathbb{P}_{\mathbb{P}_Y^X}\right); x \in g_{\eta(\cdot\mid ls)}^{-1}\left(\mathbb{P}_Y \setminus \delta\mathbb{P}_Y\right)$ uncert. on meta and ord. level (V.3)†

2 Power-conferring norms

In Definition II.2.1', norms g' map from a set of legal facts X' to the set of simple probability distributions over the *power set* of the set of consequences Y',

---

[2] In the case of bounded standards and corner solutions, there is an exception to this rule (necessity). The classifications in Table 2.1 depends on the decision mechanism if $M = M_1$ and the standard $g_F \in \mathbb{P}_{\{0,L\}}^{\,X}$ and accident technology induces equilibrium precaution investment equal to the upper support boundary, $x*(1) = \bar{x}$. In this case, the abstract norm locally functions as a rule: $_t x*(1) \in g_F^{-1}\left(\delta\mathbb{P}_{\{0,L\}}\right)$. However, if $m \geq 1$, due to the transformation of *marginal aspects* of legal uncertainty, equilibrium effects of panels imply that transformed norms locally are standards, $_t x*(2m+1)\big|_{m\geq 1} \in g_{F,2m+1}^{-1}\left(\mathbb{P}_{\{0,L\}} \setminus \delta\mathbb{P}_{\{0,L\}}\right)$, see Remarks IV.4.2 and IV.5.1.

Local uncertainty is not sufficient for mechanisms to have an effect. In the case of M_{2m+1} and dichotomous outcomes $\{y^{k'}, y^k\}$, say, the fixed point $\maltese$ such that $!g_j^{\,ls,\eta}\left(y^k \mid \maltese\right) = \frac{1}{2}$, or the fixed point $\maltese$ such that $g_{\eta(\cdot\mid ls)}\left(y^{k'} \mid \maltese\right) = \frac{1}{2}$ by Proposition III.2.2 is reproduced by all mechanisms ($h_{2m+1}\left(\frac{1}{2}\right) \equiv \frac{1}{2}$). By the same arguments, corresponding fixed points exists (that vary with mechanism parameters) for all $M_{[v;n]}$, as defined in probability space by the intersection of the $45°$-line and $H_{[v;n]}\left(!g_j^{\,ls,\eta}\left(y^k \mid \maltese_{[v;n]}\right)\right)$ and $H_{[v;n]}\left(g_{\eta(\cdot\mid ls)}\left(y^k \mid \maltese_{[v;n]}\right)\right)$, respectively, in Figure III.2.2. The argument generalizes to other mechanism.

$\wp\, Y'$. The set of such mappings, $g': X' \to \mathbb{P}_{\wp Y'}$, is denoted $\mathbb{P}_{\wp Y'}{}^{X'}$ and is intended to represent norm-based discretion (protected choice) over elements in Y' and, at the same time, possible uncertainty about discretion range. This section extends Y' to be a set of (ordinary) norms, $\mathbb{P}_Y{}^X$. In this case, the power-set $\wp\, Y' = \wp\, \mathbb{P}_Y{}^X$ is a *family* of norm sets, and g' is a mapping from the set of legal facts to the set of simple distributions over this family, $g': X' \to \mathbb{P}_{\wp \mathbb{P}_Y{}^X}$.[3] The intended interpretation of norms in $\mathbb{P}_{\wp \mathbb{P}_Y{}^X}{}^{X'}$ is as power-conferring, see Remark II.2.8. Because the situation, in principle, is one of "norms over norms (norm sets)", they are denoted η' (see Section II.5).[4]

> Definition 2.1   A power-conferring norm is a triple $\left(X', \mathbb{P}_{\wp \mathbb{P}_Y{}^X}, \eta': X' \to \mathbb{P}_{\wp \mathbb{P}_Y{}^X} \right)$. The set of possible power-conferring norms is denoted $\mathbb{P}_{\wp \mathbb{P}_Y{}^X}{}^{X'}$.

Remark 2.1 If power-conferring norms represent norms on the level of sources of law, $X' = LS = LS_1 \times \cdots \times LS_l$ and $\eta' \equiv \eta$. □

Power-conferring norms endow legislators and contract parties with the ability to create new norms, that is, objects that become elements in the source classes, $LS_1 \times \cdots \times LS_l$. The enacted norms become input to new court decisions, giving

---

[3] To avoid technical difficulties it is assumed that $\mathbb{P}_Y{}^X$ is finite. If $\#\left| \mathbb{P}_Y{}^X \right| = J$, $\#\left| \wp\, \mathbb{P}_Y{}^X \right| = 2^J$ (Section II.1).

[4] For a motivation, consider the ECtHR chamber judgment *Gillow v. United Kingdom*: "A law which confers a discretion is not in itself inconsistent with the requirement of foreseeability, provided that the scope of the discretion and the manner of its exercise are indicated with sufficient clarity, having regard to the legitimate aim of the measure in question, to give the individual adequate protection against arbitrary interference [---]" (Nov. 24, 1986, Sec. 51).

legal systems their characteristic dynamic character: η maps from an updated set

of legal sources LS .[5]

Example 2.1 (power-conferring norms). To explicate the notion of power-conferring

norms, consider a set of potential norms, $\{g_1, g_2, \ldots, g_J\} \subseteq \mathbb{P}_Y^{\ X}$, say. In "normal" circum-

stances, $X^n \subseteq X'$ power-conferring norm $\eta' \in \mathbb{P}_{\wp\{g_1,\ldots,g_J\}}^{\ X'}$ allows a choice from the

whole set, $\{g_1, g_2, \ldots, g_J\}$:

$$\eta'(\cdot\,|\,x')\big|_{x'\in X^n} = \big\langle 0,0,\ldots,0,1;\varnothing,\{g_1\},\{g_2\},\ldots,\{g_J\},\{g_1,g_2\},\ldots,\{g_1,\ldots,g_J\}\big\rangle.$$

In Definition II.3.3-notation, if $x' \in X^n$, η', in effect, maps to the unique and full set of

legislative or contract options, $!\{g_1, g_2, \ldots, g_J\}^{x',\eta'} \in \wp\{g_1,\ldots,g_J\}$. Under "extraordinary

circumstances" $x' \in X^e \subseteq X'$, η' implies incompetence or void contracts

$$\eta'(\cdot\,|\,x')\big|_{x'\in X^e} = \big\langle 1,0,\ldots,0,;\varnothing,\{g_1\},\{g_2\},\ldots,\{g_J\},\{g_1,g_2\},\ldots,\{g_1,\ldots,g_J\}\big\rangle,$$

η', in effect, maps to the unique empty set of options set $!\{\cdot\}^{x',\eta'} \in \wp\{g_1,\ldots,g_J\}$.

---

[5] "All modern legal systems a well-developed series of norms concerning the various modes of procedure for the establishment of formulated law, the reciprocal relation between the various levels of enacted law from the constitution to the private contract, the promulgation and coming into force of statutes, the delegation of legislative power, voidability and judicial control, and so on. These all constitute elements of the doctrine of the sources of law even though they may not be collected under this heading, but included under constitutional law, administrative law and the law of contracts" (Ross [1959:103]).

Procedural norms endowing courts and arbitrational tribunals with ability to generate binding decisions are power-conferring, but (generally) in the more limited sense of specific ordered pairs (Section I.1). Also, norm content judicial review is typically done in deciding individual cases and not in general, abstract terms: judgments do not take the form $\langle ls, g \rangle^M$. The question of contract content restriction is broad, an includes questions about enforceability, interpretation and civil and criminal law sanctions (see Hermalin, Katz & Craswell [2007]). See also Ayres & Gertner [1989] on preceptive (immutable) versus declaratory (default) rules.

The possibility of successful enactments, in the sense of being mapped to themselves under η, may be discussed using fixed point theorems (convexity and continuity ensures the possibility). The question has analogies in jurisprudential disputes about predictions (but accentuate both Arrow-type impossibility and the warnings in Aubert [1984]).

Similarly, η' singles out non-empty, proper subsets of $\{g_1,\ldots,g_J\}$, with probability 1 in $n'^{-1}\left(\delta\mathbb{P}_{\wp\{g_1,g_2,\cdots,g_J\}}\right)\backslash X^n \cup X^e$. In $X'\backslash n'^{-1}\left(\delta\mathbb{P}_{\wp\{g_1,g_2,\cdots,g_J\}}\right)$, the span of competence is uncertain: η' locally functions as a standard, giving a non-degenerate distribution over the family of norms, such as at $\overline{\overline{x}}'$, in Figure 2.1. ∎
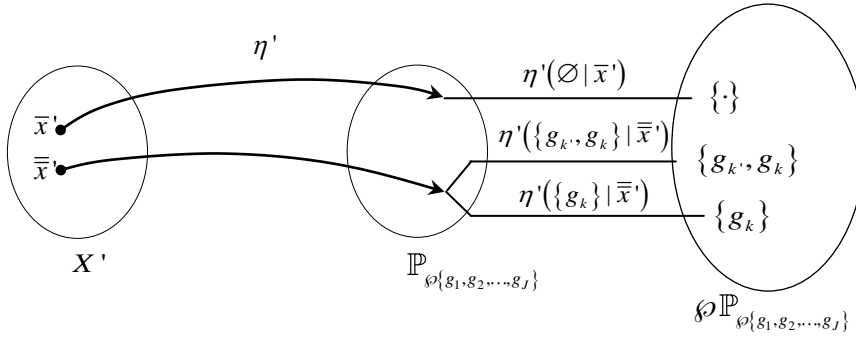
Figure 2.1 Global representation of power-conferring norms (node $\eta'(\cdot\,|\,\overline{x}')$ degenerate)

Two definitions are introduced to give "snapshot" descriptions of established positions. In this regard, note that at $\overline{\overline{x}}'$, the normative ability extent is indeterminate. Consider the *family of sets,* defined by $\operatorname{supp}\eta'\left(\cdot\,|\,\overline{\overline{x}}'\right)=\left\{\{g_k\},\{g_{k'},g_k\}\right\}$.

The *intersection of the family* at $\overline{\overline{x}}'$ is the *largest set* contained in all family members, defined by:[6]

$$\bigcap\operatorname{supp}\eta'\left(\cdot\,|\,\overline{\overline{x}}'\right)=\bigcap\left\{\{g_{k'},g_k\},\{g_k\}\right\}=\{g_{k'},g_k\}\cap\{g_k\}=\{g_k\}.$$

The set identifies the norms that can be selected for sure given $\overline{\overline{x}}'$, corresponding to a certain competence span (CS).

<u>Definition 2.2</u> At $x'\in X'$, the certain competence span $\delta CS^{\eta'}(x')$ is given by: $\delta CS^{\eta'}(x')=\bigcap\operatorname{supp}\eta'(\cdot\,|\,x').$[7]

---

[6] Let $\mathfrak{I}=\{F_1,F_2,\ldots\}$ be a family of sets. The intersection of the family, denoted $\bigcap\mathfrak{I}$, is the set of all things that are members of all sets in the family (see Suppes [1972:39]).

In a similar manner, the *union of the family* at $\bar{\bar{x}}'$ is the smallest set containing all the members, defined by:[8]

$$\bigcup \operatorname{supp} \eta'\left(\cdot \mid \bar{\bar{x}}'\right) = \bigcup \left\{\left\{g_k\right\}, \left\{g_{k'}, g_k\right\}\right\} = \left\{g_k\right\} \cup \left\{g_{k'}, g_k\right\} = \left\{g_{k'}, g_k\right\}.$$

The set identifies the set of norms that may be available at $\bar{\bar{x}}'$.

> Definition 2.3 At $x' \in X'$, the range of possible competence span $CS^{\eta'}(x')$ is given by $CS^{\eta'}(x') = \bigcup \operatorname{supp} \eta'(\cdot \mid x')$.[9]

Remark 2.2 In the context of constitutions, systems of power-conferring norms induce hierarchally ordered norms, another characteristic feature of legal systems (Ross [1959], Sec. 44 and 48). Both rank and range dimensions of competence are important in *dynamic strategic* contexts, see Section 3.2 below. □

Remark 2.3 The demarcation between meta-norms and ordinary norms is relative. Constitutional norms typically function as meta-norms in legal discourse, but may function as ordinary norms in specific situations.[10] □

---

[7] More generally, by letting the conditioning fact x' run through a region R in X', a more global notion $\delta CS^{\eta'}(R) = \bigcap_{x' \in R} \bigcap \operatorname{supp} \eta'(\cdot \mid x')$ may be defined. $\delta CS^{\eta'}(R)$ is the set of norms always available on R. In Figure 2.1, $\delta CS\left(\left\{\bar{x}', \bar{\bar{x}}'\right\}\right) = \varnothing$.

[8] The *union of the family*, denoted $\bigcup \mathfrak{I}$, is the set of all things that belongs to some member of the family (see Suppes [1972:37]).

[9] Clearly, $\delta CS^{\eta'}(x') \subseteq CS^{\eta'}(x')$ (Suppes [1972], Theorem 79). The set difference defines options only available with a positive probability, $CS^{\eta'}(x') \setminus \delta CS^{\eta'}(x') = {}^{\mathbb{P}}CS(x')$, say. In Figure 2.1, ${}^{\mathbb{P}}CS(x') = CS\left(\bar{\bar{x}}'\right) \setminus \delta CS\left(\bar{\bar{x}}'\right) = \left\{g_{k'}, g_k\right\} \setminus \left\{g_k\right\} = \left\{g_{k'}\right\}$.

[10] See for example the *Sunday Times v. United Kingdom* judgment referred to in Section I.1. Similarly, the principle of *lex superior* functions as a meta-norm in legal argumentation. However, the extent of constitutionally admissible delegation may be the object of litigation.

Mixes and transformation of power-conferring norms may arise, due to meta-level uncertainty and to direct norm-maintenance in judicial panels. However, because these situations are not accentuated in the applications below, analytical dimensions are outlined in Appendix A.2 and A.3.

3 Configurations of norm-based uncertainty and law-in-force notions II: discretion

Discretion can be present at the meta-level (corresponding to optional norm selection) or at the level of ordinary norms (corresponding to optional legal consequence selection). The general formulation $\eta \in \mathbb{P}_{\wp\mathbb{P}_{\wp Y}X}{}^{LS}$ admits discretion and norm-based uncertainty about discretion span, at both levels. The next two examples illustrate challenges that arise under collective legal decision-making in discretionary environments. The examples abstract from norm-based uncertainty and isolate discretion to either the ordinary or meta-level, respectively.

<u>Example 3.1</u> (ordinary-level discretion and a Condorcet-style paradox). Assume that $_t ls \in \eta^{-1}\left(\delta\mathbb{P}_{\delta\mathbb{P}_{\wp Y}X}\right)$, such that η, in effect, maps to the single element set of enforceable norms $!\{g_k\}^{t\,ls,\,\eta}$, and that the global rule $g_k \in \delta\mathbb{P}_{\wp Y}{}^{X}$ at $_t x \in X$, in effect, maps to $!\{y',y'',y'''\}^{t\,x,\,g_k}$. Consider M_3, and assume that judges order outcomes under g_k conditioned on $_t x$ as follows:

Judge 1	$y' \succ^1 y'' \succ^1 y'''$
Judge 2	$y'' \succ^2 y''' \succ^2 y'$
Judge 3	$y''' \succ^3 y' \succ^3 y''$

If protected choices are made from the set $!\{y',y'',y'''\}^{t\,x,\,g_k}$ by sequential majority decisions, $y' \succ^{M_3} y''$, and $y'' \succ^{M_3} y'''$. Hence, a court satisfying basic rationality expectations (compare Section I.1) *should* find $y' \succ y'''$. However, the court expresses a circular or-

206

dering when presented with a sequence of binary choices, $y''' \succ^{M_3} y'$: so-called *Condorcet paradox* has occurred. The rendered judgment depends on how the agenda is set (even assuming sincere voting). Starting with y' versus y'', $\langle_t x, y''' \rangle^{M_3}$ results; starting with y' versus y''', $\langle_t x, y'' \rangle^{M_3}$ results; starting with y'' versus y''', $\langle_t x, y' \rangle^{M_3}$ results.[11] ∎

Example 3.2 (Ex. II.3.5 continued: meta-level discretion and a generalized doctrinal paradox). Assume $_t ls \in \eta^{-1}\left(\delta\mathbb{P}_{\wp\,\delta\mathbb{P}_{\{0,L\}}^X} \right)$, the meta-norm, in effect, mapping to the set of enforceable norms $!\left\{ g_{\delta\infty}, g_{\delta\underline{c}}, g_{\delta\overline{c}} \right\}^{_t ls, \eta}$. The global rules $g_{\delta\infty}$, $g_{\delta\underline{c}}$, and $g_{\delta\overline{c}}$ represent strict liability, a lenient negligence rule, and a rigorous negligence rule, respectively (see Example II.3.3 and Section V.2). Assume that $\underline{c} <\, _t x < \tilde{x} < \overline{c}$ ($\tilde{x}$ is defined in Figures IV.4.1 and V.2.1).

Judges, envisioned to *apply* the law, do not vote on $g_{\delta\infty}$, $g_{\delta\underline{c}}$, and $g_{\delta\overline{c}}$.[12] Instead, judgments are aggregated analogously to the OBV and PBV regimes considered in Section III.4. It is instructive to apply the framework introduced by Landa & Lax [2009], discussed in Example II.3.5: Let $C = \{0,1\} \times \{0,1\} \times \{0,1\}$ be the case space, and let the first, second, and third dimensions correspond to whether the criterion for strict liability, rigorous negligence, or lenient negligence is satisfied, respectively. Because judges observe the same conditioning fact $\underline{c} <\, _t x < \overline{c}$, $c = \langle 1,1,0 \rangle$. They select the following base rules (mapping to L iff $c \cdot rd^j \geq \tau^j$):

---

[11] The profile corresponds to Arrows [2012:3] example. Stressing the role of transitivity, Arrow remarks "Independence of the final choice from the path to it" makes legal systems "capable of full adaption to varying environments" (p. 120).

[12] If they did, an implied notion of judges forming preferences (complete transitive orderings) over norms is accentuated (see Section VIII.2). With three or more norms and three or more judges, Condorcet cycles may appear as exemplified in Example 3.1.

$$\text{Judge 1 } (g_{\delta\infty}) \quad br^1 = \langle rd, \tau \rangle^1 = \langle\langle 1,0,0\rangle, 1\rangle$$
$$\text{Judge 2 } (g_{\delta\underline{c}}) \quad br^2 = \langle rd, \tau \rangle^2 = \langle\langle 0,0,1\rangle, 1\rangle$$
$$\text{Judge 3 } (g_{\delta\overline{c}}) \quad br^3 = \langle rd, \tau \rangle^3 = \langle\langle 0,1,0\rangle, 1\rangle$$

If voting directly on outcomes $Y = \{0, L\}$ ("case-by-case adjudication"), in M_3 Judge 1 and Judge 3 vote for liability ($c \cdot \langle 1,0,0 \rangle = c \cdot \langle 0,1,0 \rangle = 1 \geq 1$). Hence, the judgment $\langle {}_t x, L \rangle^{M_3}$ is generated. Alternatively, the court might proceed using what Lada and Lax call a doctrinal *aggregation method,* involving a so-called collegial factor rule (CFR). This implies (i) generation of a collective base rule, $br^{CFR, M} : C \to \{0, L\}$ ($Y = \{'no', 'yes'\}$ in their framework) and (ii) application of $br^{CFR, M}$ to the case, $c \in C$. The collegial factor rule $br^{CFR, M}$ is constructed by majority vote on each factor dimension, combined with the median threshold. Hence:

$$br^{CFR, M_3} = \langle\langle 0,0,0\rangle, 1\rangle.^{13}$$

It follows that $c \cdot rd^{CFR, M_3} = \langle 1,1,0 \rangle \cdot \langle 0,0,0 \rangle = 0$, generating judgment $\langle {}_t x, 0 \rangle^{M_3}$.

Generation of $\langle {}_t x, L \rangle^{M_3}$ and $\langle {}_t x, 0 \rangle^{M_3}$ means that what Landa and Lax call a *generalized doctrinal paradox* arises.[14] They analyze the paradox from a global perspective and reveal logical difficulties inherent in case-by-case decision-making that challenge visions of law as coherent and determinate.[15] ∎

---

[13] Note that $br^{CFR, M_3} = \langle\langle 0,0,0\rangle, 1\rangle$ generated by the collective is not equal to any of the individual judges' base rules. This contrasts to the *classical rule,* see Remark III.1.1.

[14] The paradox has similarity to the (regular) doctrinal paradox illustrated in Example III.4.1 where all judges share the *same* base norm (corresponding to the doctrinal constraint $c \leftrightarrow a \wedge b$), but form different views on ultimate facts (the case), see Remark III.4.1.B.

[15] All judges observe the same case $c \in C$ and are identified with base rules (mappings) $br : C \to \{'no', 'yes'\}$. A rule can be associated with its outcome set ("decision set"), the ordered set of pairs of cases and case outcomes ("case dispositions"). In Section I.1 terminology, the set is a functional relation in $C \times \{'no', 'yes'\}$. Under reasonable assumptions about the outcome set and base functions (keyword: monotonicity), Landa and Lax investigate to which extent outcome sets generated by collectives (called the collegial

Legal decision-making under discretion *and* norm-based uncertainty, from the ex ante perspective, is conceived as follows: At point in time t, judges $j \in \{1, 2, \ldots, n\}$ independently:

I. *observe* data constellation $\langle {}_t ls, {}_t x \rangle \in LS \times X$;

II. *apply* (power-conferring) meta-norm $\eta \in \mathbb{P}_{\wp\mathbb{P}_{\wp Y}{}^X}{}^{LS}$ to ${}_t ls$, leading to the (simple) distribution $\eta(\cdot \mid {}_t ls) \in \mathbb{P}_{\wp\mathbb{P}_Y{}^X}$ over the family of norm sets in $\wp\mathbb{P}_{\wp Y}{}^X$;

III. (i) *select* norms from $\operatorname{supp} \eta(\cdot \mid {}_t ls) \subseteq \wp\mathbb{P}_{\wp Y}{}^X$;

(ii) *apply* selected norms to ${}_t x \in X$; and

(iii) *vote* in $M \in \mathcal{M}$, according to the selected norm and (possible) re-solved uncertainty.

The process is illustrated in Figure 3.1 and simplified using Definitions 2.2 and 2.3. At stage II, the set of possibly available norms is given by $CS^\eta({}_t ls)$ $= \bigcup \operatorname{supp} \eta(\cdot \mid {}_t ls)$.[16] At stage III, the set of options possibly available to a judge, conditional on selection of g_k, is $CS^{g_k}({}_t x) = \bigcup \operatorname{supp} g_k(\cdot \mid {}_t x)$.[17] The set of possible conclusions conditioned on ${}_t x$ is, therefore, spanned by $\bigcup_{g_k \in CS^\eta({}_t ls)} CS^{g_k}({}_t x)$.

This leads to the following suggested law-in-force definition (reflecting the lack

---

decision set) can be induced by the individual base rules when decisions are generated by majority rule and median thresholds. In this setting they develop impossibility theorems.

[16] The set of certainly available norms is given by $\delta CS^\eta({}_t ls)$ $= \bigcap \operatorname{supp} \eta(\cdot \mid {}_t ls)$ $\subseteq CS^\eta({}_t ls)$, a correspondence $CS^\eta : LS \rightrightarrows \mathbb{P}_{\wp Y}{}^X$.

[17] The set of certainly available consequences is given by $\delta CS^{g_k}({}_t x)$ $= \bigcap \operatorname{supp} g_k(\cdot \mid {}_t x) \subseteq CS^{g_k}({}_t x)$, a correspondence $CS^{g_k} : X \rightrightarrows Y$.

of structure from norm-based uncertainty and discretion at both the meta- and ordinary level):
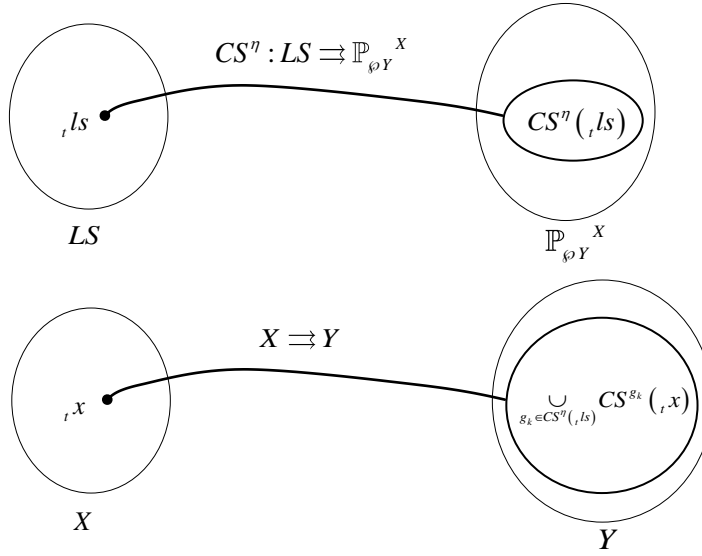
Figure 3.1 Enforceable norms and formally enforceable judgments under discretion and legal uncertainty (global representation)

<u>Definition 3.1</u> At time t, conditional on data $\langle {}_t ls, {}_t x \rangle \in LS \times X$, $\eta \in \mathbb{P}_{\wp \mathbb{P}_{\wp Y}{}^X}{}^{LS}$, and $M \in \mathcal{M}$, the set of *enforceable norms* is $CS^{\eta}\left({}_t ls \right) = \bigcup \mathrm{supp}\, \eta \left(\cdot \mid {}_t ls \right) \subseteq \mathbb{P}_{\wp Y}{}^X$. The set of *formally enforceable judgments* is the set of ordered pairs:

$$
\begin{aligned}
&\left\{ \langle {}_t x, y \rangle^M \in X \times Y \middle| y \in \bigcup_{g_k \in CS^{\eta}\left({}_t ls \right)} CS^{g_k}\left({}_t x \right) \right\} \\
&= \left\{ \langle {}_t x, y \rangle^M \in X \times Y \middle| y \in \bigcup_{g_k \cup \mathrm{supp}\, \eta \left(\cdot \mid {}_t ls \right)} \bigcup \mathrm{supp}\, g_k \left(\cdot \mid {}_t x \right) \right\}
\end{aligned}.
$$

Discretion means (protected) *choice*. The normative sources (conditioning facts and norm structures) do not determine a probability distribution over formally enforceable judgments, but only delineate a binary relation counter-domain. As Ross [1959:21–22] pertinently points out:

> Any presentation of [law-in-force] confined to a definite date is [---] a question as to what will happen tomorrow. A co-determining factor for this calculation is what happened yesterday. [Law-in-force] is never a historical fact but a calculation, with regard to the future. This gives to the propositions of present-day study of law a fundamental element of uncertainty and results, as the certainty of the calculation diminishes, in a peculiar fusing together of problems of the law in force with problem of legal politics […].

Sen [1967] fundamentally distinguishes between basic and non-basic value judgments. Basic value judgments are normative propositions, that will not be revised under any conceivable variation in factual premises. However, as he points out, at closer inspection, many seemingly basic propositions turn out to be factually conditioned, and, as such, can be subject to positive analysis. Furthermore, a set of inconsistent normative propositions cannot all be basic (Sen [1979:67–70]). The unavoidable logical problems occurring in collective legal decision-making (see Section I.1, Examples 3.1, 3.2, and III.4.1), thus, independently suggest a role for functional analyses. As Arrow [2011:25] notes:

> Hume famously distinguished between normative and descriptive propositions […]; one can never deduce a proposition that imposes an obligation (an "ought") from a series of statements that describe the world (statements of what "is"). Obviously, in some sense, this must be right, but the examination of social choice theory suggests that the dichotomy is more blurred than it seems.

