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Hermit Points On A Box

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Hermit Points on a Box

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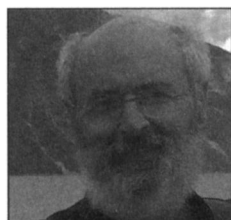
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Once upon a time, a mathematician named Herman decided to leave the hustle and
bustle of Earth and become a hermit on Bachs, a newly-discovered asteroid. This as-
teroid had been named for the famous composer and his sons by a music-loving as-
tronomer. Its shape was that of a giant rectangular parallelepiped (see Figure 1). One
can imagine Herman's surprise when, upon arriving, he discovered that another her-
mit, also named Herman and also a mathematician, had already taken up residence on
this world. To avoid confusion, in what follows we will refer to these two hermits as
Hesse and Melville (in the obvious order).

Being hermits, they decided to build their huts as far from one another as possible.
As we do on Earth, they took the distance between two points on Bachs to be the length
of the shortest path between them on the surface. Thus, they wanted to find two points

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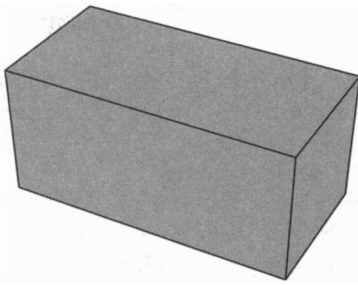
Figure 1. The asteroid Bachs.

on the asteroid that were farthest apart. We will refer to such pairs of points as *hermit points*.

At first glance, the obvious answer would seem to be that Hesse and Melville should live at antipodal corners of Bachs (two corners are antipodal if they lie on a line that contains the center of Bachs). However, it turns out that antipodal corners are not always hermit points. Consider, for example, the case where the edge lengths of Bachs are 1, 1, and 2 miles (see Figure 2). A path between a pair of antipodal corners that lies on two adjacent 1×2 faces has length $\sqrt{8}$ miles, so this is an upper bound on the distance between these points (there may be another path that is shorter). Now let P and Q be center points on opposite 1×1 faces. To get from P to Q we must travel at least $1/2$ mile to leave P's face, at least 2 miles to get to Q's face, and then at least $1/2$ mile more to reach Q, for a total distance of at least 3 miles. Thus the distance from P to Q is clearly greater than the distance between antipodal corners.
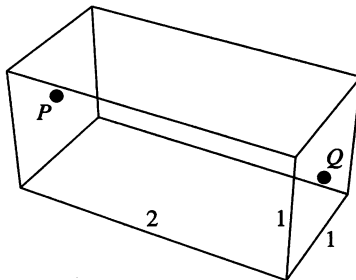
Figure 2. A $1 \times 1 \times 2$ box.

Hesse and Melville found all pairs of hermit points for all rectangular parallelepipeds. They transmitted to us their solution and a proof of its correctness. We thank them for giving us permission to publish their results.

Similar problems have been studied for over a hundred years [7]. According to Singmaster [7], Dudeney posed the Spider and Fly Problem in 1932. In this problem, a spider and a fly are on different walls of a rectangular room with given dimensions. The problem is to determine the shortest path along the walls from the spider to the fly. Kotani's Ant Problem is a generalization of this; it asks for the point farthest from a corner of a $1 \times a \times b$ box, for general a and b. Knuth [4] asked for the pair of points that are farthest apart on a $1 \times 1 \times 2$ box. This problem, for general boxes, has been explored and partially solved by Bottomley [1] and Hess [3]. Bottomley's website

gives a solution along with great graphics and interactive applets, but does not prove that the solution is correct.

Hesse and Melville thought of their problem in the following way: Assume that the huts are on two specified faces of the asteroid, with coordinates (x_h, y_h) and (x_m, y_m) (these coordinates are rectangular, and are relative to the faces on which the huts sit). For each such pair of points, calculate the lengths of all possible paths between them, and decide which one is the shortest; this shortest length is the distance between the two points. Finally, maximize the distance over the set of all pairs of points. Thought of in this way, this is a maximization problem involving four variables.

The hermits were ecstatic to learn of the following result of Vilcu [8], which answered a question posed by Propp [5].

