6-1-2009

Cutting A Pie Is Not A Piece Of Cake

J. B. Barbanel

S. J. Brams

Walter Stromquist
Swarthmore College, wstromq1@swarthmore.edu

Recommended Citation

J. B. Barbanel, S. J. Brams, and Walter Stromquist. (2009). "Cutting A Pie Is Not A Piece Of Cake". *American Mathematical Monthly*. Volume 116, Issue 6. 496-514. DOI: 10.4169/193009709x470407
https://works.swarthmore.edu/fac-math-stat/31

Cutting a Pie Is Not a Piece of Cake

Julius B. Barbanel, Steven J. Brams, and Walter Stromquist

We dedicate this paper to the memory of David Gale.

1. INTRODUCTION. The general problem of fair division and the specific problem of cutting a cake fairly have received much attention in recent years (for overviews, see [**1**], [**2**], [**3**], [**5**], [**6**], [**10**]). Cutting a pie into wedge-shaped sectors has received far less attention, though it would seem that the connection between cake-cutting and pie-cutting is close ([**4**], [**14**]). Roughly speaking, if a cake is represented by a line segment, then it becomes a pie when its endpoints are connected to form a circle. Pie-cutting might be applied to the division of an island's shoreline into connected lots, to the allocation of land around an oil strike, or to the assignment of "on call" periods in a daily cycle.

Suppose each of n players attributes values to pieces of cake or to sectors of pie. We ask whether it is always possible to divide a cake into n connected pieces with parallel, vertical cuts, or a pie into n sectors with radial cuts from the center, and assign one piece or one sector to each player in a way that is "envy-free," whereby each player thinks he or she receives at least a tied-for-largest portion and so does not envy any other player. If so, can we arrange that the resulting envy-free allocation is "undominated," meaning that there is no other allocation that is better for at least one player and not worse for the other players? These questions were posed by David Gale ([**8**]) about fifteen years ago. While he gave an affirmative answer for cakes (assuming a strong kind of continuity), he asked whether envy-free, undominated allocations are also possible for pies.

We begin by discussing cake-cutting in Section 2, where we present Gale's result and then show that when there are three or more players, an envy-free allocation of a cake may fail to be undominated unless this strong version of continuity is assumed. (This assumption is implicit, but not explicit, in Gale's result.) Even with this assumption, however, we show in Section 3 (again for three or more players) that there may be no envy-free, undominated allocation for a pie, which makes pie-cutting harder—not a piece of cake. We extend our results for pie-cutting in Sections 4 and 5 and draw some conclusions in Section 6, ending with two open questions.

To make the questions we pose precise, we introduce some mathematical formalism. We represent a cake by the half-open interval $[0, 1)$, and we represent pieces of the cake by subintervals $[\alpha, \beta)$ with $0 \le \alpha \le \beta \le 1$. Let's deem a pie mathematically equivalent to the circle $S^1 = R/Z$ or, equivalently, to the interval $[0, 1]$ with its endpoints identified. (It will sometimes be more convenient for us to consider the pie to be some other interval.) We wish to introduce notation for sectors of the pie. For any α and β with $0 \le \alpha \le 1$ and $\alpha \le \beta \le \alpha + 1$, we let $[a, \beta)$ denote the sector from α to β, with the value of β being interpreted mod 1. Thus, for example, $[1/3, 2/3) = \{x \in S^1 : 1/3 \le x < 2/3\}$ and $[2/3, 4/3) = \{x \in S^1 : 2/3 \le x < 4/3\} = \{x \in S^1 : 2/3 \le x < 1\} \cup \{x \in S^1 : 0 \le x < 1/3\}$. We note that for any α, $[\alpha, \alpha)$ denotes the empty sector and $[\alpha, \alpha + 1)$ denotes the entire pie, and the complement of the sector $[\alpha, \beta)$ is the sector $[\beta, \alpha + 1)$ if $\beta \le 1$, and $[\beta - 1, \alpha)$ if $\beta > 1$. For ease of notation, we will always use $[\beta, \alpha + 1)$ to denote the complement of $[\alpha, \beta)$, even if $\beta > 1$.

In order to assess the values of pieces of cake or sectors of pie, let's assume that player i uses a finitely additive measure v_i, so that $v_i(S)$ is the value of piece S to player i. If $S = [\alpha, \beta)$, then we write $v_i(S) = v_i(\alpha, \beta)$. We always assume that every measure v_i is continuous as a function of α and β. This corresponds to the intuitive notion that, as one endpoint of some sector moves continuously through the interval $[0, 1)$, each player views the value of that sector as changing continuously. We also assume that every measure assigns value 1 to the whole cake or to the whole pie. It is understood that different players may operate with different measures.

We say that players' measures are *absolutely continuous with respect to one another* if, whenever one player assigns value 0 to a particular piece of a cake or to a sector of a pie, all players do so. We do not always make this assumption, but when we do, we might as well contract to a point each piece or sector to which all the players assign value 0, so there is no piece or sector of positive length to which any player assigns value 0.

To state Gale's question, let's call an allocation of pieces of a cake or sectors of a pie among players

- *envy-free* if no player prefers another piece or sector to his or her own;
- *undominated* if no other allocation gives each player at least as much value according to his or her measure as he or she had in the original allocation, and gives one player strictly more value.

If the players' measures are absolutely continuous with respect to one another, then we can give a stronger definition of "undominated." An allocation is *undominated* if and only if no other allocation gives *every* player strictly more value according to his or her measure than he or she had in the original allocation. (If one player receives strictly more value, the absolute continuity of the measures with respect to one another allows that player to "spread" some of its value to all the other players to make a new allocation that gives each player a slightly larger piece.) Without assuming that the measures are absolutely continuous with respect to one another, this strengthening does not work.

We emphasize that "undominated" here means "undominated with respect to other allocations into connected pieces (for cakes) or sectors (for pies), one per player," which are the only types of allocations we consider in this paper. It is certainly possible that an allocation can be dominated by an allocation that assigns to some player disjoint pieces or sectors.

Gale's query ([**8**]) is simple: Does there always exist an envy-free and undominated allocation of a cake or pie? We answer this question for cakes in Section 2 and for pies in later sections.

For two players, the answer to Gale's question with respect to cake and with respect to pie is affirmative. We prove this for cake in Section 2 and for pie in Section 4.

For three or more players, the answer to Gale's question is:

- affirmative for cake, if the players' measures are absolutely continuous with respect to one another. This result follows from Gale's result, and we present it in Section 2.
- negative for cake, if we do not require that the players' measures be absolutely continuous with respect to one another. We give examples in Section 2 for different cases with three or more players.
- negative for pie, regardless of any assumption about the absolute continuity of the players' measures with respect to one another. We give examples in Section 3 for all cases with three or more players.

While our main focus is on the existence of allocations that are envy-free and undominated, we shall also consider the existence of allocations that are equitable: an allocation is *equitable* if all players assign exactly the same value (in their respective measures) to the pieces or sectors they receive (and so no player envies another's "degree of happiness").

Finding procedures, sometimes only approximate ([**13**]), for producing certain desirable allocations—as opposed to merely demonstrating that such allocations exist—is a central concern in the fair-division literature. In Section 5 we present two "moving-knife" procedures for pie-cutting.

2. CAKE-CUTTING. In this section we cut cakes. Our starting point is Gale's theorem [**8**] that when players' measures are absolutely continuous with respect to one another, *every* envy-free allocation is also undominated. This result, combined with well-known existence results for envy-freeness, tells us that if the players' measures are absolutely continuous with respect to one another, then there exists an allocation that is both envy free and undominated.

