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On Compactness and Closed-Rangeness of Composition Operators

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Abstract

Let ϕ be an analytic self-map of the unit disk $\mathbb{D} := \{z : |z| < 1\}$. The composition operator C_ϕ defined by $C_\phi(f) = f \circ \phi$ is a bounded linear operator on the Hardy space $H^2(\mathbb{D})$. It is well-known that if C_ϕ is compact on $H^2(\mathbb{D})$ then $\|\phi^n\|_{H^2(\mathbb{D})} \to 0$ as $n \to \infty$. But the converse doesn't necessarily hold. We discuss the decay rate of $\|\phi^n\|_{H^2(\mathbb{D})}$ in the case when ϕ maps the unit disk to a domain whose boundary touches the unit circle exactly at one point. We also investigate inheritance of closed-rangeness property of C_ϕ from a Banach space of analytic functions on $\mathbb{D}$ to a weighted subspace.

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1 Introduction

The Hardy Space $H^2(\mathbb{D})$

Let us denote the unit disk by $\mathbb{D} := \{z : |z| < 1\}$ and let $\mathbb{T} := \partial\mathbb{D} = \{z : |z| = 1\}$ be the unit circle. Let $\mathcal{H}(\mathbb{D})$ denote the space of all analytic functions on $\mathbb{D}$. The Hardy space $H^2(\mathbb{D})$ consists of functions in $H(\mathbb{D})$ whose power series coefficients are square-summable, i.e.

$$H^2(\mathbb{D}) := \left\{ f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{H}(\mathbb{D}) : \sum_{n=0}^{\infty} |a_n|^2 < \infty \right\}.$$

The norm of $f \in H^2(\mathbb{D})$ is defined to be $\|f\|_{H^2(\mathbb{D})} = \left(\sum_{n=0}^{\infty} |a_n|^2 \right)^{\frac{1}{2}}$. This definition of norm gives a vector space isomorphism between $H^2(\mathbb{D})$ and l^2, the Hilbert space of square summable complex sequences. $H^2(\mathbb{D})$ can also be related to the space $L^2(\mathbb{T})$, another Hilbert space of functions. Under this correspondence,

$$H^2(\mathbb{D}) := \left\{ f \in \mathcal{H}(\mathbb{D}) : \sup_{0<r<1} \int_{\mathbb{T}} |f(r\zeta)|^2 dm(\zeta) < \infty \right\}$$

where m denotes normalized Lebesgue measure on $\mathbb{T}$. The norm of any $f \in H^2(\mathbb{D})$ is defined to be

$$\|f\|^2_{H^2(\mathbb{D})} = \sup_{0<r<1} \int_{\mathbb{T}} |f(r\zeta)|^2 dm(\zeta) = \lim_{r \to 1^-} \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{it})|^2 dt.$$

Also the inner product between two functions f and g on $H^2(\mathbb{D})$ is defined as:

$$\langle f, g \rangle := \lim_{r \to 1^-} \frac{1}{2\pi} \int_0^1 f(re^{it}) \overline{g(re^{it})} dt.$$

and the connection between $H^2(\mathbb{D})$ and a closed subspace of $L^2(\mathbb{T})$ was shown very clearly in the following Fatou's Radial Limit Theorem.

Theorem 1 (Fatou's Radial Limit Theorem[44]). *Suppose* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *belongs to* $H^2(\mathbb{D})$*, and* f^* *is a function in* $L^2(\mathbb{T})$ *with Fourier series* $\sum_{n=0}^{\infty} a_n e^{int}$*. Then*

$$\lim_{r\to 1^-} f(re^{it}) = f^*(e^{it})$$

for almost every $e^{it} \in \mathbb{T}$*, and* $\|f\|^2_{H^2(\mathbb{D})} = \|f^*\|^2_{L^2(\mathbb{T})}$.

A detailed discussion and proof of this theorem is available in [44, 42].

Composition Operators on $H^2(\mathbb{D})$

Let ϕ be an analytic self-map of $\mathbb{D}$. Define the composition operator C_ϕ on $H^2(\mathbb{D})$ by

$$C_\phi(f) = f \circ \phi$$

for $f \in H^2(\mathbb{D})$. The following Littlewood's Subordination Principle shows that C_ϕ maps H^2 into H^2 and does so boundedly, i.e. C_ϕ takes bounded subset of $H^2(\mathbb{D})$ to bounded subset of $H^2(\mathbb{D})$.

Theorem 2 (Littlewood's Subordination Principle [44]). *Let* ϕ *be an analytic self-map of* $\mathbb{D}$ *with* $\phi(0) = 0$*. Then for each* $f \in H^2(\mathbb{D}), C_\phi(f) \in H^2(\mathbb{D})$ *and* $\|C_\phi(f)\| \leq \|f\|$.

Though Littlewood's subordination principle only proves the case when ϕ fixes the origin, the general case, where ϕ can be any analytic self-map of $\mathbb{D}$, can be proven by showing the composition operator C_{σ_a} induced by the conformal automorphism $\sigma_a := \frac{a-z}{1-\bar{a}z}$, for $a \in \mathbb{D}$, is bounded on $H^2(\mathbb{D})$.

Theorem 3 (Littlewood's Theorem [44]). *Let* $\phi : \mathbb{D} \to \mathbb{D}$ *be an analytic function. Then* C_ϕ *is bounded on* $H^2(\mathbb{D})$ *, and*

$$\|C_\phi\|_{H^2(\mathbb{D})} \leq \sqrt{\frac{1+|\phi(0)|}{1-|\phi(0)|}}.$$

A proof of this theorem can also be found in [44]. Please see [53, 15, 34, 55, 35, 48] to learn more about boundedness of composition operators in other Banach spaces of analytic functions.

2 Preliminaries

In this section we provide some background concepts related to the study of composition operators.

Angular Derivative

For $\zeta \in \mathbb{T}$ and $\alpha > 1$, the region

$$\Gamma(\zeta, \alpha) = \{z \in \mathbb{D} : |z - \zeta| < \alpha(1 - |z|)\}$$

is called *non-tangential approach region* at ζ. This cone shaped region is asymptotic to a sector with vertex at ζ and angle less than π and is symmetric about the radius at ζ. A function f is said to have a *non-tangential limit* L at ζ if $\lim_{z \to \zeta} f(z) = L$ in each non-tangential approach region $\Gamma(\zeta, \alpha)$, denoted as $\angle \lim_{z \to \zeta} f(z) = L$. An analytic self-map ϕ of $\mathbb{D}$ has an *angular derivative* at $\zeta \in \mathbb{T}$ if for some $\eta \in \mathbb{T}$, the following limit

$$\angle \lim_{z \to \zeta} \frac{\eta - \phi(z)}{\zeta - z}$$

exists (finitely). We denote the angular derivative of ϕ at ζ as $\phi^{'}(\zeta)$ whenever the above limit exists [44, 15].

One very important result concerning the existence of angular derivative is the Julia-Carathéodory theorem.

Theorem 4 (Julia-Carathéodory Theorem [44])**.** *Let $\phi : \mathbb{D} \to \mathbb{D}$ be an analytic function and $\zeta \in \mathbb{T}$. Then the following statements are equivalent:*

1. $\liminf_{z \to \zeta} \frac{1 - |\phi(z)|}{1 - |z|} = \delta < \infty$,

2. $\angle \lim_{z \to \zeta} \frac{\eta - \phi(z)}{\zeta - z}$ *exists for some* $\eta \in \mathbb{T}$,

3. $\angle \lim_{z\to\zeta} \phi'(z)$ *exists, and* $\angle \lim_{z\to\zeta} \phi(z) = \eta \in \mathbb{T}$.

Moreover:

- $\delta > 0$ *in (1),*
- *the boundary points* η *in (2) and (3) are the same, and*
- *the limit of the difference quotient in (2) coincides with that of the derivative in (3), with both equal to* $\overline{\zeta}\eta\delta$.

For a beautiful proof of this classical theorem please refer to [44]. Before stating another major theorem on the existence of angular derivative we need to introduce the concept of angular derivative in the upper half-plane $\Im z > 0$ setting. This discussion is taken from [50]. Let Δ be a simply connected domain on the $w = \xi + i\eta$ plane, bounded by a Jordan curve C, which passes through $w = 0$ and touches the real axis at $w = 0$ and its inner normal at $w = 0$ coincides with the positive imaginary axis. We map Δ conformally on the upper half plane $\Im z > 0$ of $z = x + iy$ plane by $w = w(z)$, $w(0) = 0$.

If $\angle \lim_{z\to 0} \frac{w(z)}{z} = \angle \lim_{z\to 0} w'(z) = \gamma$ exists, then γ is called the angular derivative of $w(z)$ at $z = 0$. Here is a niceness condition on the behavior of C.

Theorem 5 (Warschawski's Theorem [50])**.** *Let* Δ *be a simply connected domain on the* $w = \xi + i\eta$*-plane, bounded by a Jordan curve* C*, which passes through* $w = 0$ *and touches the real axis at* $w = 0$ *and its inner normal at* $w = 0$ *coincides with the positive imaginary axis.*

We assume that in a neighborhood of $w = 0$*,* C *lies between two curves* H *an* $\tilde{H}$*, each of which lies symmetric to the imaginary axis and whose part on the right of the imaginary axis as follows:*

$H : \eta = h(\xi)$, $\tilde{H} : \eta = -h(\xi)$ $(0 \leq \xi \leq 1)$ *and* $h(0) = 0$,*where* $h(t) \geq 0$ *is a continuous increasing function of* t.

If we map Δ conformally on the upper half-plane $\Im z > 0$ of the z-plane by $w = w(z)$, $w(0) = 0$, then

$$\lim_{z\to 0} \frac{w(z)}{z} = \gamma$$

exists uniformly, if $z \to 0$ from the inside of any fixed nontangential approach region whose vertex is at $z = 0$ if and only if

$$\int_0^1 \frac{h(t)}{t^2} dt$$

is finite.

Warschawski's theorem gives a necessary and sufficient condition for the existence of the angular derivative. For a proof of necessity and sufficiency of the above condition please refer to Theorem IX.10 in [50].

J.H.Shapiro [44] restated Warschawski's theorem for the case when the map ϕ from $\mathbb{D}$ to the simply connected domain Δ is univalent and touches $\mathbb{T}$ at exactly one point.

Corollary 6 ([44]). *Suppose Δ is a Jordan domain in $\mathbb{D}$ whose boundary curve in a neighborhood of 1 is a curve of the form*

$$1 - r = h(|t|)$$

where $h : [0, 1] \to [0, 1]$ is a continuous, increasing, function with $h(0) = 0$. Let ϕ be a univalent map of $\mathbb{D}$ onto Δ, with $\phi(1) = 1$. Then ϕ has an angular derivative at 1 if and only if

$$\int_0^1 \frac{h(t)}{t^2} dt$$

is finite.

Compact Operator and Approximation Numbers

Before we explore compactness of composition operators, let us first refresh our memory with the definition of a compact operator: a linear operator T on a Hilbert space S is said to be compact if it maps every bounded set into a relatively compact one (one whose closure in S is compact). It is a known fact that on an infinite dimensional Hilbert space, if a bounded operator has finite dimensional range then it is also compact. It can also be argued that on an infinite dimensional Hilbert space compact operators can be approximated in operator norm by such finite rank operators and every compact operator arise in this way. The following theorem restates this as a property of compact operators on an infinite dimensional Hilbert space, whose proof can be found in [44].

Theorem 7 (Finite Rank Approximation Property)**.** *Suppose T is a bounded linear operator on a Hilbert space S. Then T is compact if and only if there is a sequence $\{R_n\}$ of finite rank bounded operators such that $\|T - R_n\| \to 0$, as $n \to \infty$.*

Let us denote the distance in operator norm between T and the set of bounded operators on S with rank $\leq n$ as $a_n(T)$. From the above theorem it is clear that T is compact if and only if $a_n(T) \to 0$, as $n \to \infty$. We call these $a_n(T)$'s *approximation numbers*. Later we will discuss some recent results relating the decay rate of these approximation numbers and compact composition operators.

Schatten Class Operators

This section is taken largely from K. Zhu's book [53]. Let H be any Hilbert space and T be any continuous linear operator on H. As a consequence of the Riesz representation theorem there exists an unique continuous linear operator T^* such that

$$\langle Tx, y\rangle = \langle x, T^*y\rangle \qquad \forall x, y \in S$$

and $\|T\| = \|T^*\|$. Now we say that a continuous linear operator T on H is self-adjoint if $T^* = T$. It can easily be seen that T is self-adjoint when and only when the inner product $\langle Tx, x\rangle$ is real for all $x \in S$. If $\langle Tx, x\rangle$ is non-negative then we call T a positive operator. For example, for any operator T on H, T^*T is positive and hence self-adjoint.

Any continuous (bounded) linear operator T on H can be decomposed as

$$T = UP$$

where P is the positive operator $(T^*T)^{\frac{1}{2}}$ and U is a partial isometry defined as $\|Ux\| = \|x\|$ for all x in the closure of the range of $(T^*T)^{\frac{1}{2}}$. This decomposition of T is called *polar decomposition*.

The Spectral theorem for compact self-adjoint operators states : if T is any self-adjoint compact operator on H, then there exists a sequence of nonzero real numbers $\{\lambda_n\}$, either finitely many or $\{\lambda_n\}$ tends to 0 and an orthonormal sequence $\{e_n\}$ in the closure of the range of T such that

$$Tx = \sum_{n=1}^{\infty} \lambda_n \langle x, e_n\rangle e_n$$

for all $x \in H$. These $\{\lambda_n\}$ are eigenvalues of T and $\{e_n\}$ are corresponding eigenvectors. If in addition T is positive then these $\{\lambda_n\}$ are also positive for each n.

So in the case T is only compact but not necessarily self-adjoint we consider the positive operator $(T^*T)^{\frac{1}{2}}$. Then by the Spectral theorem we have the following decomposition of $(T^*T)^{\frac{1}{2}}$:

$$(T^*T)^{\frac{1}{2}} x = \sum_{n=1}^{\infty} \lambda_n \langle x, e_n\rangle e_n$$

where $\{\lambda_n\}$ are eigenvalues of $(T^*T)^{\frac{1}{2}}$ and $\{e_n\}$ are corresponding eigenvectors. From the polar decomposition of T if we take $Ue_n = \sigma_n$ for each n then $\{\sigma_n\}$ is also an orthonormal sequence in H. Now we have the following decomposition of T:

$$Tx = \sum_{n=1}^{\infty} \lambda_n \langle x, e_n \rangle \sigma_n \qquad \forall x \in H.$$

Indeed, any compact operator on a Hilbert space can be decomposed in this form. The non-negative real values $\{\lambda_n\}$ are called nth singular values of T.

For $0 < p < \infty$, the *Schatten p-class*, $S_p(H)$), consists of compact operators T for which the sequence of singular values $\{\lambda_n\}$ belongs to l^p. It is equivalent as saying T is in $S_p(H)$ when the sequence of approximation numbers $\{a_n(T)\}$ of T is in l^p, which implies $\sum_{n=0}^{\infty} a_n^p(T) < \infty$. There are several characterizations of Schatten class operators. We mention a couple of these characterizations for future use.

Theorem 8 ([53])**.** *Suppose T is a compact operator on a Hilbert space H. Then the following are true:*

1. *For $p \geq 1$, T is in $S_p(H)$ if and only if for all orthonormal sequences $\{e_n\}$ in S,*

$$\sum_{n=1}^{\infty} |\langle Te_n, e_n \rangle|^p < \infty.$$

2. *For $p \geq 2$, T is in $S_p(H)$ if and only if for all orthonormal sequences $\{e_n\}$ in S,*

$$\sum_{n=1}^{\infty} \|Te_n\|^p < \infty.$$

Nevanlinna Counting Function

The Nevanlinna Counting Function is a heavily used tool in characterizing properties of composition operators. For an analytic self-map of $\mathbb{D}$ and $w \in \phi(\mathbb{D}) \backslash \{\phi(0)\}$, the Nevanlinna Counting Function for ϕ is defined as:

$$N_\phi(w) := \sum_{z\in\phi^{-1}(w)} \log\frac{1}{|z|},$$

counting multiplicities of the zeros of $\phi(z) = w$.