In conclusion, in situations where decision are not logical derivations under (locally) consistent and determinative norms, there is room for functional or repercussion analysis of law ("legal politics" in the sense of Ross), as constrained by the conditioning sources and legal system meta-norms.[18]

---

[18] In the Ross quote the inserted "law-in-force" term replaces "valid law" (the latter caused confusion, see Ross [1962] and Tur [1998]). The suggested legal norm and law-in-force notions make use of decision theory elements. As stressed by Gilboa [2009:55–64], an advantage of formal representations is that questions regarding proposition status (modality) can be left to interpretation. Discussions about the "nature of law" and "true meaning of definitions" (Merryman & Pérez-Perdomo [2007:63]) can be avoided.

  Paul Dirac points out: "When you ask what are electrons and protons I ought to answer that this question is not a profitable one to ask and does not really have a meaning. The important thing about electrons and protons is not what they are but how they behave – how they move. I can describe the situation by comparing it to the game of chess. In chess, we have various chessmen, kings, knights, pawns, and so on. If you ask what a chessman is, the answer would be [that] it is a piece of wood, or a piece of ivory,

Mattei & Cafaggi [1998:350] argue that as state monopoly on legal production is eroded, and agents increasingly are able to choose law, competitive forces should put more focus on a means-end, forward-looking approach to law.[19] Fon & Parisi [2007:147] emphasize that "incomplete legal precepts can be purposefully enacted as a way to optimize the lawmaking and adjudication functions, transferring to the judiciary some of the tasks that would otherwise have to be carried out ex ante by the legislature." On the meta-norm level, means-end approach is used by Shavell [2006] (contract interpretation[20]) and Macey [1998] (precedent[21]). This perspective is adopted below.

<u>Remark 3.1</u> In absence of norm-based uncertainty on the meta-level, ordinary-level, or both levels, the sets of formally enforceable judgments reduce to:

$$\left\{ \langle _t,x,y\rangle^M \in X\times Y \,\middle|\, y \in \bigcup_{g_k \in \delta CS^n(_t ls)} CS^{g_k}(_t x)\right\}, \quad \left\{ \langle _t,x,y\rangle^M \in X\times Y \,\middle|\, y \in \bigcup_{g_k \in CS^n(_t ls)} \delta CS^{g_k}(_t x)\right\},$$

---

or perhaps just a sign written on paper, [or anything whatever]. It does not matter. Each chessman has a characteristic way of moving and this is all that matters about it. The whole game of chess follows from this way of moving the various chessmen [...]" (quoted from Farmelo [2009:354]).

　　Interestingly, the chess metaphor is similarly used by Feller [1968:1–6] regarding probability and by Ross [1959:11–18] regarding law-in-force notions. (See Popper [1964] for a critical account of essentialism.)

[19] See also Posner [1990], especially pp. 105–8.

[20] See Example II.5.2 and Remark IV.1.2.

[21] "A richer conception of precedent would have judges pursue the general societal goals articulated in previous cases, even when those general goals are no longer being reached by the particular rules articulated in individual cases. This approach permits judges to remain faithful to the ideal of precedent by following general principles reflected in previous cases while ignoring the specific holdings when such specific holdings are inconsistent with these principles. In a world of flux, the most desirable legal systems are those that offer credible promises to legal actors that they will continue to maximize social welfare by quickly responding to new information and changing conditions" (pp. 72–73).

and $\left\{ \left\langle {}_t x, y \right\rangle^M \in X \times Y \middle| y \in \underset{g_k \in \delta CS^n({}_t ls)}{\cup} \delta CS^{g_k}\left({}_t x \right) \right\}$, respectively. The sets are contracting.[22] □

3.1 Legal politics I: parametric environments (controllability)

A general framework for discussion of "legal politics" may be conceived as follows: Let S be the set of possible social system states in the general sense of Arrow [2012:17], and individual states be described by $s = \left\langle s_1, \ldots, s_n \right\rangle$. Assume that (democratically elected) law *empowers* an agent to use a vector of instruments $\mu = \left\langle \mu_1, \ldots, \mu_k \right\rangle \in M \subseteq \mathbb{R}^k$, but at the same time, *obligates* the agent to ensure that the target vector $\bar{s} = \left\langle \bar{s}_1, \ldots, \bar{s}_m \right\rangle \in \bar{S} \subseteq S$, $m \leq n$ is reached. A positive theory about the working of the system is represented by a system of n (independent) equations:

$$
\begin{aligned}
\phi^1\left(s_1, \ldots, s_n; \mu_1, \ldots, \mu_k \right) &= 0 \\
&\vdots \\
\phi^n\left(s_1, \ldots, s_n; \mu_1, \ldots, \mu_k \right) &= 0
\end{aligned}
$$

Generally, the system can be solved for $n - m$ variables (not of direct concern) and be written:

$$
\begin{aligned}
\Phi^1\left(s_1, \ldots, s_m; \mu_1, \ldots, \mu_k \right) &= 0 \\
&\vdots \\
\Phi^m\left(s_1, \ldots, s_m; \mu_1, \ldots, \mu_k \right) &= 0
\end{aligned}
\tag{3.1}
$$

---

[22] In a simple binary situation (without meta-level uncertainty), conditioned on the possible configurations of a fixed panel of 19 apex judges, Nordén [2010] calculates the probability of outcomes in smaller panels (M_3, M_5 and M_{11}) *deviating* from the M_{19} * majority benchmark. The calculation of deviation probabilities assume of random ("impartial") allocation of the 19 judges to the smaller panels. It is a simple exercise in hypergeometric probabilities to demonstrate serious risk of contrary judgments appropriatly *conditioned* on plenary-panel configurations. The deviation-risk, not surprisingly, increases in panel size reductions, peaking at more than 44 percent.

Locally, the transformation $\Phi : \bar{S} \times M \rightarrow \mathbb{R}^m$ can be described by its linear approximation (assuming the Jacobian matrix $\Phi_{m \times m}$ has full rank, m) and is given by:[23]

$$\Phi_{m \times m} \, s + \underline{\Phi}_{m \times k} \, \mu = 0 \leftrightarrow s = -\Phi_{m \times m}^{-1} \underline{\Phi}_{m \times k} \, \mu \, . \qquad (3.2)$$

It follows:

- for an arbitrary target $\bar{s} \in \bar{S} \subseteq S$ to be reachable, at least as many independent instruments as independent targets in $\langle \bar{s}_1, \ldots, \bar{s}_m \rangle$ are needed: If $k = m$, a *unique* instrument vector (control) is defined for each $\bar{s} \in \bar{S}$ [24]

- if $k > m$ (more targets than instruments), the system has $k - m$ *degrees of freedom*: For a given target vector $\bar{s}$, $k - m$ instruments can be selected in M, while the remaining instruments are uniquely determined

From (3.2), if $k = m$, the law *implicitly* defines a *global rule* $g^{\Phi} : \bar{S} \rightarrow M$, $\mu = -\underline{\Phi}_{m \times m}^{-1} \Phi_{m \times m} \bar{s}$. If $k > m$, it defines a *correspondence* $\gamma^{\Phi} : \bar{S} \rightrightarrows M$. The two situations, conditioned on Φ, track the distinction between obligation (to do exactly $\mu = g^{\Phi}(\bar{s}) \in M$) and constrained option (select one of the instruments $\mu \in \gamma^{\Phi}(\bar{s}) \subseteq M$). If *structural* (epistemic) uncertainty is added to (3.1), corresponding to random matrix elements in (3.2), and the target is described as a loss function around $\bar{s}$, the optimal instrument is generally uniquely determined,

---

[23] The inverse matrix $\Phi_{m \times m}^{-1}$ exists by the full rank assumption (Sydsæter [1981:92–98]).

[24] The result concerns so-called static controllability. The number of independent instruments corresponds to the dimension of the column-space of $\underline{\Phi}_{m \times k}$: it has dimension k and hence spans $\mathbb{R}^m$. In general, *each* instrument is a function of *all* targets. In some circumstances (keyword: block-recursive structures) the constraint on instrument use is less rigid (admitting delegation of instrument use): see generally Johansen [1978], Ch. 7.

even if the number of instruments is larger than the number of variables in the minimand ($k > m$), closing the discretion.[25]

Remark 3.2 The controllability results generally hold in dynamic settings. The system of equations is then written $\Phi(s, \dot{s}; \mu) = 0$, where $s = s(t)$ is a vector of *continuous* state variables and $\dot{s}(t)$ its derivative with respect to time.[26] □

The next two examples concern the use of Pareto-efficiency (PE) as a (necessary) criterion for norm selection. The concept's centrality in functionally oriented analyses of law and appearance *in* formal sources (both legislation and international conventions), accentuate its importance. Combined with a positive theory linking norm structures to (equilibrium) outcomes, PE is a powerful concept:[27] Add to the description of S a set of $i = \{1, \ldots, I\}$ agents, represented by the profile $\langle \succsim^1, \ldots, \succsim^I \rangle$ of weak (complete, transitive) preference orderings on S (a set of "culmination outcomes in Sen's [2009:215-7] terminology). A movement from state $s' \in S$ to $s'' \in S$ is a *Pareto-improvement* (PI) if all agents weakly prefer s'' to s' ($s'' \succsim^i s'$ for all i), and at least one agent strictly prefers s'' (for at

---

[25] Under standard regularity assumptions a unique instrument vector $\mu^{*\Phi}$ (as a function of system parameters, including the ones describing the loss function) results, see Brainard [1967] and Example 3.6 below.
    Craswell & Calfee [1986] demonstrate that as uncertainty increases (the standard deviation in normally distributed standards) the effect of a change in legal policy (the distribution mean) is blunted.

[26] In Leibniz' notation (Devlin [1994:83–87])! See generally Murata [1977].

[27] See generally Kaplow & Shavell [2002]. Anti-trust law is a classic example of law prescribing target states $\bar{s} \in \bar{S}$; functional analysis ("infiltration of economics" Whinston [2006:6]) seems unavoidable. Similar arguments can be made regarding corporate law (see, generally, Easterbrook & Fischel [1991]). Normative issues have been intensively discussed, see e.g. Coleman [2003].

least one i, $s'' \succsim^i s'$ and not $s' \succsim^i s''$). The set of Pareto-efficient states, $S_{PE} \subseteq S$, is the set of states from which no PI is possible.[28]
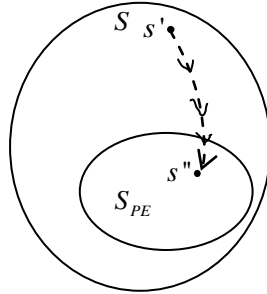
Figure 3.2 The set of possible states S and Pareto-efficient states S_{PE}

States in S_{PE} do not waste resources, in the sense that states allowing PI-moves have been filtered out. However, not all moves from $S \setminus P_{PE}$ to S_{PE} can be designed as PI-moves. In lack of backing in formal sources $ls \in LS$, it will be argued that use of the PE concept regarding coerced moves in mechanisms $M \in \mathcal{M}$ (in contradistinction to legislatures), should rely on a positive analysis demonstrating a PI move. This leads to a distinction between contract and tort law. Exchange economies are used as a framework for the analysis.

It is helpful to define the set of states that agent i, relatively to state $\tilde{s}$, strictly prefers, is indifferent to, and finds strictly worse, respectively:

$$P^i(\tilde{s}) = \left\{ s \in S \middle| s \succsim^i \tilde{s} \text{ and not } \tilde{s} \succsim^i s \right\}$$
$$I^i(\tilde{s}) = \left\{ s \in S \middle| s \succsim^i \tilde{s} \text{ and } \tilde{s} \succsim^i s \right\} \quad .$$
$$W^i(\tilde{s}) = \left\{ s \in S \middle| \tilde{s} \succsim^i s \text{ and not } s \succsim^i \tilde{s} \right\}$$

Example 3.3 (contracts in general equilibrium). Consider a convex exchange economy, with L commodities available in fixed quantities $\langle \bar{\omega}_1, \ldots, \bar{\omega}_L \rangle$.[29] Let x_j^i be the quantity

---

[28] In defining S, basic notions of property rights and mechanisms for their transfers are implicit and invariant under the analysis. The definition of PE is a so-called *strong* version.

216

of commodity j allocated to agent i, and $x^i = \langle x_1^i, \ldots, x_L^i \rangle$ the L-tuple of commodities

allocated to i, respectively. An *allocation* is a $L \times I$-tuple $x = \langle x^1, \ldots, x^L \rangle$, such that

$\sum_{i=1}^{I} x_j^i = \bar{\omega}_j$, $j = 1, \ldots, L$ (the equations define S). An initial assignment of entitlements

to the commodities is denoted $\omega = \langle \omega^1, \ldots, \omega^I \rangle$.[30] The situation is illustrated in the 2×2

economy in Figure 3.3, with the added assumption that agent i's preferences effectively

only depend on $x^i = \langle x_1^i, \ldots, x_L^i \rangle \in \mathbb{R}_+^L$, which correspons to an assumption of no exter-

nalities.[31]

Agents endowed with (transferable) entitlements may form coalitions and trade.

The Core defines a theoretical benchmark: A *coalition* of agents K is a subset of

$\{1, \ldots, I\}$, including the whole set and the I sets of single agents, $\{i\}$.[32] An allocation

$x \in S$ is *blocked* by K if, by using the resources available to it, the coalition can make

all members better off: K blocks x if there is an allocation $\breve{x}$ such that

$\sum_{i \in K} \breve{x}^i = \sum_{i \in K} \omega^i$ and $\breve{x}^i \succsim^i x^i$ for all $i \in K$. The *core, $C(\omega)$,* is the set of allocations

which no coalition of agents can block.

In Figure 3.3, indifference sets (curves) $I^A(\omega)$ and $I^B(\omega)$ pass through the

initial endowment point $\omega \in S$. A move to the north-east of A's indifference curve is a

move to $P^A(\omega)$: an improvement for A. (A move to the south-west into $W^A(\omega)$ is

blocked by $\{A\}$ and is said to be to be "individually irrational".) Using the same reason-

---

[29] The discussion is conceptual. For full details, see Maulinvaud [1985], Hildenbrand & Kirman [1998], and Mas-Colell, Whinston & Green [1995].

[30] The initial assignment of entitlements must be distinguished from their protection (Ayres [2005:14], Example II.5.1, and Remark II.5.1). A keyword for the present example is property rule protection.

[31] The illustrated downward sloping *lines*, convex toward origo, represent indifference sets and reflect local non-satiation and falling relative marginal valuation. The preference map orientations are indicated. (Under uncertainty, convexity expresses risk-aversion.)

[32] There are $2^I - 1$ coalitions (the empty set is not counted), see Section II.1.

ing with respect to B, the set of mutually beneficial agreements available from ω lie in

the lens defined by $P^A(\omega) \cap P^B(\omega)$. The coalition $\{A, B\}$ blocks ω because, by mov-

ing into the lens, both are better off (a PI-move). However, if an allocation such as $\ddot{x}$ is

suggested as final, a new lens is defined, and a second PI-move is possible. In conclusion,

the Core consists of the points of tangency as illustrated, end-points of all possible (path-

dependent) sequences of PI-moves: The elements in $C(\omega)$ are PI-reachable from ω.

The set of Pareto-efficient allocations, S_{PE}, is the set that cannot be blocked by

(and only by) the coalition of *all* individuals $\{1,\ldots,I\}$. It follows that $C(\omega) \subseteq S_{PE}$: PE

only allows for blocking by $\{A, B\}$, not $\{A\}$ and $\{B\}$.[33]
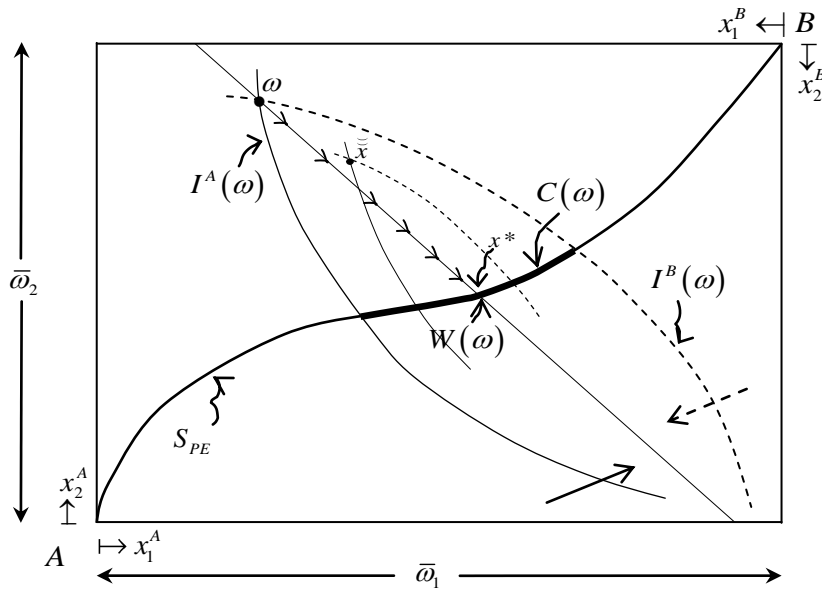
Figure 3.3 The Pareto-set S_{PE}, Core $C(\omega)$, and Walrasian equilibrium set $W(\omega)$ in the

Edgeworth-Bowley box

In Walrasian (general competitive) equilibrium, an L-dimensional vector q of (relative)

prices, *perceived* as a datum by all agents, induces sets of affordable vectors (budget sets

---

[33] The PE-concept has no individual rationality constraint, and is given by the curve from
the south-west to the north-east corner in Figure 3.3. In the two-dimensional case, the
subset of S_{PE} defined by $C(\omega)$ is also called the *contract set*.

B^i), from which agents choose optimal bundles $x^i\left(q;q\omega^i\right)$. In equilibrium, $q*$ clears all markets: As illustrated in Figure 3.3, the final state $x* = \left\langle x^1*,\ldots,x^I *\right\rangle \in W\left(\omega\right)$ is induced. It is a fundamental theorem of economics that $W\left(\omega\right) \subseteq C\left(\omega\right)$.[34]

The two transition mechanisms take the economy from an(y) initial $\omega \in S$ to end states in S : The Core correspondence $C : S \rightrightarrows S$ assumes unlimited communication and negotiations between freely formed coalitions of agents. The dynamics are implicit and path dependent.[35] The Walrasian correspondence $W : S \rightrightarrows S$ supposes a decentralized mechanism, where agents make decisions knowing only the tuple of anonymous prices. Both mechanisms assume the existence of enforceable contracts in entitlements, and may be considered as meta-theorems for *pacta sundt servanda,* in relevant environments.[36]

Both transition mechanisms ensure PI-movements. The operational Walrasian mechanism has additional properties of normative interest.[37] ∎

---

[34] More precisely, the $L\left(I+1\right)$ -tuple $\left\langle q*,x^1*,\ldots,x^I *\right\rangle$ is a Walrasian equilibrium if the price vector $q* = q\left(\omega\right)$ and demand functions $x^i* = x^i\left(q\left(\omega\right);q\left(\omega\right)\omega^i\right)$ satisfy $\sum_{i=1}^{I} x^i\left(q\left(\omega\right);q\left(\omega\right)\omega^i\right) = \sum_{i=1}^{I}\omega^i$ (vector equality), and $x^i\left(q\left(\omega\right);q\left(\omega\right)\omega^i\right) \succsim^i x^i$ for all $x^i \in B^i\left(q\left(\omega\right);q\left(\omega\right)\omega^i\right) = \left\{x^i \in \mathbb{R}_+^L \middle| q\left(\omega\right)x^i \leq q\left(\omega\right)\omega^i\right\}$, $i = 1,\ldots,I$. $W\left(\omega\right)$ is a single-element set only under additional strong assumptions, but can be shown to be non-empty and have a finite (in fact, odd) number of elements under weak regularity assumptions.

[35] See Hahn [1974].

[36] See Remark 3.5. (Enforcement means injunctions or sufficiently high penalty clauses or option excise prices.)

[37] Keywords regarding the *transition mechanism* $W : S \rightarrow S$ are: equal trading opportunities, anonymity, non-manipulability, and low informational requirements. Regarding end states, symmetric initial wealth distributions (symmetric initial entitlements distributions are special cases) give rise to so-called envy-free (even coalitional envy-free) equilibria. See Varian & Thomsen [1985].

In large economies, the core in fact inherits these properties, due to the famous Core convergence theorem: as the economy grows (the number of individuals I increases) the Core shrinks to the set of Walrasian equilibria. The theorem is remarkable, because the two equilibrium notions have completely different motivations (unrestricted cooperation and decentralized decisions).

In contract law, due to ensured PI-moves, use of PE for norm selection in legal decision-mechanisms M is arguably suggested by a reasonable interpretation of the institution of contracting itself. However, in more complex environments than discussed in Example 3.3 (keywords: trading over time under uncertainty and asymmetric information), design questions quickly become complex.[38]

Example 3.4 (liability in general equilibrium). Let $x_2^A = e^A \in [0, \overline{e}]$ be an index of A's use of resource 2 (such as clean air), assumed to directly affect B adversely (B's access to clean air is defined by $x_2^B = \overline{e} - e^A$).[39] An initial entitlement allocation defines $\left\langle \left\langle \omega^A, {}^e\omega^A \right\rangle, \left\langle \omega^B, {}^e\omega^B \right\rangle \right\rangle \equiv {}^e\underline{\omega}$, say. As remarked above, initial entitlement assignments should be distinguished from their protection. Figure 3.4 illustrates ${}^0\underline{\omega} = \left\langle \left\langle \omega^A, 0 \right\rangle, \left\langle \omega^B, \overline{e} \right\rangle \right\rangle$, and ${}^{\overline{e}}\underline{\omega} = \left\langle \left\langle \omega^A, \overline{e} \right\rangle, \left\langle \omega^B, 0 \right\rangle \right\rangle$. Letting τ increase in $(0, \overline{e})$, any point ${}^\tau\underline{\omega}$ between ${}^0\underline{\omega}$ and ${}^{\overline{e}}\underline{\omega}$ is defined on the vertical dashed line.

Assume that the initial entitlement distribution is ${}^0\underline{\omega}$, and that A is identified as a tortfeasor. If B's entitlement is protected by *strict liability*, A has the option to move along $I^B\left({}^0\underline{\omega}\right)$, fully compensating B in units of the physical commodity. A forces a move to $C\left({}^0\underline{\omega}\right) \cap I^B\left({}^0\underline{\omega}\right) \subseteq S_{PE}$.[40]

---

[38] Optimal contracts are complex and sensitive to changes in information structures. Arguably, power-conferring norms are well suited to describe theory-prescribed authority allocations. See Laffont [1989], Laffont & Martimort [2002], and Bolton & Dewatripont [2005] on modern theories of information and *institutions* (theories that are only hinted at in Sections I.2 and 3.2).

[39] This is a much simplified way of introducing (intentional) externalities in general equilibrium (see Arrow [1970] for a general discussion).

[40] Damages or losses, $L^B\left(e^A\right)$, is measured in units of commodity x, defined by $\left\{ x^B\left(e^A; {}^0\underline{\omega}\right) \in \left[0, \omega^A + \omega^B\right] \middle| \left\langle x^B\left(e^A; {}^0\underline{\omega}\right), \overline{e} - e^A \right\rangle \in I^B\left({}^0\underline{\omega}\right) \right\}$. The loss function is strictly increasing and concave (from convexity of preferences $\succsim^B$).

Figure 3.4 Liability rules in the exchange economy

If B's entitlement is protected by a *negligence rule*, mapping to liability for A if $e^A > \underline{c}$ under a rigorous rule or if $e^A > \overline{c}$ under a lenient rule, A has the option to move *along* the vertical dashed line from $^0\underline{\omega}$ until $^{\underline{c}}\underline{\omega}$ or in $^{\overline{c}}\underline{\omega}$, respectively, followed by a "horizontal jump" from point $^{\underline{c}}\underline{\omega}$ or $^{\overline{c}}\underline{\omega}$, respectively, to $I^B\left(^0\underline{\omega}\right)$ (reflecting the dichotomizing nature of negligence rules).[41] A forces a move to $^{\underline{c}}\underline{\omega}$ and $^{\overline{c}}\underline{\omega}$, respectively.[42] If legally permissible, parties can negotiate from $^{\underline{c}}\underline{\omega}$ and $^{\overline{c}}\underline{\omega}$. Under zero transaction-cost "Coasian" bargaining, they reach an efficient point in $C\left(^{\underline{c}}\underline{\omega}\right) \subseteq S_{PE}$ or $C\left(^{\overline{c}}\underline{\omega}\right) \subseteq S_{PE}$,

---

Let the set of legal facts be given by $E = [0, \overline{e}]$. The global rule of strict liability, $r_s : E \to \mathbb{R}_+$, is defined by $r_s\left(e^A\right) = x^B\left(e^A; {}^0\underline{\omega}\right)$.

[41] A global negligence rule $r_c : E \to \mathbb{R}_+$ is defined by

$$r_s\left(e^A\right) = \begin{cases} 0 \text{ if } e^A \leq c \\ x^B\left(e^A; {}^0\underline{\omega}\right) \text{ if } e^A > c \end{cases}.$$

[42] Compare Sections IV.4, V.2, and V.4.

respectively.[43] If parties do not negotiate (and in the absence of markets in the externali-ty), efficiency under negligence rule protection requires a rule defined by $c = c^{PE}$, deter-mined by the intersection of the vertical dashed line and the Pareto-set. ∎

Remark 3.3 The identification of A as a tortfeasor, in Example 3.4, may be arbi-trary (a key insight from Coasian style analysis). The analysis is symmetric if B is identified as the tortfeasor.[44] Irrespective of whose initial entitlement is pro-tected, additional methods include property rules and put-options (see, generally, Ayres [2005], Example II.5.1, and Remark II.5.1). □

The slection of liability regime under Coase theorem assumptions, corresponds to a choice between different culmination outcomes in S_{PE} from initial entitlement assignments ${}^{e}\underline{\omega} \in S \setminus S_{PE}$. In contrast to Example 3.3 concerning PI-moves, Ex-ample 3.4 illustrates that a choice between liability protection of entitlements based on the PE criterion concerns income distribution (see also Parts IV–V). Arguably, coerced income-shifting selection of entitlement protection belongs in legislatures.[45] As opposed to courts, legislatures have access to a wide set of in-struments that can be used to compensate adverse income effects (compare the

---

[43] Arrow [1970] considers the possibility of competitive markets in externalities (imply-ing transition to S_{PE} via the Walrasian correspondence), illustrating the relationship be-tween the externality notion and market and property right structures.

[44] In *symmetric situations* (agents are just as likely to produce as be affected by externali-ties), arguments for PE as a legitimate design criterion in courts are strengthened (see the discussion of reciprocal situations in Kaplow & Shavell [2002:100–154]).

[45] This is not a descriptive statement. Parisi [1992] for a comprehensive study of court developed liability regimes.

controllability results, discussed above). There are also inherent problems regarding identification of externalities.[46]

3.2 *Legal politics II: strategic environments (rules v. discretion)*

A basic corollary of the general controllability properties discussed in Section 3.1 is that, over time, discretion *dominates* rule bound policy: flexible instruments use can replicate *any* asserted optimal rule and react optimally to unforeseen circumstances (rule gaps or insufficient state sensitivity in the Ayres & Gertner [1989:92] sense).[47] However, if instruments are implemented in *strategic environments*, the conceptual framework changes fundamentally.

