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A GENERALIZATION OF CLASS OF TESTS FOR EXPONENTIALITY AGAINST INCREASING FAILURE RATE AVERAGE ALTERNATIVES

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Abstract

The class of increasing failure rate average distributions plays a central role in the statistical theory of reliability. It is the smallest class of probability distributions which contains the exponential distribution and is closed under formation of coherent systems. Also these distributions arise as life distributions from various useful shock models. Deshpande (1983) proposed a class of tests for testing exponentiality against the increasing failure rate average class of non exponential probability distributions. He investigated some properties of these tests and compared its performance with Hollander-Proschan test and cumulative time on test statistic in terms of Pitman asymptotic relative efficiency. A generalization of the class of tests proposed by Deshpande (1983) based on sub sample minima is considered in this paper. Some properties of the tests are investigated and the performances in terms of Pitman asymptotic relative efficiency are evaluated. An alternative expression in terms of ranks is given and the actual cut-off points (10% and 5%) are tabulated to facilitate the application of tests. Small sample comparisons are also made.

Key words : ARE, Cumulative time on test, Hollander-Proschan statistic, New better than used, U-statistics, IFRA.

1. Introduction

Increasing failure rate average (IFRA) distributions naturally arise when coherent systems are formed from components with independent increasing failure rate distributions. Also, it naturally arises when one considers cumulative damage shock models. The class of increasing failure rate average distributions contains the class of increasing failure rate distributions as subclass. IFRA class of distributions is the smallest class of distributions which contains the exponential distribution and is closed under formation of coherent systems.

Let F be a probability distribution such that F(0)=0. Then F is an increasing failure rate average distribution (IFRA) if $\left[\bar{F}(t)\right]^{1/t}$ is decreasing in $t>0$, or equivalently, for $x>0$, $0<b<1$,

$$\bar{F}(bx) \geq \left\{\bar{F}(x)\right\}^{b} \tag{1.1}$$

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where $\overline{F} = 1 - F$. The equality in (1.1) holds if and only if F is an exponential distribution.

Tests for exponentiality designed to detect the alternative hypotheses relevant in reliability theory include those of Proschan and Pyke (1967), Bickel and Doksum (1969), Ahmed (1975), Hollander and Proschan (1972, 1975) and Koul (1977, 78). However, the test that have been developed specifically for testing for increasing failure rate average alternative only is due to Deshpande (1983). The following Lemma gives a characterization of IFRA distribution which is useful in the development of our class of test statistics.

Lemma 1.1: F is said to be IFRA distribution if and only if

$$\overline{F}^k(bx) \geq \overline{F}^{kb}(x), \text{ for } x > 0,\ 0 < b < 1,\ k \geq 1.$$

Proof: Let F be a IFRA distribution. Then by (1.1)

$$\overline{F}^k(bx) \geq \overline{F}^b(x),\ x > 0,\ 0 < b < 1.$$

Hence, for $k \geq 1$,

$$\overline{F}^k(bx) \geq \overline{F}^{kb}(x).$$

The converse holds if we take k=1.

In section 2, we propose new class of statistics which are U-statistics whose kernel depends on sub sample minima. We also give an alternative expression based on ranks for the class of statistics. In section 3, asymptotic normality and consistency of the proposed class of tests is presented. Section 4 deals with Pitman ARE comparisons. Section 5 deals with unbiasedness of these tests. Also empirical powers of the tests are considered in section 6.

2. Proposed class of test statistics

Let $X_1, X_2, ..., X_n$ be a random sample from a continuous probability distribution function F such that F(0)=0. We wish to test the null hypothesis

$$H_0 = \overline{F}(bx) = \overline{F}^b(x),\ x > 0,\ 0 \leq b \leq 1$$

against

$$H_1 = \overline{F}(bx) \geq \overline{F}^b(x),\ x > 0,\ 0 \leq b \leq 1$$

with strict inequality for some x.