Also it should be noted that $N_\phi(w) = 0$ whenever $w \notin \phi(\mathbb{D})\backslash\{\phi(0)\}$ to make sure it is defined on the whole disk $\mathbb{D}$.

Induced Measure

Let ϕ be an analytic self-map of $\mathbb{D}$. Then the radial and nontangential limits of ϕ exist almost everywhere $[m]$ on $\mathbb{T}$. We denote the boundary limit function as ϕ^*. Define the *induced measure* of ϕ on the Borel subsets E of $\overline{\mathbb{D}}$ as

$$\mu_\phi(E) = m(\{\zeta \in \mathbb{T} : \phi^*(\zeta) \in E\}).$$

C. Sundberg [47] provided some useful results involving induced measure and answered a more than a decade old question posed by W. Rudin.

Harmonic Measure

The concept of harmonic measure plays an important role in our work. Though harmonic measure is discussed in several books, this section is largely taken from [19]. Let Δ be a domain in the extended complex plane in which the Dirichlet problem is solvable, i.e. given a continuous function $f(\zeta)$ on the boundary $\partial\Delta$, we can find an unique function $u(z)$, harmonic in Δ and continuous on $\overline{\Delta}$ such that $u(\zeta) = f(\zeta)$ for all $\zeta \in \partial\Delta$. It is shown in [19] that we can associate a harmonic function $Hf(z)$, the solution to the Dirichlet problem in Δ with the boundary function $f(\zeta)$. If $z \in \Delta$ fixed , then there is a linear mapping

$$H_z : C(\partial\Delta) \to \mathbb{R}$$

where $C(\partial\Delta)$ denotes the space of all continuous real valued functions on $\partial\Delta$, defined by

$$H_z(f) = Hf(z).$$

Additionally if we take f to be non-negative then by the Maximum Principle, $H_z(f)$ is a positive, linear functional on $C(\partial\Delta)$. By the Riesz Representation Theorem [42] there exists a unique (probability) measure μ_z defined on $\partial\Delta$ such that

$$Hf(z) = \int_{\partial\Delta} f(\zeta)d\mu_z(\zeta).$$

Definition. *Suppose Δ is any domain. Let E be a Borel set on the boundary $\partial\Delta$ of Δ. The* harmonic measure *of E with respect to Δ is defined as:*

$$\omega(z, E, \Delta) := \int_E d\mu_z(\zeta) = \mu_z(E).$$

An important feature of harmonic measure is conformal invariance: If ϕ is a conformal map from $\mathbb{D}$ to some domain Δ with its boundary consits of finitely connected Jordan arcs and in addition, ϕ is also continuous and injective on $\mathbb{T}$ then, for any Borel set $E \subset \mathbb{T}$, $\omega(z, E, \mathbb{D}) = \omega(\phi(z), \phi(E), \Delta)$; see [19]. It is well-known that if ϕ is a conformal map from the unit disk $\mathbb{D}$ onto a Jordan domain Δ then ϕ has a continuous extension to $\overline{\mathbb{D}}$ and the extension map is an one-to-one correspondence between $\overline{\mathbb{D}}$ and $\overline{\Delta}$. See Theorem 3.1 in [20] or Theorem IX.2 in [50]. Using this fact it can be shown that :

Proposition 9. *Suppose ϕ is a univalent map from $\mathbb{D}$ onto a Jordan domain Δ which is bounded by a rectifiable curve. Let $\phi(0) = \alpha$ and $\omega(\alpha, ., \Delta)$ be the harmonic measure on $\partial\Delta$ at α. Then*

$$d\omega = |\psi^{'}|d\xi$$

where $\psi = \phi^{-1}$, $d\xi$ denotes the arclength measure on $\partial\Delta$ and $\psi^{'}$ is defined as *non-tangential or angular limit.*

This is a known result in harmonic measure theory and for a proof of this proposition, please refer to [50, 20, 14, 22]

General Hardy spaces $H^p(\mathbb{D})$

For $1 \leq p < \infty$, the general Hardy spaces $H^p(\mathbb{D})$, are defined as follows:

$$H^p(\mathbb{D}) = \left\{ f \in \mathcal{H}(\mathbb{D}) : \sup_{0<r<1} \int_{\mathbb{T}} |f(r\zeta)|^p dm(\zeta) < \infty \right\}.$$

These are all Banach spaces under the norm $\|f\|_{H^p}^p = \sup_{0<r<1} \int_{\mathbb{T}} |f(r\zeta)|^p dm(\zeta)$. The Banach space $H^\infty(\mathbb{D})$ is called the space of bounded analytic functions on $\mathbb{D}$ and it is defined as:

$$H^\infty(\mathbb{D}) = \left\{ f \in \mathcal{H}(\mathbb{D}) : \sup_{z\in\mathbb{D}} |f(z)| < \infty \right\}.$$

Weighted Bergman Spaces $\mathbb{A}_\alpha^p$

For $\alpha > -1$, let λ_α denote the finite measure defined on $\mathbb{D}$ by

$$d\lambda_\alpha(z) = (1 - |z|^2)^\alpha dA(z).$$

where A denotes normalized Lebesgue area measure on $\mathbb{D}$.

For $0 < p < \infty$ the weighted Bergman spaces $\mathbb{A}_\alpha^p$ are defined by

$$\mathbb{A}_\alpha^p = \left\{ f \in \mathcal{H}(\mathbb{D}) : \int_{\mathbb{D}} |f|^p d\lambda_\alpha < \infty \right\}.$$

For $p \geq 1$, the weighted Bergman spaces $\mathbb{A}_\alpha^p$ are Banach spaces under the norm $\|f\|_{\mathbb{A}_\alpha^p}^p = \int_{\mathbb{D}} |f|^p d\lambda_\alpha$. When $p = 2$, $\mathbb{A}_\alpha^2$ are Hilbert spaces.

Weighted Dirichlet Spaces D^2_α

The weighted Dirichlet spaces D^2_α, $\alpha > -1$, is the collection of analytic functions of $\mathbb{D}$ such that f' is in $\mathbb{A}^2_\alpha$. D^2_α is a Hilbert space in the following norm:

$$\|f\|^2_{D^2_\alpha} = |f(0)|^2 + \int_{\mathbb{D}} |f'|^2 d\lambda_\alpha$$

for $f \in D^2_\alpha$. For $\alpha = 0$, D^2_0 is called the classical Dirichlet space. Also for $\alpha = 1$, D^2_1 is the Hardy space $H^2(\mathbb{D})$.

Carleson Measure

Carleson measure plays a crucial role in the composition operator theory. For $e^{i\theta_0} \in \mathbb{T}$ and $h > 0$, a Carleson window or square is a square-shaped region near the boundary of unit disk $\mathbb{D}$ defined as:

$$S_h(e^{i\theta_0}) = \{re^{i\theta} : 1 - h \leq r < 1, |\theta - \theta_0| \leq h\}.$$

A positive Borel measure μ is called a Carleson measure if and only if there exists a constant $K > 0$ such that $\mu(S_h(e^{i\theta_0})) \leq Kh$ for all $e^{i\theta} \in \mathbb{T}$ and $h > 0$. μ is called a compact or vanishing Carleson measure if

$$\lim_{h \to 0} \frac{\mu(S_h(e^{i\theta_0}))}{h} = 0$$

uniformly for $e^{i\theta_0} \in \mathbb{T}$ [53].

For the weighted Bergman spaces $\mathbb{A}^p_\alpha$, a positive Borel measure on $\mathbb{D}$ is an $\alpha-$Carleson measure if and only if there exists a constant $K > 0$ such that $\mu(S_h(e^{i\theta_0})) \leq Kh^{\alpha+2}$ for all $e^{i\theta} \in \mathbb{T}$ and $h > 0$. A compact or vanishing Carleson measure on these spaces is defined in the same way we defined it earlier. The definition of Carleson measure would not be complete without the famous theorem of L. Carleson:

Theorem 10 ([53]). *For $1 \leq p < \infty$, a positive Borel measure μ on $\mathbb{D}$ is a Carleson measure if and only if there exists a constant $K > 0$ such that*

$$\int_{\mathbb{D}} |f|^p d\mu \leq K \|f\|^p_{H^p(\mathbb{D})}$$

for each $f \in H^p(\mathbb{D})$, where $H^p(\mathbb{D})$ are the general Hardy spaces.

This characterization of Carleson measure can also be rephrased in the setting of weighted Bergman spaces $\mathbb{A}^p_\alpha$. More information on Carleson measure can be found in [53].

Bloch Space

An analytic function f on $\mathbb{D}$ belongs to the Bloch space $\mathcal{B}$ if

$$\|f\|_{\mathcal{B}^\sharp} = \sup_{z \in \mathbb{D}} (1 - |z|^2)|f^{'}(z)| < \infty.$$

The norm $\|f\|_{\mathcal{B}} = |f(0)| + \|f\|_{\mathcal{B}^\sharp}$ makes $\mathcal{B}$ a Banach space. One very important feature of $\mathcal{B}$ is its Möbius invariance. In particular, if $f \in \mathcal{B}$ and $\sigma_a (= \frac{a-z}{1-\bar{a}z}, a \in \mathbb{D})$ is a Möbius transformation of $\mathbb{D}$ then $f \circ \sigma_a \in \mathcal{B}$. One can easily verify that

$$\|f \circ \sigma_a\|_{\mathcal{B}^\sharp} = \|f\|_{\mathcal{B}^\sharp}.$$

It is known that bounded analytic functions on $\mathbb{D}$ are in $\mathcal{B}$. Associated to Bloch space there is little Bloch space $\mathcal{B}_0$, consists of analytic functions of $\mathbb{D}$ for which

$$\lim_{|z| \to 1^-} (1 - |z|^2)|f^{'}(z)| = 0.$$

$\mathcal{B}_0$ is also Möbius invariant and a closed subspace of $\mathcal{B}$. In fact, $\mathcal{B}_0$ is the closure of polynomials in $\mathcal{B}$. For a detailed discussion on Bloch spaces and the above results please refer to [53]. It should also be noted that bounded analytic functions are not properly contained in $\mathcal{B}_0$. So one may ask which bounded analytic functions are in $\mathcal{B}_0$. Detailed information on the history of this question and answers can be found in the article [8] by C. J. Bishop.

Besov Space

For $1 < p < \infty$, the Besov space B_p is the space of analytic functions f on $\mathbb{D}$ such that

$$\|f\|_{B_p^\sharp}^p = \int_{\mathbb{D}} |f^{'}(z)|^p (1-|z|^2)^{p-2} dA(z) < \infty.$$

The norm $\|f\|_{B_p}^p = |f(0)| + \|f\|_{B_p^\sharp}^p$ makes B_p a Banach space. It can be easily shown that each B_p is a Möbius invariant Banach space. Note that for $p = 2$, the space B_2 is the classical Dirichlet space D_0^2 discussed earlier. Another interesting fact is, for $1 < p < q < \infty$, $B_p \subset B_q \subset \mathcal{B}$.

For $1 < p < \infty$ and $\alpha > -1$, the Besov type spaces $B_{p,\alpha}$ are defined as:

$$B_{p,\alpha} = \left\{ f \in \mathcal{H}(\mathbb{D}) : \int_{\mathbb{D}} |f'(z)|^p d\lambda_\alpha < \infty \right\}$$

which are Banach spaces under the norm: $\|f\|_{B_{p,\alpha}}^p = |f(0)|^p + \int_{\mathbb{D}} |f'(z)|^p d\lambda_\alpha$. More information on Besov space and Besov type spaces can be found in [53, 48, 49].

$S^2(\mathbb{D})$

The space $S^2(\mathbb{D})$ is another Banach space of analytic functions on $\mathbb{D}$, defined as follows:

$$S^2(\mathbb{D}) := \{f \in \mathcal{H}(\mathbb{D}) : f' \in H^2(\mathbb{D})\}.$$

The norm on this space is given by: $\|f\|_{S^2(\mathbb{D})}^2 = |f(0)|^2 + \frac{1}{2\pi} \int_0^{2\pi} |f'(e^{it})|^2 dt$.

BMOA

An analytic function f on $\mathbb{D}$ is in BMOA if

$$\|f\|_{BMOA^\sharp} = \sup_{a\in\mathbb{D}} \|f \circ \sigma_a(z) - f(a)\|_{H^2(\mathbb{D})} < \infty.$$

The norm on BMOA can be defined by $\|f\|_{BMOA} = |f(0)| + \|f\|_{BMOA^\sharp}$. BMOA also a Möbius invariant Banach space. An interesting containment relation is:

$$B_p \subset BMOA \subset \mathcal{B}.$$

Some good references on BMOA (including Bloch space and Besov space) are [53, 48, 10, 38].

Determining Functions

Determining functions and Nevanlinna counting functions discussed earlier are similar in nature. Determining functions were introduced in [55]. For $\alpha > -1$ and ϕ an analytic self-map of $\mathbb{D}$, define

$$\tau_{\phi,\alpha+2}(w) = \frac{\sum_j (1 - |z_j(w)|)^{\alpha+2}}{(1 - |w|)^{\alpha+2}}$$

where $w \in \phi(\mathbb{D})$, $\{z_j(w)\}$ is the set of all preimages of w, counting multiplicities, and $\tau_{\phi,\alpha+2}(w) = 0$ when $w \notin \phi(\mathbb{D})$. $\tau_{\phi,\alpha+2}(w)$ is called the determining functions for the composition operator C_ϕ on $D_{\alpha+2}$. Note that the numerator in the above expression looks very similar to a generalized version of the Nevanlinna counting function we discussed earlier.

Absolutely Monotonic Functions

A function $f(x)$ is *absolutely monotonic* in the interval $a \leq x \leq b$ if it is continuous on $[a, b]$ and all of its derivatives of all orders are non-negative on (a, b) (see [51]). For example, $f(x) = c$, where c is any non-negative constant, is an absolutely monotonic function on $\mathbb{R}$. Another class of examples are functions which can be represented as powers series of the form, $f(x) = \sum_{k=0}^{\infty} a_k x^k$, where $0 \leq x \leq 1$ and $a_k \geq 0$. Also sum, product and composition of absolutely monotonic functions are absolutely monotonic; see Theorem 2a in [51].

Absolutely monotonic functions are necessarily analytic. The following theorem points out the analyticity of absolutely monotonic functions.

Theorem 11 ([51])**.** *If $f(x)$ is absolutely monotonic in $a \leq x < b$, then it can be extended analytically into the complex plane, and the function $f(z)$ will be analytic in the circle*

$$|z - a| < b - a.$$

To learn more about absolutely monotonic functions please see chapter 4 of [51].

Closed-Range Operators on Banach Spaces

Closed-range operators are the ones whose range is a closed subspace of the image space. To characterize closed-range operators on any Banach space, first we need to introduce the concept of bounded below operators. This discussion here is taken largely from [1].

An operator $T : \mathfrak{X} \to \mathfrak{Y}$ between two Banach spaces is said to be *bounded below* if there exists a constant $\varepsilon > 0$ such that

$$\|Tx\| \geq \varepsilon \|x\|$$

for each $x \in \mathfrak{X}$.

This following theorem completely characterizes bounded below operators on Banach spaces.

Theorem 12. *A continuous operator $T : \mathfrak{X} \to \mathfrak{Y}$ between Banach spaces is bounded below if and only if T is injective and has closed-range.*

The theorem above is a consequence of *open mapping theorem* and its proof can be found in [1]. The above characterization can also be interpreted as: for any bounded operator $T : \mathfrak{X} \to \mathfrak{Y}$ between two Banach spaces, there exists a constant $\varepsilon > 0$ such that for each $y \in \text{range}(T)$ there exists some $x \in \mathfrak{X}$ satisfying $y = Tx$ and $\|x\| \leq \varepsilon\|y\|$ if and only if T has closed range. For detailed discussion on closed-range operators including this section please refer to [16, 1].

