Journal of the Arkansas Academy of Science

Volume 68 Article 9

2014

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Running Title: Synchronized Limits of Chaotic Circuits

Abstract

Through system modeling with electronic circuits, two circuits were constructed that exhibit chaos over a wide ranges of initial conditions. The two circuits were one that modeled an algebraically simple "jerk" function and a resistor-inductor-diode (RLD) circuit where the diode was reverse-biased on the positive voltage cycle of the alternating current source. Using simulation data from other experiments, the waveforms, bifurcation plots, and phase space plots of the concrete circuit were verified. Identical circuits were then built containing variable components and coupled to their original, matching circuits. The variable components were used to observe a wide range of conditions to establish the desynchronization parameters and the range of synchronization.

Introduction

History

Ever since the conception of chaos became a subject to be studied, the list of systems that can be modeled by chaotic equations has been growing. Many of these systems are of great importance, featuring such scientific irritations as the weather, noise, and the precise movement of fluids. While chaos cannot be defined by specific sets of equations as waves can be, the study of chaos has revealed several characteristics of chaotic systems that help to define the term and show its potential for engineering applications.

The most obvious trait of chaos is its aperiodic behavior, never repeating a solution. With his computer-made weather model and simulator based on iterative mapping in the 1960's, Edward Lorenz stumbled upon another important feature of chaos, *sensitive dependence on initial conditions.* Due to computer rounding, Lorenz found that different initial conditions that vary by only a slight decimal difference will result in drastically different outcomes after only a few iterations. Lorenz also discovered that for some initial conditions the solutions never repeated (indicating chaos) but they did tend to be similar taking advantage of the universe that lies between 1 and 0. That is to say, for some initial conditions, the solutions would be incredibly close to each other but off by miniscule decimal places when evaluated quantitatively. When the solutions were plotted, they displayed a shape that became known as the Lorenz attractor. Since then, many other systems have been found with unique attractors of a variety of shapes (reviewed in Gleick 1987). Later in the 1970's, Mitchell Feigenbaum studied iterative mapping of several non-linear equations using a wide range of initial conditions (reviewed in Gleick 1987). He found initial conditions that when mapped converged on a single solution, and that when the value of the initial conditions are increased to a certain point, the mapping converges to two solutions. As the initial conditions are increased the number of solutions continues to double (now called *period-doubling*, or *bifurcation*) until the solutions diverge in chaos. The period-doubling alone is remarkably useful for identifying chaos, but Feigenbaum also discovered a chaotically, universal constant (reviewed in Gleick 1987). Through the study of numerous non-linear systems, Feigenbaum found that the range of initial values that yield a specific number of solutions compared to the range of the next period-doubling converges on the number 4.67. The mathematical statement for this idea is,

$$\frac{f_2 - f_1}{f_3 - f_2} = 4.67 \qquad (1)$$

Where f is a bifurcation point. This constant holds true for all systems that approach chaos through period-doubling. The universality of this number allows for predictions of period-doublings and another way to verify chaos within a system (reviewed in Gleick 1987).

More recently, J.C. Sprott (2011) discovered several simple functions (simple meaning they contain few terms) that still exhibit chaos. Equations containing a third order differential (*Jerk functions*) can achieve chaos with only one non-linear term. Having only one non-linear term allows a function to be easily modeled with electrical components where the signal can be viewed on an oscilloscope so that measurements such as chaotic and periodic ranges can be made. Sprott analyzed several of these functions by modeling them with circuits, however, we now report the analysis of a physical circuit that Sprott has only measured through computer simulation (Sprott 2011).

Theory

The simplicities of the Sprott circuit and the resistor-inductor-diode (RLD) circuit are useful for producing the same exact signal in two nearly identical circuits, but trying to create the exact chaotic signal in two separate circuits is a rather difficult task due to the sensitive dependence on initial conditions described earlier. It is difficult to control every possible initial condition in a real world system, but through synchronization, exact replication of chaotic signal is possible. *Synchronization* is the process of allowing one circuit to drive another circuit through circuit coupling, and when used with chaotic waveforms, the use of synchronization is very powerful. In synchronization, the secondary circuit is driven so that the exact signal in the primary circuit appears in the secondary. Even with differing initial conditions, the primary and secondary circuits can still exhibit identical behavior provided that the initial conditions are similar. The nonspecific term “similar” is used because the question addressed in this manuscript is to define how "similar" the two circuits must be.