The theory of games sets the stage for analysis. Generally formulated, a game is defined by a set of players $i \in \{1,...,I\}$, each choosing a *strategy* σ_i from a set of possible strategies, $\Sigma_i(\sigma_{-i})$, the notation expressing the possibility of strategy-sets, depending on the the $I - i$ other players' strategies, σ_{-i}. The description is completed by defining payoff-functions $u_i : \Sigma_i \times \Sigma_{-i} \to \mathbb{R}$. Strategies are elements in appropriately defined spaces, and correspond to *complete contin-*

---

[46] "There is one deep problem in the interpretation of externalities which can only be signaled here. What aspects of others' behavior do we consider as affecting a utility function? If we take a hard-boiled revealed preference attitude, then if an individual expends resources in supporting legislation regulating another's behavior , it must be assumed that that behavior affects his utility. Yet in the cases that students of criminal law call "crimes without victims," such as homosexuality or drug-taking, there is no direct relation between the parties. Do we have to extend the concept of externality to all matters that an individual cares about? Or, in the spirit of John Stuart Mill, is there a second-order value judgment which excludes some of these preferences from the formation of social policy as being illegitimate infringements of individual freedom?" Arrow [1970:16].

[47] See e.g. Fischer [1990].

gent plans (algorithms), describing what a player does in any possible situation throughout the game, as defined by physical and legal-institutional constraints.[48]

Let a policy implementing agency's strategy-space be denoted Σ_M. The next two examples illustrate the change in perspective on "legal politics" arising from strategic considerations.

Example 3.5 (commitment in an extensive form game). In the Figure 3.5 two-stage dynamic game (of complete and perfect information), a first-moving agent considers sinking irreversible investments in a jurisdiction. The agent's strategy set is $\{\neg i, i\}$, i and $\neg i$ denoting investment and no investment, respectively. If investment is undertaken, a policy implementing agency has a choice between levying "low" or "high" capital taxes as defined by $\Sigma_M = \{\mu_{\underline{\tau}}, \mu_{\bar{\tau}}\}$, say. Payoffs are as illustrated below:
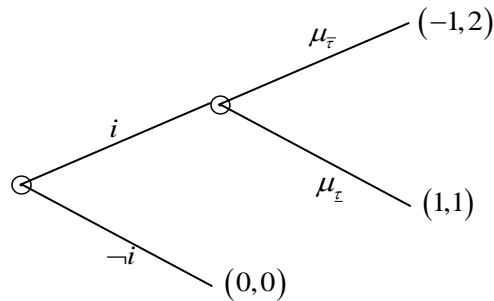
Figure 3.5 Implementation of legal instruments in a strategic environment

Rationality dictates the backward induction equilibrium $\langle \neg i, \mu_{\bar{\tau}} \rangle$: the agent realizes that the agency has an incentive to choose high taxes $\mu_{\bar{\tau}}$, given investment i. The corresponding $(0,0)$ outcome is Pareto-dominated by outcome $(1,1)$ corresponding to strategy-profile $\langle i, \mu_{\underline{\tau}} \rangle$. Because the choice of $\mu_{\underline{\tau}}$ at the second stage is *incentive incompati-*

---

[48] A detailed discussion is not attempted (but see Section IV.2). Maulinvaud [1985:150–83] provides a general introduction (see Gibbons [1992] and Mas-Colell, Whinston & Green [1995] on non-cooperative theory).

ble, the $(1,1)$ outcome is not supportable as an equilibrium. However, if the agency credibly could limit its strategy-set from $\Sigma_M = \{\mu_{\underline{\tau}}, \mu_{\bar{\tau}}\}$, to $\{\mu_{\underline{\tau}}\}$, a PI move is ensured (corresponding to a Figure 3.2 move from $S \setminus S_{PE}$ to S_{PE} in utility-space). ∎

In parametric environments, a reduction in the set of admissible instruments M would (weakly) impair the agent's ability to implement policy. In strategic environments, focus shifts from selection of instruments to design of *institutions* capable of implementing preferred strategic equilbria. Law, partly constituting the game structure, defines or shape strategy sets and payoff functions. If the power-conferring norm $n': X' \to \wp\Sigma_M$ is such that η' in effect (credibly) maps to $!\{\mu_{\underline{\tau}}\}^{x',\eta'}$, an improved equilibrium is obtained. In this case, law functions as a commitment mechanism.

The next, paradigmatic example combines game-and control-theoretic dimensions.

Example 3.6 (monetary policy and strategic delegation). A stripped version of the "work-horse model" used in political economics, is given by the following equations:[49]

$$
\begin{aligned}
\pi &= \mu + \upsilon \\
y &= \theta + \left(\pi - E\{\pi \mid I_{RE}^\theta\}\right) - \varepsilon \\
E\{L(y,\pi)\} &= E\{(\pi - \pi^*)^2 + \alpha(y - y^*)^2\}, \quad y^* > E\{\theta\}
\end{aligned}
\qquad (3.3)
$$

The first equation is the demand side, π inflation, μ base money growth rate, and the stochastic variable υ demand shocks. The second equation is the supply side, y output growth, θ a stochastic parameter for non-inflationary output growth, $E\{\pi \mid I_{RE}^\theta\}$ private sector inflation growth expectations, and ε supply side shocks. $E\{L(\cdot)\}$ is the govern-

---

[49] The model is from Persson & Tabellini [1999]. Focus in the summary discussion is on the strategic issues and trade-offs in regime designs, not on the economic model content.

ment loss function. All stochastic variables are i.i.d.'s: $E\{\langle\theta,\upsilon,\varepsilon\rangle\} = \langle\bar{\theta},0,0\rangle$ and the

covariance matrix given by $Var\{\langle\theta,\upsilon,\varepsilon\rangle\} = \mathrm{diag}\left(\sigma_\theta^2,\sigma_\upsilon^2,\sigma_\varepsilon^2\right)$.

The sequential structure is as follows: At time

t_0 the government chooses policy-regime (RE);

t_1 θ is realized and observed by the government and private sector;

t_2 private sector expectation $E\{\pi\,|\,I_{RE}^\theta\}$ is formed, conditioned on θ and

RE;

t_3 υ and ε are realized and observed by the private sector and govern-

ment; and

t_4 the government chooses instrument $\pi \in \mathrm{M}$.

Crucially, private sector expectations are formed rationally at t_2, conditioned on the in-

formation available, as expressed by the information set I_{RE}^θ. The policy regime—the

legal rule defining downstream use of instrument μ at t_4—is fully known when expec-

tations are formed. The private sector also has complete information about the model

(3.3) and the stochastic structure of $\langle\theta,\upsilon,\varepsilon\rangle$.

Assume that the rule for instrument use is designed depending on the whole vec-

tor $\langle\theta,\upsilon,\varepsilon\rangle$ (in Example II.3.4 terminology, a complete state contingent rule, CSC). The

optimal rule (among the class of linear ones) turns out to be

$\mu_{CSC} = r(\theta,\upsilon,\varepsilon) = \pi* -\upsilon + \frac{\alpha}{1-\alpha}\varepsilon$, giving first-best equilibrium values $\langle\pi_{CSC},y_{CSC}\rangle$.[50] Due

to the strategic environment, the rule $\mu_{CSC}(\cdot)$ *dominates* discretion. However, given the

expectations $E\{\pi\,|\,I_{\mu_{CSC}}^\theta\}$ formed at t_2, the shocks υ,ε realized at t_3, evaluated at t_4:

$$\frac{d}{d\mu}\{L(\pi_{CSC},y_{CSC})\}\Big|_{\substack{E\{\pi|I_{\mu_{CSC}(\cdot)}^\theta\} \\ \theta,\upsilon,\varepsilon}} = -2\alpha(y*-\theta) \neq 0. \tag{3.4}$$

---

[50] Persson & Tabellini [1999] provide explicit solutions $\langle\pi_{CSC},y_{CSC}\rangle$.

Equation (3.4), means that that the government, at t_4, has an incentive to deviate from the announced rule, $\mu_{CSC}(\cdot)$ (the derivative is non-zero with probability one): the rule is dynamically inconsistent. If the private sector believes that instruments will be set ex post optimally (discretionary, D), selection of μ_D at t_4, in equilibrium gives $\mu_D = \pi^* + \alpha(y^* - \theta) - \upsilon + \frac{\alpha}{1-\alpha}\varepsilon$, with equilibrium values $\langle \pi_D, y_D \rangle$ (expectations are formed conditioned on $I^\theta_{\mu_D}$).[51] The lack of ability to commit to $\mu_{CSC}(\cdot)$ is costly, as expressed by the difference:

$$E\{L(\pi_D, y_D)\} - E\{L(\pi_{CSC}, y_{CSC})\} = \alpha^2 \left(\sigma_\theta^2 + (y^* - \bar\theta)^2\right).$$

It may be possible to commit fully to a simpler, partially state contingent (PSC) rule $\mu_{PSC}(\cdot)$. Conditioned on e.g. $\langle \theta, \upsilon \rangle$, and expectations formed based on $I^\theta_{\mu_{PSC}(\cdot)}$, the optimal (linear) rule is given by $\mu_{PSC} = r(\theta, \upsilon) = \pi^* - \upsilon$, and the corresponding equilibrium $\langle \pi_{PSC}, y_{PSC} \rangle$.[52]

However, equilibrium under μ_{PSC} and μ_D cannot be ranked unconditionally:

$$E\{L(\pi_{PSC}, y_{PSC})\} - E\{L(\pi_D, y_D)\} = \frac{\alpha^2}{1-\alpha}\sigma_\varepsilon^2 - \alpha^2 \left(\sigma_\theta^2 + (y^* - \bar\theta)^2\right). \tag{3.5}$$

Based on the tradeoffs between (fully committed) rules and discretion expressed in (3.5), Flood & Isard [1989] suggest an optimal combination of a rule bound and discretionary regime: use of a $\mu_{PSC}(\cdot)$ under "normal circumstances" and under large, costly shocks

---

[51] Persson & Tabellini [1999] provide explicit solutions $\langle \pi_D, y_D \rangle$.

[52] Persson & Tabellini [1999] provide explicit solutions $\langle \pi_{PSC}, y_{PSC} \rangle$. The PSC rule is also incentive incompatible ($\frac{d}{d\mu}\{L(\pi_{PSC}, y_{PSC})\}\big|_{\substack{E\{\pi|I^\theta_{\mu_{PSC}(\cdot)}\}\\ \theta, \upsilon, \varepsilon}} = -2\alpha\left((y^* - \theta) + \varepsilon\right) \neq 0$ with probability one). If the problem of credibility can be solved, $\mu_{CSC}(\cdot)$ dominates $\mu_{PSC}(\cdot)$ (a direct consequence of solving a free versus constrained minimization problem).

that the $\mu_{PSC}(\cdot)$ does not accommodate (ε), activation of an exit clause and discretionary instrument use.[53] ∎

It has been illustrated that mechanisms limiting discretion may improve the ability to implement (democratically elected) policies in strategic environments. The ideas illustrated in Examle 3.6 have influenced actual institutions.[54]

Design of *system-based* legal *commitment mechanisms* concern power-conferring norms $\eta' \in \mathbb{P}_{\wp \mathbb{P}_Y^X}^{X'}$.[55] As explicated in Example 3.5 (and in reference to Section 2), under "normal circumstances" $X^n \subseteq X'$ the legal system fully commits to a (dynamically inconsistent) partially state-contingent rule $(\text{supp } \eta'(\cdot \mid x')\big|_{X^n} = \{\{r_{PSC}\}\})$, and in "extraordinary circumstances", $X^e = X \setminus X^n$, switches to a discretionary regime, $\text{supp } \eta'(\cdot \mid x')\big|_{X^e} = \wp \mathbb{P}_Y^X$. Commitment is only possible if:

- it is credible that n' is not changed over the relevant time span, and
- it can be verified that η' is followed.

---

[53] In the present example, the design concerns subsets $(-\infty, \underline{\varepsilon}] \cup [\overline{\varepsilon}, \infty)$ for the shocks ε that are costly under μ_{PSC}, cf. (3.5).

[54] Arguably, Bank of England (BoE) is a leading example. A simple (transparent) partially stage contingent rule is delegated to the relatively independent BoE as a method to obtain commitment to the rule The Bank of England Act gives BoE full operational independence, while the Treasury specifies targets (Sec. 10–12). The Treasury can only instruct BoE regarding monetary policy instruments if "required in the public interest and by extreme economic circumstances" (Sec. 19(1)), see Cranston [1998]. Nordén [2002] uses the logics of a strategic delegation regime with a discretionary exit clause to suggest interpretations *de lege lata* of the Norwegian Central Bank Act.

[55] For private sector agents, *contracts* and arbitrational mechanisms are essential for relaxing incentive constraints, and may be thought of as transforming the pay-off structure in Figure 4.1. A profound literature cannot be discussed here (see Watson [2002] for an introduction). Government contracts are problematic as commitment devices due to the difficulties in enforcing rights against sovereign states and constitutional prohibitions of limitations on future legislation (Hermalin, Katz & Craswell [2007:127]).

The first bullet point concerns commitment to the legal regime. Strategic delegation regimes may be defined in ordinary legislation, in international conventions, or at the constitutional level, reflecting different costs of regime change. In this regard, hierarchical structure of legal norms, maintained by independent courts and arbitrational panels, help constituting commitment mechanisms (see Remark 2.2).[56]

The second bullet point concerns the mapping $n': X' \to \mathbb{P}_{\wp \mathbb{P}_Y^X}$ (partitioning of X' into X^n and X^e) and (simultaneously) the choice $r_{PSC} \in \delta \mathbb{P}_Y^X$ (keywords: observability and transparency). Both bullet points presume that enacted norms are interpreted in a stable and predictable manner, including—if applicable—in downstream courts. It concerns the structure of meta-norms and the legal system understood as an *institution*.

Remark 3.4 Even in strategic settings, where logical arguments can be made for constraints on legislative power, the use of commitment mechanisms raises challenging questions. Fischer [1990:1181] warns against irreversible designs based on stylized theories.[57] □

Remark 3.5 In dynamic systems (Remark 3.2), so-called model consistent expectations are introduced by partitioning $s(t)$ into subsets of "forward" and "prede-

---

[56] However, courts are traditionally reluctant to intervene in governmental inter-branch disputes (see Ferejohn & Pasquino [2004]).

International courts and tribunals can constitute valuable commitment mechanisms. In this context, however, independence and efficiency takes on important additional dimensions as demonstrated by Posner & Yoo [2005].

[57] Rigid constitutions and substantive judicial review are controversial (see Merryman & Pérez-Perdomo [2007:134–60] on the historical development, including direct access to review and increasing use of international instruments). La Porta *et al.* [2004] argues that it is a predictor of economic and political freedom. Elster [2013:282] argues that courts should not screen decisions of properly elected and organized legislatures on substantive grounds.

termined" state variables. The forward variables are allowed discontinuous moves at points in time t_I, when new information (I) about parameter paths (μ_t) become available, ensuring that the resumed continuous solution trajectory leads to a *stationary point* as $t \to +\infty$. Through the forward variables, (probability weighted) information about *future* shifts in μ_t influence *current* states.[58] In particular, it gives a framework for analyzing the impact of expected changes in law, *lex futura*.[59] □

---

[58] The methodology "leads to notions of stability and to solution theory that is different than much of that in the natural sciences" (Brock & Malliaris [1989:263]). In terms of the Section IV.1 equilibrium correspondence, solution discontinuity at t_I avoids chaos (under continuity, perturbation of system parameters would have lead to explosive development, see the Section I.1 reference to Ekeland [1988] and Buiter [1984] on linear systems).

[59] In fact, it has been suggested that law expected to be promulgated, should count as a formal source (Tvarnø & Nielsen [2014:268], discussing EU-directives awaiting implementation). Similarly, some dimensions of courts' transnational engagement (as advocated e.g. by Jackson [2010]), may challenge legal systems' institutional ability to function as commitment mechanisms.

PART SEVEN

EPISTEMIC UNCERTAINTY

1   Introduction

In Parts II and III, conditioning source complexes $ls \in LS$ and facts $x \in X$ are *exogenously* given. Parts IV and V *endogenize* facts and establish an equilibrium correspondence from ordered pairs of (mixed) abstract norms and decision mechanisms to the set of solutions, $x^*: \mathbb{P}_Y^X \times \mathcal{M} \rightrightarrows X$ (also reflected in Part VI law-in-force notions). In all situations, judges were assumed to observe variables correctly. This assumption is now relaxed with respect to ordinary facts, $x \in X$. Judges' observations of these facts will have a systematic component and an error term, representing epistemic uncertainty. Norm application in mechanisms $M \in \mathcal{M}$ leads to confluence of legal and epistemic uncertainty that systematically affect probability distributions over legal consequences. Section 2 starts with an analysis of M_1, which a "doubly" stochastic environment is not an identity transformation. Adding epistemic uncertainty further accentuates the question of decision mechanism evaluation. Section 3 suggests a definition, enables mechanism comparison, in terms of *error generation*. In this perspective, Section 4 analyzes the impact of mechanism size and super-majority rules under joint legal and epistemic uncertainty. Section 5 evaluates the impact of separation of decisions on law and facts, and it includes a discussion of optimal design.

2 Epistemic competence and M_1-transformation

A judge observes conditional fact $\mathbf{x} = x + \varepsilon \in X$, ε denoting a stochastic variable, interpreted as an *error term,* independent of the actual or "true" state $x \in X$.[1]

The error term has cdf $F_\varepsilon(t)$, with expectation $\mu_{F_\varepsilon} = E\{\varepsilon\} = 0$, and variance $E\{\varepsilon^2\} = \sigma_\varepsilon^2$. These assumptions define the judge's *epistemic* competence.[2]

Remark 2.1 The law and economics literature considers (pure) epistemic uncertainty (situations in which an independent criterion for a correct decision exist). It includes Shavell [1987] Sec. 4.A.3.1 and Diamond [1974]. □

In Example II.2.1, $g_{F_S} \in \mathbb{P}_{\{\oplus,\otimes\}}^X$, the abstract legal standard is defined by the stochastic variable S with cdf $F_S(t)$, liability ($\otimes$) occurring iff $S \leq x$. The norm is given by $g(\cdot \,|\, x)\big|_{x \in X} = \langle 1 - F_S(x); F_S(x); \oplus, \otimes \rangle\big|_{x \in X}$. With a stochastic fact $\mathbf{x}$, liability is concluded in M_1 iff $S \leq \mathbf{x}$. It follows:

> Proposition 2.1 If $g_{F_S} \in \mathbb{P}_{\{\oplus,\otimes\}}^X$, liability occurring iff $S \leq \mathbf{x}$, with
>
> $\mathbf{x} = x + \varepsilon$, the error term ε having cdf F_ε and the legal standard stochastic variable S having cdf F_S, the M_1 transformed norm $g_1 \in \mathbb{P}_{\{\oplus,\otimes\}}^X$ is
>
> given by:

---

[1] The reservation regarding "true" concerns situations in which an external criterion for occurrence of a phenomenon and/or classification does not exists (individuation and system aspects of norms make this relevant: output from one non-determinative norm may be input to another).

[2] The Parts III through VI analyses correspond to situations where judges never make factual mistakes ($\sigma_\varepsilon^2 \to 0$ or cdf $F_\varepsilon(t) = F_{\delta 0}(t)$, the discontinuity at $t = 0$ corresponding to a jump from 0 to 1 (see Example II.3.2).

$$g_1(\cdot \mid x)\big|_{x \in X} = \left\langle 1 - F_{S-\varepsilon}(x); F_{S-\varepsilon}(x); \oplus, \otimes \right\rangle\big|_{x \in X}, \qquad (2.1)$$

with $F_{S-\varepsilon}(x) = \int \{1 - F_\varepsilon(s-x)\} dF_S$.

<u>Proof.</u> The liability component in (2.1), $g_1(\otimes \mid x) = P\{S \leq \mathbf{x}\} = P\{S - \varepsilon \leq\} = F_{S-\varepsilon}(x)$, is

obtained by conditioning on S : $P\{S \leq \mathbf{x}\} = \int P\{S < \mathbf{x} \mid S = s\} dF_S = \int \{1 - F_{\mathbf{x}}(s;x)\} dF_S$

$= \int \{1 - F_\varepsilon(s-x)\} dF_S .^3$ ◄

To focus conceptual issues, it is assumed that legal and epistemic uncertainty can be described by normal distributions. In addition to considerable analytical simplifications compared to (2.1), it leads to instructive comparative statics that will be demonstrated to have (approximate) relevance, for all underlying distributions. Hence, let $F_{S-\varepsilon}(t; \mu_S, \sigma_S^2 + \sigma_\varepsilon^2)$ denote the cdf from $N(\mu_S, \sigma_S^2 + \sigma_\varepsilon^2)$.

<u>Proposition 2.2</u> If the legal standard and error distributions are independent and normal, $S \sim N(\mu_S, \sigma_S^2)$ and $\varepsilon \sim N(0, \sigma_\varepsilon^2)$,

$$g_1(\cdot \mid x)\big|_{x \in X} =$$
$$\left\langle 1 - F_{S-\varepsilon}(x; \mu_S, \sigma_S^2 + \sigma_\varepsilon^2), F_{S-\varepsilon}(x; \mu_S, \sigma_S^2 + \sigma_\varepsilon^2); \oplus, \otimes \right\rangle\big|_{x \in X} \qquad (2.2)$$

with a strictly increasing cdf $F_{S-\varepsilon}(t; \mu_S, \sigma_S^2 + \sigma_\varepsilon^2)$, symmetric around the median equal to mode and expectation $\mu_S = F_{S-\varepsilon}^{-1}\left(\frac{1}{2}; \mu_S, s_S^2 + \sigma_\varepsilon^2\right)$, and strictly convex on $(-\infty, \mu_S]$ and strictly concave on $[\mu_S, \infty)$.

---

[3] $F_{\mathbf{x}}(t;x)$ is the cdf of $\mathbf{x}$ and given by $F_\varepsilon(t-x)$ ($F_{\mathbf{x}}(t;x) = P\{\mathbf{x} \leq t \mid x\} = P\{\varepsilon \leq t - x\}$). Alternatively, it can be conditioned on $\mathbf{x}$.

<u>Proof.</u> Because the sum of independent normal variables is normal, $E\{S-\varepsilon\}=\mu_S$, $var\{S-\varepsilon\}=\sigma_S^2+\sigma_\varepsilon^2$, $S-\varepsilon\sim N\left(\mu_S,\sigma_S^2+\sigma_\varepsilon^2\right)$ (Bartoszyński & Niewiadomska-Bugaj [1996], Theorem 9.10.3.). From normality, $F_{S-\varepsilon}^{-1}\left(\tfrac{1}{2};\mu_S,s_S^2+\sigma_\varepsilon^2\right)=\mu_S$.[4] See Example IV.2.2 regarding the cdf first and second order derivatives, $F_{S-\varepsilon}{}'$, $F_{S-\varepsilon}{}''$. ◄

Norm-based uncertainty is measured by σ_S^2 and epistemic uncertainty by σ_ε^2:

> <u>Proposition 2.3</u> Legal uncertainty σ_S^2 and epistemic uncertainty σ_ε^2 impact liability probability $g_1\left(\otimes\,|\,x;\mu,\sigma_2^S+\sigma_\varepsilon^2\right)$ positively if $x<\mu_S$ and negatively if $x>\mu_S$. The impact is measured by:
>
> $$\partial g_1\left(\otimes\,|\,x;\mu_S,\sigma_S^2,\sigma_\varepsilon^2\right)\big/\partial\sigma_S^2=\partial g_1\left(\otimes\,|\,x;\mu_S,\sigma_S^2,\sigma_\varepsilon^2\right)\big/\partial\sigma_\varepsilon^2=$$
> $$-\frac{1}{2\sqrt{2\pi}}\exp\left[-\frac{1}{2}\left\{\frac{x-\mu_S}{\sqrt{\sigma_S^2+\sigma_\varepsilon^2}}\right\}\right]\frac{x-\mu_S}{\sqrt[3]{\sigma_S^2+\sigma_\varepsilon^2}}\qquad.$$
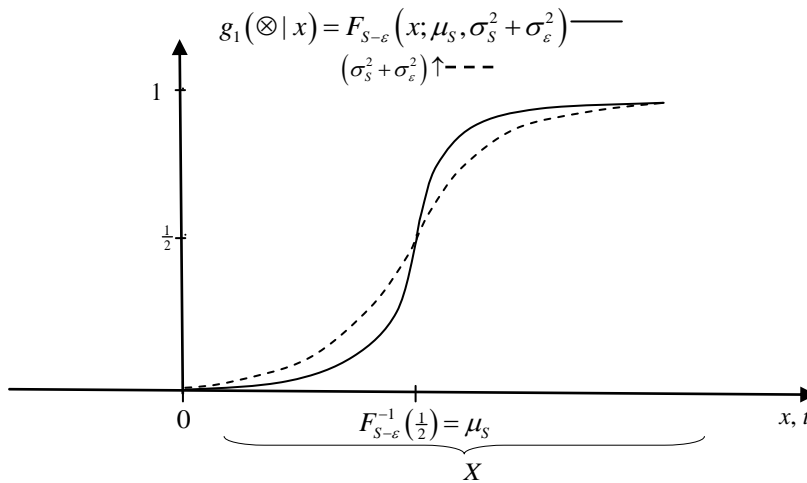
Figure 2.1 Impact of increased legal or epistemic uncertainty on liability probability in

M_1 under normality assumptions

---

[4] Normal distributions are unbounded, uni-modal, symmetric, and have median and mean equal to the mode. (Equality of the median and mean applies to all symmetric distributions.)

234

<u>Proof.</u> $g_1\left(\otimes \mid x; \mu_S, \sigma_S^2, \sigma_\varepsilon^2\right) = F_{S-\varepsilon}\left(t; \mu_S, s_S^2 + \sigma_\varepsilon^2\right)\big|_{t=x}$. The event $S - \varepsilon \leq x$ is equivalent to

the event $\left(S - \varepsilon - \mu_S\right)\left(\sqrt{\sigma_S^2 + \sigma_\varepsilon^2}\right)^{-1} \leq \left(x - \mu_S\right)\left(\sqrt{\sigma_S^2 + \sigma_\varepsilon^2}\right)^{-1}$. Because the left hand side

standardized normal variable is distributed $N(0,1)$ (has cdf $\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{1}{2}\tau} d\tau$ [5]),

$F_{S-\varepsilon}\left(x; \mu_S, s_S^2 + \sigma_\varepsilon^2\right) = \Phi\left(\left(x - \mu_S\right)\left(\sqrt{\sigma_S^2 + \sigma_\varepsilon^2}\right)^{-1}\right)$. Hence, given $x \in X$, from Lebniz's

formula for differentiation of integrals [6],

$$\frac{\partial F_{S-\varepsilon}\left(x; \mu_S, s_S^2 + \sigma_\varepsilon^2\right)}{\partial \sigma_S^2} = \frac{\partial F_{S-\varepsilon}\left(x; \mu_S, s_S^2 + \sigma_\varepsilon^2\right)}{\partial \sigma_\varepsilon^2} = -\frac{1}{2\sqrt{2\pi}} e^{-\frac{1}{2}\left\{\frac{x-\mu_S}{\sqrt{\sigma_S^2 + \sigma_\varepsilon^2}}\right\}} \frac{x - \mu_S}{\sqrt[3]{\sigma_S^2 + \sigma_\varepsilon^2}}. \quad \blacktriangleleft$$

Figure 2.1 illustrates curvature properties of the liability probability component

$g_1\left(\otimes \mid x; \mu_S, \sigma_S^2, \sigma_\varepsilon^2\right) = F_{S-\varepsilon}\left(t; \mu_S, s_S^2 + \sigma_\varepsilon^2\right)$, under Proposition 2.2 normality as-

sumptions. The cdf is unbounded and strictly increasing, with convex and con-

cave regions, as described in Proposition 2.2. The dotted line illustrates the effect

of increased legal or epistemic uncertainty. A more lenient standard (a higher

μ_S) can be represented in the figure by a negative vertical shift of the cdf curve. [7]

---

[5] Bartoszyński & Niewiadomska-Bugaj [1996], Theorem 9.10.1.