Theorem 2.1.

a. (*Gale* [**8**]). *Any envy-free allocation of a cake among two or more players whose measures are absolutely continuous with respect to one another is also undominated.*

b. *For two or more players and any cake and corresponding measures, if the measures are absolutely continuous with respect to one another, then there exists an allocation that is envy-free and undominated.*

Proof. For part a, let $\langle S_1, S_2, \dots, S_n\rangle$ be an envy-free allocation (where, for each $i = 1, 2, \dots, n$, player i receives piece S_i), and let $\langle T_1, T_2, \dots, T_n\rangle$ be any other allocation. If $\langle T_1, T_2, \dots, T_n\rangle$ consists of the same intervals as $\langle S_1, S_2, \dots, S_n\rangle$ but allocated to different players, then it is impossible that $v_i(T_i) > v_i(S_i)$ for any i (where v_i denotes player i's measure). This is because T_i is identical to some S_j, and $v_i(S_j) \le v_i(S_i)$ by envy-freeness. Therefore $\langle T_1, T_2, \dots, T_n\rangle$ does not dominate $\langle S_1, S_2, \dots, S_n\rangle$.

Now suppose that the intervals of $\langle T_1, T_2, \dots, T_n\rangle$ differ from the intervals of $\langle S_1, S_2, \dots, S_n\rangle$. In this case, some interval of $\langle T_1, T_2, \dots, T_n\rangle$ must be strictly contained in some interval of $\langle S_1, S_2, \dots, S_n\rangle$. (This statement is false for pies!) Suppose that $T_j \subseteq S_i$ with $T_j \ne S_i$. Now $v_j(T_j) < v_j(S_i)$ by the absolute continuity of the measures with respect to one another, and $v_j(S_i) \le v_j(S_j)$ by envy-freeness. It follows that $v_j(T_j) < v_j(S_j)$, and hence the allocation $\langle T_1, T_2, \dots, T_n\rangle$ does not dominate the allocation $\langle S_1, S_2, \dots, S_n\rangle$.

Part b follows from part a and the well-known fact (see, for example, [**11**], [**13**], and references therein) that, in this context, envy-free allocations always exist. ∎

The assumption that the players' measures are absolutely continuous with respect to one another is necessary for Gale's theorem. Envy-free allocations exist in any case, but if the measures are not absolutely continuous with respect to one another, then they need not be undominated.

Theorem 2.2. *For three or more players with measures that are not absolutely continuous with respect to one another, there need not exist an allocation that is envy-free and undominated.*

Proof. First suppose that there are three players, and let the cake be the interval $[0, 1)$. Consider measures for players A, B, C as follows (where we observe that the 0 entry for player A shows that players' measures are not absolutely continuous with respect to one another):

	$[0, 1/6)$	$[1/6, 1/3)$	$[1/3, 1)$
Player A	$\frac{1}{3}$	0	$\frac{2}{3}$
Player B	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{3}$
Player C	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{3}$

Each player's measure is uniform on each of the three segments shown. For example, $v_A(0, 1/12)$, $v_B(0, 1/12)$, and $v_C(0, 1/12)$ are $1/6$, $1/12$, and $1/12$, respectively. Note that B's and C's measures are uniform on the entire cake, and that all of the measures are the same on $[1/3, 1)$.

Consider an allocation $\langle S_A, S_B, S_C \rangle$. We will show that $\langle S_A, S_B, S_C \rangle$ is not both envy-free and undominated.

First note that for $\langle S_A, S_B, S_C \rangle$ to be envy-free, we must have $v_i(S_i) \geq 1/3$ for each $i = \text{A}, \text{B}, \text{C}$. That is, each player must receive at least $1/3$ according to its own measure.

Now suppose that player A receives the leftmost piece: $S_A = [0, x)$ for some x. If $x > 1/3$, then there is not enough cake left over for B and C to have at least $1/3$ each. If $x < 1/3$, then B and C must divide the remainder of the cake equally in order that the allocation be envy-free; if, say, C receives the rightmost piece, then $v_A(S_A) \leq 1/3$, but $v_A(S_C) = (1 - x)/2 > 1/3$, again violating envy-freeness. If $x = 1/3$, then the pieces are $[0, 1/3)$, $[1/3, 2/3)$, and $[2/3, 1)$. This allocation is envy-free, but it is dominated by the allocation $\langle T_A, T_B, T_C \rangle$ where $T_A = [0, 1/6)$, $T_B = [1/6, 7/12)$, and $T_C = [7/12, 1)$, since $v_A(S_A) = 1/3$, $v_B(S_B) = 1/3$, and $v_C(S_C) = 1/3$, but $v_A(T_A) = 1/3$, $v_B(T_B) = 5/12$, and $v_C(T_C) = 5/12$.

If some other player has the leftmost piece, then again envy-freeness requires that the pieces be $[0, 1/3)$, $[1/3, 2/3)$, and $[2/3, 1)$, and again the allocation is dominated by the one given above.

If there are n players for some $n > 3$, the same approach works, with the players' measures given by the following table:

	$[0, 1/2n)$	$[1/2n, 1/n)$	$[1/n, 1)$
Player A	$\frac{1}{n}$	0	$\frac{n-1}{n}$
All other players	$\frac{1}{2n}$	$\frac{1}{2n}$	$\frac{n-1}{n}$

The proof is analogous to that for three players. ■

The case of two players is excluded from the preceding theorem, and indeed it is special.

Theorem 2.3. *For two players and any cake and corresponding measures,*

a. *there exists an allocation that is both envy-free and undominated;*
b. *there exists an allocation that is both envy-free and equitable; and*
c. *if the measures are absolutely continuous with respect to one another, then there exists an allocation that is envy-free, undominated, and equitable.*

Proof. Choose x so that $v_A(0, x) = v_B(x, 1)$. We can always find such an x, because when $x = 0$, $v_A(0, x) < v_B(x, 1)$, and when $x = 1$, the reverse inequality holds. Continuity and the intermediate value theorem then guarantee that for some x, $v_A(0, x) = v_B(x, 1)$. Giving one of these pieces to each player (and denoting player A's piece by S_A and player B's piece by S_B), we can assume that $v_A(S_A) = v_B(S_B) \geq 1/2$, because if not we can simply exchange pieces to make this true. This establishes part b. Part c then follows from Theorem 2.1a.

For part a, we note that if the allocation $\langle S_A, S_B \rangle$ above is not undominated, then Theorem 2.1a implies that the players' measures are not absolutely continuous with respect to one another. In particular, if $v_A(S_A) = v_B(S_B) > 1/2$, then it must be possible to move x to the left or to the right some positive distance and have one player's valuation of its piece increase while the other player's valuation of its piece does not change. If $v_A(S_A) = v_B(S_B) = 1/2$, then either the move just described is possible, or else it is possible after the players exchange pieces. In either case (after exchanging pieces if necessary), if we move x from its original position so as to produce the greatest increase in one player's valuation of its piece while not changing the other player's valuation of its piece, then the resulting allocation will be envy-free and undominated. This establishes part a. ∎

For two players with measures that are not absolutely continuous with respect to one another, the methods used in the proof of Theorem 2.3 make it clear that

- an envy-free and undominated allocation may fail to be equitable,
- an envy-free and equitable allocation may fail to be undominated, and
- there may be no allocation that is both undominated and equitable.

3. PIE-CUTTING: THREE OR MORE PLAYERS. In this section we show that when a pie is to be divided among three or more players, it may be impossible to find an allocation that is both envy-free and undominated, or one that is both envy-free and equitable. But one that is both equitable and undominated always exists. These results hold even if the players' measures are absolutely continuous with respect to one another.

Theorem 3.1. *For three or more players, there exist a pie and corresponding measures for which no allocation is envy-free and undominated.*

Proof. We give an example involving measures that are nearly uniform. We show that with these measures, no envy-free, undominated allocation is possible.

