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ON TESTING AGAINST POSITIVE QUADRANT DEPENDENCE BASED ON SUB-SAMPLE ORDER STATISTICS

Parameshwar V. Pandit and Savitha Kumari

Department of Statistics
Bangalore University
Bangalore - 560056, India
e-mail: panditpv12@gmail.com

Department of Statistics
SDM College
Ujire - 574240, India
e-mail: savi_rrao@yahoo.co.in

Abstract

A new class of tests based on convex combination of the two statistics is proposed. These are functions of sub-sample order statistics. The classes of tests proposed by Kochar and Gupta [6], Shetty and Pandit [16], Pandit and Kumari [11] and Kendall's test lie in the proposed class of test statistics. The asymptotic normality of the proposed class of tests is established. It has been observed that some members of the class perform better than the existing tests. Unbiasedness and consistency of the proposed class of tests are established.

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1. Introduction

One of the widely studied concepts in probability and statistics is the dependence relations between random variables. The relation between random variables X and Y is explored by designing some studies such as determining whether X and Y are independent or dependent. For example, a doctor may be interested in studying the relationship between obesity and blood pressure. Particularly, he may be interested in testing whether obesity (X) and blood pressure (Y) are independent against the alternative that they are positively associated. Testing the hypothesis that the time until it takes an infant to walk alone (X) is independent of the infant's IQ at a later age (Y) versus the alternative that children who learn to walk early tend to have higher IQs may be a problem of interest to the psychologists.

Let (X, Y) be absolutely continuous random variable with joint distribution function $F(x, y)$ and survival function

$$\overline{F}(x, y) = P[X > x, Y > y].$$

Let F and G denote the marginal distribution functions of X and Y, respectively, with corresponding survival functions $\overline{F} = 1 - F$ and $\overline{G} = 1 - G$. The definition of particular type of dependence namely positive quadrant dependence (PQD) due to Lehmann [9] is given below.

Definition 1.1. A random vector (X, Y) is said to be *positive quadrant dependent* (PQD) if $F(x, y) \geq F(g)G(y)$, for all (x, y). or $\overline{F}(x, y) \geq$ $P[X > x, Y > y]$, for all (x, y). The dependence is strict if the inequality holds at least for one pair (x, y).

A concept that is symmetric to PQD is the concept of negative quadrant dependence (NQD), which swaps the inequality in the definition of PQD. The relation between both the concepts can be seen in terms of monotonic transformations. However, if an increasing function is applied to one random variable and a decreasing function to the other random variable, then the quadrant dependence of the transformed couple of random variables is changed. The probabilistic approach says that the probability that random variables be jointly large is greater than that when are looked separately.

Positive quadrant dependence might be a very realistic assumption in many situations. For example, in the study of life expectancies among men and women in various countries, one would expect that a higher life expectancy for men in one country goes along with a higher life expectancy for women in that country. Recently, there is more attention on the effects of positive dependence among risks. Positive dependence may lead to substantial deviations in the stop-loss premiums, compared to independence case. In mathematical finance, positive (or negative) dependence is an important concern. For instance, one wishes to know whether certain stocks are negatively dependent in order to build a well-balanced portfolio.

A measure of general positive association between two random variables X and Y is defined in terms of covariance between every pair of non-decreasing real functions f and g as $Cov[f(x), g(Y)] \geq 0.$ However, if a pair (X, Y) is PQD, then $Cov(X, Y) \geq 0.$ Equality holds if X and Y are independent. Furthermore, if the pair of functions f, g are real and non-decreasing, then (X, Y) is PQD implies that $[f(X), g(Y)]$ is PQD which, in turn, implies that $Cov[f(X), g(Y)] \geq 0.$ Consequently, general positive association and PQD are equivalent. The concept of PQD has been used to construct conservative confidence intervals for the components of the mean vector in bivariate normal distribution. Many applications of this concept may be founding the study of contaminated independence models, slippage problems, tests of symmetry, etc.

In this paper, we consider the problem of testing the null hypothesis of independence $H_0 : F(x, y) = F(x)G(y),$ for all (x, y) against the alternative of PQD $H_1 : F(x, y) \geq F(x)G(y),$ for all (x, y) with strict inequality on a set of non-zero probabilities. It should be noted here that since F and G are unknown, H_0 and H_1 are both composite. Moreover, the alternative H_1 is ordered which gives rise to general difficulties associated with ordered restricted inference.

