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A priori bounds and existence of positive solutions for semilinear elliptic systems

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Abstract

We provide a-priori L^∞ bounds for classical positive solutions of semilinear elliptic systems in bounded convex domains when the nonlinearities are below the power functions v^p and u^q for any (p, q) lying on the critical Sobolev hyperbola. Our proof combines moving planes method and Rellich-Pohozaev type identities for systems. Our analysis widens the known ranges of nonlinearities for which classical positive solutions of semilinear elliptic systems are a priori bounded.

Using these a priori bounds, and local and global bifurcation techniques, we prove the existence of positive solutions for a corresponding parametrized semilinear elliptic system.

Key words: A priori estimates, semilinear elliptic systems, critical Sobolev hyperbola, moving planes method, Rellich-Pohozaev identity, biparameter bifurcation.
2010 MSC: 35J47, 35J60, 35J61

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1. Introduction

We consider the following semilinear elliptic system

$$\begin{cases} -\Delta u = \dfrac{v^p}{\big[\ln(e+v)\big]^{\alpha}}, & \text{in } \Omega, \\ -\Delta v = \dfrac{u^q}{\big[\ln(e+u)\big]^{\beta}}, & \text{in } \Omega, \\ u = 0, \quad v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 3$, is a bounded, convex domain with a smooth boundary $\partial\Omega$ (at least of class C^3), and $1 < p, q < \infty$, $\alpha, \beta > 0$. The purpose of this paper is to establish a-priori estimates for positive classical solutions of (1.1) and subsequently prove an existence result for the parametrized version for the system. By a positive classical solution of (1.1), we mean (u, v) that satisfies (1.1) and both components are positive. Let us mention that when the exponents $\alpha = \beta = 0$, we have the system

$$\begin{cases} -\Delta u = v^p, & \text{in } \Omega, \\ -\Delta v = u^q, & \text{in } \Omega, \\ u = 0, \quad v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

that is usually referred to as the Lane-Emden system. This problem arises in modeling spatial phenomena in a variety of biological and chemical problems. Naturally positive solutions of system (1.2) is of particular interest, and there have been a significant studies of positive solutions of (1.2) where Ω is either a bounded, smooth subset of $\mathbb{R}^N$, a half space, or the entire space $\mathbb{R}^N$, see [2, 3, 5, 9, 11, 15, 17, 30, 31, 33, 37, 38, 40] and references therein.

It is known that the pair of exponents (p, q) plays a crucial role in the questions of existence and nonexistence of positive solutions of (1.2). For instance, it has been shown that on a bounded smooth star-shaped domain $\Omega \subset \mathbb{R}^N$, the (Sobolev) hyperbola

$$\frac{1}{p+1} + \frac{1}{q+1} = \frac{N-2}{N} \tag{1.3}$$

is precisely the dividing curve on the pq-plane between existence and nonexistence of positive solutions of (1.2), see [9, 30, 31].

In [9], the authors established a-priori estimates and proved the existence of positive solutions of (1.2) when (p, q) is subcritical (i.e. (p, q) lies below

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the critical Sobolev hyperbola), that is, $\dfrac{1}{p+1} + \dfrac{1}{q+1} > \dfrac{N-2}{N}$. Moreover, in [30], the author proved that if (p, q) is critical (i.e. (p, q) lies on critical Sobolev hyperbola) or supercritical (i.e. (p, q) lies above the critical Sobolev hyperbola), namely if $\dfrac{1}{p+1} + \dfrac{1}{q+1} \leq \dfrac{N-2}{N}$ then (1.2) has no positive solution.

However, when $\Omega = \mathbb{R}^N$, it has been conjectured that the hyperbola (1.3) is also the dividing curve between existence and nonexistence for (1.2). The conjecture has been completely proved for radial positive solutions (see e.g. [30, 36]), that is, if (p, q) is subcritical then there are no radial positive classical solution to (1.2), see [30] (for $p > 1$, $q > 1$) and it has been extended in [36] for the case $p > 0$ and $q > 0$. Furthermore, if (p, q) is critical or supercritical, system (1.2) does admit (bounded) positive radial solutions (see e.g. [30, 36]). But the question for the more general case, i.e. without assuming radial symmetry has not been completely answered yet. Partial answers are known for nonexistence of positive entire solutions of (1.2) when the pair of exponents are subcritical, for example, it has been proved the nonexistence in certain space dimensions [30, 38] or in certain subregions, below the critical hyperbola in the $(p, q)-$plane, (see e.g. [5, 38]). For $\Omega = \mathbb{R}_+^N$ (i.e. the half space), we refer to [2] for the study of nonexistence of positive solutions. These nonexistence results in $\mathbb{R}^N$ or $\mathbb{R}_+^N$ allow to prove a-priori bounds for positive solutions of semilinear elliptic equations in bounded domains via the blow-up method (see e.g. [21, 2, 40]).

In the present paper, we use the method of moving planes and the Rellich-Pohazev identity for systems to establish the a-priori L^∞ bounds when the pair of exponents (p, q) lies on the critical Sobolev hyperbola (1.3) and α, $\beta > \dfrac{2}{N-2}$, and then subsequently prove an existence result for the parametrized version for the system, see system (1.6) below. We shall point out that our nonlinearities are not pure powers, they are below the powers functions v^p and u^q. Problems of type (1.1) has been considered by several authors, we refer to [16, 10]. In [16, Theorem 1.3] the authors study the existence of solutions of (1.1) when the pair of exponents (p, q) lies on the critical Sobolev hyperbola (1.3) and $\alpha < 0$, $\beta > \dfrac{q+1}{p+1} |\alpha|$, that is, one nonlinearity is above the power function. Whereas in [10, Theorem 2.7] the authors study related nonlinearities when the pair of exponents (p, q) lies below the critical Sobolev

hyperbola (1.3), using variational approaches.

Throughout this paper we assume that $\Omega \subset \mathbb{R}^N$ is a smooth bounded, convex domain. The hypothesis on convexity of the domain is needed in order to establish a priori bounds in a neighborhood of the boundary, via the moving planes method, see Lemma 2.1. This idea was introduced in [15] for scalar equations. In [39, Lemma 4.3] the author develop the moving planes method for systems assuming that both nonlinearities are nondecreasing and do not depend explicitly on the spatial variable x.

For general bounded domains, not necessarily convex, de Figueiredo, Lions and Nussbaum [15] applied the moving planes method on the Kelvin transform in order to avoid the difficulty of an empty *cap* (see [20, 6] for details and the definition of a cap). In that situation, it turns out that the (transformed) nonlinearity depends on the spatial variable x. Then, they obtained a priori bounds in a neighborhood of the boundary for classical positive solutions of scalar equations on non-convex domains. To the best of our knowledge, the moving planes method for systems is not yet developed for nonlinearities depending also on the variable x. Hence, we focus on convex domains.

We now state our main results.

Theorem 1.1. *(a-priori L^∞ bounds)*
Suppose that $p, q > 1$, $\alpha, \beta > \dfrac{2}{N-2}$, and

$$\frac{1}{p+1} + \frac{1}{q+1} = \frac{N-2}{N}, \tag{1.4}$$

and assume that

$$\min\left\{ \frac{p}{\alpha}, \ \frac{q}{\beta} \right\} \geq \max_{t \geq 0}\left[\frac{t}{(e+t)\,\ln(e+t)} \right]. \tag{1.5}$$

Then there exists a uniform constant C, depending only on Ω and p, q, α, β but not on (u, v), such that

$$\|u\|_{L^\infty(\Omega)} \leq C, \quad and \quad \|v\|_{L^\infty(\Omega)} \leq C,$$

for all positive solutions (u, v) of (1.1).