[6] Bartle [1976], Theorem 31.8.

[7] The shift is given by μ_S impacts liability probability negatively globally on X and is given by

$$\frac{\partial g_1\left(\otimes \mid x; \mu_S, \sigma_S^2, \sigma_\varepsilon^2\right)}{\partial \mu_S} = \frac{\partial F_{S-\varepsilon}\left(x; \mu_S, s_S^2 + \sigma_\varepsilon^2\right)}{\partial \mu_s} = -\frac{1}{\sqrt{\sigma_S^2 + \sigma_\varepsilon^2}\sqrt{2\pi}} e^{-\frac{1}{2}\left\{\frac{x-\mu_S}{\sqrt{\sigma_S^2 + \sigma_\varepsilon^2}}\right\}},$$

see the proof of Proposition 2.3.

3   Evaluative criteria for decision mechanisms under joint legal and epistemic uncertainty

Part III demonstrates that abstract norms $g_{F_S} \in \mathbb{P}_{\{\oplus,\otimes\}}{}^X$ are systematically trans-

formed in mechanisms $M \in \mathcal{M}$. Adding a second layer of uncertainty—as illus-

trated in Section 2—further accentuates questions about mechanism evaluation.

However, an independent criterion is not readily available, with respect to legal

variables (with respect to $\mathbf{x}$, the external benchmark is given by state $x \in X$).

However, Table I.1 suggests using the outcome in large majority mechanisms, as

a benchmark for the correct decision regarding legal variables, using M_{2m+1} or

$M_{2m+1}{}^T$.[8]

From the Condorcet theorem (Section III.2) and Proposition III.6.1.iii,

the benchmark transformation of $g_{F_S} \in \mathbb{P}_{\{\oplus,\otimes\}}{}^X$ is given by:

$$\lim_{m \to \infty} \left\{ g_{F_S,2m+1}\left(\cdot \mid x\right)\Big|_{x \in X} \right\} = \lim_{m \to \infty} \left\{ g_{F_S,2m+1}^T\left(\cdot \mid x\right)\Big|_{x \in X} \right\} = \begin{cases} \left\langle 1,0;\oplus,\otimes \right\rangle\Big|_{x < F^{-1}\left(\frac{1}{2}\right)} \\ \left\langle \frac{1}{2},\frac{1}{2};\oplus,\otimes \right\rangle\Big|_{x = F^{-1}\left(\frac{1}{2}\right)} \\ \left\langle 0,1;\oplus,\otimes \right\rangle\Big|_{x > F^{-1}\left(\frac{1}{2}\right)} \end{cases}. \qquad (3.1)$$

Based on the asymptotic argument, the benchmark partitions X into a region of

no liability $X^{\oplus}$ and liability $X^{\otimes}$ by:

$$\begin{aligned} X^{\oplus} &= \left\{ x \in X \mid x \le F_S^{-1}\left(\tfrac{1}{2}\right) \right\} \\ X^{\otimes} &= X \setminus X^{\oplus} = \left\{ x \in X \mid x > F_S^{-1}\left(\tfrac{1}{2}\right) \right\} \end{aligned}, \qquad (3.2)$$

---

[8] The mechanisms are functionally equivalent, see Remark III.6.C. In legal hierarchies, ultimate decisions on (pure) law are typically made in large collectives under majority rule. Remarkably, in the absence of effective double jeopardy protection, majority rules also apply to criminal procedure, even if mixed questions of law and fact (liability) is determined under super-majority rules in downstream courts or juries.

    In the case of multi-dimensional norms (Examples II.2.2 and III.4.1), the definition arguably should reflect outcome-based voting, not premise-based regimes (see Nordén [2015]).

respectively. In the context of adverse decisions, the state of no liability ($\oplus$) may be defined as a *null hypothesis* H_0, and Type I *errors* as rejection of H_0 if $x \in X^{\oplus}$. The *alternative hypothesis* H_A denotes liability. Correspondingly, Type II *errors* may be defined as *not* rejecting H_0 if $x \in X^{\otimes}$.[9]

Table 3.1 Decision matrix

	state $x \in X^{\oplus}$	state $x \in X^{\otimes}$
decision *not reject* H_0	$\sqrt{}$	type II error
decision *reject* H_0	type I error	$\sqrt{}$

By calculating the liability probability in finite, real world mechanisms $M \in \mathcal{M}$, the distribution of errors is obtained globally on $X = X^{\oplus} \cup X^{\otimes}$. Let $F_{\varepsilon,i}(t)$ denote the error term cdf characterizing judge *i*. In completing Table 3.1, generation of Type I and II errors depends on:

- the abstract legal standard, as defined by $g_{F_S} \in \mathbb{P}_{\{\oplus,\otimes\}}{}^X$ (hence, implicitly on the meta-level factors $\langle ls, \eta \rangle \in LS \times \mathbb{P}_{\mathbb{P}_{\{\oplus,\otimes\}}{}^X}{}^{LS}$);

- judges' epistemic competencies, as given by $\left\{ F_{\varepsilon,i}(t) \right\}_{i=1}^{n}$;

- the actual state $x \in X$ ($x^* \in X$); and

- the parameters defining $M \in \mathcal{M}$.

The next two sections investigate these dependencies.

---

[9] On hypothesis testing in scientific contexts (with analogues to law), see Larsen & Marx [1986:286–304].

<u>Remark 3.1</u> In case of a normal standard distribution $S \sim N\left(\mu_S, \sigma_S^2\right)$, X is partitioned into $\left\{X^\oplus \vdots X^\otimes\right\}$ by $\mu_S = F_S^{-1}\left(\frac{1}{2}; \mu_S, \sigma_S^2\right)$. It follows directly from Proposition 2.3, that increased legal and/or epistemic uncertainty (as measured by σ_S^2 and σ_ε^2) in the single-judge contexts, implies a global increase in Type I errors on $X^\oplus$ and a global increase in Type II errors on $X^\otimes$, see Figure 2.1. and 4.1. □

<u>Remark 3.2</u> Criminal procedure typically involves lay judges and jurors selected from lists representative of larger populations. They determine liability in bifurcated trials. Their *general verdict* is protected by double jeopardy rules. In such contexts, a representative perspective on agency and mechanism legitimacy seems pertinent. As suggested in Nordén [2015], it motivates an adjustment in the analytical framework: Norms, envisioned as incomplete at points or in regions of X, are completed by properly sampled adjudicators entitled to their own (representative) judgments.[10] Specializing Arrovian axioms to dichotomous situations, Kenneth O. May demonstrated that majority rule is the only method satisfying *anonymity* (all individuals are given equal weight in the aggregation), *neutrality* (equal treatment of the outcomes), and *monotonicity* (more support for an outcome cannot imperil its election).[11] The majority decision over $\{\oplus, \otimes\}$, in the population of *potential adjudicators,* may, therefore, be *defined* as the

---

[10] Kornhauser & Sager [1986] distinguish between several purposes that may be imputed to legal decision mechanisms, including representation. Even stratified randomization is used (Elster [1989:96–98]). In this context, a properly sampled adjudicator convicts with a probability equal to the population fraction (see Bartoszyński & Niewiadomska-Bugaj [1996], Theorem 9.6.1, on sampling without replacement from large populations). In norm-theoretical terms, the substantive norm should be considered power-conferring.

[11] The axioms give necessary and sufficient conditions. See Moulin [1988], Theorem 11.1, for particularly apt formulations (indifference between alternatives is not permitted).

benchmark against which actual (finite) mechanisms can be measured. On a more profane level, invoking Beccaria's legality principle (Section I.1), it should be added that substantive criminal law itself is promulgated through representative majority-voting mechanisms.[12] This benchmark may also be motivated by "modest objectivity" as suggested for the legal domain by Coleman & Leiter [1993], and which is analogous to the notion of "closed impartiality" in Sen [2009].[13] □

4 $M_{[v;n]}$-transformations and Type I and II errors

In panels, errors $\{\varepsilon_i\}_{i=1}^{n}$ are assumed to be i.i.d.'s from $F_\varepsilon(t)$. That is, judges are assumed to have the same (homogenous) epistemic competence and to make decisions on facts, independently of each other, conditioned on facts revealed in the formal legal proceedings.[14]

---

[12] Kelsen [1955:25] remarks: "The principle of majority, the greatest possible approximation to the idea of freedom in political reality, presupposes as an essential condition the principle of equality. [---] This synthesis of freedom and equality is at the basis of the democratic idea concerning the relationship between the social order (as the collective will) and the individual will, between the subject and the object of domination […]."

    In reference to H. L. A. Hart and John Austin, Thomas & Pollack [1992:8] point out that "the sovereign in a democracy is ultimately the electorate. [---] a jury is simply a sample of the larger universe of the electorate, and a jury verdict can be compared to the result the electorate would have reached had it judged the case." They call it a "broad positive conception of guilt" in contradistinction to procedural truth concepts. Note that in some recent constitutions, referenda ("direct democracy") are given legal status (Merryman & Pérez-Perdomo [2007:25]).

[13] Under "ideal epistemic conditions" for legal judgments Coleman & Leiter [1993:629–32] suggest the majority outcome among all judges as defining the right outcome. Their form of "legal cognitivism" does not imply imposition of external truths to normative questions. Arrow [2012], requiring positive association between unrestricted individual values (universal domain) and social choice, contrasts his approach to "idealistic positions" (including some interpretations of the Condorcet theorem), see esp. pp. 28–30, 81–86, and 96–97. Although stressing law application as a form of cognition, Heckscher [1892] rejects judges tracking independent truths (Remark 5.1).

[14] If judges are randomly allocated to the mechanisms from sufficiently large reference populations, uniform competences equal to the population average follow from the sampling method. Independence arguably is a fair trial component paralleling assumptions made with respect to legal variables (Section I.2).

<u>Remark 4.1.A</u> Remark III.6.1.C implies that results developed for $M_{[v;n]}$ also

apply to $M_{[v;n]^{\uparrow\tau}}$. $\square$

<u>Remark 4.1.B</u> A large literature considers collective decision-making under

(pure) epistemic uncertainty, combining social choice, team theory (see

Nitzan 2010), and reliablilty theory (see Boland [1989] and Arnold, Balakrishnan

& Nagaraja [2008]). Extensions to logical aggregation theory include List [2005]

and Bovens & Rabonowicz [2006], who discuss outcome versus premise-based

protocols. Hecksher [1892] also contains many insights (see Remark 5.1). $\square$

<u>Proposition 4.1</u> Let $g_{F_S} \in \mathbb{P}_{\{\oplus,\otimes\}}{}^X$ and judges vote on $Y = \{\oplus,\otimes\}$ in

$M_{[v;n]}$ under default $H_0 = \oplus$, individually concluding liability iff

$S \leq \mathbf{x} = x + \varepsilon$, S with cdf F_S and ε with cdf F_ε. Under independence:

$$
\begin{aligned}
P\left\{\text{type I error} \mid x, M_{[v;n]}\right\} &= g_{[v;n]}\left(\otimes \mid x\right)\Big|_{x \in X^\oplus} = H_{[v;n]}\left(F_{S-\varepsilon}\left(x\right)\right)\Big|_{x \in X^\oplus} \\
P\left\{\text{type II error} \mid x, M_{[v;n]}\right\} &= g_{[v;n]}\left(\oplus \mid x\right)\Big|_{x \in X^\otimes} = 1 - H_{[v;n]}\left(F_{S-\varepsilon}\left(x\right)\right)\Big|_{x \in X^\otimes}
\end{aligned}
, (4.1)
$$

with $F_{S-\varepsilon}$ defined in (2.1) and $H_{[v;n]}\left(\cdot\right)$ defined in Proposition III.2.1.

<u>Proof.</u> The proposition proof follows from Propositions 2.1 and III.2.1. ◄

While Part III's "outer" aggregation results can be applied without specific dis-

tributional assumptions, this and the next section proceed on the basis of the

Proposition 2.2 normality assumptions. The liability component $g_{[v;n]}\left(\otimes \mid x\right)$ is

focused without loss of generality ($g_{[v;n]}\left(\oplus \mid x\right) = 1 - g_{[v;n]}\left(\otimes \mid x\right)$). Qualitatively,

the transformed norm curvature properties $g_1\left(\otimes \mid x\right): X \to [0,1]$, illustrated in

Figure 2.1, are preserved in majority mechanisms and, to a large extent, in super-

majority mechanisms. This is due to the properties of the outer aggregation function $H_{[v;n]} : [0,1] \to [0,1]$ discussed Section III.2.

Proposition 4.2 Under Proposition 2.2 normality assumptions, the strictly increasing functions:

(i) $g_{[v;n]}(\otimes|\cdot) : X \to (0,1)$ is strictly convex on $X^{\oplus}$ and strictly concave

on $X^{\otimes} \cap \left[F_{S-\varepsilon}^{-1}\left(\frac{v-1}{n-1}; \mu_S, \sigma_S^2 + \sigma_\varepsilon^2\right), \infty \right)$.

(ii) $g_{2m+1}(\otimes|\cdot) : X \to (0,1)$ is strictly convex on $X^{\oplus}$ and strictly concave

on $X^{\otimes}$.

Proof. $g_{[v;n]}(\otimes|\cdot) : X \to (0,1)$ is   defined   by   the   composite   function

$H_{[v;n]} \circ F_{S-\varepsilon} : X \to (0,1)$. $F_{S-\varepsilon}\left(t; \mu_S, \sigma_S^2 + \sigma_\varepsilon^2\right)$ is   strictly   convex   on $(-\infty, \mu_S] \cap X$ and

$F_{S-\varepsilon}\left(\mu_S; \mu_S, \sigma_S^2 + \sigma_\varepsilon^2\right) = \frac{1}{2}$ (Proposition 2.2). From Proposition III.2.2, $H_{[v;n]}(\cdot)$ is strictly

increasing and strictly convex to the point of inflection $\left[0, \breve{F}\right]$, which covers the range of

$F_{S-\varepsilon}$ on $(-\infty, \mu_S] \cap X$ because $\breve{F} = (v-1)/(n-1) > \frac{1}{2} = F_{S-\varepsilon}(\mu_S)$. A strictly increasing

convex transformation of a strictly convex function is strictly convex (Sydsæter [1981],

Theorem 5.14.iv and Note 1), hence $H_{[v;n]} \circ F_{S-\varepsilon}$ is strictly convex on $(-\infty, \mu_S] \cap X$.

Similarly, $F_{S-\varepsilon}(\cdot)$ is strictly concave on $[\mu_S, \infty)$, and $H_{[v;n]}(\cdot)$ is a strictly increasing,

strictly   concave   function   on $\left[\breve{F}, 1\right]$ (Proposition   III.2.2).   On

$\left[F_{S-\varepsilon}^{-1}\left(\frac{v-1}{n-1}; \mu_S, \sigma_S^2 + \sigma_\varepsilon^2\right), \infty \right) \cap X$, $H_{[v;n]} \circ F_{S-\varepsilon}$ is a strictly increasing and strictly concave

transformation   of   a   strictly   concave   function,   and,   hence,   strictly   concave

241

(Sydsæter [1981], Theorem 5.14.iv). In majority mechanisms M_{2m+1}, the inflection point

is $\breve{F} = \frac{1}{2}$, hence the region of concavity extends to $X^{\otimes}$.[15] ◀

The impact of size on errors is easily described in the context of majority mechanisms:

> Proposition 4.3. In M_{2m+1} , increasing size m implies that:

> (i) Type I error probability falls monotonically globally on $X^{\oplus} \setminus \{\mu_S\}$

> and is equal to $\frac{1}{2}$ for all m if $x = \mu_S$; and

> (ii) Type II error probability falls monotonically globally on $X^{\otimes}$.

Proof. Because $F_{S-\varepsilon}\left(x; \mu_S, \sigma_S^2 + \sigma_\varepsilon^2\right) < \frac{1}{2}$ if $x < \mu_S$, $F_{S-\varepsilon}\left(\mu_S; \mu_S, \sigma_S^2 + \sigma_\varepsilon^2\right) = \frac{1}{2}$ and

$F_{S-\varepsilon}\left(x; \mu_S, \sigma_S^2 + \sigma_\varepsilon^2\right) > \frac{1}{2}$ if $x > \mu_S$, the proposition follows from Proposition 4.1 and the

Condorcet theorem (Section III.2). ◀

Increasing the requirement v for rejection of H_0 for a fixed mechanism size n,

shifts the graph of $g_{[v;n]}\left(\otimes \mid \chi\right)$ down for all $x \in X$ (fewer terms are added in

$H_{[v;n]}\left(F_{S-\varepsilon}\left(x; \mu_S, \sigma_S^2 + \sigma_\varepsilon^2\right)\right) = \sum_{i=v}^{n} \binom{n}{i}\left[F_{S-\varepsilon}\left(\cdot\right)\right]^i \left[1 - F_{S-\varepsilon}\left(\cdot\right)\right]^{n-i}$). It follows directly

from Proposition 4.1 that an increase in v, for a given size n, protects against

Type I error probability globally on $X^{\oplus}$, at the cost of a higher Type II error

probability globally on $X^{\otimes}$.

---

[15] The proof can be used to establish strict convexity of the Example III.2.1 liability component $g_{[3,3]}\left(\otimes \mid x\right)$ globally on X and strict convexity of $g_{[2,3]}\left(\otimes \mid x\right)$ —and more generally of $g_{2m+1}\left(\otimes \mid x\right)$—on $\left(-\infty, F_S^{-1}\left(\frac{1}{2}\right)\right] \cap X$ and strict concavity on $\left[F_S^{-1}\left(\frac{1}{2}\right), \infty\right) \cap X$ given a linear cdf as given by the uniform distributions (see Example III.6.1): linear functions are both convex and concave.

Increasing *both* n and v in a way that ensures a super-majority rule is still in place, creates a mix of effects. As illustrated in Table III.2.1, it tends to protect against Type I errors globally on $X^\oplus$ and to protect against Type II errors towards the upper part of $X^\otimes$, at the cost of increased Type II error probability on initial parts of $X^\otimes$ (see Figure 4.1). However, generalizations can be made in larger mechanisms. A q-rule ($M_{[\lceil nq\rceil;q]}$, see Section III.2) implies increasing v and n, in such a way that the ratios v/n and inflection points $(v-1)/(n-1)$ are stabilized.
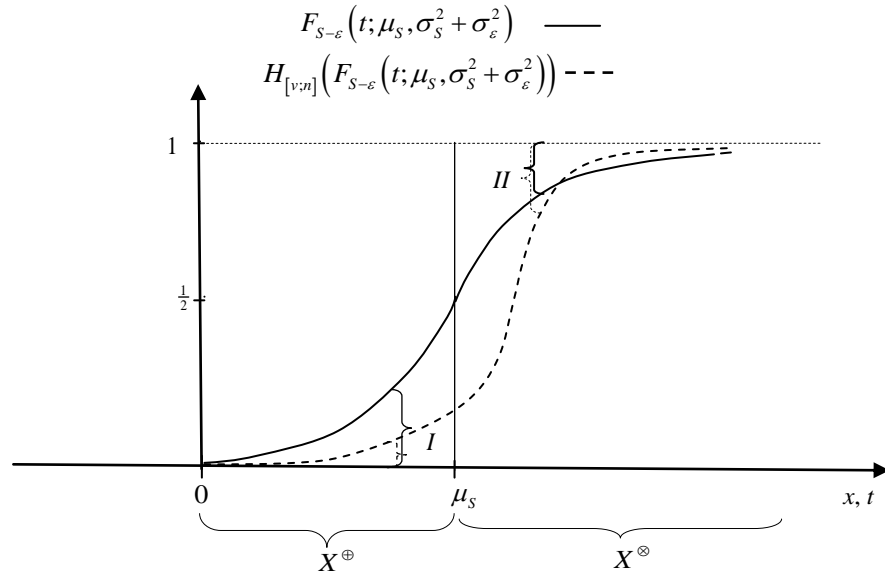
Figure 4.1 Type I and II error probabilities generated by M_1 and $M_{[v;n]}$ (legal and epistemic uncertainty from normal distributions)

<u>Proposition 4.4</u> Assume a q-rule $q \geq \frac{1}{2}$ and a sufficiently large n. It follows that $g_{[\lceil nq\rceil,n]}(\otimes \mid x) = F_{S-\varepsilon, \lceil nq\rceil:n}(x; q, n, \mu_S, \sigma_S^2 + \sigma_\varepsilon^2)$, the cdf approximately from:

$$N\left(F_{S-\varepsilon}^{-1}(q; \mu_S, \sigma_S^2 + \sigma_\varepsilon^2), \frac{q(1-q)}{n\left[f_{S-\varepsilon}\left(F_{S-\varepsilon}^{-1}(q; \mu_S, \sigma_S^2 + \sigma_\varepsilon^2); \mu_S, \sigma_S^2 + \sigma_\varepsilon^2\right)\right]^2}\right).$$

<u>Remark 4.2</u> The proposition does not require normality of the underlying sub-stantive norm distribution, but its cdf must be continuous around $F_{S-\varepsilon}^{-1}(q)$, and q in the support of $F_{S-\varepsilon}$ (the assumptions are satisfied under Proposition 2.2 nor-mality assumptions). $\square$

<u>Proof.</u> By Remark III.6.1.C, the composite $H_{\left[\lceil nq\rceil;n\right]} \circ F_{S-\varepsilon} : X \to (0,1)$ can be considered *as if* resulting fom i.i.d.s from $F_{S-\varepsilon}(t)$ in $M_{\left[\lceil nq\rceil;n\right]^{0,T}}$ corresponding to the central order statistic $(S-\varepsilon)_{\lceil nq\rceil;n}$. It has cdf $F_{S-\varepsilon,\lceil nq\rceil;n}(t) = H_{\left[\lceil nq\rceil;n\right]}\left(F_{S-\varepsilon}(t)\right)$.[16] $F_{S-\varepsilon,\lceil nq\rceil;n}(t)$ is approx-imately from $N\left(F_{S-\varepsilon}^{-1}(q), \frac{q(1-q)}{n\left[f_{S-\varepsilon}\left(F_{S-\varepsilon}^{-1}(q)\right)\right]^2}\right)$, see Proposition III.6.2 and Remark III.6.2. ◄

<u>Remark 4.3</u> Under Proposition 2.2 normality assumptions and $n = 2m+1$, $q = \frac{1}{2}$,

$$F_{S-\varepsilon}^{-1}\left(\tfrac{1}{2}; \mu_S, \sigma_S^2 + \sigma_\varepsilon^2\right) = \mu_S \qquad \text{and} \qquad f_{S-\varepsilon}\left(\mu_S; \mu_S, \sigma_S^2 + \sigma_\varepsilon^2\right) = \left[\sqrt{2\pi\left(\sigma_S^2 + \sigma_\varepsilon^2\right)}\right]^{-1}.$$

Hence, $F_{S-\varepsilon, m+1:2m+1}\left(t; \mu_S, \sigma_S^2 + \sigma_\varepsilon^2\right)$ is approximately from $N\left(\mu_S, \dfrac{\pi\left(\sigma_S^2 + \sigma_\varepsilon^2\right)}{2(2m+1)}\right)$.

The fixed mean and reduction of variance from increased size m corroborate the second order stochastic dominance characterization, in Proposition III.7.4. $\square$

The following is an immediate consequence of Proposition 4.4:

<u>Proposition 4.5</u> Increasing the size of $M_{\left[\lceil nq\rceil;q\right]}$ and $q > \frac{1}{2}$ fixed reduces

Type I error probability, globally on $X^{\oplus}$, and reduces Type II error

---

[16] This distribution is complex (and non-normal) even when drawn from underlying nor-mal populations (Arnold Balakrishnan & Nagaraja [2008:86–94]).

probability on $\left(F_{S-\varepsilon}^{-1}\left(q;\mu_S,\sigma_S^2+\sigma_\varepsilon^2\right),\infty\right)\cap X^\otimes$. In the interim region,

$$\left(\mu_S,F_{S-\varepsilon}^{-1}\left(q;\mu_S,\sigma_S^2+\sigma_\varepsilon^2\right)\right)\subset X^\otimes \text{ the Type II error probability increases.}$$

Proof. The Proposition follows directly from the fact that the asymptotic distribution in Proposition 4.4 is normal, and that expectation $F_{S-\varepsilon}^{-1}(q)$ is independent of size, while the variance is falling in n. The effect on error probabilities, therefore, follows from Proposition 2.3. ◄

Remark 4.4. In the majority case, $q=\frac{1}{2}$, and $F_{S-\varepsilon}^{-1}\left(\frac{1}{2}\right)=\mu_S$. In this case, the interim interval collapses and a global reduction of error probabilites on $X=X^\oplus\cup X^\otimes$ result, as observed above. □

An increased super-majority requirement q for a given mechanism size n from Proposition 4.4 works through two channels (dropping the function parameters $\mu_S,\sigma_S^2+\sigma_\varepsilon^2$ for notational simplicity): (i) $q\uparrow$ impacts the mean as measured by:

$$F_{S-\varepsilon}^{-1}{}'(q)=\left\{f_{S-\varepsilon}\left(F_{S-\varepsilon}^{-1}(q)\right)\right\}^{-1}>0 .{}^{17} \tag{4.2}$$

Equation (4.2) measures the *horizontal* shift in the graph of $g_{\lceil\lceil nq\rceil;n\rceil}(\otimes\,|\,x)$ at $F_{S-\varepsilon}^{-1}(q)$. Hence, an increase in q, ceteris paribus, contributes to a global decrease of Type I error probability on $X^\oplus$, and a global increase of Type II error probability on $X^\otimes$. The horizontal shift at the expectation is independent of

---

[17] The identity $F_{S-\varepsilon}\left(F_{S-\varepsilon}^{-1}(q)\right)\equiv q$ implies $F_{S-\varepsilon}{}'\left(F_{S-\varepsilon}^{-1}(q)\right)F_{S-\varepsilon}^{-1}{}'(q)=1$, hence $F_{S-\varepsilon}^{-1}{}'(q)=\left[F_{S-\varepsilon}{}'\left(F_{S-\varepsilon}^{-1}(q)\right)\right]^{-1}=\left[f_{S-\varepsilon}\left(F_{S-\varepsilon}^{-1}(q)\right)\right]^{-1}$, which is strictly positive under Proposition III.6.2 assumptions.

mechanism size n. (ii) Letting $N(q) = q(1-q)$ denote the variance numerator, $q \uparrow$ impacts the distribution variance as measured by:

$$\frac{\partial}{\partial q} \left\{ \frac{q(1-q)}{n \left[f_{S-\varepsilon} \left(F_{S-\varepsilon}^{-1}(q) \right) \right]^2} \right\} =$$

$$\frac{N'(q) \left[f_{S-\varepsilon} \left(F_{S-\varepsilon}^{-1}(q) \right) \right]^2 - 2N(q) f_{S-\varepsilon}{}' \left(F_{S-\varepsilon}^{-1}(q) \right)}{n \left[f_{S-\varepsilon} \left(F_{S-\varepsilon}^{-1}(q) \right) \right]^4} \tag{4.3}$$

$N(q)$ is a strictly concave function with a global maximum at $q = \frac{1}{2}$, with $N(\frac{1}{2}) = \frac{1}{4}$ and $N(0) = N(1) = 0$. Hence, the impact of a marginal increase in q on the variance numerator is zero, if $q = \frac{1}{2}$ and negative if $q > \frac{1}{2}$.