The driving force behind the design of our experiment is the use of synchronization for noise cancellation. From a practical sense, the noise appearing in a machine will not be the same every time it is used, and more realistically will vary depending on the settings of the machine and how it is used. To meet the demands of a wide range of chaotic possibilities, the cancellation circuit needs to be robust, requiring a wide range of parameters over which it is chaotic. “Jerk” functions have been shown numerically to be very robust, and these third order differentials can exhibit chaos with minimal terms making them algebraically simple. For instance, the following functions achieve chaos with only two non-linear terms and four total terms,

$$\dddot{x} + Ax\ddot{x} - (\dot{x})^2 + x = 0 \quad (2)$$

$$\dddot{x} + Ax\ddot{x} - x\dot{x} + x = 0 \quad (3)$$

The simplistic functions allow for better predictions of what the waveform might do and are easy to model with electrical components in a circuit. Simulations done have shown that chaotic jerk functions are very robust (Sprott 2011). In this experiment, a circuit was constructed to model the equation,

$$\dddot{x} + A\ddot{x} + x + (\dot{x})^2 = 0 \quad (4)$$

The value of the parameter *A* changes the initial conditions allowing for bifurcations and chaos to be observed and evaluated. It will be shown later that *A* can be controlled with a potentiometer (Sprott 2011). The equation of an RLD circuit also contains a bifurcation parameter but this time it is controlled by varying the amplitude of the voltage source. The amplitude parameter can easily be seen in the following equation found with Kirchoff’s voltage loop rule (Hammill 1993).

$$\mathrm{A}e^{j(\omega t-\varphi)} - i(t)R + L\frac{di}{dt} - nV_T\ln\left(\frac{i(t)}{Is} + 1\right) = 0 \quad (5)$$

Here $Ae^{j(\omega t-\varphi)}$ represents the oscillating voltage source with a controllable amplitude, $L\frac{di}{dt}$ is the voltage drop across the inductor, $i(t)R$ is the voltage drop across the resistor as the current changes with the voltage, and $nV_T\ln(\frac{I}{Is} + 1)$ is the voltage drop across the diode according to Shockley’s theorem. However, Shockley’s theorem does not include the capacitive effects of the diode at high frequencies. V_T is the thermal voltage characteristic of the diode, I_s is the reverse bias saturation current, and *n* is another characteristic of the diode called the ideality factor. This is a much more complex equation but the circuit is much simpler and easier to construct. Easier construction reduces build time and simplifies the synchronization process (Hammill 1993).

Synchronization

Mathematically, synchronized systems can be defined as a situation where one system determines the behavior of another.

$$f(x) = x + \dot{x} \quad (6)$$

$$g(x,y) = (\mu x + (1-\mu)y) + (\mu\dot{x} + (1-\mu)\dot{y}) \quad (7)$$

Here, μ represents the amount of the output that is governed by x and $1 - \mu$ is the remainder that is governed by y. Thus $0 \leq \mu \leq 1$ as it is a fraction of the whole. When $\mu = 0$, $g(x,y)$ is no longer dependent on $f(x)$ making the two uncoupled with a loss in synchronization.

Materials and Methods

A jerk circuit was constructed to model Equation (4) using three integrating sub-circuits to produce the third order differential, and an inverting sub-circuit to bring about a positive first order term. The circuit schematic is shown below in Figure 1. The diode was responsible for the non-linear term. The circuit was constructed using 5% tolerance 1 kΩ resistors and 1.0 μF capacitors, OP27 amplifiers, and a 1N4001 silicon rectifier diode on a standard prototyping board (breadboard as they're colloquially known). The waveforms were observed in an uncoupled jerk circuit using an Agilent Technologies DSO1002A digital oscilloscope, and the bifurcations were measured by substituting R^* with a 10 kΩ potentiometer. The bifurcations were measured by steadily increasing the potentiometer while watching the oscilloscope for period doubling at the positive first differential due to the clarity of the doublings at this point. The potentiometer was then switched to a 1 MΩ potentiometer to examine the higher values of the parameter with decreased accuracy.

