5-2013

The Effect of Symmetry on the Riemann Map

Jeanine Louise Myers
University of Arkansas, Fayetteville

THE EFFECT OF SYMMETRY
ON THE RIEMANN MAP

THE EFFECT OF SYMMETRY ON THE RIEMANN MAP

A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy in Mathematics

By

Jeanine L. Myers
Baylor University
Bachelor of Science in Mathematical Sciences, 1992
Clemson University
Master of Science in Mathematical Sciences, 1994

May 2013
University of Arkansas

Abstract

The Riemann mapping theorem guarantees the existence of a conformal mapping or Riemann map in the complex plane from the open unit disk onto an open simply-connected domain, which is not all of $\mathbb{C}$. Although its existence is guaranteed, the Riemann map is rarely known except for special domains like half-planes, strips, etc. Therefore, any information we can determine about the Riemann map for any class of domains is interesting and useful.

This research investigates how symmetry affects the Riemann map. In particular, we define domains with symmetries called Rectangular Domains or RDs. The Riemann map of an RD has real-valued coefficients, as opposed to complex-valued, and therefore we can determine the sign of the coefficients of the Taylor series about the origin of the Riemann map, $f(z)$, from the unit disk onto RDs determined by $f(0) = 0$ and $f'(0) > 0$. We focus on the form of the Riemann map for specific RD polygons. These include rhombi, rectangles and non-equilateral octagons which have 2-fold symmetries. We also investigate equilateral polygons with more than 2-fold symmetries such as squares (rotated diamond), regular polygons, and equilateral octagons.

This dissertation is approved for recommendation
to the Graduate Council.

Dissertation Directors:

Loredana Lanzani, Ph.D.

Andrew Raich, Ph.D.

Dissertation Committee:

Mark Arnold, Ph.D.

Phillip Harrington, Ph.D.

Dissertation Duplication Release

I hereby authorize the University of Arkansas Libraries to duplicate this dissertation when needed for research and/or scholarship.

Agreed ____

Jeanine L. Myers

Acknowledgements

Without my dissertation advisors, Dr. Loredana Lanzani and Dr. Andrew Raich, this dissertation would not have been possible. Dr. Lanzani has been a mentor, encourager, and inspiration to me as a professional women in mathematics. I am deeply grateful for her exceptional knowledge and insight into different aspects of my dissertation. I am also grateful for the time I have worked with Dr. Raich. Over the last five years, he has sharpened and improved my mathematical preciseness and correctness in written and oral communication and as a result, I have become a better mathematician because of his instruction. I sincerely thank both of my dissertation advisors for everything they have done for me. I especially appreciate their time, knowledge, and investment in my mathematical career and in me as a person.

I would also like to thank my committee members, Dr. Mark Arnold for his help with the numerical computation aspects of my dissertation and Dr. Phillip Harrington for his invaluable input and comments.

The unconditional love of my family has been my greatest support. I would like to thank my parents, Bob and Jackie Boyd, who have always supported and encouraged me throughout my life and through these years of writing my dissertation. I would like to thank my three wonderful children, Rachel, Ben, and Anna whose love, hugs, sense of humor, and offers to help with my math "homework" are always of great encouragement to me. Most importantly, I would like to thank my beloved husband of twenty years, Dr. Matt Myers, for his unconditional love, support, encouragement, and willingness to take over some family responsibilities so that I could have time to pursue my doctorate. I am deeply grateful for our relationship in which I can daily expel to him new findings in my research.

Finally, I would like to thank God most of all who made me with the unique ability to understand and enjoy mathematics. Phillipians 4:13 states, "I can do all things through Christ who strengthens me,"... even a Ph.D. in mathematics at age 42.

Dedication

This dissertation is dedicated to my wonderful husband Matt, and my three wonderful children, Rachel, Ben and Anna Myers.

Table of Contents

Notation	Use
$\mathbb{C}$	the complex plane
D	a domain in $\mathbb{C}$
$\mathbb{D}$	the open unit disk
$\overline{\mathbb{D}}$	closed unit disk
$\mathbb{H}^+$	upper half-plane
$\mathbb{H}^-$	lower half-plane
Ω	an RD
$\mathcal{E}_\xi$	an ellipse with foci at ± 1
$\mathcal{P}$	a general polygon
Λ	a regular polygon
$\mathcal{P}_\Omega$	an RD polygon
R_Ω	an RD rectangle
Rh_Ω	an RD rhombus with foci at ± 1
S_Ω	an RD square with sides parallel to the axes
Λ_Ω	a regular RD polygon with one vertex on the positive real axis
O_Ω	an RD octagon
$O_{\star_\Omega}$	an equilateral RD octagon with four vertices on the axes
$O_{\mathcal{E}_\Omega}$	an RD octagon with foci at ± 1 with four vertices on the axes
f_Ω	conformal mapping from $\mathbb{D} \to \Omega$
$f_{\mathcal{E}_\xi}$	Schwarz-Christoffel mapping from $\mathbb{D} \to \mathcal{E}_\xi$
$f_{\mathcal{P}}$	Schwarz-Christoffel mapping from $\mathbb{D} \to \mathcal{P}$
$f_{\mathbb{H}^+}$	Schwarz-Christoffel mapping from $\mathbb{H}^+ \to \mathcal{P}$
$f_{\mathcal{P}_\Omega}$	Schwarz-Christoffel mapping from $\mathbb{D} \to \mathcal{P}_\Omega$
f_{R_Ω}	Schwarz-Christoffel mapping from $\mathbb{D} \to R_\Omega$
f_{Rh_Ω}	Schwarz-Christoffel mapping from $\mathbb{D} \to Rh_\Omega$
f_{S_Ω}	Schwarz-Christoffel mapping from $\mathbb{D} \to S_\Omega$
f_{O_Ω}	Schwarz-Christoffel mapping from $\mathbb{D} \to O_\Omega$
$f_{O_{\star_\Omega}}$	Schwarz-Christoffel mapping from $\mathbb{D} \to O_{\star_\Omega}$
$f_{O_{\mathcal{E}_\Omega}}$	Schwarz-Christoffel mapping from $\mathbb{D} \to O_{\mathcal{E}_\Omega}$

Notation	Use
f_{Λ}	Schwarz-Christoffel mapping from $\mathbb{D} \to \Lambda$
$f_{\Lambda_{\Omega}}$	Schwarz-Christoffel mapping from $\mathbb{D} \to \Lambda_{\Omega}$

CHAPTER 1

Introduction

1.1 Background and Method

This thesis investigates the nature of coefficients of the Taylor series representation about the origin of the Riemann map, $f(z)$, from the unit disk onto symmetric domains, called rectangular domains or RDs, determined by $f(0) = 0$ and $f'(0) > 0$. In a paper in 2006 **[9]**, Kanas and Sugawa noted that the necessary conditions for a one-to-one function in the unit disk to have non-negative Taylor coefficients about the origin are that $f(0) \geq 0$, $f'(0) > 0$ and that the image domain is symmetric in the real axis. It follows from these necessary conditions that f has a Taylor series representation about the origin with real coefficients. Conversely, however, it is difficult to determine the sufficient geometric conditions for positivity of the Taylor coefficients. Kanas and Sugawa determined that the ellipse $\mathcal{E}_\xi$ defined by

$$\left(\frac{u}{\cosh \xi}\right)^2 + \left(\frac{v}{\sinh \xi}\right)^2 = 1, \ \xi > 0$$

had sufficient geometric conditions for positivity of the odd Taylor coefficients. The ellipse $\mathcal{E}_\xi$ has foci at ± 1 and any arbitrary ellipse is similar to $\mathcal{E}_\xi$ for some $\xi > 0$. In fact, since $\cosh \xi$ and $\sinh \xi$ have the property

$$\cosh^2 \xi - \sinh^2 \xi = 1 \quad \text{for} \quad \xi > 0$$

we know that there exists a $\xi > 0$ such that any ellipse with foci at ± 1 can be written as ellipse $\mathcal{E}_\xi$. Kanas and Sugawa proved the following theorem:

Theorem 1.1. *For any given $\xi > 0$, let $f_{\mathcal{E}_\xi}$ be the conformal mapping of the unit disk onto the ellipse $\mathcal{E}_\xi$ determined by $f_{\mathcal{E}_\xi}(0) = 0$ and $f'_{\mathcal{E}_\xi}(0) > 0$. Then $f_{\mathcal{E}_\xi}$ has positive odd Taylor coefficients about the origin.*

In the same spirit of the 2006 Kanas and Sugawa paper, we will investigate the sufficient geometric conditions that guarantee positivity of odd Taylor coefficients about the origin of the conformal mapping from the unit disk onto symmetric domains. We first define a class of domains, called rectangular domains or RDs, that are centered at the origin and have x and y axes symmetries and nice boundary characteristics. We prove that a conformal mapping from the unit disk onto an RD is unique, odd, and has a Taylor representation about the origin with real coefficients. We then investigate a sub-class of RD domains called RD polygons and define what it means for a polygon to have foci at ± 1. To investigate the nature of the coefficients of the Taylor series about the origin of the conformal map from the unit disk onto RD polygons, we implement a method similar to the method presented by Kanas and Sugawa:

1) Define an explicit form of the conformal mapping from the unit disk onto RD polygons.
2) Determine a differential equation for which the integrand of the conformal mapping is a solution.
3) Determine a recursion formula for the coefficients of the Taylor series representation of the mapping using the Taylor series solution method for differential equations.
4) Determine the sufficient geometric conditions to ensure positivity of the odd coefficients of the Taylor series representation about the origin. If none exists, then determine negativity of at least one odd coefficient. It should be noted that this step differs from the Kanas and Sugawa method.

The explicit conformal mapping from the unit disk onto a polygon is known as the Schwarz-Christoffel mapping.

Theorem 1.2. *The function $f_{\mathcal{P}}(z)$ which maps the unit disk conformally onto a polygon with angles $\alpha_k\pi$ where $k = 1, 2, ...n$ is of the form*

$$f_{\mathcal{P}}(z) = A + C \int^{z} \prod_{k=1}^{n} (1 - \frac{\xi}{z_k})^{-\beta_k} d\xi \tag{1.1}$$

where $\beta_k = 1 - \alpha_k$, $z_k \in \partial\mathbb{D}$ is a finite prevertex with $f_{\mathcal{P}}(z_k) = w_k$, and A, C are complex constants.

Under the assumptions $f_{\mathcal{P}}(0) = 0$ and $f'_{\mathcal{P}}(0) > 0$ and implementing theoretical and numerical methods, we will determine the parameters of the Schwarz-Christoffel mapping from the unit disk onto RD domains. We will then apply the Schwarz-Christoffel mapping and the four steps above to determine the nature of the coefficients for the following sub-classes of RD polygons: RD rhombi with foci at ± 1, RD rectangles, RD squares with sides parallel to the axes, regular RD polygons with one vertex on the positive real axis, equilateral RD octagons with four vertices on the axes, and non-equilateral RD octagons with foci at ± 1 with four vertices on the axes.

1.2 Main Results

The following six theorems are the main results proved in this thesis.

THEOREM 1.3. *Let Rh_Ω be an RD rhombus with foci at ± 1 . If f_{Rh_Ω} is the conformal mapping from the unit disk onto Rh_Ω determined by $f_{Rh_\Omega}(0) = 0$ and $f'_{Rh_\Omega}(0) = 1$, then the Taylor series of f_{Rh_Ω} about the origin has positive odd coefficients.*

THEOREM 1.4. *Let R_Ω be an RD rectangle. If f_{R_Ω} is the conformal mapping from the unit disk onto R_Ω determined by $f_{R_\Omega}(0) = 0$ and $f'_{R_\Omega}(0) = 1$, then the Taylor series of f_{R_Ω} about the origin has at least one negative odd coefficient.*

THEOREM 1.5. *Let S_Ω be an RD square with sides parallel to the axes. If $f_{S_\Omega} : \mathbb{D} \to S_\Omega$ is the conformal map determined by $f_{S_\Omega}(0) = 0$ and $f'_{S_\Omega}(0) = 1$, then the Taylor series of f_{S_Ω} about the origin has non-zero odd coefficients that alternate in sign.*

THEOREM 1.6. *Let Λ_Ω be a regular RD polygon with one vertex on the positive real axis. If f_{Λ_Ω} is the conformal mapping of the unit disk onto Λ_Ω determined by $f_{\Lambda_\Omega}(0) = 0$ and $f'_{\Lambda_\Omega}(0) = 1$, then the Taylor series of f_{Λ_Ω} about the origin has non-negative odd coefficients.*

THEOREM 1.7. *Let $O_{\star_\Omega}$ be an equilateral RD octagon with four vertices on the axes. If $f_{O_{\star_\Omega}}$ is the conformal mapping from the unit disk onto $O_{\star_\Omega}$ determined by $f_{O_{\star_\Omega}}(0) = 0$ and $f'_{O_{\star_\Omega}}(0) = 1$ with $\beta_1 \geq 1/4$, then the Taylor series of $f_{O_{\star_\Omega}}$ about the origin has non-negative odd coefficients.*

THEOREM 1.8. *Let $O_{\mathcal{E}_\Omega}$ be an RD octagon with foci at ± 1 and with four vertices on the axes. If $f_{O_{\mathcal{E}_\Omega}}$ is the conformal mapping from the unit disk onto an $O_{\mathcal{E}_\Omega}$ determined by $f_{O_{\mathcal{E}_\Omega}}(0) = 0$ and $f'_{O_{\mathcal{E}_\Omega}}(0) > 0$, then there exists a $\xi > 0$ such that the Taylor series of $f_{O_{\mathcal{E}_\Omega}}$ about the origin has at least one negative odd coefficient.*

CHAPTER 2

Preliminaries

2.1 Hyperbolic Sine, Cosine, and Tangent Functions

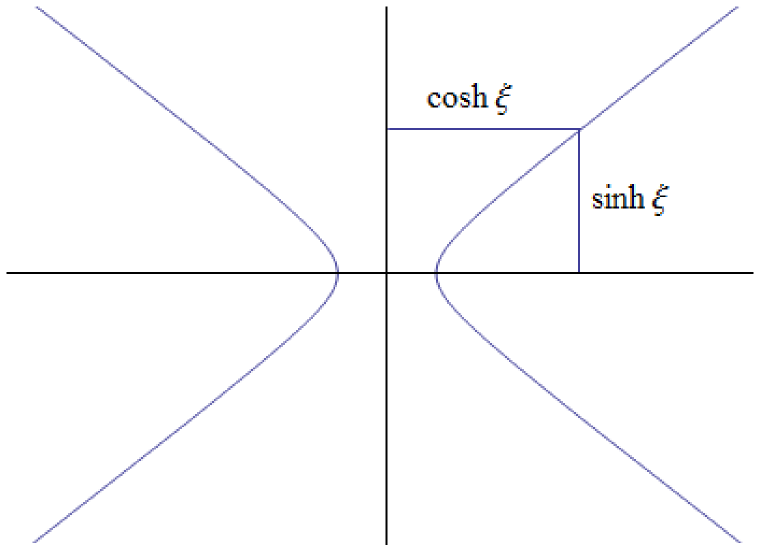

FIGURE 2.1. Relationship of $\cosh\xi$ and $\sinh\xi$. This picture was produced using Wolfram Mathematica Version 8.0 for Students [**26**].

For any real number ξ, $\sinh\xi$ and $\cosh\xi$ are the hyperbolic sine and hyperbolic cosine functions, respectively. As in Figure 2.1 $(\cosh\xi, \sinh\xi)$ is a point on the hyperbola $x^2 - y^2 = 1$ whereas $(\cos\xi, \sin\xi)$ is a point on the unit circle $x^2 + y^2 = 1$. We now present the definitions and properties of the hypebolic functions $\sinh\xi$, $\cosh\xi$, and $\tanh\xi$ [**21**].

DEFINITION 2.1. For any real number ξ, we define the hyperbolic sine as

$$\sinh\xi = \frac{e^{\xi} - e^{-\xi}}{2},$$

the hyperbolic cosine as

$$\cosh\xi = \frac{e^{\xi} + e^{-\xi}}{2},$$

and the hyperbolic tangent as

$$\tanh\xi = \frac{\sinh\xi}{\cosh\xi} = \frac{e^{\xi} - e^{-\xi}}{e^{\xi} + e^{-\xi}}.$$

The French mathematician Johann Lambert and the Italian mathematician, Vincenzo Riccati, independently published their work on hyperbolic functions in the 1760s. Although it is clear

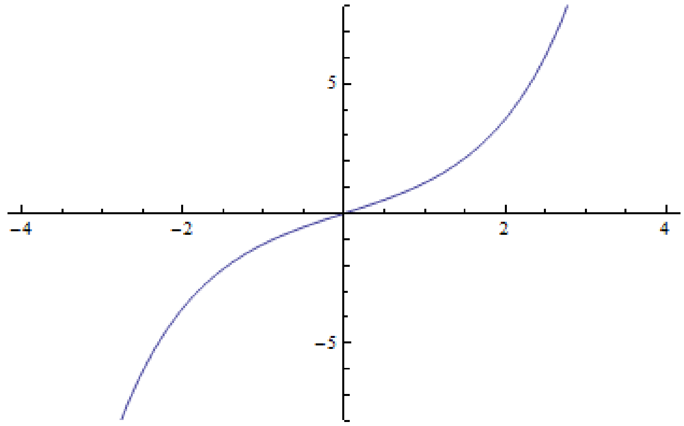

Figure 2.2. Graph of $\sinh \xi$.

from evidence that Lambert and Riccati developed their hyperbolic functions independently, the fact remains that Riccati did publish his work on hyperbolic functions first. However, Lambert's name is the one associated with the earliest developments of hyperbolic functions as his motivations for his work were more central to mathematical interests and his works are more widely available, written in different languages and with notation [**2**].

Now we will investigate the relationships between $\sinh \xi$, $\cosh \xi$ and $\tanh \xi$.

Proposition 2.2. *For any real number* ξ, $\cosh^2 \xi - \sinh^2 \xi = 1$.

Proof. Using the definition of $\cosh \xi$ and $\sinh \xi$, squaring both terms, and cancelling terms we have

$$\begin{aligned}\cosh^2 \xi - \sinh^2 \xi &= \left(\frac{e^{\xi} + e^{-\xi}}{2}\right)^2 - \left(\frac{e^{\xi} - e^{-\xi}}{2}\right)^2 \\ &= \frac{1}{4}(e^{2\xi} + 2e^{\xi}e^{-\xi} + e^{-2\xi}) - \frac{1}{4}(e^{2\xi} - 2e^{\xi}e^{-\xi} + e^{-2\xi}) \\ &= \frac{1}{4} \cdot (4) \\ &= 1.\end{aligned}$$

□

Now we will prove whether the hyperbolic sine, cosine, and tangent are even or odd functions.

Proposition 2.3. *For any real number* ξ, $\sinh \xi$ *is an odd function,* $\cosh \xi$ *is an even function, and* $\tanh \xi$ *is an odd function.*

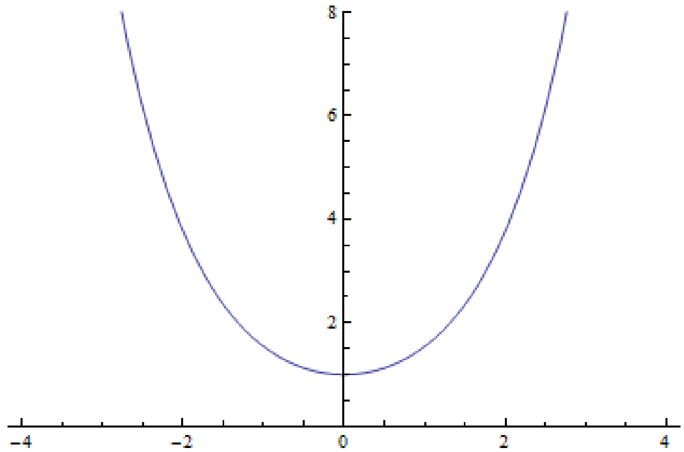

FIGURE 2.3. Graph of $\cosh \xi$.

PROOF. By evaulating the hyperbolic functions at $-x$ and using their definitions we have that

$$\sinh(-\xi) = \frac{1}{2}(e^{-\xi} - e^{\xi}) = -\sinh \xi,$$
$$\cosh(-\xi) = \frac{1}{2}(e^{-\xi} + e^{\xi}) = \cosh \xi,$$

and

$$\tanh(-\xi) = \frac{\sinh(-\xi)}{\cosh(-\xi)} = \frac{-\sinh(\xi)}{\cosh(\xi)} = -\tanh \xi.$$

Hence, $\sinh \xi$ is an odd function, $\cosh \xi$ is an even function, and $\tanh \xi$ is an odd function. □

The even and odd function properties of $\sinh \xi$, $\cosh \xi$ and $\tanh \xi$ are visually represented in Figures 2.2, 2.3, and 2.4.

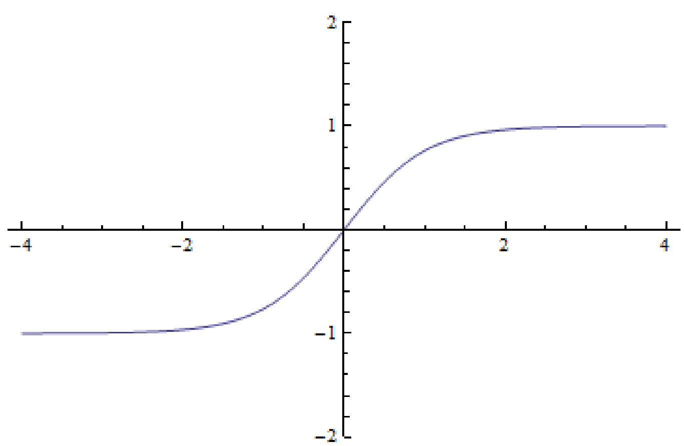

FIGURE 2.4. Graph of $\tanh \xi$.

Proposition 2.4. *For any real number ξ, the derivatives of the hyperbolic functions are given by*

$$\frac{d}{d\xi}\sinh\xi = \cosh\xi,$$

$$\frac{d}{d\xi}\cosh\xi = \sinh\xi,$$

and

$$\frac{d}{d\xi}\tanh\xi = \operatorname{sech}^2\xi.$$

Proof. By taking the derivatives of the definitions of the hyperbolic functions we have

$$\frac{d}{d\xi}\sinh\xi = \frac{d}{d\xi}\frac{1}{2}(e^{\xi} - e^{-\xi}) = \cosh\xi,$$

$$\frac{d}{d\xi}\cosh(\xi) = \frac{d}{d\xi}\frac{1}{2}(e^{\xi} + e^{-\xi}) = \sinh\xi,$$

and

$$\begin{aligned}\frac{d}{d\xi}\tanh(\xi) = \frac{d}{d\xi}\frac{\sinh\xi}{\cosh\xi} &= \frac{\cosh\xi\cosh\xi - \sinh\xi\sinh\xi}{\cosh^2\xi} = \frac{\cosh^2\xi - \sinh^2\xi}{\cosh^2\xi}\\ &= \frac{1}{\cosh^2\xi} = \operatorname{sech}^2\xi.\end{aligned}$$

$\square$

Proposition 2.5. *For any real number ξ, the derivatives of inverse hyperbolic functions are given by*

$$\frac{d}{d\xi}\sinh^{-1}\xi = \frac{1}{\sqrt{1+\xi^2}},$$

$$\frac{d}{d\xi}\cosh^{-1}\xi = \frac{1}{\sqrt{\xi^2-1}},$$

and

$$\frac{d}{d\xi}\tanh^{-1}\xi = \frac{1}{1-\xi^2}.$$

Proof. For $\sinh^{-1}\xi$, we make the substitution $\xi = \sinh u$, $d\xi = \cosh u\,du$ and evaluate the following integral,

$$\int \frac{1}{\sqrt{1+\xi^2}}\,d\xi = \int \frac{\cosh u}{\sqrt{1+\sinh^2 u}}\,du = \int \frac{\cosh u}{\cosh u}\,du = \int du = u + c = \sinh^{-1}\xi + c.$$

Therefore,

$$\frac{d}{d\xi}\sinh^{-1}\xi = \frac{1}{\sqrt{1+\xi^2}}.$$

For $\cosh^{-1}\xi$, we make the substitution $\xi = \cosh u$, $d\xi = \sinh u\, du$ and evaluate the following integral,

$$\int \frac{1}{\sqrt{\xi^2-1}}\,d\xi = \int \frac{\sinh u}{\sqrt{\cosh^2 u - 1}}\,du = \int \frac{\sinh u}{\sinh u}\,du = \int du = u + c = \cosh^{-1}\xi + c.$$

Therefore,

$$\frac{d}{d\xi}\cosh^{-1}\xi = \frac{1}{\sqrt{\xi^2-1}}.$$

For $\tanh^{-1}\xi$, we make the substitution $\xi = \tanh u$, $d\xi = \operatorname{sech}^2 u\, du$ and evaluate the following integral,

$$\int \frac{1}{1-\xi^2}\,d\xi = \int \frac{\operatorname{sech}^2 u}{1-\tanh^2 u}\,du = \int \frac{\operatorname{sech}^2 u}{\operatorname{sech}^2 u}\,du = \int du = u + c = \tanh^{-1}\xi + c.$$

Therefore,

$$\frac{d}{d\xi}\tanh^{-1}\xi = \frac{1}{1-\xi^2}.$$

□

Proposition 2.6. *As* $\xi \to \infty$, $\tanh\xi \to 1$.

Proof. Notice that as $\xi \to \infty$, $e^{-\xi} \to 0$, and we have

$$\lim_{\xi\to\infty} \frac{\sinh\xi}{\frac{1}{2}e^{\xi}} = \lim_{\xi\to\infty} \frac{\frac{1}{2}(e^{\xi}-e^{-\xi})}{\frac{1}{2}e^{\xi}} = 1,$$

$$\lim_{\xi\to\infty} \frac{\cosh\xi}{\frac{1}{2}e^{\xi}} = \lim_{\xi\to\infty} \frac{\frac{1}{2}(e^{\xi}+e^{-\xi})}{\frac{1}{2}e^{\xi}} = 1,$$

and

$$\lim_{\xi\to\infty}\tanh\xi = \lim_{\xi\to\infty}\frac{\sinh\xi}{\cosh\xi} = \lim_{\xi\to\infty}\left(\frac{\sinh\xi}{\frac{1}{2}e^{\xi}}\right)\left(\frac{\frac{1}{2}e^{\xi}}{\cosh\xi}\right) = 1.$$

Thus,

$$\lim_{\xi\to\infty}\frac{\sinh\xi}{\cosh\xi} = 1.$$

□

Proposition 2.7. *As $\xi \to 0$, $\tanh\xi \to 0$.*

Proof. Notice that as $\xi \to 0$, $e^{-\xi} \to 1$, and we have

$$\lim_{\xi\to 0}\frac{\sinh\xi}{\frac{1}{2}e^{\xi}} = \lim_{\xi\to 0}\frac{\frac{1}{2}(e^{\xi}-e^{-\xi})}{\frac{1}{2}e^{\xi}} = 0,$$

$$\lim_{\xi\to 0}\frac{\cosh\xi}{\frac{1}{2}e^{\xi}} = \lim_{\xi\to 0}\frac{\frac{1}{2}(e^{\xi}+e^{-\xi})}{\frac{1}{2}e^{\xi}} = 1,$$

and

$$\lim_{\xi\to 0}\tanh\xi = \lim_{\xi\to 0}\frac{\sinh\xi}{\cosh\xi} = \lim_{\xi\to\infty}\left(\frac{\sinh\xi}{\frac{1}{2}e^{\xi}}\right)\left(\frac{\frac{1}{2}e^{\xi}}{\cosh\xi}\right) = 0.$$

Thus,

$$\lim_{\xi\to\infty}\frac{\sinh\xi}{\cosh\xi} = 0.$$

□

These observations are visually represented by Figure 2.5.

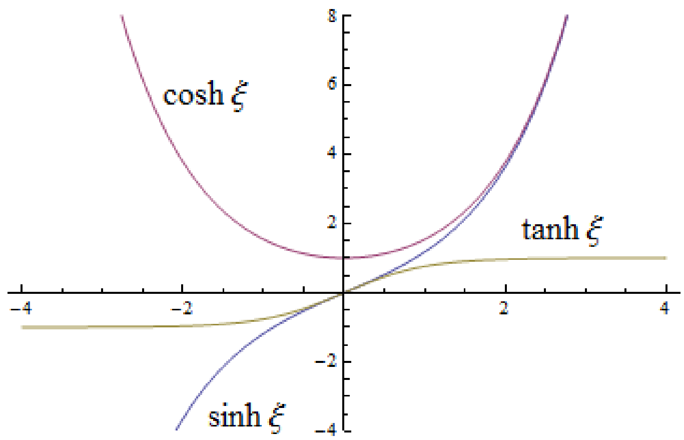

Figure 2.5. Graph of $\sinh\xi$, $\cosh\xi$, and $\tanh\xi$.