Theorem 1. *Let S be a convex surface with a center of symmetry O. If the points x and y on S are as far apart as possible, then they are antipodal, that is, symmetric through O.*

This result immediately reduces the problem to one involving only two variables, because once Hesse's hut is placed, they knew that Melville's hut must be placed at the antipodal point, so they needed only to calculate the distance to this antipodal point. Because they wanted to solve the problem for general parallelepipeds, the problem has three parameters, namely the edge lengths of the parallelepiped, in addition to the two variables. It is easy to see how to reduce the number of parameters by one; simply divide all three by the smallest edge length. Thus, they assumed that Bachs had edge lengths $1, a$, and b, with $1 \leq a \leq b$. In what follows, we will call such a box a *standard box*.

Determining distances

A *geodesic* between two points on a surface is a path between the two points whose length equals the distance between the two points. Hesse noted that on any rectangular box, any section of a geodesic that lies on just one face will be a straight line.

Melville noted that if the box is unfolded in the right way, the geodesic will be a straight line in the plane. To see why this is true, consider a geodesic that passes from one face to another. If we flatten these two faces into the plane, and the geodesic is not a straight line on the flattened faces, then we can make the path shorter by straightening it out. Thus, given a geodesic, we can travel along it, flattening into the plane the faces we encounter. There are many ways to unfold a box, however, so care is required to determine an unfolding that yields a geodesic.

After playing with many different models of their asteroid, Hesse conjectured that the hermit points always lie on the smallest faces, that is, the $a \times 1$ faces. (In an incredible coincidence, when working on this problem here on Earth, the first author made exactly the same conjecture!) When Melville heard this conjecture, he went off onto a corner to think about it.

Melville realized that he could determine the distance between antipodal corners, and he could show that pairs of antipodal points on faces other than the smallest ones were closer to one another than this distance. This shows that either pairs of antipodal corners are hermit points or else the pairs of hermit points lie in the interiors of the smallest faces. We now give his proofs of these facts.

Lemma 1. *The squared distance between antipodal corners on a standard box is* $(a + 1)^2 + b^2$.

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Proof. If we begin at a corner of the box and proceed towards the antipodal corner along a geodesic, then this geodesic goes along a face (allowing the possibility that the geodesic lies on an edge of this face). At the point where the geodesic leaves this first face, the antipodal corner is in view along a face. Thus, the geodesic must proceed in a straight line towards the antipodal corner. For this reason, geodesics between antipodal corners can be realized by proceeding on exactly two faces. Figure 3 shows the three possible geodesics between arbitrary antipodal corners P and Q. They have squared lengths $(a + 1)^2 + b^2$, $(b + 1)^2 + a^2$, and $(a + b)^2 + 1^2$. Since we have assumed $1 \leq a \leq b$, the first value is the minimum. ∎
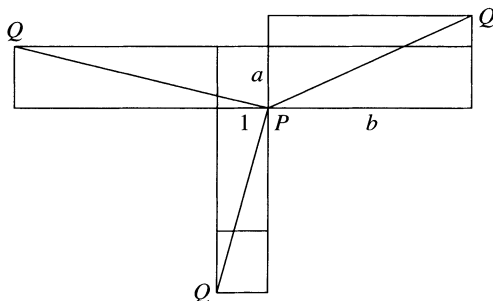
Figure 3. Possible geodesics between antipodal corners.

Theorem 2. *Pairs of hermit points on a box always lie on the smallest faces.*

Proof. Let P and Q be antipodal points that lie in the interiors of the $a \times b$ faces. Figure 4 shows a path from P to Q with squared distance equal to $(a + 1)^2 + t^2$, where $t < b$. By Lemma 1, P and Q are not as far apart as pairs of antipodal corners and thus are not hermit points. An analogous argument applies for points on the $1 \times b$ face. ∎
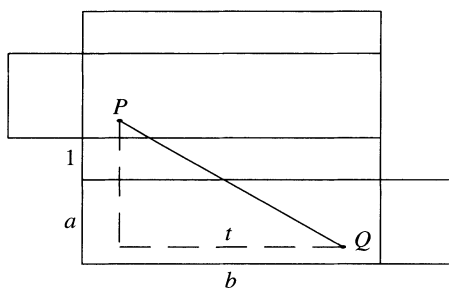
Figure 4. Antipodes on large faces are not hermit points.