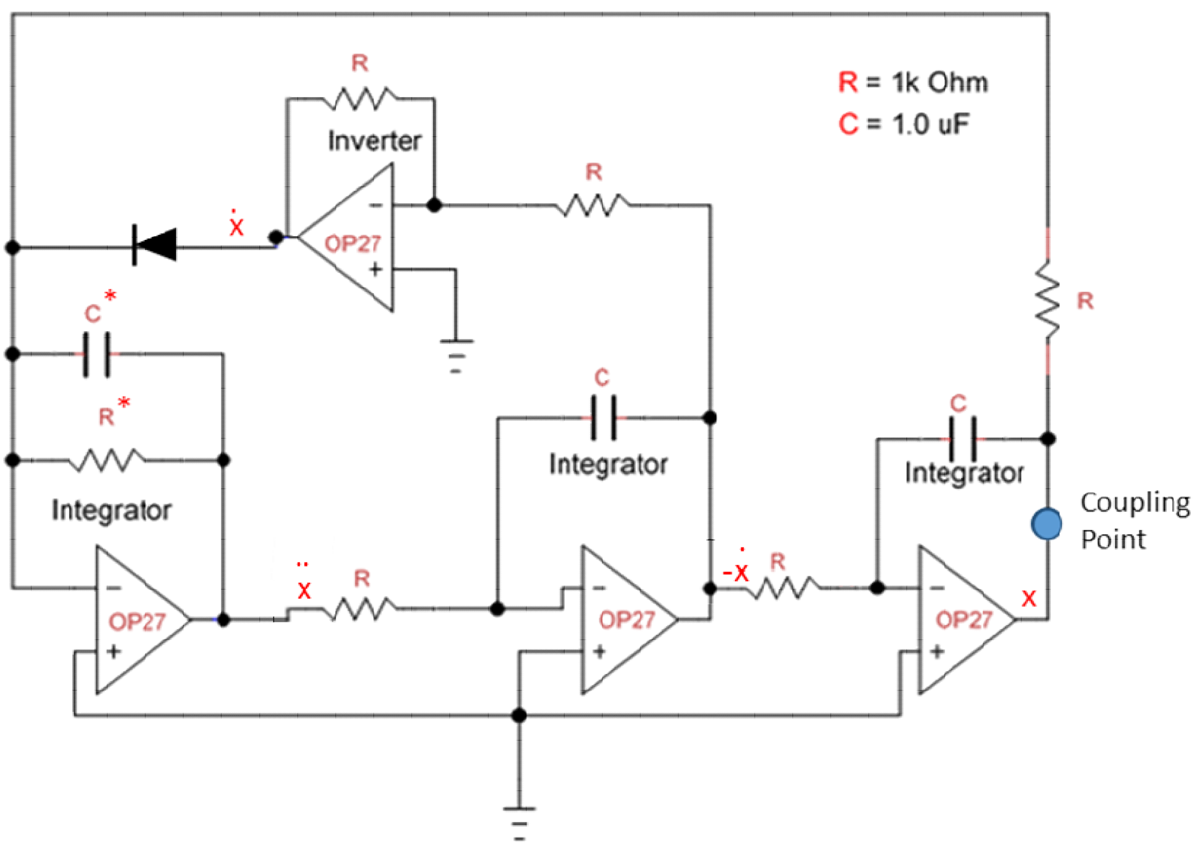


Figure 1. Chaotic circuit modeling a jerk function. It contains three integrating sub-circuits (to create the "jerk" term) and an inverting sub-circuit.

A fixed-value RLD circuit is shown in Figure 2 and was constructed and hardwired to a circuit board with a 5% tolerance 1 Ω resistor, a 1N4001 diode, and a BK Precision 2 MHz signal generator. The inductor for the fixed value circuit was a single 1 mH inductor. In this circuit, bifurcations were found by increasing the voltage output of the signal generators. It is also worth noting that the resistance of the signal generator is 50 ohms and the internal resistance of the inductor is likely and order of magnitude higher than the resistor.