2.2 Analytic Functions and Taylor Series

Analytic functions, also known as holomorphic functions, are central concepts in complex analysis. An function f is analytic at z_0 if it is complex-valued and complex differentiable in every neighborhood of z_0 in domain D. That is, $f'(z_0) \neq 0$ for every $z_0 \in D$. This implies that an analytic function is infinitely differentiable and equal to its convergent Taylor series.

Theorem 2.8. *For each power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ a real (finite or infinite) number $R \geq 0$ can be determined, called the radius of convergence of the series. If $R = \infty$ then the series converges for* *any $z \in \mathbb{C}$; if $R = 0$, then the series converges only at one point $z = z_0$. If $0 < R < \infty$, then the series converges at $|z - z_0| < R$ and diverges at $|z - z_0| > R$. The domain $|z - z_0| < R$ is called the circle of convergence of the power series. Then sum of the power series is an analytic function within the circle of convergence. The inverse statement is also true: if the function $f(z)$ is analytic in some circle $|z-z_0| < R$, then in this circle it can be expanded into the power series called the Taylor's series:*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \tag{2.1}$$

where $a_0 = f(z_0)$ and $a_n = f^{(n)}(z_0)/n!$. The radius of convergence of the Taylor's series equals the distance from the point z_0 to the nearest singular point of the function $f(z)$ **[11]**.

Analytic functions are preserved under addition, multiplication, and compositions. They are also preserved under division provided that the function in the denominator does not vanish at any point in D. Hence, if a conformal mapping $f(z)$ is analytic on the unit disk with z_0=0, then it has a convergent Taylor series representation about the origin.

2.3 Schwarz's Lemma and Reflection Principle

Two main contributions to the field of complex analysis attributed to H.A. Schwarz in his work published in the 19th century are Schwarz's Lemma and the Schwarz Reflection Principle. These results are useful tools in proving theorems in complex analysis and will be used to prove results in this thesis.

Schwarz's Lemma is a result in complex analysis about holomorphic functions from the unit disk to itself. The formal statement of the Lemma is as follows [**1**]:

Lemma 2.9. *If f(z) is analytic for $|z| < 1$ and satisfies the conditions $f(z) \leq 1$, $f(0) = 0$, then $|f(z)| \leq |z|$ and $|f'(0)| \leq 1$. If $|f(z)| = |z|$ for some $z \neq 0$, or if $|f'(0)| = 1$, then $f(z) = cz$ with $|c| = 1$.*

The Schwarz Reflection Principle is an effective tool in extending a function's domain where it is analytic. A straight arc or segment of a straight line is a part of the straight line that is bounded by two distinct end points and contains every point on the straight line between its end points. Thus, if an analytic function f extends continuously to a straight arc and maps the boundary arc to another straight arc, then f can be analytically continued across the arc by reflection [**6**]. Thus, we can formally state the Schwarz Reflection Principle for the real axis [**1**].

Theorem 2.10. *Let V^+ be the part in the upper half plane of a symmetric region V, and let σ be the part of the real axis in V. Suppose that $v(x)$ is continuous in $V^+ \cup \sigma$, harmonic in V^+, and zero on σ. Then v has a harmonic extension to V which satisfies the symmetry relation $v(\bar{z}) = -v(z)$. In the same situation, if v is the imaginary part of an analytic function $f(z)$ in V^+, then $f(z)$ has an analytic extension which satisfies $f(z) = \overline{f(\bar{z})}$.*

A similar theorem is true for the unit disk. A circular boundary arc or segment of a curve is a part of the curve that is bounded by two distinct end points and contains every point on the curve between its end points. If an analytic function f extends continuously to a circular boundary arc or curve and maps the boundary arc to another circular arc, then f can be analytically continued across the arc by reflection [**6**].

2.4 Conformal Mapping

The notion of mapping two domains conformal to each other developed from motiviations in cartography and geography. This notion started with nonconformal map projections in 600 B.C., continued with Ptolemy's stereographic projection discovery around 1500 A.D., and continued with Mercator's conformal map from a sphere onto a plane strip called the Mercator's Projection. Soon after this, C.F. Gauss invented the idea of general conformal mapping. He completely solved the problem of finding all conformal transformations from a small domain on an analytic surface to a region in the plane area. Riemann's doctoral dissertation in 1851 provided the foundation for the field of conformal mappings as it included the statement of the Riemann mapping theorem. Soon came the Schwarz-Christoffel mapping independently developed and published by both Schwarz and Christoffel [**10**].

In complex function theory, a conformal mapping f maps a domain D, an open subset of $\mathbb{C}$, to $f(D)$ with the properties of preserving angles and orientation as shown in Figure 2.6.

Definition 2.11. A conformal mapping f is a function that preserves angles (and orientation) at a point $z_0 \in D$. That is, f is conformal if and only if it is analytic and $f'(z_0) \neq 0$ for every $z_0 \in D$. Thus, if two curves intersect at z_0 at angle θ, then the image curves under the conformal map f will intersect at the point $f(z_0)$ at the same angle θ.

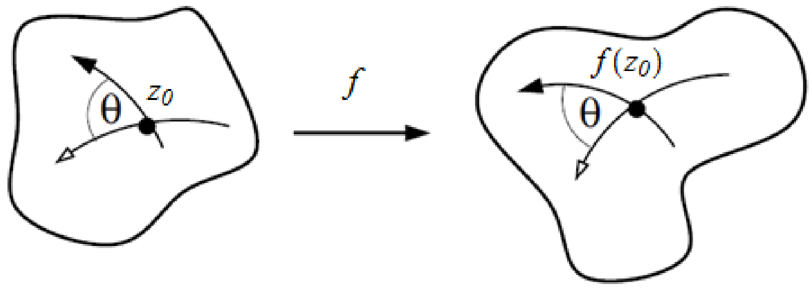

FIGURE 2.6. Conformal mappings preserve angles (and orientation). Adapted from figure produced by Christopher J. Bishop [3].

Conformal mappings are preserved under translations, dilations, rotations, inversions and a combination of these, Möbius transformations, as shown in Figures 2.7, 2.8, 2.9, 2.10 and 2.11, respectively [13].

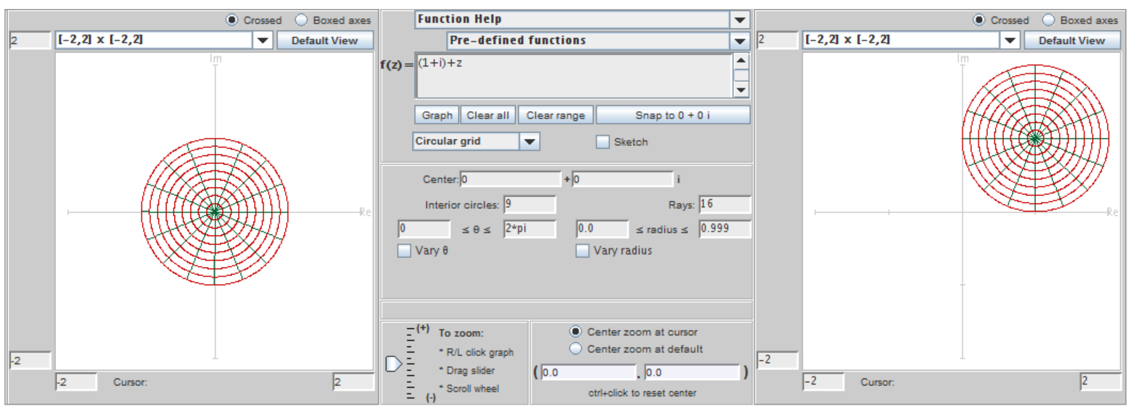

FIGURE 2.7. Conformal mapping of a disk under the translation $f(z) = A + z = (1 + i) + z$. This picture was produced with Jim Rolf's Complex Tools applet [17].

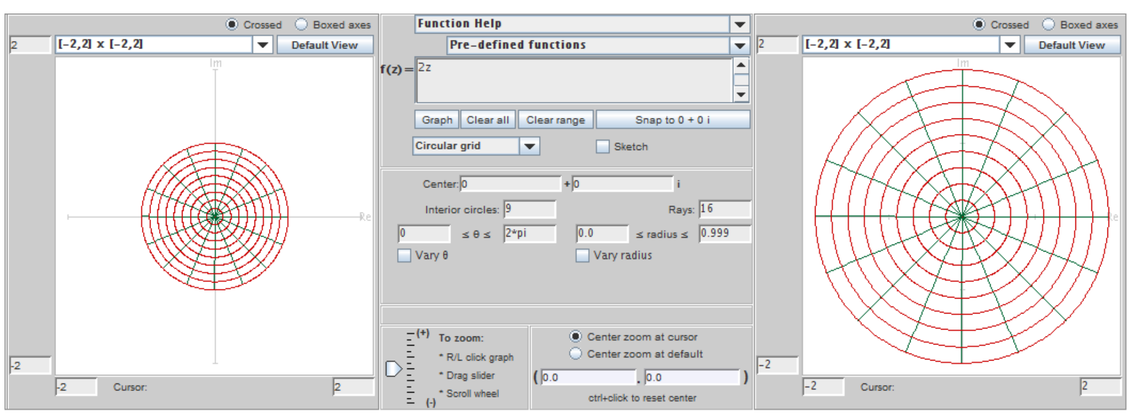

FIGURE 2.8. Conformal mapping of a disk under the dilation $f(z) = 2z$.

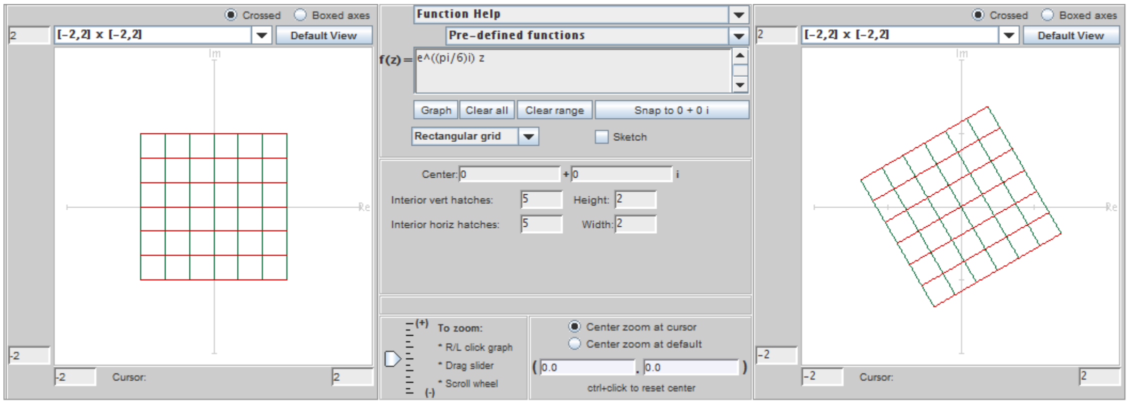

FIGURE 2.9. Conformal mapping of a square under the rotation $f(z) = e^{(\pi/6)i}z$.

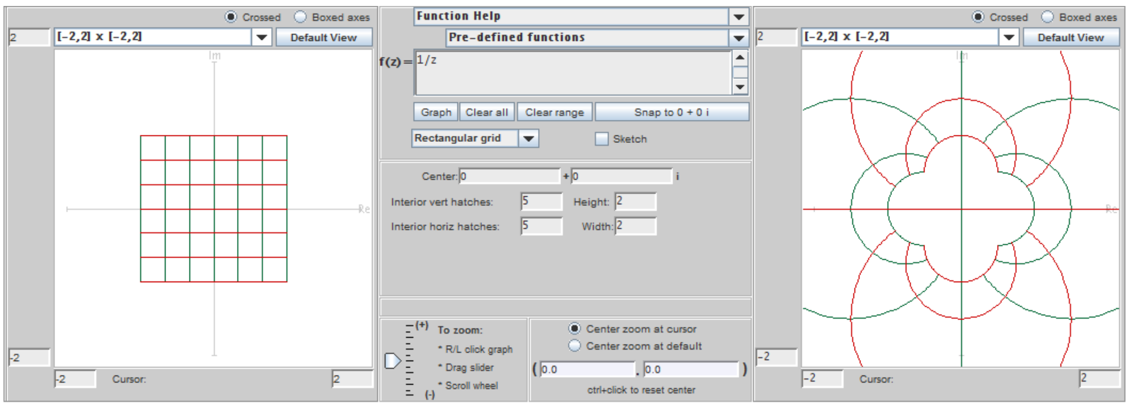

FIGURE 2.10. Conformal mapping of a square under the inversion $f(z) = 1/z$.

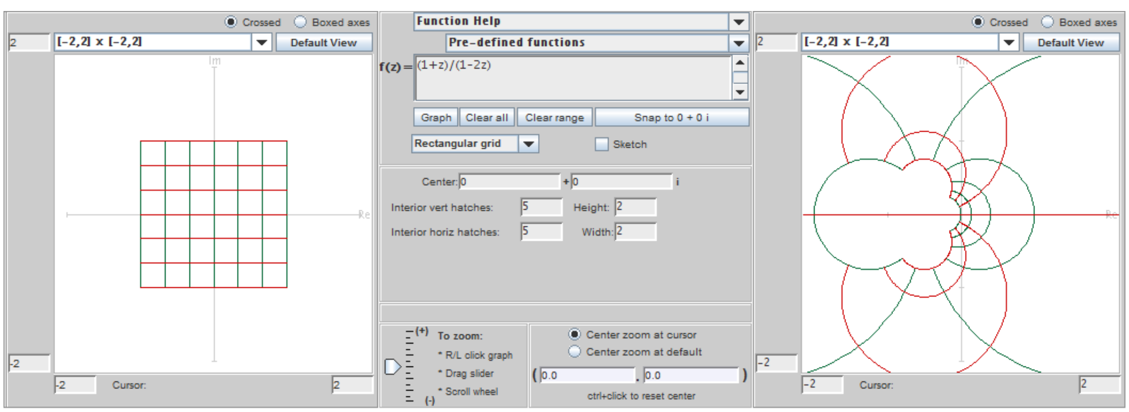

FIGURE 2.11. Conformal mapping of a square under the Möbius transformation $f(z) = (1 + z)/(1 - 2z)$.

2.5 The Riemann Mapping Theorem

Georg Friedrich Bernhard Riemann was born September 17, 1826 in Breselenz, Germany and died of tuberculosis-related health issues on July 20, 1866. During his lifetime, Bernhard Riemann studied at the University of Göttingen under Gauss and studied under Steiner, Jacobi, and Dirchlet at Berlin University [**14**]. Riemann first stated the Riemann mapping theorem in his doctoral thesis, supervised by Gauss and submitted in 1851 [**16**]. It guarantees that any simply connected region in the complex plane can be conformally mapped onto any other, provided that neither is the entire plane. Although Riemann's proof of his theorem was incomplete in his doctoral thesis, others such as Koebe, Osgood, Carathéodory, and Hilbert provided rigous proofs of the theorem in the 20^{th} century [**6**].

Theorem 2.12. *Let D be a nonempty simply connected subset of $\mathbb{C}$ but not all of $\mathbb{C}$ and let $z_0 \in D$. Then there exists a unique biholomorphic conformal mapping (analytic and bijective) $f : D \to \mathbb{D}$ with $f(z_0) = 0$ and $f'(z_0) > 0$.*

The importance of the Riemann mapping theorem is that it guarantees conformal equivalence to the unit disk. Thus, for domains with irregular boundaries we can transform otherwise complicated computations onto the unit disk where computations are easier because of its nice geometric properties, and then transform them back to the original domain. Moreover, the theorem implies there exists biholomorphisms between any two simply connected domains in the complex plane since any two such domains are conformally equivalent to the unit disk. However, it is an existence result and not a constructive theorem. Therefore, the Riemann map for a general domain D is usually not known explicitly. Lastly, it should also be noted that the Riemann map only exists in the complex plane. It does not hold in higher dimensional $\mathbb{C}^n$.

2.6 The Schwarz-Christoffel Mapping for $\mathbb{H}^+$

Soon after the Riemman mapping theorem of 1851, the Schwarz-Christoffel formula was discovered independently by Christoffel in 1867 and Schwarz in 1869. Edwin Bruno Christoffel (1829-1900) was born in Moschau, Germany and studied mathematics in Berlin under Dirchlet and others. In a paper in 1867 [**4**], Christoffel published the Schwarz-Christoffel formula. His motivation was the problem of heat conduction in which he used Green's function. In the case of a polygonal domain, Green's function could be obtained by a conformal map from the half-plane. In later papers he extended these ideas to exteriors of polygons and to curved boundaries.

Herman Amandus Schwarz (1843-1921) was also influenced by Riemann and in the late 1860s discovered the Schwarz-Christoffel formula independent of Christoffel. He published three papers [**18**], [**19**], [**20**] which included similar material found in Christoffel's work including generalizations to curved boundaries and circular polygons. However, Schwarz's emphasis was more numerical and concerned with triangles and quadrilaterals. Schwarz also was the first to publish a plot of the Schwarz-Christoffel map.

The Schwarz-Christoffel formula has had a significant impact in complex analysis, especially as a tool for proving the Riemann mapping theorem and similar results. However, because of the Schwarz-Christoffel parameter problem, the practical use of the formula was limited to simple special cases until the invention of computers [**6**].

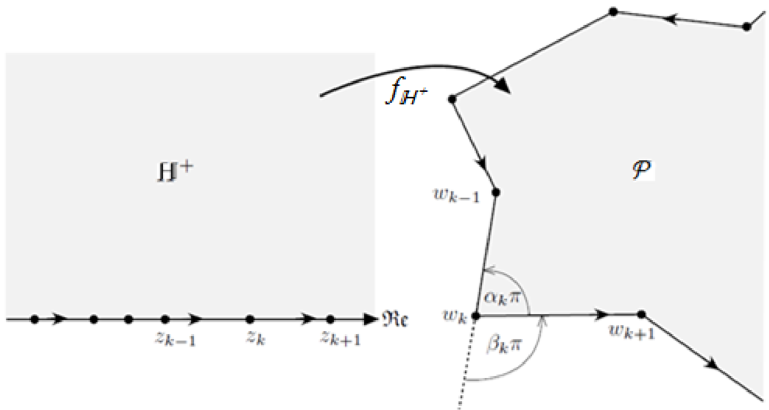

FIGURE 2.12. The Schwarz-Christoffel mapping from the upper half-plane onto a polygon. Adapted from figure produced by Landstorfer [**10**].

In Figure 2.12, $\mathcal{P}$ is a polygon, $\mathbb{H}^+$ is the upper half-plane and $f_{\mathbb{H}^+}(z)$ is the Schwarz-Christoffel mapping $f_{\mathbb{H}^+} : \mathbb{H}^+ \to \mathcal{P}$. The vertices of the polygon $\mathcal{P}$ on $\partial\mathcal{P}$ are $w_1, w_2, ..., w_n$, given in counterclockwise order, with interior angles $\alpha_1\pi$, $\alpha_2\pi$, ..., $\alpha_n\pi$ where $\alpha_k \in (0, 2)$ for each k, and exterior angles or turning angles $\beta_1\pi$, $\beta_2\pi$, ..., $\beta_n\pi$ where $\beta_k = 1 - \alpha_k$ for each k. The prevertices of $\mathbb{H}^+$ on $\partial\mathbb{H}^+$ are $z_1, z_2, ..., z_n$, given in counterclockwise order, where $f^{-1}_{\mathbb{H}^+}(w_k) = z_k$.

Now we present the formula for the Schwarz-Christoffel mapping from the upper half-plane onto a polygon.

THEOREM 2.13. *The function $f_{\mathbb{H}^+}(z)$ which maps the upper half-plane, $\mathbb{H}^+$, conformally onto a polygon with angles $\alpha_k\pi$ where $k = 1, 2, ...n$ is of the form*

$$f_{\mathbb{H}^+}(z) = A + C \int^z \prod_{k=1}^{n} (1 - \frac{\xi}{z_k})^{-\beta_k} d\xi \tag{2.2}$$

where $\beta_k = 1 - \alpha_k$, z_k is a finite prevertex on the real axis of H^+ with $f_{\mathbb{H}^+}(z_k) = w_k$, and A, C are complex constants.

Note that this version of the S-C mapping for the upper half-plane assumes each prevertex is finite and $z_n \neq \infty$. If $z_n = \infty$ then the product index only goes to n-1 to take into account that there is one less finite vertex. The following proof is given by **[6]** and **[10]** .

Proof. In proving the Schwarz-Christoffel mapping, we will formulate the geometric properties that result from conformally mapping the upper half-plane onto a polygon. Next, we will determine a canonical form of $f_{\mathbb{H}_k^+}$ which satisfies these geometric properties. We will then apply the Schwarz Reflection Principle and consider the pre-Schwarzian derivative $f''_{\mathbb{H}^+}(z)/f'_{\mathbb{H}^+}(z)$ which is invariant under linear transformations and show that it is single-valued and analytic. Finally, we will calculate the pre-Schwarzian derivative using residues and then use calculus and algebraic properties to complete the derivation of the S-C formula.

Before we discuss the geometric properties of the mapping, recall the notation where $\mathcal{P}$ is a polygon, $\mathbb{H}^+$ is the upper half-plane and $f_{\mathbb{H}^+}(z)$ is the Schwarz-Christoffel mapping $f_{\mathbb{H}^+}$: $\mathbb{H}^+ \to \mathcal{P}$. The vertices of the polygon $\mathcal{P}$ on $\partial\mathcal{P}$ are $w_1, w_2, ..., w_n$, given in counterclockwise order, with interior angles $\alpha_1\pi$, $\alpha_2\pi$, ..., $\alpha_n\pi$ where $\alpha_k \in (0, 2)$ for each k, and exterior angles or turning angles $\beta_1\pi$, $\beta_2\pi$, ..., $\beta_n\pi$ where $\beta_k = 1 - \alpha_k$ for each k. The prevertices of $\mathbb{H}^+$ on $\partial\mathbb{H}^+$ are $z_1, z_2, ..., z_n$, given in counterclockwise order, where $f^{-1}_{\mathbb{H}^+}(w_k) = z_k$. It is easy to see from these definitions that the following geometric properties hold:

i) $\alpha_k\pi + \beta_k\pi = \pi \Rightarrow \alpha_k + \beta_k = 1 \Rightarrow \beta_k = 1 - \alpha_k$
ii) $\sum_{k=1}^{n} \beta_k\pi = 2\pi \Rightarrow \sum_{k=1}^{n} \beta_k = 2$.

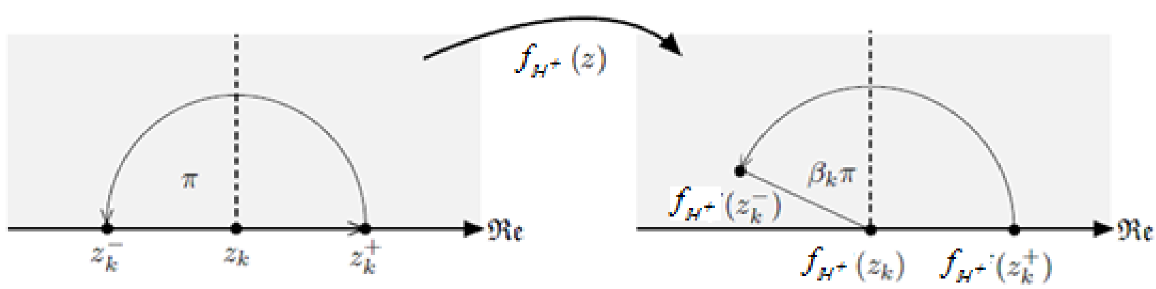

Figure 2.13. A diagram of the argument. Adapted from figure produced by Landstorfer **[10]**.

As shown in Figure 2.13, as the polygon is traversed in a counterclockwise manner, $f_{\mathbb{H}^+}$ is continued analytically along the segment (z_{k-1}, z_k). So $f'_{\mathbb{H}^+}$ exists along this segment and *arg* $f'_{\mathbb{H}^+}$

must be constant there until it reaches the vertex $z = z_k$ where its turning angle is given by

$$arg\, f'_{\mathbb{H}^+}(z)\big|_{z_k^-}^{z_k^+} = \beta_k \pi. \tag{2.3}$$

This holds for all segments so we are interested in determining a canonical form $f_{\mathbb{H}_k^+}$ that satifies the following three conditions:

i) analytic on $\mathbb{H}^+$,

ii) satisfies equation (2.3)

iii) $arg\, f_{\mathbb{H}_k^+}$ is constant on each segment (z_{k-1}, z_k) for $k = 1, 2, ..., n$.

In determining the canonical form of $f_{\mathbb{H}_k^+}$ for each k, let us first consider the well-known mapping, $g(z) = z^\alpha$ that maps $\mathbb{H}^+$ to the sector $\{z : 0 < arg\, z < \alpha\pi\}$ with $0 < \alpha < 2$. So, $g'(z) = \alpha z^{\alpha-1} = \alpha z^{-\beta}$. Notice that

$$z^\alpha = g(z) = \int_0^z g'(\xi) d\xi = \alpha \int_0^z \xi^{-\beta} d\xi$$

Since $g(z)$ maps $[0, \infty) \to [0, \infty)$ and $(-\infty, 0] \to (e^{i\pi\infty}, 0]$, the vertex of this sector is at the origin and it follows that for the sector with vertex at z_k we have

$$(z - z_k)^\alpha = g(z - z_k) = \int_0^z g'(\xi - z_k) d\xi = \alpha \int_0^z (\xi - z_k)^{-\beta} d\xi.$$

Thus, if we let $f_{\mathbb{H}_k^+} = (z - z_k)^{-\beta}$ then clearly $f_{\mathbb{H}_k^+}$ meets the three conditions previously mentioned. Moreover, for $f_{\mathbb{H}_k^+}$, we will not use the common branch cut for logarithms which is the negative real axis. Instead, we will use the branch cut $\{z_k + iy : y \leq 0\}$ to take into consideration the negative exponent of $f_{\mathbb{H}_k^+}$.

Now we will show that the pre-Schwarzian $f''_{\mathbb{H}^+}(z)/f'_{\mathbb{H}^+}(z)$ is single-valued and holomorphic. By the Schwarz Reflection Principle, the conformal mapping $f_{\mathbb{H}^+}$ can be analytically continued into the lower half-plane. The image continues into the reflection of the interior of $\mathcal{P}$ about one of the sides, or a straight arc, on $\partial\mathcal{P}$. By reflecting again about a side of the new polygon, we can return analytically to the upper half-plane. The same can be done for any even number of reflections of the interior of $\mathcal{P}$, each time creating a new branch of $f_{\mathbb{H}^+}$. Therefore, the image of each branch must be translated and rotated copy of the interior of $\mathcal{P}$, given by $A + Cf_{\mathbb{H}^+}(z)$. Thus, by applying an even number of reflections to a polygon, we again get a translated and rotated copy of the original polygon.

Now if A and C are any complex constants and we want to show $f_{\mathbb{H}^+}$ is single-valued then we must consider the pre-Schwarzian derivative $f''_{\mathbb{H}^+}(z)/f'_{\mathbb{H}^+}(z)$, which is invariant under linear transformations. Thus, we have

$$\frac{f''_{\mathbb{H}^+}(A+Cz)}{f'_{\mathbb{H}^+}(A+Cz)} = \frac{f''_{\mathbb{H}^+}(z)}{f'_{\mathbb{H}^+}(z)}.$$

Hence, the pre-Schwarzian derivative $f''_{\mathbb{H}^+}(z)/f'_{\mathbb{H}^+}(z)$ remains the same for all reflections and therefore is single-valued and holomorphic on $\mathbb{C}\setminus\{z_1, ..., z_n\}$ as the derivative may not exist at the prevertices.

Now we will calculate $f''_{\mathbb{H}^+}(z)/f'_{\mathbb{H}^+}(z)$. We know that

$$arg\, f'_{\mathbb{H}^+}(z) = \sum_{k=1}^{n} arg\, f_{\mathbb{H}^+_k}$$

By properties of arg, we have

$$arg\, f'_{\mathbb{H}^+}(z) = arg \prod_{k=1}^{n} f_{\mathbb{H}^+_k}$$

Thus,

$$f'_{\mathbb{H}^+}(z) = C \prod_{k=1}^{n} f_{\mathbb{H}^+_k} = C \prod_{k=1}^{n} (z - z_k)^{-\beta_k}$$

where C is a rotation and dilation constant.

Now we can write

$$f'_{\mathbb{H}^+}(z) = (z - z_k)^{-\beta_k} \cdot \psi(z)$$

for a function $\psi(z)$ analytic in a neighborhood of z_k. Thus, $f''_{\mathbb{H}^+}(z)/f'_{\mathbb{H}^+}(z)$ has a simple pole at z_k with residue $-\beta_k$ and

$$\frac{f''_{\mathbb{H}^+}(z)}{f'_{\mathbb{H}^+}(z)} - \sum_{k=1}^{n} \frac{-\beta_k}{z - z_k} \tag{2.4}$$

is an entire function. Since all the prevertices are finite and $f_{\mathbb{H}^+}$ is analytic at $z = \infty$, we have a Laurent series expansion at $z = \infty$ which implies that $f''_{\mathbb{H}^+}(z)/f'_{\mathbb{H}^+}(z) \longrightarrow 0$ as $z \longrightarrow \infty$.