Since the measures are nearly uniform, it is easy to find allocations of the pie that are *almost* envy-free and *almost* undominated. We don't know whether it is possible to find examples in which the discrepancies are much larger. Another example for the case of $n = 3$ is given in **[12]**, but the measures are still nearly uniform and the discrepancies are still very small.

Fix $n \geq 3$ and label the players $1, 2, \ldots, n$. For this proof, we represent the pie as the interval $[0, n]$, with its endpoints identified. We also relax the requirement that each player's valuation of the entire pie be 1. (The requirement could be restored, at the cost of complicating our calculations, by rescaling our valuations.) We specify two constants for use throughout this section: $C = n^{-8}$, and $\varepsilon = n^{-16}$.

Figure 1 illustrates the players' measures. For $i = 2, \ldots, n$, define the ith player's *window* to be the interval $\left[i - \frac{7}{6}, i + \frac{10}{6}\right)$. The endpoints are defined mod n, so the last

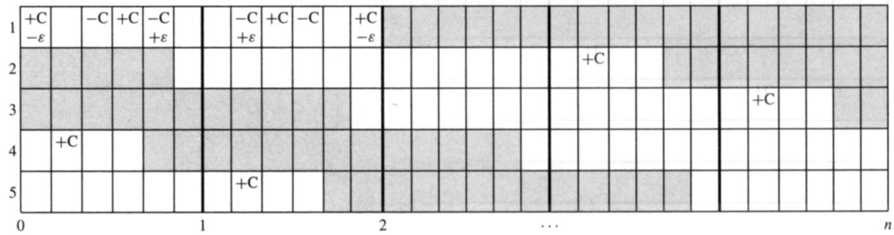

Figure 1. With these measures, no envy-free and undominated division is possible.

two windows actually end at $\frac{4}{6}$ and $\frac{10}{6}$. Define the first player's window to be $[0, 2)$. The windows are the white spaces in Figure 1.

With certain exceptions, the value of a piece to a player is

- 1 per unit length inside the player's window, and
- $1 - n^{-4}$ per unit length outside the player's window.

The exceptions are as follows:

For $i = 2, \ldots, n$, player i assigns an extra value of $C = n^{-8}$ to the segment $[i + \frac{7}{6}, i + \frac{8}{6})$, spread uniformly over that interval. We call this interval the player's *bonus cell.*

Player 1 has several positive and negative adjustments:

- $+C - \varepsilon$, uniformly over $[0, \frac{1}{6})$
- $-C$, uniformly over $[\frac{2}{6}, \frac{3}{6})$
- $+C$, uniformly over $\left[\frac{3}{6}, \frac{4}{6}\right)$
- $-C + \varepsilon$, uniformly over $\left[\frac{4}{6}, \frac{5}{6}\right)$
- $-C + \varepsilon$, uniformly over $\left[\frac{7}{6}, \frac{8}{6}\right)$
- $+C$, uniformly over $\left[\frac{8}{6}, \frac{9}{6}\right)$
- $-C$, uniformly over $\left[\frac{9}{6}, \frac{10}{6}\right)$
- $+C - \varepsilon$, uniformly over $\left[\frac{11}{6}, 2\right)$

The adjustments are in addition to the normal value of 1 per unit length. For example, the actual value of the segment $\left[0, \frac{1}{6}\right)$ to player 1 is $\frac{1}{6} + C - \varepsilon$, uniformly spread over the interval. We note that the players' measures are absolutely continuous with respect to one another.

By way of contradiction, suppose we have found an allocation $\langle S_1, S_2, \ldots, S_n \rangle$ that is both envy-free and undominated. As usual, S_i is the piece assigned to the ith player, and $v_i(S_i)$ is player i's valuation of his or her own piece. By the *values vector* we mean the vector $\langle v_1(S_1), \ldots, v_n(S_n) \rangle$, and by the *total value* of an allocation we mean the sum $v_1(S_1) + \cdots + v_n(S_n)$. We write d_i for the length of S_i.

We first show that the pieces must have a certain form. Each player's piece must be (mostly) within the player's window and have length (about) 1, and the pieces must be in order around the pie; that is, S_{i+1} is always immediately to the right of S_i. We make this precise in Lemmas 1 through 5.

Lemma 1. *Each piece has both value and length greater than $1 - n^{-4}$. That is, for each i, we have $d_i > 1 - n^{-4}$ and $v_i(S_i) > 1 - n^{-4}$. It follows that the total length L of any k consecutive pieces must satisfy $k - n^{-3} < L < k + n^{-3}$.*

Proof. Player 1's valuation of the entire pie is $v_1(0, n) = n - (n-2)(n^{-4}) = n - n^{-3} + 2n^{-4}$. Player 1's piece must have value

$$v_1(S_1) \geq \frac{v_1(0, n)}{n} = 1 - n^{-4} + 2n^{-5},$$

or else some other player's piece would have greater value than S_1 by player 1's measure. No piece can have value more than $C = n^{-8}$ greater than its length, so both the value and the length must be greater than $1 - n^{-4}$. The same reasoning applies to players $2, \ldots, n$, whose valuations of the entire pie are slightly larger. ■

Lemma 2. *The total value satisfies* $v_1(S_1) + \cdots + v_n(S_n) \geq n$.

Proof. This follows from the assumption that the allocation is undominated. Suppose by way of contradiction that $v_1(S_1) + \cdots + v_n(S_n) < n$. Define a new allocation $\langle T_1, \ldots, T_n \rangle$ by starting at 0 and making each piece T_i have length $v_i(S_i)$:

$$T_1 = [0, v_1(S_1))$$

and

$$T_i = [v_1(S_1) + \cdots + v_{i-1}(S_{i-1}), v_1(S_1) + \cdots + v_i(S_i))$$

for each i, except that to avoid wasting any of the pie we extend T_n to

$$T_n = [v_1(S_1) + \cdots + v_{n-1}(S_{n-1}), 1).$$

It follows from Lemma 1 that each T_i is within player i's window and avoids bonus cells (except for player 1, whose adjustments add to 0). So $v_i(T_i)$ is equal to the length of T_i, and in each case that means that $v_i(T_i) = v_i(S_i)$, except that T_n is longer, so $v_n(T_n) > v_n(S_n)$, and $\langle T_1, T_2, \ldots, T_n \rangle$ dominates $\langle S_1, S_2, \ldots, S_n \rangle$. ■

Lemma 3. *If any players' pieces include parts outside of the players' windows, then the total length of these parts cannot exceed* n^{-3}.

Proof. Define the *excess value* of each piece S_i to be $v_i(S_i)$ minus d_i, the length of S_i. From Lemma 2, the sum of the excess values must be at least 0. But no piece can contribute excess value greater than $C = n^{-8}$. Parts outside the players' windows contribute an excess of $-n^{-4}$ per unit length, so the total of all these lengths cannot exceed n^{-3}. ■

Lemma 4. *The players' pieces are in order, with* S_{i+1} *immediately to the right of* S_i *for each* i.

Proof. The ordering is forced by the arrangement of the windows. ■

Lemma 5. *For each* $i = 2, \ldots, n$*, player* i*'s piece (i.e.,* S_i*) cannot include any part of player* i*'s bonus cell.*

Proof. Lemma 1 implies that the pieces have length about 1, and hence they have about the same relative position within the players' windows. If player i's piece includes a bonus cell, then player 1's piece is forced to extend too far outside player 1's window. ■

We are now ready to isolate the possibilities for $\langle S_1, \ldots, S_n \rangle$ and to eliminate them one by one. We do this in Lemmas 6 through 9. We recall that, by assumption, $\langle S_1, S_2, \ldots, S_n \rangle$ is envy-free and undominated.

Lemma 6. *S_1 must have one of these forms:*

- $[0, x)$, *or*
- $[y, 2)$, *or*
- $\left[\frac{1}{2} + x, \frac{3}{2} + y\right)$, *where* $|x| + |y| \leq \frac{1}{3}n^{-8}$.