3 History on the Compactness of Composition Operators

The study of Composition Operators is a delightful subject which has its origin in 1960s in the works of such mathematicians as E. Nordgren[37] and H. J. Schwartz[41]. They have been studied extensively on several Banach spaces of analytic functions on different types of simply connected domains in the complex plane. We already know, from the Littlewood's theorem, that every composition operator on $H^2(\mathbb{D})$ is bounded. So now it is natural to be curious about the compactness of composition operator. The following result is first of its kind and can be proven in a straightforward way. For a proof refer to [44].

Theorem 13 ([44])**.** *Suppose ϕ is an analytic self -map of $\mathbb{D}$. If $\|\phi\|_\infty < 1$ then C_ϕ is a compact operator on $H^2(\mathbb{D})$.*

So it tells us if the image of the unit disk $\mathbb{D}$ under the map ϕ is merely relatively compact then C_ϕ is compact on $H^2(\mathbb{D})$. Shapiro and Taylor[45, 44] improved the first compactness theorem by showing that if $\sum_{n=0}^{\infty}\|\phi^n\|^2 < \infty$ then C_ϕ is compact on $H^2(\mathbb{D})$.

Theorem 14 (Hilbert-Schmidt Theorem for composition operators [44])**.** *Suppose ϕ is an analytic self -map of $\mathbb{D}$. If $\int_{\mathbb{T}} \frac{1}{1-|\phi(\zeta)|^2} dm(\zeta) < \infty$ then C_ϕ is a compact operator on $H^2(\mathbb{D})$.*

Composition operators which satisfy the above condition in Theorem 14 are called *Hilbert-Schmidt operators.* Shapiro and Taylor also gave an example of a new class of maps which induce Hilbert-Schmidt composition operators.

Theorem 15 ([44])**.** *Suppose ϕ is an analytic self -map of $\mathbb{D}$. If $\phi(\mathbb{D})$ is contained in a polygon inscribed in $\mathbb{T}$, then C_ϕ is Hilbert-Schmidt on $H^2(\mathbb{D})$.*

The operator-theoretic definition of compactness for Hilbert space operators involves the concept of *weak convergence*: A sequence $\{s_n\}$ in a Hilbert space S is said to converge *weakly* to $s \in S$ if $\langle s_n, u\rangle \to \langle s, u\rangle$, as $n \to \infty$, for every $u \in S$. A compact operator T on a Hilbert space S takes a weakly convergent sequence $\{s_n\}$ into a norm convergent sequence. Here is a version of this statement in the case of composition operators.

Theorem 16 ([34]). *Suppose ϕ is an analytic self-map of $\mathbb{D}$. Then a necessary and sufficient condition for C_ϕ to be a compact operator on $H^2(\mathbb{D})$ is the following: for each sequence $\{f_n\}$ bounded in $H^2(\mathbb{D})$ and uniformly convergent to 0 on compact subsets of $\mathbb{D}$, the sequence $\{C_\phi(f_n)\}$ also converges to 0 in the $H^2(\mathbb{D})$ metric.*

With the help of the Theorem 16 it has been shown that the composition operator C_ϕ can fail to be compact if $\phi(e^{it})$ approaches boundary $\mathbb{T}$ too quickly, even if it happens at only one point. For example, let $0 < \lambda < 1$ and $\phi(z) = \lambda z + (1 - \lambda)$. Then C_ϕ is not compact on $H^2(\mathbb{D})$ [44]. So it seems reasonable that if a self-map of the unit disk induces a non-compact composition operator, then any map whose values approach the boundary $\mathbb{T}$ faster should also induce a non-compact operator. This intuition gives rise to another compactness theorem.

Theorem 17 (Comparison Principle [44]). *Suppose ϕ and ψ are analytic self-maps of $\mathbb{D}$, with ϕ univalent and $\psi(\mathbb{D}) \subset \phi(\mathbb{D})$. If C_ϕ is a compact operator on $H^2(\mathbb{D})$, then so is C_ψ.*

Theorem 17 gives birth to an important corollary which characterizes a class of non-compact composition operators.

Corollary 18 ([44]). *Suppose ϕ is an univalent analytic self -map of $\mathbb{D}$, and that the image of the unit disk under the map ϕ contains a disk that is tangent to $\mathbb{T}$. Then C_ϕ is not compact.*

A necessary and sufficient condition for compactness of C_ϕ when ϕ is univalent, was proved by B. MacCluer and J. Shapiro [34].

Theorem 19 (Univalent Compactness Theorem[44]). *Suppose ϕ is an univalent analytic self -map of $\mathbb{D}$. Then C_ϕ is compact on $H^2(\mathbb{D})$ if and only if*

$$\lim_{|z|\to 1^-} \frac{1 - |\phi(z)|}{1 - |z|} = \infty.$$

Now with the help of the Julia-Carathéodory theorem the above theorem can be restated as follows:

Corollary 20 ([44]). *Suppose ϕ is an univalent analytic self -map of $\mathbb{D}$. Then C_ϕ is compact on $H^2(\mathbb{D})$ if and only if ϕ has no angular derivative at any point of $\mathbb{T}$.*

Please note that that the univalence criteria of ϕ is necessary only for the reverse direction in Corollary 20. Now if the univalent analytic self-map ϕ satisfies all the conditions in Warschawski's theorem on angular derivative then C_ϕ is compact if and only if $\int_0^1 \frac{h(t)}{t^2} dt$ diverges.

So far we have a necessary and sufficient condition for compactness of composition operator in the case when the inducing map ϕ is univalent. But what happens in the case of arbitrary analytic self-map ϕ ? The following result is due to B. D. MacCluer [32].

Corollary 21 ([15]). *Suppose ϕ is an analytic self -map of $\mathbb{D}$. Then C_ϕ is compact on $H^2(\mathbb{D})$ if and only if*

$$\lim_{h \to 0} \frac{\mu_\phi(S_h(e^{i\theta}))}{h} = 0$$

where μ_ϕ is the *induced measure* of ϕ and $S_h(e^{i\theta_0}) = \{re^{i\theta} : 1 - h \leq r < 1, |\theta - \theta_0| \leq h\}$. It is shown in [53] that the above condition is satisfied only when

$$\lim_{|p| \to 1^-} \int_{\mathbb{D}} \frac{1 - |p|^2}{|1 - \overline{p}\xi|^2} d\mu_\phi(\xi) = 0$$

which is equivalent to the following condition:

$$\lim_{|p| \to 1^-} \int_{\mathbb{T}} \frac{1 - |p|^2}{|1 - \overline{p}\phi(\zeta)|^2} dm(\zeta) = 0. \qquad (\spadesuit)$$

Shapiro [43] also gave a necessary and sufficient condition for compactness of composition operators in the case when the inducing map ϕ is any analytic self-map of $\mathbb{D}$ by computing the *essential norm* of the composition operator, where essential norm of a composition operator is defined to be its distance in the operator norm from the space of compact operators on $H^2(\mathbb{D})$.

Theorem 22 ([43])**.** *Suppose ϕ is an analytic self-map of $\mathbb{D}$. Let $\|C_\phi\|_e$ denote the essential norm of C_ϕ. Then*

$$\|C_\phi\|_e^2 = \limsup_{|w|\to 1^-} \frac{N_\phi(w)}{\log \frac{1}{|w|}}.$$

In particular, C_ϕ is compact on $H^2(\mathbb{D})$ if and only if $\lim_{|w|\to 1^-} \frac{N_\phi(w)}{\log \frac{1}{|w|}} = 0.$

J. A. Cima and A. L. Matheson [12] observed the connection between essential norm of a composition operator and condition (♠), which can be stated as an identity as follows:

Theorem 23. *Suppose ϕ is an analytic self-map of $\mathbb{D}$. Let $\|C_\phi\|_e$ denote the essential norm of C_ϕ. Then*

$$\|C_\phi\|_e^2 = \limsup_{|p|\to 1^-} \int_{\mathbb{T}} \frac{1-|p|^2}{|1-\overline{p}\phi(\zeta)|^2} dm(\zeta).$$

J. R. Akeroyd [2] gave a direct function-theoretic proof of the above identity.

In 1988, D. Sarason asked, "do there exist compact composition operators which do not belong to any of the Schatten p-classes ? " C. Cowen and T. Carroll [11] gave an affirmative answer to this question by constructing an explicit analytic self-map of the unit disk which induces a compact composition operator on $H^2(\mathbb{D})$ but does not belong to any of the Schatten p-classes, $S_p(H^2(\mathbb{D}))$ for $0 < p < \infty$. They used the following Luecking Criterion [31] to verify the membership of the compact composition operator in the Schatten p-classes, $S_p(H^2(\mathbb{D}))$ for $0 < p < \infty$.

Theorem 24 ([31])**.** *For $0 < p < \infty$, $C_\phi \in S_p(H^2(\mathbb{D}))$ if and only if $\frac{N_\phi(w)}{\log \frac{1}{|w|}} \in L^{\frac{p}{2}}(d\lambda)$ where $d\lambda = \frac{dA}{(1-|z|)^2}$ is a measure defined on $\mathbb{D}$.*

Several other examples were given, respectively, in [54, 23, 25] and all of these examples rely on Luecking Criterion as stated in Theorem 24.

By the Finite Rank Approximation property of compact operators, C_ϕ is compact on $H^2(\mathbb{D})$ if and only if the approximation numbers $a_n(C_\phi)$ goes to 0 as $n \to \infty$. D. Li, H. Queffélec and L. Rodríguez-Piazza [29] estimated the decay rates of approximation numbers of compact composition operators on $H^2(\mathbb{D})$ for different types of analytic self-maps of the unit disk $\mathbb{D}$. They were able to estimate the lower and upper bounds for the approximation numbers in the case where $\phi(\mathbb{D})$ is contained in a polygon and in the case where the image $\phi(\mathbb{D})$ is a cusp. Their main results are summarized in the following theorem:

Theorem 25 ([29])**.** *Suppose ϕ is an analytic self-map of $\mathbb{D}$.*

1. *If the image $\phi(\mathbb{D})$ is contained in a polygon with vertices on $\mathbb{T}$. Then, there exist positive constants α, β (depending only on ϕ) such that*

$$a_n(C_\phi) \leq \alpha e^{-\beta\sqrt{n}}.$$

2. *If ϕ is a cusp map, then there exist positive constants α_1, α_2 such that*

$$e^{-\frac{\alpha_1 n}{\log n}} \lesssim a_n(C_\phi) \lesssim e^{-\frac{\alpha_2 n}{\log n}}.$$

One major limitation of the Theorem 25 is that it does not tell us much about the approximation numbers in case when $\phi(\mathbb{D})$ touches $\mathbb{T}$ "smoothly" exactly at one point. Also it fails to provide a precise estimate on the approximation numbers in the case when $\phi(\mathbb{D})$ falls in between the two extreme cases, smooth tangency at exactly one point on $\mathbb{T}$ and the cusp maps. Queffélec and Seip[40] gave precise estimates for both of the above mentioned cases. They showed that a composition operator with any slow rate of decay of approximation numbers can be constructed. For simplification a new class of functions are defined.

Definition ([40])**.** *Let ϕ be an analytic self-map of $\mathbb{D}$ of the form $\phi = \mathrm{e}^{u - i\tilde{u}}$, where u is real* *valued, belongs to* $C(\mathbb{T})$ *, satisfies* $u(z) = u(\overline{z})$*, and is smooth everywhere but not necessarily at* $z = 1$ *and* $\tilde{u}$ *is the harmonic conjugate of* u*. An even function* $U(t) := u(e^{it})$ *belongs to class* $\mathcal{U}$ *if it is increasing on* $[0, \pi]$*,* $U(0) = 0$ *and the integral function*

$$h_U(t) := \int_t^{\pi} \frac{U(x)}{x^2} dx \to \infty \qquad \textit{when} \qquad t \to 0^+.$$

First, Queffélec and Seip considered two extreme cases : one when the integral function $h_U(t)$ grows very slowly, implying there is a smooth tangency at 1 and another one when $U(t) \to 0$ very slowly at $t = 0$, implying there is a sharp cusp at 1. The following theorem covers both of these cases entirely.

Theorem 26 ([40])**.** *Suppose that* U *belongs to* $\mathcal{U}$*.*

1. *If* $\frac{tU'(t)}{U(t)} \leq 1 + \frac{c}{|\log t|}$ *and* $\frac{U(t)}{th_U(t)} \leq \frac{C}{|\log t| \log|\log t|}$ *for* $c > 1$*,* $C > 0$*, and sufficiently small* $t > 0$*, then*

$$a_n(C_\phi) = \frac{e^{O(1)}}{\sqrt{h_U(e^{-\sqrt{n}})}} \qquad \textit{as} \qquad n \to \infty.$$

2. *Suppose* $U(t) = e^{\eta_U(|\log t|)}$ *whenever* $0 < t \leq 1$ *and* $U(t) \leq \frac{1}{e}$*. Let* $\omega_U(x) = \eta_U(\frac{x}{\omega_U(x)})$ *for* $x \geq 0$ *such that* $\eta_U(x) \geq 1$*. If* $\frac{\eta_U'(x)}{\eta_U(x)} = o(\frac{1}{x})$ *as* $x \to \infty$*, then*

$$a_n(C_\phi) = \mathrm{e}^{-\frac{(\frac{\pi^2}{2}+o(1))n}{\omega_U(n)}} \qquad \textit{as} \qquad n \to \infty.$$

Second, they considered maps that fall between the above mentioned two extreme cases including the maps that have a corner at a boundary point. These maps lie in the interface of two types of maps discussed earlier.

Theorem 27 ([40])**.** *Let* $\phi(z)$ *be the holomorphic self-maps of* $\mathbb{D}$ *of the form* $\phi(z) := \frac{1}{1+(1-z)^\alpha}$ *where* $0 < \alpha < 1$*. Then*

$$\mathrm{e}^{-\pi(1-\alpha)\sqrt{\frac{2n}{\alpha}}} \ll a_n(C_\phi) \ll \mathrm{e}^{-\pi(1-\alpha)\sqrt{\frac{n}{2\alpha}}}.$$

Here, by $f(n) \ll g(n)$, we mean $f(n) \leq c \cdot g(n)$ for all n.

Recall that if $\sum_{n=0}^{\infty} \|\phi^n\|_{H^2(\mathbb{D})}^2$ converges then C_ϕ is compact on $H^2(\mathbb{D})$. It is also evident from the Theorem 16 that if C_ϕ is compact on $H^2(\mathbb{D})$ then $\|\phi^n\|_{H^2(\mathbb{D})}$ decreases to 0, as $n \to \infty$. J.R. Akeroyd [2] showed a new way of constructing self-maps of $\mathbb{D}$, univalent or otherwise, for which C_ϕ is compact on $H^2(\mathbb{D})$, such that $\|\phi^n\|_{H^2(\mathbb{D})}$ decreases to 0 at an arbitrarily slow rate, as $n \to \infty$.

Theorem 28 ([2])**.** *Let $\{s_n\}_{n=1}^{\infty}$ be a sequence of real numbers in the interval $(0,1)$ such that $\lim_{n\to\infty} s_n = 0$. Then there exists a holomorphic self-map ϕ of $\mathbb{D}$, where C_ϕ is compact on $H^2(\mathbb{D})$, such that $\|\phi^n\|_{H^2(\mathbb{D})} \geq s_n$ for all n. Furthermore, ϕ can be univalent.*

Akeroyd's proof, for the non-univalent case, relies heavily on a famous result of C. J. Bishop [9], which can be stated as follows:

Theorem 29 ([9])**.** *Suppose ϕ is a holomorphic self-map of the unit disk such that $\phi(0) = 0$ and μ_ϕ is the induced measure of ϕ. Then $\int_{\mathbb{T}} \phi^n \overline{\phi}^m dt = 0$ whenever $n \neq m$ if and only if $\mu_\phi(E) = \mu_\phi(e^{it}E)$ for every measurable set E, supported in $\overline{\mathbb{D}}$, satisfying*

$$\int_{\mathbb{D}} \log \frac{1}{|z|} d\mu_\phi(z) < \infty.$$

Moreover, given any measure μ satisfying above conditions there exists ϕ with the above mentioned characteristics such that $\mu = \mu_\phi$.