So, the expression in (2.4) is analytic and vanishes at infinity. Thus, it is entire and bounded and by applying Liouville's Theorem is identically 0. Therefore,

$$\frac{f''_{\mathbb{H}^+}(z)}{f'_{\mathbb{H}^+}(z)} = \sum_{k=1}^{n} \frac{-\beta_k}{z - z_k}.$$

Now observing that $f''_{\mathbb{H}^+}(z)/f'_{\mathbb{H}^+}(z) = (log f'_{\mathbb{H}^+}(z))'$ and integrating twice we get the formula in (2.2). ☐

Let us make some observations about this form of the Schwarz-Christoffel mapping. First note that this version assumes each prevertex is finite and $z_n \neq \infty$. If $z_n = \infty$ then the product index only goes to $n-1$ to take into account that there is one less finite vertex. Also, notice that the integral does not have a lower bound. The lower bound is typically 0 but other lower bounds may be chosen, affecting the values of the constants A and C. Moreover, we used $\alpha \in (0, 2)$ where 0 and 2 were not included. The values of α includes 0 and 2 if we consider an infinite vertex and a slit. Lastly, it is important to observe that the Schwarz-Christoffel mapping applies to polygons with both convex and nonconvex vertices with arbitrary side lengths.

2.7 The Schwarz-Christoffel Mapping for $\mathbb{D}$

Using the unit disk as the canonical domain may be preferred to the half-plane as it has the computational advantage of being a bounded domain and has no naturally distinguished boundary point. The Schwarz-Christoffel mapping from the unit disk onto a polygon is developed by adding some Möbius transformation from the unit disk onto the half-plane. For the Schwarz-Christoffel formula for the unit disk presented below the Möbius transformation $1/(1+z)$ is used [**6**]. A direct proof of the Schwarz-Christoffel mapping from the unit disk onto a polygon is found in [**5**].

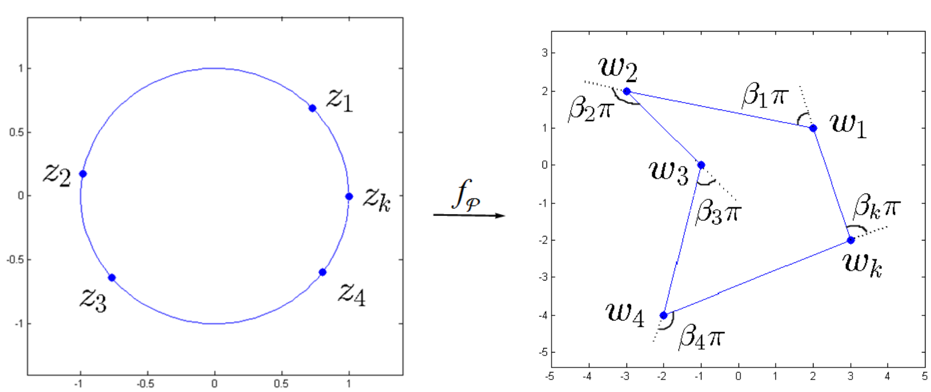

FIGURE 2.14. The Schwarz-Christoffel mapping from the unit disk onto a polygon.

Theorem 2.14. *The function $f_{\mathcal{P}}(z)$ which maps the unit disk conformally onto a polygon with angles $\alpha_k\pi$ where $k = 1, 2, ...n$ is of the form*

$$f_{\mathcal{P}}(z) = A + C\int^{z}\prod_{k=1}^{n}(1-\frac{\xi}{z_k})^{-\beta_k}d\xi \tag{2.5}$$

where $\beta_k = 1-\alpha_k$, $z_k \in \partial\mathbb{D}$ is a finite prevertex with $f_{\mathcal{P}}(z_k) = w_k$, and A, C are complex constants.

Proposition 2.15. *Let $\mathcal{P}$ be a polygon. Suppose $f_{\mathcal{P}} : \mathbb{D} \to \mathcal{P}$ is the Schwarz-Christoffel mapping from the unit disk onto a polygon determined by $f_{\mathcal{P}}(0) = 0$ and $f'_{\mathcal{P}}(0) > 0$. Then $f_{\mathcal{P}}(z)$ is conformal only on the interior of $\mathcal{P}$.*

Proof. The Schwarz-Christoffel mapping $f_{\mathcal{P}}(z)$ which maps the unit disk conformally onto a polygon with angles $\alpha_k\pi$ where $k = 1, 2, ...n$ is of the form

$$f_{\mathcal{P}}(z) = \int_0^{z}\prod_{k=1}^{n}(1-\frac{\xi}{z_k})^{-\beta_k}d\xi$$

where $\beta_k = 1-\alpha_k$, $z_k \in \partial\mathbb{D}$ is z finite prevertex with $f_{\mathcal{P}}(z_k) = w_k$, and $C > 0$. $f_{\mathcal{P}}(0) = 0$ implies that $A = 0$.

Since the integrand is the product of binomials to the exponent of $-\beta_k$ and not -1 we can apply the Fundamental Theorem of Calculus to $f_{\mathcal{P}}(z)$ to give us the derivative

$$f'_{\mathcal{P}} = \prod_{k=1}^{n}(1-\frac{z}{z_k})^{-\beta_k} = C\prod_{k=1}^{n}\frac{1}{(1-\frac{z}{z_k})^{\beta_k}}.$$

Notice that for all z such that $|z| < 1$, $f'_{\mathcal{P}}(z)$ exists and thus $f_{\mathcal{P}}(z)$ is conformal on the interior of $\mathcal{P}$. However, for a particular prevertex on the boundary of $\mathbb{D}$, say $z_{k'}$ which is mapped to the vertex $w_{k'}$ with $\beta_{k'} > 0$ on the boundary of $\mathcal{P}$, we have

$$f'_{\mathcal{P}}(z_{k'}) = C\frac{1}{(1-\frac{z_1}{z_{k'}})^{\beta_1}}\frac{1}{(1-\frac{z_2}{z_{k'}})^{\beta_2}}\cdots\frac{1}{(1-\frac{z_{k'}}{z_{k'}})^{\beta_{k'}}}\cdots\frac{1}{(1-\frac{z_{n-1}}{z_{k'}})^{\beta_{n-1}}}\frac{1}{(1-\frac{z_n}{z_{k'}})^{\beta_n}}.$$

Notice that the $z_{k'}$ term is undefined and hence $f'_{\mathcal{P}}(z_{k'})$ does not exist. Thus, $f_{\mathcal{P}}(z)$ does not preserve angles on the boundary of $\mathcal{P}$. □

Therefore, when we say we that the function f conformally maps from a canonical domain onto another domain, it is understood that we are mapping onto the interior of the image domain.

2.8 The Schwarz-Christoffel Parameter Problem

In the previous sections we introduced the formula for the Schwarz-Christoffel mapping from the unit disk onto a polygon. However, without knowledge of the prevertex z_k, we cannot use the integral in (2.5) to compute values. The image $f(\mathbb{R} \cup \{\infty\})$ of the extended real line will be some polygon whose interior angles $\alpha_k \pi$ and hence exterior angles $\beta_k \pi$ will match those of polygon $\mathcal{P}$, however, the prevertices influence the side lengths of $f(\mathbb{R} \cup \{\infty\})$. Thus, a "distortion" may occur if the prevertices are not chosen appropiately. Determining the correct values of the prevertices is known as the Schwarz-Christoffel parameter problem. The solution to the parameter problem is the primary step in using the Schwarz-Christoffel mapping for the unit disk. Furthermore, the Schwarz-Christffel parameter problem applies to the other Schwarz-Christoffel formulas for the half-plane, strip, rectangle, etc.

Since the prevertices depend nonlinearly on the side lengths of $\mathcal{P}$, in most problems, there is no analytic solution for the prevertices. There do exist some classical cases for which the parameter problem can be solved explicitly. However, for most cases, numerical methods are not only needed to determine the prevertices but to estimate the integral in (2.5) and to invert the map. The main current computer program that contains algorithms that address the classical Schwarz-Christoffel parameter problem and produce nontrivial conformal maps is the SC Toolbox for MATLAB [**7**] developed by Driscoll. We will be using this package to determine some results involving conformal mappings onto polygons [**6**].

2.9 The Ellipse

In the complex plane, the equation of an ellipse is given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{2.6}$$

where $(a, 0)$ and $(-a, 0)$ are the points where the ellipse intersects the real axis and $(0, -b)$ and $(0, b)$ are the points where the ellipse intersects the imaginary axis. The foci of the ellipse are given by $(0, -c)$ and $(0, c)$ where $c = \sqrt{a^2 - b^2}$. If $a > b$ then c is real and the foci are located at $(-c, 0)$ and $(c, 0)$. If $a < b$ then c is imaginary and the foci are located at $(0, -c)$ and $(0, c)$. If we let $a = \cosh \xi$ and $b = \sinh \xi$ where $\xi > 0$, then the foci of the ellipse are given by $c = \sqrt{\cosh^2 \xi - \sinh^2 \xi}$ or $c^2 = \cosh^2 \xi - \sinh^2 \xi$. By Proposition 2.2 we know that $1 = \cosh^2 \xi - \sinh^2 \xi$ and therefore it follows that an ellipse and all similar ellipses with foci at ± 1 have boundaries that cross the x-axis at $(\pm\cosh\xi, 0)$ and cross the y-axis at $(0, \pm\sinh\xi)$. We should note that two ellipses are similar if their ratio of major to minor axis are the same.

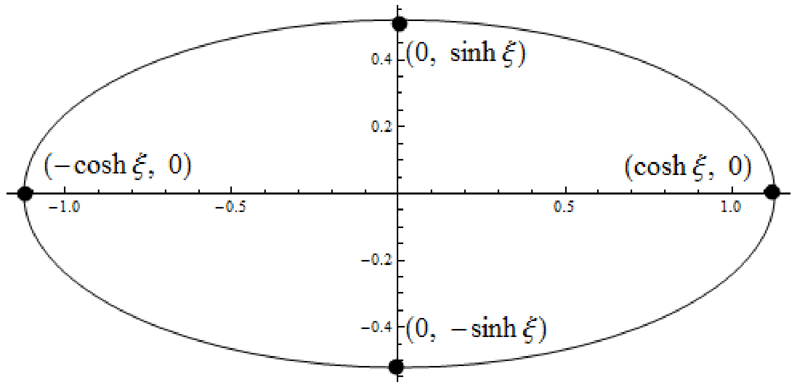

FIGURE 2.15. Example of an ellipse with foci at ± 1 where $\xi = .5$.

LEMMA 2.16. *Let $(\mathcal{E}, ue^{i\theta_0}, ve^{i(\theta_0+\pi/2)})$ with $u > v > 0$ and $0 < \theta_0 < \pi/2$ be the ellipse with axes U defined by the angle θ_0 and V defined by $\theta_0 + \pi/2$ and suppose the foci, denoted by F_1, F_2 are given by the two square roots of $F \in \mathbb{C}\ /\{0\}$ defined by the equation*

$$F^2 = (ue^{i\theta_0})^2 + (ve^{i(\theta_0+\pi/2)})^2.$$

Then there is an ellipse $(\mathcal{E}', u', v'e^{i\pi/2})$ such that

i) $\mathcal{E}'$ has foci at ± 1

ii) $\mathcal{E}'$ is similar to $\mathcal{E}$.

PROOF. i) To prove that $\mathcal{E}'$ has foci at ± 1, i.e. $F_1', F_2' = \pm 1 + i0$, we want to show

$$(u')^2 - (v')^2 = 1$$

where

$$u' = \frac{u}{|F|} \quad \text{and} \quad v' = \frac{v}{|F|}.$$

Now suppose $u > v > 0$ and $0 < \theta_0 < \pi/2$. Observe that

$$\begin{aligned} F^2 &= (ue^{i\theta_0})^2 + (ve^{i(\theta_0+\pi/2)})^2 \\ &= u^2e^{2i\theta_0} + v^2e^{2i(\theta_0+\pi/2)}. \end{aligned}$$

Since $e^{i\pi} = -1$,

$$F^2 = (u^2 - v^2)e^{2i\theta_0}.$$

Taking the norm of both sides we have

$$|F|^2 = |(u^2 - v^2)e^{2i\theta_0}| = u^2 - v^2.$$

Thus,

$$(u')^2 - (v')^2 = \frac{u^2}{|F|^2} - \frac{v^2}{|F|^2} = \frac{1}{|F|^2}(u^2 - v^2) = \frac{(u^2 - v^2)}{(u^2 - v^2)} = 1.$$

ii) We want to show

$$\frac{u'}{v'} = \frac{u}{v}.$$

To see this, observe

$$\frac{u'}{v'} = \frac{\frac{u}{|F|}}{\frac{v}{|F|}} = \frac{u}{v}.$$

$\square$

CHAPTER 3

Rectangular Domains in $\mathbb{C}$

3.1 Definitions and Examples

We now apply the 2-fold symmetry characteristic of an ellipse to a class of domains called rectangular domains.

Definition 3.1. Given a point $z \in \mathbb{C} \setminus \{\mathbb{R} \cup i\mathbb{R}\}$, we define the domain $R(-z, z)$ to be the rectangle with vertices at the four points: $(\pm z, \ \pm \overline{z})$.

Definition 3.2. Let Ω be a domain in $\mathbb{C}$. Ω is *rectangular with respect to 0* if $R(-z, z) \subset \Omega$ for every $z \in \overline{\Omega}$.

Henceforth, Ω will denote an RD domain. See Figures 3.16 and 3.17.

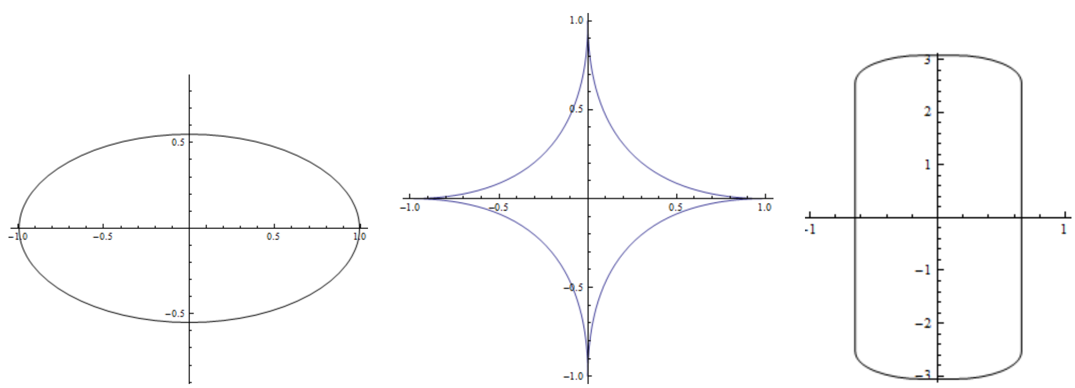

Figure 3.16. Three examples of RDs.

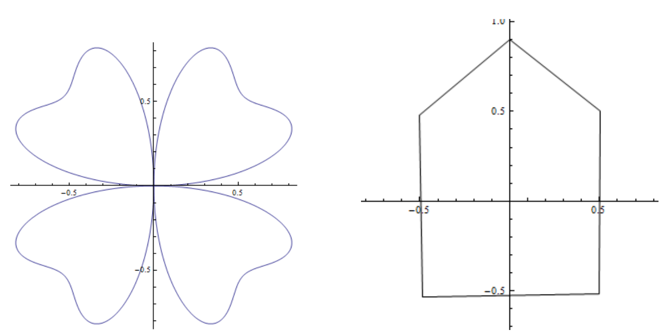

Figure 3.17. Two examples that are not RDs.

Definition 3.3. We say an RD polygon $\mathcal{P}_\Omega$ has foci at ± 1 if $\mathcal{P}_\Omega$ can be inscribed within an ellipse with foci at ± 1.

The phrase "inscribed within" means that the vertices of a polygon $\mathcal{P}_\Omega$ are located on the boundary of an ellipse with foci at ± 1. Thus, if a polygon $\mathcal{P}_\Omega$ has vertices on the axes they are located at $(\pm\cosh\xi, 0)$ and $(0, \pm\sinh\xi)$ where $\xi > 0$.

3.2 Properties of the Conformal Map from $\mathbb{D}$ onto RDs

Proposition 3.4. *Let f_Ω be a conformal mapping $f_\Omega : \mathbb{D} \to \Omega$ determined by $f_\Omega(0) = 0$ and $f'_\Omega(0) > 0$. Then the conformal mapping f_Ω is unique.*

Proof. Suppose $G, g : \mathbb{D} \to \Omega$ are conformal mappings with $G(0) = 0, G'(0) \geq 0, g(0) = 0$ and $g'(0) \geq 0$. We will show $G(z) = g(z)$ for $z \in \mathbb{D}$. Suppose $h(z) = (g^{-1} \circ G)(z)$ for $z \in \mathbb{D}$. So, $h : \mathbb{D} \to \mathbb{D}$ with $h(0) = 0$ and $h'(0) \geq 0$. Since h is a conformal self-mapping of the unit disk, there exists $\theta_0 \in [0, 2\pi], |a_0| < 1$ such that

$$h(z) = e^{i\theta_0}\frac{z - a_0}{1 - \bar{a}_0 z}.$$

Since $h(0) = 0$, we have $h(0) = e^{i\theta_0}(-a_0) = 0$. Moreover, $e^{i\theta} \neq 0$ implies that $a_0 = 0$ and hence, $h(z) = e^{i\theta_0}z$. Notice that $h'(z) = e^{i\theta}$ for every $z \in \mathbb{D}$ and $h'(0) = e^{i\theta_0} \geq 0$. So, $h'(0)$ is real and positive which implies $\theta_0 = 0$. Thus, $h(z) = z$ and hence, $(g^{-1} \circ G)(z) = z$ which implies $G(z) = g(z)$ for all $z \in \mathbb{D}$.

□

Proposition 3.5. *Let f_Ω be the conformal mapping $f_\Omega : \mathbb{D} \to \Omega$ determined by $f_\Omega(0) = 0$ and $f'_\Omega(0) > 0$. Then the conformal mapping f_Ω is an odd function.*

Proof. Let $z \in \mathbb{D}, w \in \Omega$, and define $H(z) = h \circ g \circ f_\Omega(w)$ where f_Ω is defined above, $g : \Omega \to \Omega$ is the mapping $g(w) = -w$ for $w \in \Omega$ with $g(0) = 0$, and $h : \Omega \to \mathbb{D}$ is the mapping $h = f_\Omega^{-1}$ with $h(0) = 0$. So, $H : \mathbb{D} \to \mathbb{D}$ is holomorphic with $H(0) = 0$. Since $H(z) = h(-f_\Omega(z))$,

$$H'(z) = h'(-f_\Omega(z)) \cdot (-f'_\Omega(z)) = h'(-f_\Omega(z)) \cdot \frac{-1}{h'(h^{-1}(z))}.$$

Now $h = f_\Omega^{-1}$ implies that $h^{-1}(0) = 0$, so

$$H'(0) = h'(0) \cdot \frac{-1}{h'(0)} = -1.$$

Thus, $|H'(0)| = |-1| = 1$. By Schwarz's Lemma, we have $H(z) = e^{i\theta_0}z$ and $H'(0) = e^{i\theta_0}$. Since $H'(0) = -1$, we have $\theta_0 = \pi$ for $\theta_0 \in [0, 2\pi]$ and hence $H(z) = -z$. Combining this fact with $H(z) = h(-f_\Omega(z)) = f_\Omega^{-1}(-f_\Omega(z))$ yields $-z = f_\Omega^{-1}(-f_\Omega(z))$ for every $z \in \mathbb{D}$. Consequently, $f_\Omega(-z) = f_\Omega(f_\Omega^{-1}(-f_\Omega(z))) = -f_\Omega(z)$. □

Proposition 3.6. *Let f_Ω be the conformal mapping $f_\Omega : \mathbb{D} \to \Omega$ determined by $f_\Omega(0) = 0$ and $f_\Omega'(0) > 0$. Then the Taylor series about the origin of the conformal mapping f_Ω has real coefficients.*

Proof. To prove that f_Ω has real coefficients it suffices to show that $f_\Omega(z) = \overline{f_\Omega(\overline{z})}$. Observe that if $z \in \mathbb{D}$ then $\bar{z} \in \mathbb{D}$. Define $H(z) = h \circ g \circ f_\Omega(z)$ where f_Ω is defined above, $g : \Omega \to \Omega$ is the mapping $g(w) = \overline{w}$ for $w \in \Omega$ with $g(0) = 0$, and $h : \Omega \to \mathbb{D}$ is the mapping $h = f_\Omega^{-1}$ with $h(0) = 0$. So, $H(z) = h \circ g \circ f_\Omega(z) = h(\overline{f_\Omega(z)})$ is anti-holomorphic. Thus, $\overline{H} : \mathbb{D} \to \mathbb{D}$ is holomorphic with $\overline{H}(0) = 0$. Notice that

$$\left|\frac{\partial \overline{H}(0)}{\partial z}\right| = \left|\frac{\partial H(0)}{\partial \overline{z}}\right|.$$

Now observe that $\xi = \overline{f_\Omega}(z)$ implies $\overline{\xi} = f_\Omega(z)$. Thus,

$$\frac{\partial H}{\partial \overline{z}}(z) = \frac{\partial}{\partial \overline{z}}(h \circ \overline{f_\Omega})(z) = \frac{\partial h}{\partial \xi}(\xi) \cdot \frac{\partial \overline{f_\Omega}}{\partial \overline{z}}(z) + \frac{\partial h}{\partial \overline{\xi}}(\xi) \cdot \frac{\partial f_\Omega}{\partial \overline{z}}(z).$$

Since f_Ω is holomorphic,

$$\frac{\partial f_\Omega}{\partial \overline{z}} = 0$$

and

$$\frac{\partial H}{\partial \overline{z}} = \frac{\partial h}{\partial \xi}(\xi) \cdot \frac{\partial \overline{f_\Omega}}{\partial \overline{z}}(z).$$

For $z = 0$, we have $f_\Omega(0) = 0$ which implies $\overline{f_\Omega}(0) = 0$. Thus,

$$\frac{\partial H}{\partial \overline{z}}(0) = \frac{\partial h}{\partial \xi}(0) \cdot \overline{\frac{\partial f_\Omega}{\partial z}(0)} = \frac{\left(\overline{\frac{\partial f_\Omega}{\partial z}(0)}\right)}{\left(\frac{\partial f_\Omega}{\partial z}(0)\right)} = \frac{re^{-i\theta_0}}{re^{i\theta_0}} = e^{-2i\theta_0}.$$

Hence

$$\left|\frac{\partial H}{\partial \overline{z}}(0)\right| = \left|\frac{\partial \overline{H}}{\partial z}(0)\right| = 1.$$

By Schwarz's Lemma, $H(z) = e^{i\theta_0}\overline{z}$. Thus, for every $z \in \mathbb{D}$,

$$\frac{\partial H}{\partial \overline{z}}(z) = e^{i\theta_0}$$

and

$$\frac{\partial H}{\partial \bar{z}}(0) = 1.$$

Therefore, $e^{i\theta_0} = 1$. This implies that $H(z) = \bar{z}$. Thus, $f_\Omega^{-1}(\overline{f_\Omega(z)}) = \bar{z}$ and it follows that $\overline{f_\Omega(z)} = f(\bar{z})$ and hence $f_\Omega(z) = \overline{f_\Omega(\bar{z})}$. $\square$

Proposition 3.7. *Let f_Ω be the conformal mapping $f_\Omega : \mathbb{D} \to \Omega$ determined by $f_\Omega(0) = 0$ and $f_\Omega'(0) > 0$. If $z \in \mathbb{D}$ with $f_\Omega(z) = w$, $w \in \Omega$, then*

i) $-z = f_\Omega^{-1}(-w)$

ii) $\bar{z} = f_\Omega^{-1}(\overline{w})$

iii) $-\bar{z} = f_\Omega^{-1}(-\overline{w})$.

Proof. (i) Suppose $f_\Omega(z) = w$. Multiplying both sides by -1,

$$-f_\Omega(z) = -w$$

and using the fact that the biholomorphic function f_Ω is odd we have,

$$f_\Omega(-z) = -w.$$

Taking f_Ω^{-1} of both sides yields,

$$-z = f_\Omega^{-1}(-w).$$

(ii) Suppose $f_\Omega(z) = w$. Taking the conjugate of both sides,

$$\overline{f_\Omega(z)} = \overline{w}$$

and using the fact that the biholomorphic function f_Ω has real Taylor coefficients we have,

$$f_\Omega(\bar{z}) = \overline{w}.$$

Taking f_Ω^{-1} of both sides yields,

$$\bar{z} = f_\Omega^{-1}(\overline{w}).$$

(iii) The result follows from (*i*) and (*ii*). $\square$

Proposition 3.8. *Let f_Ω be the conformal mapping $f_\Omega : \mathbb{D} \to \Omega$ determined by $f_\Omega(0) = 0$ and $f'_\Omega(0) > 0$. Suppose $f_\Omega(z) = w$ where $z \in \mathbb{D}$, $w \in \Omega$. If w lies on the imaginary axis then z lies on the imaginary axis.*

Proof. Suppose that $bi = w$ where $b \in \mathbb{R}$. Since Ω is an RD we also know $-bi = \overline{w}$. Thus, by 3.7 we have

i) $bi = w = f_\Omega(z)$
ii) $-bi = \overline{w} = f_\Omega(\overline{z})$.

Multiplying (ii) by -1 and using the fact that the biholomorphic function f_Ω is odd we get

$$bi = -\overline{w} = f_\Omega(-\overline{z}).$$

Combining (i) and (ii) we have

$$bi = f_\Omega(z) \quad \textit{and} \quad bi = f_\Omega(-\overline{z})$$

which implies that

$$f_\Omega^{-1}(bi) = z \quad \textit{and} \quad f_\Omega^{-1}(bi) = -\overline{z}.$$

Therefore $z = -\overline{z}$ and it follows that z lies on the imaginary axis. □

Proposition 3.9. *Let f_Ω be the conformal mapping $f_\Omega : \mathbb{D} \to \Omega$ determined by $f_\Omega(0) = 0$ and $f'_\Omega(0) > 0$. Suppose $f_\Omega(z) = w$ where $z \in \mathbb{D}$, $w \in \Omega$. If w lies on the real axis then z lies on the real axis.*

Proof. Suppose that $a = w$ where $a \in \mathbb{R}$. Since Ω is an RD we also know $-a = -\overline{w}$. Thus, by Proposition 3.7 we have

i) $a = w = f_\Omega(z)$
ii) $-a = -\overline{w} = f_\Omega(-\overline{z})$.

Multiplying (ii) by -1 and using the fact that the biholomorphic function f_Ω is odd yields

$$a = \overline{w} = f_\Omega(\overline{z}).$$

Combining (i) and (ii) we have

$$a = f_\Omega(z) \quad \textit{and} \quad a = f_\Omega(\overline{z})$$

which implies that

$$f_{\Omega}^{-1}(a) = z \quad and \quad f_{\Omega}^{-1}(a) = \bar{z}.$$

Therefore $z = \bar{z}$ and it follows that z lies on the real axis. $\square$

3.3 The Schwarz-Christoffel Mapping from $\mathbb{D}$ onto RD Polygons

Theorem 3.10. *The function $f_{\mathcal{P}_\Omega}(z)$ which maps the unit disk, $\mathbb{D}$, conformally onto an RD polygon, denoted by $\mathcal{P}_\Omega$, with angles $\alpha_k\pi$ where $k = 1, 2, \ldots n$ is of the form*

$$f_{\mathcal{P}_\Omega}(z) = C\int_0^z \prod_{k=1}^{n}(1 - \frac{\xi}{z_k})^{-\beta_k} d\xi$$

where $\beta_k = 1 - \alpha_k$, $z_k \in \partial\mathbb{D}$ is a finite prevertex with $f_{\mathcal{P}_\Omega}(z_k) = w_k$, and $C > 0$.

Proof. Since $f_{\mathcal{P}_\Omega} : \mathbb{D} \to \Omega$ is determined by $f_{\mathcal{P}_\Omega}(0) = 0$ and $f'_{\mathcal{P}_\Omega}(0) > 0$, we can determine the Schwarz-Christoffel parameters A and C and develop the Schwarz-Christoffel mapping for $\mathcal{P}_\Omega$. So, let $f_{\mathcal{P}_\Omega}(z)$ be the Schwarz-Christoffel mapping for the unit disk with lower bound of 0 be given by

$$f_{\mathcal{P}_\Omega}(z) = A + C\int_0^z \prod_{k=1}^{n}(1 - \frac{\xi}{z_k})^{-\beta_k} d\xi.$$

Evaluating $f_{\mathcal{P}_\Omega}(z)$ for $z = 0$ we have

$$f_{\mathcal{P}_\Omega}(0) = A + C\int_0^0 \prod_{k=1}^{n}(1 - \frac{\xi}{z_k})^{-\beta_k} d\xi \; = A.$$

Thus, $f_{\mathcal{P}_\Omega}(0) = 0$ implies that $A = 0$.