We now know that all pairs of hermit points lie on the $1 \times a$ faces. By symmetry we need only consider points P in the lower left quadrant of one face. The next challenge is to find the actual distance between such a point P and its antipode Q. Figure 5

presents a schematic way to consider all possible geodesics from P to Q. There are twenty-eight copies of Q that correspond to paths crossing no more than six faces of the box. Two copies of Q are labelled as Q_1 and Q_2, and these will be used below. Only one $1 \times b$ and $a \times b$ face are shown, but their images can be imagined to realize any path from P to Q that crosses six or fewer faces. Numerous elementary arguments, examples of which will be given later in this section, allow us to reduce the twenty-eight paths to only five possible geodesics, none of which crosses six faces.
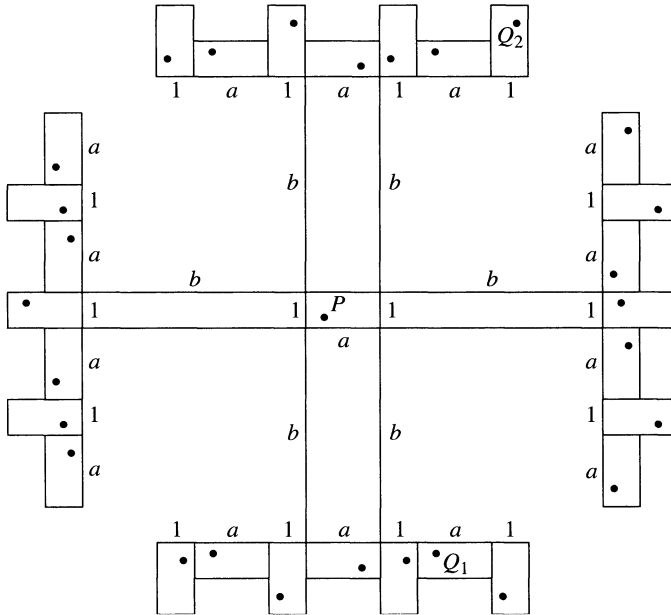

Figure 5. Schematic drawing suggesting all 28 paths between two antipodal points.
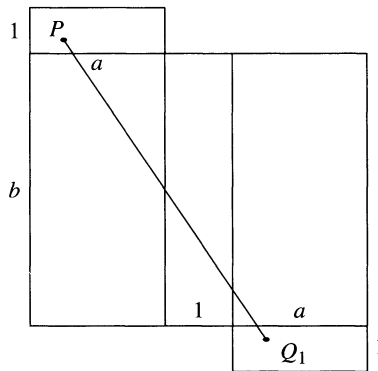
Figure 6. A five-face path.

In Figure 6, we show the path from P to Q_1, together with the faces it crosses. This figure should be compared with Figure 7, which shows a "path" from P to Q_2. In the

latter case, the path does not exist, because the straight line from P to Q_2 "misses" the fifth face. All four 3-face paths in Figure 5 clearly exist. By a simple comparison of slopes it is not hard to show that all eight 4-face paths exist as well. Five-face paths are trickier. There are boxes with pairs of hermit points that have 5-face geodesic paths as suggested by Figure 5. We will show that in this case, a 4-face geodesic also exists. Thus we need not be concerned that the 5-face geodesic may not really exist.
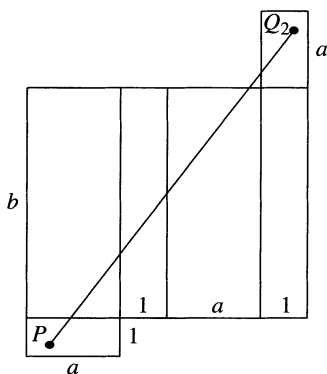
Figure 7. A path that doesn't exist.