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Remark 1.2. *Observe that condition* (1.4) *relates the exponents p and q to the Sobolev hyperbola, and condition* (1.5) *ensures that our nonlinearities are nondecreasing since it will be needed in the proof.*

Notice that

$$\max_{t \geq 0} \left[\frac{t}{(e+t)\,\ln(e+t)} \right] = \frac{e}{e+t^*},$$

where t^* is the solution of the logarithmic equation $e\ln(e+t) = t$.

In the next theorem, we take up on the existence of positive solutions for the parametrized version of the elliptic system (1.1).

Theorem 1.3. *(Existence)*

Consider the biparameter elliptic system

$$\begin{cases} -\Delta u = \lambda\,v + \dfrac{v^p}{\big[\ln(e+v)\big]^\alpha}, & in\ \Omega, \\[2ex] -\Delta v = \mu\,u + \dfrac{u^q}{\big[\ln(e+u)\big]^\beta}, & in\ \Omega, \\[2ex] u = 0, \quad v = 0 \quad on\ \partial\Omega, \end{cases} \tag{1.6}$$

where the exponents p, q, α, β are as defined in Theorem 1.1, and the parameters λ and μ are non-negative real parameters.

Then (1.6) *has a positive solution (u,v) if and only if $\lambda\mu < \lambda_1^2$, where λ_1 is the principal eigenvalue associated with the linear eigenvalue problem with homogeneous Dirichlet boundary conditions $-\Delta\phi = \lambda\,\phi$ in Ω; $\phi = 0$ on $\partial\Omega$.*

The proof is based on local bifurcation techniques [12], combined with global bifurcation theorem [13, 24, 35], and the a-priori estimates obtained therein. From the seminal works of Crandall and Rabinowitz, see [12, 35], there are a significant amount of references on bifurcation theory. Let us mention Alexander and Antman's Theorem [1] on global multiparameter bifurcation techniques, looking for a change of fixed point index, and providing a manifold of solutions of topological dimension at least the number of parameters, [25] on local multiparameter bifurcation techniques on elliptic systems, [18, 24, 25, 26, 27, 28, 29] on combination of local and global multiparameter bifurcation techniques on elliptic systems, and [19] on the multiparameter bifurcation for the p-laplacian.

Before we take up on the study of system (1.1), let us briefly explain why we are interested in the nonlinearities of type (1.1) by recalling some known facts about a priori bounds of positive solutions of the scalar equation

$$\begin{cases} -\Delta u = f(u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.7}$$

From the classical literature, see the well known results of Gidas and Spruck in [21] and Figueiredo, Lions and Nussbaum in [15], there are recent advances in this area, see [6]. The results in [21] depend heavily on the blow up method which requires f to be essentially of the form $f(x,s) = h(x)s^p$ with $p \in (1, \frac{N+2}{N-2})$ and $h(x)$ continuous and strictly positive. In [15], using the moving plane method [20], and Rellich-Pohozaev identities [32], the authors show the existence of a-priori L^∞ bounds for classical positive solutions of equation (1.7) when the nonlinearity f is assumed to satisfy

$$\liminf_{s \to +\infty} \frac{\theta F(s) - sf(s)}{s^2 f(s)^{2/N}} \geq 0, \qquad \text{for some} \quad \theta \in [0, 2^\star),$$

where $F(s) := \int_0^s f(t)\, dt$. They conjecture that this condition is not necessary, but it is essential in proving their result. It can be seen that for $f_1(s) = s^{2^*-1}/\ln(2+s)^\alpha$ with $\alpha > 0$

$$\liminf_{s \to +\infty} \frac{\theta F_1(s) - sf_1(s)}{s^2 f_1(s)^{2/N}} = -\infty, \qquad \text{for any} \quad \theta \in [0, 2^\star),$$

where $F_1(s) := \int_0^s f_1(t)\, dt$. In [6] the authors prove the existence of a-priori bounds when $f(s) = s^{\frac{N+2}{N-2}}/\ln(2+s)^\alpha$, with $\alpha > 2/(N-2)$, see [6, Corollary 2.2]. Combining a priori bounds with degree theory, they obtain existence results for parametrized versions with $f = f(\lambda, u) = \lambda u + u^{\frac{N+2}{N-2}}/\ln(2+u)^\alpha$, see [7, 8].

In view of this recent advance, it is natural to ask whether it is possible to obtain the corresponding results for systems. In this paper, we extend the results of [6] from scalar equations to systems. The existence of a-priori bounds for the system (1.1) is proved as much as in the same lines of [6], that is, using the Rellich-Pohozaev identity and the method of moving planes as in [9], combined with Morrey's Theorem. The moving planes method is used to obtain L^∞ bounds in a neighborhood of the boundary for classical positive

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solutions of (1.1), whereas the Rellich-Pohozaev identity is used to get two bounded integrals in Ω. Furthermore, Morrey's Theorem is used to estimate the radius R, (R'), of a ball where the function u, (v), exceeds half of its L^∞ bound (see fig 1), allowing us to reach a contradiction on the lower bounds of the above integrals.
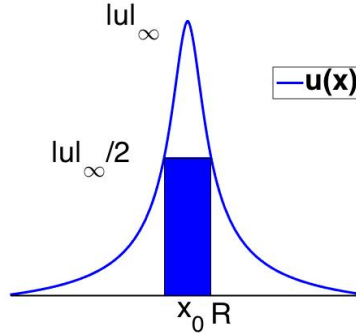

Figure 1: Let u be a solution of (1.1), we plot $u(x)$, its L^∞ norm, and the estimate of the radius R such that $u(x) \geq \dfrac{\|u\|_\infty}{2}$ for all $x \in B(x_0, R)$, where x_0 is such that $u(x_0) = \|u\|_\infty$.

This paper is organized as follows. In Section 2, we state some preliminary and known results that are needed for the proof of our main results, which include the moving plane method, and an extension of Rellich-Pohozaev type identity for systems. Section 3 is devoted to the proofs of Theorem 1.1 and Theorem 1.3.

2. Preliminaries and known Results

In this Section, we state two lemmas that are relevant in order to obtain the a priori estimates. The first lemma provides L^∞ a priori bounds for any positive solution of (1.1) in a neighborhood of the boundary, see [15]. The hypothesis of convexity of the domain is needed in order to establish these a priori bounds in a neighborhood of the boundary. Whereas the second lemma provides a Rellich-Pohozaev-Mitidieri type identity, see [30].

Lemma 2.1. *Let (u, v) be a positive classical solution of the system* (1.1). *Assume that the hypotheses of Theorem 1.1 are satisfied, then there exists a*

constant $\delta > 0$ depending only on Ω and not on p, q, α, β or (u, v), and a constant C depending only on Ω and p, q, α, β but not on (u, v), such that

$$\max_{\Omega \backslash \Omega_\delta} u \leq C \quad and \quad \max_{\Omega \backslash \Omega_\delta} v \leq C \tag{2.1}$$

where $\Omega_\delta := \{x \in \Omega \ : \ d(x, \partial\Omega) > \delta\}$.