Proof. From Lemma 5 we know that no other player's piece has value greater than its length. From Lemma 2, the values must sum to at least n. This means that $v_1(S_1)$ must be at least equal to the length of S_1. Equivalently, the sum of the adjustments that influence $v_1(S_1)$ must be at least 0. The cases listed are the only possibilities.

In the third case, the piece $\left[\frac{1}{2}, \frac{3}{2}\right)$ has excess value $+2\varepsilon$, and moving either boundary by an amount t reduces that excess by $6tC = 6tn^{-8}$. Therefore, if the piece is to have excess at least 0, we must have $6tn^{-8} \leq 2\varepsilon = 2n^{-16}$. Therefore $t \leq \frac{1}{3}n^{-8}$. ■

Lemma 7. $S_1 \neq [0, 1)$ *and* $S_1 \neq [1, 2)$.

Proof. If $S_1 = [0, 1)$ or $S_1 = [1, 2)$, then envy-freeness demands that each piece have length 1. Then, the values vector is $\langle v_1(S_1), \ldots, v_n(S_n) \rangle = \langle 1, 1, \ldots, 1 \rangle$, and the allocation is dominated by the allocation in which $T_1 = [1/2, 3/2)$ and all pieces have length 1, which has values vector $\langle v_1(T_1), \ldots, v_n(T_n) \rangle = \langle 1 + 2\varepsilon, 1, \ldots, 1 \rangle$. This contradicts our assumption that $\langle S_1, S_2, \ldots, S_n \rangle$ is undominated. ■

Lemma 8. $S_1 \neq [0, x)$ *for any* $x \neq 1$ *and* $S_1 \neq [y, 2)$ *for any* $y \neq 1$.

Proof. Let $S_1 = [0, x)$. (The other case is symmetrical.) As usual, let d_i denote the length of piece S_i. For each $i = 2, \ldots, n-1$, both S_{i+1} and S_i are within player i's window, so the envy-freeness requirement $v_i(S_{i+1}) \leq v_i(S_i)$ forces $d_{i+1} \leq d_i$. The same reasoning for player n shows that $d_1 \leq d_n$, so we have ultimately $d_1 \leq d_n \leq \cdots \leq d_2$. Since $x \neq 1$, the pieces are not all the same length, and hence we must have $d_1 < 1 < d_2$.

We will show that $v_1(S_2) > v_1(S_1)$, contradicting the envy-freeness of $\langle S_1, S_2, \ldots, S_n \rangle$. It is easy to check that $v_1(S_1) = d_1 < 1$. But it is not necessarily the case that $v_1(S_2) = d_2$, since the right end of S_2 might extend beyond player 1's window, and even if it does not, the value $v_1(S_2)$ is affected by player 1's adjustments. We must therefore do some computing, using the fact that player 1's value just to the left of 2 is $1 + 6(C - \varepsilon)$ per unit length. Therefore:

$$\begin{aligned}
v_1(S_2) &= v_1(d_1, d_1 + d_2) \\
&> v_1(d_1, d_1 + 1) \\
&= v_1(0, 2) - v_1(0, d_1) - v_1(d_1 + 1, 2) \\
&= 2 - d_1 - (2 - (d_1 + 1))(1 + 6(C - \varepsilon)) \\
&= 1 + (1 - d_1)(6(C - \varepsilon)) \\
&> 1 \\
&> v_1(S_1).
\end{aligned}$$

■

Lemma 9. $S_1 \neq \left[\frac{1}{2}+x, \frac{3}{2}+y\right)$ *for any x and y with* $|x|+|y| \leq \frac{1}{3}n^{-8}$.

Proof. Suppose $S_1 = \left[\frac{1}{2}+x, \frac{3}{2}+y\right)$ where $|x|+|y| \leq \frac{1}{3}n^{-8}$. For each $i = 2, \ldots, n-1$, both S_{i+1} and S_i are within player i's window, and S_{i+1} includes player i's bonus cell. Envy-freeness forces

$$d_{i+1} + C \leq d_i$$

for each such i, and the same reasoning for player n forces $d_1 + C \leq d_n$. We calculate:

$$\begin{aligned} d_1 &\geq d_1 \\ d_n &\geq d_1 + C \\ d_{n-1} &\geq d_1 + 2C \\ &\vdots \\ d_2 &\geq d_1 + (n-1)C. \end{aligned}$$

The sum of the lengths is n, so adding these inequalities gives

$$n \geq nd_1 + \left[\frac{n(n-1)}{2}\right]C,$$

or

$$d_1 \leq 1 - \frac{(n-1)}{2}C \leq 1 - C = 1 - n^{-8}.$$

But $1 - n^{-8} < 1 - \frac{1}{3}n^{-8} \leq d_1$, which implies that $d_1 < d_1$, a contradiction. ∎

We have eliminated all of the possibilities for $\langle S_1, \ldots, S_n \rangle$, and the theorem follows. ∎

Theorem 3.2. *For three or more players, there exists a pie and corresponding measures for which no allocation is envy-free and equitable.*

Proof. First suppose that there are three players, and let the pie be the interval [0, 6], with its endpoints identified. Consider measures for players A, B, and C to be the following:

	[0, 1)	[1, 2)	[2, 3)	[3, 4)	[4, 5)	[5, 6)
Player A	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$
Player B	$\frac{7}{36}$	$\frac{1}{36}$	$\frac{7}{36}$	$\frac{7}{36}$	$\frac{7}{36}$	$\frac{7}{36}$
Player C	$\frac{7}{36}$	$\frac{7}{36}$	$\frac{7}{36}$	$\frac{7}{36}$	$\frac{1}{36}$	$\frac{7}{36}$

Each player's measure is uniform on each of the six segments shown. Suppose, by way of contradiction, that $\langle S_A, S_B, S_C \rangle$ is an allocation that is envy-free and equitable.

It will be convenient to refer to the *zone* of player B or C as the connected segment of pie of length 5 to which that player assigns value 35/36. Thus, for example, player B's zone is the segment $[0, 1) \cup [2, 6)$. (This is a connected segment, since the points 0 and 6 are identified.)

Consider player A's piece, S_A. We claim that this piece cannot be a subset of the zone of one of the other players. Suppose, for example, that S_A is a subset of player B's zone. Then $v_B(S_A) > v_A(S_A)$. But equitability demands that $v_A(S_A) = v_B(S_B)$. Hence, $v_B(S_A) > v_B(S_B)$. This contradicts the envy-freeness of the allocation, and thus S_A cannot be a subset of the zone of player B or C. We note that any segment of length at most 2 is a subset of the zone of player B or C, so S_A must have length greater than 2.

Any segment of length greater than 2 must include a segment of length at least 2 within either player B's zone or player C's zone (or both). Assume, without loss of generality, that S_A includes a segment of length at least 2 within player B's zone. Then $v_B(S_A) \geq (2)(7/36) = 7/18$. But envy-freeness implies that $v_B(S_B) \geq v_B(S_A)$, and hence $v_B(S_B) \geq 7/18$. The equitability of the allocation tells us that $v_C(S_C) \geq 7/18$.

Finally, we observe that $v_B(S_B) \geq 7/18$ implies that S_B has length at least 2, and $v_C(S_C) \geq 7/18$ implies that S_C has length at least 2. We have already shown that S_A has length greater than 2. This is a contradiction, since the pie has length 6. We conclude that there is no allocation of this pie that is envy-free and equitable.