Now we get down to the business of calculating distances. Working with Figure 5, we define a coordinate system with the origin at the lower left corner of the $a \times 1$ face containing $P(x, y)$. Note that $0 \leq x \leq a/2$ and $0 \leq y \leq 1/2$. The twenty-eight squared distances between P and each copy of Q can then be written analytically. By symmetry, each squared distance occurs twice on the list, which reduces the total to fourteen. Further elementary comparisons reduce the list to five:

- $S_1 = (b + 2x)^2 + (1 + a)^2$
- $S_2 = (b + x + y)^2 + (1 + a - x - y)^2$
- $S_3 = (b + a)^2 + (1 - 2y)^2$
- $S_4 = (a + 1)^2 + (b + 2y)^2$
- $S_5 = (a - 2x)^2 + (b + 1)^2$.

Here are some sample arguments for eliminating the other nine squared distances. Remember that for given a, b, x, and y we want the smallest squared distance. One of the 6-face paths has squared distance $(a + b + 1 - x - y)^2 + (2 + 2a - x - y)^2$. If we compare this expression to S_2, we see that S_2 is always smaller, so the given 6-face path cannot be a geodesic.

One of the 5-face paths has squared distance equal to $(a + b + 1 - x - y)^2 + (-x - y)^2$. If we subtract S_2 from this, we are left with $2b(1 - 2y + a - 2x)$, which is always non-negative because of the constraints on x and y. Thus, if this 5-face path is a geodesic, then it has the same length as S_2, which corresponds to a 4-face path, so this 5-face path can be ignored.

Arguments such as these can be used to show that the squared distance between antipodal points $P = (x, y)$ and Q on a standard box will always be one of the five values above. Note also that when both x and y are nonzero, S_1 and S_4 correspond to 5-face paths, S_2 corresponds to a 4-face path, and S_3 and S_5 correspond to 3-face paths.
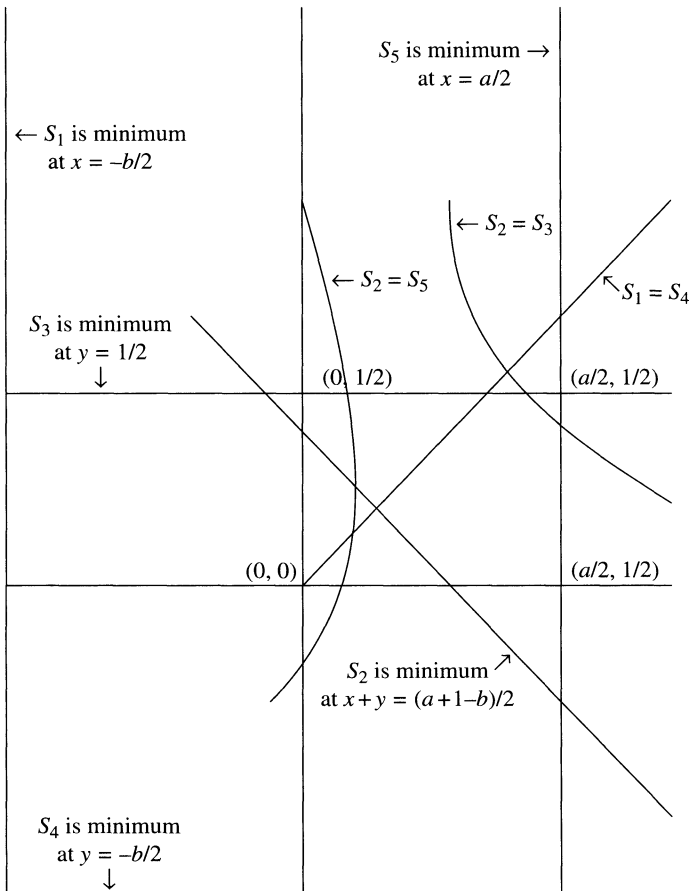
Finding the hermit points

For fixed a and b, we can find the hermit points in the following way:

1. Let $P(x, y)$ be in the lower left quadrant of an $a \times 1$ face (so $0 \le x \le a/2$ and $0 \le y \le 1/2$), and let Q be the antipode of P.
2. Find the smallest value of S_1, S_2, S_3, S_4, and S_5, thereby finding the squared distance between P and Q.
3. Vary x and y to maximize the squared distance between P and Q.