The proof is done in a similar way as step 2 in the proof of Theorem 1.1 in [15], using the moving planes method for systems [39, Lemma 4.3]. See also step 1 and 2 in the proof of Theorem 2.1 in [9].

Lemma 2.2. *(Rellich-Pohozaev-Mitidieri type identity)*
Let u and v be in $C^2(\bar{\Omega})$, where Ω is a C^1 domain in $\mathbb{R}^N$, and $u = v = 0$ on $\partial\Omega$. Then

$$\int_\Omega \Delta u \, (x \cdot \nabla v) + \Delta v \, (x \cdot \nabla u) = (N - 2) \int_\Omega (\nabla u \cdot \nabla v) + \int_{\partial\Omega} \frac{\partial u}{\partial n} \, (x \cdot \nabla v)$$
$$+ \int_{\partial\Omega} \frac{\partial v}{\partial n} \, (x \cdot \nabla u) - \int_{\partial\Omega} (\nabla u, \nabla v) \, (x \cdot n),$$

where n denotes the exterior normal, and $(x \cdot n)$ denotes the inner product.

For the proof we refer to [30].

3. Proof of main results

In this section we prove Theorem 1.1 and Theorem 1.3. The proof of Theorem 1.1 (L^∞ a-priori bounds) is based on a version for systems of moving planes arguments [39] and an extension of Rellich-Pohozaev type identity [30], combined with Morrey's Theorem. Whereas the proof of Theorem 1.3 (existence of positive solutions for the parametrized system) is based on the local and global bifurcation techniques [12, 13, 14, 24, 35] and the a-priori estimates obtained therein.

Let $\theta \in (0, 1)$ be such that

$$\frac{1}{p+1} = \theta \frac{N-2}{N} \qquad \text{and} \qquad \frac{1}{q+1} = (1-\theta) \frac{N-2}{N} \tag{3.1}$$

which is possible by (1.4).

Set $F(t) := \displaystyle\int_0^t f(s) \, ds \qquad$ and $\quad G(t) := \displaystyle\int_0^t g(s) \, ds, \quad$ where

$$f(s) = \frac{s^p}{\left[\ln(e+s)\right]^\alpha}, \qquad g(s) = \frac{s^q}{\left[\ln(e+s)\right]^\beta}, \tag{3.2}$$

Integrating by parts and taking into account (3.1) we have that

$$F(t) - \theta \left(\frac{N-2}{N}\right) t \, f(t) = \frac{\alpha}{p+1} \int_0^t \frac{s^{p+1}}{\ln(e+s)^{\alpha+1}} \frac{ds}{e+s}, \tag{3.3}$$

Likewise

$$G(t) - (1-\theta) \left(\frac{N-2}{N}\right) t \, g(t) = \frac{\beta}{q+1} \int_0^t \frac{s^{q+1}}{\ln(e+s)^{\beta+1}} \frac{ds}{e+s}. \tag{3.4}$$

Now, if we set $W(s, t) := F(t) + G(s)$ then $W_s = g(s)$ and $W_t = f(t)$. Therefore, for solutions $u > 0$ and $v > 0$ of (1.1),

$$\int_\Omega - [\Delta u \, (x \cdot \nabla v) + \Delta v \, (x \cdot \nabla u)] = \int_\Omega \sum_j x_j \left(\frac{\partial W}{\partial v} \frac{\partial v_j}{\partial x_j} + \frac{\partial W}{\partial u} \frac{\partial u_j}{\partial x_j} \right)$$

$$= \int_\Omega \sum_j x_j \frac{\partial W}{\partial x_j}$$

$$= -N \int_\Omega W + \int_\Omega \text{div}(W \, \vec{x})$$

$$= -N \int_\Omega [F(v) + G(u)] + \int_{\partial\Omega} (x \cdot n) \, W(u, v),$$

and

$$\int_\Omega \nabla u \nabla v = (1-\theta) \int_\Omega u g(u) + \theta \int_\Omega v f(v).$$

Applying Lemma 2.2 (Pohozaev-Rellich-Mitidieri type identity) we get that

$$N \int_\Omega [F(v) + G(u)] - (N-2) \int_\Omega [\theta \, v \, f(v) + (1-\theta) \, u \, g(u)]$$

$$= \int_{\partial\Omega} (x \cdot n) \, W(u,v) - \int_{\partial\Omega} (\nabla u \cdot \nabla v) \, (x \cdot n) \quad (3.5)$$

$$+ \int_{\partial\Omega} \frac{\partial u}{\partial n} \, (x \cdot \nabla v) + \int_{\partial\Omega} \frac{\partial v}{\partial n} \, (x \cdot \nabla u).$$

It follows from (2.1) and de Giorgi-Nash type Theorems for systems, see [23, Theorem 3.1, p. 397] that

$$\|(u,v)\|_{C^{0,\alpha}(\Omega_{\delta/8} \setminus \Omega_{7\delta/8})} \le C, \quad \text{for any} \quad \alpha \in (0,1),$$

where $\Omega_t := \{x \in \Omega \ : \ d(x, \partial\Omega) > t\}$, and $\|(u,v)\| := \|u\| + \|v\|$.
Using Schauder interior estimates, see [22, Theorem 6.2]

$$\|(u,v)\|_{C^{2,\alpha}(\Omega_{\delta/4} \setminus \Omega_{3\delta/4})} \le C.$$

Finally, combining L^p estimates with Schauder boundary estimates, see [4, 22, 23]

$$\|(u,v)\|_{W^{2,p}(\Omega \setminus \Omega_{\delta/2})} \le C, \qquad \text{for any} \quad p \in (1,\infty).$$

By the Sobolev embedding for $p > N$, we have that there exists two constants $C, \delta > 0$ independent of u such that

$$\|(u,v)\|_{C^{1,\alpha}(\Omega \setminus \Omega_\delta)} \le C, \qquad \text{for any} \quad \alpha \in (0,1). \quad (3.6)$$

Therefore, it follows from (3.5) and (3.6) that

$$\left| N \int_\Omega [F(v) + G(u)] - (N-2) \int_\Omega [\theta \, v \, f(v) + (1-\theta) \, u \, g(u)] \right| \le C. \quad (3.7)$$

Using (3.3), (3.4), and (3.7), we obtain that

$$\left| \frac{\alpha}{p+1} \int_\Omega \left(\int_0^{v(x)} \frac{s^{p+1}}{\ln(e+s)^{\alpha+1}} \frac{ds}{e+s} \right) dx \right.$$

$$\left. + \frac{\beta}{q+1} \int_\Omega \left(\int_0^{u(x)} \frac{s^{q+1}}{\ln(e+s)^{\beta+1}} \frac{ds}{e+s} \right) dx \right| \le C. \quad (3.8)$$

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Moreover,

$$\lim_{t\to\infty} \frac{\displaystyle\int_0^t \left(\frac{1}{\ln(e+s)}\right)^{\alpha+1} \frac{s^{p+1}}{e+s}ds}{\dfrac{t^{p+1}}{\ln(e+t)^{\alpha+1}}} = \frac{1}{p+1},$$

and

$$\lim_{t\to\infty} \frac{\displaystyle\int_0^t \left(\frac{1}{\ln(e+s)}\right)^{\beta+1} \frac{s^{q+1}}{e+s}ds}{\dfrac{t^{q+1}}{\ln(e+t)^{\beta+1}}} = \frac{1}{q+1}.$$