With the help of Theorem 29, Akeroyd showed that there exists a non-univalent analytic self-map ϕ of $\mathbb{D}$ with $\phi(0) = 0$ which induces a measure μ_ϕ with the above mentioned characteristics in terms of normalized Lebesgue measure on the union of circles of the form $\{|z| = r_k : \lim_{k\to\infty} r_k = 1\}$.

For the univalent case, he used harmonic measure to construct a simply connected region Δ of $\mathbb{D}$ with multiple radial slits removed so that, if ϕ is a conformal mapping from

$\mathbb{D}$ to Δ with $\phi(0) = 0$, then ϕ has no angular derivative at any point of $\mathbb{T}$ and $\omega(\{z : r < |z| < 1\})$ tends to 0 at an arbitrarily slow rate as $r \to 1^-$. For detailed discussion on the proof of Theorem 28 see [2].

So far we have concentrated on compactness of composition operators on the Hardy space $H^2(\mathbb{D})$. A curious mind would naturally ask what happens to compactness of composition operators in other spaces of analytic functions. B. D. MacCluer and J. Shapiro [34] gave a necessary and sufficient condition for compactness of composition operators on the (weighted) Bergman spaces.

Theorem 30 ([34]). *Suppose $0 < p < \infty$ and $\alpha > -1$. Let ϕ be an analytic self-map of $\mathbb{D}$. Then C_ϕ is compact on $\mathbb{A}^p_\alpha$ if and only if ϕ has no angular derivative at any point in $\mathbb{T}$.*

Please note that the angular derivative criterion alone is not sufficient in the Hardy space $H^2(\mathbb{D})$, where an additional condition of ϕ being univalent (or boundedly valent) is necessary in order to guarantee compactness [refer to section 3]. B. D. MacCluer and J. Shapiro also gave another complete characterization of compact composition operators on $\mathbb{A}^p_\alpha$ in terms of Carleson measure. Please see section 2 for a definition of Carleson measure.

Theorem 31 ([34]). *Suppose $0 < p < \infty$ and $\alpha > -1$. Let ϕ be an analytic self-map of $\mathbb{D}$. Then C_ϕ is compact on $\mathbb{A}^p_\alpha$ if and only if $\lambda_\alpha \phi^{-1}$ is a compact α-Carleson measure.*

As a corollary a similar necessary and sufficient condition was obtained in the case of the weighted Dirichlet spaces. For a discussion on Dirichlet spaces, see section 2.

Corollary 32 ([34]). *Suppose $\alpha > -1$ and ϕ be an analytic self-map of $\mathbb{D}$ such that $\phi \in D^2_\alpha$. Also define a measure ν_α on $\mathbb{D}$ as*

$$d\nu_\alpha(z) = |\phi^{'}(z)|^2 d\lambda_\alpha(z).$$

Then C_ϕ is compact on D^2_α if and only if $\nu_\alpha \phi^{-1}$ is a compact α-Carleson measure.

It should also be noted that the angular derivative criterion is not sufficient enough to guarantee compactness in the weighted Dirichlet space setting. An additional condition, as in the case of the Hardy space $H^2(\mathbb{D})$, is required to guarantee compactness of composition operators on D^2_α. The following theorem is the "main" theorem in this context, as indicated by MacCluer and Shapiro [34].

Theorem 33 ([34])**.** *Suppose $\alpha > -1$. Let ϕ be an analytic self-map of $\mathbb{D}$. If C_ϕ is compact on D_β then ϕ does not have any angular derivative at any point of $\partial\mathbb{D}$. If ϕ does not have any angular derivative at any point of $\partial\mathbb{D}$ and if in addition C_ϕ is bounded on D_γ for some $-1 < \gamma < \beta$, then C_ϕ is compact on D_β.*

The additional condition that C_ϕ is bounded on D^2_β for some $-1 < \beta < \alpha$ is only necessary for the converse direction of the above statement. The reason behind this, as argued by MacCluer and Shapiro, is that if C_ϕ bounded on D^2_β the it is also bounded on D^2_α for $-1 < \beta < \alpha$.

In 1995, K. Madigan and A. Matheson formulated the following necessary and sufficient condition for compactness of composition operators in the Bloch spaces.

Theorem 34 ([35])**.** *Let ϕ be an analytic self-map of $\mathbb{D}$. Then,*

- *C_ϕ is compact on $\mathcal{B}_0$ if and only if*

$$\lim_{|z|\to 1} \frac{1-|z|^2}{1-|\phi(z)|^2}|\phi^{'}(z)| = 0.$$

- *C_ϕ is compact on $\mathcal{B}$ if and only if for every $\varepsilon > 0$, there exists r, $0 < r < 1$, such that*

$$\frac{1-|z|^2}{1-|\phi(z)|^2}|\phi^{'}(z)| < \varepsilon$$

whenever $|\phi(z)| > r$.

The authors [35] remarked that if the angular derivative of ϕ exists at any point of $\mathbb{T}$ then C_ϕ is not compact on Bloch spaces. They presented several example scenarios where C_ϕ is non-compact or compact in the context of little Bloch Space $\mathcal{B}_0$. In particular, if ϕ is an univalent self-analytic map of $\mathbb{D}$ and the image of ϕ touches $\mathbb{T}$ at exactly one point, but is not a cusp at that point, then C_ϕ is non-compact on $\mathcal{B}_0$; on the other hand, if the image of ϕ is a nontangential cusp at that point then C_ϕ is compact on $\mathcal{B}_0$.

Shortly after, in 1996, M. Tjani [48] proved several new and interesting results about compactness of composition operators in Besov spaces and Bloch space. One of these results is about a complete characterization of compact composition operators on these spaces.

Theorem 35 ([48])**.** *Let ϕ be an analytic self-map of $\mathbb{D}$ and $X = B_p(1 < p < \infty)$, BMOA, or $\mathcal{B}$. Then $C_\phi : X \to \mathcal{B}$ is compact if and only if*

$$\lim_{|a|\to 1} \|C_\phi \sigma_a\|_{\mathcal{B}} = 0$$

where $\sigma_a(z) = \frac{a-z}{1-\bar{a}z}$ is the basic disk automorphism for $a \in \mathbb{D}$.

In addition to the previous theorem, Tjani also gave Carleson measure type characterization of compact composition operators on the Besov spaces B_p $(1 < p < \infty)$ and Bloch space $\mathcal{B}$ and a necessary and sufficient condition for compactness of C_ϕ on B_p when C_ϕ is bounded on smaller Besov space B_q, $1 < p \leq q < \infty$. For detailed discussion and proofs of these results see [48].

Later in 1999, P. S. Bourdon, J. A. Cima, and A. L. Matheson [10] came up with a necessary and sufficient condition for compactness of composition operators on BMOA in terms of Carleson measure, which can be stated as follows: C_ϕ is compact on BMOA if and only if for every $\varepsilon > 0$ there is an r, $0 < r < 1$, such that

$$\int_{S(I)} \chi_r (1-|z|^2)|f^{'}(\phi(z))|^2|\phi^{'}(z)|^2 dA(z) \leq \epsilon |I|$$

for each arc $I \subset \mathbb{T}$ and each $f \in BMOA$ with $\|f\| \leq 1$, where $S(I)$ is the Carleson square at I and χ_r is the characteristic function on $\{z \in \mathbb{D} : |\phi(z)| > r\}$.

W. Smith, in [46], provided an improved condition, as compared to the complicated nature of the previous condition, to characterize compact composition operators on BMOA. Smith's characterization of compact composition operators on BMOA uses the classical Nevanlinna counting function of ϕ.

Theorem 36 ([46])**.** *Let ϕ be an analytic self-map of $\mathbb{D}$. Then C_ϕ is compact on BMOA if and only if*

$$\lim_{|\phi(a)|\to 1} \sup_{0<|w|<1} |w|^2 N_{\sigma_{\phi(a)}\circ\phi\circ\sigma_a}(w) = 0$$

and for all $0 < R < 1$

$$\lim_{t\to 1} \sup_{\{a:|\phi(a)|\leq R\}} m(\sigma_a(E(\phi,t))) = 0$$

where $\sigma_a(z) = \frac{a-z}{1-\bar{a}z}$ is the basic disk automorphism for $a \in \mathbb{D}$ and
$E(\phi,t) = \{e^{i\theta} : |\phi(e^{i\theta})| > t\}$, $0 < t < 1$.

Another complete characterization of compactness in the Dirichlet spaces was given by N. Zorboska [55] in terms of *determining functions* for composition operators. Determining functions are discussed in section 2.

Theorem 37 ([55])**.** *Suppose $\alpha > -1$. Let ϕ be an analytic self-map of $\mathbb{D}$. Then C_ϕ is compact on $D^2_{\alpha+2}$ if and only if there exists δ, $0 < \delta < 1$, such that*

$$\lim_{a\to\partial D} \frac{1}{A(D(a,\delta))} \int_{D(a,\delta)} \tau_{\phi,\ \alpha+2}(w) dA(w) = 0$$

where $D(a,\delta) = \left\{ z \in \mathbb{D} : |\frac{a-z}{1-\bar{a}z}| < \delta \right\}$ is called pseudohyperbolic disk and $\tau_{\phi,\ \alpha+2}$ is the *determining function for C_ϕ on $D^2_{\alpha+2}$.*

A lot of important work have been done on compactness of composition operators on different spaces of analytic functions. For example, D. Li, H. Queffélec, L. Rodriguez-Piazza [28] computed the decay rate of approximation numbers of compact composition operators acting on the weighted Bergman spaces discussed earlier; recently K. Seip and H. Queffélec [39] discussed the approximation numbers of composition operators on the H^2 space of the Dirichlet series; shortly after that, P. Lefévre, D. Li, H. Queffélec, L. Rodriguez-Piazza [27] studied the decay rate of approximation numbers of composition operators on the Dirichlet spaces. For more information on recent compactness results of composition operators acting on different types of Banach spaces of analytic functions please refer to [18, 36, 26, 17, 7, 52]. We would also recommend [15] for some interesting information on composition operators on Banach spaces of analytic functions.

4 Estimates for the Decay Rate of $\|\phi^n\|_{H^2(\mathbb{D})}$

We already know that if C_ϕ is compact on $H^2(\mathbb{D})$ then $\|\phi^n\|_{H^2(\mathbb{D})}$ decreases to 0, as $n \to \infty$. But the converse of the last statement doesn't necessarily hold since there exists ϕ for which $\|\phi^n\|_{H^2(\mathbb{D})} \to 0$, yet C_ϕ is not compact on $H^2(\mathbb{D})$. The following serves as a simple counter-example to the converse.

Example: Suppose ϕ is an analytic self-map of $\mathbb{D}$ given by $\phi(z) = \frac{z+1}{2}$. Then

$$\|\phi^n\|_{H^2(\mathbb{D})} = \frac{1}{\sqrt[4]{\pi n}}.$$

By definition,

$$\begin{aligned}
\|\phi^n\|^2_{H^2(\mathbb{D})} &= \int_{\mathbb{T}} \left| \frac{1+\zeta}{2} \right|^{2n} dm(\zeta) \\
&= \frac{1}{2^{2n}} \int_{\mathbb{T}} |1+\zeta|^{2n} dm(\zeta) \\
&= \frac{1}{2^{2n}} \int_{\mathbb{T}} (1+\zeta)^n (1+\overline{\zeta})^n dm(\zeta) \\
&= \frac{1}{2^{2n}} \int_{\mathbb{T}} (2+\zeta+\overline{\zeta})^n dm(\zeta) \\
&= \frac{1}{2^{2n}} \cdot \frac{1}{2\pi} \int_0^{2\pi} (2+2\cos\theta)^n d\theta \\
&= \frac{1}{2^{2n}} \cdot \frac{2^n}{2\pi} \int_0^{2\pi} (1+\cos\theta)^n d\theta \\
&= \frac{1}{2^{2n}} \cdot \frac{2^n}{2\pi} \int_0^{2\pi} (2\cos^2\frac{\theta}{2})^n d\theta \\
&= \frac{1}{2^{2n}} \cdot \frac{2^{2n}}{2\pi} \int_0^{2\pi} (\cos^2\frac{\theta}{2})^n d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} (\cos^2\frac{\theta}{2})^n d\theta \\
&= \frac{1}{2\pi} \cdot \frac{2\pi}{2^{2n}} \binom{2n}{n}
\end{aligned}$$

where the last equality is an well-known identity. Now by Stirling's formula,

$$\begin{aligned}\|\phi^n\|^2_{H^2(\mathbb{D})} &= \frac{1}{2\pi}\cdot\frac{2\pi}{2^{2n}}\binom{2n}{n}\\ &= \frac{1}{2^{2n}}\frac{(2n)!}{(n!)^2}\\ &\sim \frac{\sqrt{4\pi n}\left(\frac{2n}{e}\right)^{2n}}{2^{2n}(2\pi n)\left(\frac{n}{e}\right)^{2n}}\\ &= \frac{1}{\sqrt{\pi n}}.\end{aligned}$$

H. J. Schwartz observed that the map $\phi(z) = \frac{1+z}{2}$ induces a non-compact composition operator on $H^2(\mathbb{D})$ [45]. Now let us consider the map $\Psi(z) = \frac{z}{2}$ which induces a compact composition operator on $H^2(\mathbb{D})$ [44, 45]. A straightforward computation shows that $\|\Psi^n\|_{H^2(\mathbb{D})} \simeq \frac{1}{\sqrt{n}}$, which goes to 0 much faster compared to $\|\phi^n\|_{H^2(\mathbb{D})}$ where $\phi(z) = \frac{1+z}{2}$.

H. Wulan, D. Zheng, K. Zhu [52] gave a proof for the converse direction in the Bloch space and BMOA settings. They showed that convergence of Bloch or BMOA (semi-)norm of $\{\phi^n\}$ to 0 is necessary and sufficient for C_ϕ to be compact on these spaces.

Theorem 38 ([52])**.** *Let X= BMOA or $\mathcal{B}$ and ϕ be an analytic self-map of $\mathbb{D}$. Then C_ϕ is compact on X if and only if $\|\phi^n\|_{X^\sharp} \to 0$, as $n \to \infty$.*

O. El-Fallah, K. Kellay, M. Shabankhah, H. Youssfi [18] proved the same in the classical Dirichlet space D^2_0 setting. Next we compute the decay rate of $\|\phi^n\|_{H^2(\mathbb{D})}$ for the Schatten class composition operators.