By computing partial derivatives for each squared distance, it is easy to check that none achieves a maximum in the interior of the lower left quadrant. Thus, the maximum distance can occur only for P on the boundary of the quadrant or at interior points where at least two of the five squared distances are equal.

To help us visualize further how each squared distance behaves as x and y vary, consider a box with $a = 1.4$ and $b = 1.6$. Figure 8 shows the lower left quadrant of the $a \times 1$ face. Lines are drawn showing where each of S_1, S_2, S_3, S_4, and S_5 has its

S_5 is minimum $\rightarrow$
at $x = a/2$

$\leftarrow S_1$ is minimum
at $x = -b/2$

$\leftarrow S_2 = S_3$

$\leftarrow S_2 = S_5$

$\nwarrow S_1 = S_4$

S_3 is minimum
at $y = 1/2$
$\downarrow$

$(0, 1/2)$

$(a/2, 1/2)$

$(0, 0)$

$(a/2, 1/2)$

S_2 is minimum $\nearrow$
at $x + y = (a + 1 - b)/2$

S_4 is minimum
at $y = -b/2$
$\downarrow$

Figure 8. Curves showing minima and some intersections of squared distances.

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minima. Also plotted are the curves where $S_1 = S_4$, $S_2 = S_3$, and $S_2 = S_5$, the significance of which will be clear shortly. These three curves have the following forms:

- $S_1 = S_4 : x = y$
- $S_2 = S_3 : y^2 - y(b + 1 - a + 2x) + x(a + 1 - b - x) + ab - a = 0$
- $S_2 = S_5 : y^2 + y(b - a - 1 + 2x) + x(b + a - 1 - x) - b + a = 0$.

Notice also that at $x = y = 0$, $S_1 = S_2 = S_4$.

Now we imagine ourselves in the quadrant at a point $P(x, y)$ whose distance to its antipode is S_i for some i. We make a small change in our position in order to increase the value of S_i. Because no S_i has a local maximum inside the quadrant, this will always be possible until we run into a curve where $S_i = S_j$ for some $j \neq i$ or until we reach the boundary of the quadrant. Specifically, we start at the upper right corner where $x = a/2$ and $y = 1/2$. At this point, S_5 is never larger than the other squared distances, so it gives a geodesic from P to its antipode. S_5 can be increased by moving left along the top edge. Because y remains constant and x is decreasing, this motion decreases S_1 and S_2 while leaving S_3 and S_4 unchanged. We move along the top edge until we reach the point where $S_2 = S_5$ (point A in Figure 9). It is straightforward to show that throughout this motion, S_1 and S_4 will remain greater than S_5, if the 5-face paths corresponding to these squared distances even exist. Thus at point A we have $y = 1/2$, and we solve $S_2 = S_5$ for x to get

$$x = \frac{a + b - \sqrt{(a + b)^2 - 2b + 2a - 1}}{2},$$

and the squared distance simplifies to

$$S = 3b^2 + a^2 + 2ab + 2a - 2b\sqrt{(a + b)^2 - 2b + 2a - 1}. \tag{1}$$
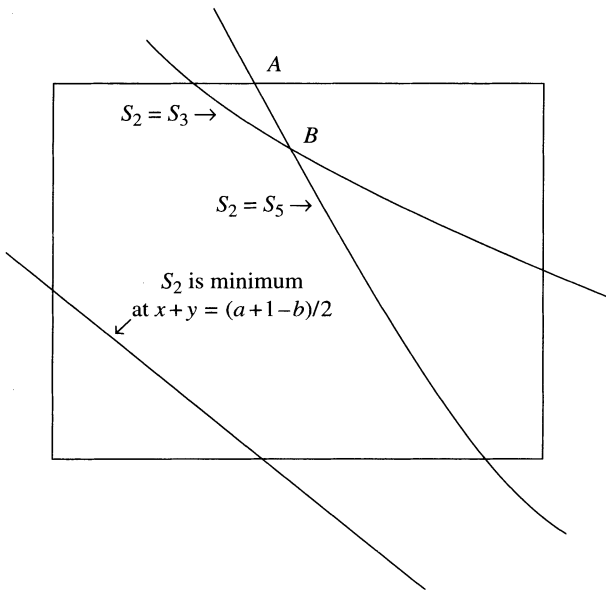
Figure 9. The triple point.