Therefore, for any $\varepsilon > 0$ there exists a constant t_ε such that if $t > t_\varepsilon$ then

$$\frac{1}{p+1} - \varepsilon < \frac{\displaystyle\int_0^t \left(\frac{1}{\ln(e+s)}\right)^{\alpha+1} \frac{s^{p+1}}{e+s}ds}{\dfrac{t^{p+1}}{\ln(e+t)^{\alpha+1}}},$$

and

$$\frac{1}{q+1} - \varepsilon < \frac{\displaystyle\int_0^t \left(\frac{1}{\ln(e+s)}\right)^{\beta+1} \frac{s^{q+1}}{e+s}ds}{\dfrac{t^{q+1}}{\ln(e+t)^{\beta+1}}}.$$

Let us choose $\varepsilon = \frac{1}{2}\min\left\{\frac{1}{p+1}, \frac{1}{q+1}\right\}$, then there exists a constant $C > 0$ such that for any $t > 0$

$$\frac{t^{p+1}}{\ln(e+t)^{\alpha+1}} \leq C\left(1 + \int_0^t \left(\frac{1}{\ln(e+s)}\right)^{\alpha+1} \frac{s^{p+1}}{e+s}ds\right),$$

and

$$\frac{t^{q+1}}{\ln(e+t)^{\beta+1}} \leq C\left(1 + \int_0^t \left(\frac{1}{\ln(e+s)}\right)^{\beta+1} \frac{s^{q+1}}{e+s}ds\right).$$

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Hence, applying the above inequalities for any $v(x)$ and $u(x)$ solving (1.1) respectively, integrating in Ω and using (3.8), we have that

$$\left| \int_\Omega \frac{v^{p+1}}{\ln(e+v)^{\alpha+1}} + \int_\Omega \frac{u^{q+1}}{\ln(e+u)^{\beta+1}} \right| \leq C.$$

This implies that

$$\int_\Omega \frac{vf(v)}{\ln(e+v)} = \int_\Omega \frac{v^{p+1}}{\ln(e+v)^{\alpha+1}} \leq C,$$
$$\int_\Omega \frac{ug(u)}{\ln(e+u)} = \int_\Omega \frac{u^{q+1}}{\ln(e+u)^{\beta+1}} \leq C. \tag{3.9}$$

As pointed out in Remark 1.2, condition (1.5) ensures that f and g are nondecreasing.

Now, let us fix $r, r' \in (\frac{N}{2}, N)$ such that $r \geq 1 + \frac{1}{p}$ and $r' \geq 1 + \frac{1}{q}$. Then using (3.9) and the fact that f and g are nondecreasing for $\|v\|_\infty$ large enough, we get that

$$\int_\Omega |f(v)|^r = \int_\Omega \frac{v^{pr}}{\ln(e+v)^{\alpha r}}$$
$$= \int_\Omega \frac{v^{pr-p-1}}{\ln(e+v)^{\alpha(r-1)-1}} \frac{v^{p+1}}{\ln(e+v)^{\alpha+1}}$$
$$\leq C \frac{\|v\|_\infty^{pr-(p+1)}}{\ln(e+\|v\|_\infty)^{\alpha(r-1)-1}}$$
$$= C \left(\frac{\|v\|_\infty^p}{\ln(e+\|v\|_\infty)^\alpha} \right)^{r-1-\frac{1}{p}} \frac{1}{\ln(e+\|v\|_\infty)^{\frac{\alpha}{p}-1}},$$

and similarly

$$\int_\Omega |g(u)|^{r'} \leq C \left(\frac{\|u\|_\infty^q}{\ln(e+\|u\|_\infty)^\beta} \right)^{r'-1-\frac{1}{q}} \frac{1}{\ln(e+\|u\|_\infty)^{\frac{\beta}{q}-1}}. \tag{3.10}$$

By the regularity of elliptic equations (see [22, 23]) it follows that

$$\|u\|_{W^{2,r}(\Omega)} \leq C \left(\frac{\|v\|_\infty^p}{\ln(e+\|v\|_\infty)^\alpha} \right)^{1-\frac{1}{r}-\frac{1}{rp}} \frac{1}{\ln(e+\|v\|_\infty)^{\frac{\alpha}{pr}-\frac{1}{r}}},$$

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$$\|v\|_{W^{2,r'}(\Omega)} \leq C \left(\frac{\|u\|_\infty^q}{\ln(e + \|u\|_\infty)^\beta} \right)^{1 - \frac{1}{r'} - \frac{1}{r'q}} \frac{1}{\ln(e + \|u\|_\infty)^{\frac{\beta}{qr'} - \frac{1}{r'}}}.$$

From the Sobolev embeddings, for $\dfrac{1}{r^*} = \dfrac{1}{r} - \dfrac{1}{N}$ with $r^* > N$, and $\dfrac{1}{r'^*} = \dfrac{1}{r'} - \dfrac{1}{N}$ with $r'^* > N$, we have

$$\|u\|_{W^{1,r^*}(\Omega)} \leq C \left(\frac{\|v\|_\infty^p}{\ln(e + \|v\|_\infty)^\alpha} \right)^{1 - \frac{1}{r} - \frac{1}{rp}} \frac{1}{\ln(e + \|v\|_\infty)^{\frac{\alpha}{pr} - \frac{1}{r}}}, \qquad (3.11)$$

$$\|v\|_{W^{1,r'^*}(\Omega)} \leq C \left(\frac{\|u\|_\infty^p}{\ln(e + \|u\|_\infty)^\beta} \right)^{1 - \frac{1}{r'} - \frac{1}{r'p}} \frac{1}{\ln(e + \|u\|_\infty)^{\frac{\beta}{qr'} - \frac{1}{r'}}}. \qquad (3.12)$$

From Morrey's Theorem, (see [4, Corollary 9.14]), there exists a constant C that only depends on Ω, r and N such that $\forall\, x_1, x_2 \in \Omega$

$$\left| u(x_1) - u(x_2) \right| \leq C|x_1 - x_2|^{1 - N/r^*} \|u\|_{W^{1,r^*}(\Omega)}. \qquad (3.13)$$

Hence, for all $x \in B(x_1, R) \subset \Omega$

$$\left| u(x_1) - u(x) \right| \leq C\, R^{2 - \frac{N}{r}} \|u\|_{W^{2,r}(\Omega)}. \qquad (3.14)$$

Similarly, there exists a constant C only dependent on Ω, r' and N such that $\forall\, x_1', x_2' \in \Omega$

$$\left| v(x_1') - v(x_2') \right| \leq C|x_1' - x_2'|^{1 - N/r'^*} \|v\|_{W^{1,r'^*}(\Omega)}. \qquad (3.15)$$

Hence, for all $x \in B(x_1', R') \subset \Omega$

$$\left| v(x_1') - v(x) \right| \leq C\, (R')^{2 - \frac{N}{r'}} \|v\|_{W^{2,r'}(\Omega)}. \qquad (3.16)$$

For now on, we shall argue by contradiction. Assume that there is a sequence of classical positive solutions $\{(u_k, v_k)\}$ to the equation (1.1) such that

$$\lim_{k \to \infty} \|(u_k, v_k)\| = \infty, \quad \text{where} \quad \|\cdot\| := \|\cdot\|_\infty.$$

Observe that from (3.11) and (3.12) it follows that $\{u_k\}$ is bounded if and only if that the sequence $\{v_k\}$ is bounded. Hence, the unboundedness

of the sequence $\{(u_k, v_k)\}$ implies that both sequences $\{u_k\}$ and $\{v_k\}$ are unbounded.