For testing independence against PQD, Kochar and Gupta [6, 7] proposed some competitors of Kendall's sample tau coefficient. Schriever [13] contained a large number of tests, available in the literature for the problem of independence. Shetty and Pandit [15-17] proposed distribution-free tests for this problem based on the ordering of observations in sub-samples. The statistic proposed by Kochar and Gupta [7] is a member of the test proposed by Shetty and Pandit [17]. In this paper, we propose a class of distribution-free tests for this problem.

In Section 2, we propose a new class of distribution-free tests based on U-statistics for testing H_0 against H_1. The distribution of the test statistics is considered in Section 3. Section 4 is devoted to asymptotic relative efficiency properties. Some remarks and conclusions are given in Section 5.

2. The Proposed Class of Test Statistics

Let the random sample $(X_1, Y_1), \ldots, (X_n, Y_n)$, $n \geq 2$ be drawn from the distribution H, the problem is to test

$$H_0 : F(x, y) = F(x) \cdot G(y), \text{ for at least one } (x, y)$$

against

$$H_1 : F(x, y) \geq F(x) \cdot G(y), \text{ for all } (x, y) \text{ and with strict inequality for at least one } (x, y).$$

Let $k \geq 2$ be a fixed positive integer. Then the proposed test statistic is based on U statistic with kernel defined by

$$h_k[(x_1, y_1), \ldots, (x_k, y_k)]$$

$$= \delta h_k^{(1)}[(x_1, y_1), \ldots, (x_k, y_k)] + (1 - \delta) h_k^{(2)}[(x_1, y_1), \ldots, (x_k, y_k)],$$

$$h_k^{(1)}[(x_1, y_1), \ldots, (x_k, y_k)]$$

$$= \begin{cases} 1 & \text{if } r\text{th smallest of } (x_1, \ldots, x_k) \text{ and if } r\text{th smallest} \\ & \text{of } (y_1, \ldots, y_k) \text{ belongs to the same pair} \\ 0 & \text{otherwise,} \end{cases}$$

$$h_k^{(2)}[(x_1,\ y_1),\ ...,\ (x_k,\ y_k)]$$

$$= \begin{cases} 1 & \text{if } (k-r+1)\text{th smallest of } (x_1,\ ...,\ x_k) \text{ and } (k-r+1)\text{th} \\ & \text{smallest of } (y_1,\ ...,\ y_k) \text{ belongs to the same pair} \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

Then the test statistic $U(k,\ r)$ corresponding to the kernel h_k is

$$U(k,\ r) = \binom{n}{k}^{-1} \sum h_k[(X_{i_1},\ Y_{i_1}),\ ...,\ (X_{i_k},\ Y_{i_k})],$$

where the summation is over all combinations of k integers $(i_1,\ i_2,\ ...,\ i_k)$ chosen out of n integers $(1,\ 2,\ ...,\ n)$ without replacement.

The statistic $U(k,\ r)$ can be easily computed as follows. Without loss of generality, assume that $X_1 \le \cdots \le X_n$ and denote the corresponding Ys by $Y_{[1]},\ ...,\ Y_{[n]}$.

Let $L_{(j)} = rank$ of $Y_{(j)}$ among $Y_{[1]}, ..., Y_{[j]}$. and $M_{(j)} =$ number of $Y_{[i]}$s greater than or equal to $Y_{[j]},\ Y_{[j+1]},\ ...,\ Y_{[n]}$. Then $U(k,\ r)$ can be written as

$$U(k,\ r) = \frac{\delta \sum_{j=1}^{n} \binom{L_{(j)} - 1}{r-1}\binom{M_{(j)} - r}{k-r} + (1-\delta)\sum_{j=1}^{n}\binom{L_{(j)} - 1}{k-1}\binom{M_{(j)} - k + r - 1}{r-1}}{\binom{n}{k}}.$$

Woodworth [18] called $L_{(j)}$ the *third quadrant layer rank* of $(X_j,\ Y_{[j]})$.