In *majority mechanisms*, a marginal increase in q (from $\frac{1}{2}$) has no effect on the variance term because $N'(q) \big|_{q=\frac{1}{2}} = 0$ and $f_{S-\varepsilon}{}' \left(F_{S-\varepsilon}^{-1}(q) \right) \big|_{q=\frac{1}{2}}$ $= f_{S-\varepsilon}{}'(\mu_S) = 0$. Hence, from (i) it can be concluded that a small increase in q leads to a global reduction in Type I error probability globally on $X^{\oplus}$ and a global reduction of Type I error probability globally on $X^{\otimes}$, for all parameter constellations $\left(\mu_S, \sigma_S^2, \sigma_\varepsilon^2 \right)$. The *horizontal* shift at μ_S is measured by

$\left\{ f_{S-\varepsilon} \left(F_{S-\varepsilon}^{-1}(q) \right) \right\}^{-1} \Big|_{q=\frac{1}{2}} = \left\{ f_{S-\varepsilon} \left(\mu_S; \mu_S, \sigma_S^2 + \sigma_\varepsilon^2 \right) \right\}^{-1} = \sqrt{2\pi \left(\sigma_S^2 + \sigma_\varepsilon^2 \right)}$, thus it is in-

creasing in legal and epistemic variance.

In super-majority mechanisms $M_{\left[\lceil nq \rceil ; n \right]}$, it is difficult to predict the total effect of an increase in q, for a fixed mechanism size n, due to ambiguous effects on the variance, as seen in (4.3): If $q > \frac{1}{2}$, $F_{S-\varepsilon}^{-1}(q) > \mu_S$, hence

$f_{S-\varepsilon}{}'\left(F_{S-\varepsilon}^{-1}(q)\right) < 0$, contributes to a positive effect on the variance. But

$N'(q) < 0$.

The *total* effect on liability probability of an increase in q, for a given

mechanism size n, includes the effect on expectation (i) and variance (ii). It is

given by the *vertical shift* of $g_{\left[\lceil nq\rceil,n\right]}\left(\otimes \mid x; \mu_S, \sigma_S^2 + \sigma_\varepsilon^2\right)$ on X ;

$$\frac{\partial g_{\left[\lceil nq\rceil,n\right]}\left(\otimes \mid x; \mu_S, \sigma_S^2 + \sigma_\varepsilon^2\right)}{\partial q} = \frac{\partial F_{S-\varepsilon, \lceil nq\rceil n}\left(x; \mu_S, \sigma_S^2 + \sigma_\varepsilon^2\right)}{\partial q}.$$

Define:

$$\psi\left(x; q, \mu_S, \sigma_S^2 + \sigma_\varepsilon^2\right)$$
$$= \sqrt{n} f_{S-\varepsilon}\left(F_{S-\varepsilon}^{-1}\left(q; \mu_S, \sigma_S^2 + \sigma_\varepsilon^2\right)\right)\left\{x - F_{S-\varepsilon}^{-1}\left(q; \mu_S, \sigma_S^2 + \sigma_\varepsilon^2\right)\right\}\left[N(q)\right]^{-\frac{1}{2}},$$

and let ψ_q denote the partial derivative of ψ w.r.t. q. It follows (dropping the

parameters in the statement):

<u>Proposition 4.6</u> In $M_{\left[\lceil nq\rceil;n\right]}$, increasing the (super-)majority requirement

$q \geq \frac{1}{2}$, for a fixed size n, has the following (approximate) effect on lia-

bility probability:

A) $\dfrac{\partial g_{\left[\lceil nq\rceil,n\right]}\left(\otimes \mid x; \mu_S, \sigma_S^2 + \sigma_\varepsilon^2\right)}{\partial q} = \dfrac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\psi\left(\chi; q, \mu_S, \sigma_S^2 + \sigma_\varepsilon^2\right)} \psi_q\left(x; q, \mu_S, \sigma_S^2 + \sigma_\varepsilon^2\right),$

$\psi_q \equiv \sqrt{n}\left(N(q)\right)^{-\frac{1}{2}}\left\{\left[\dfrac{f_{S-\varepsilon}{}'\left(F_{S-\varepsilon}^{-1}(q)\right)}{f_{S-\varepsilon}\left(F_{S-\varepsilon}^{-1}(q)\right)} - \dfrac{N'(q)}{2N(q)} f_{S-\varepsilon}\left(F_{S-\varepsilon}^{-1}(q)\right)\right]\left\langle x - F_{S-\varepsilon}^{-1}(q)\right\rangle - 1\right\}$

B)   In   the   majority   case   ($q = \frac{1}{2}$), $\psi_q = -2\sqrt{n}$ and

$\psi\left(x; \frac{1}{2}\right) = \dfrac{\sqrt{2n}}{\sqrt{\pi\left(\sigma_S^2 + \sigma_\varepsilon^2\right)}}\left\{x - \mu_S\right\}$, giving

$$\left.\frac{\partial g_{\left[\lceil(2m+1)q\rceil;2m+1\right]}\left(\otimes\mid x;\mu_S,\sigma_S^2+\sigma_\varepsilon^2\right)}{\partial q}\right|_{q=\frac{1}{2}}=-\sqrt{2m+1}\sqrt{\frac{2}{\pi}}\,e^{-\frac{\sqrt{2m+1}}{\sqrt{2\pi\left(\sigma_S^2+\sigma_\varepsilon^2\right)}}\{x-\mu_S\}}<0$$

on $\quad X=X^{\oplus}\cup X^{\otimes}$.

<u>Proof.</u> The event $\left(S-\varepsilon\right)_{\lceil nq\rceil:n}\le x$ is equivalent to the event (function arguments

$\mu_S,\sigma_S^2+\sigma_\varepsilon^2$ dropped)

$$\frac{\left(S-\varepsilon\right)_{\lceil nq\rceil:n}-F_{S-\varepsilon}^{-1}(q)}{\frac{\sqrt{q(1-q)}}{\sqrt{n}f_{S-\varepsilon}\left(F_{S-\varepsilon}^{-1}(q)\right)}}\le\frac{\chi-F_{S-\varepsilon}^{-1}(q)}{\frac{\sqrt{q(1-q)}}{\sqrt{n}f_{S-\varepsilon}\left(F_{S-\varepsilon}^{-1}(q)\right)}}\equiv\psi\left(x;q\right).$$

From Proposition 4.4 and Bartoszyński & Niewiadomska-Bugaj [1996], Theorem 9.10.1,

the event has probability (Φ denoting the cdf of the standard normal):

$$\Phi\left(\psi\left(x;q\right)\right)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\psi(x;q)}e^{-\frac{1}{2}\tau}d\tau.$$

From Lebniz's formula for differentiation of integrals:[18]

$$\frac{\partial\Phi\left(\psi\left(x;q\right)\right)}{\partial q}=\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\psi(x;q)}\psi_q\left(x;q\right).$$

$\psi_q\left(x;q\right)$ is given by:

$$\sqrt{n}\Big\{f_{S-\varepsilon}{}'\left(F_{S-\varepsilon}^{-1}(q)\right)F_{S-\varepsilon}^{-1}{}'(q)\{\chi-F_{S-\varepsilon}^{-1}(q)\}\left[N(q)\right]^{-\frac{1}{2}}$$

$$-f_{S-\varepsilon}\left(F_{S-\varepsilon}^{-1}(q)\right)F_{S-\varepsilon}^{-1}{}'(q)\left[N(q)\right]^{-\frac{1}{2}}-\tfrac{1}{2}f_{S-\varepsilon}\left(F_{S-\varepsilon}^{-1}(q)\right)\{\chi-F_{S-\varepsilon}^{-1}(q)\}\left[N(q)\right]^{-\frac{3}{2}}N{}'(q)\Big\}$$

Using $F_{S-\varepsilon}^{-1}{}'(q)=\left\{f_{S-\varepsilon}\left(F_{S-\varepsilon}^{-1}(q)\right)\right\}^{-1}$ and rearranging gives

$$\psi_q\left(x;q\right)=\sqrt{n}\left(N(q)\right)^{-\frac{1}{2}}\left\{\left[\frac{f_{S-\varepsilon}{}'\left(F_{S-\varepsilon}^{-1}(q)\right)}{f_{S-\varepsilon}\left(F_{S-\varepsilon}^{-1}(q)\right)}-\frac{N{}'(q)}{2N(q)}f_{S-\varepsilon}\left(F_{S-\varepsilon}^{-1}(q)\right)\right]\langle x-F_{S-\varepsilon}^{-1}(q)\rangle-1\right\}.$$

At $q=\frac{1}{2}$, $F_{S-\varepsilon}^{-1}\left(\frac{1}{2}\right)=\mu_S$, $N{}'\left(\frac{1}{2}\right)=f_{S-\varepsilon}{}'(\mu_S)=0$, and $N\left(\frac{1}{2}\right)=\frac{1}{4}$, giving $\psi_q\left(x;\frac{1}{2}\right)$ and

$\psi\left(x;\frac{1}{2}\right)$ in the proposition. ◄

---

[18] Bartle [1976], Theorem 31.8.

It follows that the majority mechanism *vertical shift* (reduction in liability proba-bility), at the expectation $x = \mu_S$, is independent of σ_S^2 and σ_ε^2 and is larger the larger the mechanism. Otherwise, the liability probability impact is a complex function of size and the parameters describing uncertainty. The impact, however, is—as observed above—a global reduction of the probability of Type I errors on $X^\oplus$ and a global increase in the probability of Type II errors on $X^\otimes$.

In super-majority mechanisms, the discussion of Table III.2.1 suggests that it should be possible to demonstrate a negative sign in Proposition 6.4.A, in regions of $X^\oplus$ sufficiently close to the lower boundary, and a positive sign in regions in $X^\otimes$, sufficiently far from μ_S.

5   Separation of collective decisions on law and facts

Section 4 assumes that each judge in $M_{[v;n]}$ observes conditioning facts and makes a decision regarding liability, based on the legal standard concluding lia-bility ($\otimes$) iff $S \leq \mathbf{x} = x + \varepsilon$. In the context of collectives, it is relevant to investi-gate effects of separating decisions on law and facts. The separation can take many forms. *Vertical separation* takes place in legal hierarchies, with pure legal questions finally determined at higher echelons. *Horizontal separation* takes place between courts and juries, especially in civil law.[19] Horizontal separation may also take place in judicial panels, which sequentially split legal and factual determinations.

---

[19] In criminal law, however, protection of the *general verdict* prohibits separation of law and fact (see LaFave, Israel & King [2004] Sec. 24.10. Double jeopardy rules constrain vertical separation. However, there are many variants over these themes, law-fact separa-tion often a fuzzy notion even from a descriptive point of view.

It is assumed that adverse decisions with respect to law are made in $M_{\left[v^S;n^S\right]^{\Uparrow,T}}$ and with respect to facts in $M_{\left[v^{\mathbf{x}};n^{\mathbf{x}}\right]^{\Downarrow,T}}$, see Section III.6. Both classes of mechanisms, ceteris paribus, prioritize H_0 (a default state of no liability, $\oplus$). The resulting combined mechanisms are denoted $M_{\left[v^S;n^S\right]^{\Uparrow,T}\wedge\left[v^{\mathbf{x}};n^{\mathbf{x}}\right]^{\Downarrow,T}}$. The corresponding majority mechanisms $M^T_{2m^S+1}$ and $M^T_{2m^{\mathbf{x}}+1}$, ceteris paribus, treat H_0 and H_A symmetrically. Combined majority mechanisms are denoted $M^T_{2m^S+1\,\wedge\,2m^{\mathbf{x}}+1}$. The mechanisms are evaluated from an ex ante perspective. Judges vote independently in and across mechanisms, both with respect to law and facts, reflecting normative commitments (Section I.2) and the assumptions about epistemic competence discussed in Sections 2 and 4.

In $M_{\left[v^S;n^S\right]^{\Uparrow,T}\wedge\left[v^{\mathbf{x}};n^{\mathbf{x}}\right]^{\Downarrow,T}}$, H_0 is rejected iff $S_{v^S;n^S}\leq\mathbf{x}_{n^{\mathbf{x}}-v^{\mathbf{x}}+1:n^{\mathbf{x}}}$, or, $S_{v^S;n^S}-\varepsilon_{n^{\mathbf{x}}-v^{\mathbf{x}}+1:n^{\mathbf{x}}}\leq x$. The transformed standard $S_{v^S;n^S}$ is composed of i.i.d.'s from $F_S\left(t;\mu_S,\sigma_\varepsilon^2\right)$ and has cdf $H_{[v;n]}\left(F_S\left(t;\mu_S,\sigma_S^2\right)\right)$.[20] Similarly, the transformed error term $\varepsilon_{n^{\mathbf{x}}-v^{\mathbf{x}}+1:n^{\mathbf{x}}}$ is composed of i.i.d.'s from $F_\varepsilon\left(t;0,\sigma_\varepsilon^2\right)$, with cdf $H_{[n-v+1;n]}\left(F_\varepsilon\left(t;0,\sigma_\varepsilon^2\right)\right)=1-H_{[v;n]}\left(1-F_\varepsilon\left(t;0,\sigma_\varepsilon^2\right)\right)$.[21] However, it is complex to derive $P\left\{S_{v^S;n^S}-\varepsilon_{n^{\mathbf{x}}-v^{\mathbf{x}}+1:n^{\mathbf{x}}}\leq x\right\}\equiv g_{\left[v^S;n^S\right]^{\Uparrow,T}\wedge\left[v^{\mathbf{x}};n^{\mathbf{x}}\right]^{\Downarrow,T}}\left(\otimes\,|\,x\right)$, even under Proposition 2.2 normality assumptions. Hence, the analysis is limited to mechanisms sufficiently large to allow approximation from asymptotic theory.[22] Let $q^S\in\left[\tfrac{1}{2},1\right)$

---

[20] See Proposition III.6.1.i.

[21] See Proposition III.6.ii and Proposition III.3.2.

[22] The requirements on n^S (m^S) and $n^{\mathbf{x}}$ ($m^{\mathbf{x}}$) for accuracy depend on the structure of the underlying population distributions.

and $q^{\mathbf{x}} \in \left(0, \frac{1}{2}\right]$. The parameter values correspond to q-rules, which refer to laws and facts, respectively.

Proposition 5.1 In $M_{\left[\left[n^S q^S\right];n^S\right]^{\Uparrow,T} \wedge \left[\left[n^{\mathbf{x}} q^{\mathbf{x}}\right];n^{\mathbf{x}}\right]^{\Downarrow,T}}$, transformation of abstract

norm $g = g_{F_S} \in \mathbb{P}_{\{\oplus,\otimes\}}^X$ gives:

$$g_{\left[\left[n^S q^S\right];n^S\right]^{\Uparrow,T} \wedge \left[\left[n^{\mathbf{x}} q^{\mathbf{x}}\right];n^{\mathbf{x}}\right]^{\Downarrow,T}} (\cdot \mid x)\Big|_{x\in X} = \Big\langle 1-\Gamma(x), \Gamma(x); \oplus,\otimes \Big\rangle\Big|_{x\in X}$$

with cdf $\Gamma\left(x; q^S, n^S, q^{\mathbf{x}}, n^{\mathbf{x}}; \mu_S, \sigma_S^2 + \sigma_\varepsilon^2\right)$ approximately from:

$$N\left(F_S^{-1}\left(q^S\right) - F_\varepsilon^{-1}\left(q^{\mathbf{x}}\right), \frac{q^S\left(1-q^S\right)}{n^S\left[f_S\left(F_S^{-1}\left(q^S\right)\right)\right]^2} + \frac{q^{\mathbf{x}}\left(1-q^{\mathbf{x}}\right)}{n^{\mathbf{x}}\left[f_\varepsilon\left(F_\varepsilon^{-1}\left(q^{\mathbf{x}}\right)\right)\right]^2} \right).$$

Proof. Following the Proposition 4.4 proof strategy, the cdf of $S_{\left\lceil n^S q^S\right\rceil:n^S}$ is approximately

$$N\left(F_S^{-1}\left(q^S\right), \frac{q^S\left(1-q^S\right)}{n^S\left[f_S\left(F_S^{-1}\left(q^S\right)\right)\right]^2} \right)$$ and $\varepsilon_{\left\lceil n^{\mathbf{x}} q^{\mathbf{x}}\right\rceil:n^{\mathbf{x}}}$ approximately $N\left(F_\varepsilon^{-1}\left(q^{\mathbf{x}}\right), \frac{q^{\mathbf{x}}\left(1-q^{\mathbf{x}}\right)}{n^{\mathbf{x}}\left[f_\varepsilon\left(F_\varepsilon^{-1}\left(q^{\mathbf{x}}\right)\right)\right]^2} \right)$. As a

linear combination of (approximately) normal variables, $S_{\left\lceil n^S q^S\right\rceil:n^S} - \varepsilon_{\left\lceil n^{\mathbf{x}} q^{\mathbf{x}}\right\rceil:n^{\mathbf{x}}}$ is (approxi-

mately) $\quad N\left(F_S^{-1}\left(q^S\right) - F_\varepsilon^{-1}\left(q^{\mathbf{x}}\right), \frac{q^S\left(1-q^S\right)}{n^S\left[f_S\left(F_S^{-1}\left(q^S\right)\right)\right]^2} + \frac{q^{\mathbf{x}}\left(1-q^{\mathbf{x}}\right)}{n^{\mathbf{x}}\left[f_\varepsilon\left(F_\varepsilon^{-1}\left(q^{\mathbf{x}}\right)\right)\right]^2} \right)$ (Bartoszyński &

Niewiadomska-Bugaj [1996], Theorem 9.10.2). ◄

Discussion. Focusing on the liability components in direct and combined mecha-

nisms, $g_{\left[\lceil nq\rceil:n\right]}(\otimes \mid x)$ and $g_{\left[\left[n^S q^S\right];n^S\right]^{\Uparrow,T} \wedge \left[\left[n^{\mathbf{x}} q^{\mathbf{x}}\right];n^{\mathbf{x}}\right]^{\Downarrow,T}}(\otimes \mid x)$ respectively, it follows

from Propositions 4.4 and 5.1 that all *majority mechanisms* have the same expec-

tation component. If $q = q^S = q^{\mathbf{x}} = \frac{1}{2}$, $n = 2m+1$: $F_{S-\varepsilon}^{-1}\left(\frac{1}{2}\right) = F_S^{-1}\left(\frac{1}{2}\right) - F_\varepsilon^{-1}\left(\frac{1}{2}\right)$

$= \mu_S$. The variance components are given by $\frac{\pi}{2}\left\{\frac{\sigma_S^2 + \sigma_\varepsilon^2}{2m+1}\right\}$ and $\frac{\pi}{2}\left\{\frac{\sigma_S^2}{2m^S+1} + \frac{\sigma_\varepsilon^2}{2m^{\mathbf{x}}+1}\right\}$, re-

spectively. Hence, separation of decisions on law and fact is neutral, if

$m^S = m^{\mathbf{x}} = m$. Increasing the size of any mechanism (mechanism component) m, m^S or $m^{\mathbf{x}}$ reduces variance, and, therefore, by Proposition 2.3 globally reduces Type I error probability on $X^{\oplus}$ and globally reduces Type II error probability on $X^{\otimes}$. To the extent that separation effectively means commitment to smaller mechanisms, more errors will be generated, unless it can be designed to the specifics of the legal and epistemic uncertainties:

Example 5.1 (mechanism design under joint legal and epistemic uncertainty). Consider the problem of minimizing $\frac{\pi}{2}\left\{\frac{\sigma_S^2}{2m^S+1}+\frac{\sigma_\varepsilon^2}{2m^{\mathbf{x}}+1}\right\}$ with respect to m^S and $m^{\mathbf{x}}$ subject to the constraint $m^S + m^{\mathbf{x}} \leq m$. In sufficiently large mechanisms, m^S and $m^{\mathbf{x}}$ can be taken as continuous variables. The minimization problem is convex[23] and has necessary condition (the constraint is binding):

$$\frac{\sigma_S}{2m^S+1} = \frac{\sigma_\varepsilon}{2m^{\mathbf{x}}+1}.$$

It implies that an optimal mechanism $M^T_{2m^{S}*+1 \,\wedge\, 2m^{\mathbf{x}}*+1}$ (satisfying the constraint $m^S* + m^{\mathbf{x}}* = m$) has the design:

$$\begin{cases} m^S* = \dfrac{1}{2\left[\sigma_S + \sigma_\varepsilon\right]}\left\{\sigma_S\left(2m+1\right)-\sigma_\varepsilon\right\} \\[2mm] m^{\mathbf{x}}* = \dfrac{1}{2\left[\sigma_S + \sigma_\varepsilon\right]}\left\{\sigma_\varepsilon\left(2m+1\right)-\sigma_S\right\} \end{cases}.$$

It follows that a mechanism globally minimizing Type I error probability globally on $X^{\oplus}$ and Type II error probability globally on $X^{\otimes}$ has the relative size of the two mechanisms depend on standard-deviations:

---

[23] To minimize the variance equals minimizing $\left\{\frac{\sigma_S^2}{2m^S+1}+\frac{\sigma_\varepsilon^2}{2m^{\mathbf{x}}+1}\right\}$. The minimand is the sum of two convex functions, hence itself convex (Sydsæter [1981], Theorem 5.14.i and Note 1).

$$\frac{m^S *}{m^{\mathbf{x}} *} = \frac{\left\{\sigma_S\left(2m+1\right)-\sigma_\varepsilon\right\}}{\left\{\sigma_\varepsilon\left(2m+1\right)-\sigma_S\right\}}.$$

If the solution cannot be conditioned on standard deviation information, that is, $\sigma_S = \sigma_\varepsilon$, $m^S * = m^{\mathbf{x}} * = \frac{1}{2}m$. This mechanism is functionally equivalent to the "direct" mechanism $M_{2\left(\frac{1}{2}m\right)+1}$. It follows that the optimal combined mechanism, under incomplete information, is dominated by any direct mechanism $M_{2m'+1}$ with $m' > \frac{1}{2}m$. ∎

The Example 5.1 type of analysis can varied and be much refined (for example, errors caused by legal and epistemic uncertainty might not be given equal weight), but it points to problematic aspects of splitting a pool of decision makers into sub-groups in the absence of specific information.[24]

In $M_{\left[\left\lceil n^S q^S\right\rceil;n^S\right]^{\Uparrow,T} \wedge \left[\left\lceil n^{\mathbf{x}} q^{\mathbf{x}}\right\rceil;n^{\mathbf{x}}\right]^{\Downarrow,T}}$ with $q^S > \frac{1}{2}$ or $q^{\mathbf{x}} < \frac{1}{2}$, protection against Type I error probability is ensured through the expectation component $F_S^{-1}\left(q^S\right) - F_\varepsilon^{-1}\left(q^{\mathbf{x}}\right)$ both by $F_S^{-1}\left(q^S\right) > \mu_S$ and $F_\varepsilon^{-1}\left(q^{\mathbf{x}}\right) < 0$, pushing down the liability probability globally on $X^\oplus$, at the cost of higher Type II error probability globally on $X^\otimes$. Hence, increasing component sizes (n^S or $n^{\mathbf{x}}$), for a given a constellation $\left(q^S, q^{\mathbf{x}}\right)$, also decreases variance, and, hence, from normality and Propositions 2.3 and 4.5, ceteris paribus, reduces Type I error probability globally on $X^\oplus$ and Type II error probability globally on $\left(F_S^{-1}\left(q^S\right) - F_\varepsilon^{-1}\left(q^{\mathbf{x}}\right), \infty\right) \cap X \subset X^\otimes$, at the cost of higher Type II error probability in the interim region $\left(\mu_S, F_S^{-1}\left(q^S\right) - F_\varepsilon^{-1}\left(q^{\mathbf{x}}\right)\right) \subset X^\otimes$.

---

[24] See Boland [1989], Theorem 7, on direct and indirect majority systems and, more generally, Nitzan [2010] part III on mechanism design under epistemic uncertainty.

A detailed comparison of composite and joint mechanisms is complicated by the mixed effect on variance from a change in q, q^S, and $q^{\mathbf{x}}$ (Proposition 4.6.A). Limiting the discussion to the case of a marginal *increase* in q and q^S from $\frac{1}{2}$ and a marginal *decrease* in $q^{\mathbf{x}}$ from $\frac{1}{2}$, variance term effects are eliminated. In this case, the $g_{[\lceil nq \rceil, n]}(\otimes | \chi)$ graph shifts to the right as, measured by

$$F_{S-\varepsilon}^{-1}{}'(q)\big|_{q=\frac{1}{2}} = \sqrt{2\pi}\sqrt{\sigma_S^2 + \sigma_\varepsilon^2}.$$ Similarly, $g_{[[\lceil n^S q^S \rceil; n^S]^{\Uparrow, T} \wedge [\lceil n^{\mathbf{x}} q^{\mathbf{x}} \rceil; n^{\mathbf{x}}]^{\Downarrow, T}}(\otimes | \chi)$ shifts to

the   right,   as   measured   by $F_S^{-1}{}'(q^S) - F_\varepsilon^{-1}{}'(q^{\mathbf{x}})\big|_{q^S = q^{\mathbf{x}} = \frac{1}{2}}$

$$= \sqrt{2\pi\sigma_S^2} + \sqrt{2\pi\sigma_\varepsilon^2} = \sqrt{2\pi}\left\{\sqrt{\sigma_S^2} + \sqrt{\sigma_\varepsilon^2}\right\}.$$ These shifts imply a global reduction

of Type I error on $X^\oplus$ and an increase in Type II error on $X^\otimes$, sufficiently far from μ_S.