For certain values of a and b, S_3 will be less than or equal to S_2 at point A. This occurs when

$$a \leq \frac{b\sqrt{b^4 + 2b^3 + b^2 - 2b - 1} - b^3}{b^2 - 1}.$$

We let $f(b)$ denote the expression on the right. If the above inequality holds, we move down along the curve $S_2 = S_5$ as long as S_3 increases, stopping at the point B where $S_2 = S_3 = S_5$. Figure 9 shows such a case, with $a = 1.05$, $b = 1.6$, and B marking the point of triple intersection. We call B a *triple point*. It is straightforward to show that B lies above the line $y = x$, the line where $S_1 = S_4$. Above this line $S_1 \leq S_4$, so we just check that $S_2 (= S_5)$ is still smaller than S_1, allowing us to ignore both S_1 and S_4 at the triple point.

If the line where S_2 is a minimum goes through the quadrant, it is also necessary to check whether S_2 is larger at $(0, 0)$ than at A or B. First we look at A. When $a > f(b)$, S_2 at A is given by the expression S in (1) above. (If $a \leq f(b)$, then only B need be considered.) At $(0, 0)$, S_2 equals $(1 + a)^2 + b^2$. We compare these two distances and find that S_2 is larger at the corner than at the point A if $a > b - 1/2 + 1/(4b)$. The example in Figure 9 is just such a case, with the hermit points occurring at $(0, 0)$.

Before we move on to comparisons with the triple point B, it will help to look at some more examples. Figure 10 shows several plots, each for fixed values of a and b. Each plot shows the squared distance for points (x, y) in the quadrant, so the highest point on the surface corresponds to the hermit point.
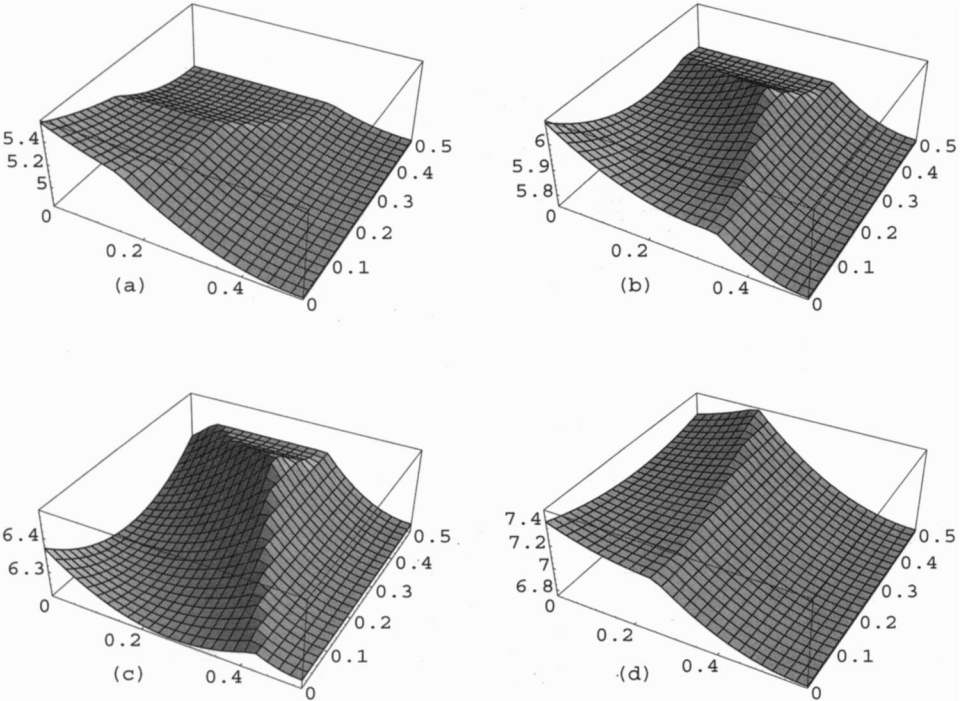

Figure 10. Sample hermit points.