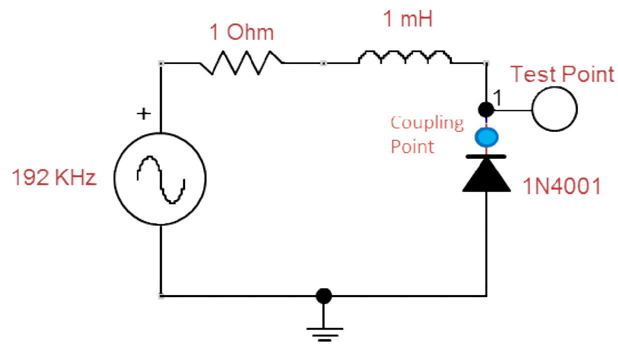


Figure 2. RLD circuit that exhibits chaos when the diode is reverse-biased

After bifurcations were recorded and waveforms observed for each circuit, similar circuits with variable components were built for synchronization purposes. Another Sprott jerk circuit was built with all fixed components except for a 100 kΩ potentiometer at *R**. The two jerk circuits were then coupled with an ordinary jumper wire at the *x* signal points of each circuit, and the second differential waveform of each circuit was probed by the digital oscilloscope. The oscilloscope displayed the waveforms from each circuit as well as the waveform produced by subtracting the fixed circuit waveform from the variable circuit waveform.

The synchronized waveform in Figure 3 has a line in the middle that is the subtraction voltage of the variable circuit voltage from the fixed circuit voltage. The subtraction waveform is not shown in the desynchronized figure because its presence makes the figure very confusing. However, the subtraction waveform was still used when the circuit was desynchronized. The subtraction waveform allowed for desynchronization to be observed easily (Figure 3) due to the abruptness of desynchronization. The desynchronization of two circuits is a rapid event that occurs in a matter of a couple of ohms making it possible to record the synchronization limits with little uncertainty (±5 ohms) when watching an oscilloscope. The 100 kΩ potentiometer was positioned around 1 kΩ and gently increased until either desynchronization or loss of chaos occurred. The value of the potentiometer was then measured using a Fluke multi-meter. The same was done for decreasing the potentiometer from 1 kΩ. The potentiometer again was set to 1 kΩ and the value of *C** was increased by adding more capacitors in parallel with the initial 1.0 µF capacitor including a variable capacitor that allowed for more precise measurements of the desynchronization or loss of chaos parameters. The lower limit was found by adding capacitors in series.

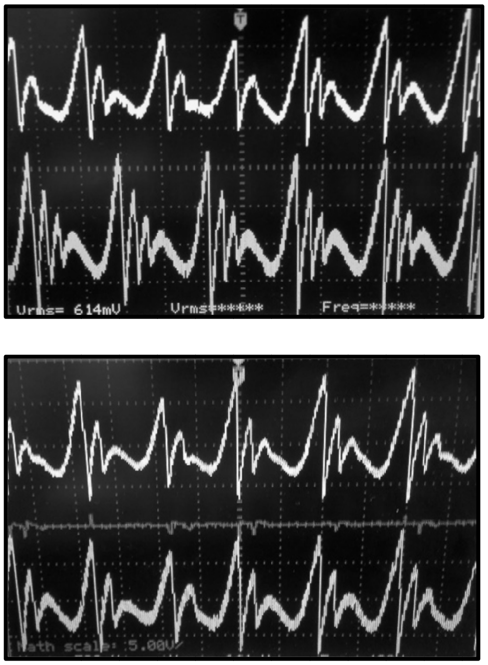


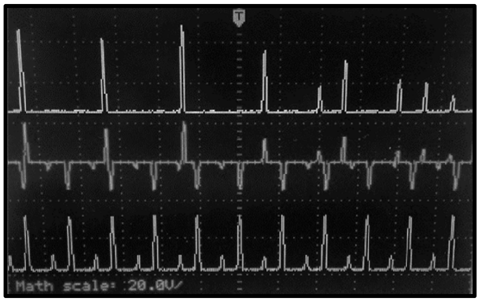
Figure 3. Observed jerk circuit waveforms of desynchronized (above) and synchronized (below)

Next, a similar RLD circuit was built with variable components including, a 10k potentiometer. The inductor for the variable circuit was an array of 8.0 mH inductors added in parallel to achieve 0.998 mH. The oscilloscope probe was attached just before the diode for measurements, and the signal generators were both set to 192 kHz to more accurately match the waveforms seen in previous works.