3. Distributional Properties of $U(k,\ r)$

The expectation of $U(k,\ r)$ is given by

$$\gamma(F) = E\{U(k,\ r)\}$$

$$= E\{h_k[(x_1,\ y_1),\ ...,\ (x_k,\ y_k)]\}$$

$= \delta P\{r$th smallest of $(x_1, ..., x_k)$ and rth smallest of $(y_1, ..., y_k)$ belong to the same pair$\} + (1-\delta)P\{(k-r+1)$th smallest of $(x_1, ..., x_k)$ and $(k-r+1)$th smallest of $(y_1, ..., y_k)$ belong to the same pair$\}$

$$= k\binom{k-1}{r-1}[\delta P\{\max(x_1, ..., x_{r-1}) < x_r < \min(x_{r+1}, ..., x_k),$$

$$\max(y_1, ..., y_{r-1}) < y_r < \min(y_{r+1}, ..., y_k)\}$$

$$+ (1-\delta)P\{\max(x_1, ..., x_{k-r}) < x_{k-r+1} < \min(x_{k-r+2}, ..., x_k),$$

$$\max(y_1, ..., y_{k-r}) < y_{k-r+1} < \min(y_{k-r+2}, ..., y_k)\}]$$

$$= k\binom{k-1}{r-1}\left\{\delta\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F^{r-1}(x, y)\overline{F}^{k-r}dF(x, y) + (1-\delta)\right.$$

$$\left.\cdot \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F^{k-r}\overline{F}^{r-1}(x, y)dF(x, y)\right\}.$$

Under H_0, $\gamma_0(F) = k\binom{k-1}{r-1}[\beta(r, k-r+1)]^2$, where

$$\beta(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx.$$

Theorem 1. *Under the alternative* H_1, $\gamma(F) > \gamma_0(F)$.

Proof. Under H_1,

$$\gamma(F) = E[U(k, r)] \geq k\begin{bmatrix}k-1\\ r-1\end{bmatrix}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} B(x, y)dF(x, y),$$

where

$$B(x, y) = \delta[F^{r-1}(x)\overline{F}^{k-r}(x)G^{r-1}(y)\overline{G}^{k-r}(y)]$$

$$+ (1-\delta)[F^{k-r}(x)\overline{F}^{r-1}(x)G^{k-r}(y)\overline{G}^{r-1}(y)]$$

and $\overline{F}(x) = 1 - F(x)$, $\overline{G}(y) = 1 - G(y)$.

Since $B(-\infty,\ y) \equiv 0 \equiv B(x,\ -\infty)$ and B is of bounded variation on finite intervals, we can integrate by parts and obtain

$$k\begin{bmatrix} k-1 \\ r-1 \end{bmatrix}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} B(x,\ y)dF(x,\ y)$$

$$= k\begin{bmatrix} k-1 \\ r-1 \end{bmatrix}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(x,\ y)dB(x,\ y)$$

$$> k\begin{bmatrix} k-1 \\ r-1 \end{bmatrix}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(x)G(y)dB(x,\ y)$$

$$= k\begin{bmatrix} k-1 \\ r-1 \end{bmatrix}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} B(x,\ y)dF(x)G(y) = \gamma_0(F).$$

Hence, under H_0, $\gamma(F) > \gamma_0(F)$.

The asymptotic distribution of $U(k,\ r)$ is emphasized in Theorem 2, the proof of which is the consequence of Hoeffding [5].

Theorem 2. *Under the assumed model and* $0 < \xi_{1u} < \infty$, $\sqrt{n}[U(k,\ r) - E(U(k,\ r))]$ *converges in distribution to* $N(0,\ k^2\xi_{1u})$ *random variable as* $n \to \infty$, *with* $\xi_{1u} = Var\{h_k[(x_1,\ y_1), \ldots, (x_k,\ y_k)|(x_1,\ y_1)]\}$, *where* $h_k(\cdot)$ *is as defined by* (2.1).

