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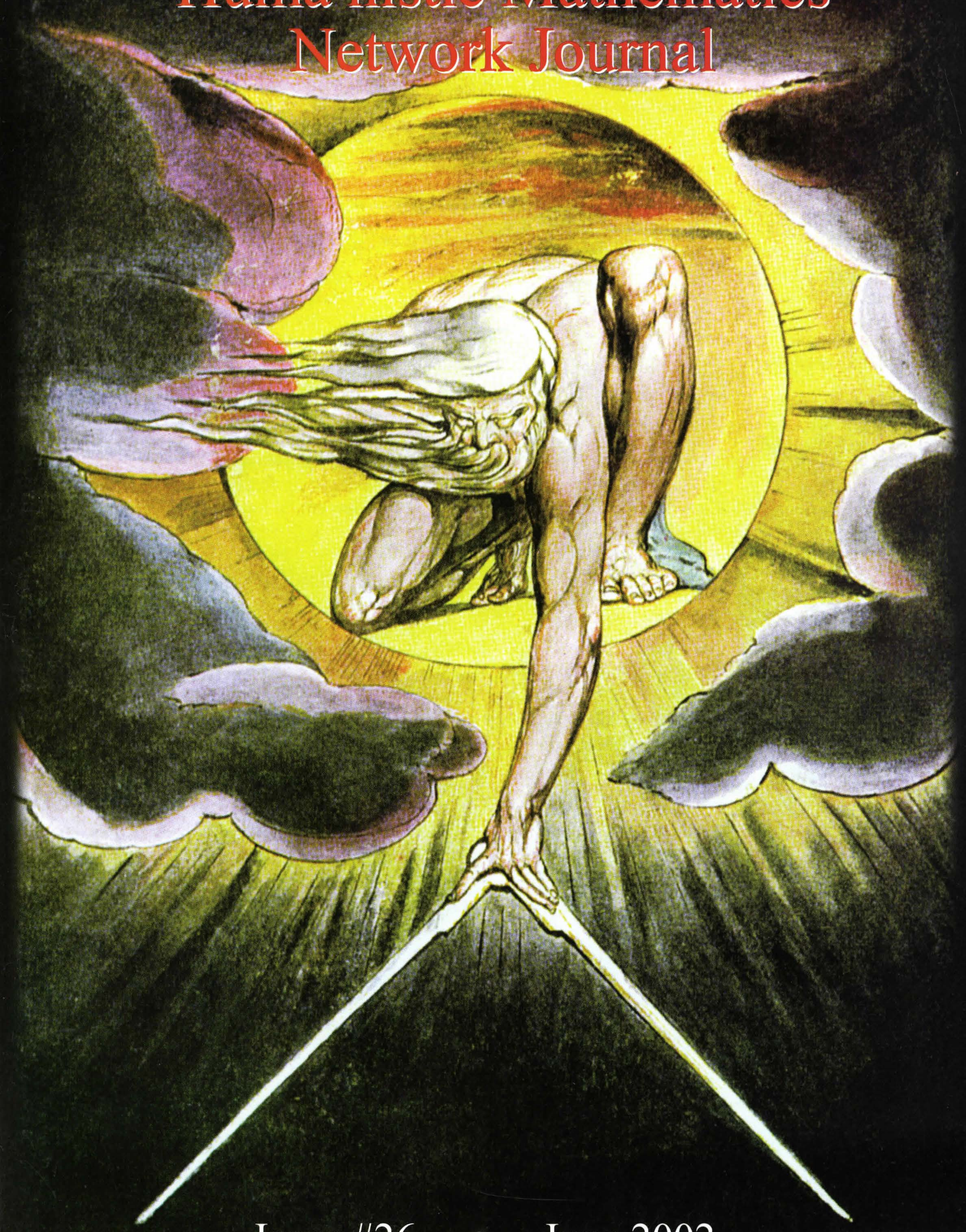

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COVER

The Ancient of Days Striking the First Circle of the Earth by William Blake. This image appeared in black and white in Humanistic Mathematics Network Newsletter #4, the first printed issue of this publication.

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Humanistic Mathematics Network Journal #26
June 2002

From the Editor

When newsletter #1 was published in 1987 it was an act of hope. That hope has been fulfilled by the support and good will of many readers, authors and friends worldwide as well as several program officers of the Exxon Education Foundation and its successors. I am grateful to all. I am also grateful to the presidents and deans of Harvey Mudd College, the chairs of the math department, and the technical staff for their support and encouragement. My gratitude also extends to the generations of Harvey Mudd students who have created the format, typed the articles to be sent to the printer, and performed the essential tasks that converted the authors' ideas into print.

The journal is in library collections all over the world. Future issues of HMNJ will be on the website created by the college at http://www2.hmc.edu/hmnj/

Although HMNJ will continue as an online publication, this last printed issue is a special occasion that should be marked. Thanks to the Exxon Mobil Foundation for support all of these years. Special thanks to Bob Witte, my program officer for most of the years, and to the other helpful officers of the Foundation. Also, special thanks to everyone who supported the journal with their articles and advice.

Starting and editing the journal has been rewarding and educational. I learned a few things and met (by mail and in person) many wonderful people.

My apologies to colleagues, friends and others whose articles could not be included or fully printed in the hard-copy journals. I look forward to continuing our associations and friendships through meetings and correspondence.

Comments on Zal Usiskin's Article in the Humanistic Mathematics Network Journal for May, 2001

Bob Stein

Zal Usiskin's article certainly brings out some important issues, but I think that with regard to the situation in California some additional comments are in order.

California's K-12 educational system was once a source of pride, but first the Serrano Priest decision cut local control of school finance, and the infamous Proposition 13 cut the flow of state dollars to schools. Ever since, the schools have been in desperate straights. Like the pigeons used by B.F. Skinner in his famous behavioral experiments, California schools are kept at 80% of normal body weight and their behavior is thus easily manipulated. For years now, California's K-12 education expenditures per child have been among the lowest of any state as a fraction of per capita income. Many of California's schools today are depressing places, overcrowded and poorly maintained. The supply of qualified teachers, especially in areas like mathematics, cannot keep up with the demand, because schools do not have the money or the contractual freedom to bid market price for the services of the best qualified people. Instead, shockingly large numbers of teachers are on one kind or another of "emergency" credential. (The term "emergency credential" is being replaced, which should fool nobody.) Naturally, it is particularly difficult to hire and retain good teachers in neighborhoods with other severe social problems. Teachers get tenure more or less automatically, unless they prove clearly incompetent in their first two years. Districts where it is hard to hire good people have little incentive to evaluate beginners too closely, because of all the expense and trouble needed to find a replacement who may turn out to be no better than the teacher who was released. Indeed, only by intense special efforts can we keep new teachers from leaving in their early years on the job. We produce too few good teachers, we have trouble keeping the best ones in the classroom, and we too easily give tenure to others who should not stay.