This result generalizes to more than three players in a natural way. For n players, we let the pie be the interval $[0, n(n-1)]$ and assign measures to the players in a manner entirely analogous to the above. For four players, A, B, C, and D, the measures would be as follows:

	$[0,1)$	$[1,2)$	$[2,3)$	$[3,4)$	$[4,5)$	$[5,6)$
Player A	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$
Player B	$\frac{13}{144}$	$\frac{1}{144}$	$\frac{13}{144}$	$\frac{13}{144}$	$\frac{13}{144}$	$\frac{13}{144}$
Player C	$\frac{13}{144}$	$\frac{13}{144}$	$\frac{13}{144}$	$\frac{13}{144}$	$\frac{13}{144}$	$\frac{1}{144}$
Player D	$\frac{13}{144}$	$\frac{13}{144}$	$\frac{13}{144}$	$\frac{13}{144}$	$\frac{13}{144}$	$\frac{13}{144}$

	$[6,7)$	$[7,8)$	$[8,9)$	$[9,10)$	$[10,11)$	$[11,12)$
Player A	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$
Player B	$\frac{13}{144}$	$\frac{13}{144}$	$\frac{13}{144}$	$\frac{13}{144}$	$\frac{13}{144}$	$\frac{13}{144}$
Player C	$\frac{13}{144}$	$\frac{13}{144}$	$\frac{13}{144}$	$\frac{13}{144}$	$\frac{13}{144}$	$\frac{13}{144}$
Player D	$\frac{13}{144}$	$\frac{13}{144}$	$\frac{13}{144}$	$\frac{1}{144}$	$\frac{13}{144}$	$\frac{13}{144}$

We omit the details showing that there is no envy-free and equitable allocation of this pie, and the details of generalizing to more than four players. ■

So far we have shown that for three or more players, there are pies for which no allocation is both envy-free and undominated, and there are also pies for which no allocation is both envy-free and equitable. However, for the third pair of properties, equitable and undominated, the story is different: there always is such an allocation for pie (as well as for cake; see **[3]**), assuming that the measures are absolutely continuous with respect to one another. It is not hard to see that there are many equitable allocations. To show that one of these equitable allocations must also be undominated, one can use the continuity of the measures and the fact that the pie is a compact set to show that there is a "best" equitable allocation P (i.e., one in which the common value that players assign to their pieces in allocation P is at least as great as it is in any other equitable allocation). If some other allocation Q dominated P, then it would be possible (using the absolute continuity of the measures with respect to one another) to shift the boundaries in Q to find an equitable allocation that dominated P, thus obtaining an equitable allocation in which the common value to each player is bigger than in P, which is a contradiction.

To illustrate an equitable and undominated allocation in the example used to prove Theorem 3.2 for three players (it is not unique), give player C the first segment and 17/19 of the second segment, player A 2/19 of the second segment, the third and fourth segments, and 2/19 of the fifth segment, and player B 17/19 of the fifth segment and the sixth segment. Each player thereby obtains a value of 7/19, but this equitable allocation is not envy-free, because players B and C think player A has obtained 47/114 and so will envy player A because they each perceive player A to have $(47/114) - (7/19) = 5/114$ more than they do. By the same token, an envy-free and undominated allocation would be to give player C the first two segments, player A the next two, and player B the last two (7/18 to A, 1/3 to B, 7/18 to C), but this allocation is obviously not equitable.

4. PIE-CUTTING: TWO PLAYERS. We next analyze pie-cutting for two players, for which there are more positive results than we found for pie-cutting when there are three or more players. Consider two players A and B who receive pieces $[\alpha, \beta)$ and $[\beta, \alpha + 1)$ and assign to them the values $v_A(\alpha, \beta)$ and $v_B(\beta, \alpha + 1)$, respectively. The pair $(v_A(\alpha, \beta), v_B(\beta, \alpha + 1))$ is a point in the closed unit square $[0, 1] \times [0, 1]$. Let Ψ denote the set of points that arise this way. Thus,

$$\Psi = \{(v_A(\alpha, \beta), v_B(\beta, \alpha + 1)) : 0 \leq \alpha \leq 1 \text{ and } \alpha \leq \beta \leq \alpha + 1\}.$$

The points $(1, 0)$ and $(0, 1)$, for instance, are always points of Ψ. Let

$$\Phi = \{[\alpha, \beta) \mid 0 \leq \alpha \leq 1 \text{ and } \alpha \leq \beta \leq \alpha + 1\}$$

be the set of possible pieces for A; then Ψ is the continuous image of Φ under the map $[\alpha, \beta) \mapsto (v_A(\alpha, \beta), v_B(\beta, \alpha + 1))$. Any point of Ψ lying on the main diagonal of the square corresponds to an equitable allocation, and any point in the upper-right closed quarter $[1/2, 1] \times [1/2, 1]$ corresponds to an envy-free allocation. The construction of Ψ resembles the IPS construction in [**1**].

To determine whether a point P of Ψ corresponds to an undominated allocation, draw horizontal and vertical axes centered at P and label the four (closed) quadrants I_P, II_P, III_P, and IV_P, as illustrated in Figure 2. If the only point of Ψ contained in quadrant I_P is P, then P corresponds to an undominated allocation. Our goal is to determine whether points with some, or all, of these properties exist.

The continuity of players' measures shows that Ψ possesses four key features:

Lemma 10.

a. *Ψ is closed.*
b. *Ψ is path connected.*
c. *Ψ is symmetric with respect to the point* $(1/2, 1/2)$.
d. *If P and Q are two points of Ψ with the same x-coordinate or the same y-coordinate, then line segment $\overline{PQ}$ lies in Ψ.*

Proof. Parts a and b follow from the fact that Ψ is a continuous image of the compact, path-connected set Φ, and is, therefore, itself compact and path connected. Part c follows from the fact that A and B can exchange pieces, and

$$(v_A(\alpha, \beta), v_B(\beta, \alpha + 1)) = (1 - v_A(\beta, \alpha + 1), 1 - v_B(\alpha, \beta)).$$

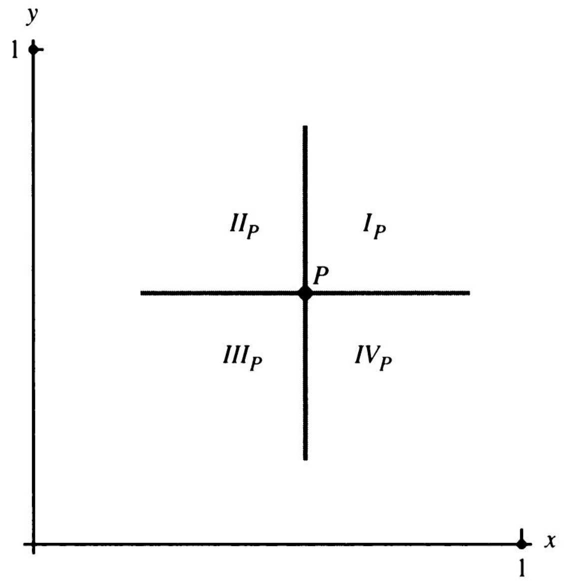

Figure 2. Quadrants used to determine whether an allocation is undominated.

To prove d, without loss of generality suppose P and Q have the same x-coordinate a. Let (a, b) be any point on the segment $\overline{PQ}$, and define U and V as follows:

$$U = \{\alpha \in [0, 1] : \text{for some } \beta \text{ with } \alpha \leq \beta \leq \alpha + 1,$$
$$v_A(\alpha, \beta) = a \text{ and } v_B(\beta, \alpha + 1) \geq b\}$$
$$V = \{\alpha \in [0, 1] : \text{for some } \beta \text{ with } \alpha \leq \beta \leq \alpha + 1,$$
$$v_A(\alpha, \beta) = a \text{ and } v_B(\beta, \alpha + 1) \leq b\}$$

Then U and V are both closed.

For any $\alpha \in [0, 1]$, $v_A(\alpha, \beta) = a$ for some β with $\alpha \leq \beta \leq \alpha + 1$, and thus either $\alpha \in U$ or $\alpha \in V$. It follows that $U \cup V = [0, 1]$.