Under H_0, $\sigma^2_{U_0}(k,\ r) = k^2\xi_{1U_0}$, where

$$\xi_{1U_0} = E[h_k^2\{(X_1,\ Y_1),\ (X_2,\ Y_2), \ldots, (X_k,\ Y_k)\}] - \gamma_0^2(F)$$

and

$$\gamma_0(F) = k\binom{k-1}{r-1}[\beta(r,\ k-r+1)]^2.$$

Now,

$$E[h_k^2\{(X_1, Y_1), (X_2, Y_2), ..., (X_k, Y_k)\}] - \gamma_0^2(F)$$

$$= E\left[\delta I_1 + (1-\delta) I_2\right]^2 - \gamma_0^2(F)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [\delta I_1 + (1-\delta) I_2]^2 dF(x) dG(y) - \gamma_0^2(F),$$

where

$$I_1 = \binom{k-1}{r-1} F^{r-1}(x_1)\overline{F}^{k-r}(x_1)G^{r-1}(y_1)\overline{G}^{k-r}(y_1) + (k-1)$$

$$\cdot\left\{\binom{k-2}{r-2}\int_{y_1}^{\infty}\int_{x_1}^{\infty} F^{r-1}(x)\overline{F}^{k-r}(x)G^{r-1}(y)\overline{G}^{k-r}(y) dF(x) dG(y)\right.$$

$$\left.+\binom{k-2}{r-1}\int_{-\infty}^{y_1}\int_{-\infty}^{x_1} F^{r-1}(x)\overline{F}^{k-r-1}(x)G^{r-1}(y)\overline{G}^{k-r-1}(y) dF(x) dG(y)\right\},$$

$$I_2 = \binom{k-1}{r-1} F^{k-r}(x_1)\overline{F}^{r-1}(x_1)G^{k-r}(y_1)\overline{G}^{r-1}(y_1) + (k-1)$$

$$\cdot\left\{\binom{k-2}{r-1}\int_{y_1}^{\infty}\int_{x_1}^{\infty} F^{k-r-1}(x)\overline{F}^{r-1}(x)G^{k-r-1}(y)\overline{G}^{r-1}(y) dF(x) dG(y)\right.$$

$$\left.+\binom{k-2}{r-2}\int_{-\infty}^{y_1}\int_{-\infty}^{x_1} F^{k-r}(x)\overline{F}^{r-2}(x)G^{k-r}(y)\overline{G}^{r-2}(y) dF(x) dG(y)\right\}.$$

The corresponding asymptotic variances of $U(k, r)$ for $k = 4, 5, 6$ and $r = 2$ under H_0 are

$$\sigma_{U_0}^2(4, 2) = 0.006349\delta^2 - 0.006349\delta + 0.13528,$$

$$\sigma_{U_0}^2(5, 2) = 0.011999\delta^2 - 0.011999\delta + 0.021676,$$

$$\sigma_{U_0}^2(6, 2) = 0.013536\delta^2 - 0.013536\delta + 0.014576.$$

4. Asymptotic Relative Efficiencies

Pitman introduced a method for asymptotic comparison of test procedures popularly known as Pitman asymptotic relative efficiency (ARE). It is defined as the limiting ratio of sample sizes of the two test procedures required to attain the same limiting power for the sequence of alternatives converging to the null hypothesis.

For the sequence of Pitman alternatives, the efficacy of $U(k, r)$ is defined as

$$Eff[U(k, r)] = \lim_{n\to\infty} \frac{\left(\frac{d}{d\theta} E[U(k, r)]\right)_{\theta=0}}{\sqrt{nVar_{H_0}[U(k, r)]}}.$$

The asymptotic relative efficiency of $U(k, r)$ with respect to any other test T is given by $ARE[U(k, r), T] = \left[\frac{Eff[U(k, r)]}{Eff(T)}\right]^2$.

For Pitman asymptotic relative efficiency comparisons, three models namely Morgenstern [10] distribution, Woodworth [18] family of distributions and Block and Basu [2] distributions are considered.

First we consider Woodworth [18] family of distributions, which is given by the pdf

$$f(x, y) = 1 + \theta[1 - (m+1)x^m][1 - (m+1)y^m], \quad 0 \le \theta \le 1/m^2, \quad m \ge 1.$$

Woodworth [18] family of distributions contains Morgenstern [10] distribution as a particular case when $m = 1$.

The asymptotic relative efficiencies (AREs) of the $U(k, r)$ relative to Kendall's tau T_n for different values of k, r and m are presented in Tables 1.1 to 1.4, for $m = 2, 3, 4$ and 5.

Also, AREs of $U(k, r)$ relative to Kendall's tau T_n for Morgenstern [10] distribution are given in Table 2.