> **"...California schools are kept at 80% of normal body weight and their behavior is thus easily manipulated.**

California has term limits, which only encourage politicians to meddle in amateurish and often destructive ways that allow them to posture about their concern for education without having to be around to face the consequences of their actions. (I saw this first hand when, as a school board member, I inquired about the law which determines the annual dollar amount the state pays to each district per student. I thought this would be just a "word problem," but after an hour I was totally confused. Nobody at the school district office or the County Office of Education could figure it out either, nor could anyone at the California School Boards Association. They finally put me in touch with their legislative specialist, who explained that when the law was modified some years earlier, certain parts of it were not removed, although they must now be disregarded!) To compound the political circus, the Governor, then a Republican, implements his education policies through a State Board of Education, which he appoints, and whose CEO is independently elected and happens to be a Democrat, some say with gubernatorial ambitions. The politics of education in California, like the state's geology, is unstable, and a lot of time and money goes into finding faults. The resulting swings of the educational pendulum, which Zal Usiskin so ably describes, have a greater amplitude in California than in most other places, and as any student of seismic forces knows, the amplitude is directly related to the energy of the quake. Not only did California endorse "whole language," for example, but it effectively labelled those who insisted on teaching phonics as somehow backward and recalcitrant.

In view of the situation, it was hardly surprising that somewhere in California some parents would get upset about the mediocre mathematics education their children were getting. One district, for example, adopted as the seventh grade text a book which had been written not as a text but as a supplement to a text. This supplement, whatever its other merits, contained no explicit mathematical content at all. Parents grew irate when they were stonewalled by school officials who insisted that there was nothing wrong with the way mathematics was being taught, when in fact virtually no mathematics was being taught. These upset parents, now suspicious of the education establishment, organized into political pressure groups which found ready allies among the back to basics crowd and the political right wing. When it was announced that the California Mathematics Framework would be revised in 1997, two years ahead of schedule, the political jockeying turned into a full scale battle.

The Superintendent of Public Instruction, Delaine Easton, drew up a list of people to serve on the Framework Revision Committee, and the State Board of Education, reversing what they saw as a bias toward the status quo, quickly substituted its own choices for several of hers and convened the committee while she cried foul. I was not on Delaine Easton's original committee, but I was put on by the state board. (I do not know how I was chosen. I have a lifetime of experience as well as a Ph.D. in mathematics education, but I am not particularly conservative as an educator. My only book in mathematics education, Mathematics, An Exploratory Approach, explicitly opposes the drill and kill approach to teaching basic math. Maybe I was chosen because I was on a school board.) I accepted appointment to the committee, thinking it an honor, thinking I could help, and thinking I would get an education in the process. Only on the last count was I correct.

The committee of 22 was selected to be diverse geographically (rural, urban, suburban, north, south), ethnically, and in mathematical level (kindergarten through grad school). At best it would be difficult for such a group to reach consensus, but in this case it was made nearly impossible by California's naive public meeting law, which made it an illegal meeting if any three or more of us communicated by phone or e-mail outside of an official committee meeting!

If all this wasn't enough, the state legislature, acting entirely on its own, had set up a committee to draft mathematics standards for California, which were to be included in the Framework. However, the standards committee had no membership in common with the framework committee and the two groups never met together or communicated in any way. The Framework committee was to produce its document by a deadline several months earlier than the deadline for the standards, which the Framework was to include!

From day one, the framework committee was divided into camps of reformers and antireformers who viewed each other as enemies and would not listen to each other. Through the long, hot Sacramento summer, discussions became something to win or lose rather than to learn from, and it was difficult to get consensus on even the smallest details. I was one of very few who tried to be independent and in fact remained on relatively cordial, even friendly, terms with members of both camps, but I found the committee so frustrating that I publicly suggested that the state lock us away for several days with little beyond writing implements, a large stash of beer, and instructions not to come out until we could agree. Of course, nobody took that seriously, but it might have helped.

The committee produced a draft framework that had a lot of rough edges and really satisfied nobody. At least one committee member refused to sign it. Later, the mathematics standards were written into the Framework, which was extensively edited. The final result contains some atrocities and some valuable ideas. Among the latter, the curriculum is organized into five strands rather than eight, and a statement is included that all strands should not be given equal attention at every grade (This is at least in part a reaction to the amount of elementary school class time spent making histograms of things like ice cream flavors, which are not even ordinal variables, when the kids still can't add fractions.). I would like to report that the recommendation to make algebra a standard course in grade 8 is another outstanding achievement of the 97 Framework. At the time this seemed like a reasonable goal, since it is already done in many schools around the globe. The Framework Committee agreed that it would take California several years of careful work and planning to make algebra a standard course in grade 8, but they also realized that without a recommendation in this area no change would occur. How wrong we were about the effect of our recommendation. The Governor insisted on testing all 8th graders in algebra, whether or not they had a course in it, and school districts up and down the state mandated grade 8 algebra for all even though most of their seventh graders were not ready for algebra in grade 8 and most of their grade 8 teachers were not ready to teach algebra. This is a prescription for trouble, and we have already seen plenty of it.

As you can see, the political circus they call education in California goes well beyond any question of the merits of reform in the math curriculum. They say California often sets the trends for the rest of the nation to follow. Decades ago that was the case when bumper stickers were all the rage. Maybe it's time to revive the one which said, "Don't follow me, I'm lost."

Loopy

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As a mathematician and sculptor, I try to create artworks that are infused with an underlying sense of pattern and structure. I hope that viewers of my sculpture will consciously or unconsciously develop an appreciation for the power of mathematical ideas to enrich art. Ideally, they will be inspired to try their own explorations in the area of mathematical art. This paper considers one sculpture in some detail and provides instructions for a paper model that lets one explore its structure.

Figure 1 shows a sculpture I call *Loopy*, made of painted aluminum. It is five feet in diameter, and stands eight feet tall on its steel base. I created it for an outdoor setting at a local hospital; it is intentionally colorful and joyous. There are thirty identically shaped loops, six in each of five bright colors: red, orange, yellow, green, and blue. My vision of the form is abstract; some viewers tell me they see it as a giant brain, but that was not a conscious intention of mine. ("No, it isn't a self-portrait," I tell people.)

On examination, most viewers can discover and articulate essential aspects of the structure and color pattern: Each loop crosses four loops of the four other colors. The ends of the loops always meet in groups of three of the same color. Around each five-sided opening, there is a spiral of five loops, one in each of the five colors. No loop crosses another of the same color. Each bolt is both the vertex of a regular pentagon and the midpoint of the side of an equilateral triangle.

Each loop began as a ten-foot long strip of aluminum that was drilled for bolt holes, rolled into a loop, primed, painted, and bolted into the assembly. Thus, the thirty strips together sum to 300 feet, the length of a football field. While this arithmetic may be impressive in its own way, there is a different sort of mathematical analysis that I wish to emphasize.