Since P and Q are both in Ψ, we know that $U \neq \emptyset$ and $V \neq \emptyset$. The interval $[0, 1]$ cannot be expressed as the union of two disjoint closed sets, and hence $U \cap V \neq \emptyset$. Choose any $\alpha \in U \cap V$. Then, for some $\beta_1, \beta_2 \in [0, 1]$, $v_A(\alpha, \beta_1) = v_A(\alpha, \beta_2) = a$ and $v_B(\beta_1, \alpha + 1) \leq b \leq v_B(\beta_2, \alpha + 1)$. If $\beta_1 = \beta_2$, set β equal to this common value. If $\beta_1 \neq \beta_2$, then the continuity of v_B and the intermediate value theorem imply that for some β between β_1 and β_2, $v_B(\beta, \alpha + 1) = b$. In either case, we must have $v_A(\alpha, \beta) = a$. Thus, $(a, b) \in \Psi$. This establishes that line segment $\overline{PQ}$ lies in Ψ. ■

Parts b, c, and d of Lemma 10 imply that $(1/2, 1/2)$ is always a point of Ψ. It is also not difficult to see that $(1, 0)$ and $(0, 1)$ are the only points of Ψ that belong to the boundary of the unit square if, and only if, the measures of the two players are absolutely continuous with respect to one another. In this case, each horizontal and each vertical cross-section of Ψ away from $(1, 0)$ and $(0, 1)$ has endpoints in the interior of the unit square.

Figure 3 illustrates four possibilities for the set Ψ. If the players' measures are equal, then Ψ consists only of the diagonal between $(1, 0)$ and $(0, 1)$, as shown in Figure 3a. The Ψ's in Figures 3a and 3b correspond to measures that are absolutely

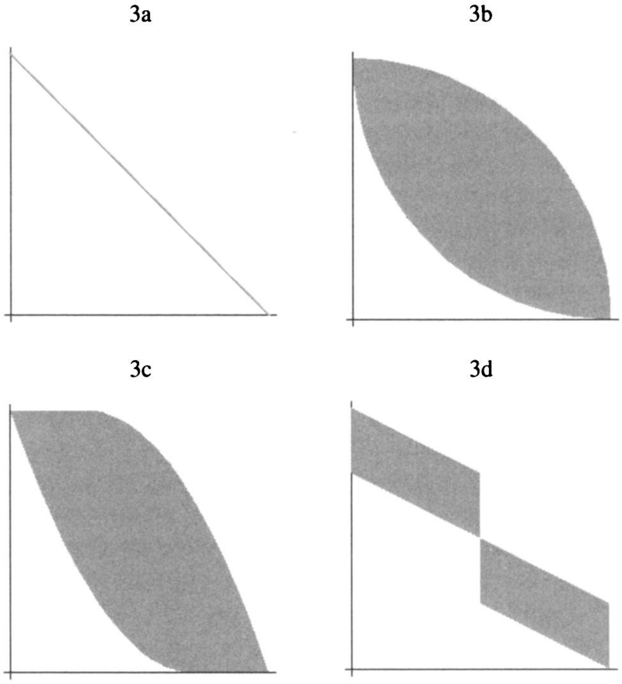

Figure 3. Some possibilities for the set Ψ.

continuous with respect to one another, while the Ψ's in Figures 3c and 3d correspond to measures that are not.

Curiously, for two players the result for pies is exactly the same as the result for cakes. (See Theorem 2.3.)

Theorem 4.1. *For two players and any pie and corresponding measures,*

a. *there exists an allocation that is both envy-free and undominated;*
b. *there exists an allocation that is both envy-free and equitable; and*
c. *if the measures are absolutely continuous with respect to one another, then there exists an allocation that is envy-free, undominated, and equitable.*

Proof. For part a, let Ψ^{ur} denote the intersection of Ψ with the closed upper-right quarter of the unit square (i.e., $[1/2, 1] \times [1/2, 1]$). Then Ψ^{ur} is a closed set. The function $F : (x, y) \mapsto x + y$ is a continuous function from Ψ^{ur} to $\mathbf{R}$. By the extreme value theorem, there exists a point P in Ψ^{ur} for which F attains a maximal value. Any allocation that corresponds to this point is envy-free and undominated.

We have already shown that $(1/2, 1/2)$ is in Ψ. This point corresponds to an allocation that is envy-free and equitable, and this establishes part b.

Finally, for part c, assume that players' measures are absolutely continuous with respect to one another, and let P be the rightmost point of Ψ on the main diagonal of the unit square. Clearly, any allocation that corresponds to P is envy-free and equitable. Our goal is to show that any such allocation is also undominated.

Suppose, to the contrary, there is a point Q of Ψ different from P in quadrant I_P. This point does not lie on the main diagonal of the unit square. Assume, without loss of generality, that it lies above the main diagonal.

Now $Q = (v_A(\alpha, \beta), v_B(\beta, \alpha + 1))$ for some sector $[\alpha, \beta)$. Using continuity, and the fact that the measures are absolutely continuous with respect to one another, we can adjust the values of α and β to produce a second point lying in the interior of I_P with x-coordinate larger than the x-coordinate of Q (and y-coordinate necessarily smaller). Thus, without loss of generality, we can assume Q has x-coordinate larger than P's.

Since Ψ is path connected, there is a path λ from $(1/2, 1/2)$ to $(1, 0)$ consisting of points from Ψ. As P is the rightmost point of Ψ on the diagonal, λ does not cross the main diagonal in I_P. Consequently there is a point R on λ with the same x-coordinate as Q lying below the main diagonal. That $\overline{QR} \subseteq \Psi$ provides a contradiction to the choice of P. This establishes part c and completes the proof of the theorem. ∎

It follows from the theorem that Gale's question has an affirmative answer for pie-cutting involving two players, regardless of whether the measures are absolutely continuous with respect to one another.

The final three paragraphs of this proof can be readily modified to show that when the measures are absolutely continuous measures with respect to one another, the rightmost point along any horizontal cross-section of Ψ corresponds to an undominated allocation of sectors, as does the topmost point of any vertical cross-section. Loosely speaking, every point along the upper boundary of Ψ corresponds to an undominated allocation. (The upper boundary might be called the "efficient frontier.")

The assumption that the measures are absolutely continuous with respect to one another is necessary for the proof of part c of Theorem 4.1. Suppose, for example, that B's measure is uniform (that is, $v_B(\alpha, \beta) = \beta - \alpha$ always) but that A's measure is concentrated uniformly on the subset $[0, 1/4] \cup [1/2, 3/4]$ (that is, $v_A(\alpha, \beta)$ is twice the length of the intersection of $[\alpha, \beta)$ with that set). These measures produce the set Ψ shown in Figure 3d, in which no point corresponds to an envy-free, equitable, and undominated allocation. In fact (again, precisely as is the case in two-player cake division), this example illustrates that, in general, if the measures are not absolutely continuous with respect to one another, then there need not be an allocation that is both undominated and equitable.

We close this section by considering a special case of pie-cutting for two players. Suppose we insist that the two players, A and B, are allowed to cut only along a diameter of the pie. Construct the subset Ψ_d of the closed unit square given as the set of all points of the form $(v_A(\alpha, \alpha + 1/2), v_B(\alpha + 1/2, \alpha + 1))$ for $\alpha \in [0, 1)$. Then Ψ_d is a (possibly self-intersecting) loop within the closed unit square, symmetric about the center of the square. As before, one is guaranteed allocations that are envy-free and equitable, or envy-free and undominated, but not all three properties. Suppose, for example, that B's measure is uniform but that A prefers the first half of the pie. Specifically, A assigns a value of $3/2$ per unit length in the interval $[0, 1/2)$, but only $1/2$ per unit length in the interval $[1/2, 1)$. In this case, Ψ_d is the horizontal line segment between $(1/4, 1/2)$ and $(3/4, 1/2)$, and no point of Ψ_d corresponds to an allocation that is simultaneously envy-free, equitable, and undominated. Note, too, that these players' measures are absolutely continuous with respect to one another.