Figure 10(a) shows that for $a = 1.03$ and $b = 1.2$, the hermit point is clearly at the corner of the box. Figure 10(b) shows that when we keep $a = 1.03$ but increase b to 1.4, the hermit point still appears to be at the corner, though in this case the triple point is also visible. Figure 10(c) shows that increasing b further to 1.5 gives us the hermit point at the triple point. Finally, Figure 10(d) shows that for $a = 1.2$ and $b = 1.6$, the hermit point occurs at point A along the upper boundary of the quadrant.

To find the coordinates of B, we set the three squared distances S_2, S_3, and S_5 equal, obtaining the following fourth-degree equation

$$16x^4 + 16x^3(b - 2a) + 4x^2(5a^2 - 8ab + 2b + 1)$$
$$+ 4x(2b^2 - 2ab^2 + 4a^2b + b - a - 3ab - a^3)$$
$$+ (b^2 - 2ab^3 + 2b^3 - 2ab + 3a^2b^2 - 4ab^2 - 2a^3b + 4a^2b) = 0.$$

While one can solve this quartic using a computer, the results are too scary to give here explicitly.

We know that at B, we have $S = S_5 = S_2 = S_3$. As pointed out earlier, we need to compare S_2 at B with S_2 at the corner ($x = y = 0$). The point with the larger squared distance will be the hermit point in the quadrant. We first find the value of x that makes S_5 at B equal to S_2 at the corner. Evaluating S_5 for this x, we set the result equal to S_3 and solve for y. The results are

$$x = \frac{a - \sqrt{2a + a^2 - 2b}}{2} \quad \text{and} \quad y = \frac{1 - \sqrt{1 + 2a - 2ab}}{2}.$$

Finally, we substitute x and y into S_2 and subtract the value of S_5 from the result. The values of a and b for which the distances at the triple point B and the corner are equal are those for which

$$-a - \sqrt{2a + a^2 - 2b}\sqrt{1 + 2a - 2ab} + b\left(\sqrt{2a + a^2 - 2b} + \sqrt{1 + 2a - 2ab}\right) = 0.$$

This curve thus defines the boundary between between those standard boxes with hermit points at the corners and those standard boxes with hermit points at the triple points.

Figure 11 summarizes our analysis, showing how the general solution separates into three cases. Values of a and b determine a point (a, b) in the plot. Because $a \leq b$, the point will lie on or above the line $a = b$. If the point lies in the region labeled V, then a standard box will have eight hermit points at the corners of the box (hence the label V for vertices). In region C the standard box will have four hermit points at the centerline points (previously labeled A) at distance x from the unit edge on the $a \times 1$ face where

$$x = \frac{a + b - \sqrt{(a + b)^2 - 2b + 2a - 1}}{2}.$$

In region F the standard box will have eight hermit points on the $a \times 1$ face at points determined by the fourth-degree equation given previously. We note that for points along the left boundary of region F, where $a = 1$, boxes will have hermit points along the 1×1 diagonal at distance $x = y$ from both edges, where x and y have the much more accessible form

$$x = y = \frac{1 - b + \sqrt{b^2 - 1}}{2}.$$
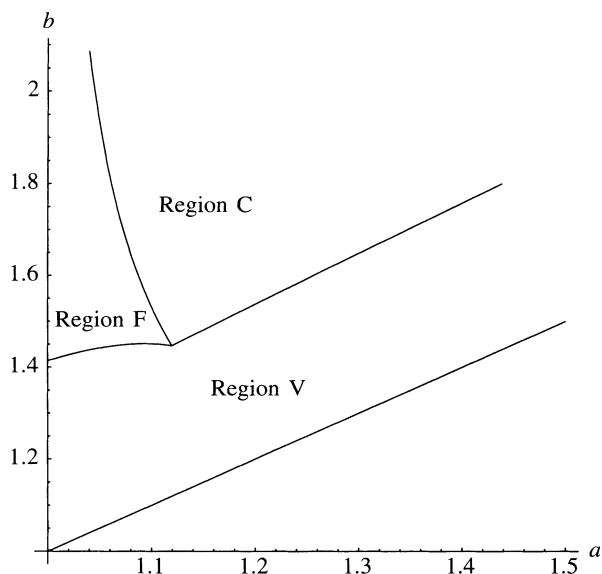
Figure 11. Regions corresponding to different hermit points locations for the $1 \times a \times b$ box.