To synchronize the RLD circuits, the two circuits were coupled by a jumper wire at the same points where the probes were connected. Examples of the synchronized and desynchronized waveforms are shown in Figure 4. The voltages of the generators were varied with respect to one another to find upper and lower synchronization limits and a potentiometer in the variable circuit was used to find the synchronization limits when the resistance is varied. The array of inductors was also varied.

Results

Both the RLD waveforms and the jerk circuit waveforms shown in Figures 5 and 6 are appropriately scaled to match the simulation data on both time and



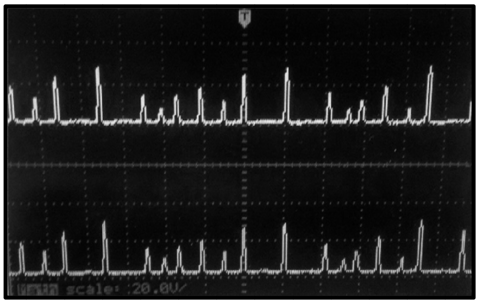


Figure 4. Observed RLD waveforms and subtraction (middle line) of synchronized (top) and desynchronized (bottom) circuits

voltage. The observed waveforms are very similar to those seen in previous works (Sprott 2011, Hammill 1993). While bifurcations were easily observed, there was some uncertainty when defining the instant that a bifurcation occurred. Bifurcations were recorded for the parameter value when the new waveform dominated with little to none of the previous waveform being visible. The sparse amount of data points for the jerk circuit was due to the circuit achieving chaos after three bifurcations. The RLD also produced only three data points due to the bifurcations becoming too minute to observe. It is still useful to see that the ratios from equation (1) are within the range of the Feigenbaum constant with the uncertainty accounted for. The bifurcations and their ratios are in Table 1 below.

The attractor for the Sprott circuit was observed on an analog oscilloscope (Figure 7) by putting the original signal on the x axis and the first derivative on the y axis because it had more time divisions allowing for a better view of the attractor. This observed attractor is very similar to the one found by Sprott (Sprott 2011). The attractor for the RLD circuit was found using the digital oscilloscope (Figure 7) with the signal from coupling point put on the x axis and signal from the signal generator placed on the y axis and appears to provide further evidence of chaos based on the very similar patterns that never repeat.

The desynchronization parameters in Table 1 indicate the high and low values at which the variable components were too far from the fixed value components in the other circuit and caused desynchronization. The desynchronization values show a wide range of conditions where synchronization can occur. The window of capacitance is on the order of a few microFarads. The window for the resistor is on the order of a few thousand ohms for both circuits and the input amplitude difference between the two RLD circuits is around ten volts before synchronization is

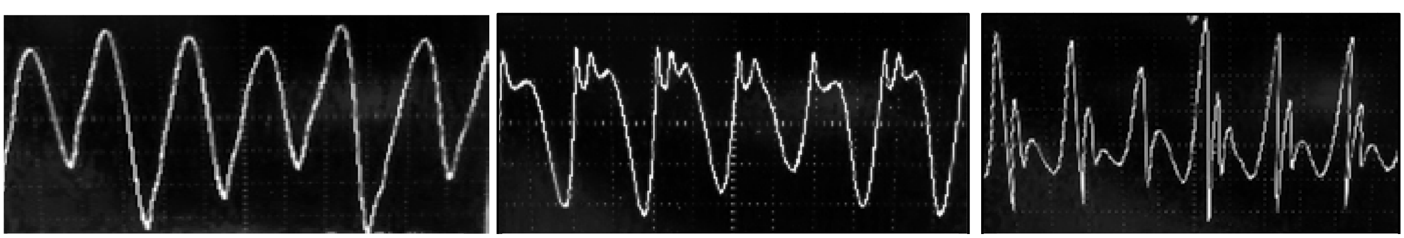


Figure 5. Waveforms from the Sprott circuit observed in this experiment. R* for these waveforms was 1k ohm and C* was 1.0 microFarads.