Therefore

$$\lim_{k \to \infty} \|(u_k, v_k)\| = \infty \implies \lim_{k \to \infty} \|u_k\| = \infty \text{ and } \lim_{k \to \infty} \|v_k\| = \infty.$$

From (2.1), we have that there exist $C, \delta > 0$ such that

$$\max_{\Omega \setminus \Omega_\delta} u_k \leq C \qquad \text{and} \qquad \max_{\Omega \setminus \Omega_\delta} v_k \leq C. \tag{3.17}$$

Let $x_k, x'_k \in \overline{\Omega}_\delta$ be such that

$$u_k(x_k) = \max_{\Omega_\delta} u_k = \max_{\Omega} u_k,$$

and

$$v_k(x'_k) = \max_{\Omega_\delta} v_k = \max_{\Omega} v_k,$$

By taking a subsequence if needed, we may assume that there exist $x_0, x'_0 \in \overline{\Omega}_\delta$ such that

$$\lim_{k \to \infty} x_k = x_0 \in \overline{\Omega}_\delta, \qquad and \quad d_0 := dist(x_0, \partial \Omega) \geq \delta > 0. \tag{3.18}$$

$$\lim_{k \to \infty} x'_k = x'_0 \in \overline{\Omega}_\delta, \qquad and \quad d'_0 := dist(x'_0, \partial \Omega) \geq \delta > 0. \tag{3.19}$$

Let us choose R_k and R'_k such that $B_k = B(x_k, R_k) \subset \Omega$, $B'_k = B(x'_k, R'_k) \subset \Omega$, and

$$u_k(x) \geq \frac{1}{2} \|u_k\| \quad \forall x \in B_k, \qquad \text{and} \qquad v_k(x) \geq \frac{1}{2} \|v_k\| \quad \forall x \in B'_k,$$

moreover, there exist $y_k \in \partial B_k$ and $y'_k \in \partial B'_k$ such that

$$u_k(y_k) = \frac{1}{2} \|u_k\|, \qquad v_k(y'_k) = \frac{1}{2} \|v_k\|. \tag{3.20}$$

Let us denote by

$$m_k := \min_{[\|v_k\|/2, \|v_k\|]} f, \qquad M_k := \max_{[0, \|v_k\|]} f,$$
$$m'_k := \min_{[\|u_k\|/2, \|u_k\|]} g, \qquad M'_k := \max_{[0, \|u_k\|]} g. \tag{3.21}$$

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Since f and g are increasing for all s, see (1.2), we have

$$m_k = f(\|v_k\|/2), \qquad M_k = f(\|v_k\|) = \frac{\|v_k\|^p}{\ln(e + \|v_k\|)^\alpha},$$

$$m_k' = g(\|u_k\|/2), \qquad M_k' = g(\|u_k\|) = \frac{\|u_k\|^q}{\ln(e + \|u_k\|)^\beta}. \tag{3.22}$$

Moreover, there exists a constant $C > 0$ such that

$$\frac{m_k}{M_k} \geq C, \qquad \frac{m_k'}{M_k'} \geq C. \tag{3.23}$$

Using (3.10) we obtain

$$\left(\int_\Omega |f(v_k)|^r \right)^{1/r} \leq C \left(\frac{\|v_k\|^p}{\ln(e + \|v_k\|)^\alpha} \right)^{1 - \frac{1}{r} - \frac{1}{pr}} \frac{1}{\ln(e + \|v_k\|)^{\frac{\alpha}{pr} - \frac{1}{r}}}$$

$$= C\, M_k^{1 - \frac{1}{r} - \frac{1}{pr}} \frac{1}{\ln(e + \|v_k\|)^{\frac{\alpha}{pr} - \frac{1}{r}}}$$

$$\left(\int_\Omega |g(u_k)|^{r'} \right)^{1/r'} \leq C \left(\frac{\|u_k\|^q}{\ln(e + \|u\|_\infty)^\beta} \right)^{1 - \frac{1}{r'} - \frac{1}{qr'}} \frac{1}{\ln(e + \|u_k\|)^{\frac{\beta}{qr'} - \frac{1}{r'}}}$$

$$= C\, (M_k')^{1 - \frac{1}{r'} - \frac{1}{qr'}} \frac{1}{\ln(e + \|u_k\|)^{\frac{\beta}{qr'} - \frac{1}{r'}}}.$$

It follows from elliptic regularity, see (3.11) and (3.12), that

$$\|u_k\|_{W^{2,r}(\Omega)} \leq C\, M_k^{1 - \frac{1}{r} - \frac{1}{rp}} \frac{1}{\ln(e + \|v_k\|)^{\frac{\alpha}{pr} - \frac{1}{r}}} \tag{3.24}$$

$$\|v_k\|_{W^{2,r'}(\Omega)} \leq C\, (M_k')^{1 - \frac{1}{r'} - \frac{1}{r'q}} \frac{1}{\ln(e + \|u_k\|)^{\frac{\beta}{qr'} - \frac{1}{r'}}} \tag{3.25}$$

Using Morrey's Theorem, (3.24) and (3.25), we get

$$\left| u_k(x) - u_k(x_k) \right| \leq C\, \frac{R_k^{2 - \frac{N}{r}} M_k^{1 - \frac{1}{r} - \frac{1}{rp}}}{\ln(e + \|v_k\|)^{\frac{\alpha}{pr} - \frac{1}{r}}}, \qquad \text{for any } x \in B_k, \tag{3.26}$$

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and

$$\left| v_k(x) - v_k(x_k) \right| \leq C \, \frac{(R'_k)^{2-\frac{N}{r'}} \, (M'_k)^{1-\frac{1}{r'}-\frac{1}{r'q}}}{\ln(e + \|u_k\|)^{\frac{\beta}{qr'}-\frac{1}{r'}}}, \qquad \text{for any } x \in B'_k \, . \quad (3.27)$$

Taking $x = y_k$ in the inequality (3.26) and from (3.20) we obtain

$$C \, R_k^{2-\frac{N}{r}} \, M_k^{1-\frac{1}{r}-\frac{1}{rp}} \, \frac{1}{\ln(e + \|v_k\|)^{\frac{\alpha}{pr}-\frac{1}{r}}} \geq \left| u_k(y_k) - u_k(x_k) \right| = \frac{1}{2} \|u_k\|,$$

hence,

$$R_k^{2-\frac{N}{r}} \geq C \frac{\|u_k\|}{M_k^{1-\frac{1}{r}-\frac{1}{rp}}} \, \ln(e + \|v_k\|)^{\frac{\alpha}{pr}-\frac{1}{r}}. \quad (3.28)$$