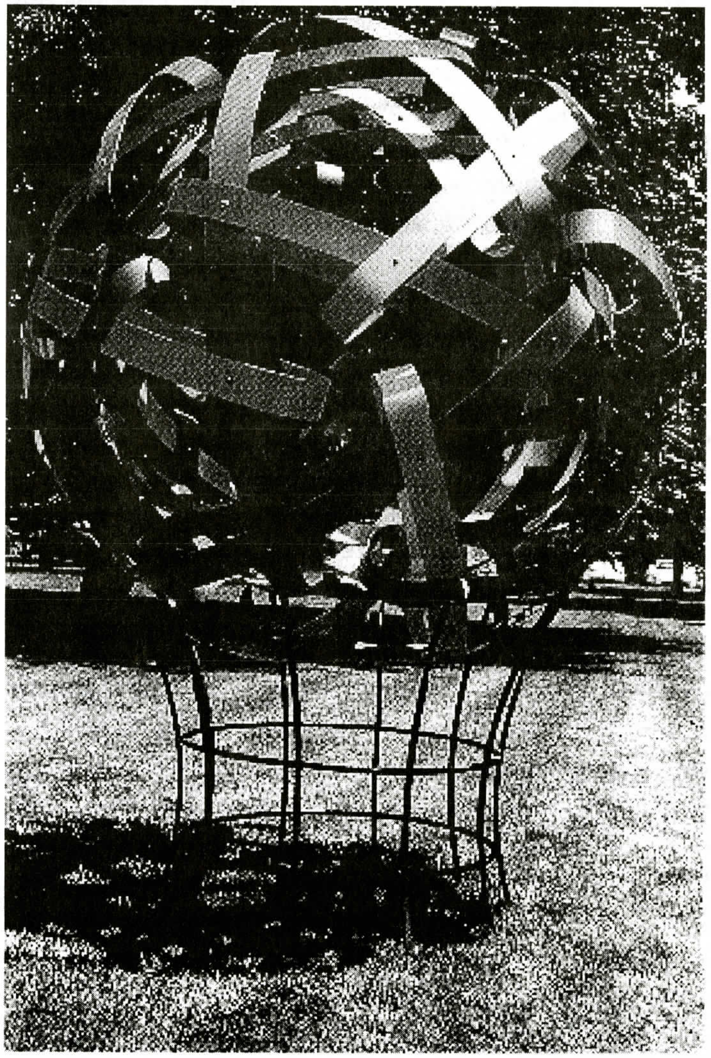

Figure 1
Loopy

Another sort of calculation one could make concerning the sculpture is to determine the angle at which the loops cross. Knowing only that the bolt holes are equally spaced and each loop is part of a great circle, with a bit of spherical trigonometry it is possible to calculate the angle at which loops meet. Pleasantly, it is not necessary to solve this, as the components automatically position themselves at the proper angle. This is analogous to cutting out twenty equilateral triangles and taping them together into an icosahedron. One does not need to know the dihedral angle of an icosahedron beforehand; the model automatically rigidifies to the necessary dihedral angles as the last edges are joined.

The mathematical properties that I consider most worthy of attention involve the structure and coloring. I do not know that I can articulate the intuitive reasons why I feel they are worth study. Perhaps the sculpture itself addresses that ineffable issue. But I do predict that if the reader or a student makes the following paper model, they will come to share some of this understanding. *Loopy* has a two-layer structure, which for architectural simplicity, this paper model simplifies to just one layer.

Cut out thirty strips of heavy paper, e.g., poster stock. Eleven inches by one-half inch is a convenient size if one starts with U.S. standard 8.5x11 inch paper. A paper cutter makes quick work of this. There should be six strips in each of five different colors. Use a hole puncher to make four equally spaced holes in each, as shown in Figure 2. Paper fasteners will be used to join the strips. They allow free rotation, and the proper angle will form on its own.

Begin by making a fivefold spiral as in Figure 3, using five different colors. The placement of the next five strips is indicated by the dotted lines. Note that the pentagon edges are one unit long but the triangle edges are two units long. Each strip has two "end holes" and two "inner holes." Each paper fastener connects one end hole and one inner hole. For a maximally attractive model, consistently place the end hole inside (underneath) the inner hole it joins. The assembly and coloring of the remainder of the model is then determined by logically keeping in mind three rules:

(1) Every opening is either a pentagon with edge length one or a triangle with edge length two;

(2) Each pentagon is surrounded by five triangles;

(3) The three strips that touch any triangle's edge midpoints are of a single color. See Figure 4.

In building this model, one will appreciate that there is a fundamental structure inherent in it. Why does it work? For example, why is one never forced into a situation where a strip touches another of the same color? Or why doesn't one come to a point around back where a triangle's midpoints contact endpoints of more than one color? A full explanation would involve a discussion of how the group of tetrahedral rotations is a subgroup of the icosahedral group. If

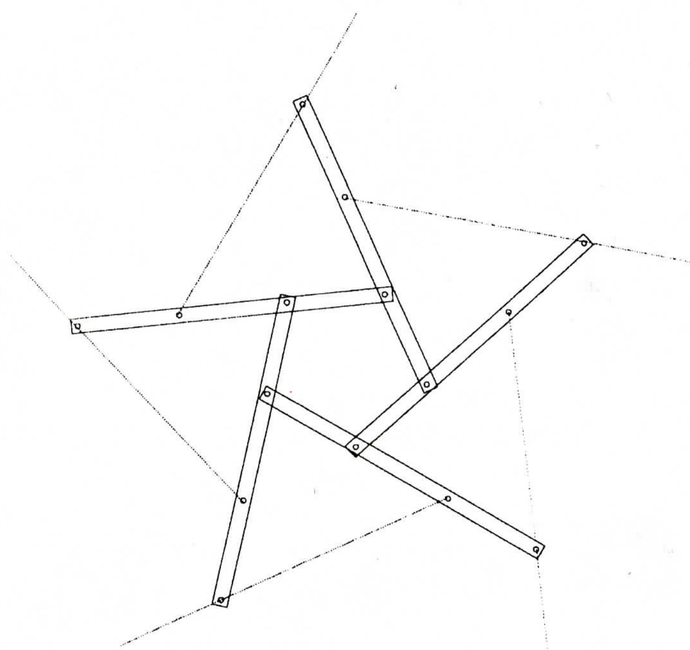

Figure 3
Start with five strips.
Dotted lines indicate position of next five strips.

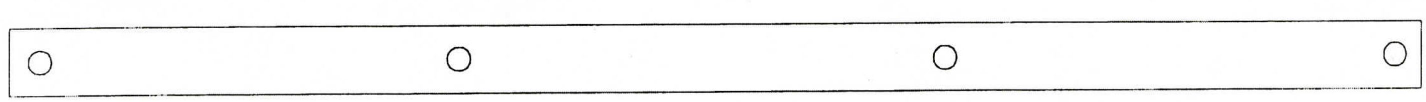

Figure 2
One strip. Space holes equally.

the geometric model known as "the compound of five tetrahedra" were better known, a study of its thirty edges could serve as a basis for understanding *Loopy*; each presents five groups of six. Similarly, one might wonder at how the paper strips work together as an analog computer, to solve the equation of the angle crossing. As I came to the form intuitively before doing any mathematical analysis, my feeling is that thinking about these questions goes hand in hand with just seeing that the form is beautiful.

For a well-rounded education, it is important that students develop the ability to discover new truths on their own. Let me suggest a few ideas for those who wish to start with this model and then explore further: *Loopy* has small pentagons and large triangles; try making the pentagons twice the edge length of the triangles. Notice you can start with a left-handed or right-handed spiral of five strips; what happens if you turn your model inside out? Or, try making a four-color structure with squares instead of pentagons. Or, design an analogous structure in which all the openings are triangles. Or, explore other color patterns. Can you generalize any of these models to a structure with hundreds of components? These questions might engage curious students.