Remark 5.1 Heckscher [1892], in discussing the *classical rule*, argues that a court will not use a an aggregator such as the mean, because it contradicts the purpose of finding a collective *meaning* (according to the rule, equal to at least one of the individual judgments). If, on the other hand, the main purpose of the collective were to track an independent truth ("det Sande"), the minority should have an impact on the outcome (see Remark III.1.1). □

Remark 5.2 If courts use the mean for factual aggregation, liability is concluded iff $S_{v^S:n^S} \leq \bar{\mathbf{x}}_{n^{\mathbf{x}}}$, or, $S_{v^S:n^S} - \bar{\varepsilon}_{n^{\mathbf{x}}} \leq x$, $\bar{\varepsilon}_{n^{\mathbf{x}}} = \frac{1}{n^{\mathbf{x}}}\sum_{i=1}^{n^{\mathbf{x}}} \varepsilon_i$. With $\{\varepsilon_i\}_{i=1}^{n^{\mathbf{x}}}$ i.i.d.'s from $F_\varepsilon(t; 0; \sigma_\varepsilon^2)$, $\bar{\varepsilon}_{n^{\mathbf{x}}}$ has the *exact* distribution $N\left(0, [n^{\mathbf{x}}]^{-1}\sigma_\varepsilon^2\right)$. Due to the complex distribution of the order statistic $S_{v^S:n}$, even under normality assumptions (Arnold, Balakrishnan & Nagaraja [2008:86-94] and Section III.6 above), the analy-

sis must be based on approximations (or simulations). It should be noted that (for large $n^{\mathbf{x}}$) $\bar{\varepsilon}_{2m+1}$ has a lower variance than $\varepsilon_{m+1:2m+1}$. Because both are have expected values equal to $\mu_{F_\varepsilon} = 0$, the former is a more efficient estimator of epistemic uncertainty.[25] □

To determine error generation, it is important to study the distribution of cases over $X^\oplus \cup X^\otimes$. This is a question with many interesting dimensions. Equilibrium analysis is relevant and should include case costs.[26] In criminal procedure, case screening is essential.[27]

---

[25] $\operatorname{var}\{\bar{\varepsilon}_{2m+1}\} = (2m+1)^{-1}\sigma_\varepsilon^2$; $\operatorname{var}\{\varepsilon_{m+1:2m+1}\} = \frac{\pi}{2}(2m+1)^{-1}\sigma_\varepsilon^2$.   On efficiency, see Bartoszyński & Niewiadomska-Bugaj [1996] in and at Definition 12.5.2.

[26] For studies endogenizing the stream of cases, see Borchrevink [2011] with further references.

[27] See Posner [1990:216].

PART EIGHT

EXTENSIONS

1 Further applications

It has been demonstrated that norm structures and distributions over final outcomes under (joint) legal and epistemic uncertainty vary systematically with mechanisms $M \in \mathcal{M}$. Part VII suggests that large mechanisms have positive effects on Type I and II error distributions. The notions of mean-preserving risk reduction and second-order stochastic dominance, which characterize the mechanisms in Section III.7, suggest that large panels are valuable to risk-averse agents (see also the shrinking contract zones from increasing arbitration panels in Section IV.2). However, instead of establishing large, one-shot mechanisms, many jurisdictions have built elaborate, hierarchical authority structures that typically are *jus cogens* before a claim has materialized (see Damaška [1986], contrasting bureaucratic and coordinate structures). Fuzzy law-fact distinctions and discretionary trial structuring (Section III.5) imply uncertainty regarding which mechanism applies. Lack of an effective *vertical stopping rule*, combined with compulsory fee-shifting ("English") rules, arguably compounds risk, adversely affecting risk-averse agents' access to court.[1]

---

[1] The notion and terminology of a stopping rule is inspired by Urfalino [2010]. Access difficulties may be further compounded by prohibition of contingent fee contracts between litigants and their representatives. These are complex questions (with many dimensions, including parties' incentives and risk spreading through markets for claims) that have been intensively studied in the context of coordinate systems (see Prichard [1988]).

Hansmann [1996] points out that intrinsic difficulties concerning collective decision-making is a main determinant of how enterprises are structured. Because these institutions are exposed to the forces of market selection, they are suggested to be important models to study (although they have received much less attention than the heavily scrutinized political institutions). This idea seems transferable to legal decision mechanisms; arbitrational mechanisms, in contrast to courts, are exposed to competitive forces. Characteristically, decisions are final, barring procedural errors.

These fundamental design questions warrant further analyses (under joint legal and epistemic uncertainty).

Several references have been made to the essential role of contracts, including corporate forms, collective agreements, and international conventions. An integration of modern contract (incentive) theory, with the judicial panel analysis, would likely be rewarding.[2]

The equilibrium analyses presume that agents have complete information about legal sources, the structure of meta-norms, and the decision mechanism in place. The informational burden and discretionary elements in mechanism selection accentuate a relaxation of these assumptions.[3]

Norms are (locally) represented as prospects. The complexities involved in norm combinations and norm transformations may mean that optimizing

The American Bar Association approved contingent fees in its 1908 Canon of Ethics (Yeazell [2001]).

[2] As noted by Bolton & Dewatripont [2005:3], enforcement mechanisms play an implicit role in the theory.

Hierarchical courts systems may also be analyzed as incentive structures. Cabrillo & Fitzpatrick [2008:9] note that "judicial institutions have hidden behind the notion of independence or the singular nature of the service they provide, to shield themselves from the organizational analysis that other institutions are commonly submitted to."

[3] In this context, it may be relevant that the norm representations (pure or mixed, abstract or transformed) have similarities to information structures, see Remark II.2.1.

agents do not consider compound and reduced simple prospects as equivalent for cognitive reasons. This makes insights from prospect theory relevant, since that theory treats perception errors in a systematic way.[4]

2   Framework extensions

The analysis of (joint) legal and epistemic uncertainty assumes homogenous, epistemic competencies and a single-dimensional substantive norm, defined over a dichotomous outcome space. Extensions of the framework to multi-dimensional norms and to sequential decision-making more generally, are relevant.[5] Extensions should include consideration of group agency.[6]

Judges have been assumed not to vote on complete norms (based on the notion that they *apply* the law). Constitutional courts may, however, generate decisions, $\langle x, g \rangle^M \in X \times \mathbb{P}_Y{}^X$, and even rule on norms in abstraction from individual cases, $\langle g \rangle^M \in \mathbb{P}_Y{}^X$, challenging such notions as independence (Shetreet [1985:636]), legitimacy, and doctrines such as *res judicata.* Further analysis is merited.

Experimental studies suggest that so-called *hindsight bias* is a systematic problem for decision-makers, who evaluate ex ante precaution with access to ex post adverse outcome information. The tendency is for decision-makers to inflate the probability of an adverse outcome.[7] Procedural law measures have been sug-

---

[4] See, generally, Wakker [2010].

[5] See List [2005] and Bovens & Rabonowicz [2006] on epistemic uncertainty in complex situations.

[6] See Nordén [2015] and, more generally, List & Pettit [2011].

[7] The tendency is present even if decision makers are informed about the bias' typical presence. See Christensen-Szalanski & Willham [1991] and LaBine & LaBine [1996].

gested to alleviate this problem (including bifurcation). Such issues deserve close attention (under joint legal and epistemic uncertainty).[8]

The possibility that judges form preferences for norms arises naturally from discretion (power-conferring norms), as illustrated in Examples VI.3.1–2. However, it is a complex notion, studied by Kornhauser [1992a,1992b] and Landa & Lax [2009]. Noting that Lax [2007] defines the case space as a hyper-cube (each case dimension is coded in $[0,1]$), and that $\mathbb{P}_Y^X$ is a so-called mixture space (see Kreps [1988] Theorem 5.11), merging of the present probabilistic representation of norms with these analyses might simplify the representation of preference orderings.

It may also be observed that, given the preferences related to the set of consequences $Y = \{0, L\}$ in Example VI.3.2, Judge 1 and Judge 3, when voting on doctrinal factor dimensions in the construction of what Landa & Lax [2009] call the collegial factor rule, have an incentive to *misrepresent* their preferences. Even though the judges are envisioned as committed to not voting strategically (Section I.2), it is germane to design robust mechanisms. In lieu of paradoxes and asymmetric information, such questions are challenging, both positively and normatively.[9]

---

[8] See LaBine & Labine [1996:514] and more generally Fienberg [1989:10–15] (which includes recommendations for legal education).

[9] See generally Nizan [2010], who discusses situations when decision makers have differing preferences, and situations where agents have the same preferences but differ in decisional capabilities in uncertain environments.

PART NINE

CONCLUSIONS

Courts are obliged to decide cases from pools of cases they do not control, typically in all-or-nothing fashion. As described by the *Panel on Statistical Assessments as Evidence in the Courts,* if courts are no longer affiliated with philosophical determinism, they are likely to under-communicate uncertainty as arbitrators of individual cases.[1] Hindsight bias, or creeping determinism, may well affect doctrinal studies of law that reconstruct norm patterns, based on previous case dispositions (norms in the extensive sense).[2] However, as Bishop Hoadly points out:[3]

> Nay whoever hath an absolute authority to interpret any written or spoken laws it is he who is the lawgiver to all intents and purposes and not the person who first wrote or spake them.

---

[1] "In every society with a formal legal system, the adjudicative power rests its authority on the assumption that it will do justice. To the extent that this assumption is open to question, the legitimacy of the entire regime may be called into question [---] Certainty, and even the appearance of certainty, are important in law" (Fienberg [1989:139]). In civil law jurisdictions, and even in constitutional courts, it is still the prevailing norm or rule to suppress uncertainty in reaching a decision, not admitting concurring or dissenting opinions (Section I.1).

[2] "When we attempt to understand past events, we implicitly test the hypotheses or rules we use to both interpret and anticipate the world around us. If, in hindsight, we systematically underestimate the surprises which the past held and holds for us, we are subjecting those hypotheses to inordinately weak tests and, presumably, finding little reason to change them. Thus, the very outcome knowledge which give us the feeling that we understand the past was all about may prevent us from learning anything from it" Fischoff [1975:298–99].

[3] Quoted from Hart [1961:137].

If legal sources do not determine outcomes uniquely (or epistemic uncertainty is present), court design affects the pattern of decisions, accentuating U.S. Supreme Court Justice Robert Jackson's observation that "[w]e are not final because we are infallible, but we are infallible only because we are final."[4]

This work has endeavored to demonstrate that formal representation of norms enables the study of fallible final lawgivers. Whatever the specific assumptions, it is conjectured that functional (repercussion) analysis of law will follow the lines suggested by the Part I epigraph quotes and, thus, concerns abstract patterns of law. To the extent that the representations preserve essential properties of the phenomena studied, new insights are gained.[5] Explicit formulation of assumptions facilitates critique and opens the study of law to contributions from other disciplines, hopefully advancing a *cumulative* legal theory. Functional analyses contribute to the enterprise Ackerman [1984] calls "lawyering in the active state." It seems a worthwhile enterprise, given the interests at stake. Democratically determined policies are promulgated through norms maintained by decision mechanisms that are used across large classes of cases and that affect, not only the parties directly involved, but also society at large.

---

[4] Quoted from James, Hazard & Leubsdorf [2001:361].

[5] See Cohen & Nagel [1964], esp. pp. 371–75.

REFERENCES

Ackerman, B. A. 1984. *Reconstructing American Law.* Cambridge, MA: Harvard University Press.

Allan, T. R. S. Rule of law. In *The New Palgrave Dictionary of Economics and the Law,* vol. 3, ed. P. Newman. London: Macmillan.

Allen, D. W. 2012. *The Institutional Revolution: Measurement and the Economic Emergence of the Modern World.* Chicago, IL: University of Chicago Press.

Anscombe, F. J. and R. J. Auman. 1963. A definition of subjective probability. *The Annals of Mathematics and Statistics* 34(1), 199–205.

Apostol, T. M. 1969. *Calculus: Multi-Variable Calculus and Linear Algebra, with Applications to Differential Equations and Probability.* Vol. 2, 2nd Ed. New York, NY: Wiley.

Arnold, B. C., N. Balakrishnan and H. N. Nagaraja. 2008. *A First Course in Order Statistics.* Philadelphia, PA: Society for Industrial and Applied Mathematics.

Arrow, K. J. 1968. Economic equilibrium. *International Encyclopedia of the Social Sciences.* Vol. 4. New York, NY: Crowell Collier & Macmillan.

———. 1970. Political and economic evaluation of social effects and externalities. In *The Analysis of Public Output,* ed. J. Margolis. New York, NY: Columbia University Press.

———. 2011. The classification of social choice propositions. In *Handbook of Social Choice and Welfare,* vol. 2, eds. K. J. Arrow, A. Sen and K. Suzumura. Amsterdam: North-Holland.

———. 2012. *Social Choice and Individual Values.* 3rd Ed. New Haven, CT: Yale University Press.

Aubert, K. E. 1984. Spurious mathematical modelling. *The Mathematical Intelligencer* 6(3), 54–60.

Aumann, R. J. and S. Hart, eds. 1992. *Handbook of Game Theory.* Vol. 1. Amsterdam: North-Holland.

Ayres, I. 2005. *Optional Law: The Structure of Legal Entitlements.* Chicago, IL: University of Chicago Press.

Ayres, I. and R. Gertner. 1989. Filling gaps in incomplete contracts: an economic theory of default rules. *Yale Law Journal* 99(1), 87–130.

Bartle, R. G. 1976. *The Elements of Real Analysis.* 2nd Ed. New York, NY: Wiley.

Bartle, R. G. and D. R. Sherbert. 1982. *Introduction to Real Analysis.* New York: Wiley.

Bartoszyński, R. and M. Niewiadomska-Bugaj. 1996. *Probability and Statistical Inference.* New York, NY: Wiley.

Berlinski, D. 2000. *The Advent of the Algorithm: The 300-Year Journey from an Idea to the Computer.* San Diego, CA: Harcourt.

Bickel, P. J. and K. A. Doksum. 2001. *Mathematical Statistics.* Vol. 1, 2nd Ed. Upper Saddle River, NJ: Prentice Hall.

Boland, P. J. 1989. Majority systems and the Condorcet jury theorem. *The Statistician* 38(3), 181–89.

Bolton, P. and M. Dewatripont. 2005. *Contract Theory.* Cambridge, MA: MIT Press.

Borch, K. H. 1968. *The Economics of Uncertainty.* Princeton, NJ: Princeton University Press.

Borchgrevink, H. 2011. Three essays on civil litigation. Ph.D. dissertation, Dept. of Economics, University of Oslo.

Bovens, L. and W. Rabonowicz. 2006. Democratic answers to complex questions – an epistemic perspective. *Synthese* 150(1), 131–53.

Brainard, W. 1967. Uncertainty and the effectiveness of policy. *American Economic Review* 57(2), 1421–31.

Brennan, G. 2001. Collective coherence? *International Review of Law and Economics* 21(2), 197–211.

Brock, W. A. and A. G. Malliaris. 1989. *Differential Equations, Stability and Chaos in Dynamic Economics.* Amsterdam: North-Holland.

Brooks, R. R. W and W. F. Schwartz. 2005. Legal uncertainty, economic efficiency, and the premliminary injuction doctrine. *Stanford Law Review* 58(2), 381–409.

Buiter, W. H. 1984. Saddlepoint problems in continuous time rational expectations models: a general method and some macroeconomic examples. *Econometrica* 52(3), 665–80.

Cabrillo, F. and S. Fitzpatrick. 2008. *The Economics of Courts and Litigation.* Cheltenham, UK: Edward Elgar.

Calabresi, G. and D. Melamed. 1972. Property rules, liability rules, and inalienability: one view of the cathedral. *Harvard Law Review* 85(6), 1089–128.

Cameron, P. J. 1998. *Sets, Logic and Categories.* London, UK: Springer.

Camiker, E. H. 1999. Sincere and strategic voting norms on multimember courts. *Michigan Law Review* 97(8), 2297–380.

Carter, M. 2001. *Foundations of Mathematical Economics.* Cambridge, MA: MIT Press.

Christensen-Szalanski, J. J. J. and C. F. Willham. 1991. The hindsight bias: a meta-analysis. *Organizational Behavior and Human Decision Processes* 48(1), 147–68.

Cohen, M. R. and E. Nagel. 1964. *An Introduction to Logic and Scientific Method.* Compl. Ed. London: Routledge & Kegan Paul.

Coleman, J. L. 1987. Competition and cooperation. *Ethics* 98(1), 76–90.

———. 2003. The grounds of welfare. *Yale Law Journal* 112(6), 1510–543.

Coleman, J. L. and B. Leiter. 1993. Determinacy, objectivity, and authority. *University of Pennsylvania Law Review* 142(2), 549–637.

Coons, J. E. 1964. Approaches to court imposed compromise – the uses of doubt and reason. *Northwestern University Law Review* 58(6), 750–94.

Cooter. R. 1985. Unity of tort, contract and property: the model of precaution. *California Law Review* 73(1), 1–51.

———. 1998. Liability rights as contingent claims. In *The New Palgrave Dictionary of Economics and the Law,* vol. 2, ed. P. Newman. London: Macmillan.

Cranston, R. 1998. Monetary policy. In *Blackstone's Guide to the Bank of England Act 1998,* eds. M. Blair *et al.* London: Blackstone Press.

Craswell, R. and J. E. Calfee. 1986. Deterrence and uncertain legal standards. *Journal of Law, Economics, and Organization* 2(2), 279–303.

Damaška, M. R. 1986. *The Faces of Justice and State Authority: A Comparative Approach to the Legal Process.* New Haven, CT: Yale University Press.

Devlin, K. J. 1994. *Mathematics: The Science of Patterns.* New York, NY: Scientific American Library.

Diamond, P. A. 1974. Single activity accidents. *Journal of Legal Studies* 3(1), 107–64.

Diggins, J. P. 1987. Science and the American experiment: how Newton's laws shaped the Constitution. *The Sciences* 27(6), 28–31.

Dudnik, R. 1965. Special verdicts: Rule 49 of the Federal rules of civil procedure. *Yale Law Journal* 74(3), 483–523.

Easterbrook, F. H. and D. R. Fischel. 1991. *The Economic Structure of Corporate Law.* Cambridge, MA: Harvard University Press.

Ehrlich, I. and R. A. Posner. 1974. An economic analysis of legal rulemaking. *Journal of Legal Studies* 3(1), 257–86.

Ekeland, I. 1988. *Mathematics and the Unexpected.* Chicago, IL: University of Chicago Press.

Elster, J. 1989. *Solmonic Judgements: Studies in the Limitations of Rationality.* Cambridge, UK: Cambridge University Press.

———. 2013. *Securities Against Misrule: Juries, Assemblies, Elections.* Cambridge, UK: Cambridge University Press.

Eng, S. 2000. Does the doctrine of precedent apply to judicial arguments concerning the doctrine of sources of law? In *The Reasonable as Rational? On Legal Argumentation and Justification,* eds. W. Krawietz, R. Summers, O. Weinberger and G. H. von Wright. Berlin: Dunker & Humblot.

Farber, H. S. 1980. An analysis of final-offer arbitration. *Journal of Conflict Resolution* 24(4), 683–705.

Farmelo, G. 2009. *The Strangest Man: The Hidden Life of Paul Dirac, Mystic of the Atom.* New York, NY: Basic Books.

Feller, W. 1968. *An Introduction to Probability Theory and Its Applications*. Vol. 1, 3rd Ed. New York, NY: Wiley.

Ferejohn, J. and P. Pasquino. 2004. Constitutional adjudication: lessons from Europe. *Texas Law Review* 82(7), 1671–704.

Ferris, T. 2010. *The Science of Liberty: Democracy, Reason, and the Laws of Nature.* New York, NY: HarperCollins Publishers.

Fey, M. 2003. A note on the Condorcet jury theorem with supermajority voting rules. *Social Choice and Welfare* 20(1), 27–32.

Feynman, R. 1994. *The Character of Physical Law.* Modern Library Edition. New York, NY: Random House.

Fienberg, S. E., ed. 1989. *The Evolving Role of Statistical Assessments as Evidence in the Courts.* Report from the Panel on Statistical Assessments as Evidence in the Courts. New York, NY: Springer.

Fischer, S. 1990. Rules versus discretion in monetary policy. In *Handbook of Monetary Economics,* vol. 2, eds. B. M. Friedman and F. H. Hahn. Amsterdam: North-Holland.

Fischoff, B. 1975. Hindsigt ≠ foresight: the effect of outcome knowledge on judgment under uncertainty. *Journal of Experimental Psychology: Human Perception and Performance* 1(3), 288–99.

Flood, R. P. and P. Isard. 1989. Monetary policy strategies. *IMF Staff Papers* 36, 612–32.

Fon, V. and F. Parisi. 2007. On the optimal specificity of legal rules. *Journal of Institutional Economics* 3(2), 147–64.

French, A. 1964. *The Growth of the Athenian Economy.* London: Routledge & Keagan Paul.

Fuller, L. L. 1978. The forms and limits of adjudication. *Harvard Law Review* 92(2), 353–409.

Gibbons, R. 1992. *Game Theory for Applied Economists.* Princeton, NJ: Princeton University Press.

Gilboa, I. 2009. *Theory of Decision under Uncertainty.* New York, NY: Cambridge University Press.

Gilmore, G. 1961. Circular priority systems. *Yale Law Journal* 71(1), 53–74.

Hacking, I. 1975. *The Emergence of Probability.* Cambridge, UK: Cambridge University Press.

Hahn, F. H. 1974. On the notion of equilibrium in economics. In *Equilibrium and Macroeconomics.* Oxford: Basil Blackwell 1984.

Hansmann, H. 1996. *The Ownership of Enterprise.* Cambridge, MA: Belknap, Harvard University Press.

Hart, H. L. A. 1961. *The Concept of Law.* Oxford, UK: Clarendon Press.

Heckscher, A. 1892. *Bidrag til Grundleggelse af en Afstemningslære*. [Contributions to a theory of voting.] Copenhagen: Universitetsboghandler G. E. C. Gad.

Hermalin, B. A. W. Katz and R. Craswell. 2007. Contract law. In *Handbook of Law and Economics,* vol. 1, eds. A. M. Polinsky and S. Shavell. Amsterdam: North-Holland.

Hildenbrand, W. and A. P. Kirman. 1988. *Equilibrium Analysis.* Amsterdam: North-Holland.

Isaac, R. 1995. *The Pleasures of Probability.* New York, NY: Springer.

Jackson, V. C. 2010. *Constitutional Engagement in a Transnational Era.* Oxford, NY: Oxford University Press.

James, F. G. C. Hazard and J. Leubsdorf. 2001. *Civil Procedure.* 5th Ed. New York, NY: Foundation Press.

Johansen, L. 1978. *Lectures on Macroeconomic Planning. Part Two: Centralization, Decenralization and Planning under Uncertainty.* Amsterdam: North-Holland.

Kaplow, L. 1992. Rules versus standards: an economic analysis. *Duke Law Journal* 42(3), 557–629.

Kaplow, L. and S. Shavell. 2002. *Fairness versus Welfare.* Cambridge, MA: Harvard University Press.

Kelsen, H. 1955. Foundations of democracy. *Ethics* 66(1), 1–101.

Kornhauser, L. A. 1992a. Modeling collegial courts I: path-dependence. *International Review of Law and Economics* 12(2), 169–85.

———. 1992b. Modeling collegial courts II: legal doctrine. *Journal of Law, Economics, & Organization* 8(2), 441–70.

———. 1998. Stare decisis. In *The New Palgrave Dictionary of Econom*ics and the Law, vol. 3, ed. P. Newman. London: Macmillan.

Kornhauser, L. A. and Sager, L. G. 1986. Unpacking the court. *Yale Law Journal* 96(1), 82–117.

Kraakman, R. R., P. Davies, H. Hansmann, G. Hertig, K. Hopt, H. Kanada and E. Rock. 2004. *The Anatomy of Corporate Law: A Comparative and Functional Approach.* Oxford, UK: Oxford University Press.

Kreps, D. M. 1988. *Notes on the Theory of Choice.* Boulder, CO: Westview Press.

LaBine, S. and G. Labine. 1996. Determinations of negligence and the hindsight bias. *Law and Human Behavior* 20(5), 501–16.

LaFave, W. R., J. H. Israel and N. J. King. 2004. *Criminal Procedure*. 4th Ed. St. Paul, MN: West Publishing.

Laffont, J.-J. 1989. *The Economics of Uncertainty and Information.* Cambridge, MA: MIT Press.

Laffont, J.-J. and D. Martimort. 2002. *The Theory of Incentives: The Principal-Agent Model*. Princeton, NJ: Princeton University Press.

Lagerspetz, E. 2014. Albert Heckscher on collective decision-making. *Public Choice* 159(3), 327–39.

Landa, D. and J. R. Lax. 2009. Legal doctrine on collegial courts. *Journal of Politics* 71(3), 946–63.

Landes, W. M. 1998. Sequential and bifurcated trials. In *The New Palgrave Dictionary of Economics and the Law*, vol. 3, ed. P. Newman. London: Macmillan.

Langbein, J. H., R. L. Lerner and B. P. Smith. 2009. *History of the Common Law: The Development of Anglo-American Legal Institutions.* Austin, TX: Wolters Kluwer.

La Porta, R., F. López-de-Silanes, C. Pop-Eleches and A. Schleifer. 2004. Judicial checks and balances. *Journal of Political Economy* 112(2), 445–70.

Larsen, R. J. and M. L. Marx. 1986. *An Introduction to Mathematical Statistics and Its Applications*. Englewood Cliffs, NJ: Prentice-Hall.

Lax, J. R. 2007. Constructing legal rules on appellate courts. *American Political Science Review* 101(3), 591–604.

Ledebur, L. E. 2009. Plurality rule: concurring opinions and a divided Supreme Court. *Penn State Law Review* 113(3): 899–921.

List, C. 2005. The probability of inconsistencies in complex collective decisions. *Social Choice and Welfare* 24(1), 3–32.

List, C. and P. Pettit. 2002. Aggregating sets of judgments: an impossibility result. *Economics and Philosophy* 18(1), 89–110.

———. 2011. *Group Agency: The Possibility, Design and Status of Corporate Agents.* Oxford, UK: Oxford University Press.

Llewellyn, K. 1989. *The Case Law System in America*. Chigaco, IL: University of Chicago Press.

Macey, J. R. 1998. Precedent. In *The New Palgrave Dictionary of Economics and the Law*, vol. 3, ed. P. Newman. London: Macmillan.

Mackaay, E. 1998. Emergence of legal rules. In *The New Palgrave Dictionary of Economics and the Law*, vol. 2, ed. P. Newman. London: Macmillan.

Mackenzie, R., C. P. R. Romano, Y. Shany and P. Sands. 2010. *Manual on International Courts and Tribunals.* 2nd Ed. Oxford, UK: Oxford University Press.

Malinvaud, E. 1985. *Lectures on Microeconomic Theory.* Rev. Ed. Amsterdam: North-Holland.

Mas-Colell, A. M. D. Whinston and G. Green. 1995. *Microeconomic Theory.* New York, NY: Oxford University Press.

Maskin, E. S. 2012. Foreword to *Social Choice and Individual Values,* 3rd Ed, by K. J. Arrow. New Haven, CT: Yale University Press.

Mattei, U. and F. Cafaggi. 1998. Comparative law and economics. In *The New Palgrave Dictionary of Economics and the Law*, vol. 1, ed. P. Newman, London: Macmillan.

Merryman, J. H. and R. Péres-Perdomo. 2007. *The Civil Law Tradition: An Introduction to the Legal Systems of Europe and Latin America.* 3rd Ed. Stanford, CA: Stanford University Press.

Monateri, P. G. and R. Sacco. 1998. Legal formants. In *The New Palgrave Dictionary of Economics and the Law,* vol. 2, ed. P. Newman. London: Macmillan.

Mongin, P. 2012. The doctrinal paradox, the discursive dilemma, and logical aggregation theory. *Theory and Decision* 73(3), 315–55.

Moulin, H. 1988. *Axioms of Cooperative Decision Making.* Cambridge, UK: Cambridge University Press.

Murata, Y. 1977. *Mathematics for Stability and Optimization of Economic Stystems.* New York, NY: Academic Press.

Nitzan, S. 2010. *Collective Preference and Choice.* Cambridge, UK: Cambridge University Press.

Nordén, G. 2002. Norges Banks rettslige stilling: en systemorientert analyse med implikasjoner på nivået for juridisk metode. *Tidsskrift for Rettsvitenskap* 115(2-3), 710–75.