Similarly,

$$(R'_k)^{2-\frac{N}{r'}} \geq C \frac{\|v_k\|}{(M'_k)^{1-\frac{1}{r'}-\frac{1}{qr'}}} \, \ln(e + \|u_k\|)^{\frac{\beta}{qr'}-\frac{1}{r'}}. \quad (3.29)$$

Now, using (3.9), (3.29) and since definition of m_k, we have that

$$C \geq \int_\Omega v_k \frac{f(v_k)}{\ln(e + v_k)} \geq \int_{B'_k} v_k \frac{f(v_k)}{\ln(e + v_k)} \geq C \, \|v_k\| \frac{m_k}{\ln(e + \|v_k\|)} (R'_k)^N$$

$$\geq C \, \|v_k\| \frac{m_k}{\ln(e + \|v_k\|)} \left(\frac{\|v_k\|}{(M'_k)^{1-\frac{1}{r'}-\frac{1}{qr'}}} \, \ln(e + \|u_k\|)^{\frac{\beta}{qr'}-\frac{1}{r'}} \right)^{\frac{1}{\frac{2}{N}-\frac{1}{r'}}}$$

$$= C \, \frac{\|v_k\|}{\ln(e + \|v_k\|)} \frac{m_k}{M'_k} \left(\frac{\|v_k\|}{(M'_k)^{1-\frac{2}{N}-\frac{1}{qr'}}} \, \ln(e + \|u_k\|)^{\frac{\beta}{qr'}-\frac{1}{r'}} \right)^{\frac{1}{\frac{2}{N}-\frac{1}{r'}}} . \quad (3.30)$$

Similarly,

$$C \geq \int_\Omega u_k \frac{g(u_k)}{\ln(e + u_k)} \geq \int_{B_k} u_k \frac{g(u_k)}{\ln(e + u_k)} \geq C \, \|u_k\| \frac{m'_k}{\ln(e + \|u_k\|)} (R_k)^N$$

$$\geq C \, \|u_k\| \frac{m'_k}{\ln(e + \|u_k\|)} \left(\frac{\|u_k\|}{M_k^{1-\frac{1}{r}-\frac{1}{pr}}} \, \ln(e + \|v_k\|)^{\frac{\alpha}{pr}-\frac{1}{r}} \right)^{\frac{1}{\frac{2}{N}-\frac{1}{r}}}$$

$$= C \, \frac{\|u_k\|}{\ln(e + \|u_k\|)} \frac{m'_k}{M_k} \left(\frac{\|u_k\|}{M_k^{1-\frac{2}{N}-\frac{1}{pr}}} \, \ln(e + \|v_k\|)^{\frac{\alpha}{pr}-\frac{1}{r}} \right)^{\frac{1}{\frac{2}{N}-\frac{1}{r}}} . \quad (3.31)$$

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Multiplying both inequalities (3.30) and (3.31), using the definitions of m_k, m'_k, M_k and M'_k, and also taking into account (3.23), we have that

$$C \geq \|u_k\|^a \|v_k\|^b \ln(e + \|u_k\|)^c \ln(e + \|v_k\|)^d, \qquad (3.32)$$

where

$$a := 1 + \frac{1}{\frac{2}{N} - \frac{1}{r}} - \frac{q(1 - \frac{2}{N}) - \frac{1}{r'}}{\frac{2}{N} - \frac{1}{r'}}, \qquad b := 1 + \frac{1}{\frac{2}{N} - \frac{1}{r'}} - \frac{p(1 - \frac{2}{N}) - \frac{1}{r}}{\frac{2}{N} - \frac{1}{r}},$$

$$c := -1 + \frac{\beta(1 - \frac{2}{N} - \frac{1}{r'q}) + \frac{\beta}{r'q} - \frac{1}{r'}}{\frac{2}{N} - \frac{1}{r'}}, \qquad d := -1 + \frac{\alpha(1 - \frac{2}{N} - \frac{1}{rp}) + \frac{\alpha}{rp} - \frac{1}{r}}{\frac{2}{N} - \frac{1}{r}}.$$

straightforward calculations lead to

$$a = \frac{1}{\frac{2}{N} - \frac{1}{r}} - \frac{q(\frac{N-2}{N}) - \frac{2}{N}}{\frac{2}{N} - \frac{1}{r'}}, \qquad b = \frac{1}{\frac{2}{N} - \frac{1}{r'}} - \frac{p(\frac{N-2}{N}) - \frac{2}{N}}{\frac{2}{N} - \frac{1}{r}},$$

$$c = \frac{\beta(1 - \frac{2}{N}) - \frac{2}{N}}{\frac{2}{N} - \frac{1}{r'}}, \qquad d = \frac{\alpha(1 - \frac{2}{N}) - \frac{2}{N}}{\frac{2}{N} - \frac{1}{r}}.$$

Observe that

$$a = 0 \iff \frac{2}{N} - \frac{1}{r'} = \left(\frac{2}{N} - \frac{1}{r} \right) \left[q \left(\frac{N-2}{N} \right) - \frac{2}{N} \right]$$

$$= \left(\frac{2}{N} - \frac{1}{r} \right) \left[(q+1) \frac{N-2}{N} - 1 \right]$$

$$= \left(\frac{2}{N} - \frac{1}{r} \right) \left[\frac{1}{1 - \theta} - 1 \right]$$

$$= \left(\frac{2}{N} - \frac{1}{r} \right) \frac{q+1}{p+1}.$$

Fix a small $\delta_0 > 0$ to be specified later. Let us choose r such that $\frac{2}{N} - \frac{1}{r} = \delta_0 > 0$. Next, let us choose r' such that $\frac{2}{N} - \frac{1}{r'} = \delta_0 \frac{q+1}{p+1}$. Taking $\delta_0 = \frac{1}{2} \min \left\{ \frac{1}{N}, \left(\frac{p+1}{q+1} \right) \frac{1}{N} \right\} > 0$, then $r, r' \in (\frac{N}{2}, N)$ and satisfy the inequalities $r > 1 + \frac{1}{p}$, $r' > 1 + \frac{1}{q}$, as required, and $a = 0$.

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With similar calculations as above, we get that $b = 0$ if and only if $\frac{2}{N} - \frac{1}{r} = \left(\frac{2}{N} - \frac{1}{r'}\right)\frac{p+1}{q+1}$, which holds if and only if $a = 0$.

Since by assumption $\alpha > \frac{2}{N-2}$ and $\beta > \frac{2}{N-2}$, one can see that $c > 0$ and $d > 0$.

Taking the limit as $k \to \infty$, we get that the right-hand side of inequality (3.32) goes to ∞, which leads to a contradiction. $\square$

3.2. **Proof of Theorem 1.3**

The proof is divided in two parts. We first prove that if there exists a positive solution $\big((\lambda, \mu), (u, v)\big)$ of equation (1.6) with $\lambda, \mu \geq 0$, then $\lambda\mu < \lambda_1^2$. In the second part, we prove the converse.