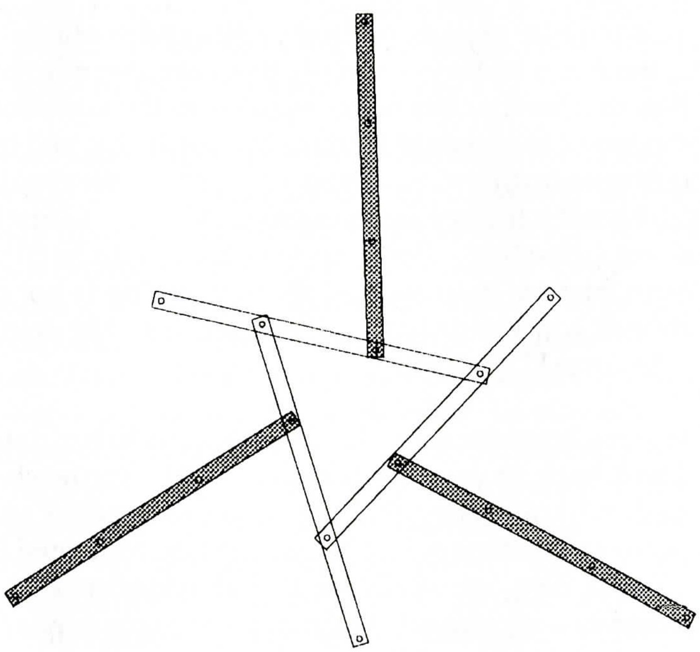

Figure 4
Three strips of same color end at midpoints of edges of triangles.

For me, one key aspect of the math-art connection is that both can concern space, and patterns or structures in space. Both also involve aesthetics in a hard to define manner, which involves much subjectivity. But a theorem or proof of Euclid's can be simply beautiful, and a mathematician sees this as clearly as an artist sees that a line by Leonardo can be simply beautiful. Both fields also involve creative individuals who find satisfaction in creating and communicating new forms that are the outward expression of their inner visions. I feel *Loopy* and my other constructive geometric sculptures are somewhere in this common tradition, and I hope viewers can appreciate the synthesis. Mathematics is about patterns, structures, and relationships of all kinds. Art as much as anything can bring this out.

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History of Mathematics, an Intuitive Approach

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ABSTRACT
The main goal of this essay is to discuss, informally, an intuitive approach to the history of mathematics as an academic discipline. The initial point of departure includes the analysis of some traditional definitions of the concept of 'history' taken from standard dictionaries. This concise dissection attempts to suggest the complexity of the discipline.

KEY WORDS: HISTORY, HISTORY OF MATHEMATICS, HISTORIOGRAPHY, METHODOLOGY, CHRONOLOGY.

The term 'history' is familiar to almost everyone, and most people believe they know its meaning intuitively. The lay person sometimes thinks of history in terms of dates, names, places and of colorful anecdotes on interesting characters. History provides a record of where someone lived and what he did. In short, history is the repository of the past. But what exactly is the history of mathematics as an academic discipline?

The events that took place yesterday are now part of history. Photographs are history: they reflect the way we once were. History is studied in elementary school, especially that of student's native countries. Teachers, who often seem old enough to have taken part in some of the historical events, teach -over and over again-anecdotes, names, places, dates, and so on. Historical movies, TV programs and books are popular. The media affects the way people understand history as a discipline. Unfortunately, sometimes the public's knowledge of important historical issues is derived from the popular media (especially movies), and not from professional sources. Thus, their understanding of these events can be distorted.

Unlike the word mathematics, the term history is used on a daily basis by the news media. Some reporters may believe that history is actually being made when they type or read the news. On occasions, reporters and anchormen have trouble attempting to disassociate themselves from the event they are reporting on.

Of course, in most cases, the term history is misused by professional reporters. Often, for example, TV sports commentators discuss player's individual records while narrating a baseball game. These statistics include the player's batting average, number of times at bat in the game, number of stolen bases, etc. Sometimes, the commentators also narrates the player's background. They mention the college the player attended; where he played in the minor leagues, his previous professional teams and so on. The commentator may also discuss some of the player's qualities as a human being (e.g., generosity, sportsmanship). Then, the sportscaster may try to explain why the player was motivated to become a professional. After the commentator has finished with these statistics and anecdotes, he often says: '... and, ladies and gentlemen, ... the rest is history'.

This popular expression 'and, ladies and gentlemen, ... the rest is history'- suggests that once the information or anecdote has been revealed to the audience, the rest of the account is common knowledge and requires no further explanation. Or, perhaps, the phrase '... the rest is history' is synonymous with '... the rest is unimportant' or '... the rest is well-known'. In fact, the term 'history' may suggest that something is not of current concern or lacks importance (e.g., My youth is history) [2, p. 614].

History, however, is much more than long lists of data. Think back to your student days. Did your teacher demand knowledge that was more substantial and profound that mere factual information contained in a list of data, especially on a test? Sometimes they asked essay questions which required some explanation of historical events. Consider an often asked question on why XV century navigators attempted to find a new route to India. Students, apparently, fail to distinguish between the factual aspects of historical data and the interpretation of historical evidence. In general, young students do not understand that the words 'history' and 'chronology' are not synonymous.

The word history may be defined as "[...] a narrative of events; [...]" [2, p. 614]. This definition seems to indicate that there is indeed a subtle difference between the terms chronology and history. History develops a narrative and, therefore, it is not simply a list of dates and names arranged sequentially. History, viewed from this perspective, is familiar to us. Historical works (e.g., a movie, a book, a play) contain more than just a simple list of names and dates. In many cases, the presentation (or form) of the material is as important as the raw material (or historical data) itself. For example, the movie industry attempts to produce and sell good films to entertain an audience, not necessarily to reproduce good history. For this, producers, directors and writers pay special attention to the presentation of the narrative. In most occasions, they even add external elements to produce a more interesting movie or a more attractive story. In fact, almost any historical movie or TV show contains a disclaimer asserting that some characters, situations and dialogues were added for dramatic purposes. Similarly, a scientist -being not a professional historian- may judge a historical textbook by its literary and entertainment qualities and not necessarily by its historical accuracy and objectivity. Some classical texts used for many years by the mathematical community may provide excellent examples [in particular, items 3 and 7 of the references].

Before attempting to refine the definition of history, it is important to understand that a historical scholarship does not necessarily arise from a description of past events -analogously, doing mathematics involves much more than dealing with numbers. Some people, other than historians, are constantly interested in the past. Take, for instance, a private investigator. He may be trying to solve a homicide case. His task is to reconstruct how a murder took place. Depending on the case, the private eye will have to answer certain questions. Some may deal with 'historical' factual inquiries: for example, attempting to answer when, where and how the event took place. Other questions may involve non-factual issues; for example, the state of mind of the killer at the moment of the crime. Is the private eye an historian?

He is trying to reconstruct the past, presumably the goal of the historian. No doubt there are some similarities between the two professions. Every private detective must be acquainted with some of the methodological techniques used by historians. He may have visited a newspaper archive and read old items. Perhaps, he may have studied autobiographical sources (e.g., diaries, address books, old photographs, unpublished correspondence, etc.) to determine the activities of the person he is investigating. On the other hand, the historian may enjoy the detective work of his profession. A wonderful example is remarkably illustrated by Reid's attempts to determine where Eric T. Bell spent his childhood [12]. Furthermore, it is always an intellectual challenge to find a difficult source, or to devise new ways of understanding or interpreting historical data, or to prove a point in a more effective way.