Proposition 39. *Let ϕ be an analytic self-map of $\mathbb{D}$ and $p \geq 2$. If C_ϕ belongs to any of the Schatten p-classes, $S_p(H^2(\mathbb{D}))$, then*

$$\|\phi^n\|_{H^2(\mathbb{D})} = o(\frac{1}{\sqrt[p]{n}}).$$

Proof. First of all, since C_ϕ is in $S_p(H^2(\mathbb{D}))$ it is compact. So the weak convergence theorem implies $\|\phi^n\|_{H^2(\mathbb{D})} \to 0$ as $n \to \infty$. Theorem 8 implies that for Schatten p-class composition operators $\sum_{n=1}^{\infty} \|\phi^n\|^p_{H^2(\mathbb{D})}$ converges. Now since $\sum_{n=1}^{\infty} \|\phi^n\|^p_{H^2(\mathbb{D})}$ is a series of positive, monotonic decreasing terms, $\lim_{n\to\infty} n \cdot \|\phi^n\|^p_{H^2(\mathbb{D})} = 0$ (see [24]). Thus for $C_\phi \in S_p(H^2(\mathbb{D}))$, where $p \geq 2$,

$$\|\phi^n\|_{H^2(\mathbb{D})} = o(\frac{1}{\sqrt[p]{n}}).$$

■

J. R. Akeroyd showed that we can construct analytic self-map of $\mathbb{D}$ such that the composition operator C_ϕ is compact on $H^2(\mathbb{D})$ yet $\|\phi^n\|_{H^2(\mathbb{D})}$ converges to 0 in an arbitrarily slow rate, see [2]. But the image of ϕ (may be univalent), in his construction, touches the unit circle $\mathbb{T}$ at multiple points. We study the decay rate of $\|\phi^n\|_{H^2(\mathbb{D})}$ for the composition operator C_ϕ on the Hardy space of unit disk $H^2(\mathbb{D})$, where C_ϕ is defined by $C_\phi = f \circ \phi$ and ϕ is an univalent analytic map of unit disk $\mathbb{D}$ onto itself. We want to identify as precisely as possible the rate of decay for the $\|\phi^n\|_{H^2(\mathbb{D})}$ when $\phi(\mathbb{D})$ touches the unit circle $\mathbb{T}$ at just one point. For simplicity we consider ϕ which maps the unit disk $\mathbb{D}$ to a Jordan domain Δ whose boundary $\partial\Delta$ has an equation $1 - r = h(t)$, where $h : [0, 1] \to [0, 1]$ is a continuous, increasing, convex function with $h(0) = 0$ and $0 \leq h(t) \leq M \cdot t$, for some constant $M > 0$. The functions h that satisfy these conditions will be said to belong to the class $\mathscr{H}$.

We begin our work with a few lemmas and observations concerning the behavior of functions in class $\mathscr{H}$.

Lemma 40. *Suppose that h belongs to $\mathscr{H}$. Then*

$$(1 - h(t))^k \geq 1 - kh(t)$$

on $[0, 1]$ for any large k.

Proof. Let

$$\rho(t) = (1-h(t))^k - (1-kh(t))$$

Notice that $\rho(0) = 0$, so it suffices to show that $\rho^{'}(t)$ is positive on $(0,1)$.

$$\rho^{'}(t) = k(1-h(t))^{k-1}(-h^{'}(t)) + kh^{'}(t) \quad = \quad kh^{'}(t)[1-(1-h(t))^{k-1}]$$

Since h is increasing and convex, $h^{'}(t) > 0$ on $(0,1)$. Also from the definition of h, $1-(1-h(t))^{k-1}$ is positive on $(0,1]$. Thus $\rho^{'}(t)$ is positive on $(0,1)$. ■

Observation: Choose t_k, $0 < t_k < 1$, such that $h(t_k) = \frac{1}{2k}$.

Since $h(t)$ is increasing on $[0,1]$, $h(t) \leq \frac{1}{2k}$ on $[0,t_k]$. So then

$$1 - kh(t) \geq \frac{1}{2}$$

and it is clear that $(1-h(t))^k \leq 1$ on $[0,1]$. From which we have

$$\frac{1-kh(t)}{(1-h(t))^k} \geq \frac{1}{2}$$

on $[0,t_k]$, for any k. We call $\{t_k\}_{k=1}^{\infty}$ the *cutoff sequence* for $h(t)$.

The following lemma is an important feature of the functions that belong to class $\mathscr{H}$ and also a key tool that will help us prove our main results concerning composition operators.

Lemma 41. *Suppose that h belongs to $\mathscr{H}$.Then there exists an $\varepsilon > 0$ such that*

$$\frac{\int_0^{t_k}(1-h(t))^k dt}{\int_0^1 (1-h(t))^k dt} \geq \varepsilon$$

for large k, where $\{t_k\}_{k=1}^{\infty}$ is the cutoff sequence for $h(t)$.

Proof. First of all, for large $k > 0$, choose the same $0 < t_k < 1$ such that $h(t_k) = \frac{1}{2k}$ as in the discussion above. Now

$$\begin{aligned} \int_0^1 (1-h(t))^k dt &:= \int_0^{t_k} (1-h(t))^k dt + \int_{t_k}^1 (1-h(t))^k dt && (1) \\ &:= I + II. && (2) \end{aligned}$$

Notice that the first integral (I) in the above expression is boundedly equivalent to t_k, that is, $\int_0^{t_k} (1-h(t))^k dt \asymp t_k$. To see that,

$$\begin{aligned} t_k &\geq \int_0^{t_k} (1-h(t))^k dt \\ &\geq (1-h(t_k))^k \cdot t_k \\ &= (1-\frac{1}{2k})^k \cdot t_k \\ &\sim \frac{1}{\sqrt{e}} \cdot t_k. \end{aligned}$$

Now choose a subinterval of $[t_k, 1]$, with a partition $t_k = {t_k}^{(1)} < {t_k}^{(2)} < \cdots < {t_k}^{(j)}$, where $j \leq \lfloor 4\log(k) \rceil$ such that

$$\begin{aligned} h(t_k^{(1)}) &= \frac{1}{2k} \\ h(t_k^{(2)}) &= \frac{2}{2k} \\ &\vdots \\ h(t_k^{(j)}) &= \frac{j}{2k}. \end{aligned}$$

Then,

$$h(t_k^{(j)}) - h(t_k^{(j-1)}) = \frac{1}{2k} \qquad (3)$$

for all j.

By Mean Value Theorem , there exists a point s_k^j between $t_k^{(j-1)}$ and $t_k^{(j)}$ such that

$$h(t_k^{(j)}) - h(t_k^{(j-1)}) \;\; = \;\; h^{'}(s_k^{(j)}) \cdot (t_k^{(j)} - t_k^{(j-1)}). \tag{4}$$

Let $s_k^{(j-1)} \in (t_k^{(j-2)}, t_k^{(j-1)})$. Then combining (3) and (4) and by applying Mean Value Theorem again we have,

$$h^{'}(s_k^{(j)}) \cdot (t_k^{(j)} - t_k^{(j-1)}) = h^{'}(s_k^{(j-1)}) \cdot (t_k^{(j-1)} - t_k^{(j-2)}).$$

Now from the definition of $h(t)$ we know that $h'(t)$ is increasing and never zero on $(0, 1)$. So,

$$h^{'}(s_k^{(j)}) \geq h^{'}(s_k^{(j-1)})$$

which implies

$$t_k^{(j)} - t_k^{(j-1)} \leq t_k^{(j-1)} - t_k^{(j-2)} \tag{5}$$

for any $j \leq \lfloor 4\log(k) \rceil$ for any large k.

Now if we integrate $(1 - h(t))^k$ on the subinterval $[t_k^{(1)}, t_k^{(4\log(k))}]$ for any large k then by (5) above, we have

$$\int_{t_k^{(1)}}^{t_k^{(4\log(k))}} (1-h(t))^k dt \sim \sum_{j=1}^{4\log(k)} (t_k^{(j+1)} - t_k^{(j)})(1-h(t_k^{(j)}))^k$$
$$\leq t_k^{(1)} \left[\sum_{j=1}^{4\log(k)} (1-h(t_k^{(j)}))^k \right]$$
$$= t_k^{(1)} \left[\sum_{j=1}^{4\log(k)} (1-\frac{j}{2k})^k \right].$$

The sum on the right hand side of above inequality converges uniformly and equals to some constant $L > 0$ because $(1-\frac{j}{2k})^k \sim \frac{1}{e^{\frac{j}{2}}}$ unformly for all $j > 0$ growing upto $\lfloor 4\log(k) \rceil$ for any large k. In other words, $(1-\frac{j}{2k})^k$ nearly equals the value $\frac{1}{e^{\frac{j}{2}}}$ for all $j \leq \lfloor \log(k) \rceil$ no matter how large k gets. The following claim explains this in more detail.

Claim: $e^{\frac{j}{2}}(1-\frac{j}{2k})^k \to 1$ uniformly on $1 \leq j \leq \lfloor 4\log(k) \rceil$ as $k \to \infty$.

Proof of claim: First of all note that the sequence $e^{\frac{j}{2}}(1-\frac{j}{2k})^k$ approaches to 1 uniformly for all $1 \leq j \leq \lfloor 4\log(k) \rceil$ as $k \to \infty$ if and only if the sequence $\log(e^{\frac{j}{2}}(1-\frac{j}{2k})^k)$ approaches 0 uniformly on $1 \leq j \leq \lfloor 4\log(k) \rceil$ as $k \to \infty$. So it suffices to show that $\log(e^{\frac{j}{2}}(1-\frac{j}{2k})^k) \to 0$ uniformly on $1 \leq j \leq \lfloor 4\log(k) \rceil$ as $k \to \infty$.

For $1 \leq j \leq \lfloor 4\log(k) \rceil$, let $m = \frac{2k}{j}$. Then

$$\log(e^{\frac{j}{2}}(1-\frac{j}{2k})^k) = \frac{j}{2} + k \cdot \log(1-\frac{j}{2k})$$
$$= \frac{j}{2} + m \cdot \frac{j}{2} \cdot \log(1-\frac{1}{m})$$
$$= \frac{j}{2} \cdot (1 + m\log(1-\frac{1}{m})).$$

Now since $1 \leq j \leq \lfloor 4\log(k) \rceil$, $0 \lesssim \frac{1}{m} < 1$ from which by the logarithmic inequalities/identities, we have

$$\begin{aligned}
\frac{1}{m} &< -\log(1-\frac{1}{m}) \\
&= \log(\frac{1}{1-\frac{1}{m}}) \\
&= \log(\frac{m}{m-1}) \\
&= \log(\frac{(m-1)+1}{m-1}) \\
&= \log(1+\frac{1}{m-1}) \\
&\lesssim \frac{1}{m-1} \qquad \text{since} \quad \log(1+x) \sim x \qquad \text{as} \qquad x \to 0
\end{aligned}$$

from which it follows,

$$0 < -\log(e^{\frac{j}{2}}(1-\frac{j}{2k})^k) \lesssim \frac{j}{2(m-1)}.$$

Now if k is very large , then m also is very large. So as $k \to \infty$,

$$\begin{aligned}
\frac{j}{2(m-1)} &= \frac{j}{2} \cdot \frac{m}{m-1} \cdot \frac{1}{m} \\
&= \frac{j}{2} \cdot \frac{m}{m-1} \cdot \frac{j}{2k} \\
&= \frac{m}{m-1} \cdot \frac{j^2}{4k} \\
&\leq \frac{1}{1-\frac{1}{m}} \cdot \frac{4(\log(k))^2}{k}
\end{aligned}$$

approaches 0 .

Now $|\log(e^{\frac{j}{2}}(1-\frac{j}{2k})^k) - 0| < \epsilon \iff \frac{4(\log(k))^2}{k} < \epsilon \iff \frac{4}{k} < \epsilon \iff k > \frac{4}{\epsilon}$.

Choose $K(\epsilon) = \frac{4}{\epsilon}$. Thus for every $\epsilon > 0$, there exists $K(\epsilon)$, independent of j, such that $k \geq K(\epsilon)$ implies

$$|\log(e^{\frac{j}{2}}(1-\frac{j}{2k})^k) - 0| < \epsilon$$

on $1 \leq j \leq \lfloor 4\log(k) \rceil$.

From the above discussion we have,

$$\int_{t_k^{(1)}}^{t_k^{(4\log(k))}} (1-h(t))^k dt \leq t_k^{(1)} \cdot L \tag{6}$$

$$\lessapprox L^* \cdot \int_0^{t_k} (1-h(t))^k dt \tag{7}$$

where $L^* > 0$ is some constant.

(7) implies that the integral of $(1-h(t))^k$ on $[t_k^{(1)}, t_k^{(4\log(k))}]$ is a constant multiple of the integral of $(1-h(t))^k$ on $[0, t_k]$. Now since k is very large and $(1-h(t))^k$ is a decreasing function on $[0,1]$ then by the above claim ,

$$\int_{t_k^{(4\log k)}}^{1} (1-h(t))^k dt \leq (1 - \frac{4\log(k)}{2k})^k \sim \frac{1}{k^2} \qquad on \qquad [t_k^{(4\log k)}, 1]$$

which is very negligible compared to the integral $\int_0^{t_k^{(4\log(k))}} (1-h(t))^k dt$ due to the hypothesis $h(t) \leq M \cdot t$ on $[0,1]$ which implies $t_k \geq \frac{M}{2k}$ and $\frac{1}{k^2}$ converges to 0 faster than $\frac{1}{k}$ as $k \to \infty$. So we can conclude that

$$\int_0^1 (1-h(t))^k dt \sim \int_0^{t_k^{(4\log(k))}} (1-h(t))^k dt$$

Set $\varepsilon = \frac{1}{1+L^*}$. Thus by (2),

$$\frac{\int_0^{t_k} (1-h(t))^k dt}{\int_0^1 (1-h(t))^k dt} \geq \varepsilon.$$

■

We discussed in section 3 for a compact composition operator C_ϕ induced by a self-map ϕ of $\mathbb{D}$, the H^2-norm of $\{\phi^n\}$ decreases to zero as $n \to \infty$. The following proposition tells us that for a compact composition operator, induced by a univalent self-map of $\mathbb{D}$ whose image touches the boundary $\mathbb{T}$ at exactly one point, the H^2-norm of $\{\phi^n\}$ decreases to zero faster than the sequence $\{\sqrt{t_n}\}$ as $n \to \infty$.

Proposition 42. *Suppose Δ is a Jordan domain in $\mathbb{D}$ bounded by a smooth boundary curve C which has an equation $1 - r = h(t)$, where h belongs to $\mathscr{H}$. Let ϕ be a univalent map of $\mathbb{D}$ onto Δ, which fixes 1. If C_ϕ is compact then*

$$\|\phi^n\|_{H^2(\mathbb{D})} = o(\sqrt{t_n})$$

where $\{t_n\}_{n=1}^{\infty}$ is the cutoff sequence for $h(t)$.

Proof. It is given that boundary curve C is smooth; hence rectifiable. Now suppose $\alpha \in \Delta$ and let $\omega(\alpha, ., \Delta)$ be harmonic measure on $\partial\Delta$ at α. It is clear from Proposition 9 that $d\omega = |\psi^{'}||d\xi$; where $\psi = \phi^{-1}$ and ψ' exists in terms of non-tangential limit and $d\xi$ is the arc-length.

Now Choose an r where $0 < r < 1$. Then from the above discussion,

$$\begin{aligned}
\|\phi^n\|^2_{H^2(\mathbb{D})} &= \frac{1}{2\pi}\int_{\mathbb{T}}|\phi(\zeta)|^{2n}|d\zeta| \\
&= \frac{1}{2\pi}\int_{\partial\Delta}|\xi|^{2n}|\psi'(\xi)||d\xi| \qquad \left(= \int_{\partial\Delta}|\xi|^{2n}d\omega(\xi)\right) \\
&= \frac{1}{2\pi}\left[\int_{\partial\Delta\cap|\xi|\le r}|\xi|^{2n}|\psi'(\xi)||d\xi| + \int_{\partial\Delta\cap|\xi|>r}|\xi|^{2n}|\psi'(\xi)||d\xi|\right].
\end{aligned}$$

Since $h(t) \le M \cdot t$ for some positive M and by the Lemma 41 above the first term in the above inequality, as $n \to \infty$, as we choose r close enough to 1, tends to 0 faster than t_n. That is , for all $\epsilon > 0$ there exists an N such that $\int_{\partial\Delta\cap|\xi|\le r}|\xi|^{2n}|\psi'(\xi)||d\xi| \le \pi\epsilon \cdot t_n$ whenever $n \ge N$.

Also since C_ϕ is compact, as r is close enough to 1, $|\psi'(\xi)|$ gets smaller. That is for every $\epsilon > 0$ there exists an N' such that $\int_{\partial\Delta\cap|\xi|>r}|\xi|^{2n}|\psi'(\xi)||d\xi| \le \frac{\pi\epsilon}{2} \cdot \int_{\partial\Delta\cap|\xi|>r}|\xi|^{2n}|d\xi|$ whenever $n \ge N'$.