———. 2010. Legal uncertainty and collective decision mechanisms: an analysis of the Norwegian supreme court. In *Law and Economics*, eds. E. Røsæg, H-B. Schäfer and E. Stavang. Oslo: Cappelen Akademisk Forlag.

———. 2012. Kollektivbeslutninger under vage standarder. *Scandinavian Insurance Quarterly* 91(1). 1–14.

———. 2015. Reasoned judgments and the rule of law: juries v. mixed-courts under norm-based legal uncertainty. (Rev. draft.)

North, D. C. 1994. Economic performance through time. *American Economic Review* 84(3), 359–68.

Parisi, F. 1992. *Liability for Negligence and Judicial Discretion.* 2nd Ed. Berkeley, CA: University of California at Berkeley.

Persson, T. and G. Tabellini. 1999. Political economics and macroeconomic policy. In *Handbook of Macroeconomics*, vol. 1c, eds. J. B. Taylor and M. Woodford. Amsterdam: North-Holland.

Polinsky, A. M. 2003. *Introduction to Law and Economics.* 3rd Ed. New York, NY: Aspen Publishers.

Popper, K. R. 1964. *The Poverty of Historicism.* 3rd Ed. New York, NY: Harper & Row.

Posner, R. A. 1990. *The Problems of Jurisprudence.* Cambridge, MA: Harvard University Press.

Posner, E. A. and J. C. Yoo. 2005. Judicial independence in international tribunals. *California Law Review* 93(1), 1–74.

Potts, H. E. 1913. An application of mathematics to law. *Nature* 91(2272), 270.

Prichard, J. R. S. 1988. A systematic approach to comparative law: the effect of cost, fee, and financing rules on the development of the substantive law. *Journal of Legal Studies* 17(2), 451–75.

Reese, R. L. 2000. *University Physics.* Pasific Grove, CA: Brooks/Cole Publishing Company.

Rodriguez, D. B. Legislative intent. In *The New Palgrave Dictionary of Economics and the Law*, vol. 2, ed. P. Newman. London: Macmillan.

Ross, A. 1947. *A Textbook of International Law*. London, UK: Longmans, Green & Co.

———. 1959. *On Law and Justice*. Berkeley, CA: University of California Press.

———. 1962. Review of *The Concept of Law* by H. L. A. Hart. *Yale Law Journal* 71(6), 1185–190.

Saari, D. G. 2001. *Decisions and Elections: Explaining the Unexpected.* Cambridge, UK: Cambridge University Press.

Sacco, R. 1991a. Legal formants: a dynamic approach to comparative law. *American Journal of Comparative Law* 39(1), 1–34.

———. 1991b. Legal formants: a dynamic approach to comparative law. *American Journal of Comparative Law* 39(2), 343–401.

Salzberger, E. M. 1993. A positive analysis of the doctrine of separation of powers, or: why do we have an independent judiciary? *International Review of Law and Economics* 13(4), 349–79.

Sen, A. K. 1967. The nature and classes of prescriptive judgments. *Philosophical Quarterly* 17(66), 46–62.

———. 1979. *Collective Choice and Social Welfare.* Amsterdam: North-Holland.

———. 2009. *The Idea of Justice.* Cambridge, MA: Belknap.

Schauer, F. and R. Zeckhauser. 1996. On the degree of confidence for adverse decisions. *Journal of Legal Studies* 25(1), 27–52.

Shavell, S. 2006. On the writing and interpretation of contracts. *Journal of Law, Economics, & Organization* 22(2), 289–314.

———. 1987. *Economic Analysis of Accident Law*. Cambridge, MA: Harvard University Press.

Shetreet, S. 1985. Judicial independence: new conceptual dimensions and contemporary challenges. In *Judicial Independence: The Contemporary Debate*, eds. S. Shetreet and J. Deschênes. Dordrecht: Martinus Nijhoff Publishers.

Stein, P. 1992. Foreword to *Liability for Negligence and Judicial Discretion*, 2nd Ed., by F. Parisi. Berkeley, CA: University of California at Berkeley.

Stremitzer, A. 2005. *Agency Theory: Methodology, Analysis.* Frankfurt am Main: Peter Lang.

Suppes, P. 1972. *Axiomatic Set Theory.* New York, NY: Dover.

Sydsæter, K. 1981. *Topics in Mathematical Analysis for Economists.* London, UK: Academic Press.

Taroni, F., C. Aitken, P. Garbolino and A. Biedermann. 2006. *Bayesian Networks and Probabilistic Inference in Forensic Science.* Chichester, UK: Wiley.

Thomas, G. C. and B. S. Pollac. 1992. Rethinking guilt, juries, and jeopardy. *Michigan Law Review* 91(1), 1–33.

Thomson, W. and H. R. Varian. 1985. Theories of justice based on symmetry. In *Social Goals and Social Organization*, eds. L. Hurwicz, D. Schmeidler, and H. Sonnenschein. Cambridge, UK: Cambridge University Press.

Tur, R. H. S. 1998. Rule of recognition. In *The New Palgrave Dictionary of Economics and the Law*, vol. 3, ed. P. Newman. London: Macmillan.

Tvarnø, C. D. and R. Nielsen. 2014. *Retskilder og retsteorier*. Copenhagen: Jurist- og Økonomforbundets forlag.

Urfalino, P. 2010. Deciding as bringing deliberation to a close. *Social Science Information* 49(1), 111–40.

Vermeule, A. 2005. Submajority rules: forcing accountability upon majorities. *Journal of Political Philosophy* 13(1), 74–98.

von Mehren, A. T. and J. R. Gordley. 1977. *The Civil Law System: An Introduction to the Comparative Study of Law*. 2nd Ed. Boston, MA: Little, Brown & Company.

Wakker, P. P. 2010. *Prospect Theory.* Cambridge, UK: Cambridge University Press.

Waldron, J. 1994. Vagueness in law and language: some philosophical issues. *California Law Review* 82(3), 509–40.

Wallis, W. A. and H. V. Roberts. 1956. *Statistics: A New Approach.* New York, NY: The Free Press 1956.

Watson, J. 2002. *Strategy: An Introduction to Game Theory.* New York, NY: Norton.

Wedberg, A. 1951. Some problems in the logical analysis of legal science. *Theoria* 17(1-3), 246–75.

Whinston, M. D. 2006. *Lectures on Antitrust Economics.* Cambridge, MA: MIT Press.

Whitman, J. Q. 2008. No right answer? In *Crime, Procedure and Evidence in a Comparative and International Context*, eds. J. Jackson, M. Langer and P. Tillers. Portland, OR: Hart Publishing.

Wissler, K., A. Rector and M. J. Saks. 2001. The impact of jury instructions on the fusion of liability and compensatory damages. *Law and Human Behavior* 25(2), 125–39.

Yeazell, S. C. 2001. Re-financing civil litigation. *DePaul Law Review* 51(2), 183–217.

APPENDICES

A.1 Simulation results

Table A.1 reports simulation results that are discussed in Section IV.5: Equilibrium solutions $\chi(2m+1;\underline{x})$ are obtained for $m \in \{0,1,\ldots,9\}$ and $\underline{x} \in (0,e]$, with $L = e^e$, $S \sim U(e-\underline{x}, e+\underline{x})$, and $p(x) = \exp[-x]$ (compare Figure IV.5.4). The w-variables correspond to inputs $\underline{x}$, (w010 equals $\underline{x} = 0.10$, w010 equals $\underline{x} = 0.11$, etc.; wexp equals $\underline{x} = e$).

The solutions, and values reported in Tables III.4.1, III.7.1–3, and IV.5.1, were calculated using Portable TROLL (Intex Solutions, Inc.).

Table A.1 Equilibrium precaution investments

	m=0	m=1	m=2	m=3	m=4	m=5	m=6	m=7	m=8	m=9
w010	2, 8183	2, 8107	2, 7996	2, 7917	2, 7860	2, 7815	2, 7779	2, 7749	2, 7724	2, 7702
w011	2, 8283	2, 8190	2, 8065	2, 7979	2, 7916	2, 7867	2, 7828	2, 7796	2, 7769	2, 7745
w012	2, 8383	2, 8271	2, 8133	2, 8039	2, 7971	2, 7919	2, 7877	2, 7842	2, 7813	2, 7787
w013	2, 8483	2, 8350	2, 8198	2, 8097	2, 8025	2, 7969	2, 7924	2, 7887	2, 7856	2, 7829
w014	2, 8583	2, 8427	2, 8262	2, 8154	2, 8077	2, 8018	2, 7970	2, 7931	2, 7898	2, 7869
w015	2, 8683	2, 8502	2, 8324	2, 8210	2, 8128	2, 8065	2, 8015	2, 7974	2, 7939	2, 7909
w016	2, 8783	2, 8575	2, 8385	2, 8264	2, 8178	2, 8112	2, 8059	2, 8016	2, 7979	2, 7948
w017	2, 8883	2, 8646	2, 8444	2, 8316	2, 8226	2, 8157	2, 8102	2, 8057	2, 8019	2, 7986
w018	2, 8983	2, 8714	2, 8501	2, 8368	2, 8273	2, 8202	2, 8144	2, 8097	2, 8057	2, 8023
w019	2, 9083	2, 8781	2, 8556	2, 8417	2, 8320	2, 8245	2, 8186	2, 8137	2, 8095	2, 8059
w020	2, 9183	2, 8845	2, 8610	2, 8466	2, 8365	2, 8288	2, 8226	2, 8175	2, 8132	2, 8095
w021	2, 9283	2, 8906	2, 8662	2, 8513	2, 8409	2, 8329	2, 8265	2, 8213	2, 8169	2, 8130
w022	2, 9383	2, 8966	2, 8712	2, 8559	2, 8451	2, 8369	2, 8304	2, 8250	2, 8204	2, 8165
w023	2, 9483	2, 9023	2, 8761	2, 8604	2, 8493	2, 8409	2, 8342	2, 8286	2, 8239	2, 8199
w024	2, 9583	2, 9077	2, 8808	2, 8647	2, 8534	2, 8448	2, 8379	2, 8322	2, 8273	2, 8232
w025	2, 9683	2, 9130	2, 8854	2, 8689	2, 8574	2, 8485	2, 8415	2, 8356	2, 8307	2, 8264
w026	2, 9783	2, 9179	2, 8898	2, 8730	2, 8612	2, 8522	2, 8450	2, 8391	2, 8340	2, 8296
w027	2, 9883	2, 9227	2, 8941	2, 8770	2, 8650	2, 8558	2, 8485	2, 8424	2, 8372	2, 8328
w028	2, 9983	2, 9272	2, 8982	2, 8809	2, 8687	2, 8593	2, 8519	2, 8457	2, 8404	2, 8358
w029	3, 0083	2, 9314	2, 9021	2, 8846	2, 8722	2, 8628	2, 8552	2, 8489	2, 8435	2, 8389
w030	3, 0183	2, 9354	2, 9059	2, 8882	2, 8757	2, 8661	2, 8584	2, 8520	2, 8465	2, 8418
w031	3, 0283	2, 9392	2, 9096	2, 8918	2, 8791	2, 8694	2, 8616	2, 8551	2, 8495	2, 8447
w032	3, 0383	2, 9427	2, 9131	2, 8952	2, 8824	2, 8726	2, 8647	2, 8581	2, 8525	2, 8476
w033	3, 0483	2, 9459	2, 9164	2, 8985	2, 8856	2, 8757	2, 8677	2, 8610	2, 8553	2, 8504
w034	3, 0583	2, 9490	2, 9197	2, 9016	2, 8887	2, 8787	2, 8706	2, 8639	2, 8581	2, 8531
w035	3, 0683	2, 9518	2, 9227	2, 9047	2, 8917	2, 8817	2, 8735	2, 8667	2, 8609	2, 8558
w036	3, 0625	2, 9543	2, 9256	2, 9077	2, 8947	2, 8846	2, 8763	2, 8695	2, 8636	2, 8585
w037	3, 0535	2, 9567	2, 9284	2, 9105	2, 8975	2, 8874	2, 8791	2, 8722	2, 8662	2, 8611
w038	3, 0448	2, 9588	2, 9311	2, 9133	2, 9003	2, 8901	2, 8818	2, 8748	2, 8688	2, 8636
w039	3, 0363	2, 9607	2, 9336	2, 9159	2, 9029	2, 8928	2, 8844	2, 8774	2, 8714	2, 8661
w040	3, 0279	2, 9623	2, 9359	2, 9185	2, 9055	2, 8953	2, 8870	2, 8799	2, 8739	2, 8686

```
w041 3, 0198   2, 9637   2, 9381   2, 9209   2, 9081   2, 8979   2, 8895   2, 8824   2, 8763   2, 8710
w042 3, 0118   2, 9650   2, 9402   2, 9233   2, 9105   2, 9003   2, 8919   2, 8848   2, 8787   2, 8733
w043 3, 0039   2, 9660   2, 9422   2, 9255   2, 9128   2, 9027   2, 8943   2, 8872   2, 8810   2, 8756
w044 2, 9963   2, 9668   2, 9440   2, 9277   2, 9151   2, 9050   2, 8966   2, 8895   2, 8833   2, 8779
w045 2, 9888   2, 9674   2, 9457   2, 9297   2, 9173   2, 9072   2, 8989   2, 8917   2, 8856   2, 8801
w046 2, 9814   2, 9679   2, 9473   2, 9317   2, 9194   2, 9094   2, 9011   2, 8939   2, 8878   2, 8823
w047 2, 9742   2, 9681   2, 9488   2, 9335   2, 9214   2, 9115   2, 9032   2, 8961   2, 8899   2, 8844
w048 2, 9671   2, 9682   2, 9501   2, 9353   2, 9234   2, 9136   2, 9053   2, 8982   2, 8920   2, 8865
w049 2, 9601   2, 9681   2, 9513   2, 9370   2, 9252   2, 9155   2, 9073   2, 9002   2, 8940   2, 8886
w050 2, 9533   2, 9678   2, 9524   2, 9385   2, 9270   2, 9174   2, 9093   2, 9022   2, 8960   2, 8906
w051 2, 9466   2, 9673   2, 9534   2, 9400   2, 9288   2, 9193   2, 9112   2, 9042   2, 8980   2, 8925
w052 2, 9401   2, 9667   2, 9543   2, 9414   2, 9304   2, 9211   2, 9130   2, 9061   2, 8999   2, 8944
w053 2, 9336   2, 9660   2, 9550   2, 9428   2, 9320   2, 9228   2, 9148   2, 9079   2, 9018   2, 8963
w054 2, 9273   2, 9651   2, 9557   2, 9440   2, 9335   2, 9245   2, 9166   2, 9097   2, 9036   2, 8982
w055 2, 9210   2, 9640   2, 9562   2, 9451   2, 9350   2, 9261   2, 9183   2, 9114   2, 9054   2, 9000
w056 2, 9149   2, 9629   2, 9566   2, 9462   2, 9363   2, 9276   2, 9199   2, 9131   2, 9071   2, 9017
w057 2, 9089   2, 9616   2, 9569   2, 9472   2, 9376   2, 9291   2, 9215   2, 9148   2, 9088   2, 9034
w058 2, 9030   2, 9601   2, 9572   2, 9481   2, 9389   2, 9305   2, 9231   2, 9164   2, 9105   2, 9051
w059 2, 8972   2, 9586   2, 9573   2, 9489   2, 9400   2, 9319   2, 9245   2, 9180   2, 9121   2, 9067
w060 2, 8914   2, 9570   2, 9573   2, 9496   2, 9412   2, 9332   2, 9260   2, 9195   2, 9136   2, 9083
w061 2, 8858   2, 9552   2, 9572   2, 9503   2, 9422   2, 9344   2, 9274   2, 9210   2, 9152   2, 9099
w062 2, 8803   2, 9534   2, 9571   2, 9509   2, 9432   2, 9356   2, 9287   2, 9224   2, 9166   2, 9114
w063 2, 8748   2, 9514   2, 9568   2, 9514   2, 9441   2, 9368   2, 9300   2, 9238   2, 9181   2, 9129
w064 2, 8694   2, 9494   2, 9565   2, 9518   2, 9449   2, 9379   2, 9312   2, 9251   2, 9195   2, 9144
w065 2, 8641   2, 9473   2, 9561   2, 9522   2, 9457   2, 9389   2, 9324   2, 9264   2, 9209   2, 9158
w066 2, 8589   2, 9451   2, 9556   2, 9525   2, 9464   2, 9399   2, 9335   2, 9276   2, 9222   2, 9171
w067 2, 8538   2, 9428   2, 9550   2, 9527   2, 9471   2, 9408   2, 9346   2, 9288   2, 9235   2, 9185
w068 2, 8487   2, 9405   2, 9543   2, 9529   2, 9477   2, 9417   2, 9357   2, 9300   2, 9247   2, 9198
w069 2, 8438   2, 9381   2, 9536   2, 9529   2, 9482   2, 9425   2, 9367   2, 9311   2, 9259   2, 9211
w070 2, 8389   2, 9356   2, 9528   2, 9530   2, 9487   2, 9433   2, 9376   2, 9322   2, 9271   2, 9223
w071 2, 8340   2, 9331   2, 9519   2, 9529   2, 9491   2, 9440   2, 9385   2, 9333   2, 9282   2, 9235
w072 2, 8293   2, 9305   2, 9509   2, 9528   2, 9495   2, 9446   2, 9394   2, 9343   2, 9293   2, 9247
w073 2, 8246   2, 9279   2, 9499   2, 9526   2, 9498   2, 9453   2, 9402   2, 9352   2, 9304   2, 9258
w074 2, 8199   2, 9252   2, 9488   2, 9524   2, 9501   2, 9458   2, 9410   2, 9361   2, 9314   2, 9269
w075 2, 8154   2, 9225   2, 9476   2, 9521   2, 9503   2, 9464   2, 9417   2, 9370   2, 9324   2, 9280
w076 2, 8109   2, 9198   2, 9464   2, 9517   2, 9504   2, 9468   2, 9424   2, 9379   2, 9333   2, 9290
w077 2, 8064   2, 9170   2, 9452   2, 9513   2, 9505   2, 9473   2, 9431   2, 9387   2, 9342   2, 9300
w078 2, 8020   2, 9142   2, 9438   2, 9509   2, 9506   2, 9476   2, 9437   2, 9394   2, 9351   2, 9309
w079 2, 7977   2, 9114   2, 9424   2, 9503   2, 9506   2, 9480   2, 9443   2, 9401   2, 9360   2, 9319
w080 2, 7934   2, 9085   2, 9410   2, 9498   2, 9505   2, 9483   2, 9448   2, 9408   2, 9368   2, 9328
w081 2, 7892   2, 9056   2, 9395   2, 9491   2, 9504   2, 9485   2, 9453   2, 9415   2, 9376   2, 9336
w082 2, 7850   2, 9027   2, 9380   2, 9485   2, 9503   2, 9487   2, 9457   2, 9421   2, 9383   2, 9345
w083 2, 7809   2, 8998   2, 9364   2, 9477   2, 9501   2, 9489   2, 9461   2, 9427   2, 9390   2, 9353
w084 2, 7769   2, 8968   2, 9347   2, 9469   2, 9499   2, 9490   2, 9465   2, 9432   2, 9397   2, 9361
w085 2, 7729   2, 8939   2, 9331   2, 9461   2, 9496   2, 9491   2, 9468   2, 9437   2, 9403   2, 9368
w086 2, 7689   2, 8909   2, 9313   2, 9452   2, 9493   2, 9491   2, 9471   2, 9442   2, 9409   2, 9375
w087 2, 7650   2, 8879   2, 9296   2, 9443   2, 9489   2, 9491   2, 9474   2, 9446   2, 9415   2, 9382
w088 2, 7612   2, 8849   2, 9278   2, 9434   2, 9485   2, 9491   2, 9476   2, 9451   2, 9421   2, 9389
w089 2, 7574   2, 8820   2, 9260   2, 9424   2, 9480   2, 9490   2, 9477   2, 9454   2, 9426   2, 9395
w090 2, 7536   2, 8790   2, 9241   2, 9413   2, 9475   2, 9489   2, 9479   2, 9458   2, 9431   2, 9401
w091 2, 7499   2, 8760   2, 9222   2, 9402   2, 9470   2, 9487   2, 9480   2, 9461   2, 9435   2, 9407
w092 2, 7462   2, 8730   2, 9203   2, 9391   2, 9464   2, 9485   2, 9481   2, 9463   2, 9439   2, 9412
w093 2, 7426   2, 8700   2, 9183   2, 9379   2, 9458   2, 9483   2, 9481   2, 9466   2, 9443   2, 9417
w094 2, 7390   2, 8669   2, 9163   2, 9367   2, 9451   2, 9480   2, 9481   2, 9468   2, 9447   2, 9422
w095 2, 7354   2, 8639   2, 9143   2, 9355   2, 9444   2, 9477   2, 9481   2, 9470   2, 9450   2, 9426
w096 2, 7319   2, 8610   2, 9123   2, 9342   2, 9437   2, 9473   2, 9480   2, 9471   2, 9453   2, 9431
w097 2, 7285   2, 8580   2, 9102   2, 9329   2, 9429   2, 9470   2, 9479   2, 9472   2, 9456   2, 9435
w098 2, 7250   2, 8550   2, 9081   2, 9316   2, 9421   2, 9466   2, 9478   2, 9473   2, 9458   2, 9438
w099 2, 7216   2, 8520   2, 9060   2, 9302   2, 9413   2, 9461   2, 9476   2, 9473   2, 9461   2, 9442
w100 2, 7183   2, 8490   2, 9039   2, 9288   2, 9404   2, 9456   2, 9474   2, 9474   2, 9462   2, 9445
```

wl01	2, 7150	2, 8461	2, 9018	2, 9274	2, 9396	2, 9451	2, 9472	2, 9474	2, 9464	2, 9448
wl02	2, 7117	2, 8431	2, 8996	2, 9259	2, 9386	2, 9446	2, 9470	2, 9473	2, 9465	2, 9451
wl03	2, 7084	2, 8402	2, 8974	2, 9244	2, 9377	2, 9440	2, 9467	2, 9473	2, 9466	2, 9453
wl04	2, 7052	2, 8372	2, 8952	2, 9229	2, 9367	2, 9434	2, 9464	2, 9472	2, 9467	2, 9455
wl05	2, 7021	2, 8343	2, 8930	2, 9214	2, 9357	2, 9428	2, 9460	2, 9471	2, 9468	2, 9457
wl06	2, 6989	2, 8314	2, 8908	2, 9198	2, 9346	2, 9421	2, 9456	2, 9469	2, 9468	2, 9459
wl07	2, 6958	2, 8285	2, 8886	2, 9183	2, 9335	2, 9414	2, 9452	2, 9467	2, 9468	2, 9460
wl08	2, 6927	2, 8256	2, 8863	2, 9166	2, 9324	2, 9407	2, 9448	2, 9465	2, 9468	2, 9461
wl09	2, 6897	2, 8227	2, 8841	2, 9150	2, 9313	2, 9400	2, 9444	2, 9463	2, 9467	2, 9462
wl10	2, 6867	2, 8199	2, 8818	2, 9134	2, 9301	2, 9392	2, 9439	2, 9460	2, 9467	2, 9463
wl11	2, 6837	2, 8170	2, 8796	2, 9117	2, 9290	2, 9384	2, 9434	2, 9458	2, 9466	2, 9463
wl12	2, 6808	2, 8142	2, 8773	2, 9100	2, 9278	2, 9376	2, 9428	2, 9455	2, 9464	2, 9464
wl13	2, 6778	2, 8114	2, 8750	2, 9083	2, 9265	2, 9367	2, 9423	2, 9451	2, 9463	2, 9464
wl14	2, 6750	2, 8086	2, 8727	2, 9066	2, 9253	2, 9358	2, 9417	2, 9448	2, 9461	2, 9463
wl15	2, 6721	2, 8058	2, 8704	2, 9048	2, 9240	2, 9349	2, 9411	2, 9444	2, 9459	2, 9463
wl16	2, 6693	2, 8030	2, 8681	2, 9031	2, 9227	2, 9340	2, 9405	2, 9440	2, 9457	2, 9462
wl17	2, 6665	2, 8003	2, 8658	2, 9013	2, 9214	2, 9331	2, 9398	2, 9436	2, 9455	2, 9461
wl18	2, 6637	2, 7975	2, 8635	2, 8995	2, 9201	2, 9321	2, 9391	2, 9431	2, 9452	2, 9460
wl19	2, 6609	2, 7948	2, 8612	2, 8977	2, 9187	2, 9311	2, 9384	2, 9427	2, 9449	2, 9459
wl20	2, 6582	2, 7921	2, 8589	2, 8959	2, 9174	2, 9301	2, 9377	2, 9422	2, 9446	2, 9457
wl21	2, 6555	2, 7894	2, 8566	2, 8941	2, 9160	2, 9291	2, 9370	2, 9416	2, 9443	2, 9456
wl22	2, 6528	2, 7867	2, 8543	2, 8923	2, 9146	2, 9280	2, 9362	2, 9411	2, 9439	2, 9454
wl23	2, 6502	2, 7840	2, 8520	2, 8904	2, 9131	2, 9269	2, 9354	2, 9406	2, 9436	2, 9452
wl24	2, 6476	2, 7814	2, 8497	2, 8886	2, 9117	2, 9258	2, 9346	2, 9400	2, 9432	2, 9449
wl25	2, 6450	2, 7788	2, 8474	2, 8867	2, 9102	2, 9247	2, 9338	2, 9394	2, 9428	2, 9447
wl26	2, 6424	2, 7761	2, 8452	2, 8849	2, 9088	2, 9236	2, 9329	2, 9388	2, 9423	2, 9444
wl27	2, 6399	2, 7735	2, 8429	2, 8830	2, 9073	2, 9224	2, 9320	2, 9381	2, 9419	2, 9441
wl28	2, 6373	2, 7710	2, 8406	2, 8811	2, 9058	2, 9213	2, 9311	2, 9375	2, 9414	2, 9438
wl29	2, 6348	2, 7684	2, 8383	2, 8792	2, 9043	2, 9201	2, 9302	2, 9368	2, 9409	2, 9434
wl30	2, 6324	2, 7659	2, 8360	2, 8773	2, 9027	2, 9189	2, 9293	2, 9361	2, 9404	2, 9431
wl31	2, 6299	2, 7633	2, 8337	2, 8754	2, 9012	2, 9177	2, 9284	2, 9354	2, 9399	2, 9427
wl32	2, 6275	2, 7608	2, 8315	2, 8735	2, 8997	2, 9164	2, 9274	2, 9346	2, 9393	2, 9423
wl33	2, 6251	2, 7583	2, 8292	2, 8716	2, 8981	2, 9152	2, 9264	2, 9339	2, 9388	2, 9419
wl34	2, 6227	2, 7558	2, 8269	2, 8696	2, 8965	2, 9139	2, 9254	2, 9331	2, 9382	2, 9415
wl35	2, 6203	2, 7534	2, 8247	2, 8677	2, 8949	2, 9127	2, 9244	2, 9323	2, 9376	2, 9411
wl36	2, 6179	2, 7509	2, 8224	2, 8658	2, 8933	2, 9114	2, 9234	2, 9315	2, 9370	2, 9406
wl37	2, 6156	2, 7485	2, 8202	2, 8639	2, 8917	2, 9101	2, 9224	2, 9307	2, 9363	2, 9401
wl38	2, 6133	2, 7461	2, 8180	2, 8619	2, 8901	2, 9088	2, 9213	2, 9299	2, 9357	2, 9396
wl39	2, 6110	2, 7437	2, 8157	2, 8600	2, 8885	2, 9074	2, 9202	2, 9290	2, 9350	2, 9391
wl40	2, 6088	2, 7413	2, 8135	2, 8581	2, 8869	2, 9061	2, 9192	2, 9281	2, 9343	2, 9386
wl41	2, 6065	2, 7389	2, 8113	2, 8561	2, 8853	2, 9048	2, 9181	2, 9273	2, 9336	2, 9380
wl42	2, 6043	2, 7365	2, 8091	2, 8542	2, 8836	2, 9034	2, 9169	2, 9264	2, 9329	2, 9375
wl43	2, 6021	2, 7342	2, 8069	2, 8522	2, 8820	2, 9020	2, 9158	2, 9254	2, 9322	2, 9369
wl44	2, 5999	2, 7319	2, 8047	2, 8503	2, 8803	2, 9006	2, 9147	2, 9245	2, 9314	2, 9363
wl45	2, 5977	2, 7296	2, 8025	2, 8484	2, 8786	2, 8992	2, 9135	2, 9236	2, 9307	2, 9357
wl46	2, 5955	2, 7273	2, 8003	2, 8464	2, 8770	2, 8978	2, 9124	2, 9226	2, 9299	2, 9351
wl47	2, 5934	2, 7250	2, 7982	2, 8445	2, 8753	2, 8964	2, 9112	2, 9217	2, 9291	2, 9345
wl48	2, 5913	2, 7228	2, 7960	2, 8425	2, 8736	2, 8950	2, 9100	2, 9207	2, 9283	2, 9338
wl49	2, 5892	2, 7205	2, 7938	2, 8406	2, 8720	2, 8936	2, 9088	2, 9197	2, 9275	2, 9332
wl50	2, 5871	2, 7183	2, 7917	2, 8387	2, 8703	2, 8922	2, 9076	2, 9187	2, 9267	2, 9325
wl51	2, 5850	2, 7161	2, 7896	2, 8367	2, 8686	2, 8907	2, 9064	2, 9177	2, 9258	2, 9318
wl52	2, 5830	2, 7139	2, 7875	2, 8348	2, 8669	2, 8893	2, 9052	2, 9166	2, 9250	2, 9311
wl53	2, 5809	2, 7117	2, 7853	2, 8329	2, 8652	2, 8878	2, 9039	2, 9156	2, 9241	2, 9304
wl54	2, 5789	2, 7095	2, 7832	2, 8310	2, 8635	2, 8863	2, 9027	2, 9145	2, 9232	2, 9297
wl55	2, 5769	2, 7074	2, 7811	2, 8290	2, 8618	2, 8849	2, 9014	2, 9135	2, 9223	2, 9289
wl56	2, 5749	2, 7052	2, 7791	2, 8271	2, 8601	2, 8834	2, 9002	2, 9124	2, 9214	2, 9282
wl57	2, 5730	2, 7031	2, 7770	2, 8252	2, 8584	2, 8819	2, 8989	2, 9113	2, 9205	2, 9274
wl58	2, 5710	2, 7010	2, 7749	2, 8233	2, 8567	2, 8804	2, 8976	2, 9102	2, 9196	2, 9266
wl59	2, 5691	2, 6989	2, 7729	2, 8214	2, 8550	2, 8789	2, 8963	2, 9091	2, 9187	2, 9258
wl60	2, 5671	2, 6968	2, 7708	2, 8195	2, 8533	2, 8774	2, 8950	2, 9080	2, 9177	2, 9250