5. PIE-CUTTING: PROCEDURAL RESULTS.

Establishing the existence of an allocation with certain properties is not the same as producing it. In this section we consider procedures, or algorithms, for finding desirable allocations of a pie. These procedures assume that players move knives continuously, whereas discrete procedures assume that players make choices at discrete times. For examples of each type of procedure, see **[5]**, **[10]**.

Moving-knife procedures were first applied to dividing a cake among n players using $n-1$ parallel, vertical cuts (the minimal number). Two minimal-cut envy-free moving-knife procedures for three players have been found ([**2**], [**11**]), but no minimal-cut four-player envy-free procedure is known. An envy-free procedure for four players that requires up to five cuts—two more than the minimal number of three—is known, but it may necessitate that players receive disconnected pieces ([**2**]; see also [**7**]).

For pie, as we have seen, there is always an envy-free, undominated allocation for two players, but for three players, we may have to settle for an envy-free allocation that is dominated. Next we address how to find allocations that are envy-free, undominated, or equitable.

We present two procedures, one for two players and one for three players. The two-player procedure will produce an allocation that is envy-free and undominated but need not be equitable, and the three-player procedure will produce an allocation that is envy-free but need not be either undominated or equitable.

Two-player procedure. This procedure produces an envy-free and undominated allocation for two players. First assume that their measures are absolutely continuous with respect to one another. Call the two players player A and player B, whom we shall refer to as "she" and "he," respectively. Player A holds two radial knives above the pie in such a way that, in her view, the two sectors of pie determined by these knives each have value $1/2$. She then rotates these knives continuously, all the way around the pie, maintaining this $1/2$ value of the sectors until the knives return to their original positions. After observing this process, player B identifies the position that, in his view, gives the maximum value to one of the two sectors so determined. (Ties can be broken randomly. The continuity of the players' measures, along with the extreme value theorem, guarantees that there is such a maximum-value sector.) Player B takes this sector, and player A receives the other sector. Call this allocation P.

We claim that P is envy-free and undominated. To see that P is envy-free, we first observe that player A certainly believes that her sector has value exactly $1/2$, and so she will not envy player B. Player B does not envy player A since, if he does, then he must have picked the smaller sector, rather than the larger sector, at his chosen position. Thus, P is envy-free.

Suppose, by way of contradiction, that some allocation Q dominates P. Then, both players receive at least as much pie in allocation Q as in allocation P (in each player's own view), and at least one player receives strictly more. The absolute continuity of the measures with respect to one another allows us to alter Q, if necessary, so as to give player A less pie and player B more pie, and in this way to obtain an allocation R such that player A receives the same value of pie (in her view) in allocation R as in allocation P (i.e., value $1/2$), and player B receives strictly more value of the pie (in his view) in allocation R than in allocation P. Then the sector that player B obtains in allocation R is one of the sectors that he would have seen as player A rotated the knives around the pie. This contradicts the fact that player B chose the largest sector that he saw.

This establishes that the allocation is envy-free and undominated. It will be equitable if and only if any sector of pie that player A considers to be half of the pie is also considered to be half of the pie by player B. In general, of course, this will not be the case.

Without absolute continuity, the same procedure works, but more care must be taken. The problem is that the location of one of A's knives may not determine the location of the other one, and the procedure may not reveal all of the relevant choices to B. To remedy this, we must refine the procedure. We imagine player A moving one of the knives slowly around the pie, and using the other knife to maintain the 1/2-1/2 values of the two sectors so determined, in her view. If the second knife comes to a sector that player A values at 0, player A immediately (i.e., discontinuously) moves this second knife to the other end of this sector, and then she continues to move both knives as before. Thus, one position of the first knife can correspond to two positions of the second knife, and the sectors so determined by each of these positions are possibilities from which player B can choose. If both of A's knives reach zero-value sectors at the same time, A must reveal to B all four extreme possibilities before proceeding.

What happens if one or both players are not truthful? For example, player A could rotate the knives in such a way as to maintain a 1/3-2/3 balance in her view and, if player B then chooses the 1/3 sector, player A ends up with what she thinks is 2/3 of the pie instead of 1/2. But, of course, player A could also end up with 1/3 of the pie instead of 1/2. Thus, we see that players can do better, or worse, by not being truthful. What is true, however, is that by being truthful, each player is guaranteed a sector of size at least 1/2 (in each player's own view) and, hence, will not envy the other player, regardless of whether or not the other player is truthful. Players who are risk-averse will presumably like this procedure, because it maximizes the minimum-value sectors that players A and B can ensure for themselves.

To relate this procedure to the set Ψ in Section 4, note that the procedure moves along the line segment that is the intersection of the line $x = 1/2$ and the set Ψ. In picking his largest sector, player B is identifying an allocation that corresponds to the point on the line segment with greatest y coordinate. (There is a largest such point, because Ψ is a closed set.)

We do not know whether there is a moving-knife procedure to produce an allocation that is envy-free, undominated, and equitable. If we insist that the measures be absolutely continuous with respect to one another, such an allocation is known to exist (see Theorem 4.1c and [**9**]).

Three-Player Procedure. This procedure produces an envy-free allocation for three players. We call the three players player A, player B, and player C and refer to them as "she," "he," and "it," respectively. We assume that the three players' measures are absolutely continuous with respect to one another.

Player A rotates three radial knives continuously around the pie, maintaining what she believes to be 1/3-1/3-1/3 sectors. Player B calls "stop" when he thinks two of the sectors are tied for largest, which must occur for at least one set of positions in the rotation (see below). The players then choose sectors in the order C first, B second, and A third.

We must show that at some point player B will think that two of the sectors are tied for most-valued, and that the allocation produced by this procedure is envy-free.

To show that there must be at least one set of knife positions in the rotation at which player B thinks there are two sectors that tie for most-valued, let us call the three sectors determined by the beginning positions of the knives sector *i*, sector *ii*, and sector *iii*. (These sectors will change as player A rotates the knives.) Let player B specify his most-valued sector at the start of the rotation. If there is a tie, then we are done. If not, then player A begins rotating the three radial knives. We assume, without loss of generality, that player B's most-valued sector at the start of the rotation is sector *i*, and that player A rotates the three knives in such a way that sector *i* moves toward the original position of sector *ii*. Because, in player A's view, each of the three sectors is 1/3 of the pie, sector *i* will eventually occupy the position of the original sector *ii*, which we make a requirement of the procedure. At this point, sector *iii* occupies the original position of sector *i*, which we also make a requirement, and hence player B must think that this new sector *iii* is the largest sector. Because, in player B's view, sector *i* starts out largest and another sector becomes largest as the rotation proceeds, it follows from the continuity of the players' measures and the intermediate value theorem that there must be a position in the rotation when player B views two sectors as tied for largest.

To see that the procedure gives an envy-free allocation, note that the first player to choose, player C, can take a most-valued sector, so it will not be envious. If player C takes one of player B's tied-for-most-valued sectors, player B can take the other one; otherwise, player B can choose either of his two tied-for-most-valued sectors. Because player A values all three sectors equally, it does not matter which sector she gets.

Finally, we make two observations. First, this procedure may fail to give an allocation that is either undominated or equitable, just as the two-player procedure may not give an allocation that is equitable. Also, like the two-player procedure, the three-player procedure does not rely on the players' being truthful. In other words, if any player misrepresents her or his or its valuations of sectors of pie (for example, if player A moves the knives in such a way that the sectors are not maintained at value $1/3$-$1/3$-$1/3$ in her view, or if player B calls "stop" at some time other than when there is a tied-for-largest sector in his view), it is still the case that any player that is truthful will not envy any other player's portion in the resulting allocation, regardless of the truthfulness of the other players.