From the above discussion and by lemma 41,

$$
\begin{aligned}
\|\phi^n\|^2_{H^2(\mathbb{D})} &\leq \frac{\epsilon}{2}\cdot t_n + \frac{\epsilon}{4}\int_{\partial\Delta\cap|\xi|>r} |\xi|^{2n}|d\xi| \\
&\leq \frac{\epsilon}{2}\cdot t_n + \frac{\epsilon}{4}\int_{\partial\Delta} |\xi|^{2n}|d\xi| \\
&\sim \frac{\epsilon}{2}\cdot t_n + \frac{\epsilon}{4}\int_0^1 (1-h(t))^{2n}dt \\
&\leq \frac{\epsilon}{2}\cdot t_n + \frac{\epsilon}{4}\int_0^{t_n} (1-h(t))^{2n}dt \\
&\leq \frac{\epsilon}{2}\cdot t_n + \frac{\epsilon}{2}\int_0^{t_n} (1-2nh(t))dt \\
&= \frac{\epsilon}{2}\cdot t_n + \frac{\epsilon}{2}(t_n - 2n\int_0^{t_{2n}} h(t)dt) \\
&\leq \frac{\epsilon}{2}\cdot t_n + \frac{\epsilon}{2}(t_n - 2n\cdot h(0)\cdot t_n) \\
&= \frac{\epsilon}{2}\cdot t_n + \frac{\epsilon}{2}\cdot t_n \\
&= \epsilon\cdot t_n.
\end{aligned}
$$

From which it follows that

$$\|\phi^n\|_{H^2(\mathbb{D})} = o(\sqrt{t_n}).$$

■

Remarks:

- It was noted in [35] and [10] that the map $\phi(z) = 1-\sqrt{1-z}$ which maps $\mathbb{D}$ to a tear-drop shaped region in $\mathbb{D}$ induces a non-compact composition operator on the little Bloch space $\mathcal{B}_0$ and BMOA. But C_ϕ is compact on $H^2(\mathbb{D})$. K. Madigan and A. Matheson [35] also proved : if ϕ is univalent and the image of ϕ touches $\mathbb{T}$ at exactly one point and doest not have a cusp at that point then C_ϕ is not compact on $\mathcal{B}_0$. But we know from Theorem 15 that C_ϕ on $H^2(\mathbb{D})$ is Hilbert-Schmidt in the case when the image of ϕ has a cusp at the touching point. Now by Proposition 39 the decay rate of

$\|\phi^n\|_{H^2(\mathbb{D})}$ in the Hilbert-Schmidt operator case (when $p = 2$) is much faster than $\frac{1}{\sqrt{n}}$.

- In [18] El-Fallah et.al. noticed that if $\|\phi^n\|_{H^2(\mathbb{D})} = o(\frac{1}{\sqrt{n}})$ then C_ϕ is compact on $H^2(\mathbb{D})$. In light of proposition 42 above taking $h(t) = \frac{t}{\log(\frac{1}{t})}$ gives us $\|\phi^n\|_{H^2(\mathbb{D})} = o(\sqrt{\frac{\log(n)}{n}})$ and we know that C_ϕ is compact in this case.

- Also if we assume C_ϕ is compact and $\|\phi^n\|_{H^2(\mathbb{D})} = o(\frac{1}{\sqrt{n}})$ with the same hypothesis as in Proposition 42 then C_ϕ is Hilbert-Schmidt on $H^2(\mathbb{D})$. To see that notice C_ϕ is compact in this case. So by Proposition 42, since t_n is unique up to a constant multiple, $t_n = \frac{1}{n}$. Now since $h(t_n) = \frac{1}{2n}$ and $h(t)$ is an increasing, injective function, $h(t) = \frac{t}{2}$. So the image of ϕ is contained in a polygon which implies C_ϕ is Hilbert-Schmidt.

  It should also be noted that this result is not true in general for any analytic self-map of $\mathbb{D}$ with $\|\phi^n\|_{H^2(\mathbb{D})} = o(\frac{1}{\sqrt{n}})$. For example, if we choose $\|\phi^n\|_{H^2(\mathbb{D})} = \frac{1}{\sqrt{n\log n}}$, then $\lim_{n\to\infty} \frac{\frac{1}{\sqrt{n\log n}}}{\frac{1}{\sqrt{n}}} = 0$, but $\sum_{n=1}^{\infty}\|\phi^n\|^2_{H^2(\mathbb{D})} = \sum_{n=1}^{\infty} \frac{1}{n\log n}$ diverges. Thus C_ϕ is not Hilbert-Schmidt in this case. Theorem 28 guarantees the existence of such an analytic self-map ϕ of $\mathbb{D}$ which may not be univalent and $\phi(\mathbb{D})$ touches $\mathbb{T}$ at multiple points.

The above estimate for the decay rate of $\|\phi^n\|_{H(\mathbb{D})}$ in the case of compact composition operator induced by a univalent analytic self-map ϕ of $\mathbb{D}$ with $\phi(1) = 1$ gets better as we choose ϕ whose image approaches the boundary $\mathbb{T}$ smoothly or "faster" as opposed to sharply or "slower", yet induces a compact composition operator. Our next proposition gives us a precise estimate on the decay rate of $\|\phi^n\|_{H^2(\mathbb{D})}$ in the case when the inducing map ϕ maps $\mathbb{D}$ onto a domain Δ whose boundary touches $\mathbb{T}$ very smoothly and as a consequence induces a non-compact composition operator.

Proposition 43. *Suppose Δ is a Jordan domain in $\mathbb{D}$ bounded by a smooth boundary curve C, represented by the equation $1 - r = h(t)$, where h belongs to $\mathscr{H}$. Let ϕ be a univalent map of $\mathbb{D}$ onto Δ, which fixes 1. Then C_ϕ is not compact on $H^2(\mathbb{D})$ if and only if*

$$\|\phi^n\|^2_{H^2(\mathbb{D})} \asymp t_n$$

where $\{t_n\}_{n=1}^{\infty}$is the cutoff sequence for $h(t)$.

Proof. ($\Longleftarrow$) If $\|\phi^n\|^2_{H^2(\mathbb{D})} \asymp t_n$ then $\|\phi^n\|_{H^2(\mathbb{D})} \neq o(\sqrt{t_n})$. Thus by Lemma 42, C_ϕ is not compact on $H^2(\mathbb{D})$.

($\Longrightarrow$) As in the proof of previous proposition, it is given that boundary curve C is rectifiable. Now let $\alpha \in \Delta$ and $\omega(\alpha, ., \Delta)$ be the harmonic measure on $\partial\Delta$ at α. It is clear from Proposition 9 that $d\omega = |\psi^{'}|d\xi$; where $\psi = \phi^{-1}$ and ψ' exists in terms of non-tangential limit and $d\xi$ is the arc-length.

Since C_ϕ is not compact, by univalent compactness theorem in Section 3, ϕ does have finite angular derivative at some point on $\mathbb{T}$, which implies $\psi^{'} \neq 0$. Also since the boundary curve C is smooth, $\phi^{'}$ has a continuous extension on $\overline{\mathbb{D}}$. Thus on $\partial\Delta$, $C_1 < |\psi^{'}| < C_2$ for some positive constants C_1 and C_2, which is equivalent as saying $d\omega \asymp d\xi$ on $\partial\Delta$. So we have,

$$\begin{aligned}
\|\phi^n\|^2_{H^2(\mathbb{D})} &= \frac{1}{2\pi}\int_{\mathbb{T}} |\phi(\zeta)|^{2n}|d\zeta| \\
&= \frac{1}{2\pi}\int_{\partial\Delta} |\xi|^{2n}|\psi'(\xi)||d\xi| \qquad \left(= \int_{\partial\Delta} |\xi|^{2n} d\omega(\xi)\right) \\
&\asymp \int_{\partial\Delta} |\xi|^{2n}|d\xi| \\
&\sim \int_0^1 (1-h(t))^{2n} dt \\
&\leq const. \int_0^{t_n} (1-h(t))^{2n} dt \\
&\asymp t_n.
\end{aligned}$$

∎

Remark 1. If we Let Δ be a Jordan domain on the w-plane, bounded by a rectifiable Jordan curve C represented by $w = w(\xi)$ $(0 \leq \xi \leq l)$, where l is the length of C and ξ the arc length of C. Also that C has a tangent at every point, which varies continuously and $w^{'}(\xi)$ satisfies the following Hölder's condition:

$$|w^{'}(\xi_1) - w^{'}(\xi_2)| \leq K|\xi_1 - \xi_2|^{\lambda} \qquad (0 < \lambda < 1)$$

where K is some constant, then by Kellogg's theorem [50], $d\omega \asymp d\xi$. So this particular scenario resembles the "smooth" criterion mentioned in Proposition 43 and the result holds.

The following theorem is our main result. It gives a necessary and sufficient condition for the compactness of the composition operator C_ϕ in the case when the the image of the inducing map ϕ touches $\mathbb{T}$ at exactly one point.

Theorem 44. *Suppose Δ is a Jordan domain in $\mathbb{D}$ bounded by a smooth boundary curve C, represented by the equation $1 - r = h(t)$, where h belongs to $\mathscr{H}$. Let ϕ be a univalent map of $\mathbb{D}$ onto Δ, which fixes 1. Then C_ϕ is compact on $H^2(\mathbb{D})$ if and only if*

$$\sum_{n=1}^{\infty} \frac{1}{n} \left[\frac{1}{\|\phi^{n+1}\|^2_{H^2(\mathbb{D})}} - \frac{1}{\|\phi^{n}\|^2_{H^2(\mathbb{D})}} \right]$$

diverges.

Proof. $(\Longleftarrow)$ Assume that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n} \left[\frac{1}{\|\phi^{n+1}\|^2_{H^2(\mathbb{D})}} - \frac{1}{\|\phi^{n}\|^2_{H^2(\mathbb{D})}} \right]$ diverges. Also for the sake of contradiction assume that C_ϕ is not compact on $H^2(\mathbb{D})$.

Since C_ϕ is not compact , by Warschawski's Theorem $\int_0^1 \frac{h(t)}{t^2} dt$ converges. Now we know that $h(t)$ is a continuous, increasing function on $[0, 1]$ and $h(t_n) = \frac{1}{2n}$ and $t_{n+1} < t_n$. So $h(t) \leq \frac{1}{2n}$ on $[0, t_n]$ and $h(t_{n+1}) < h(t_n)$, which implies $\int_0^1 \frac{h(t)}{t^2} dt = \displaystyle\sum_{n=1}^{\infty} \int_{t_{n+1}}^{t_n} \frac{h(t)}{t^2} dt \asymp \sum_{n=1}^{\infty} \frac{1}{2n} \int_{t_{n+1}}^{t_n} \frac{dt}{t^2}$ converges.

Now from Proposition 43 above $\|\phi^n\|^2_{H^2(\mathbb{D})} \asymp t_n$. Then,

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{2n} \int_{t_{n+1}}^{t_n} \frac{dt}{t^2} &= \sum_{n=1}^{\infty} \frac{1}{2n} \left[\frac{1}{t_{n+1}} - \frac{1}{t_n} \right] < \infty \\ &\asymp \sum_{n=1}^{\infty} \frac{1}{n} \left[\frac{1}{\|\phi^{n+1}\|^2_{H^2(\mathbb{D})}} - \frac{1}{\|\phi^n\|^2_{H^2(\mathbb{D})}} \right] < \infty\end{aligned}$$

which contradicts our assumption.

($\Longrightarrow$) Suppose C_ϕ is compact. Let $\|\phi^n\|^2_{H^2(\mathbb{D})} = s_n$ for all n. Since C_ϕ is compact $\|\phi^n\|^2_{H^2(\mathbb{D})} \to 0$, which implies $s_{n+1} < s_n$ for all n. Now by proposition 42, $\|\phi^n\|^2_{H^2(\mathbb{D})} = s_n = o(t_n)$. Define a piecewise linear function $g(s)$ such that $g(s_n) = \frac{1}{2n}$ for all n. Since $h(t)$ is convex, $g(s) \geq h(t)$ for all s, t in $[0, 1]$. Now since $g(s) \geq h(t)$ for all s, t in $[0, 1]$, $\int_0^1 \frac{g(s)}{s^2} ds \geq \int_0^1 \frac{h(t)}{t^2} dt$.

Since C_ϕ is compact, by Warschawski's theorem , $\int_0^1 \frac{h(t)}{t^2} dt$ diverges, which implies $\int_0^1 \frac{g(s)}{s^2} ds$ diverges. From which and with the same argument as in the previous case, we conclude

$$\begin{aligned}\int_0^1 \frac{g(s)}{s^2} ds &= \sum_{n=1}^{\infty} \int_{s_{n+1}}^{s_n} \frac{g(s)}{s^2} ds \\ &\asymp \sum_{n=1}^{\infty} \frac{1}{2n} \int_{s_{n+1}}^{s_n} \frac{ds}{s^2} \\ &= \sum_{n=1}^{\infty} \frac{1}{2n} \left[\frac{1}{s_{n+1}} - \frac{1}{s_n} \right]\end{aligned}$$

diverges.

Thus C_ϕ is compact if and only if $\sum_{n=1}^{\infty} \frac{1}{n} \left[\frac{1}{\|\phi^{n+1}\|^2_{H^2(\mathbb{D})}} - \frac{1}{\|\phi^n\|^2_{H^2(\mathbb{D})}} \right]$ diverges. ■

An easy and simple example of Theorem 44 can be given by considering the analytic self-map of $\mathbb{D}$, discussed earlier, given by $\phi(z) = \frac{z+1}{2}$ whose image touches $\mathbb{T}$ at exactly one point and does so smoothly. Notice that, the series

$$\sum_{n=1}^{\infty} \frac{1}{n} \left[\frac{1}{\|\phi^{n+1}\|^2_{H^2(\mathbb{D})}} - \frac{1}{\|\phi^{n}\|^2_{H^2(\mathbb{D})}} \right] = \sum_{n=1}^{\infty} \frac{\sqrt{\pi}}{n} \left[\sqrt{n+1} - \sqrt{n} \right]$$

converges by the Comparison Test. Thus the composition operator C_ϕ, in this case, is not compact on $H^2(\mathbb{D})$.

5 Closed-Range Composition Operators

We know, from section 2, what it means for an operator on any Banach space to be closed-range. In the context of composition operators, we have the following characterization which is just the Banach-space version of Proposition 3.30 in [15].

Theorem 45. *A bounded (and one-to-one) composition operator C_ϕ on any Banach space $\mathfrak{B}$ of analytic functions on $\mathbb{D}$ has closed-range if and only if there exists an $\varepsilon > 0$ so that*

$$\|C_\phi(f)\|_{\mathfrak{B}} \geq \varepsilon \|f\|_{\mathfrak{B}}$$

for all f in $\mathfrak{B}$.

In 1974, J. A. Cima, J. Thomson and W. Wogen [13] obtained a necessary and sufficient condition for closed-rangeness of composition operators on $H^2(\mathbb{D})$. Their condition focuses on the boundary behavior of the analytic self-map ϕ of $\mathbb{D}$.

Theorem 46 ([13]). *Let ϕ be a nonconstant analytic self-map of $\mathbb{D}$. Then C_ϕ has closed-range if and only if $\frac{d\mu_\phi}{dm}$ is essentially bounded away from zero, where μ_ϕ is the induced measure on $\overline{\mathbb{D}}$ as defined in section 2.*

Cima, Thomson and Wogen also posed the problem of obtaining a necessary and sufficient condition for closed-rangeness of composition operators in terms of the range of the inducing analytic self-map ϕ on $\mathbb{D}$ rather than $\mathbb{T}$. Approximately twenty years later, N. Zorboska [56] gave a complete characterization of closed-range composition operators on $H^2(\mathbb{D})$ in terms of the properties of the range of the inducing analytic self-map ϕ on $\mathbb{D}$ instead of $\mathbb{T}$.

