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Charles H. Harbison
University of Arkansas at Little Rock

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CHARLES H. HARBISON
University of Arkansas at Little Rock, Little Rock, Arkansas

ABSTRACT

A proof that: for any given polyhedron so shaped that every closed non-self intersecting broken line composed of edges of the polyhedron divides the surface of the polyhedron into precisely two disjoint regions each of which is bounded by the closed broken line, $v - e + f = 2$, where v is the number of vertices of the polyhedron, e the number of edges and f the number of faces.

PROOF

Step I. Call the two disjoint regions mentioned in the theorem complements of each other. Visualize the construction of a duplicate of the polyhedron by mentally positioning a set of edges (line segments) of appropriate lengths to construct a network of the same size and shape as that network formed by the edges of the given polyhedron. Call an edge that has been placed in our construction an in position edge and let an in position vertex refer to the point of intersection of two or more in position edges. As the construction progresses, let $e =$ the number of in position edges, $v =$ the number of in position vertices and $f =$ the number of faces for which the complete boundaries are in position. Upon completion of the construction, v, e and f will certainly take on the values assigned them in the theorem. Define an incompleted region of the surface S of the polyhedron being constructed as a region whose complete boundary is in position but which lacks all of its interior edges of which there is at least one. Let u equal the number of incompleted regions as the construction develops - thus the value of u may vary. Consider the boundary of one face in position. Now by hypothesis the boundary of this face will divide the finished surface S of the polyhedron into two regions - one the in position face and the other the now incompleted region having the same boundary as this face.

At this point v obviously $= e$, $f = 1$ and $u = 1$. Thus $v - e + f = 2 - u$.

Step II. Choose any incompleted region of our construction, say D, and place in position a continuous non-self intersecting broken line of interior edges of D reaching from a vertex, say P, on the border of D to another vertex, call it Q, on the border of D. This newly added broken line PQ may consist of only one edge if it reaches from border to border. In any event the added line PQ is to contact the border of D at both points P and Q and nowhere else. Certainly it is possible to add such a line as PQ since D by definition lacks all its interior edges of which there is at least one. Note, too, that adding the line PQ as indicated never leaves an in position edge dangling from one end. Now the points P and Q partition the border of D into two parts. Let A be one of the regions carved from D and bounded by the closed non-self intersecting broken line of edges consisting of the newly added line PQ and one of these parts of the border of D. Let B designate the region carved from D and bounded by the closed non-self intersecting broken line of edges comprised of the line PQ and the part of the border of D not bordering A.

By hypothesis A is disjoint from B since B is in the complement of A. And the part of D not in A is in B since A and its complement will comprise all of S and no part of D could have been in D's own complement. Thus the added line PQ divided D into precisely two disjoint regions. Now A is an incompleted region if it lacks at least one interior edge, otherwise it is a face with completed boundary since the added line PQ could not have passed an intervening in place edge. Similarly we can conclude the same about B. Note that adding the line PQ increases e one more than it increases v since the end points P and Q were already in position. Thus the former region D has been divided in one of the three following ways:

(a) two faces with completed borders, increasing f by 2, decreasing u by 1 and adding one more edge than vertices,

(b) two incompleted regions, not changing f, increasing u by 1 and adding one more edge than vertices, or

(c) a face with completed border and an incompleted region, increasing f by 1, leaving u unchanged and adding one more edge than vertices. In any case, the relation $v - e + f = 2 - u$ continues to hold.

Step III. Repeat step II a finite number of times, completing S and reducing u to zero - whence $v - e + f = 2$, and v, e and f have assumed the values assigned them in the theorem.

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