Define the parameter $M_k(F) = \int_0^\infty \overline{F}^k(bx)dF(x)$.

If F belongs to H_0 then $M_k(F) = (bk+1)^{-1}$, whereas for all F belonging to H_1

$$M_k(F) = \int_0^\infty \overline{F}^k(bx)dF(x) > \int_0^\infty \overline{F}^{bk}(x)dF(x) = (bk+1)^{-1}.$$

Define a kernel

$$h_b^{k+1}(X_1,\dots,X_{k+1}) = \begin{cases} 1, & if \ \min(X_1,\dots,X_k) > bX_{k+1} \\ 0, & otherwise \end{cases}$$

where b is a fixed number such that 0 < b < 1. Define V(b,k) as U-statistic depending on the symmetric version $h_b^{k+1}(X_1,\dots,X_{k+1})$, that is

$$V(b,k) = \binom{n}{k+1}^{-1} \Sigma h_b^*\left(X_{i_1},\dots,X_{i_{k+1}}\right)$$

where summation is taken over all combinations of integers (i_1, ..., i_k) chosen out of integers (1, 2, ..., n) and $h_b^*(X_1,\dots,X_{k+1}) = (k+1)^{-1}\sum_{i=1}^{k+1} h_b^i(X_1,\dots,X_{k+1})$. It can be seen that $E[V(b,k)] = M_k(F)$. Large values of V(b,k) lead to rejection of H_0 against H_1.

Alternative Expression of V(b,k)

Let $X_{(1)}$, ..., $X_{(n)}$ be the order statistics of X_1, ..., X_n and $Y_{(1)}$, ..., $Y_{(n)}$ be the order statistics of Y_1, ..., Y_n with $Y_i=bX_i$, for i = 1, 2, ..., n. Suppose $S_{(j)}$ is the rank of $Y_{(j)}$ in the combined ranking of X and Y observations. Then, following Shetty and Bhat (1994), V(b,k) can be written as

$$V(b,k) = \left[n\binom{n}{k}\right]^{-1}\left\{\sum_{j=1}^{n}\binom{n-S_{(j)}+j}{k} - \binom{n}{k}\right\}.$$

3. Asymptotic Normality and Consistency

The statistic V(b,k) is the U-statistic corresponding to the kernel h_b^*. Using the results of Hoeffding (1948), the asymptotic distribution of $\sqrt{n}\left[V(b,k) - M_k(F)\right]$ is normal with mean zero and variance $(k+1)^2\varsigma_1$, where

$$\varsigma_1 = E\left[\Psi_1^2(X_1)\right] - \left[M_k(F)\right]^2$$

and $\Psi_1(x_1) = E\{h_b^*(X_1,X_2,\dots,X_{k+1})\}$, provided $\varsigma_1 > 0$.

Under the null hypothesis, $M_k(F) = (bk+1)^{-1}$ and

$$(k+1)^2\varsigma_1 = \frac{1}{2bk+1} + \frac{2k}{b(k-1)+1}\left[\frac{1}{bk+1} - \frac{b}{bk(b+1)+1}\right] - \frac{(k+1)^2}{(bk+1)^2}$$

$$+\frac{k^2}{(b(k-1)+1)^2}\left[\frac{b}{b(2k-1)+2}-\frac{2b}{bk+1}\right]+\frac{k^2}{(b(k-1)+1)^2}$$

Since $E[V(b,k)]>(bk+1)^{-1}$, the class of test statistics is consistent against increasing failure rate average alternatives.

Now F is new better than used implies that $\overline{F}(z/c)>\overline{F}^{1/c}(z)$, for every z>0 and c = 2, 3, ... The equality holds only for the exponential distribution. Hence V(b,k) test for b = 1/c (c = 2, 3, ...) is consistent against continuous new better than used distributions.