Figure 6. Observed waveform for the RLD circuit

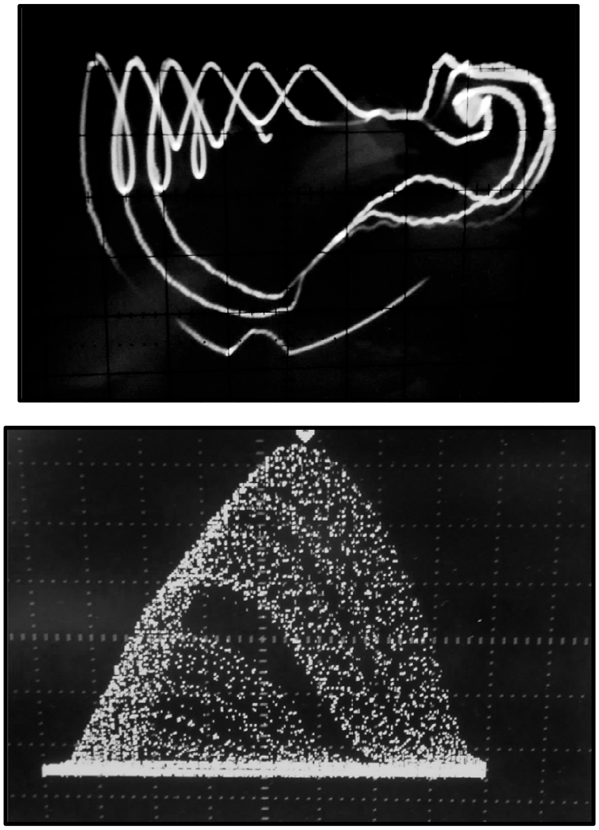


Figure 7. Observed attractors for the Sprott circuit (top) and the RLD circuit (bottom)

lost. Reducing the potentiometer completely did not cause desynchronization which is probably due to the resistances still in the circuit due to the inductors and signal generators. Interestingly, varying the inductance of the RLD circuit did not affect the synchronization.

Discussion

While the desynchronization parameters were able to be determined, the numbers could have been more accurate by using other measuring methods. For instance, running the circuit through LabVIEW would allow for precise, quantitative observation of the difference between the two signals. No anomalies from capacitance inside the prototyping board were observed and the 60 Hz signal from the lights and other external sources appeared to be minimal compared to the actual signal. Hardwiring the RLD circuit after examining its behavior on a prototyping board did not appear to improve the performance of the circuit by much, thus it is believed that leaving the jerk circuit on the prototyping board for measurements had little to no impact on the results. The chaotic range of the RLD circuit could not be fully established because the signal generator could not reach any higher in amplitude than around 10V. However, the range that could be verified is large enough that the RLD circuit is considered robust.

Conclusion

The waveforms observed on the oscilloscope verify that the signals seen in each circuit in this experiment are chaotic and are the same as those in previous works (Hammill 1993, Sprott 2011). The bifurcation points indicate that both circuits approach chaos through period-doubling in accordance with the Feigenbaum constant. The wide range of chaos in each circuit suggested that synchronized chaos would be

Table 1: Bifurcations and desynchronization parameters for each circuit

Circuits	Bifurcations	Bifurcation Ratios	Chaotic Range	Desynchronization Values		
				C (μF)	R (Ω)	V_{source} Difference
Jerk	R (±0.1 kΩ) 0.519 0.960 1.038[c]	5.65	1.038 – 3.940 kΩ	Upper: 4.522 Lower: 0.290	Upper: 3,860 Lower: 48	N/A
RLD	V_{source} (±0.10V) 1.63 3.48 3.88	4.63	5.8V-	N/A	Upper: 2,016 Lower: N/A	Upper: 5.98 Lower: -6.08

c: Chaos occurred at bifurcation point

maintained for largely varying circuits, and the suggestion was verified by the measured ranges of chaotic synchronization. While there are means for achieving more accurate numbers, the large range of variation allowed is undeniable. These findings are the first ones needed to begin an examination of synchronized chaos as a means of cancelling noise. The possibility of noise cancellation is exciting and experimental applications of chaotic noise cancellation through synchronization with these circuits can now be examined (Hammill 1993, Sprott 2011).

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