Since the integrand is the product of binomials to the exponent of $-\beta_k$ and not -1 we can apply the Fundamental Theorem of Calculus to $f_{\mathcal{P}_\Omega}(z)$ to give us

$$f'_{\mathcal{P}_\Omega}(z) = C\prod_{k=1}^{n}(1 - \frac{z}{z_k})^{-\beta_k}.$$

So we have

$$f'_{\mathcal{P}_\Omega}(0) = C\prod_{k=1}^{n}(1 - \frac{0}{z_k})^{-\beta_k} = C\prod_{k=1}^{n}(1)^{-\beta_k} = C(1^2) = C.$$

Thus, $f'_{\mathcal{P}_\Omega}(0) > 0$ implies that $C > 0$ and so C must be real and positive. $\square$

It is important to note that $C > 0$ is a dilation constant and multiplication by C will not change the results of the positivity or negativity of the Taylor coefficients. Hence, if we normalize $f'(0) = 1$, then C=1. Thus, the formula for Schwarz-Christoffel mapping from the unit disk onto $\mathcal{P}_\Omega$ when $f'(0) = 1$ is given by

$$f_{\mathcal{P}_\Omega}(z) = \int_0^z \prod_{k=1}^{n} (1 - \frac{\xi}{z_k})^{-\beta_k} d\xi. \tag{3.1}$$

Proposition 3.11. *Let $f_{\mathcal{P}_\Omega}$ be the conformal mapping $f_{\mathcal{P}_\Omega} : \mathbb{D} \to \mathcal{P}_\Omega$ determined by $f_{\mathcal{P}_\Omega}(0) = 0$ and $f'_{\mathcal{P}_\Omega}(0) > 0$. Suppose $f_{\mathcal{P}_\Omega}(z_j) = w_j$ where $z_j \in \partial\mathbb{D}$ is a prevertex on the unit disk with corresponding vertex $w_j \in \partial\Omega$ on the RD polygon. If w_j lies on the positive or negative real axis then z_j has a value of -1 or 1.*

Proof. Suppose that $= w_j$ where $\in \mathbb{R}$. Since Ω is an RD we also know $-a = -\overline{w_j}$. Thus, by Proposition 3.7 we have

i) $a = w_j = f_{\mathcal{P}_\Omega}(z_j)$

ii) $-a = -\overline{w_j} = f_{\mathcal{P}_\Omega}(\overline{-z_j}) \Rightarrow a = \overline{w_j} = f_{\mathcal{P}_\Omega}(\overline{z_j})$.

Combining (i) and (ii) yields

$$a = w_j = f_{\mathcal{P}_\Omega}(z_j) = f_{\mathcal{P}_\Omega}(\overline{z_j}) \Leftrightarrow f^{-1}_{\mathcal{P}_\Omega}(a) = f^{-1}_{\mathcal{P}_\Omega}(w_j) = z_j = \overline{z_j}.$$

Since $z_j = \overline{z_j}$, z_j is on the $\partial\mathbb{D}$, and $f_{\mathcal{P}_\Omega}$ is bijective, it follows that $z_j = -1$ or $z_j = 1$. □

Proposition 3.12. *Let $f_{\mathcal{P}_\Omega}$ be the conformal mapping $f_{\mathcal{P}_\Omega} : \mathbb{D} \to \mathcal{P}$ determined by $f_{\mathcal{P}_\Omega}(0) = 0$ and $f'_{\mathcal{P}_\Omega}(0) > 0$. Suppose $f_{\mathcal{P}_\Omega}(z_j) = w_j$ where $z_j \in \partial\mathbb{D}$ is a prevertex with corresponding vertex $w_j \in \partial\Omega$. If w_j lies on the positive or negative imaginary axis then z_j has a value of -i or i.*

Proof. Suppose that $bi = w_j$ where $b \in \mathbb{R}$. Since Ω is an RD we also know $-bi = w_j$. Thus, by Proposition 3.7 we have

i) $bi = w_j = f_{\mathcal{P}_\Omega}(z_j)$

ii) $-bi = \overline{w_j} = f_{\mathcal{P}_\Omega}(\overline{z_j}) \Rightarrow bi = -\overline{w_j} = f_{\mathcal{P}_\Omega}(-\overline{z_j})$.

Combining (i) and (ii) yields

$$bi = w_j = f_{\mathcal{P}_\Omega}(z_j) = f_{\mathcal{P}_\Omega}(-\overline{z_j})$$

which implies that

$$f^{-1}_{\mathcal{P}_\Omega}(bi) = f^{-1}_{\mathcal{P}_\Omega}(w_j) = z_j = -\overline{z_j}.$$

Since $z_j = -\overline{z_j}$, z_j is on the $\partial\mathbb{D}$, and $f_{\mathcal{P}_\Omega}$ is bijective, it follows that $z_j = -i$ or $z_j = i$. □

CHAPTER 4

The Schwarz-Christoffel Mapping from $\mathbb{D}$ onto RD Quadrilaterals

4.1 RD Rhombi

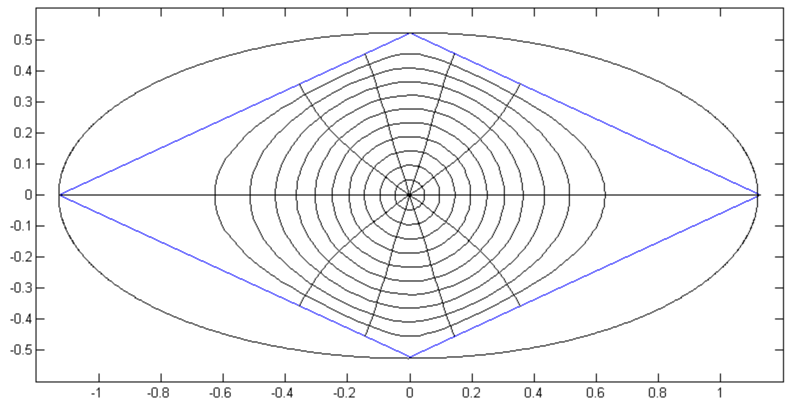

FIGURE 4.18. An RD rhombus with foci at ± 1 where $\xi = .5$. This picture was produced with SC Toolbox for MatLab [7].

4.1.1 Taylor Series Representation of RD Rhombi

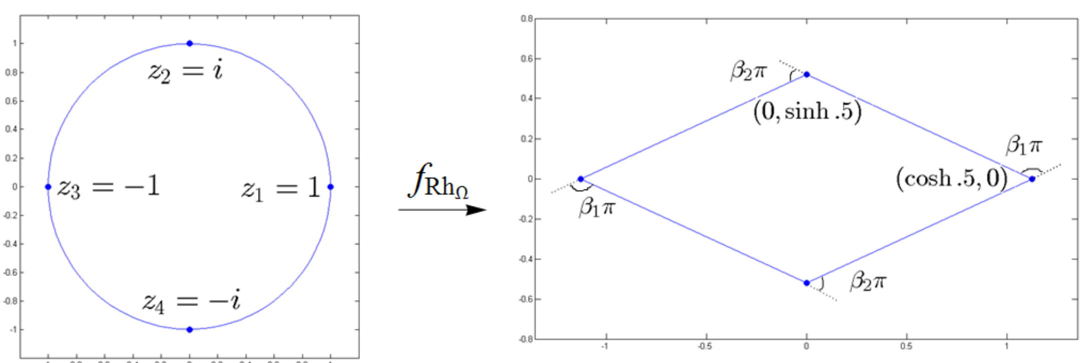

FIGURE 4.19. The conformal mapping from the unit disk onto an RD rhombus with foci at ± 1.

PROPOSITION 4.1. *Let Rh_Ω be an RD rhombus with foci at ± 1. If $f_{Rh_\Omega} : \mathbb{D} \to Rh_\Omega$ is the conformal map determined by $f_{Rh_\Omega}(0) = 0$ and $f'_{Rh_\Omega}(0) = 1$, then Rh_Ω has prevertices {1, i, -1, -i}.*

PROOF. Since $Rh_\Omega \in P_\Omega$ its vertices, denoted by w_k, lie on the real and imaginary axes. From Proposition 3.11 , the real axis is mapped to the real axis and by Proposition 3.12 the imaginary axis is mapped to the imaginary axes. Since w_k is a vertex on the boundary of Rh_Ω and z_k is a prevertex on the boundary of $\mathbb{D}$, it follows that if w_k lies on the real axis then the corresponding

z_k is 1 or -1. Similarly, if w_k lies on the imaginary axes, then the corresponding z_k is i or -i. Hence the Schwarz-Christoffel mapping is given by

$$\begin{aligned} f_{Rh_\Omega}(z) &= \int_0^z (1-\frac{t}{1})^{-\beta_1}(1-\frac{t}{i})^{-\beta_2}(1-\frac{t}{-1})^{-\beta_1}(1-\frac{t}{-i})^{-\beta_2}dt \\ &= \int_0^z (1-t^2)^{-\beta_1}(1+t^2)^{-\beta_2}dt. \end{aligned}$$

Since the sum of the exterior angles is equal to 2π, it follows that $2\beta_1 + 2\beta_2 = 2$ and hence $\beta_2 = 1 - \beta_1$. Substituting into the integral above we have

$$\begin{aligned} f_{Rh_\Omega}(z) &= \int_0^z (1-t^2)^{-\beta_1}(1+t^2)^{-(1-\beta_1)}dt \\ &= \int_0^z (1-t^2)^{-\beta_1}(1+t^2)^{\beta_1-1}dt. \end{aligned} \tag{4.1}$$

Observe that

$$f_{Rh_\Omega}(1) = \int_0^1 (1-t^2)^{-\beta_1}(1+t^2)^{\beta_1-1}dt.$$

Along the straight line path from 0 to 1, t is real and thus the integral is real. Therefore, for $0 < t < 1$, we have $(1-t^2)^{-\beta_1} > 0$ and $(1+t^2)^{\beta_1-1} > 0$ and hence the integrand is greater than 0. Therefore, the integral is greater than 0 and f_{Rh_Ω} maps 1 to the positive real axis. Thus, the vertex w_1 on the positive real axis has prevertex 1. Since f_{Rh_Ω} is odd we have $f_{Rh_\Omega}(-1) = -f_{Rh_\Omega}(1) < 0$ and it follows that f_{Rh_Ω} maps -1 to the negative real axis. Thus, the vertex w_3 on the negative real axis has prevertex -1. A similar argument can be made for w_2 and w_4 whose corresponding prevertices are $z_2 = i$ and $z_4 = -i$. □

Proposition 4.2. *Let Rh_Ω be an RD rhombus with foci at ± 1. If $f_{Rh_\Omega} : \mathbb{D} \to Rh_\Omega$ is the conformal map determined by $f_{Rh_\Omega}(0) = 0$ and $f'_{Rh_\Omega}(0) = 1$, then the Taylor series of f_{Rh_Ω} about the origin is given by*

$$f_{Rh_\Omega}(z) = \sum_{n=0}^{\infty} \frac{a_n}{2n+1} z^{2n+1}$$

where $a_0 = 1$, $a_1 = 2\beta - 1$ and a_n is given recursively by

$$a_n = \frac{(2\beta_1 - 1)a_{n-1} + (n-1)a_{n-2}}{n}, \ \ for\, n \geq 2.$$

Proof. By equation (4.1) we know

$$\begin{aligned} f_{Rh_\Omega}(z) &= \int_0^z (1-t^2)^{-\beta_1}(1+t^2)^{-(1-\beta_1)}dt \\ &= \int_0^z (1-t^2)^{-\beta_1}(1+t^2)^{\beta_1-1}dt. \end{aligned} \tag{4.2}$$

Now we need to develop a Taylor Series representation for equation (4.2). Since Taylor series whose radius of convergence is at least one are uniformly convergent on compact subsets of the open unit disk and $z \in \mathbb{D}$, we know that we can integrate along the path from 0 to z of

$$\sum_{k=0}^{\infty} c_k t^k.$$

Switching the integration and summation order we get

$$\int_0^z \sum_{k=0}^{\infty} c_k t^k dt = \sum_{k=0}^{\infty} \int_0^z c_k t^k dt = \sum_{k=0}^{\infty} \frac{c_k}{k+1} z^{k+1}.$$

Let $g(t)$ be the integrand of equation (4.2) so we have

$$g(t) = (1-t^2)^{-\beta_1}(1+t^2)^{\beta_1-1} = \frac{(1+t^2)^{\beta_1-1}}{(1-t^2)^{\beta_1}}.$$

Now we will show that g(t) satisfies a differential equation and develop a recursion formula for the coefficients a_n. Observe that

$$(1-t^4)\frac{d}{dt}\frac{(1+t^2)^{\beta_1-1}}{(1-t^2)^{\beta_1}} = (2\beta_1 - 1 + t^2)(2t)\frac{(1+t^2)^{\beta_1-1}}{(1-t^2)^{\beta_1}} \tag{4.3}$$

So $y = g(t)$ satisfies the differential equation

$$(1-t^4)y' = (2\beta_1 - 1 + t^2)(2t)y \tag{4.4}$$

or

$$(1-t^4)y' - (2\beta_1 - 1 + t^2)(2t)y = 0. \tag{4.5}$$

Writing $g(t)$ as

$$y = \sum_{n=0}^{\infty} a_n t^{2n}$$

for the Taylor series expansion of $y = g(t)$ centered at the origin we have that

$$y' = \sum_{n=1}^{\infty} 2na_n t^{2n-1}.$$

Substituting these values into equation (4.5) we have

$$\begin{aligned}
(1-t^4)\sum_{n=1}^{\infty} 2na_n t^{2n-1} - (2\beta_1 - 1 + t^2)(2t)\sum_{n=0}^{\infty} a_n t^{2n} &= 0 \\
\sum_{n=1}^{\infty} 2na_n t^{2n-1} - \sum_{n=1}^{\infty} 2na_n t^{2n+3} - \sum_{n=0}^{\infty} 2(2\beta_1 - 1)a_n t^{2n+1} - \sum_{n=0}^{\infty} 2a_n t^{2n+3} &= 0 \\
2\left(\sum_{n=1}^{\infty} na_n t^{2n-1} - \sum_{n=1}^{\infty} na_n t^{2n+3}\right) - 2\left(\sum_{n=0}^{\infty}(2\beta_1 - 1)a_n t^{2n+1} \sum_{n=0}^{\infty} a_n t^{2n+3}\right) &= 0 \\
\sum_{n=1}^{\infty} na_n t^{2n-1} - \sum_{n=1}^{\infty} na_n t^{2n+3} - \sum_{n=0}^{\infty}(2\beta_1 - 1)a_n t^{2n+1} - \sum_{n=0}^{\infty} a_n t^{2n+3} &= 0 \\
\sum_{n=0}^{\infty}(n+1)a_{n+1}t^{2n+1} - \sum_{n=2}^{\infty}(n-1)a_{n-1}t^{2n+1} - \sum_{n=0}^{\infty}(2\beta_1 - 1)a_n t^{2n+1} - \sum_{n=1}^{\infty} a_{n-1}t^{2n+1} &= 0 \\
\sum_{n=0}^{\infty}(n+1)a_{n+1}t^{2n+1} - \sum_{n=0}^{\infty}(2\beta_1 - 1)a_n t^{2n+1} - \sum_{n=1}^{\infty} a_{n-1}t^{2n+1} - \sum_{n=2}^{\infty}(n-1)a_{n-1}t^{2n+1} &= 0. \qquad (4.6)
\end{aligned}$$

Now let

$$\begin{aligned}
h(t) &= [a_1 - (2\beta_1 - 1)a_0]t + [2a_2 - (2\beta_1 - 1)a_1 - a_0]t^3 + \\
&\quad + \sum_{n=2}^{\infty}(n+1)a_{n+1}t^{2n+1} - \sum_{n=2}^{\infty}(2\beta_1 - 1)a_n t^{2n+1} - \sum_{n=2}^{\infty} a_{n-1}t^{2n+1} - \sum_{n=2}^{\infty}(n-1)a_{n-1}t^{2n+1} \\
&= [a_1 - (2\beta_1 - 1)a_0]t + [2a_2 - (2\beta_1 - 1)a_0 - a_0]t^3 + \\
&\quad + \sum_{n=2}^{\infty}[(n+1)a_{n+1} - (2\beta_1 - 1)a_n - a_{n-1} - (n-1)a_{n-1}]t^{2n+1}. \qquad (4.7)
\end{aligned}$$

Letting $C_0 = a_1 - (2\beta_1 - 1)a_0$, $C_1 = 2a_2 - (2\beta_1 - 1)a_0 - a_0$, and

$$C_n = (n+1)a_{n+1} - (2\beta_1 - 1)a_n - a_{n-1} - (n-1)a_{n-1} \quad \text{for} \quad n \geq 2$$

we can write equation (4.7) as

$$h(t) = \sum_{n=0}^{\infty} C_n t^{2n+1}.$$

From equation (4.6) we have $h(t) \equiv 0$,

$$h(t) = \sum_{n=0}^{\infty} C_n t^{2n+1} \equiv 0,$$

and $h(t)$ is analytic about the origin and is uniformly convergent on compact subsets of the unit disk, we know that all the derivatives of $h(t)$ about the origin are 0. This implies that each of the coefficients $C_n = 0$ and hence we have the following recursion formula

$$a_{n+1} = \frac{(2\beta_1 - 1)a_n + na_{n-1}}{n+1} \quad \text{for } n \geq 1 \tag{4.8}$$

which can be written in the form

$$a_n = \frac{(2\beta_1 - 1)a_{n-1} + (n-1)a_{n-2}}{n} \quad \text{for } n \geq 2. \tag{4.9}$$

Now from equation (4.2) we have

$$f_{Rh_\Omega}(z) = \int_0^z g(t)dt = \int_0^z \sum_{n=0}^{\infty} a_n t^{2n} dt = \sum_{n=0}^{\infty} \frac{a_n}{2n+1} z^{2n+1}$$

where $a_0 = g(0) = 1$, $a_1 = g''(0) = 2\beta_1 - 1$ and

$$a_n = \frac{(2\beta_1 - 1)a_{n-1} + (n-1)a_{n-2}}{n} \quad \text{for } n \geq 2.$$

□

4.1.2 Closed Form for Taylor Coefficients of RD Rhombi

To determine the closed form for a_n for the Taylor series expansion about the origin of $f_{Rh_\Omega}(z)$ we apply the binomial series expansion and observe that

$$\begin{aligned} g(t) &= \frac{(1+t^2)^{\beta_1 - 1}}{(1-t^2)^{\beta_1}} \\ &= (1+t^2)^{\beta_1 - 1} \sum_{k=0}^{\infty} (-1)^k \binom{-\beta_1}{k} t^{2k} \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} (-1)^k \binom{-\beta_1}{k} \binom{\beta_1 - 1}{n-k} t^{2n}. \end{aligned}$$

Thus, the Taylor series for $g(t)$ is given by

$$g(t) = \sum_{n=0}^{\infty} a_n t^{2n}$$

where

$$a_n = \sum_{k=0}^{n} (-1)^k \binom{-\beta_1}{k} \binom{\beta_1 - 1}{n-k}.$$

Notice that evaluating this equation for $n = 0$ and $n = 1$, we also have $a_0 = 1$ and $a_1 = 2\beta_1 - 1$.

4.1.3 Positivity of Taylor Coefficients of RD Rhombi

Theorem 4.3. *Let Rh_Ω be an RD rhombus with foci at ± 1. If f_{Rh_Ω} is the conformal mapping from the unit disk onto Rh_Ω determined by $f_{Rh_\Omega}(0) = 0$ and $f'_{Rh_\Omega}(0) = 1$, then the Taylor series of f_{Rh_Ω} about the origin has positive odd coefficients.*

Proof. We define the vertices of the RD rhombus as $(\pm\cosh\xi, 0)$ and $(0, \pm\sinh\xi)$. Note that $\cosh\xi$ and $\sinh\xi$ are positive for $\xi > 0$. Let β_1 be the coefficient of the turning angle at $(\pm\cosh\xi, 0)$ and $1 - \beta_1$ be the coefficient of the turning angle at $(0, \pm\sinh\xi)$. Notice that

$$2\beta_1 + 2(1 - \beta_1) = 2$$

which satisfies the turning angle property of the Schwarz-Christoffel mapping. We recall that $\cosh\xi$ and $\sinh\xi$ are defined as

$$\cosh\xi = \frac{e^{\xi} + e^{-\xi}}{2}$$

and

$$\sinh\xi = \frac{e^{\xi} - e^{-\xi}}{2}.$$

Moreover, we also know that $\cosh\xi$ and $\sinh\xi$ have the property

$$\cosh^2\xi - \sinh^2\xi = 1.$$

Applying this property we have that

$$\cosh\xi - \sinh\xi = \frac{e^{\xi} + e^{-\xi}}{2} - \frac{e^{\xi} - e^{-\xi}}{2} = e^{-\xi} > 0.$$

Hence, $\cosh\xi > \sinh\xi > 0$. Now let us derive the relationship between β_1 and $\cosh\xi$. From the given rhombus we have the following relationship for $\xi > 0$,

$$\begin{aligned}
\tan\left(\frac{\alpha_1\pi}{2}\right) &= \frac{\sinh\xi}{\cosh\xi} \\
\tan\left(\frac{\alpha_1\pi}{2}\right) &= \tanh\xi \\
\frac{\alpha_1\pi}{2} &= \tan^{-1}(\tanh\xi) \\
\alpha_1 &= \frac{2}{\pi}\tan^{-1}(\tanh\xi).
\end{aligned}$$

Since $\beta_1 = 1 - \alpha_1$ we have

$$\beta_1 = 1 - \frac{2}{\pi}\tan^{-1}(\tanh\xi).$$

Moreover, $\cosh\xi > \sinh\xi > 0$ yields

$$\begin{aligned}
1 > \tanh\xi > 0 \\
\frac{\pi}{4} > \tan^{-1}(\tanh\xi) > 0 \\
\frac{1}{2} > \frac{2}{\pi}\tan^{-1}(\tanh\xi) > 0 \\
-\frac{1}{2} < -\frac{2}{\pi}\tan^{-1}(\tanh\xi) < 0 \\
\frac{1}{2} < 1 - \frac{2}{\pi}\tan^{-1}(\tanh\xi) < 1 \\
\frac{1}{2} < \beta_1 < 1
\end{aligned}$$

Thus, $\cosh\xi > \sinh\xi$ implies that $1/2 < \beta_1 < 1$. Therefore, $\beta_1 > 1 - \beta_1$. It should also be noted that $\beta_1 \neq 1 - \beta_1$ since $\cosh\xi \neq \sinh\xi$. Now by equation (4.9) we have the recursion equation

$$a_n = \frac{(2\beta_1 - 1)a_{n-1} + (n-1)a_{n-2}}{n} \quad \text{for } n \geq 2.$$

From Proposition 4.2 we know that

$$a_0 = 1 > 0$$

and

$$a_1 = 2\beta_1 - 1 > 0$$

for $1/2 < \beta_1 < 1$. Moreover, we know that $n-1, n > 0$ for $n \geq 2$ and $2\beta_1 - 1 > 0$ for $1/2 < \beta_1 < 1$. Clearly, $a_2 > 0$ since a_0, $a_1 > 0$. Now assume $a_{n-2} > 0$, $a_{n-1} > 0$. Clearly, by equation (4.9) we have $a_n > 0$. Observe that

$$a_{n+1} = \frac{(2\beta_1 - 1)a_{(n+1)-1} + [(n+1) - 1]a_{(n+1)-2}}{n} = \frac{(2\beta_1 - 1)a_n + na_{n-1}}{n}.$$

Since $n > 0$, $a_{n-1} > 0$, and $a_n > 0$, we have $a_{n+1} > 0$. Hence, the Taylor series of f_{Rh_Ω} about the origin has positive odd coefficients. □

4.2 RD Rectangles

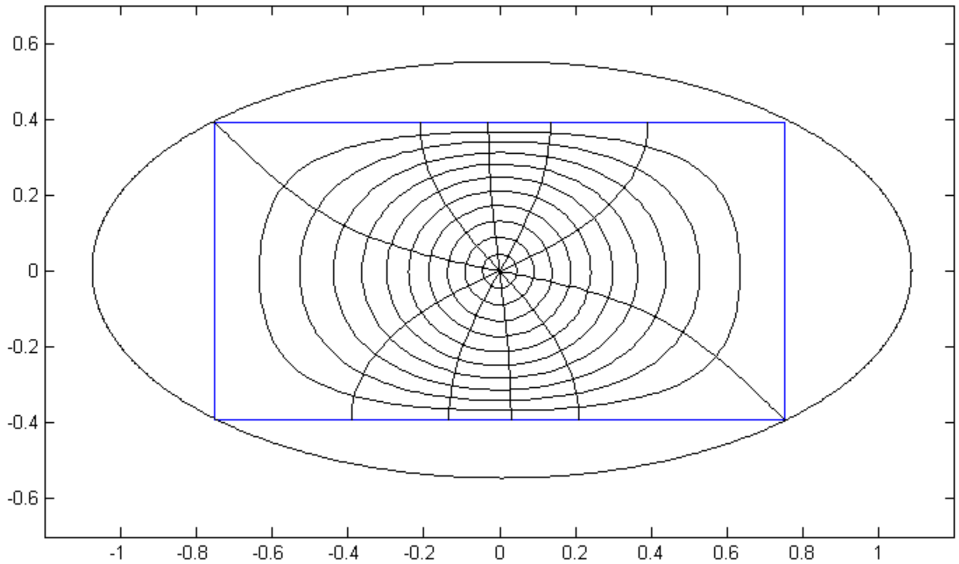

FIGURE 4.20. An RD rectangle with foci at ± 1 with $\xi = .5$.

4.2.1 Taylor Series Representation of RD Rectangles

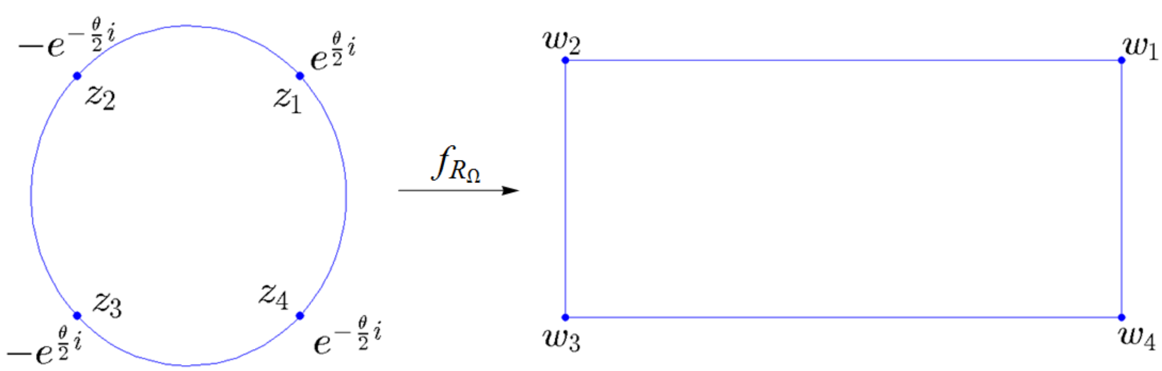

FIGURE 4.21. The conformal mapping from the unit disk onto an RD rectangle with foci at ± 1 where $\theta \in (0, \pi)$.