Table 1.1. AREs of the $U(k, r)$ for $m = 2$

δ	$ARE[U(4, 2), T_n]$	$ARE[U(4, 3), T_n]$	$ARE[U(5, 2), T_n]$	$ARE[U(5, 4), T_n]$
0	1.0840	0.8085	1.0969	0.6219
0.1	1.0720	0.8480	1.0981	0.6981
0.2	1.0561	0.8859	1.0877	0.7748
0.3	1.0362	0.9217	1.0642	0.8490
0.4	1.0127	0.9551	1.0275	0.9178
0.5	0.9855	0.9855	0.9782	0.9780
0.6	0.9551	1.0127	0.9180	1.0273
0.7	0.9217	1.0362	0.8493	1.0641
0.8	0.8859	1.0561	0.7751	1.0875
0.9	0.8480	1.0720	0.6985	1.0981
1.0	0.8085	1.0840	0.6222	1.0969

Table 1.2. AREs of the $U(k, r)$ for $m = 3$

δ	$ARE[U(4, 2), T_n]$	$ARE[U(4, 3), T_n]$	$ARE[U(5, 2), T_n]$	$ARE[U(5, 4), T_n]$
0	1.2247	0.7439	1.3343	0.5105
0.1	1.1905	0.7996	1.2992	0.6056
0.2	1.1522	0.8552	1.2491	0.7069
0.3	1.1099	0.9100	1.1840	0.8120
0.4	1.0640	0.9635	1.1050	0.9150
0.5	1.0151	1.0151	1.0142	1.0143
0.6	0.9635	1.0640	0.9150	1.1051
0.7	0.9100	1.1099	0.8111	1.1841
0.8	0.8552	1.1522	0.7068	1.2492
0.9	0.7996	1.1905	0.6056	1.2993
1.0	0.7439	1.2247	0.5105	1.3344

Table 1.3. AREs of the $U(k, r)$ for $m = 4$

δ	$ARE[U(4, 2), T_n]$	$ARE[U(4, 3), T_n]$	$ARE[U(5, 2), T_n]$	$ARE[U(5, 4), T_n]$
0	2.0353	1.0531	2.3287	0.6617
0.1	1.9543	1.1556	2.2273	0.8238
0.2	1.8670	1.2601	2.1001	1.0028
0.3	1.7740	1.3657	1.9485	1.1941
0.4	1.6763	1.4710	1.7760	1.3916
0.5	1.5750	1.5750	1.5881	1.5882
0.6	1.4710	1.6763	1.3915	1.7761
0.7	1.3657	1.7740	1.1940	1.9486
0.8	1.2601	1.8670	1.0027	2.1002
0.9	1.1556	1.9543	0.8238	2.2274
1.0	1.0531	2.0353	0.6617	2.3288

Table 1.4. AREs of the $U(k, r)$ for $m = 5$

δ	$ARE[U(4, 2), T_n]$	$ARE[U(4, 3), T_n]$	$ARE[U(5, 2), T_n]$	$ARE[U(5, 4), T_n]$
0	1.4587	0.6616	1.7288	0.3882
0.1	1.3874	0.7393	1.6330	0.5043
0.2	1.3121	0.8197	1.5186	0.6361
0.3	1.2335	0.9021	1.3873	0.7806
0.4	1.1523	0.9857	1.2428	0.9336
0.5	1.0693	1.0693	1.0896	1.0897
0.6	0.9857	1.1523	0.9336	1.2429
0.7	0.9021	1.2335	0.7806	1.3875
0.8	0.8197	1.3121	0.6361	1.5187
0.9	0.7393	1.3874	0.5043	1.6331
1.0	0.6616	1.4587	0.3882	1.7286

Table 2. AREs of the $U(k, r)$ for Morgenstern [10] distribution

δ	$ARE[U(4, 2), T_n]$	$ARE[U(5, 2), T_n]$	$ARE[U(5, 4), T_n]$
0	0.9184	0.8202	0.8202
0.1	0.9335	0.8632	0.8632
0.2	0.9456	0.8999	0.8999
0.3	0.9544	0.9280	0.9281
0.4	0.9598	0.9458	0.9459
0.5	0.9616	0.9519	0.9520
0.6	0.9598	0.9458	0.9459
0.7	0.9544	0.9280	0.9281
0.8	0.9456	0.8999	0.8999
0.9	0.9335	0.8632	0.8632
1.0	0.9184	0.8202	0.8202

We also consider the AREs of the newly proposed tests for the absolutely continuous bivariate exponential distribution of Block and Basu [2] with density function:

$$f(x, y) = \frac{1}{2}(\theta + 1)(\theta + 2)\exp[-\{\min(x, y) + (1 + \theta)\max(x, y)\}], \quad x, y \geq 0.$$

This distribution is PQD when $\theta > 0$ and the variables are independent when $\theta = 0$.