Theorem 47 ([56]). *Let ϕ be an analytic self-map of $\mathbb{D}$. Then C_ϕ has closed-range if and* *only if there exists a* $c > 0$ *such that the set*

$$G_c^\phi = \{z : \tau_\phi(z) = \frac{N_\phi(z)}{\log \frac{1}{|z|}} > c\}$$

satisfies the following condition:

There exists a constant $\delta > 0$ *such that*

$$(\star) \qquad A(G_c^\phi \cap D(\xi, r)) > \delta \cdot A(\mathbb{D} \cap D(\xi, r))$$

for all ξ in $\mathbb{T}$ and $r > 0$, where $D(\xi, r)$ is the disk with centered at ξ with radius r. Here $N_\phi(z)$ is the Nevanlinna counting function of ϕ as defined in section 2.

The condition $(\star)$ is called *reverse Carleson condition*, and was invented by D. Luecking [30] in order to answer questions related to the closed-rangeness of Toeplitz operators. It tells us about the behavior of the set G_c^ϕ at the boundary. In particular, Luecking was able to show the following interesting connection:

Luecking's Theorem ([30]). *Let* G *be a measurable subset of* $\mathbb{D}$ *and* $p > 0$. *Then there is a constant* $K > 0$ *such that for all* $f \in \mathbb{A}_0^p$, *the Bergman spaces,*

$$\int_{\mathbb{D}} |f|^p dA \leq K \int_G |f|^p dA$$

if and only if there exists a constant $\delta > 0$ *such that*

$$A(G \cap D(\xi, r)) > \delta \cdot A(\mathbb{D} \cap D(\xi, r))$$

for all ξ *in* $\mathbb{T}$ *and* $r > 0$, *where* $D(\xi, r)$ *is the disk with centered at* ξ *with radius* r.

Zorboska also proved similar results in the context of weighted Bergman spaces $\mathbb{A}_\alpha^2$, for $\alpha > -1$. But Zorboska's results make use of Nevanlinna counting function which is a complex tool to deal with. J. R. Akeroyd and P. G. Ghatage [4] provided an improved necessary and sufficient condition for closed-rangeness of C_ϕ for the classical Bergman space $\mathbb{A}_0^2$, which does not involve Nevanlinna counting function. They considered images of sets of the form $\Omega_\varepsilon(\phi) = \{z \in \mathbb{D} : \frac{1-|z|^2}{1-|\phi(z)|^2} \geq \varepsilon > 0\}$, denoted $G_\varepsilon(\phi) = \phi(\Omega_\varepsilon)$ and applied Luecking's reverse Carleson condition on these sets. The following is a restatement of their result:

Theorem 48 ([4]). *Let ϕ be a nontrivial analytic self-map of $\mathbb{D}$. Then C_ϕ closed-range on $\mathbb{A}_0^2$ if and only if there exist $\varepsilon > 0$, and $\delta > 0$, and $0 < s < 1$ such that G_ε satisfies the following condition:*

$$A(G_\varepsilon \cap D_s(z)) \geq \delta \cdot A(D_s(z))$$

for all $z \in \mathbb{D}$, where $D_s(z) = \{w \in \mathbb{D} : |\frac{z-w}{1-\overline{w}z}| < s\}$, is called the pseudo-hyperbolic disk of radius r and centered at z.

With the help of Theorem 48 Akeroyd and Ghatage were able to show that if ϕ is an univalent analytic self-map of $\mathbb{D}$ then C_ϕ is closed-range on $\mathbb{A}_0^2$ if and only if ϕ is a conformal automorphism of $\mathbb{D}$. Other characterizations of closed-range composition operators on $\mathbb{A}_0^2$ was given by Akeroyd, Ghatage and Tjani [6].

Similar results like Theorem 48, in the context of weighted Bergman spaces, are provided in [3]. P. Ghatage, D. Zhang, and N. Zorboska [21] worked on closed-range composition operators on the Bloch space. Later more results in the context of Bloch space were provided in [5]. Recently, M. Tjani [49] has studied the closed-range composition operators on Besov type spaces.

Akeroyd, Ghatage and Tjani [5, 6] also noticed an interesting implication: if C_ϕ is closed-range on $\mathbb{A}_0^2$ then it is also closed-range on $\mathcal{B}$, the Bloch space. A counterexample disproving the converse of this statement can also be found in [5]. Another implication like this was also noticed by N. Zorboska: if C_ϕ is closed-range on the Bergman space $\mathbb{A}_\alpha^2$ then it is also closed-range on $H^2(\mathbb{D})$; see Corollary 4.2 in [56]. Tjani [49] also showed that for $p > 2$, if C_ϕ is closed-range on Besov spaces $B_{p,p-1}$ then it is also closed-range on the Hardy space $H^2(\mathbb{D})$. However, all of these implications are results of complete characterization of closed-rangeness of C_ϕ on these spaces. In the next section we study this pattern from a different perspective.

6 Inheritance of Closed-Rangeness Property

So from the discussion in the preceding section one may naturally ask, does closed-rangeness of a composition operator on a larger Banach space always imply closed-rangeness on a smaller Banach subspace. In other words, if $\mathfrak{S}$ and $\mathfrak{B}$ are two Banach spaces of analytic functions on $\mathbb{D}$ such that $\mathfrak{S} \subseteq \mathfrak{B}$ and if C_ϕ is closed-range on $\mathfrak{B}$, then does it follow that C_ϕ is also closed-range on $\mathfrak{S}$? To answer this question we need a tool called absolutely monotonic radial weight functions.

Absolutely Monotonic Radial Weight

A Borel measurable function $w : \mathbb{D} \to [0, \infty)$ is called a *radial weight* on $\mathbb{D}$ if $w(z) = w(|z|)$, $\forall z \in \mathbb{D}$. In section 2, we discussed what it means for any real-valued function to be *absolutely monotonic* on an interval. If $w(z)$ is some radial weight on $\mathbb{D}$ and $w(z) = g(|z|)$ on $[0, 1)$, where $g(x)$ is an absolutely monotonic function on $[0, 1)$ then we say $w(z)$ is an *absolutely monotonic radial weight* on $\mathbb{D}$. In particular, by Theorem 11, $w(z)$ is the analytic extension of $g(x)$ on $\mathbb{D}$. Some common examples of absolutely monotonic radial weights are: for $z \in \mathbb{D}$, $\log(\frac{1}{1-|z|^2})$, $\frac{1}{1-|z|^2}$ etc. Following are some important observations regarding absolutely monotonic radial weights.

Observation 1: Let $w(z) := g(|z|)$ be an absolutely monotonic radial weight on $\mathbb{D}$ where g is defined on $[0, 1)$ as $g(x) = \log(\frac{1}{1-x})$. For $1 \le p < \infty$, if we define $w_p(z)$ on $\mathbb{D}$ as $w_p(z) := g(|z|^p)$, then w and w_p are boundedly equivalent on $\mathbb{D}$. Notice that, for $0 \le x < 1$, $g(x^p) \le g(x)$ for all p. Also, $g(x) = g(x^p) + \log(\frac{1-x^p}{1-x})$. Now we know that $\lim\limits_{x \to 1^-} \frac{1-x^p}{1-x} = p$; from which we have $\lim\limits_{x \to 1^-} \log(\frac{1-x^p}{1-x}) = \log(p)$.

Observation 2: As in the previous observation, if we consider weight $w(z) := g(|z|)$ of the form where $g(x) = \frac{1}{(1-x)^\alpha}$, $\alpha > 0$, then since $\lim\limits_{x \to 1^-} (\frac{1-x^p}{1-x})^\alpha = p^\alpha$, w and w_p are boundedly equivalent on $\mathbb{D}$.

Observation 3: If we consider rapidly increasing weights of the form $w(z) := g(|z|)$, where

$g(x) = e^{\frac{1}{(1-x)^{\alpha}}}$, $0 < \alpha \leq 1$, then we can guarantee that there exists an absolutely monotonic radial weight which is boundedly equivalent to w on $\mathbb{D}$. To verify this claim, consider the linear function $v_p(x) = \frac{1}{p}x + (1 - \frac{1}{p})$ which is clearly an absolutely monotonic function from $[0, 1)$ into itself. So the composition $l(x) = g \circ v_p(x)$ is also absolutely monotonic on $[0, 1)$. Now, for $0 \leq x < 1$, $x^{\frac{1}{p}} < v_p(x)$; from which we have $l(x^p) \geq g(x)$ for $x \in [0, 1)$. Also,

$$\frac{l(x^p)}{g(x)} = e^{\frac{p^{\alpha}}{(1-x^p)^{\alpha}}} - e^{\frac{1}{(1-x)^{\alpha}}} = e^{\frac{p^{\alpha} - (\frac{1-x^p}{1-x})^{\alpha}}{(1-x^p)^{\alpha}}}$$

By the Mean Value Theorem, for $x \in (0, 1)$, there exists $c \in (x, 1)$, depending only on p, such that, $pc^{p-1} = \frac{1-x^p}{1-x}$; which implies,

$$\begin{aligned} \frac{l(x^p)}{g(x)} &= e^{\frac{p^{\alpha} - (pc^{p-1})^{\alpha}}{(1-x^p)^{\alpha}}} \\ &= e^{\frac{p^{\alpha}(1 - c^{\alpha(p-1)})}{(1-x^p)^{\alpha}}} \\ &\leq e^{p^{\alpha}} \end{aligned}$$

Before we discuss our main results and their proofs, we would like to state our assumption throughout the rest of this section that C_{σ_a} is bounded on both spaces $\mathfrak{B}$ and $\mathfrak{S}$, where $\sigma_a(z) := \frac{a-z}{1-\bar{a}z}$, for all $a \in \mathbb{D}$, are the disk automorphisms.

Theorem 49. *Let $\mathfrak{B}$ and $\mathfrak{S}$ be two Banach spaces of analytic functions on $\mathbb{D}$, where $\mathfrak{S} \subseteq \mathfrak{B}$, defined as follows:*

$$\mathfrak{B} = \{f \in \mathcal{H}(\mathbb{D}) : \|f\|_{\mathfrak{B}}^p = \int_{\mathbb{D}} |f|^p d\mu < \infty\}$$
$$\mathfrak{S} = \{f \in \mathcal{H}(\mathbb{D}) : \|f\|_{\mathfrak{S}}^p = \int_{\mathbb{D}} |f|^p w_p d\mu < \infty\}$$

for $1 \leq p < \infty$, where $w_p(z) := w(|z|^p)$ is an absolutely monotonic radial weight on $\mathbb{D}$ and μ is some positive Borel measure defined on $\mathbb{D}$. Let ϕ be an analytic self-map of $\mathbb{D}$ and C_{ϕ}

maps $\mathfrak{B}$ into $\mathfrak{B}$ and $\mathfrak{S}$ into $\mathfrak{S}$. If C_ϕ is bounded on $\mathfrak{S}$ and closed-range on $\mathfrak{B}$ then C_ϕ is also closed-range on $\mathfrak{S}$.

Proof. By our earlier assumption, C_{σ_a} is bounded on both spaces $\mathfrak{B}$ and $\mathfrak{S}$, where, for all $a \in \mathbb{D}$, $\sigma_a(z) := \frac{a-z}{1-\bar{a}z}$. An important consequence of this assumption is that C_{σ_a} is now closed-range on both $\mathfrak{B}$ and $\mathfrak{S}$ since the inverse of σ_a is itself under function composition. So we only consider the case when $\phi(0) = 0$.

Since $w_p(z)$ is absolutely monotonic radial weight on $\mathbb{D}$ it can be written as $w_p(z) := g(|z|^p)$, where g is a real analytic function on $[0, 1)$ whose power series representation contains non-negative coefficients. In particular, $g(x) = \sum_{n=0}^{\infty} a_n x^n$, where $a_n \geq 0$ for all n.

It is given that C_ϕ is closed-range on $\mathfrak{B}$. So, by definition, there exists an $\varepsilon > 0$ such that, for $1 \leq p < \infty$,

$$\|C_\phi(f)\|_{\mathfrak{B}}^p \geq \varepsilon \|f\|_{\mathfrak{B}}^p$$

whenever $f \in \mathfrak{B}$.

Now, by the Schwarz's lemma, for $1 \leq p < \infty$ and $f \in \mathfrak{S}$,

$$\begin{aligned}
\|C_\phi(f)\|_{\mathfrak{S}}^p &= \|(f \circ \phi)(z)\|_{\mathfrak{S}}^p \\
&= \int_{\mathbb{D}} |(f \circ \phi)(z)|^p w_p(z) d\mu(z) \\
&= \sum_{n=0}^{\infty} a_n \int_{\mathbb{D}} |(f \circ \phi)(z)|^p |z|^{np} d\mu(z) \\
&\geq \sum_{n=0}^{\infty} a_n \int_{\mathbb{D}} |(f \circ \phi)(z)|^p |\phi(z)|^{np} d\mu(z) \\
&= \sum_{n=0}^{\infty} a_n \|(f \circ \phi)(z) \cdot \phi(z)^n\|_{\mathfrak{B}}^p \\
&= \sum_{n=0}^{\infty} a_n \|C_\phi(f(z) \cdot z^n)\|_{\mathfrak{B}}^p
\end{aligned}$$

$$
\begin{aligned}
&\geq \varepsilon \sum_{n=0}^{\infty} a_n \|f(z) \cdot z^n\|_{\mathfrak{B}}^p \\
&= \varepsilon \sum_{n=0}^{\infty} a_n \int_{\mathbb{D}} |f(z)|^p |z|^{np} d\mu(z) \\
&= \varepsilon \int_{\mathbb{D}} |f(z)|^p w_p(z) d\mu(z) \\
&= \varepsilon \|f\|_{\mathfrak{S}}^p
\end{aligned}
$$

which implies C_ϕ is bounded below on $\mathfrak{S}$. Thus, by Theorem 45, C_ϕ is closed-range on $\mathfrak{S}$. ■

It should be noted that Theorem 49 can be applied to any pair of Banach spaces of analytic functions which possess integral norms as mentioned above. Also the measure μ here is not restrictive at all except it is just a positive, Borel measure on $\mathbb{D}$. The importance of the weight $w(z)$ being radial shall be discussed later. Indeed, a large number of well-known Banach spaces of analytic functions on $\mathbb{D}$ discussed in various literatures do possess integral norms similar to the one defined above and are endowed with some kind of radial weights. For example, consider the weighted Bergman spaces $\mathbb{A}_\alpha^p$ ($\alpha > -1$, $1 \leq p < \infty$); C_ϕ is always bounded on these spaces (see [34]). Now if we consider the absolutely monotonic weight $w_p(z) := \frac{1}{(1-|z|^p)^{\beta-\alpha}}$, then by Theorem 49, for $-1 < \alpha < \beta$, if C_ϕ is closed-range on $\mathbb{A}_\beta^p$ then it is also closed-range on $\mathbb{A}_\alpha^p$. But there are Banach spaces of analytic functions on $\mathbb{D}$ which have integral norms defined in terms of the derivative of the functions in the spaces instead of the function itself; for example, weighted Dirichlet spaces D_α ($\alpha > -1$) or Besov type spaces. The proof above doesn't work in this case. We would need a modified approach to resolve this issue.