4. Asymptotic Relative Efficiency

For asymptotic relative efficiency comparisons, we have considered three parametric families of distributions, namely, Weibull, Makeham and Linear Failure Rate distributions. These depend upon a real parameter θ in such a way that $\theta=\theta_0$ yields a distribution belonging to null hypothesis whereas $\theta>\theta_0$ yields distribution from the alternative. These are

(i) Weibull distribution,

$$\overline{F}_\theta(x)=\exp(-x^\theta),\ x>0,\theta\ge 1,\ \theta_0=1$$

(ii) Makeham Distribution

$$\overline{F}_\theta(x)=\exp\left[-x+\theta\left(x+e^{-x}-1\right)\right],\ x>0,\ \theta\ge 0,\ \theta_0=0$$

(iii) Linear Failure Rate Distribution

$$\overline{F}_\theta(x)=\exp\left[\left(-x+\theta\frac{x^2}{2}\right)\right],\ x>0,\ \theta\ge 0,\ \theta_0=0$$

The Pitman ARE's of V(b,k) for (b=0.1, 0.25, 0.5, 0.9 and k = 2, 3) with respect to Deshpande (1983) J_b test are given in Table 1.

Table 1 : Pitman ARE's of V(b,k) for different values of b w.r.t. J_b.

	k = 2				k = 3			
b →	0.1	0.25	0.5	0.9	0.1	0.25	0.5	0.9
Makeham	1.0142	1.0574	1.1458	1.3563	1.0222	1.0795	1.1937	1.4943
LFR	0.7925	0.6134	0.6759	0.5689	0.6233	0.4005	0.4582	0.3711
Weibull	0.9312	0.8835	0.8458	0.9445	0.8703	0.7848	0.4461	0.7648

5. Unbiasedness of the test statistic

To show unbiasedness of V(b,k) test we have to know that the probability of rejection is not less than α, the size of the test whenever the alternative hypothesis is true. Let G be exponential with $\theta=1$. Let F be a distribution in the alternative hypothesis i.e., for $b\in(0,1)$,

Table 2 : Monte Carlo estimates of critical values, exact levels of significance and powers of V(.5,2), V(.5,3), V(9,2) and V(.9, 3).

b=0.5

	α	n →	5		7		9		11	
Exact α	0.10	J_b	0.130		0.057		0.052		0.070	
		V(b,2)	0.0995		0.0897		0.0983		0.0986	
		V(b,3)	0.0937		0.0994		0.0995		0.0996	
	0.05	J_b	0.043		0.035		0.033		0.048	
		V(b,2)	0.0349		0.0494		0.0483		0.0497	
		V(b,3)	0.0446		0.0434		0.0488		0.05	
Cut-off points	0.10	J_b	16		32		51		73	
		V(b,2)	25		76		168		316	
		V(b,3)	21		105		321		775	
	0.05	J_b	17		33		52		74	
		V(b,2)	30		83		181		337	
		V(b,3)	28		120		356		850	
			Weibull	**LFR**	**Weibull**	**LFR**	**Weibull**	**LFR**	**Weibull**	**LFR**
Emprical power	0.10	J_b	0.193	0.578	0.102	0.506	0.100	0.572	0.142	0.703
		V(b,2)	0.4387	0.1439	0.5767	0.1692	0.7037	0.2035	0.7860	0.2225
		V(b,3)	0.4344	0.1499	0.5264	0.1568	0.6505	0.1883	0.7374	0.1889
	0.05	J_b	0.074	0.340	0.068	0.405	0.067	0.481	0.099	0.631
		V(b,2)	0.2736	0.0676	0.4418	0.1019	0.5477	0.1238	0.6408	0.1208
		V(b,3)	0.2495	0.0777	0.3504	0.0846	0.4968	0.1082	0.5783	0.1082

b=0.9

	α	n →	5	7	9	11
Exact α	0.10	J_b	0.073	0.075	0.056	0.106
		V(b,2)	0.0544	0.0919	0.0855	0.0941
		V(b,3)	0.0961	0.0823	0.095	0.0979
	0.05	J_b	0.024	0.021	0.021	0.043
		V(b,2)	0.0306	0.0376	0.0432	0.0476
		V(b,3)	0.0282	0.0483	0.0498	0.0494
Cut-off points	0.10	J_b	12	24	40	59
		V(b,2)	14	43	101	193
		V(b,3)	8	50	164	409
	0.05	J_b	13	25	41	60
		V(b,2)	15	46	105	199
		V(b,3)	11	53	172	432

Table 2 continued...