The three boundary curves in Figure 11 meet at the common point (a, b), where

$$a = \frac{1 + (271 + 6\sqrt{633})^{1/3} + (271 - 6\sqrt{633})^{1/3}}{12} \approx 1.119458$$

and

$$b = \frac{6 + (324 + 12\sqrt{633})^{1/3} + (324 - 12\sqrt{633})^{1/3}}{12} \approx 1.446645.$$

These values are roots of the equations

$$4a^3 - a^2 - 3a - 1 = 0 \quad \text{and} \quad 4b^3 - 6b^2 + b - 1 = 0.$$

All of the boxes corresponding to points (a, b) lying on the VF-boundary in Figure 11 have eight pairs of hermit points (except at the right-hand endpoint of this boundary, where some of these pairs coalesce). For example, the point $(a, b) = (1, \sqrt{2})$ lies on the VF-boundary. The box corresponding to these values is a $1 \times 1 \times \sqrt{2}$ box with eight pairs of hermit points (four pairs of antipodal corners and four pairs of triple points). The triple points lie on the diagonals of the 1×1 face, at a distance of $\sqrt{2} - 1$ from the closest corner. The hermit points on this box are at distance $\sqrt{6}$ from their antipodes. Figure 12 shows the squared distances from points on the 1×1 face to their antipodes; the eight hermit points are clearly visible.

We end with a question that we believe to be unsolved. Given a tetrahedron in $\mathbf{R}^3$, find the pair of points on its surface that are farthest apart. The solution of this problem will involve different techniques from the ones employed in this paper, due to the paucity of right angles on the surface of a general tetrahedron. If the reader has an interest in pursuing this question (as some of the present authors are doing), we point out that Schoenberg [**6**] found necessary and sufficient conditions for a set of six positive real numbers to be the edge lengths of a tetrahedron.
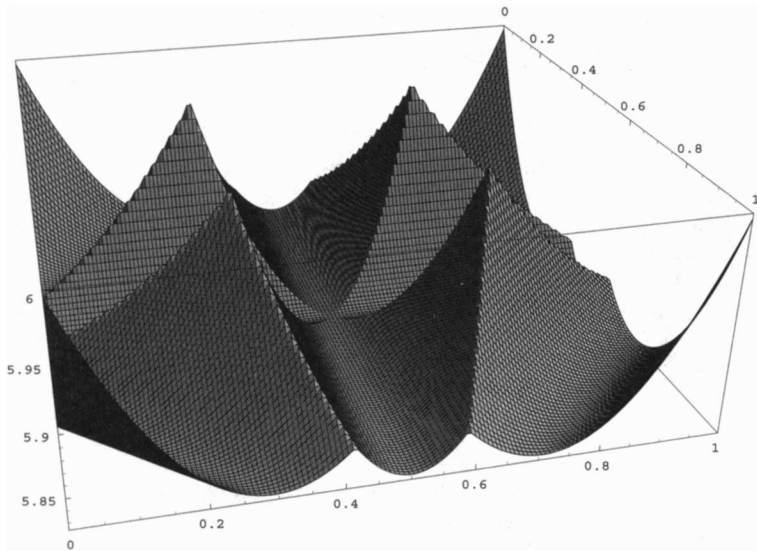
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Figure 12. Squared distances on the 1×1 face of the $1 \times 1 \times \sqrt{2}$ box.

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Teaching Tip: An Introduction to e^{ix} without Series

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When introducing the exponential function (see the article by Byungchul Cha in the September 2007 issue, pp. 288–296) as the solution of the initial value problem

$$\frac{dy}{dx} = y \text{ with } y(0) = 1,$$

tantalize your students with this:

Since both $y = e^{ix}$ and $y = \cos x + i \sin x$ are solutions to the initial value problem

$$\frac{dy}{dx} = iy \text{ with } y(0) = 1,$$

does this mean they are equal? And so does $e^{i\pi} = -1$?