Nevertheless, private investigators do not always attempt to reconstruct the past. They are usually investigating events that occurred in the immediate past, and have not yet formed a conclusion. His investigation will lead to a conclusion (e.g., a court verdict), and will likely be influenced by the actions of the detective. Furthermore, his goal is not to reconstruct the past in itself, but to use some information about the event for other purposes. For example, some detectives/reporters have written books (or reports) on police investigations attempting to reconstruct the events associated with crimes committed in the past. One of these books [18], more than three hundred pages long, attempts to reconstruct the last day in the life of Marilyn Monroe. Some authors, on occasions, criticize the original investigation, presenting new evidence that may reveal the real reason that somebody died (e.g., Marilyn Monroe and JFK, among many others [see, for example: 15, 16 and 17]).

Historians may not only find certain aspects of a detective's methodology questionable (including the private investigator's use of sources, inferences, conjectures, goals and extrapolations), but may also criticize his/her lack of objectivity. It is extremely difficult for detectives not to become personally involved in the events surrounding the case. Detectives usually receive an economic reward for their activities (except perhaps for the fictional character Mike Hammer who seems to get always involved for friendship and personal reasons) and have an obligation to get results for their clients, not necessarily to find the historical truth. On the contrary, history, mathematics and most other academic disciplines, demand the highest possible level of rigor and objectivity. For example, Frege criticized Cantor's definitions of the concept of set [menge] and cardinal number because Cantor relied on each individual's mental capacity to abstract certain properties. That sort of definition was too subjective, depending on the personal perspective of each particular mathematician. In the same way, the closer the historian is to the person, event or idea that he/she is studying, the more difficult it is for him/her to present an objective reconstruction of the event. But let us attempt, once again, to define the term 'history'.

Some biographical treatises include, as a guide, a chronology listing major events and corresponding dates in a person's live. As discussed earlier, a chronological treatment of events is not necessarily the same as a historical investigation. Furthermore, some researchers may begin an investigation intending to conduct a historical study, but may not produce results that meet contemporary professional standards. For example, consider the term 'class'. A historian may ask who first conceived this term, and when and where the concept was first used. He might examine an old reference to find out whether the term class was already in use. If it was, then he could look up an even earlier reference and eventually find out who first used the term. It is very possible that the researcher's final report will simply be a chronology itself, naming the person who initially used the term, and how its meaning may have changed over time. But this chronological narrative may not explain why the concept was formulated in the first place or modified over time.

> "
> ***...history lacks the absolute character of mathematical reasoning. Within mathematics, once the basic premises are settled, there is no space for alternative results.***

A historian may argue that a chronological description of events does not necessarily fulfill the criteria of rigor established by the professional community of historians. In fact, it is quite easy to attempt to explain facts retrospectively. A mathematician may argue along similar lines. To the general public, any text using numbers may be thought to involve mathematics. Mathematicians, however, would regard this as simplistic; in fact, most professional books or articles in mathematics contain no numbers at all, except for the page numbers.

Sometimes, the nature of a historical discipline may be partially understood by asking why some people study it. Some people want to clarify who discovered a particular idea. Hubbert Kennedy ingeniously proposed 'Boyer's law', in part because of the many occasions in which credit for a concept has been given to the wrong person. Boyer's law states that "mathematical formulas and theorems are usually not named after their original discoverers" [8, p. 67]. Boyer mentions [4], between the XVII and the XIX centuries, at least thirty cases, of mathematical results that have not been credited to the appropriate person.

Sometimes, influenced by political ideology, historians are affected by nationalist sentiment, attributing (or attempting to attribute) discoveries (previously attributed to western mathematicians) to their own countrymen [13]. There is another reason why people study history: To understand the present by studying the past. Some professional mathematicians, sooner or later, decide to investigate the origins of the concepts they use today.

At present, less emphasis is given to describing events in the lives of the great men of mathematics than to listing discoveries in the various branches of mathematics. Historians of mathematics place greater emphasis on the development of ideas. They attempt to find the key concepts that influenced the development and evolution of their discipline. In particular, they try to find the key questions that played the role of Ariadne's thread conducting mathematical research. So, some historians attempt to reconstruct how mathematical ideas originated, evolved and learn how they influenced other ideas. Historians might also focus on the development of a particular school of thought and its 'philosophical' program. Others might attend the social influences that affected the thought of mathematicians or the relationship between the mathematician's work and the society in which he lived. Others might be interested on the social conditions surrounding the academic community or, on the contrary, on the effects of mathematical ideas on society. Some historians analyze the way that institutions and governments implement scientific policy. Others study the effects of economic factors such as production and consumption on science. Indeed, historians claim that ideas do not occur in a vacuum. Knowledge is a human creation, not a preexisting entity, closely associated with specific individuals, institutions and schools of thought.

No doubt, some people are tempted to write embellished accounts of events for posterity, including sports, knowledge (in this case, mathematics), politics, and, most importantly, war. Most likely, historians belonging to opposite ideological camps would present radically different accounts of the same event. For example, a Spanish soldier and an Aztec Indian would describe the conquest of Mexico in the XVI century from very different perspectives. The same situation occurs in the history of the American Civil War, as in the description of almost any human event. The triumphant narrative of a Yankee might differ from the pessimistic account of the Southerner. They may differ on the origins of the conflict, and the importance and consequences of some events. Here rests one of the major methodological differences between mathematics and history. History provides room for alternative interpretations. In fact, if different persons have witnessed an event, there may well be as many different accounts, as beautifully illustrated by the late Akira Kurozawa's Rashomon [Film. B/W. 1950], in which four different characters, all apparently equally trustful and credible, narrate four conflicting interpretations of the same event. The nature of history provides room for diverse reconstructions of the past, as reflected by the diversity of published historical material. These reconstructions will vary in their degrees of plausibility and consistency in terms of both chronological and technical characteristics of the discipline. Thus, history lacks the absolute character of mathematical reasoning. Within mathematics, once the basic premises are settled, there is no space for alternative results. For example, if within Euclidean geometry we have shown that the sum of the inner angles of any triangle is equal to the sum of two right angles [Euclid, 1-32], then there is no room for any possible alternative, thanks to the principle of excluded middle.

The possible existence of alternate interpretations may explain why, on historical disciplines, many historical monographs are published on the same topic. For example, Drake [5] and Koyré & [10] have presented different historical accounts on whether Galileo conducted empirical experiments. Later, other historians commented on the works of Drake and Koyré, discussing each author's viewpoints [19]. Still later, some other historians, using additional sources, may discuss the works of the earlier critics. And so on. But historians are not satisfied by merely presenting an alternative view of the events in question; more importantly, they want to convince their readers that their own conceptualization is better than earlier or alternative accounts. This last point is extremely important.

Indeed, in most cases, historians are not satisfied by merely narrating past events. For example, Ferreirés [6] convincingly argues that some of Dedekind's ideas underlie Cantor's theory of transfinite numbers. Likewise, Rodriguez-Consuegra consistently disputes [14] that there is a general philosophical method inspired by Russell's philosophy of mathematics (which he developed at the turn of the century), in spite of Russell's apparently endless intellectual evolution. In current thinking, an ordered (or unordered) collection of information on a particular topic does not constitute a historical study. As May has already argued: "history arises when chronology is selected, organized, related and explained." [11, p. 28 (my emphasis)].

History is also defined as "a branch of knowledge that records and analyzes past events" [2, 614]. The key word here is 'analyzes'. Contemporary historians do not record past events in a passive manner, but evaluate the past from a critical perspective. "If error and ignorance," as Adler and van Doren say, "did not circumscribe truth and knowledge, we should not have to be critical" [1, p. 166]. By critical, at this level, one simply means a position that evaluates and judges. Certainly, every historical treatise is based on a biased framework affected by the author's ideology and background. Historians cannot disassociate themselves from previous knowledge and be completely impartial. When an author has a critical attitude, however, he/she does not intentionally supports a political or ideological position.