PROPOSITION 4.4. *Let R_Ω be an RD rectangle. Suppose $f_{R_\Omega} : \mathbb{D} \to R_\Omega$ is the conformal map determined by $f_{R_\Omega}(0) = 0$ and $f'_{R_\Omega}(0) = 1$. Then f_{R_Ω} is of the form*

$$f_{R_\Omega}(z) = \int_0^z (1 - 2\cos\theta t^2 + t^4)^{-1/2} dt.$$

Proof. By Proposition 3.7, the prevertices of R are given by $e^{\frac{i\theta}{2}}$, $-e^{-\frac{i\theta}{2}}$, $-e^{\frac{i\theta}{2}}$, and $e^{-\frac{i\theta}{2}}$ for some $\theta \in (0, \pi)$. Moreover, rectangles have exterior turning angles $\beta_k = \frac{1}{2}$ for k=1, 2, 3, 4. Applying the Schwarz-Christoffel mapping for P_Ω we have,

$$\begin{aligned}
f_{R_\Omega}(z) &= \int_0^z [(1 - \frac{t}{e^{\frac{i\theta}{2}}})(1 - \frac{t}{-e^{-\frac{i\theta}{2}}})(1 - \frac{t}{-e^{\frac{i\theta}{2}}})(1 - \frac{t}{e^{-\frac{i\theta}{2}}})]^{-1/2} dt \\
&= \int_0^z [(1 - \left(\frac{t}{e^{\frac{i\theta}{2}}}\right)^2)(1 - (e^{\frac{i\theta}{2}} t)^2)]^{-1/2} dt \\
&= \int_0^z [(1 - e^{-i\theta} t^2)(1 - e^{i\theta} t^2)]^{-1/2} dt \\
&= \int_0^z (1 - (e^{-i\theta} + e^{i\theta}) t^2 + t^4)^{-1/2} dt \\
&= \int_0^z (1 - 2\cos\theta t^2 + t^4)^{-1/2} dt. \qquad (4.10)
\end{aligned}$$

□

Proposition 4.5. *Let R_Ω be an RD rectangle. Suppose $f_{R_\Omega} : \mathbb{D} \to R_\Omega$ is the conformal map determined by $f_{R_\Omega}(0) = 0$ and $f'_{R_\Omega}(0) = 1$. Then the Taylor series about the origin of f_{R_Ω} is given by*

$$f_{R_\Omega}(z) = \sum_{n=0}^{\infty} a_n z^{2n+1}$$

where where $a_0 = 1$, and a_n is given recursively by

$$a_n = \frac{(-2n+1)a_{n-1}}{2n}, \ \ for\, n \geq 1.$$

Proof. We need to develop a Taylor Series representation for equation (4.23). Since Taylor series whose radius of convergence is at least one are uniformly convergent on compact subsets of the open unit disk and $z \in \mathbb{D}$, we know that we can integrate along the path from 0 to z of

$$\sum_{k=0}^{\infty} c_k t^k.$$

Switching the integration and summation order we get

$$\int_0^z \sum_{k=0}^{\infty} c_k t^k dt = \sum_{k=0}^{\infty} \int_0^z c_k t^k dt = \sum_{k=0}^{\infty} \frac{c_k}{k+1} z^{k+1}.$$

Let $g(t)$ be the integrand of equation (4.23) so we have

$$g(t) = (1 - 2\cos\theta t^2 + t^4)^{-1/2}.$$

Now we will show that g(t) satisfies a differential equation and develop a recursion formula for the coefficients a_n. Let $x = \cos\theta$. Observe that

$$(1 - 2xt^2 + t^4)\frac{d}{dt}(1 - 2xt^2 + t^4)^{-1/2} = (2xt - 2t^3)(1 - 2xt^2 + t^4)^{-1/2} \tag{4.11}$$

So $y = g(t)$ satisfies the differential equation

$$(1 - 2xt^2 + t^4)y' = (2xt - 2t^3)y \tag{4.12}$$

or

$$(1 - 2xt^2 + t^4)y' - (2xt - 2t^3)y = 0. \tag{4.13}$$

Writing $g(t)$ as

$$y = \sum_{n=0}^{\infty} a_n t^{2n}$$

for the Taylor series expansion of $y = g(t)$ centered at the origin we have that

$$y' = \sum_{n=1}^{\infty} 2na_n t^{2n-1}.$$

Substituting these values into equation (4.13) we have

$$(1 - 2xt^2 + t^4)\sum_{n=1}^{\infty} 2na_n t^{2n-1} + (2xt - 2t^3)\sum_{n=0}^{\infty} a_n t^{2n} = 0$$

$$\sum_{n=1}^{\infty} 2na_n t^{2n-1} - 4x\sum_{n=1}^{\infty} na_n t^{2n+1} + \sum_{n=1}^{\infty} 2na_n t^{2n+3} - 2x\sum_{n=0}^{\infty} a_n t^{2n+1} + 2\sum_{n=0}^{\infty} a_n t^{2n+3} = 0$$

$$2\left(\sum_{n=1}^{\infty} na_n t^{2n-1} - 2x\sum_{n=1}^{\infty} na_n t^{2n+1} + \sum_{n=1}^{\infty} na_n t^{2n+3} - x\sum_{n=0}^{\infty} a_n t^{2n+1} + \sum_{n=0}^{\infty} a_n t^{2n+3}\right) = 0$$

$$\sum_{n=1}^{\infty} na_n t^{2n-1} - 2x\sum_{n=1}^{\infty} na_n t^{2n+1} + \sum_{n=1}^{\infty} na_n t^{2n+3} - x\sum_{n=0}^{\infty} a_n t^{2n+1} + \sum_{n=0}^{\infty} a_n t^{2n+3} = 0$$

$$\sum_{n=0}^{\infty}(n+1)a_{n+1} t^{2n+1} - 2x\sum_{n=1}^{\infty} na_n t^{2n+1} + \sum_{n=2}^{\infty}(n-1)a_{n-1} t^{2n+1} - x\sum_{n=0}^{\infty} a_n t^{2n+1} + \sum_{n=1}^{\infty} a_{n-1} t^{2n+1} = 0$$

$$\sum_{n=0}^{\infty}(n+1)a_{n+1}t^{2n+1} - x\sum_{n=0}^{\infty}a_n t^{2n+1} - 2x\sum_{n=1}^{\infty}na_n t^{2n+1} + \sum_{n=1}^{\infty}a_{n-1}t^{2n+1} + \sum_{n=2}^{\infty}(n-1)a_{n-1}t^{2n+1} = 0. \quad (4.14)$$

Now let

$$h(t) = [a_1 - xa_0]t + [2a_2 - (3x)a_1 + a_0]t^3$$
$$+ \sum_{n=2}^{\infty}\left[(n+1)a_{n+1} - (2xn + x)a_n + na_{n-1}\right]t^{2n+1} \quad (4.15)$$

Letting $C_0 = a_1 - xa_0$,$C_1 = 2a_2 - (3x)a_1 + a_0$ and

$$C_n = (n+1)a_{n+1} - (2xn + x)a_n + na_{n-1} \text{ for } n \geq 2$$

we can write equation (4.15) as

$$h(t) = \sum_{n=0}^{\infty} C_n t^{4n+1}.$$

From equation (4.14) we have $h(t) \equiv 0$,

$$h(t) = \sum_{n=0}^{\infty} C_n t^{4n+1} \equiv 0,$$

and $h(t)$ is analytic about the origin and is uniformly convergent on compact subsets of the unit disk, we know that all the derivatives of $h(t)$ about the origin are 0. This implies that each of the coefficients $C_n = 0$ and hence we have the following recursion formula

$$a_{n+1} = \frac{(2xn + x)a_n - na_{n-1}}{n+1} \text{ for } n \geq 1. \quad (4.16)$$

which can be written in the form

$$a_n = \frac{(2x(n-1) + x)a_{n-1} - (n-1)a_{n-2}}{n} \text{ for } n \geq 2. \quad (4.17)$$

Now from equation (4.23) we have

$$f_{S_\Omega}(z) = \int_0^z g(t)dt = \int_0^z \sum_{n=0}^{\infty} a_n t^{4n} dt = \sum_{n=0}^{\infty} \frac{a_n}{4n+1} z^{4n+1}$$

where $a_0 = g(0) = 1$, $a_1 = g''(0) = xa_0$ and

$$a_n = \frac{(2x(n-1) + x)a_{n-1} - (n-1)a_{n-2}}{n} \text{ for } n \geq 2.$$

□

Since we cannot determine the nature of the Taylor series by the recursion formula of Proposition 4.5 , we need to apply an alternate method for determining the nature of the coefficients. Before we present the alternate method, we must first have knowledge of the definition and properties of orthogonal polynomials and Legendre polynomials.

4.2.2 Orthogonal Polynomials

Definition 4.6. (Orthogonal Polynomials) A sequence of orthogonal real polynomials consists of real polynomials $p_0(x)$, $p_1(x)$, ... , $p_n(x)$ such that

i) $p_m(x)$ is of degree m;

ii) $\langle p_m, p_n \rangle = 0$ for $m \neq n$

where $\langle p_m, p_n \rangle$ is the inner product defined by

$$\langle p_m, p_n \rangle = \int_a^b w(x) p_m(x) p_n(x) dx.$$

where $w(x)$ is an associated weight.

Below are some important properties of the zeros of orthogonal polynomials given by Szegö [**23**].

Theorem 4.7. *The zeros of the orthogonal polynomials $p_n(x)$, associated with the weight $w(x)$ on the interval $[a, b]$, are real and distinct and located in the interior of the interval $[a, b]$.*

Theorem 4.8. *Let $x_1 < x_2 < \cdots < x_n$ be the zeros of $p_n(x)$, $x_0 = a$, $x_{n+1} = b$. Then each interval $[x_\nu, x_{\nu+1}]$, $\nu = 1, 2, \cdots, n$, contains exactly one zero of $p_{n+1}(x)$.*

4.2.3 Legendre Polynomials

Paul Turan [**24**] developed the following results on the zeros of Legendre polynomials based on the work of Szegö and Frejer. Legendre Polynomials $P_0(x), P_1(x), ..., P_n(x), ...$ can be uniquely defined by the orthogonality property

$$\int_{-1}^{1} w(x) P_n(x) P_\nu(x) dx = 0 \tag{4.18}$$

where $w(x) = 1$ and $\nu = 0, 1, ..., n-1$, and by the normalization

$$P_n(1) = 1.$$

Legendre Polynomials can be represented by Rodriquez's formula

$$P_n(x) = \frac{1}{n!2^n} \cdot \frac{d^n}{dx^n}(x^2-1)^n \tag{4.19}$$

and are solutions to the differential equation

$$\frac{d}{dx}\left[(1-x^2)\frac{d}{dx}P_n(x)\right] + n(n+1)P_n(x) = 0.$$

They are also given explicitly by

$$P_n(x) = \sum_{k=0}^{[n/2]}(-1)^k \frac{(2n-2k)!}{2^n k!(n-k)!(n-2k)!}x^{n-2k} \tag{4.20}$$

where $[n/2]$ is equal to $n/2$ if n is even and $(n-1)/2$ if n is odd.

Listing the first ten Legendre polynomials we have:

$P_0(x) = 1$

$P_1(x) = x$

$P_2(x) = \frac{1}{2}(3x^2 - 1)$

$P_3(x) = \frac{1}{2}(5x^3 - 3x)$

$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$

$P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$

$P_6(x) = \frac{1}{16}(231x^6 - 315x^4 + 105x^2 - 5)$

$P_7(x) = \frac{1}{16}(429x^7 - 693x^5 + 315x^3 - 35x)$

$P_8(x) = \frac{1}{128}(6435x^8 - 12012x^6 + 6930x^4 - 1260x^2 + 35)$

$P_9(x) = \frac{1}{128}(12155x^9 - 25740x^6 + 18018x^4 - 4620x^2 + 315x)$

$P_{10}(x) = \frac{1}{256}(46189x^{10} - 109395x^8 + 90090x^6 - 300030x^4 + 3465x^2 - 63)$

Legendre Polynomials are orthogonal polynomials and hence Theorem 4.7 and Theorem 4.8 apply. By the orthogonality property (4.18) we know that $P_n(x)$ has n zeros that are real, simple and all lying in the interval $(-1, 1)$. Property (4.19) gives that they are symmetric about $x = 0$.

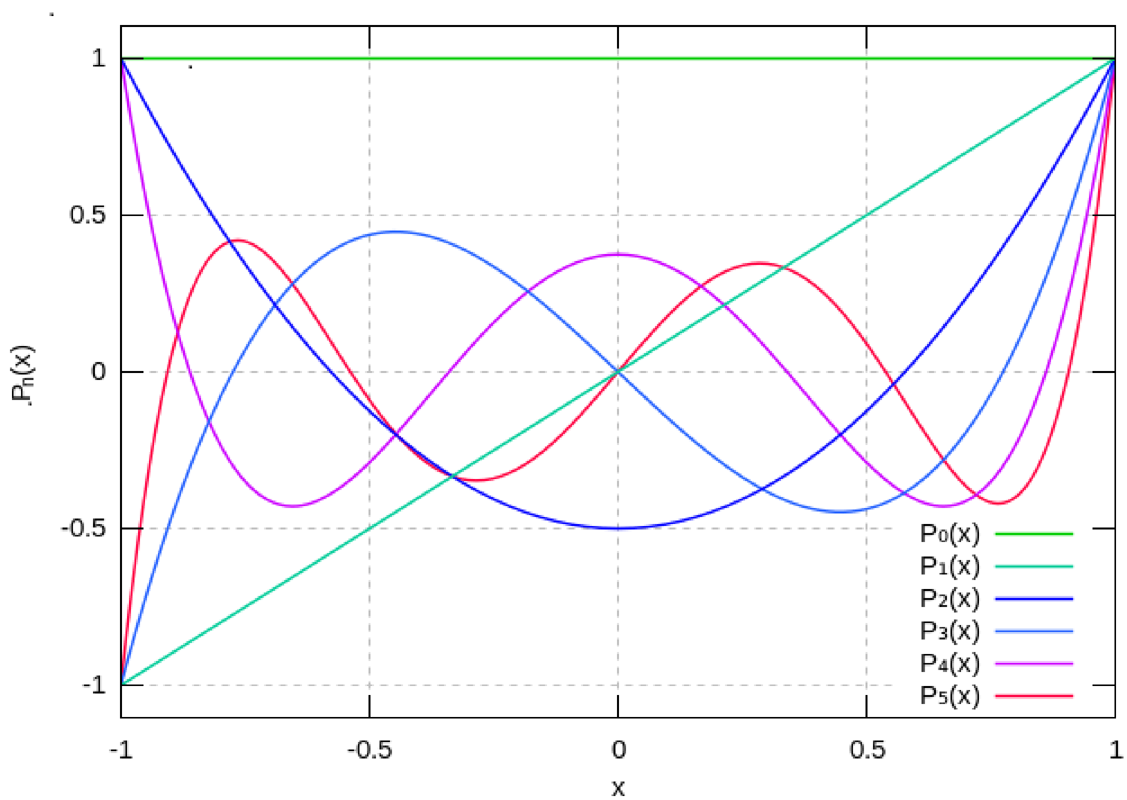

FIGURE 4.22. Legendre Polynomials [**25**]

Now let's define the zeros of Legendre polynomials by

$$x_{\nu,n} = \cos \theta_{\nu,n}$$

where $\nu = 1, 2, ..., n$ and

$$-1 < x_{1,n} < x_{2,n} < ... < x_{n,n} < 1$$

or

$$\pi > \theta_{1,n} > \theta_{2,n} > ... > \theta_{n,n} > 0.$$

Then Szegö's estimate (see [**22**], [**24**]) for $\theta_{\nu,n}$ can be written

$$\frac{n-\nu+\frac{3}{4}}{n+\frac{1}{2}}\pi < \theta_{\nu,n} < \frac{n-\nu+1}{n+1}\pi \quad (\frac{1}{2}n+1 \leq \nu \leq n). \tag{4.21}$$

In comparing the zeros of $P_{n+1}(x)$ and $P_n(x)$ we have Theorem 4.8 which states that the zeros of $P_{n+1}(x)$ separate the zeros of $P_n(x)$. In other words,

$$-1 < x_{1,n+1} < x_{1,n} < x_{2,n+1} < x_{2,n} < ... < x_{n,n} < x_{n+1,n+1} < 1$$

or

$$\pi > \theta_{1,n+1} > \theta_{1,n} > ... > \theta_{n,n} > \theta_{n+1,n+1} > 0.$$

Thus for each zero $x_{\nu,n}$ of $P_n(x)$ we can associate uniquely the zero of $x_{\nu,n+1}$ of $P_{n+1}(x)$ so that

$$x_{\nu-1,n} < x_{\nu,n+1} < x_{\nu,n}\,, \quad \nu = 1, 2, ..., n+1$$

and $x_{0,n} = -1$ and $x_{n+1,n} = 1$.

4.2.4 Alternative Method for Taylor Series Representation of RD Rectangles

LEMMA 4.9. *(Single Sum of Coefficients) Suppose*

$$\sum_{k=0}^{\infty}\sum_{n=0}^{k} d_{k,n} z^{2(k+n)+1}$$

is an absolutely convergent series. Then we can write

$$\sum_{k=0}^{\infty}\sum_{n=0}^{k} d_{k,n} z^{2(k+n)+1} = \sum_{m=0}^{\infty} b_m z^{2m+1}$$

where

$$b_m = \begin{cases} \sum_{j=\frac{m}{2}}^{m} d_{j,m-j} & for\ m\, even \\ \sum_{j=\frac{m+1}{2}}^{m} d_{j,m-j} & for\ m\, odd. \end{cases}$$

PROOF. By the absolute convergence hypothesis we can expand and regroup the double summation

$$\sum_{k=0}^{\infty}\sum_{n=0}^{k} d_{k,n} z^{2(k+n)+1}$$

and get the following,

$$\begin{aligned}\sum_{k=0}^{\infty}\sum_{n=0}^{k} d_{k,n} z^{2(k+n)+1} &= d_{0,0}z + d_{1,0}z^3 + d_{1,1}z^5 + d_{2,0}z^5 + d_{2,1}z^7 + d_{2,2}z^9 + d_{3,0}z^7 \\ &+ d_{3,1}z^9 + d_{3,2}z^{11} + d_{3,3}z^{13} + d_{4,0}z^9 + d_{4,1}z^{11} + d_{4,2}z^{13} + d_{4,3}z^{15} + d_{4,4}z^{17} + ... \\ &= d_{0,0}z + d_{1,0}z^3 + (d_{1,1} + d_{2,0})z^5 + (d_{2,1} + d_{3,0})z^7 + (d_{2,2} + d_{3,1} + d_{4,0})z^9 \\ &+ (d_{3,2} + d_{4,1} + d_{5,0})z^{11} + (d_{3,3} + d_{4,2} + d_{5,1} + d_{6,0})z^{13} + ... \\ &= \sum_{m=0}^{\infty} b_m z^{2m+1}\end{aligned}$$

where

$$b_m = \begin{cases} \sum_{j=\frac{m}{2}}^{m} d_{j,m-j} & \text{for } m \text{ even} \\ \sum_{j=\frac{m+1}{2}}^{m} d_{j,m-j} & \text{for } m \text{ odd.} \end{cases}$$

□

Proposition 4.10. *Let R_Ω be an RD rectangle. Suppose $f_{R_\Omega} : \mathbb{D} \to R_\Omega$ is the conformal map determined by $f_{R_\Omega}(0) = 0$ and $f'_{R_\Omega}(0) = 1$. Then the Taylor series of f_{R_Ω} about the origin is given by*

$$f_{R_\Omega}(z) = \sum_{n=0}^{\infty} b_n z^{2n+1}$$

where

$$b_n = \frac{P_n(x)}{2n+1},$$

$x = \cos\theta$ for some $\theta \in (0, \pi)$, and $P_n(x)$ are the Legendre Polynomials defined by equation (4.20).

Proof. By Proposition 3.7, the prevertices of R are given by $e^{\frac{i\theta}{2}}, -e^{-\frac{i\theta}{2}}, -e^{\frac{i\theta}{2}}$, and $e^{-\frac{i\theta}{2}}$ for some $\theta \in (0, \pi)$. Moreover, rectangles have exterior turning angles $\beta_k = \frac{1}{2}$ for k=1, 2, 3, 4. Applying the Schwarz-Christoffel mapping for P_Ω we have,

$$\begin{aligned} f_{R_\Omega}(z) &= \int_0^z [(1 - \frac{t}{e^{\frac{i\theta}{2}}})(1 - \frac{t}{-e^{-\frac{i\theta}{2}}})(1 - \frac{t}{-e^{\frac{i\theta}{2}}})(1 - \frac{t}{e^{-\frac{i\theta}{2}}})]^{-1/2} dt \qquad (4.22) \\ &= \int_0^z [(1 - \left(\frac{t}{e^{\frac{i\theta}{2}}}\right)^2)(1 - (e^{\frac{i\theta}{2}} t)^2)]^{-1/2} dt \\ &= \int_0^z [(1 - e^{-i\theta} t^2)(1 - e^{i\theta} t^2)]^{-1/2} dt \\ &= \int_0^z (1 - (e^{-i\theta} + e^{i\theta}) t^2 + t^4)^{-1/2} dt \\ &= \int_0^z (1 - 2\cos\theta t^2 + t^4)^{-1/2} dt. \qquad (4.23) \end{aligned}$$

Now we need to develop a Taylor series representation for integral (4.23). Since Taylor series with radius of convergence of at least one are uniformly convergent on compact subsets of the open unit disk, we know that we can integrate along a path from 0 to z of

$$\sum_{n=0}^{\infty} c_n t^n.$$

Switching the integration and summation order yields

$$\int_0^z \sum_{n=0}^{\infty} c_n t^n dt = \sum_{n=0}^{\infty} \int_0^z c_n t^n dt = \sum_{n=0}^{\infty} \frac{c_n}{n+1} z^{n+1}.$$

By Proposition 2.5, we know that

$$\frac{d}{dz} \sinh^{-1} z = (1+z^2)^{-1/2}.$$

Thus, for a function $g(z)$, the derivative of $\sinh^{-1} g(z)$ is given by

$$\frac{d}{dz} \sinh^{-1} g(z) = [1 + (g(z))^2]^{-1/2} g'(z)$$

which can be written in the form

$$\frac{1}{g'(z)} \frac{d}{dz} \sinh^{-1} g(z) = [1 + (g(z))^2]^{-1/2}. \tag{4.24}$$

Deriving the Taylor series for equation 4.24 we begin with the Taylor Series for $\sinh^{-1} z$,

$$\sinh^{-1} z = z - \frac{1}{2 \cdot 3} z^3 + \frac{3 \cdot 1}{4 \cdot 2 \cdot 5} z^5 - \frac{5 \cdot 3 \cdot 1}{6 \cdot 4 \cdot 2 \cdot 7} z^7 + \ldots \tag{4.25}$$

We know that

$$(n)!! = \begin{cases} n \cdot (n-2) \cdots 5 \cdot 3 \cdot 1 & \text{for } n > 0 \,\text{odd} \\ n \cdot (n-2) \cdots 6 \cdot 4 \cdot 2 & \text{for } n > 0 \,\text{even} \\ 1 & \text{for } n = -1, 0. \end{cases} \tag{4.26}$$

Since $2n$ is even and $2n-1$ is odd we can apply equation (4.26) and substitute into equation (4.25) to get

$$\sinh^{-1} z = \sum_{n=0}^{\infty} \frac{(-1)^n (2n-1)!!}{(2n)!!(2n+1)} z^{2n+1} \quad \text{for } |z| \leq 1.$$

Writing $(2n-1)!!$ and $(2n)!!$ without double factorials gives us the forms

$$(2n-1)!! = \frac{(2n)!}{n! 2^n}$$

and

$$(2n)!! = n! 2^n.$$

Making a substitution into the derivation of the Taylor series for $\sinh^{-1} z$ we have

$$\begin{aligned}
\sinh^{-1} z &= \sum_{n=0}^{\infty} \frac{(-1)^n \frac{(2n)!}{n!2^n}}{n!2^n(2n+1)} z^{2n+1} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{n!2^n n!2^n(2n+1)} z^{2n+1} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{4^n (n!)^2 (2n+1)} z^{2n+1} \quad \text{for } |z| \le 1.
\end{aligned}$$

Thus the Taylor Series for $\frac{d}{dz}\sinh^{-1} z$ is given by

$$\begin{aligned}
\frac{d}{dz}\sinh^{-1} z &= \sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{4^n (n!)^2 (2n+1)} (2n+1) z^{2n} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{4^n (n!)^2} z^{2n} \quad \text{for } |z| \le 1.
\end{aligned} \tag{4.27}$$

Generalizing this Taylor series to a function g(z) gives us

$$\frac{1}{g'(z)}\frac{d}{dz}\sinh^{-1} g(z) = \sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{4^n (n!)^2} (g(z))^{2n} \quad \text{for } |g(z)| \le 1. \tag{4.28}$$

We will use this result to prove our claim.

Let $\theta \in (0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi)$ and $K = -2\cos\theta = -2x$. Thus, $K \in (-2, 0) \cup (0, 2)$. The integrand of (0.2) then becomes

$$(1 + Kt^2 + t^4)^{-1/2}.$$

Letting $g(t) = (Kt^2 + t^4)^{1/2}$ we have , by equation (4.24),

$$(1 + Kt^2 + t^4)^{-1/2} = \frac{1}{g'(t)}\frac{d}{dt}\sinh^{-1} g(t) = \frac{2(Kt^2 + t^4)^{1/2}}{2Kt + 4t^3}\frac{d}{dt}\sinh^{-1}(Kt^2 + t^4)^{1/2}$$

and the Taylor series is given by

$$\begin{aligned}
\frac{2(Kt^2 + t^4)^{1/2}}{2Kt + 4t^3}\frac{d}{dt}\sinh^{-1}(Kt^2 + t^4)^{1/2} &= \sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{4^n (n!)^2} [(Kt^2 + t^4)^{1/2}]^{2n} \quad \text{for } |(Kt^2 + t^4)^{1/2}| \le 1. \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{4^n (n!)^2} (Kt^2 + t^4)^{n} \quad \text{for } |(Kt^2 + t^4)^{1/2}| \le 1.
\end{aligned}$$

Now let

$$a_n = \frac{(2n)!}{4^n (n!)^2}.$$

Clearly $a_n > 0$. Determining the Taylor series for the integrand of integral (4.23) we have

$$\begin{aligned}
(1 - 2\cos\theta t^2 + t^4)^{-1/2} &= \sum_{n=0}^{\infty} (-1)^n a_n (Kt^2 + t^4)^n \\
&= \sum_{n=0}^{\infty} (-1)^n a_n K^n t^{2n} (1 + \frac{t^2}{K})^n \\
&= \sum_{n=0}^{\infty} (-1)^n a_n K^n t^{2n} \sum_{k=0}^{n} \binom{n}{k} \frac{t^{2k}}{K^k} \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n} (-1)^n \binom{n}{k} a_n K^{n-k} t^{2n+2k} \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n} (-1)^n \frac{n!}{k!(n-k)!} \frac{(2n)!}{4^n (n!)^2} (-2x)^{n-k} t^{2n+2k}.
\end{aligned}$$

Now changing $n+k$ to n so that the exponent of t is $2n$ we have

$$\sum_{n=0}^{\infty} \sum_{k=0}^{[n/2]} (-1)^{n-k} \frac{(n-k)!}{k!(n-2k)!} \frac{(2(n-k))!}{4^{n-k}[(n-k)!]^2} (-2x)^{n-2k} t^{2n} \tag{4.29}$$

where $[n/2]$ is equal to $n/2$ if n is even and $(n+1)/2$ if n is odd. The upper bound on k is dictated by the presence of the term $(n-2k)!$ in the denominator which requires that $2k \leq n$ or $k \leq n/2$ [**12**]. Simplifying equation 4.29 we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} \sum_{k=0}^{[n/2]} (-1)^{n-k} \frac{(2n-2k)!}{4^{n-k} k!(n-k)!(n-2k)!} (-1)^{n-2k} 2^{n-2k} x^{n-2k} t^{2n} \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{[n/2]} (-1)^k \frac{(2n-2k)!}{2^n k!(n-k)!(n-2k)!} x^{n-2k} t^{2n} = \sum_{n=0}^{\infty} P_n(x) t^{2n}.
\end{aligned}$$

Therefore, the Taylor series representation about the origin for integral in (4.23) is

$$\begin{aligned}
f_{R_\Omega}(z) = \int_0^z (1 - 2\cos\theta t^2 + t^4)^{-1/2} dt &= \int_0^z \sum_{n=0}^{\infty} P_n(x) t^{2n} dt \\
&= \sum_{n=0}^{\infty} \int_0^z P_n(x) t^{2n} dt \\
&= \sum_{n=0}^{\infty} \frac{P_n(x)}{2n+1} z^{2n+1}.
\end{aligned}$$

Thus,

$$\sum_{n=0}^{\infty} b_n z^{2n+1}$$

where

$$b_n = \frac{P_n(x)}{2n+1}$$

and $P_n(x)$ is the n^{th} Legendre polynomial. This completes the proof. □

Proposition 4.11. *Let R_Ω be an RD rectangle with $\theta \in (\pi/2, \pi)$. If $f_{R_\Omega} : \mathbb{D} \to R_\Omega$ is the conformal map determined by $f_{R_\Omega}(0) = 0$ and $f'_{R_\Omega}(0) = 1$, then the Taylor series of f_{R_Ω} about the origin has at least one negative coefficient.*

Proof. Let $x = \cos\theta$ where $\theta \in (\pi/2, \pi)$. So $x \in (-1, 0)$. By Proposition 4.10, the coefficients of the Taylor series of f_{R_Ω} about the origin are given by

$$b_n = \frac{P_n(x)}{2n+1} \quad for\ n = 1, 2, ...$$

where $P_n(x)$ is the nth Legendre polynomial. The first Legendre polynomial is given by $P_1(x) = x$. Hence, $P_1(x) < 0$ for $x \in (-1, 0)$ and so $b_1 < 0$. Thus, the Taylor series for f_{R_Ω} has at least one negative coefficient. □