Theorem 50. *Let $\mathfrak{B}$ and $\mathfrak{S}$ be two Banach spaces of analytic functions on $\mathbb{D}$, where*

$\mathfrak{S} \subseteq \mathfrak{B}$*, defined as follows:*

$$\mathfrak{B} = \{f \in \mathcal{H}(\mathbb{D}) : \|f\|_{\mathfrak{B}}^p = |f(0)|^p + \int_{\mathbb{D}} |f'|^p d\mu < \infty\}$$
$$\mathfrak{S} = \{f \in \mathcal{H}(\mathbb{D}) : \|f\|_{\mathfrak{S}}^p = |f(0)|^p + \int_{\mathbb{D}} |f'|^p w_p d\mu < \infty\}$$

for $1 \leq p < \infty$*, where* $w_p(z) := w(|z|^p)$ *is an absolutely monotonic radial weight on* $\mathbb{D}$ *and* μ *is some positive Borel measure defined on* $\mathbb{D}$*. Let* ϕ *be an analytic self-map of* $\mathbb{D}$ *and* C_ϕ *maps* $\mathfrak{B}$ *into* $\mathfrak{B}$ *and* $\mathfrak{S}$ *into* $\mathfrak{S}$*. If* C_ϕ *is bounded on* $\mathfrak{S}$ *and closed-range on* $\mathfrak{B}$ *then* C_ϕ *is also closed-range on* $\mathfrak{S}$*.*

Proof. By our assumption, C_{σ_a} is bounded on both spaces $\mathfrak{B}$ and $\mathfrak{S}$, where, for all $a \in \mathbb{D}$, $\sigma_a(z) := \frac{a-z}{1-\bar{a}z}$. An important consequence of this assumption is that C_{σ_a} is now closed-range on both $\mathfrak{B}$ and $\mathfrak{S}$ since the inverse of σ_a is itself under function composition. So we only consider the case when $\phi(0) = 0$.

Since $w_p(z)$ is absolutely monotonic radial weight on $\mathbb{D}$ it can be written as $w_p(z) := g(|z|^p)$, where g is a real analytic function on $[0, 1)$ whose power series representation contains non-negative coefficients. In particular, $g(x) = \sum_{n=0}^{\infty} a_n x^n$, where $a_n \geq 0$ for all n.

Now let $f_0 = f - f(0)$. Since C_ϕ is linear and one-to-one, $C_\phi(f_0) = C_\phi(f) - f(0)$, from which we have : $\|C_\phi(f_0)\|_{\mathfrak{S}}^p = \|C_\phi(f)\|_{\mathfrak{S}}^p + |f(0)|^p$. Let $\mathfrak{S}_0 = \{f \in \mathfrak{S} : f(0) = 0\}$. Now if C_ϕ is closed-range on $\mathfrak{S}_0$ then by Theorem 45, there exists a $\delta > 0$ such that $\|C_\phi(f)\|_{\mathfrak{S}_0} \geq \delta \|f\|_{\mathfrak{S}_0}$ for all $f \in \mathfrak{S}_0$. It is now implied that if C_ϕ is closed-range on $\mathfrak{S}_0$ then it is also closed-range on $\mathfrak{S}$ and the same $\delta > 0$ does work in this case. So it suffices to show that C_ϕ is closed-range on $\mathfrak{S}_0$.

It is given that C_ϕ is closed-range on $\mathfrak{B}$. So, by Theorem 45, there exists an $\varepsilon > 0$ such that, for $1 \leq p < \infty$,

$$\|C_\phi(f)\|_{\mathfrak{B}}^p \geq \varepsilon \|f\|_{\mathfrak{B}}^p$$

whenever $f \in \mathfrak{B}$. Suppose n is some positive integer and $z \in \mathbb{D}$. Let the sequence $f_n(z) = \int_0^1 f'(tz)(tz)^n z dt$ be the analytic primitive of $f'(z)z^n$ and $f_n(0) = 0$. Now,by the Schwarz's lemma, for $1 \le p < \infty$ and $f \in \mathfrak{S}_0$,

$$\begin{aligned}
\|C_\phi(f)\|_{\mathfrak{S}}^p &= \|(f \circ \phi)(z)\|_{\mathfrak{S}}^p \\
&= \int_{\mathbb{D}} |(f \circ \phi)'(z)|^p w_p(z) d\mu(z) \\
&= \sum_{n=0}^{\infty} a_n \int_{\mathbb{D}} |(f \circ \phi)'(z)|^p |z|^{np} d\mu(z) \\
&= \sum_{n=0}^{\infty} a_n \int_{\mathbb{D}} |f'(\phi(z))\phi'(z)|^p |z|^{np} d\mu(z) \\
&\ge \sum_{n=0}^{\infty} a_n \int_{\mathbb{D}} |f'(\phi(z))\phi'(z)|^p |\phi(z)|^{np} d\mu(z) \\
&= \sum_{n=0}^{\infty} a_n \int_{\mathbb{D}} |(f_n \circ \phi)'(z)|^p d\mu(z) \\
&= \sum_{n=0}^{\infty} a_n \|C_\phi(f_n)\|_{\mathfrak{B}}^p \\
&\ge \varepsilon \sum_{n=0}^{\infty} a_n \|f_n\|_{\mathfrak{B}}^p \\
&= \varepsilon \sum_{n=0}^{\infty} a_n \int_{\mathbb{D}} |f'(z)|^p |z|^{np} d\mu(z) \\
&= \varepsilon \int_{\mathbb{D}} |f'(z)|^p w_p(z) d\mu(z) \\
&= \varepsilon \|f\|_{\mathfrak{S}}^p
\end{aligned}$$

which implies C_ϕ is bounded below on $\mathfrak{S}_0$. Thus, by Theorem 45, C_ϕ is closed-range on $\mathfrak{S}_0$. ■

As an example, consider the Besov type spaces discussed in section 2. Suppose C_ϕ is bounded on Besov type spaces $B_{p,\alpha}$ and $B_{p,\beta}$, where $-1 < \alpha < \beta$. If we consider similar weights $w_2(z) := \frac{1}{(1-|z|^2)^{\beta-\alpha}}$, as before, then if C_ϕ is closed-range on $B_{p,\beta}$, then C_ϕ is closed-range on $B_{p,\alpha}$.

Remarks:

- It should be noted that in Theorem 49, we can also consider a sequence of absolutely monotonic radial weights such as $w_{p,k}(z) = g_k(|z|^p)$, where $\{g_k\}$ is a sequence of absolutely monotonic functions on $[0,1)$. In that case, $\mathfrak{S}$ is defined as: $\mathfrak{S} := \{f \in \mathcal{H}(\mathbb{D}) : \|f\|_{\mathfrak{S}}^p = \lim_{k\to\infty} \int_{\mathbb{D}} |f|^p w_{p,k} d\mu < \infty\}$. The result still holds in this setting. To see this, let $f \in \mathfrak{S}$; then following the same proof as in Theorem 49 we get

$$\begin{aligned}
\|C_\phi(f)\|_{\mathfrak{S}}^p &= \lim_{k\to\infty} \int_{\mathbb{D}} |(f\circ\phi)(z)|^p w_{p,k}(z) d\mu(z) \\
&\geq \lim_{k\to\infty} \sum_{n=0}^{\infty} a_{k,n} \int_{\mathbb{D}} |(f\circ\phi)(z)|^p |\phi(z)|^{np} d\mu(z) \\
&= \lim_{k\to\infty} \sum_{n=0}^{\infty} a_{k,n} \|C_\phi(f(z)\cdot z^n)\|_{\mathfrak{B}}^p \\
&\geq \varepsilon \lim_{k\to\infty} \sum_{n=0}^{\infty} a_{k,n} \|f(z)\cdot z^n\|_{\mathfrak{B}}^p \\
&= \varepsilon \lim_{k\to\infty} \sum_{n=0}^{\infty} a_{k,n} \int_{\mathbb{D}} |f(z)|^p |z|^{np} d\mu(z) \\
&= \varepsilon \lim_{k\to\infty} \int_{\mathbb{D}} |f(z)|^p w_{p,k}(z) d\mu(z) \\
&= \varepsilon \|f\|_{\mathfrak{S}}^p
\end{aligned}$$

- Here's an example for the sequence case: for $1 \leq p < \infty$, if C_ϕ is closed-range on $\mathbb{A}_0^p$, then it is also closed-range on $H^p(\mathbb{D})$. To see this, note that the sequence $d\nu_k := (pk+1)r^{pk} r dr$ is weak-* convergent on [0,1] to $d\delta_{\{1\}}$, the unit point mass at 1. Thus we have:

$$\begin{aligned}
\|f\|_{H^p(\mathbb{D})}^p &= \lim_{r\to 1^-} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \\
&= \lim_{k\to\infty} \int_0^1 \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta (pk+1) r^{pk} r dr \\
&= \lim_{k\to\infty} \int_{\mathbb{D}} |f(z)|^p w_{p,k}(z) dA(z)
\end{aligned}$$

where $w_{p,k}(z) := \frac{pk+1}{2}|z|^{pk}$. Now since $w_{p,k}$ is an absolutely monotnic radial weight for each k and C_ϕ is always bounded on $\mathbb{A}_0^p$ and $H^p(\mathbb{D})$, by the above remark closed-rangeness on $\mathbb{A}_0^p$ implies closed-rangeness on $H^p(\mathbb{D})$.

- A similar argument, directly following the proof of Theorem 50, can also be provided to show that the results in Theorem 50 also hold in the case when $\mathfrak{S}$ is defined as: $\mathfrak{S} = \{f \in \mathcal{H}(\mathbb{D}) : \|f\|_{\mathfrak{S}}^p = \lim_{k\to\infty} |f(0)|^p + \int_{\mathbb{D}} |f'|^p w_{p,k} d\mu < \infty\}$, where $w_{p,k}$ is again a sequence of absolutely monotonic radial weights as defined before. Using the similar argument, as in the previous remark, it can be shown: if C_ϕ is bounded on D_0^2 and $S^2(\mathbb{D})$ and closed-range on D_0^2, then C_ϕ is closed-range on $S^2(\mathbb{D})$. For boundedness criterion for C_ϕ on $S^2(\mathbb{D})$, see [33].

The following example shows that our two assumptions: C_{σ_a} is bounded on both spaces $\mathfrak{B}$ and $\mathfrak{S}$, for all the disk automorphisms $\sigma_a := \frac{a-z}{1-\bar{a}z}$, and the weight $w(z)$ is radial play a crucial role in the theorems above and cannot be dropped.

Example: Define a measure μ on $\mathbb{D}$ by $d\mu(z) = w(z)dA(z)$, where $w(z)$ is defined on $\mathbb{D}$ as follows:

$$w(z) = \begin{cases} \frac{1}{\sqrt{1-|z|^2}} & z \in W := \{z = x+iy \in \mathbb{D} : x, y > 0\} \\ 1 & \text{elsewhere} \end{cases}$$

Obviously, w is not radial. Let $\mathfrak{B} := \mathbb{A}_0^1$ and $\mathfrak{S} := \{f \in \mathcal{H}(\mathbb{D}) : \|f\|_{\mathfrak{S}} = \int_{\mathbb{D}} |f| w dA < \infty\}$. For $z \in \mathbb{D}$, consider the region $\Gamma_z := \{\zeta : |z-\zeta| < 1-|z|\}$. If $f \in \mathfrak{S}$, then

$$\begin{aligned} |f(z)| &\leq \frac{1}{\pi(1-|z|)^2} \int_{\Gamma_z} |f| dA \\ &\leq \frac{1}{\pi(1-|z|)^2} \|f\|_{\mathfrak{S}} \end{aligned}$$

So point evaluations are continuous linear functionals on $\mathfrak{S}$, which implies that $\mathfrak{S}$ is a Banach space of analytic functions on $\mathbb{D}$. Also it is clear that $\mathfrak{S} \subseteq \mathfrak{B}$. Now let ϕ be the following Möbius transformation from $\mathbb{D}$ to itself:

$$\phi(z) = \frac{\frac{i}{2} + z}{1 - \frac{i}{2}z}$$

Note that $\phi(W) \subseteq W$ and $W \subseteq \phi^{-1}(W)$. Let $\psi = \phi^{-1}$. Now, $\phi(0) = \frac{i}{2}$ and $\frac{1-|\phi(0)|}{1+|\phi(0)|} = \frac{1}{3}$; from which we have: $\frac{1}{3} \leq |\psi'| \leq 3$. By the Schwarz-Pick lemma,

$$|\psi'(\zeta)| = \frac{1 - |\psi(\zeta)|^2}{1 - |\zeta|^2}$$

for all $\zeta \in \mathbb{D}$. From the definition of $w(z)$, we get

$$\frac{w(\psi(\zeta))}{w(\zeta)} = \sqrt{\frac{1 - |\zeta|^2}{1 - |\psi(\zeta)|^2}} = \frac{1}{\sqrt{|\psi'(\zeta)|}} \leq \sqrt{3} < 2$$

for all $\zeta \in \mathbb{D}$.

Claim: C_ϕ is bounded on $\mathfrak{B}$ and $\mathfrak{S}$. It is closed-range on $\mathfrak{B}$, but not on $\mathfrak{S}$.

Proof. It is well-established that C_ϕ is bounded on $\mathfrak{B}$. Indeed, it is bounded on any weighted Bergman spaces $\mathbb{A}^p_\alpha$, where $1 \leq p < \infty$ and $\alpha > -1$; see Proposition 3.4 in [34]. From the discussion above, for $f \in \mathfrak{S}$,

$$\begin{aligned} \|C_\phi(f)\|_{\mathfrak{S}} &= \int_{\mathbb{D}} |f(\phi(z))| w(z) dA(z) \\ &= \int_{\mathbb{D}} |f(\zeta)| w(\psi(\zeta)) |\psi'(\zeta)|^2 dA(\zeta) \\ &\leq 18 \int_{\mathbb{D}} |f| w dA \\ &= 18 \|f\|_{\mathfrak{S}} \end{aligned}$$

which establishes that C_ϕ is bounded above on $\mathfrak{S}$. It is also well-known fact that C_ϕ is closed-range on $\mathfrak{B}$; see [4] in this context. To see that C_ϕ is not closed-range on $\mathfrak{S}$, consider the following sequence of functions in $\mathfrak{S}$,

$$f_k(z) := \frac{c_k}{[(1+s_k)-z]^{\frac{3}{2}}}$$

where $s_k > 0$ for all k, decreases to 0 as $k \to \infty$ and $c_k = \frac{1}{\|[(1+s_k)-z]^{\frac{3}{2}}\|_{\mathfrak{S}}}$. Now, by our definition of μ, $f(z) = \frac{1}{(1-z)^{\frac{3}{2}}}$ does not belong to $L^1(d\mu)$; from which, $c_k \to 0$, as $k \to \infty$. So f_k converges to 0 uniformly on $\{z \in \mathbb{D} : |1-z| \geq \delta\}$, where $\delta > 0$; whence, $\{f_k \circ \phi\}_k$ converges to 0 uniformly on W. Also, since $f(z) \in L^1(dA)$, $\int_{\mathbb{D}} |f_k| dA$ converges to 0, as $k \to \infty$. We have,

$$\begin{aligned}
\|C_\phi(f_k)\|_{\mathfrak{S}} &= \int_{\mathbb{D}} |f_k(\phi(z))| w(z) dA(z) \\
&= \int_W |f_k(\phi(z))| w(z) dA(z) + \int_{\mathbb{D}\setminus W} |f_k(\phi(z))| dA(z) \\
&\leq \int_W |f_k(\phi(z))| w(z) dA(z) + \int_{\mathbb{D}} |f_k(\phi(z))| dA(z) \\
&= \int_W |f_k(\phi(z))| w(z) dA(z) + \int_{\mathbb{D}} |f_k(\zeta)| |\psi'(\zeta)|^2 dA(\zeta) \\
&\leq \int_W |f_k(\phi(z))| w(z) dA(z) + 9 \int_{\mathbb{D}} |f_k| dA
\end{aligned}$$

converges to 0, as $k \to \infty$. But, by construction, $\|f_k\|_{\mathfrak{S}} = 1$, for all k. Thus C_ϕ is not bounded below on $\mathfrak{S}$. So it is not closed-range on $\mathfrak{S}$. Furthermore, due to this, C_ψ is not bounded(above) on $\mathfrak{S}$ which violates our first assumption that C_{σ_a} is bounded on $\mathfrak{S}$ for any disk automorphism σ_a.

■

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