The reader should not misunderstand the meaning of the word judge. The historian does not make ethical or moral assertions, nor is he/she concerned about the empirical truth of a concept. His goal is to determine why a particular novel idea, theory or interpretation was accepted over other conceptualizations current at the time, or immediately beforehand. In order to do so, the historian is required to ask appropriate questions: What problem was the mathematician attempting to solve? What conceptual tools were available? What comprises a 'rigorous' solution to the problem according to the standards of the time? In the case of the history of mathematics, Wilder has asserted that: "we don't possess, and probably will never possess, any standard of proof that is independent of the time, the thing to be proved, or the person or school of thought using it" [20, p. 319].

Historical understanding does not exist independently of other kinds of knowledge. Firstly, it is highly dependent on the intellectual background and historical assumptions of the practitioner. Secondly, it is abstracted from a totality surrounding it. The historian cannot encompass this totality and necessarily omits possible associations with other intellectual disciplines; very frequently he/she omits sociological factors. The historian understands that scientific thought is profoundly influenced by science, arts, technology, philosophy and theology, among other areas. However, the relationship of mathematics to these other disciplines may be obscured by current thinking which emphasizes the independence of mathematical thought. The task of the historian is to transcend the constraints of the present and reveal these influences.

There is no single approach to studying historical questions. Historical research is strongly affected by personal values, background, interests, the surrounding environment and the characteristics of the time in question. The only way to formulate interesting historical questions is to expand the knowledge of the past. The more one knows, the more one would like to know. It is obviously necessary to read the classics and the works of great historians. While doing this, keep in mind that, there was less of a rigid distinction between mathematics, the sciences, philosophy and others branches of knowledge up to the turn of the present century than there is today. Most of the intellectuals of the past were competent in all these disciplines. In order to understand the scientific ideas of an important historical figure, it is necessary to understand the person's theological and/or philosophical thought. Descartes, for example, made important contributions to philosophy, mathematics, physics, music and medicine, and, perhaps, to other disciplines. If the historian of mathematics wishes to understand Descartes' contributions to geometry, he/she needs to study his writings in other subject areas.

To conclude, consider the following analogy to historical research. Suppose that you are attempting to put together a one thousand piece puzzle. Most people begin with the following strategy: they separate out border pieces with a straightedge to form the frame of the puzzle. Then, if possible, they try to separate out pieces with the same color, to build small sections of the puzzle. Finally, with the help of the picture on the top of the box, they attempt to assemble all of the smaller sections together, to produce the whole picture. Similarly, historians are trying to reconstruct the past by presenting a picture, story or narrative to their audience. Unlike a puzzle, however, historians do not know the shape, size or the number of pieces of the puzzle! To make matters more complicated, the historian does not even know the overall 'image' of the events he/she is trying to reconstruct, and therefore has no framework (like the pieces that form the border) to begin developing the story. Moreover, the historian knows that he will never possess all pieces of the puzzle (except in trivial cases). So, at some point, he will have to conjecture (or imagine) the potential shape, size and design of some of his pieces. But to make the situation even more difficult, others may visualize the image(s) he wishes to present in a different way. Furthermore, even if they have the same general perspective, another colleague may visualize an individual piece or sections of the puzzle in a different way. The reconstruction of the past is never final, unique or totally accurate; but a plausible and logical synthesis of the available evidence.

Most importantly, after a historian has developed a plausible interpretation of the evidence, he has the difficult task of proving the soundness of his formulation, just as a mathematician must prove a new result. It is not acceptable just to assert a new point of view. Knorr [9], in his monograph discussing the history of some pre- Euclidean concepts, provides a convincing elegant argument for the specific date for the discovery of incommensurable quantities and a general goal for the entire corpus of Euclid's Elements. Then, he goes on to argue why this is a plausible case.

We have considered several important and profound differences in the methodologies used in mathematics and historical research. One of the most important differences is apparent when the scholar begins a new project. The mathematician is usually aware of the future outcome of his/her research (e.g., a proof of the well-ordering theorem, a proof of Fermat's last theorem, etc.). It is usually extremely difficult to reach the goal (sometimes it has taken more than 300 years!), but, at least, the goal is known. On the contrary, the historian does not usually know what conclusions he/she will make. He does not yet know the key questions, concepts and results. Most importantly, the historian is unaware of the driving factors that influenced a mathematician's career. To some extent, it may be very simple to list and describe the publications of a certain individual. But it might be extremely difficult to learn what influences affected the mathematician's ideas and contributions. On some occasions, after reading dozens of articles or hundreds of manuscript unpublished folios, which may or may not be related to each other, the historian might find himself in complete darkness, without understanding the pivotal ideas behind the mathematicians's concepts and publications. As May pointed out, despite much hard work and effort in conducting his/her research (searching, finding, organizing and selecting), the historian may not be able to explain what underlay a mathematician's thinking.

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Humanizing Mathematics: The Humanistic Impression in the Course for Mathematics Teaching

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INTRODUCTION

What kind of contribution can education make in supplementing what takes place in the political arena so that future generations will be less prone to define their personal and national self- interest primarily in adversarial and military terms?

This article is based upon the assumption that education can make a difference. How much of a difference it can make and how long we will have to persist before we achieve results that are more than negligible is an issue of great concern. But we do need to make at least modest beginnings if there is any hope that we will not blow each other up in a worldwide effort to eradicate differences. As a beginning, we need to think of education in all areas from a more humanistic perspective.

My particular interest is with the subjects of mathematics, technology, and the natural sciences. In contemporary primary and secondary education they are, for the most part, presented as devoid of emotional, cultural, and humane values.

There are many reasons for concern over this amoral, ahistoric, and isolated conception of the scientific disciplines. Mathematics in particular is usually perceived by the public, and has generally been presented by teachers, as an area of knowledge characterized almost exclusively by facts and truths. What can be changed in the education of mathematics teachers that will enable them to participate in a more humanistic curriculum that acknowledges diversity?

Something of particular concern to this author is that this kind of narrow approach that does not refer to basic human values tends to ignore differences in students' religious and cultural backgrounds, as well as the traditional and cultural quality of the mathematical experience. Thus, the approach does not strive towards better understanding among students from different backgrounds, nor towards having them seek a common ground. Not only is there a total lack of connection with emotional and humane values that might enable students to appreciate their own and each others culture, but by ignoring the lengthy birth pangs of the evolution of ideas, this warped view of the scientific disciplines inaccurately portrays the nature of mathematical discovery itself.

Though in general, students are more diverse than we tend to acknowledge, I am concerned in particular with the differences among the students whom I teach, who come from two very different cultures. I teach in southern Israel, at Kaye College, a teachers' college that attracts both Jewish and Bedouin students. Kaye College was established in the 1950s as a teachers' seminary, and in 1982 was transformed into a college. Since the early 1970s, it has offered teacher training for all teaching tracks, including the Bedouin sector track. Efforts to attract Bedouin began three years ago, when the college designed a "Couples Project" course to address curricula that acknowledges the different cultures.