Lemma 4.12. *Let $x_{\nu,n} = \cos\theta_{\nu,n}$ be the zeros of the nth Legendre polynomial, $P_n(x)$, where $\nu = 1, 2, ..., n$ and $\pi > \theta_{1,n} > \theta_{2,n} > ... > \theta_{n,n} > 0$. Then $P_n(x) < 0$ for every $x \in (x_{n-1,n}\,,\, x_{n,n})$.*

Proof. By [**24**], we know the zeros $x_{1,n} < x_{2_n} < ... < x_{n,n}$ of $P_n(x)$ are simple. Moreover, the leading coefficient of every $P_n(x)$ is positive and so $P_n(x) > 0$ for every $x \in (x_{n,n}\,,\, 1)$. Thus, it follows that $P_n(x) < 0$ for every $x \in (x_{n-1,n}\,,\, x_{n,n})$. □

Proposition 4.13. *Let $x_{\nu,n} = \cos\theta_{\nu,n}$ be the zeros of the nth Legendre polynomial, $P_n(x)$, where $\nu = 1, 2, ..., n$ and $\pi > \theta_{1,n} > \theta_{2,n} > ... > \theta_{n,n} > 0$. Then*

$$\lim_{n\to\infty} x_{n,n} = 1.$$

Proof. Notice that

$$\lim_{n\to\infty} x_{n,n} = 1$$

is equivalent to

$$\lim_{n\to\infty} \theta_{n,n} = 0.$$

Using Szegö's estimate of $\theta_{\nu,n}$ given by equation (4.30) we have the following estimate for $\theta_{n,n}$,

$$\frac{\frac{3}{4}}{n+\frac{1}{2}}\pi < \theta_{n,n} < \frac{1}{n+1}\pi. \tag{4.30}$$

Applying the Squeeze Theorem we have

$$\lim_{n\to\infty} \theta_{n,n} = 0$$

which implies

$$\lim_{n\to\infty} x_{n,n} = 1.$$

□

Proposition 4.14. *Let $x_{\nu,n} = \cos\theta_{\nu,n}$ be the zeros of the nth Legendre polynomial, $P_n(x)$, where $\nu = 1, 2, ..., n$ and $\pi > \theta_{1,n} > \theta_{2,n} > ... > \theta_{n,n} > 0$. If $\hat{x} \in (-1\,,\,x_{n,n}]$ then there exists $P_k(x)$, $k \in \{1, 2, ..., n\}$ such that $P_k(\hat{x}) < 0$.*

Proof. We will prove the claim by induction on n for $\hat{x} \in (-1\,,\,x_{n,n})$. Notice that clearly k depends on $\hat{x}$. For the case $n = 1$ we have $P_1(x) = x$ with zero $x_{1,1} = 0$. By Lemma 4.12 we know that $P_1(\hat{x}) < 0$ for every $\hat{x} \in (-1\,,\,x_{1,1})$. So now we only need to consider $\hat{x} \in (0, 1)$. For the case $n = 2$ we have $P_2(x) = \frac{1}{2}(3x^2 - 1)$ with zeros $x_{1,2} = -1/\sqrt{3}$ and $x_{2,2} = 1/\sqrt{3}$. Notice that $\hat{x} \in (0\,,\,x_{2,2})$ and by Lemma 4.12, $P_2(\hat{x}) < 0$ for every $\hat{x} \in (x_{1,2}, x_{2,2})$. Now assume for every $\hat{x} \in (0\,,\,x_{n,n})\,, n > 2$, there exists $P_k(\hat{x}) < 0$ for some $k \in \{1, 2, ..., n\}$. We want to show for every $\hat{x} \in (0\,,\,x_{n+1,n+1})$ there exists $P_k(\hat{x}) < 0$ for some $k \in \{1, 2, ..., n+1\}$. If $\hat{x} \in (0\,,\,x_{n,n})$, then by our induction hypothesis we are done. Now suppose $\hat{x} \in (x_{n,n}\,,\,x_{n+1,n+1})$. By Theorem 4.8, $(x_{n,n}, x_{n+1,n+1}) \subset (x_{n,n+1},\,x_{n+1,n+1})$. By Lemma 4.12, $P_{n+1}(x) < 0$ for every $x \in (x_{n,n}, x_{n,n+1})$. It follows that $P_{n+1}(\hat{x}) < 0$. Now for the case $\hat{x} = x_{n,n}$ we have $\hat{x} \in (x_{n-1,n+1}\,,\,x_{n,n+1})$ and by Lemma 4.12 $P_{n+1}(\hat{x}) < 0$. Hence, for $\hat{x} \in (-1\,,\,x_{n,n}]$ there exists $P_k(x)$, $k \in \{1, 2, ..., n\}$ such that $P_k(\hat{x}) < 0$. □

Proposition 4.15. *Let $x_{\nu,n} = \cos\theta_{\nu,n}$ be the zeros of the nth Legendre polynomial, $P_n(x)$, where $\nu = 1, 2, ..., n$ and $\pi > \theta_{1,n} > \theta_{2,n} > ... > \theta_{n,n} > 0$. Then for every $\hat{x} \in (-1, 1)$ there exists at least one $P_n(x)$ such that $P_n(\hat{x}) < 0$.*

Proof. By Proposition 4.14, we have that the set

$$\{\hat{x} : P_k(\hat{x}) \geq 0 \text{ for all } k\} \subset \bigcap_{n=1}^{\infty} \left(x_{n,n}, 1\right).$$

Also, by Proposition 4.13, we have

$$\lim_{n\to\infty} x_{n,n} = 1,$$

so

$$\bigcap_{n=1}^{\infty} \left(x_{n,n}, 1\right) = \emptyset.$$

Hence, there exists an $N \in \mathbb{N}$ such that $x_{n,n} \in (\hat{x}, 1)$ for every $n > N$. So we can choose a $n^* > N$ such that $\hat{x} < x_{n^*,n^*}$. Thus, by Proposition 4.14 there exists a $P_k(x)$, $k \in \{1, 2, ..., n*\}$ such that $P_k(\hat{x}) < 0$. ☐

Theorem 4.16. *Let R_Ω be an RD rectangle. If f_{R_Ω} is the conformal mapping from the unit disk onto R_Ω determined by $f_{R_\Omega}(0) = 0$ and $f'_{R_\Omega}(0) = 1$, then the Taylor series of f_{R_Ω} about the origin has at least one negative odd coefficient.*

Proof. By Proposition 4.15 we know that for every $\hat{x} \in (-1, 1)$ there exists at least one $P_n(x)$ such that $P_n(\hat{x}) < 0$. By Proposition 4.10 the Taylor series coefficients are given by

$$b_n = \frac{P_n(x)}{2n+1}.$$

It follows that there is at least one $b_n < 0$. ☐

CHAPTER 5

The Schwarz-Christoffel Mapping from $\mathbb{D}$ onto Regular RD Polygons

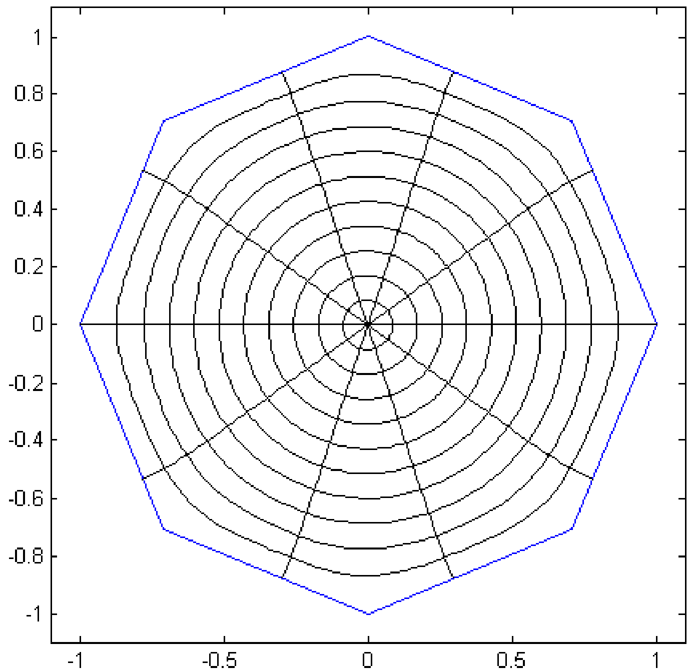

FIGURE 5.23. An 8-sided RD regular polygon with one vertex on the positive real axis.

5.1 Properties of Regular RD Polygons

A regular polygon with a vertex on the positive real axis has both x and y-axis symmetries if m is even (see Figure 5.23). If m is odd, then a regular polygon with a vertex on the positive real axis only has x-axis symmetries. Thus, a regular polygon is RD only in the case m is even. For this chapter, we assume that a regular RD m-sided, $m \geq 4$, polygon Λ_Ω has a vertex on the positive real axis.

LEMMA 5.1. *Let Λ_Ω be an m-sided, $m \geq 4$, regular RD polygon with one vertex on the real axis. If g is the counterclockwise rotation of Λ_Ω defined by $g(w) = e^{\frac{2\pi i}{m}} w$ where $w \in \Lambda_\Omega$, then Λ_Ω is invariant under g.*

PROOF. To show that Λ_Ω is invariant under the counterclockwise rotation g defined by $g(w) = e^{\frac{2\pi i}{m}} w$ where $w \in \Lambda_\Omega$, it suffices to show that g acting upon a vertex of Λ_Ω is another vertex of Λ_Ω. Let v_k be the kth vertex of an m-sided regular polygon where $k = 0, 1, ..., m - 1$.

Then $v_k = e^{\frac{2k\pi i}{m}}$ $for\, k = 1, 2, ..., m-1$. Notice that

$$g(v_k) = e^{\frac{2\pi i}{m}} v_k = e^{\frac{2\pi i}{m}} e^{\frac{2k\pi i}{m}} = e^{\frac{2(k+1)\pi i}{m}} = v_{k+1}$$

which is the $k+1$ vertex of Λ_Ω. Since v_k was arbitrarily chosen it follows that Λ_Ω is invariant under g. □

Proposition 5.2. *Let Λ_Ω be an m-sided, $m \geq 4$, regular RD polygon with one vertex on the real axis. If f_{Λ_Ω} is the conformal mapping from the unit disk onto Λ_Ω determined by $f_{\Lambda_\Omega}(0) = 0$ and $f'_{\Lambda_\Omega}(0) = 1$, then $f_{\Lambda_\Omega}(e^{\frac{2\pi i}{m}} z) = e^{\frac{2\pi i}{m}} f_{\Lambda_\Omega}(z)$ for every $z \in \mathbb{D}$.*

Proof. Let $z \in \mathbb{D}, w \in \Lambda_\Omega$, and m be the number of sides of Λ_Ω. Define $H(z) = h \circ g \circ f_{\Lambda_\Omega}(z)$ where f_{Λ_Ω} is defined above, $g : \Lambda_\Omega \to \Lambda_\Omega$ is the mapping $g(w) = e^{\frac{2\pi i}{m}} w$ for $w \in \Lambda_\Omega$ with $g(0) = 0$, and $h : \Lambda_\Omega \to \mathbb{D}$ is the mapping $h = f^{-1}_{\Lambda_\Omega}$ with $h(0) = 0$. So, $H : \mathbb{D} \to \mathbb{D}$ is holomorphic with $H(0) = 0$. Since $H(z) = h(e^{\frac{2\pi i}{m}} f_{\Lambda_\Omega}(z))$,

$$H'(z) = h'(e^{\frac{2\pi i}{m}} f_{\Lambda_\Omega}(z)) \cdot \left(e^{\frac{2\pi i}{m}} f'_{\Lambda_\Omega}(z)\right) = h'(e^{\frac{2\pi i}{m}} f_{\Lambda_\Omega}(z)) \cdot \frac{e^{\frac{2\pi i}{m}}}{h'(h^{-1}(z))}.$$

Now $h^{-1} = f_{\Lambda_\Omega}$ implies that $h^{-1}(0) = 0$, so

$$H'(0) = h'(0) \cdot \frac{e^{\frac{2\pi i}{m}}}{h'(0)} = e^{\frac{2\pi i}{m}}.$$

Thus, $|H'(0)| = |e^{\frac{2\pi i}{m}}| = 1$. By Schwarz's Lemma, we have $H(z) = e^{i\alpha_0} z$ and $H'(0) = e^{i\alpha_0}$. Since $H'(0) = e^{\frac{2\pi i}{m}}$, we have $\alpha_0 = \frac{2\pi}{m}$ for $\alpha_0 \in [0, 2\pi]$ and hence $H(z) = e^{\frac{2\pi i}{m}} z$. Combining this fact with $H(z) = h(e^{\frac{2\pi i}{m}} f_{\Lambda_\Omega}(z)) = f^{-1}_{\Lambda_\Omega}(e^{\frac{2\pi i}{m}} f_{\Lambda_\Omega}(z))$ yields $e^{\frac{2\pi i}{m}} z = f^{-1}_{\Lambda_\Omega}(e^{\frac{2\pi i}{m}} f_{\Lambda_\Omega}(z))$ which implies $f_{\Lambda_\Omega}(e^{\frac{2\pi i}{m}} z) = e^{\frac{2\pi i}{m}} f_{\Lambda_\Omega}(z)$ for every $z \in \mathbb{D}$. □

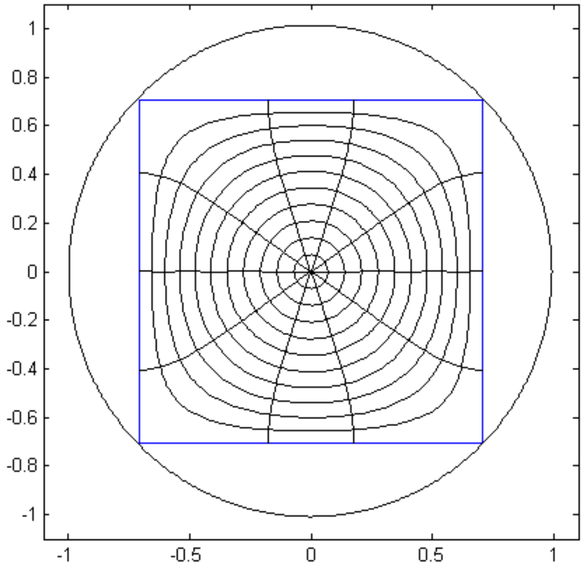

FIGURE 5.24. An RD square with sides parallel to the axes inscribed within a unit disk.

5.2.1 Taylor Series Representation of RD Squares

By Proposition 5.2 we have linearity of $f_{S_\Omega}(z)$, and hence the square is a special case of the rectangle where $\theta/2 \to \pi/4$ which implies $\theta \to \pi/2$ and thus, $\cos\theta \to 0$. By letting $\cos\theta \to 0$ in the form of the RD rectangle mapping (4.23), we have the following propostion.

PROPOSITION 5.3. *Let S_Ω be an RD square with sides parallel to the axes. If $f_{S_\Omega} : \mathbb{D} \to S_\Omega$ is the conformal map determined by $f_{S_\Omega}(0) = 0$ and $f'_{S_\Omega}(0) = 1$, then f_{S_Ω} is of the form*

$$f_{S_\Omega}(z) = \int_0^z (1 + t^4)^{-1/2} dt \tag{5.1}$$

PROPOSITION 5.4. *Let S_Ω be an RD square with sides parallel to the axes. If $f_{S_\Omega} : \mathbb{D} \to S_\Omega$ is the conformal map determined by $f_{S_\Omega}(0) = 0$ and $f'_{S_\Omega}(0) = 1$, then the Taylor series of f_{S_Ω} about the origin is given by*

$$f_{S_\Omega}(z) = \sum_{n=0}^{\infty} \frac{a_n}{4n+1} z^{4n+1}$$

where $a_0 = 1$, and a_n is given recursively by

$$a_n = \frac{(-2n+1)a_{n-1}}{2n}, \ \ for\, n \geq 1.$$

Proof. We need to develop a Taylor Series representation for equation (5.1). Since Taylor series whose radius of convergence is at least one are uniformly convergent on compact subsets of the open unit disk and $z \in \mathbb{D}$, we know that we can integrate along the path from 0 to z of

$$\sum_{k=0}^{\infty} c_k t^k.$$

Switching the integration and summation order we get

$$\int_0^z \sum_{k=0}^{\infty} c_k t^k dt = \sum_{k=0}^{\infty} \int_0^z c_k t^k dt = \sum_{k=0}^{\infty} \frac{c_k}{k+1} z^{k+1}.$$

Let $g(t)$ be the integrand of equation (5.1) so we have

$$g(t) = (1 + t^4)^{-1/2}.$$

Now we will show that g(t) satisfies a differential equation and develop a recursion formula for the coefficients a_n. Observe that

$$(1 + t^4)\frac{d}{dt}(1 + t^4)^{-1/2} = (-2t^3)(1 + t^4)^{-1/2} \tag{5.2}$$

So $y = g(t)$ satisfies the differential equation

$$(1 + t^4)y' = (-2t^3)y \tag{5.3}$$

or

$$(1 + t^4)y' + 2t^3 y = 0. \tag{5.4}$$

Writing $g(t)$ as

$$y = \sum_{n=0}^{\infty} a_n t^{4n}$$

for the Taylor series expansion of $y = g(t)$ centered at the origin we have that

$$y' = \sum_{n=1}^{\infty} 4n a_n t^{4n-1}.$$

Substituting these values into equation (5.4) we have

$$(1 + t^4) \sum_{n=1}^{\infty} 4n a_n t^{4n-1} + (2t^3) \sum_{n=0}^{\infty} a_n t^{2n} = 0$$

$$\sum_{n=1}^{\infty} 4na_n t^{4n-1} + \sum_{n=1}^{\infty} 4na_n t^{4n+3} + \sum_{n=0}^{\infty} 2a_n t^{4n+3} = 0$$

$$2\left(\sum_{n=1}^{\infty} 2na_n t^{4n-1} + \sum_{n=1}^{\infty} 2na_n t^{4n+3} + \sum_{n=0}^{\infty} a_n t^{4n+3}\right) = 0$$

$$\sum_{n=1}^{\infty} 2na_n t^{4n-1} + \sum_{n=1}^{\infty} 2na_n t^{4n+3} + \sum_{n=0}^{\infty} a_n t^{4n+3} = 0$$

$$\sum_{n=0}^{\infty} 2(n+1)a_{n+1} t^{4n+3} + \sum_{n=1}^{\infty} 2na_n t^{4n+3} + \sum_{n=0}^{\infty} a_n t^{4n+3} = 0$$

$$\sum_{n=0}^{\infty} 2(n+1)a_{n+1} t^{4n+3} + \sum_{n=0}^{\infty} a_n t^{4n+3} + \sum_{n=1}^{\infty} 2na_n t^{4n+3} = 0. \tag{5.5}$$

Now let

$$h(t) = [2a_1 + a_0]t^3 + \sum_{n=1}^{\infty} [2(n+1)a_{n+1} + a_n + 2na_n]\, t^{4n+3} \tag{5.6}$$

Letting $C_0 = 2a_1 + a_0$, and

$$C_n = 2(n+1)a_{n+1} + a_n + 2na_n \text{ for } n \geq 0$$

we can write equation (5.6) as

$$h(t) = \sum_{n=0}^{\infty} C_n t^{4n+1}.$$

From equation (5.5) we have $h(t) \equiv 0$,

$$h(t) = \sum_{n=0}^{\infty} C_n t^{4n+1} \equiv 0,$$

and $h(t)$ is analytic about the origin and is uniformly convergent on compact subsets of the unit disk, we know that all the derivatives of $h(t)$ about the origin are 0. This implies that each of the coefficients $C_n = 0$ and hence we have the following recursion formula

$$a_{n+1} = \frac{(-2n-1)a_n}{2(n+1)} \text{ for } n \geq 0. \tag{5.7}$$

which can be written in the form

$$a_n = \frac{(-2n+1)a_{n-1}}{2n} \text{ for } n \geq 1. \tag{5.8}$$

Now from equation (5.1) we have

$$f_{S_\Omega}(z) = \int_0^z g(t)dt = \int_0^z \sum_{n=0}^{\infty} a_n t^{4n} dt = \sum_{n=0}^{\infty} \frac{a_n}{4n+1} z^{4n+1}$$

where $a_0 = g(0) = 1$, and

$$a_n = \frac{(-2n+1)a_{n-1}}{2n} \text{ for } n \geq 1.$$

□

Since $a_0 = 1$ we have $a_1 < 0$, $a_2 > 0$, $a_3 > 0$ and so forth. Thus, a_n alternates in sign which can be proven easily by using the closed form of a_n in the next section.

5.2.2 Closed Form for Taylor Coefficients of RD Squares

To determine the closed form for a_n for the Taylor series expansion about the origin of $f_{Rh_\Omega}(z)$ we apply the binomial series expansion and observe that

$$\begin{aligned} g(t) &= (1+t^4)^{-1/2} \\ &= \sum_{n=0}^{\infty} (-1)^n \binom{-1/2}{n} t^{4n} \\ &= \sum_{n=0}^{\infty} (-1)^n \binom{1/2+n-1}{n} t^{4n}. \end{aligned} \tag{5.9}$$

Thus, the Taylor series for $g(t)$ is given by

$$g(t) = \sum_{n=0}^{\infty} a_n t^{4n}$$

where

$$a_n = (-1)^n \binom{1/2+n-1}{n}.$$

Notice that evaluating this equation for $n = 0$, we also have $a_0 = 1$.

5.2.3 Negativity of Taylor Coefficients of RD Squares

THEOREM 5.5. *Let S_Ω be an RD square with sides parallel to the axes. If $f_{S_\Omega} : \mathbb{D} \to S_\Omega$ is the conformal map determined by $f_{S_\Omega}(0) = 0$ and $f'_{S_\Omega}(0) = 1$, then the Taylor series of f_{S_Ω}about the origin has non-zero odd coefficients that alternate in sign.*

PROOF. By equation (5.9) we have

$$g(t) = \sum_{n=0}^{\infty} (-1)^n \binom{1/2 + n - 1}{n} t^{4n}.$$

Observe that

$$\binom{1/2 + n - 1}{n} > 0 \quad \text{for} \quad n \geq 0.$$

Thus, $g(t)$ has a Taylor series about the origin with non-zero even coefficients that alternate in sign and thus it follows that the Taylor series of $f_{S_\Omega}(z)$ about the origin has non-zero odd coefficients that alternate in sign. □

5.2.4 Alternative Method for Determining Negativity of Taylor Coefficients of RD Squares

We will use the knowledge of the rectangle mapping from Chapter 4 and determine the nature of the coefficients of the conformal mapping onto an RD square with sides parallel to the axes by letting $\cos\theta \to 0$.

PROPOSITION 5.6. *Let S_Ω be an RD square with sides parallel to the axes. If $f_{S_\Omega} : \mathbb{D} \to S_\Omega$ is the conformal map determined by $f_{S_\Omega}(0) = 0$ and $f'_{S_\Omega}(0) = 1$, then the Taylor series of f_{S_Ω} about the origin is given by*

$$f_{S_\Omega}(z) = \sum_{q=0}^{\infty} b_{2q} z^{4q+1}$$

where

$$b_{2q} = (-1)^q \frac{(2q)!}{4^q (q!)^2 (4q+1)}.$$

PROOF. From Proposition 4.10 we know that the Taylor series of f_{R_Ω} about the origin is given by

$$f_{R_\Omega}(z) = \sum_{m=0}^{\infty} b_m z^{2m+1}$$

where

$$b_m = \frac{P_m(x)}{2m+1}$$

and $P_m(x)$ is the m^{th} Legendre polynomial.

As derived in the proof of Proposition 4.10 we also know this is equivalent to

$$b_m = \begin{cases} \sum_{j=\frac{m}{2}}^{m} d_{j,m-j} & for\ m\, even \\ \sum_{j=\frac{m+1}{2}}^{m} d_{j,m-j} & for\ m\, odd. \end{cases}$$

with

$$d_{j,m-j} = (-1)^{j-m} \frac{1}{2^m(2m+1)} \binom{m}{j} \binom{2j}{m} (\cos\theta)^{2j-m}.$$

By Proposition 5.2 we have linearity of $f_{S_\Omega}(z)$, and hence the square is a special case of the rectangle where $\theta/2 \to \pi/4$ which implies $\theta \to \pi/2$ and thus, $\cos\theta \to 0$. For the case m is odd, we have

$$\lim_{\cos\theta\to 0} b_m = \lim_{\cos\theta\to 0} \sum_{j=\frac{m+1}{2}}^{m} (-1)^{j-m} \frac{1}{2^m(2m+1)} \binom{m}{j} \binom{2j}{m} (\cos\theta)^{2j-m}.$$

Since m is odd, $2j \neq m$ for all j and thus every term is a product $K_{j,m-j}(\cos\theta)^{2j-m}$ where $K_{j,m-j} \in \mathbb{R}$ defined by

$$K_{j,m-j} = (-1)^{j-m} \frac{1}{2^m(2m+1)} \binom{m}{j} \binom{2j}{m}.$$

Hence,

$$\begin{aligned} \lim_{\cos\theta\to 0} b_m &= \lim_{\cos\theta\to 0} \left[\sum_{j=\frac{m+1}{2}}^{m} (-1)^{j-m} \frac{1}{2^m(2m+1)} \binom{m}{j} \binom{2j}{m} (\cos\theta)^{2j-m} \right] \\ &= \lim_{\cos\theta\to 0} \sum_{j=\frac{m+1}{2}}^{m} K_{j,m-j}(\cos\theta)^{2j-m} \to 0. \end{aligned}$$

For the case m is even we have,

$$\lim_{\cos\theta\to 0} b_m = \lim_{\cos\theta\to 0} \sum_{j=\frac{m}{2}}^{m} (-1)^{j-m} \frac{1}{2^m(2m+1)} \binom{m}{j} \binom{2j}{m} (\cos\theta)^{2j-m}.$$

Now let $m = 2q$. Then we have

$$\lim_{\cos\theta\to 0} b_{2q} = \lim_{\cos\theta\to 0} \sum_{j=q}^{2q} (-1)^{j-2q} \frac{1}{2^{2q}(2(2q)+1)} \binom{2q}{j} \binom{2j}{2q} (\cos\theta)^{2j-2q}.$$

Now we can partition the sum as the following:

$$\lim_{\cos\theta\to 0} b_{2q} = \lim_{\cos\theta\to 0}\sum_{j=q}^{q}(-1)^{j-2q}\frac{1}{2^{2q}(4q+1)}\binom{2q}{j}\binom{2j}{2q}(\cos\theta)^{2j-2q}$$
$$+\lim_{\cos\theta\to 0}\sum_{j=q+1}^{2q}(-1)^{j-2q}\frac{1}{2^{2q}(4q+1)}\binom{2q}{j}\binom{2j}{2q}(\cos\theta)^{2j-2q}.$$

Notice that the terms in the second sum contain $\cos\theta$ as a factor. Hence,

$$\lim_{\cos\theta\to 0} b_{2q} = \lim_{\cos\theta\to 0}\sum_{j=q+1}^{2q}(-1)^{j-2q}\frac{1}{2^{2q}(4q+1)}\binom{2q}{j}\binom{2j}{2q}(\cos\theta)^{2j-2q}\to 0.$$

Thus, we are left with the first sum which is the term when $j=q$,

$$\begin{aligned}&\lim_{\cos\theta\to 0}(-1)^{q-2q}\frac{1}{2^{2q}(4q+1)}\binom{2q}{q}\binom{2q}{2q}(\cos\theta)^{2q-2q}.\\ &=(-1)^{-q}\frac{1}{4^q(4q+1)}\frac{2q!}{q!(2q-q)!}\frac{2q!}{2q!(2q-2q)!}\\ &=(-1)^{-q}\frac{2q!}{4^q q!^2(4q+1)}.\end{aligned}$$

Since $(-1)^{-q}=(-1)^q$ we have

$$\lim_{\cos\theta\to 0} b_{2q} = (-1)^q\frac{2q!}{4^q q!^2(4q+1)}.$$

Hence, for all cases of m the Taylor series for the SC mapping from the unit disk onto a square is given by

$$f_{S_\Omega}(z)=\sum_{q=0}^{\infty} b_{2q}z^{2(2q)+1}=\sum_{q=0}^{\infty} b_{2q}z^{4q+1}.$$

□

Theorem 5.7. *Let S_Ω be an RD square with sides parallel to the axes. If $f_{S_\Omega}:\mathbb{D}\to S_\Omega$ is the conformal map with $f_{S_\Omega}(0)=0$ and $f'_{S_\Omega}(0)=1$, then the Taylor series of f_{S_Ω} about the origin has non-zero odd coefficients that alternate in sign.*

Proof. From Proposition 5.6, we know that

$$f_{S_\Omega}(z) = \sum_{q=0}^{\infty} b_{2q} z^{4q+1}$$

where

$$b_{2q} = (-1)^q \frac{(2q)!}{4^q (q!)^2 (4q+1)}.$$

Notice that

$$\frac{(2q)!}{4^q (q!)^2 (4q+1)} > 0.$$

Thus, the sign of b_{2q} depends on $(-1)^q$ which is alternating since q is a non-nonegative integer. Therefore the Taylor series,

$$f_{S_\Omega}(z) = \sum_{q=0}^{\infty} b_{2q} z^{4q+1}$$

has non-zero odd coefficients that alternate in sign. □

5.3 Taylor Series Representation of Regular RD Polygons

Proposition 5.8. *Let Λ_Ω be an m-sided, $m \geq 4$, regular polygon with one vertex on the real axis. If f_{Λ_Ω} is the conformal mapping from the unit disk onto Λ_Ω determined by $f_{\Lambda_\Omega}(0) = 0$ and $f'_{\Lambda_\Omega}(0) = 1$, then the prevertices are located at the roots of unity $z_k = e^{\frac{2\pi i(k-1)}{m}}$ for $k = 1, 2, ..., m$* **[27]**.