6. CONCLUSIONS.

Our results are summarized in Table 1. In the table, "General Existence" means for any measures (i.e., with no assumption that the measures be absolutely continuous with respect to one another), and "Existence with Absolute Continuity" means that we require the measures be absolutely continuous with respect to one another. We note that this distinction is only needed for cake, not for pie.

Table 1. Existence of envy-free and undominated allocations

	Envy-Free and Undominated Allocation for Cake		Pie
Number of Players	General Existence	Existence with Abs. Cont.	Existence with or without Abs. Cont.
Two	Yes (Thm. 2.3a)	Yes (Thm. 2.1b)	Yes (Thm. 4.1a)
Three or More	No (Thm. 2.2)	Yes (Thm. 2.1b)	No (Thm. 3.1)

We close by posing two pie-cutting questions we were not able to answer.

Open Question 1. *For two players with measures that are absolutely continuous with respect to one another, is there a moving-knife procedure that produces an allocation that is envy-free, undominated, and equitable?*

If we allow a "procedure" in which the players submit their measures to a referee, the referee can determine the cuts that give such an allocation, as shown in [**4**]. But we know not even an approximate procedure by which two players can, by themselves, equalize their shares, in each player's eyes, so as to render an envy-free and undominated allocation also equitable.

Finally, we recall that for any pie and any number of players with measures that are absolutely continuous with respect to each other, there is always an allocation that is equitable and undominated. (See the discussion following the proof of Theorem 3.2.) Must there be such an allocation that is also envy-free? Theorem 3.2 tells us that the answer is "no" when there are three or more players. On the other hand, there are examples where there is such an allocation. (One trivial example is when the players' measures are all the same.) So we close by asking the following question:

Open Question 2. *For three or more players with measures that are absolutely continuous with respect to each other, are there necessary and sufficient conditions that distinguish pies for which there is an envy-free, equitable, and undominated allocation from those in which there is not?*

ACKNOWLEDGMENTS. The authors would like to thank Michael Jones, Christian Klamler, Peter Landweber, James Tanton, and William Thomson for valuable comments, and the late David Gale, whose question inspired this research. We are grateful for the opportunity to discuss some of the issues analyzed in this article at a seminar on fair division in Dagstuhl, Germany, in June 2007 (see **[12]**).

REFERENCES

1. J. Barbanel, *The Geometry of Efficient Fair Division*, Cambridge University Press, Cambridge, 2005.
2. J. Barbanel and S. Brams, Cake division with minimal cuts: Envy-free procedures for three persons, four persons, and beyond, *Math. Social Sci.* **48** (2004) 251–269.
3. S. Brams, M. Jones, and C. Klamler, Better ways to cut a cake, *Notices Amer. Math. Soc.* **53** (2006) 1314–1321.
4. ———, Proportional pie-cutting, *Internat. J. of Game Theory* **36** (2008) 353–367.
5. S. Brams and A. Taylor, *Fair Division: From Cake-Cutting to Dispute Resolution*, Cambridge University Press, Cambridge, 1996.
6. S. Brams, A. Taylor, and W. Zwicker, Old and new moving-knife schemes, *Math. Intelligencer* **17** (Dec. 1995) 30–35.
7. ———, A moving-knife solution to the four-person envy-free cake division problem, *Proc. Amer. Math. Soc.* **125** (1997) 547–554.
8. D. Gale, Mathematical entertainments, *Math. Intelligencer* **15** (Mar. 1993) 48–52.
9. M. Jones, Equitable, envy-free, and efficient cake cutting for two people and its application to divisible goods, *Math. Mag.* **75** (2002) 275–283.
10. J. Robertson and W. Webb, *Cake-Cutting Algorithms: Be Fair If You Can*, A K Peters, Natick, MA, 1998.
11. W. Stromquist, How to cut a cake fairly, this MONTHLY **87** (1980) 640–644 and addendum, **88** (1981) 613–614.
12. ———, A pie that can't be cut fairly, in *Dagstuhl Seminar Proceedings* 07261, *Seminar on Fair Division*, June 2007, available at `http://drops.dagstuhl.de/opus/frontdoor.php?source_opus=1219`.
13. F. E. Su, Rental harmony: Sperner's lemma in fair division, this MONTHLY **106** (1999) 930–942.
14. W. Thomson, Children crying at birthday parties. Why?, *Econom. Theory* **31** (2007) 501–521.

JULIUS B. BARBANEL received his B.S. from Case Western Reserve University in 1973 and his Ph.D. in mathematics from the State University of New York at Buffalo in 1979. He is a Professor of Mathematics at Union College, has published articles in both set theory and fair division, and has recently authored a book on fair division entitled *The Geometry of Efficient Fair Division* (2005). He also has interests in ancient mathematics.
Department of Mathematics, Union College, Schenectady, NY 12308
barbanej@union.edu

STEVEN J. BRAMS received his B.S. from MIT and his Ph.D. in political science from Northwestern. He is currently Professor of Politics at NYU, where he has taught since 1969. He is the author or co-author of 14 books, the most recent being *Mathematics and Democracy: Designing Better Voting and Fair-Division Procedures* (2008), and some 250 articles. He is a former president of the Peace Science Society (1990–91)

and the Public Choice Society (2004–2006), a Fellow of the American Association for the Advancement of Science, a Guggenheim Fellow, and was a Visiting Scholar at the Russell Sage Foundation.
Department of Politics, New York University, New York, NY 10012
steven.brams@nyu.edu

WALTER STROMQUIST has been teaching mathematics and statistics at Bryn Mawr College and Swarthmore College for the past six years. Previously he was employed as an analyst for the U.S. Treasury and as a mathematical consultant, primarily for Daniel H. Wagner Associates. His B.A. is from the University of Kansas and his Ph.D. from Harvard, and he has also taught at Kansas, Harvard, the George Washington University, Temple University, Eastern University, and USMA (West Point). He is editor-elect of *Mathematics Magazine*.
Department of Mathematics and Statistics, Swarthmore College, Swarthmore, PA 19081
mail@walterstromquist.com

The Laguerre Polynomials Preserve Real-Rootedness

The study of linear transformations that map polynomials with all real roots to polynomials with all real roots has been of interest for many years [**1, 2**]. We say that such transformations *preserve real-rootedness*. Notice that a linear transformation acting on polynomials can be defined by specifying its action on the polynomials x^n.

Theorem. *If $L_n(x)$ is the nth Laguerre polynomial, then the linear transformation defined by $x^n \mapsto L_n(x)$ preserves real-rootedness.*

Proof. We recall a basic result from [**2**, Part V, No. 65]: if T is the transformation $x^n \mapsto x^n/n!$, then T preserves real-rootedness. The Laguerre polynomials satisfy [**3**, p. 101]

$$L_n(x) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{x^k}{k!},$$

and this polynomial equals $T[(1-x)^n]$. Thus the linear transformation $x^n \mapsto L_n(x)$ is the composition of two linear transformations preserving real-rootedness, namely $x^n \mapsto (1-x)^n$ and T, and so preserves real-rootedness. ■

REFERENCES

1. S. Fisk, Hermite polynomials, *J. Combin. Theory Ser. A* **91** (2000) 334–336.
2. G. Pólya and G. Szego, *Problems and Theorems in Analysis*, vol. II, Springer-Verlag, New York, 1972.
3. G. Szego, *Orthogonal Polynomials*, American Mathematical Society, Providence, RI, 1975.

—Submitted by Steve Fisk, Bowdoin College, Brunswick, ME 04011