Table 2 continued...

			Weibull	LFR	Weibull	LFR	Weibull	LFR	Weibull	LFR
Empirical power	0.10	J_b	0.096	0.233	0.102	0.281	0.082	0.278	0.152	0.452
		V(b,2)	0.1758	0.0736	0.3247	0.1431	0.3551	0.1232	0.4570	0.1517
		V(b,3)	0.2263	0.1210	0.2623	0.1084	0.3251	0.1313	0.4046	0.1545
	0.05	J_b	0.032	0.092	0.030	0.117	0.034	0.139	0.066	0.279
		V(b,2)	0.1130	0.0426	0.1770	0.0647	0.2358	0.0688	0.3107	0.0804
		V(b,3)	0.1036	0.0435	0.1695	0.0654	0.2261	0.0713	0.2581	0.0760

$-log\overline{F}(bx) \le -b\log\overline{F}(x)$.

Let $X_1, ..., X_n$ be a random sample from F. Let $U_i = G^{-1}F(X_i) = -\log\overline{F}(X_i)$. Then $U_1, ..., U_n$ have the same probability distribution as a random sample $Y_1, ..., Y_n$ from G. Now, $\min(X_1, ..., X_k) \le bX_{k+1}$ implies

$$G^{-1}F(\min(X_1,...,X_k)) \le G^{-1}F(bX_{k+1}) \le bG^{-1}F(X_{k+1}),$$

So that $\min(X_1, ..., X_k) \le bX_{k+1}$ implies $\min(U_1, ..., U_k) \le bU_{k+1}$.

Therefore, $h_b^{k+1}(U_1,...,U_{k+1}) \le h_b^{k+1}(X_1,...,X_{k+1})$. But $h_b^{k+1}(Y_1,...,Y_{k+1})$ has the same distribution as $h_b^{k+1}(U_1,...,U_{k+1})$. Hence V(b,k) based on $X_1, ..., X_n$ and V(b,k) based on $Y_1, ..., Y_n$ have the same distribution. V(b,k) based on $X_1, ..., X_n$ is stochastically larger than these. Hence

$$P_F[V(b,k) \ge c^*] \ge P_G[V(b,k) \ge c^*] \tag{5.1}$$

where c^* is the cut-off point of the null distribution of V(b,k). The left hand side of (5.1) represents the power of the test at a fixed alternative F of H_1 and the right hand side is equal to α, which implies unbiasedness of the V(b,k) test.

6. Empirical Powers

The Monte-Carlo study is carried out to estimate power of V(b,k) test for two specific alternatives corresponding to a significance level α=0.05 and α=0.10. The values of b and k considered for this purpose are b=0.5, 0.9 and k=2, 3. The study is done for different values of n each value being based on 10000 samples of required size. The two alternatives to the null hypothesis H_0 of exponentiality with θ=1 for which the power has been estimated are the Weibull distribution of index 2 and Linear Failure Rate distribution with θ=1. Both these distributions are increasing failure rate distributions and hence also is increasing failure rate average distributions. Table 2 gives Monte Carlo estimates of critical values, exact levels of significance and power of V(0.5,2), V(0.5,3), V(09.2) and V(0.9,3). Monte Carlo estimates of critical values, exact levels of significance and power of n(n-1)J_b for b=0.5 and 0.9 are also included in the Table 2 for the purpose of comparison.

Table 2 indicates that V(b,k) performs better than J_b test in terms of small sample power for Weibull alternatives for both b=0.5 and 0.9 whereas J_b test beats V(b,k) for LFR alternatives.

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