Part I. Assume that there exists a positive solution $\big((\lambda, \mu), (u, v)\big)$ of equation (1.6), and that $\lambda, \mu \geq 0$. Let $\phi_1 > 0$ be the principal eigenfunction associated to λ_1 and normalized in the $L^2(\Omega)$ norm, multiplying each equation of (1.6) by ϕ_1, and integrating by parts on Ω, it yields that

$$\lambda_1 \int_\Omega u\phi_1 = \lambda \int_\Omega v\phi_1 + \int_\Omega f(v)\,\phi_1$$
$$\lambda_1 \int_\Omega v\phi_1 = \mu \int_\Omega u\phi_1 + \int_\Omega g(u)\,\phi_1.$$

Multiplying the first equation by λ_1, the second equation by λ and adding both equations we deduce

$$(\lambda_1^2 - \lambda\mu) \int_\Omega u\,\phi_1 = \int_\Omega [\lambda_1 f(v) + \lambda g(u)]\,\phi_1 > 0.$$

Thus, $\lambda\mu < \lambda_1^2$.

Part II. Assume that $\lambda, \mu \geq 0$ and $\lambda\mu < \lambda_1^2$, we will prove that there is a positive solution (u, v) of (1.6). The proof is divided in three steps. In step 1, we reformulate problem (1.6) in abstract (operators) setting. In step 2, we fix one parameter, say, $\lambda = \lambda_0 > 0$ and choosing μ as bifurcation parameter, we use Crandall-Rabinowitz's Theorem to prove that when $\sqrt{\lambda_0\mu} = \lambda_1$ there is a bifurcation phenomena from the trivial solution to positive solution. In step 3 we use the global bifurcation result stated by Rabinowitz [35] and

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completed by Dancer [13], (see also [14, 24]), to prove that for any (λ, μ) satisfying $\lambda\mu < \lambda_1^2$, with $\lambda, \mu \geq 0$, equation (1.6) has at least one positive solution. We conclude with a remark on the fact that varying λ we obtain a whole curve of non-isolated bifurcation points, and using Alexander and Antman's result [1], we can deduce that in a neighborhood of that curve, there is a bifurcating two-dimensional surface of nontrivial solution pairs $\big((\lambda, \mu), (u, v)\big)$ of equation (1.6).

Step1. We start by reformulating problem (1.6).

Let $\overline{f}$ and $\overline{g}$ be the extension of f and g, defined by

$$\overline{f}(t) = \frac{|t|^p}{\big[\ln(e + |t|)\big]^\alpha}, \qquad \overline{g}(s) = \frac{|s|^q}{\big[\ln(e + |s|)\big]^\beta}, \qquad \text{for} \quad s, t \leq 0, \quad (3.33)$$

and denote by $F(w) := \begin{pmatrix} \overline{f}(v) \\ \overline{g}(u) \end{pmatrix}$. Then (1.6) can be extended to non-positive and changing sign solutions and can be rewritten

$$-\Delta w = Aw + F(w), \quad \text{in } \Omega, \qquad w = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{on } \partial\Omega, \qquad (3.34)$$

where $A := \begin{pmatrix} 0 & \lambda \\ \mu & 0 \end{pmatrix}$, $w := \begin{pmatrix} u \\ v \end{pmatrix}$, and any positive solution (u, v) of (3.34) is a positive solution of (1.6) and conversely.

Following the same ideas used in the proof of Theorem 1.1, it can be easily checked that for any $[a, b] \times [c, d] \subset \mathbb{R}_+^2$, there exists a constant $C > 0$ such that any positive solution (u, v) of equation (3.34) satisfy

$$\|u\|_{L^\infty(\Omega)} \leq C, \quad \|v\|_{L^\infty(\Omega)} \leq C, \qquad \forall (\lambda, \mu) \in [a, b] \times [c, d]. \qquad (3.35)$$

Assume that $\lambda, \mu > 0$ and consider the Jordan canonical form of the matrix A, we can decompose $A = P^{-1}JP$, where

$$J = \begin{pmatrix} \sqrt{\lambda\mu} & 0 \\ 0 & -\sqrt{\lambda\mu} \end{pmatrix}, \ P = \frac{1}{2}\begin{pmatrix} \sqrt{\frac{\mu}{\lambda}} & 1 \\ -\sqrt{\frac{\mu}{\lambda}} & 1 \end{pmatrix}, \ P^{-1} = \begin{pmatrix} \sqrt{\frac{\lambda}{\mu}} & -\sqrt{\frac{\lambda}{\mu}} \\ 1 & 1 \end{pmatrix}. \qquad (3.36)$$

Multiplying (3.34) by P on the left and denoting by $z = Pw$, we obtain

$$-\Delta z = Jz + G(z), \quad \text{in } \Omega, \qquad z = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{on } \partial\Omega, \qquad (3.37)$$

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where $G(z) := PF(w) = PF(P^{-1}z)$.

Step 2. We check that the conditions of Crandall-Rabinowitz's Theorem [12] are satisfied. Fix $\lambda = \lambda_0 > 0$, and choose μ as the bifurcation parameter. Let $\sigma \in (0,1)$, define $E_2 = \{u \in C^{2,\sigma}(\overline{\Omega}) : u = 0 \text{ on } \partial\Omega\}$ equipped with its standard norm, E_2 is a Banach space. Set $E_0 = C^{\sigma}(\overline{\Omega})$, define the following operators $\mathbf{F} : \mathbb{R} \times (E_2)^2 \to (C^{\sigma}(\overline{\Omega}))^2$ by

$$
\mathbf{F}(\mu, w) := -\Delta w - \begin{pmatrix} 0 & \lambda_0 \\ \mu & 0 \end{pmatrix} w - F(w),
$$

$$
L_0 w := D_{(u,v)}\mathbf{F}(\mu_0, 0) w = -\Delta w - \begin{pmatrix} 0 & \lambda_0 \\ \mu_0 & 0 \end{pmatrix} w,
$$

$$
L_1 w := D_{\mu,(u,v)}\mathbf{F}(\mu_0, 0) w = \begin{pmatrix} 0 \\ -v. \end{pmatrix}, \qquad \text{where} \quad w = \begin{pmatrix} u \\ v \end{pmatrix}.
$$

Set $\mu_0 = \frac{\lambda_1^2}{\lambda_0}$, and $P_0 = P(\lambda_0, \mu_0)$. Observe that $w \in N(L_0)$ (where $N(L_0)$ is the kernel of L_0) if and only if $z = P_0 w \in N\left(-\Delta - \begin{pmatrix} \lambda_1 & 0 \\ 0 & -\lambda_1 \end{pmatrix}\right) = span\left[\begin{pmatrix} \phi_1 \\ 0 \end{pmatrix}\right]$. Therefore, $N(L_0) = span\left[\begin{pmatrix} \lambda_0\phi_1 \\ \lambda_1\phi_1 \end{pmatrix}\right]$.