Proof. In Proposition 3.11 we determined that a vertex located on the positive real axis has a prevertex at -1 or 1. Without loss of generality suppose $f(z_1) = f(1) = w_1$ where w_1 is a vertex of the Λ_Ω that lies on the positive real axis. From Proposition 5.2 we know that $e^{\frac{2\pi i}{m}} z_k = f^{-1}_{\Lambda_\Omega}(e^{\frac{2\pi i}{m}} f_{\Lambda_\Omega}(z_k))$ where the z_k's are the prevertices on the unit disk. Notice that the vertices Λ_Ω can be written as $w_k = e^{\frac{2\pi i(k-1)}{m}} w_1$ for $k = 2, 3, ..., m$. Hence,

$$z_k = f^{-1}_{\Lambda_\Omega}(w_k) = f^{-1}_{\Lambda_\Omega}(e^{\frac{2\pi i(k-1)}{m}} w_1) = f^{-1}_{\Lambda_\Omega}(e^{\frac{2\pi i(k-1)}{m}} f_{\Lambda_\Omega}(z_1)) = e^{\frac{2\pi i(k-1)}{m}} z_1 = e^{\frac{2\pi i(k-1)}{m}}$$

for $k = 1, 2, ..., m$. □

The conformal mapping $f_{\Lambda_\Omega} : \mathbb{D} \to \Lambda_\Omega$ determined by $f_{\Lambda_\Omega}(0) = 0$ and $f'_{\Lambda_\Omega}(0) = 1$ has prevertices located at the m^{th} roots of unity and exterior angles given by $2\pi/m$ or $\beta_k = 2/m$ for $k = 1, 2, ..., m$. Therefore, simplifying the product in the integrand, we have the conformal mapping

$f_{\Lambda_\Omega}(z)$ which maps the unit disk onto Λ_Ω is given by

$$f_{\Lambda_\Omega}(z) = \int_0^z \frac{dt}{(1-t^m)^{2/m}}. \tag{5.10}$$

We now can prove the following proposition.

Proposition 5.9. *Let Λ_Ω be an m-sided, $m \geq 4$, regular RD polygon with one vertex on the real axis. If f_{Λ_Ω} is the conformal mapping of the unit disk onto Λ_Ω determined by $f_{\Lambda_\Omega}(0) = 0$ and $f'_{\Lambda_\Omega}(0) = 1$, then the Taylor series of f_{Λ_Ω} about the origin is given by*

$$f_{\Lambda_\Omega}(z) = \sum_{n=0}^{\infty} \frac{a_n}{mn+1} z^{mn+1}$$

where $a_0 = 1$, and a_n is given recursively by

$$a_n = \frac{(n-1)m+2}{mn} a_{n-1}, \quad \textit{for} \quad n \geq 1.$$

Proof. Let

$$g(t) = (1-t^m)^{-2/m}.$$

Now we will show that $g(t)$ satisfies a differential equation and develop a recursion formula for the coefficients a_n. Notice that

$$(1-t^m)\frac{d}{dt}(1-t^m)^{-2/m} = 2t^{m-1}(1-t^m)^{-2/m}. \tag{5.11}$$

So $y = g(t)$ satisfies the differential equation

$$(1-t^m)y' = 2t^{m-1}y \tag{5.12}$$

or

$$(1-t^m)y' - 2t^{m-1}y = 0. \tag{5.13}$$

Writing $g(t)$ as

$$y = \sum_{n=0}^{\infty} a_n t^{mn}$$

for the Taylor series Expansion $y = g(t)$ centered at the origin we have that

$$y' = \sum_{n=1}^{\infty} mna_n t^{mn-1}.$$

Substituting these values into equation (5.13) we have

$$\begin{aligned}
(1-t^m)\sum_{n=1}^{\infty} mna_n t^{mn-1} - 2t^{m-1}\sum_{n=0}^{\infty} a_n t^{mn} &= 0 \\
\sum_{n=1}^{\infty} mna_n t^{mn-1} - \sum_{n=1}^{\infty} mna_n t^{mn+m-1} - 2\sum_{n=0}^{\infty} a_n t^{mn+m-1} &= 0 \\
\sum_{n=0}^{\infty} m(n+1)a_{n+1} t^{mn+m-1} - \sum_{n=1}^{\infty} mna_n t^{mn+m-1} - 2\sum_{n=0}^{\infty} a_n t^{mn+m-1} &= 0. \qquad (5.14)
\end{aligned}$$

Now let

$$\begin{aligned}
h(t) = {} & [2ma_2 - mna_1 - 2a_1]t^{m-1} + \\
& \sum_{n=1}^{\infty} m(n+1)a_{n+1} t^{mn+m-1} - \sum_{n=1}^{\infty} mna_n t^{mn+m-1} - 2\sum_{n=1}^{\infty} a_n t^{mn+m-1} \\
& = [2ma_2 - mna_1 - 2a_1]t^{m-1} + \\
& \sum_{n=1}^{\infty} [m(n+1)a_{n+1} - mna_n - 2a_n] t^{mn+m-1}. \qquad (5.15)
\end{aligned}$$

Letting $C_0 = 2ma_2 - (mn+2)a_1$ and

$$C_n = m(n+1)a_{n+1} - (mn+2)a_n, \text{ for } n \geq 1$$

we can write equation (5.15) as

$$h(t) = \sum_{n=0}^{\infty} C_n t^{2n+1}.$$

From equation (5.14) we have $h(t) \equiv 0$,

$$h(t) = \sum_{n=0}^{\infty} C_n t^{2n+1} \equiv 0,$$

and $h(t)$ is analytic about the origin and is uniformly convergent on compact subsets of the unit disk, we know that all the derivatives of $h(t)$ about the origin are 0. This implies that each of the coefficients $C_n = 0$ and hence we have the following recursion formula

$$a_{n+1} = \frac{mn+2}{mn} a_n, \text{ for } n \geq 0 \qquad (5.16)$$

or

$$a_n = \frac{(n-1)m+2}{mn} a_{n-1}, \text{ for } n \geq 1. \tag{5.17}$$

Now from equation (5.10) we have

$$f_{\Lambda_\Omega}(z) = \int_0^z g(t)dt = \int_0^z \sum_{n=0}^{\infty} a_n t^{mn} dt = \sum_{n=0}^{\infty} \frac{a_n}{mn+1} z^{mn+1}$$

where $a_0 = g(0) = f'_{\Lambda_\Omega}(0) = 1$, and a_n is given recursively by

$$a_n = \frac{(n-1)m+2}{mn} a_{n-1}, \text{ for } n \geq 1. \tag{5.18}$$

□

5.4 Closed Form for Taylor Coefficients of Regular RD Polygons

Since regular polygons centered at the origin with order m (m even, $m \geq 4$) are rectangular domains, we know $f_{\Lambda_\Omega}(z)$ is an odd function whose Taylor series about the origin has real coefficients. To determine the closed form for a_n for the Taylor series expansion about the origin of $f_{\Lambda_\Omega}(z)$ we apply the binomial series expansion and observe that

$$\begin{aligned} g(t) &= (1-t^m)^{-2/m} \\ &= \sum_{n=0}^{\infty} (-1)^n (-1)^n \binom{2/m+n-1}{n} t^{mn} \\ &= \sum_{n=0}^{\infty} \binom{2/m+n-1}{n} t^{mn}. \end{aligned}$$

As a result, the Taylor series expansion for $g(t)$ is given by

$$g(t) = \sum_{n=0}^{\infty} a_n t^{mn}$$

where

$$a_n = \binom{2/m+n-1}{n}.$$

Observe that evaluating this equation for $n = 0$, we also have $a_0 = 1$.

5.5 Non-negativity of Taylor Coefficients of Regular RD Polygons

THEOREM 5.10. *Let Λ_Ω be an RD regular polygon with one vertex on the positive real axis. If f_{Λ_Ω} is the conformal mapping of the unit disk onto Λ_Ω determined by $f_{\Lambda_\Omega}(0) = 0$ and $f'_{\Lambda_\Omega}(0) = 1$, then the Taylor series of f_{Λ_Ω} about the origin has non-negative odd coefficients.*

PROOF. Now by equation (5.18) we have the recursion equation

$$a_n = \frac{(n-1)m+2}{mn} a_{n-1} \quad \text{for} \quad n \geq 1.$$

From Proposition 5.9 we know that

$$a_0 = 1 > 0.$$

Clearly, $a_1 > 0$ since $a_0 > 0$ and $m > 0$ and $n \geq 1$. Now assume $a_{n-1} > 0$ and $a_n > 0$. Notice that

$$a_{n+1} = \frac{nm+2}{m(n+1)} a_n.$$

Since $n \geq 1$ and $a_n > 0$, we have $a_{n+1} > 0$. Hence, the Taylor series of f_{Λ_Ω} about the origin has non-negative odd coefficients. □

CHAPTER 6

The Schwarz-Christoffel Mapping from $\mathbb{D}$ onto RD Octagons

6.1 Equilateral RD Octagons

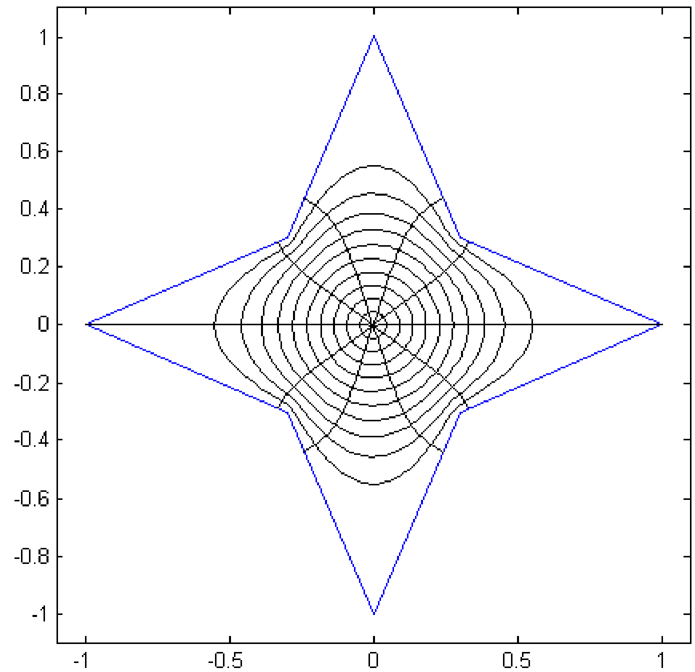

FIGURE 6.25. An equilateral RD octagon with four vertices on the axes.

6.1.1 Taylor Series Representation of Equilateral RD Octagons

Let $O_{\star_\Omega}$ be an equilateral RD octagon with four vertices on the axes as shown in Figure 6.25. Since the sides of $O_{\star_\Omega}$ are equal, the prevertices are the 8^{th} roots of unity. Hence, simplifying the product of the integrand we have the following conformal mapping. The conformal mapping $f_{O_{\star_\Omega}}$ from $\mathbb{D}$ onto $O_{\star_\Omega}$ determined by $f_{O_{\star_\Omega}}(0) = 0$ and $f'_{O_{\star_\Omega}}(0) = 1$ is given by

$$f_{O_{\star_\Omega}}(z) = \int_0^z (1 - t^4)^{-\beta_1}(1 + t^4)^{\beta_1 - 1/2} dt.$$

Proposition 6.1. *Let $O_{\star_\Omega}$ be an equilateral RD octagon with four vertices on the axes. If $f_{O_{\star_\Omega}}$ is the conformal mapping from $\mathbb{D}$ onto $O_{\star_\Omega}$ determined by $f_{O_{\star_\Omega}}(0) = 0$ and $f'_{O_{\star_\Omega}}(0) = 1$, then the Taylor series of $f_{O_{\star_\Omega}}$ about the origin is given by*

$$f_{O_{\star_\Omega}}(z) = \sum_{n=0}^{\infty} \frac{a_n}{4n+1} z^{4n+1}$$

where $a_0 = 1$, $a_1 = 2\beta_1 - 1/2$ and a_n is given recursively by

$$a_n = \frac{(4\beta_1 - 1)a_{n-1} + (2n-3)a_{n-2}}{2n}, \quad n \geq 2.$$

Proof. Let

$$g(t) = (1 - t^4)^{-\beta_1}(1 + t^4)^{\beta_1 - 1/2}.$$

Now we will show that $g(t)$ satisfies a differential equation and develop a recursion formula for the coefficients a_n. Notice that

$$(1 - t^4)(1 + t^4)\frac{d}{dt}(1 - t^4)^{-\beta_1}(1 + t^4)^{\beta_1 - 1/2} = [(2\beta_1 - 1/2 + (1/2)t^4)(4t^3)](1 - t^4)^{-\beta_1}(1 + t^4)^{\beta_1 - 1/2}. \quad (6.1)$$

So $y = g(t)$ satisfies the differential equation

$$(1 - t^8)y' = [(8\beta_1 - 2)t^3 + 2t^7]y \quad (6.2)$$

or

$$(1 - t^8)y' - [(8\beta_1 - 2)t^3 + 2t^7]y = 0. \quad (6.3)$$

Writing $g(t)$ as

$$y = \sum_{n=0}^{\infty} a_n t^{4n}$$

for the Taylor series Expansion $y = g(t)$ centered at the origin we have that

$$y' = \sum_{n=1}^{\infty} 4na_n t^{4n-1}.$$

Substituting these values into equation (6.3) we have

$$(1 - t^6)\sum_{n=1}^{\infty} 4na_n t^{4n-1} - [(8\beta_1 - 2)t^3 + 2t^7]\sum_{n=0}^{\infty} a_n t^{4n} = 0$$

$$\sum_{n=1}^{\infty} 4na_n t^{4n-1} - \sum_{n=1}^{\infty} 4na_n t^{4n+7} - (8\beta_1 - 2)\sum_{n=0}^{\infty} a_n t^{4n+3} - 2\sum_{n=0}^{\infty} a_n t^{4n+7} = 0$$

$$2\left(\sum_{n=1}^{\infty} 2na_n t^{4n-1} - \sum_{n=1}^{\infty} 2na_n t^{4n+7}\right) - 2\left((4\beta_1 - 1)\sum_{n=0}^{\infty} a_n t^{4n+3} - \sum_{n=0}^{\infty} a_n t^{4n+7}\right) = 0$$

$$\sum_{n=1}^{\infty} 2na_n t^{4n-1} - \sum_{n=1}^{\infty} 2na_n t^{4n+7} - (4\beta_1 - 1)\sum_{n=0}^{\infty} a_n t^{4n+3} - \sum_{n=0}^{\infty} a_n t^{4n+7} = 0$$

$$\sum_{n=0}^{\infty} 2(n+1)a_{n+1} t^{4n+3} - \sum_{n=2}^{\infty} 2(n-1)a_{n-1} t^{4n+3} - (4\beta_1 - 1)\sum_{n=0}^{\infty} a_n t^{4n+3} - \sum_{n=1}^{\infty} a_{n-1} t^{4n+3} = 0. \tag{6.4}$$

Now let

$$\begin{aligned} h(t) &= [2a_1 - (4\beta_1 - 1)a_0]t^3 + [4a_2 - (4\beta_1 - 1)a_1 - a_0]t^7 + \\ &+ \sum_{n=2}^{\infty} 2(n+1)a_{n+1} t^{4n+3} - \sum_{n=2}^{\infty} 2(n-1)a_{n-1} t^{4n+3} - (4\beta_1 - 1)\sum_{n=2}^{\infty} a_n t^{4n+3} - \sum_{n=2}^{\infty} a_{n-1} t^{4n+3} \\ &= [2a_1 - (4\beta_1 - 1)a_0]t^3 + [4a_2 - (4\beta_1 - 1)a_1 - a_0]t^7 + \\ &+ \sum_{n=2}^{\infty} [2(n+1)a_{n+1} - 2(n-1)a_{n-1} - (4\beta_1 - 1)a_n - a_{n-1}]t^{4n+3}. \end{aligned} \tag{6.5}$$

Letting $C_0 = 2a_1 - (4\beta_1 - 1)a_0$, $C_1 = 4a_2 - (4\beta_1 - 1)a_1 - a_0$, and

$$C_n = 2(n+1)a_{n+1} - (4\beta_1 - 1)a_n - (2n-1)a_{n-1} \text{ for } n \geq 2$$

we can write equation (6.5) as

$$h(t) = \sum_{n=0}^{\infty} C_n t^{2n+1}.$$

From equation (6.4) we have $h(t) \equiv 0$,

$$h(t) = \sum_{n=0}^{\infty} C_n t^{2n+1} \equiv 0,$$

and $h(t)$ is analytic about the origin and is uniformly convergent on compact subsets of the unit disk, we know that all the derivatives of $h(t)$ about the origin are 0. This implies that each of the coefficients $C_n = 0$ and hence we have the following recursion formula

$$a_{n+1} = \frac{(4\beta_1 - 1)a_n + (2n-1)a_{n-1}}{2(n+1)}, \text{ for } n \geq 1$$

or

$$a_n = \frac{(4\beta_1 - 1)a_{n-1} + (2n-3)a_{n-2}}{2n}, \text{ for } n \geq 2. \tag{6.6}$$

Now from equation (5.10) we have

$$f_{O_{\star_\Omega}}(z) = \int_0^z g(t)dt = \int_0^z \sum_{n=0}^{\infty} a_n t^{4n} dt = \sum_{n=0}^{\infty} \frac{a_n}{4n+1} z^{4n+1}$$

where $a_0 = g(0) = 1$, $a_1 = g''(0) = 2\beta_1 - 1/2$, and a_n is given recursively by

$$a_n = \frac{(4\beta_1 - 1)a_{n-1} + (2n-3)a_{n-2}}{2n}, \text{ for } n \geq 2. \tag{6.7}$$

$\square$

6.1.2 Closed Form for Taylor Coefficients of Equilateral RD Octagons

To determine the closed form for a_n for the Taylor series expansion about the origin of $f_{O_{\star_\Omega}}(z)$ we apply the binomial series expansion and observe that

$$\begin{aligned} g(t) &= (1-t^4)^{-\beta_1}(1+t^4)^{\beta_1 - 1/2} \\ &= (1+t^4)^{\beta_1 - 1/2} \sum_{k=0}^{\infty} (-1)^k \binom{-\beta_1}{k} t^{4k} \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} (-1)^k \binom{-\beta_1}{k} \binom{\beta_1 - 1/2}{n-k} t^{4n}. \end{aligned}$$

Thus, the Taylor series for $g(t)$ is given by

$$g(t) = \sum_{n=0}^{\infty} a_n t^{4n}$$

where

$$a_n = \sum_{k=0}^{n} (-1)^k \binom{-\beta_1}{k} \binom{\beta_1 - 1/2}{n-k}.$$

Notice that evaluating this equation for $n = 0$ and $n = 1$, we also have $a_0 = 1$ and $a_1 = 2\beta_1 - 1/2$.

6.1.3 Non-negativity of Taylor Coefficients of Equilateral RD Octagons

Theorem 6.2. *Let $O_{\star_\Omega}$ be an equilateral RD octagon with four vertices on the axes. If $f_{O_{\star_\Omega}}$ is the conformal mapping from the unit disk onto $O_{\star_\Omega}$ determined by $f_{O_{\star_\Omega}}(0) = 0$ and $f'_{O_{\star_\Omega}}(0) = 1$ with $\beta_1 \geq 1/4$, then the Taylor series of $f_{O_{\star_\Omega}}$ about the origin has non-negative odd coefficients.*

Proof. Now by equation (6.7) we have the recursion equation

$$a_n = \frac{(4\beta_1 - 1)a_{n-1} + (2n-3)a_{n-2}}{2n}, \text{ for } n \geq 2.$$

From the closed form of the Taylor series coefficients we know that

$$a_0 = 1 > 0$$

and

$$a_1 = 2\beta_1 - 1/2 \geq 0 \quad \text{if} \quad \beta_1 \geq 1/4.$$

If $\beta_1 < 1/4$, then $a_1 < 0$ and the second non-zero term of the Taylor series about the origin is negative. Clearly, $a_1 \geq 0$ if $\beta_1 > 1/4$, $a_0 = 1 > 0$ and $n \geq 2$. Now assume $a_{n-1} > 0$ and $a_n \geq 0$. Notice that

$$a_{n+1} = \frac{(4\beta_1 - 1)a_n + 2(n-1)a_{n-1}}{2n+1}, \text{ for } n \geq 2.$$

Since $n \geq 2$ and $a_n \geq 0$, we have $a_{n+1} \geq 0$. Hence, the Taylor series of $f_{O_{\star\Omega}}$ about the origin has non-negative odd coefficients. $\square$

Geometrically, $f_{O_{\star\Omega}}$ has non-negative Taylor coefficients until the vertices not on the axes pass the boundary of the regular octagon or boundary of the unit circle, and then in this case, it has at least one negative coefficient. See the following Figures 6.26, 6.27, 6.28 produced by SC Toolbox for Matlab.

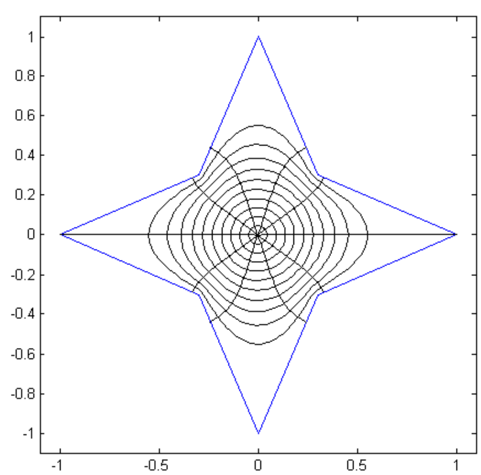

Figure 6.26. An equilateral RD octagon with $\beta_1 = .744224$. The Taylor series expansion about the origin of the conformal map from the unit disk onto this domain has non-negative coefficients.

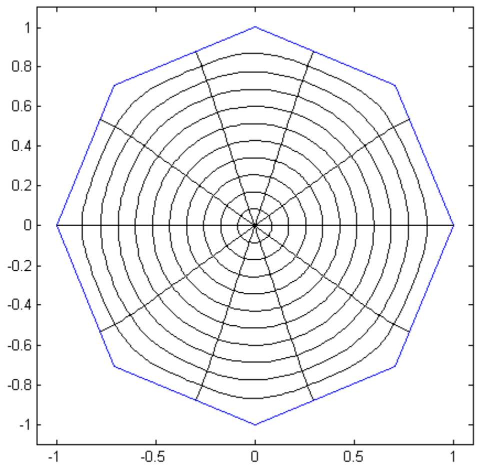

FIGURE 6.27. An equilateral RD octagon with $\beta_1 = 1/4$. The Taylor series expansion about the origin of the conformal map from the unit disk onto this domain has non-negative coefficients.

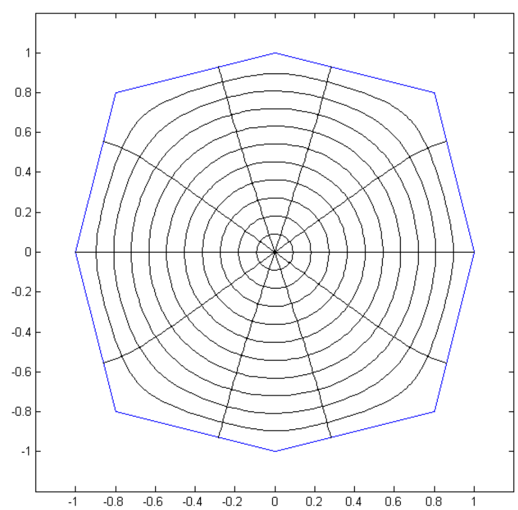

FIGURE 6.28. An equilateral RD octagon with $\beta_1 = .1559582$ which is between 0 and 1/4. The Taylor series expansion about the origin of the conformal map from the unit disk onto this domain has at least one negative coefficient.

6.2 Non-equilateral RD Octagons

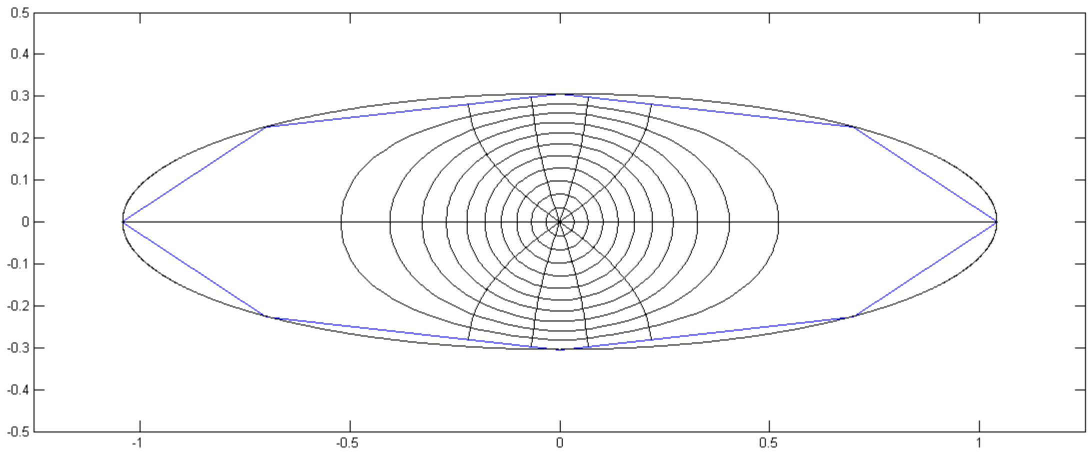

FIGURE 6.29. An RD octagon with foci at ±1 with $\xi = .5$.

6.2.1 Taylor Series Representation of RD Octagons

PROPOSITION 6.3. *The conformal mapping $f_{O_{\mathcal{E}_\Omega}}$ from unit disk onto an RD octagon with foci at* ±1 *and with four vertices on the axes determined by $f_{O_{\mathcal{E}_\Omega}}(0) = 0$ and $f'_{O_{\mathcal{E}_\Omega}}(0) = 1$ is given by*

$$f_{O_{\mathcal{E}_\Omega}}(z) = \int_0^z (1-t^2)^{-\beta_1}(1-2\cos\theta t^2 + t^4)^{-\beta_2}(1+t^2)^{-\beta_3}dt \tag{6.8}$$

where $2\beta_1 + 4\beta_2 + 2\beta_3 = 2$.