While formal education has been part of Jewish tradition for centuries, the nomadic Bedouin in this region first began sending their children to educational settings in 1948, when the State of Israel was established. Today, the majority of Israel's Bedouin have settled in permanent cities, towns, or villages and work in agriculture, construction, and education, among other fields, and the number of Bedouin students in higher education is constantly increasing. The Israeli government looks to education as means of Bedouins' integrating into Israeli society (Mofet, 2001; Ben-David, Y., 2001).

The students from both cultures come together in a course I have designed entitled "History of Mathematics and its Interlacing in Teaching Mathematics". They know very little about the development of mathematics, let alone the contributions each of their cultures has made to this development.

There is a need to address the question of what can be changed not only because of the specific cultural diversity I face with my own students, but because of cultural diversity throughout much of the world. By changing the basic principles of, and setting new goals for mathematics teaching, we can extricate ourselves from the discipline's silence on these vital matters. Teachers must be equipped with strategies for conveying mathematical knowledge that transcend skills for structuring logical foundations of scientific thought. Pupils must not only assimilate a concentration of logical phrases and ways of thought that underpin mathematical knowledge, but they must learn to do so through a process that reflects, at least to some extent, the historical ways by which humankind arrived at such knowledge.

HISTORICAL AND HUMAN PERSPECTIVES

The mathematical text, with its abundant symbols, poses numerous difficulties - even impossibilities - for many studying such a seemingly bland subject. For this reason, teaching ways of making the mathematical text come alive is of the utmost importance. One idea that has gained momentum in the past few decades is teaching mathematics through history.

Many researchers acknowledge the importance of history of mathematics in mathematics teaching and learning. Garner (1996), who reviewed the professional literature on the issue, concluded that "the study of history is essential for those who would attempt to teach mathematics." In a previous article, Garner (1995) claims that "students may be brought to a more meaningful understanding" of mathematics topics and "gain insight into how pure mathematics feeds applied mathematics in [which] seems abstract." According to Harakbi (1994), a "retrospective look at the historical development of mathematics allows the teacher to refresh and deepen both the understanding of a specific topic and didactic ways of presenting it. Engaging in the history of mathematics requires adoption of different points of view, including exposure to the obstacles and prohibitions mathematicians encountered on their way to glory. Therefore, experience in this area can be a factor in reducing anxiety about math" (Harakbi, 1994).

But history has much more to offer than merely revealing the steps through which an idea evolved - which itself can be viewed as yet another dry exposition to be absorbed. Educators often assume that there is no connection between the mathematical experience and emotional concomitants. In fact, emotional components are lacking at every turn in the development of mathematics and of personal and historical efforts to understand it. To remedy this situation, the study of an historical development of mathematics should not only reveal the evolution of ideas, but also investigate the emotional arena within which these ideas developed.

In a letter to his son, when the latter was on the verge of creating non-Euclidean geometry, Bolyai's father made a telling remark: "For God's sake, please give it up. Fear it no less than the sensual passion, because it too may take up all your time and deprive you of your health, peace of mind, and happiness in life" (Boyer, 1968). Yet Sizer (1984) wrote, "Learning is human activity, and depends absolutely on human idiosyncrasy." The latter statement is based on the idea that most prospective teachers do not begin their mathematical studies with human idiosyncrasy. Rather, it is obvious that the opposite is true. We must create an atmosphere and conditions in the classroom that are conducive to understanding, and teaching students to tolerate, what is different not only between two individuals from different cultures, but also between individuals who are apparently highly similar in say, two people from the same family.

The purpose of this paper is to show how a course in "History of Mathematics and its Interlacing in Teaching Mathematics" provides an opportunity for all involved to develop introspection about the nature of mathematics, while at the same time refreshing and strengthening the participants' espoused and non-espoused humanistic values. The paper examines how the history of mathematics can aid in the teaching of mathematics to students, in the following ways:

- Acknowledging the idiosyncratic world views of the mathematicians who developed the ideas

- Seeking an arena within which we can arrive at a better understanding of the emotional components in the development of mathematics

- Celebrating cultural diversity
- Intensifying a humanistic world view.

THE COURSE DESCRIPTION

A course such as the one we are discussing would traditionally be called "History of Mathematics," and this was the case at Kaye. However, some four years ago, the name of the course was changed to "History of Mathematics and its Interlacing in Teaching Mathematics," with the aim of emphasizing the humanistic aspect of mathematics teaching/learning. Along with the new course title came a new syllabus, which was constructed both to improve teachers' attitudes towards teaching mathematics and to expose them to diverse ways of learning mathematics.

The course participants are from two sectors: in-service teachers, ranging from novice to veteran, currently enrolled in a B.Ed program, and preservice mathematics teachers in their third year of college studies. Both groups include Jewish and Bedouin students. This diversity creates a rich opportunity for devising cooperative learning strategies in the classroom.

A discussion of the syllabus is held every year, at the beginning of the course; the discussion focuses on syllabus goals, content expectations, and terms. The discussion intensifies when course participants are informed that they will be: a) preparing teaching units during the school year; b) participating with each other in presentations; c) working in pairs or in threesomes; and d) becoming acquainted with the unit they will be preparing, which is to be included in a volume that is a compilation of their collective work over the year.

What usually occurs at this point is that the students, who have never met before, look each other over and openly or covertly seek out a partner. In this first lesson, the instructor does not dictate whom the students should choose. Later, should anyone seek advice, the instructor may make a suggestion. As for pairing up Bedouin and Jewish students, I direct the students to think crossculturally, aiming for at least some mixed pairs, in accordance with the integrated tone of Kaye College. It should be noted that although the preservice teachers are already accustomed to working in pairs during their studies, the in-service teachers may not be. Another issue is how to combine history content with mathematics content in the lesson set - that is, how to combine the science of symbols with the humanities when that science ostensibly negates any involvement with the humanities.

The first semester is devoted to familiarizing students with the chronological development of mathematics, reading and understanding texts from the history of mathematics, and attempting to make these texts more accessible to and appropriate for pupils by turning them into an integral part of the mathematics lesson. Most of the participants already have some knowledge of the better-known mathematicians and their discoveries.

I present the students with examples lesson plans compiled either by myself or by students of this course in previous years. These lesson plans give rise to intense, critical discussion, sometimes positive and sometimes negative. Often, comments such as "We'll do it differently" are heard; these comments indicate the teachers' caring and their desire to create a better product.

One lesson set, for example, is "Area: What is it?" This set has three sections, each focusing on a different sphere of the mathematics of area: geometry, geometric algebra, and algebra. Here, historical material is combined with mathematical material, and discussions of important personages in mathematics are set against solving mathematical problems (Katsap, 2000). Another example is "The Golden Section," which combines knowledge of ancient Greece, architecture, art, the works of Leonardo Da Vinci, and the beauty of nature. These elements touch on much of mathematics as well as on many humanistic areas in which mathematics plays an integral part. This first semester focuses on "classroom cohesiveness" and work in small groups. The students continually present the products of their small-group work to the class; later, these products will include units prepared by the students. This develops the students' openness and their desire to help other group members and to obtain help from them, and also fosters healthy competition and an orientation towards achievement.

At the mid-point of the first semester, the students are given guidelines for preparing their units, as follow:

- The selected unit topic is to be integrated into mathematical areas studied in school

- There is no restriction on historical eras

- Relevant areas from the history of mathematics can include: individuals who made a contribution to mathematics; mathematics problems; mathematical discoveries; and any historical material connected with mathematics

- The study unit may: a) combine an historical mathematical text with a mathematical topic relevant to teaching in the form of a complete lesson plan; or b) integrate a sequence of passages into lesson plans for the study of a particular chapter in mathematics in the school.