Now, we claim that $L_1(N(L_0)) \not\subset R(L_0)$ where $R(L_0)$ is the range of L_0. Indeed, assume that there exist $w \in N(L_0)$ and $\psi = \begin{pmatrix} \psi_1 \\ \psi_2. \end{pmatrix} \in D(L_0)$ such that $L_1 w = L_0 \psi$ or equivalently, by definition of L_0 and L_1,

$$
-\Delta\psi - \begin{pmatrix} 0 & \lambda_0 \\ \mu_0 & 0 \end{pmatrix}\psi = a\begin{pmatrix} 0 \\ -\lambda_1\phi_1 \end{pmatrix}, \qquad \text{for some} \quad a \in \mathbb{R}. \tag{3.38}
$$

Multiplying (3.38) on the left by P_0, and denoting by $\varphi = P_0\psi = \begin{pmatrix} \varphi_1 \\ \varphi_2. \end{pmatrix}$, we obtain

$$
-\Delta\varphi - \begin{pmatrix} \lambda_1 & 0 \\ 0 & -\lambda_1 \end{pmatrix}\varphi = -\frac{a\lambda_1}{2}\begin{pmatrix} \phi_1 \\ \phi_1 \end{pmatrix}. \tag{3.39}
$$

Multiplying the first (component) equation by ϕ_1, integrating on Ω, and applying Green's formulae we obtain

$$
0 = \int_\Omega (-\Delta\varphi_1 - \lambda_1\varphi_1)\phi_1 = -\frac{a\lambda_1}{2}\int_\Omega \phi_1^2.
$$

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Therefore $a = 0$. Hence, the hypotheses of Crandall Rabinowitz theorem are satisfied. Thus, there exists a neighborhood of $\left(\frac{\lambda_1^2}{\lambda_0}, (0,0) \right)$ in $\mathbb{R} \times (E_2)^2$, and continuous functions $\mu(s), \tilde{w}(s)$, $s \in (-\epsilon, \epsilon)$, such that $\mu(0) = \mu_0$, $\tilde{w}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\tilde{w} = \begin{pmatrix} \tilde{w}_1 \\ \tilde{w}_2 \end{pmatrix}$, with $\int_\Omega \tilde{w}_i \phi_1 = 0$, and the only nontrivial solutions of (1.6) for $\lambda = \lambda_0$ fixed, are

$$\left(\mu(s), s \begin{pmatrix} \lambda_0 \phi_1 \\ \lambda_1 \phi_1 \end{pmatrix} + s\,\tilde{w}(s) \right).$$

Observe that for $s > 0$ small enough, $w = \begin{pmatrix} u \\ v \end{pmatrix}$ satisfy $u > 0, v > 0$, $\frac{\partial u}{\partial n} < 0$, $\frac{\partial v}{\partial n} < 0$ on $\partial \Omega$, hence $\lambda_0 \mu < \lambda_1^2$.

Step 3. Now, we use the global bifurcation Theorem as stated by Rabinowitz [35] and as completed by Dancer [13]. Let $(-\Delta)^{-1}$ denote the inverse of $(-\Delta)$ with homogeneous Dirichlet boundary conditions. It follows from Schauder estimates that $(-\Delta)^{-1}$ maps bounded subsets of E_0 into bounded subsets of E_2, which in turn are relatively compact in E_0. Thus, $(-\Delta)^{-1} : E_0 \to E_0$ is compact.

Observe that, fixed points of the operator $(-\Delta)^{-1} [J(.) + G(.)]$ corresponds to fixed points of the operator $(-\Delta)^{-1} [A(.) + F(.)]$, that is,

$$z = (-\Delta)^{-1} [Jz + G(z)] \iff w = (-\Delta)^{-1} [Aw + F(w)].$$

Let us keep fixed $\lambda = \lambda_0 > 0$, and allow μ to vary. It follows from Rabinowitz's global bifurcation Theorem [35, Theorem 1.3] that there is a continuum of solutions, emanating from the trivial solution at $(\lambda, \mu) = \left(\lambda_0, \frac{\lambda_1^2}{\lambda_0} \right)$, which is either unbounded, or meets another bifurcation point from the trivial solution. Let

$$\mathcal{C}_{\lambda_0} := \left\{ \left((\lambda_0, \mu), (u_{\lambda_0, \mu}, v_{\lambda_0, \mu}) \right) \in \mathbb{R}^2 \times \left(C^\sigma(\overline{\Omega}) \right)^2 \right\}$$

be the continuum emanating from the trivial solution at $\mu = \mu_0 = \frac{\lambda_1^2}{\lambda_0}$ and solving (1.6) for $\lambda = \lambda_0$ fixed. By elliptic regularity, it is known that $\mathcal{C}_{\lambda_0} \subset \mathbb{R}^2 \times \left(C^{2,\sigma}(\overline{\Omega}) \right)^2$.

Considering the positive cone $\mathcal{P} := \{ u \in C^{1,\sigma}(\overline{\Omega}) : u > 0, \text{ in } \Omega, \frac{\partial u}{\partial n} < 0, \text{ on } \partial\Omega \}$, let us denote by $\mathcal{C}_{\lambda_0}^+ := \mathcal{C}_{\lambda_0} \cap \mathbb{R}^2 \times (\mathcal{P})^2 \neq \emptyset$. Since the classical positive solutions are a priori bounded, see (3.35), we have that $\mathcal{C}_{\lambda_0}^+ \cap$

$\left(\{\lambda_0\} \times \left[0, \frac{\lambda_1^2}{\lambda_0} \right) \times \left(C^\sigma(\overline{\Omega}) \right)^2 \right)$ is bounded. Assume that there exists $\mu = \mu^* \in \left[0, \frac{\lambda_1^2}{\lambda_0} \right)$ such that $\left((\lambda_0, \mu^*), (u^*, v^*) \right) \in \overline{\mathcal{C}_{\lambda_0}^+} \setminus \mathcal{C}_{\lambda_0}^+$, then either $(u^*, v^*) = (0, 0)$ or $u^* \geq 0, v^* \geq 0$ in Ω, $\frac{\partial u^*}{\partial n} \leq 0, \frac{\partial v^*}{\partial n} \leq 0$ on $\partial\Omega$, with $(u^*, v^*) \neq (0, 0)$. If $(u^*, v^*) = (0, 0)$ then $\left((\lambda_0, \mu^*), (u^*, v^*) \right)$ is a bifurcation point from the trivial solution to positive solutions. Due to the unique bifurcation point from the trivial solution to positive solutions at $\lambda = \lambda_0$ is attained at $\mu = \frac{\lambda_1^2}{\lambda_0}$, if $(u^*, v^*) = (0, 0)$ then we reach a contradiction. On the other hand, if $u^* \geq 0, v^* \geq 0$ in Ω, $(u^*, v^*) \neq (0, 0)$, from the Maximum Principle, and the Hopf Maximum Principle $u^* > 0, v^* > 0$ in Ω, and $\frac{\partial u^*}{\partial n^*} < 0, \frac{\partial v}{\partial n} < 0$, therefore $\left((\lambda_0, \mu^*), (u^*, v^*) \right) \in \mathcal{C}_{\lambda_0}^+$, which contradicts the hypothesis. $\square$

Remark 3.1. *Let us mention that when moving λ we obtain a whole curve of non-isolated bifurcation points. If $\mathcal{S}$ denote the closure of the set of nontrivial solutions pairs $\left((\lambda, \mu), w \right)$ of (3.34), and $\mathcal{F}$ denote the set of bifurcation points of (3.34) from the trivial solution. We proved in Step 2 that the set*

$$\mathcal{F}_1 := \left\{ \left(\left(\lambda, \frac{\lambda_1^2}{\lambda} \right), (0, 0) \right) : \lambda > 0 \right\}$$

is a set of bifurcation points of (3.34) from the trivial solution. All points in $\mathcal{F}_1$ are non-isolated bifurcation points. Using Alexander and Antman's result [1], we can deduce that in a neighborhood of that curve, there is a bifurcating two-dimensional surface of nontrivial solution pairs $\left((\lambda, \mu), (u, v) \right)$ of equation (1.6).

Acknowledgments

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