PROOF. Since RD octagons with foci at ±1 have vertices on the real axis with turning angle coefficient β_1 with prevertices -1 and 1, the product in the integrand of the conformal mapping $f_{O_{\mathcal{E}_\Omega}}(z)$ for these two vertices is given by

$$(1-\frac{t}{1})^{-\beta_1}(1-\frac{t}{-1})^{-\beta_1} = (1-t^2)^{-\beta_1}.$$

Since RD octagons with foci at ±1 also have vertices located on the imaginary axis with turning angle coefficient β_3 and prevertices $-i$ and i, the product in the integrand of the conformal mapping $f_{O_{\mathcal{E}_\Omega}}(z)$ for these two vertices is given by

$$(1-\frac{t}{i})^{-\beta_3}(1-\frac{t}{-i})^{-\beta_3} = (1+t^2)^{-\beta_3}.$$

Moreover, notice that RD octagons have two-fold symmetry and hence four vertices, w_i, $\bar{w}_i$,$-w_i$, and -$\bar{w}_i$

$$
\begin{aligned}
&[(1-\frac{t}{e^{\frac{i\theta}{2}}})(1-\frac{t}{-e^{-\frac{i\theta}{2}}})(1-\frac{t}{-e^{\frac{i\theta}{2}}})(1-\frac{t}{e^{-\frac{i\theta}{2}}})]^{-\beta_2}\\
&=[(1-\left(\frac{t}{e^{\frac{i\theta}{2}}}\right)^2)(1-(e^{\frac{i\theta}{2}}t)^2)]^{-\beta_2}\\
&=[(1-e^{-i\theta}t^2)(1-e^{i\theta}t^2)]^{-\beta_2}\\
&=(1-(e^{-i\theta}+e^{i\theta})t^2+t^4)^{-\beta_2}\\
&=(1-2\cos\theta t^2+t^4)^{-\beta_2}.
\end{aligned}
\tag{6.9}
$$

where $\theta \in (0,\pi)$. Since the sum of the turning angle coefficients for a polygon is 2 for the Schwarz-Christoffel mapping then it follows that $2\beta_1 + 4\beta_2 + 2\beta_3 = 2$. Thus, the conformal mapping $f_{O_{\mathcal{E}_\Omega}}(z)$ from $\mathbb{D}$ onto an RD octagon with four vertices on the axes is given by

$$f_{O_{\mathcal{E}_\Omega}}(z) = \int_0^z (1-t^2)^{-\beta_1}(1-2cos\theta t^2+t^4)^{-\beta_2}(1+t^2)^{-\beta_3}dt.$$

□

Letting $g(t)$ be the integrand of equation (6.8) with

$$f_1(t) = (1-t^2)$$

$$f_2(t) = (1-2\cos\theta t^2+t^4)$$

and

$$f_3(t) = (1+t^2)$$

we have

$$g(t) = \prod_{i=1}^{3} f_i(t)^{-\beta_i}.$$

Now taking the derivative of a product of more than two factors we have

$$\frac{d}{dt}g(t) = g(t)\sum_{i=1}^{3} -\beta_i\frac{f_i'(t)}{f_i(t)} = \frac{g(t)}{\prod_{i=1}^{3} f_i(t)}\sum_{i=1}^{3}\left(-\beta_i f_i'(t)\prod_{j=1,j\neq i}^{3} f_j(t)\right).$$

Thus,

$$\left(\prod_{i=1}^{3} f_i(t)\right)\frac{d}{dt}g(t) - \left(\sum_{i=1}^{3} \beta_i f_i'(t) \prod_{j=1,j\neq i}^{3} f_j(t)\right)g(t) = 0.$$

Letting $y = g(t)$ we have the general differential equation for which the conformal mapping (6.8) is a solution,

$$\left(\prod_{i=1}^{3} f_i(t)\right)y' + \left(\sum_{i=1}^{3} \beta_i f_i'(t) \prod_{j=1,j\neq i}^{3} f_j(t)\right)y = 0. \tag{6.10}$$

Substituting $f_1(t)$, $f_2(t)$, $f_3(t)$ into (6.8) we have

$$[(1-t^2)(1-2\cos\theta t^2+t^4)(1+t^2)]y' + \Big[\beta_1(-2t)(1-2\cos\theta t^2+t^4)(1+t^2)$$
$$+\beta_2(-4\cos\theta t+4t^3)(1-t^2)(1+t^2)+\beta_3(2t)(1-t^2)(1-2\cos\theta t^2+t^4)\Big]y = 0.$$

Letting $x = \cos\theta$ and combining like terms we have

$$(1-2xt^2+2xt^6-t^8)y' + 2\Big[(-\beta_1-2\beta_2 x+\beta_3)t$$
$$+(2\beta_1 x-\beta_1+2\beta_2-\beta_3-2\beta_3 x)t^3+(2\beta_1 x-\beta_1+2\beta_2 x+\beta_3+2\beta_3 x)t^5$$
$$+(-\beta_1-2\beta_2-\beta_3)t^7\Big]y = 0.$$

Now factoring out a negative one from the coefficient of y we have

$$(1-2xt^2+2xt^6-t^8)y' - 2\Big[(\beta_1+2\beta_2 x-\beta_3)t$$
$$+(\beta_1-2\beta_1 x-2\beta_2+\beta_3+2\beta_3 x)t^3+(\beta_1-2\beta_1 x-2\beta_2 x-\beta_3-2\beta_3 x)t^5$$
$$+(\beta_1+2\beta_2+\beta_3)t^7\Big]y = 0.$$

Since $2\beta_1+4\beta_2+2\beta_3 = 2$ we have

$$\beta_2 = \frac{1-\beta_1-\beta_3}{2}.$$

Substituting in for β_2 we have

$$(1-2xt^2+2xt^6-t^8)y' + 2\Big[(\beta_1-\beta_1 x-\beta_3-\beta_3 x+x)t$$
$$+(2\beta_1-2\beta_1 x+2\beta_3+2\beta_3 x-1)t^3+(\beta_1-\beta_1 x-\beta_3-\beta_3 x-x)t^5+t^7\Big]y = 0.$$

Now let

$b = \beta_1 - \beta_1 x - \beta_3 - \beta_3 x + x$

$c = 2\beta_1 - 2\beta_1 x + 2\beta_3 + 2\beta_3 x - 1$

$d = \beta_1 - \beta_1 x - \beta_3 - \beta_3 x - x.$

Thus,

$$(1 - 2xt^2 + 2xt^6 - t^8)y' - 2(bt + ct^3 + dt^5 + t^7)y = 0. \tag{6.11}$$

Writing $g(t)$ as

$$y = \sum_{n=0}^{\infty} a_n t^{2n}$$

for the Taylor series Expansion $y = g(t)$ centered at the origin we have that

$$y' = \sum_{n=1}^{\infty} 2na_n t^{2n-1}.$$

Substituting these values into equation (6.11) we have

$$(1 - 2xt^2 + 2xt^6 - t^8)\sum_{n=1}^{\infty} 2na_n t^{2n-1} - 2(bt + ct^3 + dt^5 + t^7)\sum_{n=0}^{\infty} a_n t^{2n} = 0$$

$$\sum_{n=1}^{\infty} 2na_n t^{2n-1} - 2x\sum_{n=1}^{\infty} 2na_n t^{2n+1} + 2x\sum_{n=1}^{\infty} 2na_n t^{2n+3} - \sum_{n=1}^{\infty} 2na_n t^{2n+7}$$
$$-2\left(\sum_{n=0}^{\infty} ba_n t^{2n+1} + \sum_{n=0}^{\infty} ca_n t^{2n+3} + \sum_{n=0}^{\infty} da_n t^{2n+3} + \sum_{n=0}^{\infty} a_n t^{2n+7}\right) = 0$$

$$2\left(\sum_{n=1}^{\infty} na_n t^{2n-1} - 2x\sum_{n=1}^{\infty} na_n t^{2n+1} + 2x\sum_{n=1}^{\infty} na_n t^{2n+3} - \sum_{n=1}^{\infty} na_n t^{2n+7}\right)$$
$$-2\left(\sum_{n=0}^{\infty} ba_n t^{2n+1} + \sum_{n=0}^{\infty} ca_n t^{2n+3} + \sum_{n=0}^{\infty} da_n t^{2n+3} + \sum_{n=0}^{\infty} a_n t^{2n+7}\right) = 0$$

$$\left(\sum_{n=1}^{\infty} na_n t^{2n-1} - 2x\sum_{n=1}^{\infty} na_n t^{2n+1} + 2x\sum_{n=1}^{\infty} na_n t^{2n+3} - \sum_{n=1}^{\infty} na_n t^{2n+7}\right)$$
$$-\left(\sum_{n=0}^{\infty} ba_n t^{2n+1} + \sum_{n=0}^{\infty} ca_n t^{2n+3} + \sum_{n=0}^{\infty} da_n t^{2n+3} + \sum_{n=0}^{\infty} a_n t^{2n+7}\right) = 0$$

$$\sum_{n=0}^{\infty}(n+1)a_{n+1}t^{2n+1} - 2x\sum_{n=1}^{\infty} na_n t^{2n+1} + 2x\sum_{n=3}^{\infty}(n-2)a_{n-2}t^{2n+1} - \sum_{n=4}^{\infty}(n-3)a_{n-3}t^{2n+1}$$

$$-\sum_{n=0}^{\infty} ba_n t^{2n+1} - \sum_{n=1}^{\infty} ca_{n-1} t^{2n+1} - \sum_{n=2}^{\infty} da_{n-2} t^{2n+1} + \sum_{n=3}^{\infty} a_{n-3} t^{2n+1} = 0$$

$$\sum_{n=0}^{\infty}(n+1)a_{n+1}t^{2n+1} - \sum_{n=0}^{\infty} ba_n t^{2n+1} - 2x\sum_{n=1}^{\infty} na_n t^{2n+1} - \sum_{n=1}^{\infty} ca_{n-1}t^{2n+1} - \sum_{n=2}^{\infty} da_{n-2}t^{2n+1}$$
$$+ 2x\sum_{n=3}^{\infty}(n-2)a_{n-2}t^{2n+1} + \sum_{n=3}^{\infty} a_{n-3}t^{2n+1} - \sum_{n=4}^{\infty}(n-3)a_{n-3}t^{2n+1} = 0. \tag{6.12}$$

Now let

$$\begin{aligned}
h(t) &= (a_1 - ba_0)t + (2a_2 - ba_1 - 2xa_1 - ca_0)t^3 + (3a_3 - ba_2 - 4xa_2 - ca_1 - da_0)t^5 \\
&+ (4a_4 - ba_3 - 6xa_3 - ca_2 - da_1 + 2xa_1 + a_0)t^7 \\
&+ \sum_{n=4}^{\infty}(n+1)a_{n+1}t^{2n+1} - \sum_{n=4}^{\infty} ba_n t^{2n+1} - 2x\sum_{n=4}^{\infty} na_n t^{2n+1} - \sum_{n=4}^{\infty} ca_{n-1}t^{2n+1} - \sum_{n=4}^{\infty} da_{n-2}t^{2n+1} \\
&+ 2x\sum_{n=4}^{\infty}(n-2)a_{n-2}t^{2n+1} + \sum_{n=4}^{\infty} a_{n-3}t^{2n+1} - \sum_{n=4}^{\infty}(n-3)a_{n-3}t^{2n+1} = \\
&= (a_1 - ba_0)t + (2a_2 - ba_1 - 2xa_1 - ca_0)t^3 + (3a_3 - ba_2 - 4xa_2 - ca_1 - da_0)t^5 \\
&+ (4a_4 - ba_3 - 6xa_3 - ca_2 - da_1 + 2xa_1 + a_0)t^7 \\
&+ \sum_{n=4}^{\infty}\Big[(n+1)a_{n+1} - (2xn+b)a_n - ca_{n-1} - \big[d - 2x(n-2)\big]a_{n-2} - (n-2)a_{n-3}\Big]t^{2n+1}.
\end{aligned} \tag{6.13}$$

Letting

$$C_0 = a_1 - ba_0$$

$$C_1 = 2a_2 - ba_1 - 2xa_1 - ca_0$$

$$C_2 = 3a_3 - ba_2 - 4xa_2 - ca_1 - da_0$$

$$C_3 = 4a_4 - ba_3 - 6xa_3 - ca_2 - da_1 + 2xa_1 + a_0$$

and

$$C_n = \Big[(n+1)a_{n+1} - (2xn+b)a_n - ca_{n-1} - \big[d - 2x(n-2)\big]a_{n-2} - (n-2)a_{n-3}\Big] \text{ for } n \geq 3$$

we can write equation (6.13) as

$$h(t) = \sum_{n=0}^{\infty} C_n t^{2n+1}.$$

Since from equation (6.12) we have $h(t) \equiv 0$,

$$h(t) = \sum_{n=0}^{\infty} C_n t^{2n+1} \equiv 0,$$

and $h(t)$ is analytic about the origin and is uniformly convergent on compact subsets of the unit disk, we know that all the derivatives of $h(t)$ about the origin are 0. This implies that each of the coefficients $C_n = 0$ and hence we have the following recursion formula

$$a_{n+1} = \frac{A(n)a_n + Ba_{n-1} + C(n)a_{n-2} + D(n)a_{n-3}}{n+1}, \quad \text{for } n \geq 3 \tag{6.14}$$

where

$A(n) = 2xn + b = 2xn + \beta_1 - \beta_1 x - \beta_3 - \beta_3 x + x$

$B = c = 2\beta_1 - 2\beta_1 x + 2\beta_3 + 2\beta_3 x - 1$

$C(n) = d - 2x(n-2) = \beta_1 - \beta_1 x - \beta_3 - \beta_3 x - x - 2x(n-2)$

and

$D(n) = n - 2$

which can be written in the form

$$a_n = \frac{A(n-1)a_{n-1} + Ba_{n-2} + C(n-1)a_{n-3} + D(n-1)a_{n-4}}{n}, \ \text{for } n \geq 4 \tag{6.15}$$

where $a_0 = g(0) = 1$,

$a_1 = g''(0) = A(0) = \beta_1 - \beta_1 x - \beta_3 - \beta_3 x + x$,

$a_2 = g^{(4)}(0) = A(1) \cdot a_1 + B \cdot a_0 = A(1) \cdot A(0) + B$,

$a_3 = g^{(6)}(0) = A(2) \cdot a_2 + B \cdot a_1 + C(0)$.

Now from equation (6.8) we have the Taylor series representation

$$f_{O_{\mathcal{E}_\Omega}}(z) = \int_0^z g(t)dt = \int_0^z \sum_{n=0}^{\infty} a_n t^{2n} dt = \sum_{n=0}^{\infty} \frac{a_n}{2n+1} z^{2n+1}$$

where a_n is defined above.

6.2.2 Closed Form for Taylor Coefficients of RD Octagons

Let $g(t)$ be the integrand of the equation (6.8). Then we can write $g(t)$ as

$$g(t) = g_1(t) \cdot g_2(t)$$

where

$$g_1(t) = (1+t^2)^{-\beta_3}(1-t^2)^{-\beta_1}$$

and

$$g_2(t) = (1 - 2\cos\theta t^2 + t^4)^{-\beta_2}.$$

Let c_n be the n^{th} Taylor coefficient of $g(t)$. To determine the closed form for c_n we apply the binomial series expansion to $g_1(t)$ and $g_2(t)$ and write c_n as the product of these two Taylor series. Deriving the closed form for the Taylor series of $g_1(t)$ we have

$$\begin{aligned} g_1(t) &= (1+t^2)^{-\beta_3}(1-t^2)^{-\beta_1} \\ &= (1+t^2)^{-\beta_3}\sum_{k=0}^{\infty}(-1)^k\binom{-\beta_1}{k}t^{2k} \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}(-1)^k\binom{-\beta_1}{k}\binom{-\beta_3}{n-k}t^{2n}. \end{aligned}$$

Thus, the Taylor series of $g_1(t)$ is given by

$$g_1(t) = \sum_{n=0}^{\infty} a_n t^{2n}$$

where

$$a_n = \sum_{k=0}^{n}(-1)^k\binom{-\beta_1}{k}\binom{-\beta_3}{n-k}.$$

Evaluating this equation for $n = 0$ and $n = 1$, we have $a_0 = 1$ and $a_1 = \beta_1 - \beta_3$.

To determine the closed form for $g(t)$ we use the Gegenbauer polynomials, denoted by $G_n(x)$. From [**8**] we know that Gegenbauer polynomials are orthogonal polynomials (see section 4.1.1) that are generated by

$$\frac{1}{(1 - 2\cos\theta t + t^2)^{\lambda}} = \sum_{n=0}^{\infty} G_n(\cos\theta)t^n.$$

Letting $x = \cos\theta$, $\beta_2 = \lambda$ and substituting in t^2 for t we have

$$\frac{1}{(1-2xt^2+t^4)^{\beta_2}} = \sum_{n=0}^{\infty} G_n(x)t^{2n}$$

The explicit form of the Gegenbauer polynomials is given by

$$G_n(x) = \sum_{k=0}^{[n/2]} (-1)^k \frac{(\beta_2)_{n-k}}{k!(n-2k)!} x^{n-2k}$$

where $(\beta_2)_{n-k} = \beta_2(\beta_2+1)(\beta_2+2)\cdots(\beta_2+n-k+1)$ and $[n/2]$ is equal to $n/2$ if n is even and $(n+1)/2$ if n is odd. Thus, the closed form for the Taylor series of $g_2(t)$ is given by

$$g_2(t) = \sum_{n=0}^{\infty} b_n t^{2n} = \sum_{n=0}^{\infty}\sum_{k=0}^{[n/2]} (-1)^k \frac{(\beta_2)_{n-k}}{k!(n-2k)!} x^{n-2k} t^{2n}.$$

where the coefficients are defined by Gegenbauer polynomials

$$b_n = G_n(x) = \sum_{k=0}^{[n/2]} (-1)^k \frac{(\beta_2)_{n-k}}{k!(n-2k)!} x^{n-2k}.$$

Since $g(t) = g_1(t)\cdot g_2(t)$, the closed form for the Taylor series of $g(t)$ is the product of the two Taylor series of $g_1(t)$ and $g_2(t)$. Thus we have,

$$\begin{aligned} g(t) = \sum_{n=0}^{\infty} c_n t^{2n} &= \sum_{n=0}^{\infty} a_n t^{2n} \sum_{\ell=0}^{\infty} b_\ell t^{2\ell} \\ &= \left(\sum_{n=0}^{\infty}\sum_{k=0}^{n} (-1)^k \binom{-\beta_1}{k}\binom{-\beta_3}{n-k} t^{2n}\right)\left(\sum_{\ell=0}^{\infty}\sum_{k=0}^{[\ell/2]} (-1)^k \frac{(\beta_2)_{\ell-k}}{k!(\ell-2k)!} x^{\ell-2k} t^{2\ell}\right). \end{aligned}$$

6.2.3 Some Observations and Partial Results of RD Octagons

Proposition 6.4. *If $A(0) < 0$ in the recursion formula* (6.15) *then the Taylor series expansion about the origin of $f_{O_{\mathcal{E}_\Omega}}$ has at least one negative coefficient.*

Proof. The initial conditions of the recursion formula (6.15) yield

$$a_0 = g(0) = 1,$$

$$a_1 = g''(0) = A(0) \cdot a_0 = A(0).$$

If $A(0) < 0$ then $a_1 < 0$. Thus, it follows that the second non-zero coefficient of the Taylor series given by will be negative. $\square$

We will determine whether the conformal mapping, $f_{O_{\mathcal{E}_\Omega}}$, from the unit disk onto an RD octagon with foci at ± 1 determined by $f_{O_{\mathcal{E}_\Omega}}(0) = 0$ and $f'_{O_{\mathcal{E}_\Omega}}(0) > 0$ has a Taylor series representation about the origin with all odd positive coefficients for $\xi > 0$ or determine if there exist at least one $\xi > 0$ such that the Taylor representation has at least one negative coefficient.

Because the sides of an RD octagon with foci at ± 1 are not all equal, the prevertices on the unit disk depend nonlinearly on the ratio of the sides of the elliptic RD octagon which is known as the Schwarz-Christoffel parameter problem mentioned in Section 2.9. Unlike the case of equilateral RD octagons where the ratio of the sides are one and the parameter problem can be solved explicitly because of the linearity of the mapping between prevertices and vertices, we need to use numerial methods to estimate the prevertices of RD octagons with foci at ± 1. The SC Toolbox for MATLAB by Driscoll produces estimates of the location of the prevertices with good precision by solving a system of nonlinear equations numerically (see [**6**]). It should be noted that since MATLAB works in 16-digit precision, values for the prevertices are entered using 16-digit precision. Also an accuracy mapping output value of e^{-9} to e^{-12} indicates our parameter estimates have good precision.

Theorem 6.5. *Let $O_{\mathcal{E}_\Omega}$ be an RD octagon with foci at ± 1 and with four vertices on the axes. If $f_{O_{\mathcal{E}_\Omega}}$ is the conformal mapping from the unit disk onto an $O_{\mathcal{E}_\Omega}$ determined by $f_{O_{\mathcal{E}_\Omega}}(0) = 0$ and $f'_{O_{\mathcal{E}_\Omega}}(0) > 0$, then there exists a $\xi > 0$ such that the Taylor series of $f_{O_{\mathcal{E}_\Omega}}$ about the origin has at least one negative odd coefficient.*

Proof. The vertices of the RD octagon with foci at ± 1 are located at $(\pm\cosh\xi, 0)$, $(0, \pm\sinh\xi)$ and $(\pm u, \pm v)$ where $0 < u < \cosh\xi$ and $v = \sqrt{\sinh^2\xi - u^2\tanh^2\xi}$. Thus letting $\xi = 3$ and $u = .7$ we get the RD octagon with vertices at $(\pm 10.06766199577777, 0)$, $(0, \pm 10.01787492740990)$, $(\pm 5, \pm 8.695083035277119)$. See Figure 6.30.

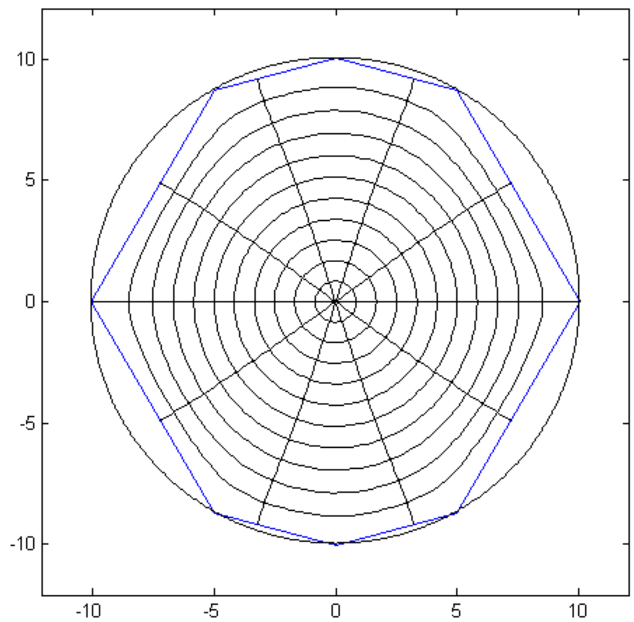

Figure 6.30. RD octagon with foci at ± 1 with $\xi = 3$ and $u = .7$.

For this mapping, SC Toolbox produces the values $x = \cos\theta = -0.6135451086590198$, $\beta_1 = .3359384077585179$, $\beta_3 = .1646509081421967$ with $C = 9.246773814211608 > 0$ and $A = 0$. Since $A(0) = \beta_1 - \beta_1 x - \beta_3 - \beta_3 x + x$ we have

$$\begin{aligned} A(0) &= .3359384077585179 - (.3359384077585179)(-0.6135451086590198) - .1646509081421967 \\ &\quad - .1646509081421967(-0.6135451086590198) + (-0.6135451086590198) \\ &= -.3359384077585179 < 0. \end{aligned}$$

Since $C > 0$, it will not affect the signs of the odd Taylor coefficients and thus by Proposition 6.4, $f_{O_{\mathcal{E}_\Omega}}$ has at least one negative odd Taylor coefficient. $\square$

Notice that by "perturbing" the vertices of the regular RD polygon we were able to elicit a negative odd coefficient.

CHAPTER 7

Undergraduate Research Experience: Regular Polygons Revisited

In Chapter 5 we investigated the conformal mapping f_{Λ_Ω} from the unit disk onto m-sided regular RD polygons with one vertex on the positive real axis determined by $f_{\Lambda_\Omega}(0) = 0$ and $f'_{\Lambda_\Omega}(0) > 0$ and discovered that the Taylor series representation about the origin of this mapping had non-negative odd coefficients.

In the summer of 2012, I directed a senior mathematics education major at UAFS in her undergraduate research for her senior project requirement. Investigating the effect of symmetry on the Riemann map involves concepts familiar to undergraduate students: symmetry, the binomial theorem, Taylor series, and the use of computer packages such as Matlab and Mathematica. Thus, this research topic exposes students to a higher level investigation of these concepts and provides numerous branches of investigation that are conducive to undergraduate research. During this particular summer, my student researched the Schwarz-Christoffel mapping from the unit disk onto n-sided regular polygons where $A = 0$ and $C > 0$. However, this was done without the knowledge of RD domains. Binomial series expansions provided the tools to investigate the nature of the coefficients. She discovered that
"...the Taylor series about the origin of the Schwarz-Christoffel mapping from the unit disk onto an n-sided regular polygon with one vertex on the positive real axis under the conditions $f(0) = 0$ and $f'(0) > 0$ has positive coefficients for the $rn + 1$ terms for $r = 0, 1, 2, \ldots$ and zero coefficients for all the other terms [**15**]."

Thus, if n is even, such as for RD regular polygons then the Taylor series expansion has non-negative odd coefficients. If the regular polygon has an odd number of sides then it has non-negative even coefficients. Therefore, because of the many symmetries of regular polygons, the conformal mapping $f_\Lambda(z)$ from the unit disk onto any regular m-sided polygons, $m \geq 4$, with a vertex on the positive real axis determined $f_\Lambda(0) = 0$ and $f'_\Lambda(0) > 0$ has a Taylor series representation about origin with non-negative coefficients.

Bibliography

[1] Ahlfors, L.V. *Complex Analysis(3rd edition)*, McGraw-Hill Inc., 1979.

[2] Barnett, J. *Stage Center: The Early Drama of the Hyperbolic Functions*, Mathematics Magazine, **77** (2004), no. 1, 15-30.

[3] Bishop, C. J. *The Riemann Mapping Theorem*, MAT 401, Seminar in Mathematics, Fall 2009. www.math.sunysb.edu/~ bishop/classes/math401.html

[4] Christoffel, E.B. *Sul problema delle temperature staxonarie e la rappresentasione di una data superficie*, Ann. Mat. Pura Appl. Serie II,, **1** (1867), 88-103.

[5] DeLillo, T.K., Elcrat, A.R., Pfaltzgraff, J.A. *Schwarz-Christoffel Mapping of the Annulus*, SIAM Review, **43** (2001), no. 3, 469-477.

[6] Driscoll, S.A., Trefethen L.N. *The Schwarz-Christoffel Mapping*, Cambridge University Press, 2002.

[7] Driscoll, S.A. *A MATLAB Toolbox for Schwarz-Christoffel Mapping*, ACM Trans. Math Soft., **22** (1995), 168-186.

[8] Horadam, A.F. *Gegenbauer Polynomials Revisted*, March 2013, http://www.fq.math.ca /Scanned/23-4/horadam-a.pdf

[9] Kanas, S., Sugawa, T. *On Conformal Representations of the Interior of an Ellipse*, Annales AcademiæScientiarum FennicæMathematica (2006), 329-348.

[10] Landstorfer, M. *Schwarz-Christoffel Mapping of Multiply Connected Domains: Proof, Discussion, and Elaborations, MathLab Implementation and Search Algorithms*, Ph. D. Thesis (2007), Fachhesch Schule Regnsburg University of Applied Sciences, Regensberg, Germany.

[11] Markushevich, A. *Theory of Functions of a Complex Variable, Second Edition*, AMS Chelsea Publishing, 1985.

[12] Moore, C. *The Legendre Polynomials*, February 2013, https://www.morehouse.edu /facstaff/cmoore/Legendre%20Polynomials.htm

[13] Nehari, Z. *Conformal Mapping*, McGraw-Hill Book Company Inc., 1952.

[14] O'Conner J., Robertson, E. *Georg Friedrich Bernhard Riemann*, September 1998, http://www-history.mcs.st-andrews.ac.uk/Biographies/Riemann.html

[15] Peer, C. *The Effects of Symmetry on the Schwarz-Christoffel Map*, Senior Project Thesis (2012), University of Arkansas Fort Smith, Fort Smith, Arkansas.

[16] Riemann, B. *Foundations of a General Theory of Functions of One Complex Variable*, Ph.D. Thesis (1951), University of Göttingen, Göttingen, Germany.

[17] Rolf, Jim *Complex Tools, Version 5.4*, October 2012, http://www.jimrolf.com/java.htm

[18] Schwarz, H. A. *Conforme Abbildung der Oberfläche eines Tetraeders auf die Oberflä einer Kugel*, J. Reine Ange. Math., **70** (1869), 121-136.

[19] Schwarz, H. A. *Über ineige Abbildungsaufgaben*, J. Reine Ange. Math., **70** (1869), 105-120.

[20] Schwarz, H. A. Gesammelte Mathematische Abhandlungen, Volume II, Springer, 1890.

[21] Sloughter, D. *Difference Equations to Differential Equations*, January 2013, http://math.furman.edu/ dcs/book/c6pdf/sec67.pdf

[22] Szegö, G. *Inequalities for the Zeros of Legendre Polynomials and Related Functions* Trans-, actions of the American Mathematical Society **99** (1936), 1-17.

[23] Szegö, G. *Orthogonal Polynomials*, The American Mathematical Society, 1939.

[24] Turan, P. *On the Zeros of the Polynomials of Legendre*, Casopis pro pestovani matematiky a fysiky, **75** (1950), no. 3, 113-122.

[25] Wikipedia: The Free Encyclopedia. Wikimedia Foundation, Inc. July 2004. November 2011. http://en.wikipedia.org/wiki/Legendre_polynomials

[26] Wolfram Research Inc. *Mathematica Edition, Version 8.0 for Students*, Champaign, IL, 2010.

[27] Wolfram, K., *Schwarz-Christoffel Mappings: Symbolic Computation of Mapping Functions for Symmetric Polygonal Domains*, Proceedings of the Workshop on Functional-Analytic Methods in Complex Analysis and Applications to Partial Differential Equations (1995), 293-305.