w161 2, 5652	2, 6947	2, 7688	2, 8176	2, 8516	2, 8759	2, 8937	2, 9069	2, 9168	2, 9242
w162 2, 5633	2, 6927	2, 7667	2, 8157	2, 8499	2, 8744	2, 8924	2, 9057	2, 9158	2, 9234
w163 2, 5615	2, 6906	2, 7647	2, 8138	2, 8482	2, 8729	2, 8911	2, 9046	2, 9148	2, 9226
w164 2, 5596	2, 6886	2, 7627	2, 8119	2, 8464	2, 8714	2, 8897	2, 9035	2, 9138	2, 9217
w165 2, 5577	2, 6866	2, 7607	2, 8100	2, 8447	2, 8699	2, 8884	2, 9023	2, 9128	2, 9209
w166 2, 5559	2, 6846	2, 7587	2, 8082	2, 8430	2, 8683	2, 8871	2, 9011	2, 9118	2, 9200
w167 2, 5541	2, 6826	2, 7567	2, 8063	2, 8413	2, 8668	2, 8857	2, 9000	2, 9108	2, 9191
w168 2, 5523	2, 6806	2, 7548	2, 8044	2, 8396	2, 8653	2, 8844	2, 8988	2, 9098	2, 9182
w169 2, 5505	2, 6786	2, 7528	2, 8026	2, 8379	2, 8638	2, 8830	2, 8976	2, 9087	2, 9173
w170 2, 5487	2, 6767	2, 7509	2, 8007	2, 8362	2, 8622	2, 8817	2, 8964	2, 9077	2, 9164
w171 2, 5469	2, 6747	2, 7489	2, 7989	2, 8345	2, 8607	2, 8803	2, 8952	2, 9066	2, 9155
w172 2, 5452	2, 6728	2, 7470	2, 7970	2, 8328	2, 8591	2, 8789	2, 8940	2, 9056	2, 9146
w173 2, 5434	2, 6709	2, 7451	2, 7952	2, 8311	2, 8576	2, 8775	2, 8928	2, 9045	2, 9137
w174 2, 5417	2, 6690	2, 7431	2, 7934	2, 8294	2, 8561	2, 8762	2, 8915	2, 9034	2, 9127
w175 2, 5400	2, 6671	2, 7412	2, 7915	2, 8277	2, 8545	2, 8748	2, 8903	2, 9024	2, 9118
w176 2, 5382	2, 6652	2, 7394	2, 7897	2, 8260	2, 8530	2, 8734	2, 8891	2, 9013	2, 9108
w177 2, 5365	2, 6634	2, 7375	2, 7879	2, 8243	2, 8514	2, 8720	2, 8878	2, 9002	2, 9098
w178 2, 5349	2, 6615	2, 7356	2, 7861	2, 8226	2, 8499	2, 8706	2, 8866	2, 8991	2, 9089
w179 2, 5332	2, 6597	2, 7337	2, 7843	2, 8210	2, 8484	2, 8692	2, 8854	2, 8980	2, 9079
w180 2, 5315	2, 6579	2, 7319	2, 7825	2, 8193	2, 8468	2, 8678	2, 8841	2, 8968	2, 9069
w181 2, 5299	2, 6560	2, 7300	2, 7807	2, 8176	2, 8453	2, 8664	2, 8828	2, 8957	2, 9059
w182 2, 5282	2, 6542	2, 7282	2, 7789	2, 8159	2, 8437	2, 8650	2, 8816	2, 8946	2, 9049
w183 2, 5266	2, 6524	2, 7264	2, 7772	2, 8143	2, 8422	2, 8636	2, 8803	2, 8934	2, 9039
w184 2, 5250	2, 6507	2, 7246	2, 7754	2, 8126	2, 8406	2, 8622	2, 8790	2, 8923	2, 9029
w185 2, 5234	2, 6489	2, 7228	2, 7736	2, 8109	2, 8391	2, 8608	2, 8778	2, 8912	2, 9018
w186 2, 5218	2, 6471	2, 7210	2, 7719	2, 8093	2, 8375	2, 8594	2, 8765	2, 8900	2, 9008
w187 2, 5202	2, 6454	2, 7192	2, 7701	2, 8076	2, 8360	2, 8580	2, 8752	2, 8889	2, 8998
w188 2, 5187	2, 6436	2, 7174	2, 7684	2, 8059	2, 8345	2, 8566	2, 8739	2, 8877	2, 8987
w189 2, 5171	2, 6419	2, 7156	2, 7667	2, 8043	2, 8329	2, 8551	2, 8726	2, 8865	2, 8977
w190 2, 5156	2, 6402	2, 7139	2, 7649	2, 8026	2, 8314	2, 8537	2, 8713	2, 8854	2, 8966
w191 2, 5140	2, 6385	2, 7121	2, 7632	2, 8010	2, 8298	2, 8523	2, 8700	2, 8842	2, 8956
w192 2, 5125	2, 6368	2, 7104	2, 7615	2, 7994	2, 8283	2, 8509	2, 8687	2, 8830	2, 8945
w193 2, 5110	2, 6351	2, 7086	2, 7598	2, 7977	2, 8268	2, 8495	2, 8674	2, 8818	2, 8934
w194 2, 5095	2, 6334	2, 7069	2, 7581	2, 7961	2, 8252	2, 8480	2, 8661	2, 8806	2, 8924
w195 2, 5080	2, 6318	2, 7052	2, 7564	2, 7945	2, 8237	2, 8466	2, 8648	2, 8794	2, 8913
w196 2, 5065	2, 6301	2, 7035	2, 7547	2, 7929	2, 8222	2, 8452	2, 8635	2, 8782	2, 8902
w197 2, 5050	2, 6285	2, 7018	2, 7531	2, 7912	2, 8207	2, 8438	2, 8622	2, 8770	2, 8891
w198 2, 5035	2, 6268	2, 7001	2, 7514	2, 7896	2, 8191	2, 8424	2, 8609	2, 8758	2, 8880
w199 2, 5021	2, 6252	2, 6984	2, 7497	2, 7880	2, 8176	2, 8409	2, 8596	2, 8746	2, 8869
w200 2, 5006	2, 6236	2, 6968	2, 7481	2, 7864	2, 8161	2, 8395	2, 8583	2, 8734	2, 8858
w201 2, 4992	2, 6220	2, 6951	2, 7464	2, 7848	2, 8146	2, 8381	2, 8570	2, 8722	2, 8847
w202 2, 4978	2, 6204	2, 6935	2, 7448	2, 7832	2, 8131	2, 8367	2, 8556	2, 8710	2, 8836
w203 2, 4963	2, 6188	2, 6918	2, 7431	2, 7817	2, 8116	2, 8353	2, 8543	2, 8698	2, 8825
w204 2, 4949	2, 6172	2, 6902	2, 7415	2, 7801	2, 8101	2, 8339	2, 8530	2, 8686	2, 8813
w205 2, 4935	2, 6156	2, 6886	2, 7399	2, 7785	2, 8086	2, 8324	2, 8517	2, 8674	2, 8802
w206 2, 4921	2, 6141	2, 6869	2, 7383	2, 7769	2, 8070	2, 8310	2, 8504	2, 8661	2, 8791
w207 2, 4908	2, 6125	2, 6853	2, 7367	2, 7754	2, 8055	2, 8296	2, 8490	2, 8649	2, 8780
w208 2, 4894	2, 6110	2, 6837	2, 7351	2, 7738	2, 8041	2, 8282	2, 8477	2, 8637	2, 8768
w209 2, 4880	2, 6095	2, 6821	2, 7335	2, 7722	2, 8026	2, 8268	2, 8464	2, 8625	2, 8757
w210 2, 4867	2, 6079	2, 6806	2, 7319	2, 7707	2, 8011	2, 8254	2, 8451	2, 8612	2, 8746
w211 2, 4853	2, 6064	2, 6790	2, 7303	2, 7691	2, 7996	2, 8240	2, 8438	2, 8600	2, 8734
w212 2, 4840	2, 6049	2, 6774	2, 7287	2, 7676	2, 7981	2, 8226	2, 8424	2, 8588	2, 8723
w213 2, 4826	2, 6034	2, 6759	2, 7272	2, 7661	2, 7966	2, 8211	2, 8411	2, 8575	2, 8711
w214 2, 4813	2, 6019	2, 6743	2, 7256	2, 7645	2, 7952	2, 8197	2, 8398	2, 8563	2, 8700
w215 2, 4800	2, 6005	2, 6728	2, 7240	2, 7630	2, 7937	2, 8183	2, 8385	2, 8550	2, 8688
w216 2, 4787	2, 5990	2, 6712	2, 7225	2, 7615	2, 7922	2, 8169	2, 8371	2, 8538	2, 8677
w217 2, 4774	2, 5975	2, 6697	2, 7210	2, 7600	2, 7907	2, 8155	2, 8358	2, 8526	2, 8665
w218 2, 4761	2, 5961	2, 6682	2, 7194	2, 7585	2, 7893	2, 8142	2, 8345	2, 8513	2, 8654
w219 2, 4748	2, 5946	2, 6667	2, 7179	2, 7570	2, 7878	2, 8128	2, 8332	2, 8501	2, 8642
w220 2, 4735	2, 5932	2, 6652	2, 7164	2, 7555	2, 7864	2, 8114	2, 8319	2, 8489	2, 8630

w221	2, 4723	2, 5918	2, 6637	2, 7149	2, 7540	2, 7849	2, 8100	2, 8305	2, 8476	2, 8619
w222	2, 4710	2, 5903	2, 6622	2, 7134	2, 7525	2, 7835	2, 8086	2, 8292	2, 8464	2, 8607
w223	2, 4697	2, 5889	2, 6607	2, 7119	2, 7510	2, 7820	2, 8072	2, 8279	2, 8451	2, 8596
w224	2, 4685	2, 5875	2, 6592	2, 7104	2, 7495	2, 7806	2, 8058	2, 8266	2, 8439	2, 8584
w225	2, 4673	2, 5861	2, 6578	2, 7089	2, 7481	2, 7792	2, 8044	2, 8253	2, 8426	2, 8572
w226	2, 4660	2, 5847	2, 6563	2, 7074	2, 7466	2, 7777	2, 8031	2, 8240	2, 8414	2, 8561
w227	2, 4648	2, 5833	2, 6548	2, 7059	2, 7451	2, 7763	2, 8017	2, 8227	2, 8402	2, 8549
w228	2, 4636	2, 5820	2, 6534	2, 7045	2, 7437	2, 7749	2, 8003	2, 8214	2, 8389	2, 8537
w229	2, 4624	2, 5806	2, 6520	2, 7030	2, 7422	2, 7735	2, 7990	2, 8200	2, 8377	2, 8525
w230	2, 4612	2, 5792	2, 6505	2, 7016	2, 7408	2, 7721	2, 7976	2, 8187	2, 8364	2, 8514
w231	2, 4600	2, 5779	2, 6491	2, 7001	2, 7393	2, 7707	2, 7962	2, 8174	2, 8352	2, 8502
w232	2, 4588	2, 5765	2, 6477	2, 6987	2, 7379	2, 7693	2, 7949	2, 8161	2, 8340	2, 8490
w233	2, 4576	2, 5752	2, 6463	2, 6972	2, 7365	2, 7679	2, 7935	2, 8148	2, 8327	2, 8479
w234	2, 4564	2, 5739	2, 6449	2, 6958	2, 7351	2, 7665	2, 7922	2, 8135	2, 8315	2, 8467
w235	2, 4553	2, 5725	2, 6435	2, 6944	2, 7336	2, 7651	2, 7908	2, 8122	2, 8302	2, 8455
w236	2, 4541	2, 5712	2, 6421	2, 6930	2, 7322	2, 7637	2, 7895	2, 8109	2, 8290	2, 8443
w237	2, 4529	2, 5699	2, 6407	2, 6916	2, 7308	2, 7623	2, 7881	2, 8096	2, 8278	2, 8432
w238	2, 4518	2, 5686	2, 6394	2, 6902	2, 7294	2, 7609	2, 7868	2, 8084	2, 8265	2, 8420
w239	2, 4507	2, 5673	2, 6380	2, 6888	2, 7280	2, 7595	2, 7854	2, 8071	2, 8253	2, 8408
w240	2, 4495	2, 5660	2, 6366	2, 6874	2, 7266	2, 7582	2, 7841	2, 8058	2, 8241	2, 8396
w241	2, 4484	2, 5647	2, 6353	2, 6860	2, 7253	2, 7568	2, 7828	2, 8045	2, 8228	2, 8385
w242	2, 4473	2, 5635	2, 6339	2, 6846	2, 7239	2, 7554	2, 7815	2, 8032	2, 8216	2, 8373
w243	2, 4462	2, 5622	2, 6326	2, 6833	2, 7225	2, 7541	2, 7801	2, 8019	2, 8204	2, 8361
w244	2, 4450	2, 5609	2, 6313	2, 6819	2, 7211	2, 7527	2, 7788	2, 8007	2, 8191	2, 8349
w245	2, 4439	2, 5597	2, 6299	2, 6805	2, 7198	2, 7514	2, 7775	2, 7994	2, 8179	2, 8338
w246	2, 4428	2, 5584	2, 6286	2, 6792	2, 7184	2, 7500	2, 7762	2, 7981	2, 8167	2, 8326
w247	2, 4417	2, 5572	2, 6273	2, 6778	2, 7171	2, 7487	2, 7749	2, 7968	2, 8155	2, 8314
w248	2, 4407	2, 5559	2, 6260	2, 6765	2, 7157	2, 7474	2, 7736	2, 7956	2, 8142	2, 8303
w249	2, 4396	2, 5547	2, 6247	2, 6752	2, 7144	2, 7460	2, 7723	2, 7943	2, 8130	2, 8291
w250	2, 4385	2, 5535	2, 6234	2, 6739	2, 7130	2, 7447	2, 7710	2, 7930	2, 8118	2, 8279
w251	2, 4374	2, 5523	2, 6221	2, 6725	2, 7117	2, 7434	2, 7697	2, 7918	2, 8106	2, 8267
w252	2, 4364	2, 5511	2, 6208	2, 6712	2, 7104	2, 7421	2, 7684	2, 7905	2, 8094	2, 8256
w253	2, 4353	2, 5498	2, 6195	2, 6699	2, 7091	2, 7408	2, 7671	2, 7893	2, 8082	2, 8244
w254	2, 4343	2, 5486	2, 6183	2, 6686	2, 7077	2, 7395	2, 7658	2, 7880	2, 8069	2, 8232
w255	2, 4332	2, 5475	2, 6170	2, 6673	2, 7064	2, 7382	2, 7645	2, 7868	2, 8057	2, 8221
w256	2, 4322	2, 5463	2, 6157	2, 6660	2, 7051	2, 7369	2, 7632	2, 7855	2, 8045	2, 8209
w257	2, 4312	2, 5451	2, 6145	2, 6647	2, 7038	2, 7356	2, 7620	2, 7843	2, 8033	2, 8197
w258	2, 4301	2, 5439	2, 6132	2, 6634	2, 7025	2, 7343	2, 7607	2, 7830	2, 8021	2, 8186
w259	2, 4291	2, 5427	2, 6120	2, 6621	2, 7012	2, 7330	2, 7594	2, 7818	2, 8009	2, 8174
w260	2, 4281	2, 5416	2, 6108	2, 6609	2, 7000	2, 7317	2, 7582	2, 7806	2, 7997	2, 8163
w261	2, 4271	2, 5404	2, 6095	2, 6596	2, 6987	2, 7304	2, 7569	2, 7793	2, 7985	2, 8151
w262	2, 4261	2, 5393	2, 6083	2, 6583	2, 6974	2, 7292	2, 7556	2, 7781	2, 7973	2, 8140
w263	2, 4251	2, 5381	2, 6071	2, 6571	2, 6961	2, 7279	2, 7544	2, 7769	2, 7961	2, 8128
w264	2, 4241	2, 5370	2, 6059	2, 6558	2, 6949	2, 7266	2, 7531	2, 7756	2, 7949	2, 8116
w265	2, 4231	2, 5358	2, 6047	2, 6546	2, 6936	2, 7254	2, 7519	2, 7744	2, 7938	2, 8105
w266	2, 4221	2, 5347	2, 6035	2, 6534	2, 6923	2, 7241	2, 7507	2, 7732	2, 7926	2, 8093
w267	2, 4211	2, 5336	2, 6023	2, 6521	2, 6911	2, 7229	2, 7494	2, 7720	2, 7914	2, 8082
w268	2, 4201	2, 5325	2, 6011	2, 6509	2, 6899	2, 7216	2, 7482	2, 7708	2, 7902	2, 8070
w269	2, 4192	2, 5313	2, 5999	2, 6497	2, 6886	2, 7204	2, 7470	2, 7696	2, 7890	2, 8059
w270	2, 4182	2, 5302	2, 5987	2, 6484	2, 6874	2, 7191	2, 7457	2, 7684	2, 7878	2, 8048
w271	2, 4172	2, 5291	2, 5976	2, 6472	2, 6861	2, 7179	2, 7445	2, 7672	2, 7867	2, 8036
wexp	2, 4164	2, 5282	2, 5966	2, 6462	2, 6851	2, 7169	2, 7435	2, 7662	2, 7857	2, 8027

A.2   Abstract mixed power-conferring norms

Consider the power-conferring norms $\eta'_1, \eta'_2 \in \mathbb{P}_{\wp \mathbb{P}_Y{}^X}{}^{X'}$ illustrated in Figure A.2

and locally (locally) defined by:

$$\eta'_1\left(\cdot\,|\,\overline{\overline{x}}\,'\right) =$$

$$\left\langle 0,\ldots,0,1,0,\ldots,0;\varnothing,\{g_1\},\ldots,\{g_{k-1}\},\{g_k\},\{g_{k+1}\},\ldots,\{g_1,\ldots,g_J\}\right\rangle;\ \text{and}$$

$$\eta'_2\left(\cdot\,|\,\overline{\overline{x}}\,'\right) =$$

$$\left\langle 0,\ldots,0,\eta'_2\left(\{g_k\}\,|\,\overline{\overline{x}}\,'\right),0,\ldots,0,\eta'_2\left(\{g_{k'},g_k\}\,|\,\overline{\overline{x}}\,'\right),0,\ldots,0;\varnothing,\{g_1\},\ldots,\{g_1,\ldots,g_J\}\right\rangle,$$

respectively. The *mixed* power-conferring norm $\eta'_\lambda = \lambda\eta'_1 \Lambda (1-\lambda)\eta'_2$ is (locally) given by:

$$\eta'_\lambda\left(\cdot\,|\,\overline{\overline{x}}\,'\right) =$$

$$\left\langle 0,\ldots,0,\lambda+[1-\lambda]\eta'_2\left(\{g_k\}\,|\,\overline{\overline{x}}\,'\right),0,\ldots,0,[1-\lambda]\eta'_2\left(\{g_{k'},g_k\}\,|\,\overline{\overline{x}}\,'\right),0,\ldots,0\right\rangle,$$

and also illustrated in Figure A2. Its support is given by a union of family of sets,

$$\operatorname{supp}\eta'_\lambda\left(\cdot\,|\,\overline{\overline{x}}\,'\right) == \operatorname{supp}\eta'_1\left(\cdot\,|\,\overline{\overline{x}}\,'\right) \cup \operatorname{supp}\eta'_2\left(\cdot\,|\,\overline{\overline{x}}\,'\right)$$
$$= \{\{g_k\}\} \cup \{\{g_k\},\{g_{k'},g_k\}\} = \{\{g_k\},\{g_{k'},g_k\}\}$$
.
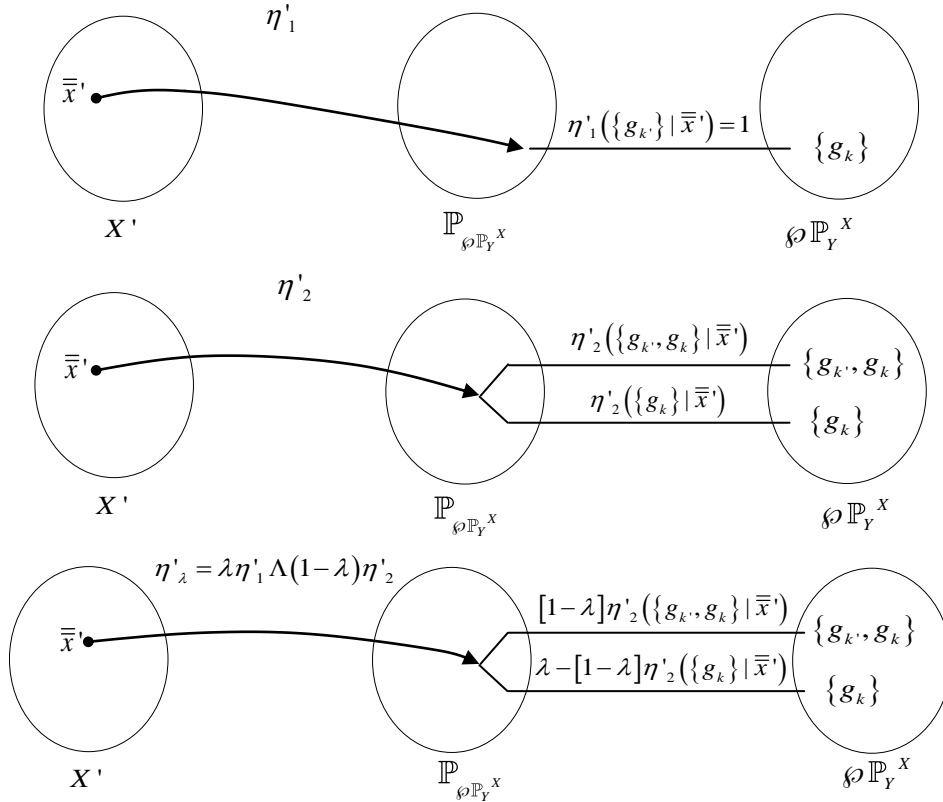
Figure A.2 Convex combinations of power-conferring norms

A.3   Transformed power-conferring norms

If power-conferring norms are directly maintained by courts, and if $\#\left|\operatorname{supp}\eta'(\cdot\mid x')\right|=2$, all binary transformation results in Part III apply. For example, consider the abstract power-conferring norm $\eta'\in\mathbb{P}_{\wp\mathbb{P}_Y x}{}^{X'}$ illustrated in Figure VI.2.1. At $\bar{\bar{x}}'$:

$$\eta'\left(\cdot\mid\bar{\bar{x}}'\right)=\Big\langle 1-\eta'\left(\{g_{k'},g_k\}\mid\bar{\bar{x}}'\right),\eta'\left(\{g_{k'},g_k\}\mid\bar{\bar{x}}'\right);\{g_k\},\{g_{k'},g_k\}\Big\rangle.$$

It follows from Proposition III.2.1 and III.2.3.B that in $M_{[v;n]}$ the norm is transformed to:

$$\Big\langle 1-H_{[v;n]}\Big(\eta'\left(\{g_{k'},g_k\}\mid\bar{\bar{x}}'\right)\Big),H_{[v;n]}\Big(\eta'\left(\{g_{k'},g_k\}\mid\bar{\bar{x}}'\right)\Big);\{g_k\},\{g_{k'},g_k\}\Big\rangle.$$

More generally, the competence span is transformed by collective mechanisms in all situations where the abstract power-conferring norm is non-degenerate (all $x'\notin\eta'^{-1}\left(\delta\mathbb{P}_{\wp\mathbb{P}_Y x}\right)$).[1] The same remarks apply to transformation of abstract mixed power-conferring norms.

If the number sets in the support is larger than two, acute problems arise: sets cannot be ordered according to the classical rule. As illustrated in Example VI.3.1, the impact of Arrow-type impossibility seems inevitable.

---

[1] In the case of majority mechanisms, the "fixed point" $\maltese'\in X':\eta'\left(\{g_{k'},g_k\}\mid\maltese'\right)=\tfrac{1}{2}$ is mapped to the same probability distribution for all m because $h_{2m+1}\left(\tfrac{1}{2}\right)\equiv\tfrac{1}{2}$ (Proposition III.2.2).