The values of the AREs of the $U(k, r)$ tests with respect to the Kendall's tau T_n are presented in Table 3 given below:

Table 3. AREs of the $U(k, r)$ for Block and Basu [2] distribution

δ	$ARE[U(4, 2), T_n]$	$ARE[U(5, 2), T_n]$	$ARE[U(5, 4), T_n]$	$ARE[U(6, 3), T_n]$	$ARE[U(6, 4), T_n]$	$ARE[U(7, 3), T_n]$
0	1.2255	1.2693	0.6892	1.1696	0.8868	1.2206
0.1	1.2079	1.2665	0.7781	1.1538	0.9248	1.2137
0.2	1.1860	1.2499	0.8681	1.1347	0.9614	1.1964
0.3	1.1596	1.2185	0.9559	1.1126	0.9963	1.1681
0.4	1.1292	1.1718	1.0381	1.0875	1.0292	1.1290
0.5	1.0948	1.1110	1.1111	1.0595	1.0598	1.0795
0.6	1.0570	1.0380	1.1719	1.0289	1.0877	1.0211
0.7	1.0160	0.9559	1.2185	0.9961	1.1129	0.9555
0.8	0.9725	0.8680	1.2500	0.9611	1.1350	0.8848
0.9	0.9269	0.7780	1.2666	0.9246	1.1540	0.8112
1.0	0.8799	0.6892	1.2694	0.8866	1.1699	0.7369

5. Some Remarks and Conclusions

(1) The proposed class of test statistics $U(k, r)$ is unbiased for testing H_0 against H_1.

(2) The proposed class of test statistics $U(k, r)$ for testing H_0 against H_1 is consistent.

Proof. Since $E_{H1}[U(k, r)] > E_{H0}[U(k, r)]$ and $U(k, r)$ is asymptotically normal, it follows from Lehmann [8] that $U(k, r)$ is consistent for testing H_0 against H_1.

(3) The AREs of members of the class are studied for three alternatives namely, Morgenstern distribution [10], Woodworth family of distributions [18] and Block and Basu distributions [2].

(4) For Block and Basu bivariate [2] alternative, the performances of $U(4, 3)$ and $U(4, 2)$ relative to Kendall's tau are better for values $\delta \le 0.3$ and $\delta \le 0.7$, respectively.

(5) It is observed that the performances of $U(5, 4)$ and $U(6, 4)$ relative to Kendall's tau are better for values $\delta \geq 0.4$. When the alternative is Block and Basu bivariate distribution [2], $U(6, 2)$ is better for $\delta \geq 0.6$.

(6) The performances of $U(5, 2)$, $U(6, 3)$ and $U(7, 3)$ are better than Kendal's tau test when $\delta \leq 0.6$ for Block and Basu bivariate distribution [2].

(7) For Woodworth's family [18], if $m = 4$, then the performances of $U(4, 2)$, $U(4, 3)$ and $U(5, 4)$ are uniformly better as compared to Kendall's tau and that of $U(5, 2)$ is better for $\delta \leq 0.8$.

(8) It is observed that, for Woodworth's family [18], the performances of $U(4, 2)$ and $U(5, 2)$ relative to Kendall's tau test are better when $\delta \leq 0.4$ and that of $U(4, 3)$ and $U(5, 4)$ relative to Kendall's tau are better for $\delta \geq 0.6$, when $m = 2$.

(9) If $m = 3$ and 5, the performances of $U(4, 2)$ and $U(5, 2)$ are better than Kendall's tau test when $\delta \leq 0.5$ and that of $U(4, 3)$ and $U(5, 4)$ are better compared to that of Kendall's tau test for $\delta \geq 0.5$, when the alternative is Woodworth's family of distributions [18].

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