The variety of options offered the students in choosing a study unit makes the choice easier for them, and gives them a greater sense of independence and self-confidence. At the same time, however, it means that they must take responsibility and demonstrate competence, criticism, and autonomy in their decision-making. All this may sound familiar to instructors of teacher education courses, but again, combining a humanistic subject such as history with a scientific subject such as mathematics is somewhat unusual. After all, when have mathematics teachers had to struggle with the question of whether to present an historical tale to the pupils and allow them to discover mathematical elements in it, or to teach them mathematical laws (as is usually done), and then tell them about the life and times of the individual who discovered the laws? Or, if an interesting historical discovery does not fit in with the mathematics curriculum but the teacher finds it fascinating and wants to present it to the pupils, how should it be combined with the regular mathematics lesson? In cases, the teacher (i.e. the students in my course) must take responsibility and use his or her autonomy to make a decision.

Towards the end of the first semester, students choose partners, and a timetable for the students' second-semester presentations of the study units is drawn up. Each student pair begins the process of selecting a topic to be investigated, prepared, and presented to the class in a 45-minute presentation. Although this process is already familiar to the students from their previous studies, the innovation here is the manner in which the students repeatedly consult with each other before approaching the course instructor for help. Most of those who do consult with the instructor need help in choosing and organizing the material collected in their search, or in finding written and electronic information sources. The Internet plays a tremendous role in knowledge sources, and learning to use it is an additional learning skill that this course helps develop. Most of the topics and content chosen by the students are accepted.

Some of the students "try out" their study units first on their own pupils in the classes they currently teach. They then can tell their fellow course participants how the pupils responded. At the end of the unit presentation, a discussion, usually quite lively, is held in the course class; usually both constructive and negative criticism is offered. Often, there is a sense that the students say what they have to say for themselves more than for others, and give themselves tips for the future.

Not surprisingly, none of the lessons presented by the students in the second semester take the form of a lecture. They all involve group class activity, whether it be collaborative work, discussion, or drawing conclusions in groups and in full class discussions—providing a welcome experience in changing teaching methods and seeking diverse ways of improving teaching.

At the end of the second semester, after all work has been presented in the class, two students are chosen to compile the material on the computer and, together with me, edit the volume. These students reported that they found it very rewarding to process the separate study units presented during the course (to be discussed later) into a structured volume. The students also participate in discussions to determine topic classifications and to organize the study units accordingly. Each of the topics includes four to five units. Some of the titles that have come up in past courses are: Mathematics in Judaism and Islam; From Primitive Counting to Probability; Numerals, Numbers, and Sets of Numbers; Science, Art, and Craft; and Journey to the Roots of Geometry. The students claim that this work gives them a sense of contributing to the course's collective effort to promote advancement and innovation in mathematics teaching in the school. To date, four volumes have been published.

What is challenging in this course is the need for the instructor to flow with the class, never knowing what topics will be raised by the students during the year. This keeps the instructor on his or her toes, and stimulates the students' and the instructor's interest and curiosity. Together, the instructor and the students form a learning community requiring constant communication and cooperation. Friere sees such a situation as the ideal learning circumstances for liberating education, in which "teacher and pupils form a community of learners, and where both sides are essential factors in the process of obtaining knowledge" (Friere & Schor, 1990).

UNIQUE TOPICS

Some extremely interesting topics arose in this course following the students' extensive searching for knowledge sources. Before I compiled the course, I discovered the book Ayil Meshulash (Stature of the Triangle) (1960), written in the 19th century by a student of the Jewish Torah scholar the Vilna Gaon (Genius from Vilna). It was based upon notes found after the Gaon's death, that derived from his oral mathematics teaching. In this book, subtitled "On the wisdom of triangles and geometry and some rules of qualities and algebra," author Shmuel Lukenik notes that the Gaon was putting the notes and explanations in book form to preserve them for generations to come.

The Vilna Gaon, or Rabbi Eliyahu was a master of Torah, Talmud, Jewish philosophy, Halacha (Jewish religious law), and Kabbalah (Jewish mysticism). The bulk of his written work concerns corrections and emendations of Talmudic texts, and interpretations of the Shulchan Aruch (code of Jewish law). In the academic field, the Vilna Gaon wrote on geography (The Form of the Earth) and grammar (Eliyahu Grammar). He was also interested in music, claiming that most of the arguments of the Torah would be incomprehensible without it (On-line Resources, 2001). According to some sources, the Vilna Gaon is the author of "The Gaon's Theorem," a principle of the mathematics of infinity (Feldman, A., 1999). Other sources claimed that this theorem was called "Kramer's Theorem," Kramer being the Vilna Gaon's family name. Gerver (1993), however, stated that this supposition was unlikely, as the author of this article did not present proof for these two opinions.

After reading Ayil Meshulash, I had the idea of including in the course Jewish and Islamic sources of mathematics history and integrating them in mathematics teaching. The result was a collection of materials included in a chapter entitled "Mathematics in Judaism and Islam," unique to this course. Every year, a number of student pairs, both heterogeneous (Jewish and Bedouin) or homogeneous, work on this chapter.

Ayil Hameshulash was frequently chosen for investigation by these students, as it is both a mathematical work including mathematical explanations and geometrical definitions, theorems, and proofs as well as selected topics in algebra, and an historical document. Although the book was written 200 years ago, it still contains explanations that cannot be found today in any other mathematical work. The students in the course maintain that this book offers better explanations than those used today, and often say that they find it exciting to study such a venerable work. In one of the lectures, the students compared the Vilna Gaon's presentation of the right-angle triangle with that of Israeli mathematics teacher Benny Goren in his book Plane Geometry - a text currently in wide use in Israeli schools. Most of the students in the class announced that they would adopt the former method in teaching this topic in their classes.

CULTURAL DIVERSITY

Not surprisingly, homogeneous Jewish student pairs usually seek material on Jews who contributed to mathematics, as well as material found in the Torah, which is full of mathematics (the Jewish tradition of interest in studying mathematics comes from the Torah).

Homogeneous Bedouin pairs of students usually seek mathematical material authored by Arabs. It is interesting to note that the mathematicians they choose are often already familiar to them, but only as clerics or poets. As one of these is the Persian mathematician, astronomer, philosopher, physician, and poet commonly known as Omar Khayyam. Khayyam may be best known for The Rubaiyat, but his book Maqalatfi al-Jabr wa al-Muqabila is a masterpiece on algebra and is of great importance in the development of algebra.

The Bedouin students' discovery of so many significant mathematicians gives them a different view of mathematics, as well as greater pride in their people. In addition to their investigation of these individuals and their contributions to mathematics, the Bedouin students bring in Koranic writings, and all the students work on mathematical problems found therein—thus imbuing them with greater pride in their Arab heritage.

Investigating the Jewish and Arab mathematicians arouses a great deal of interest among the students participating in the course. As it is the first opportunity both the Bedouin and the Jewish students have to interpret the original mathematical language of the Vilna Gaon, in the ancient Rashi Hebrew of the time, as well as the mathematical language of the Arabic Koranic text. Thus, the students unwittingly acquire humanist values, such as respect for the history and tradition of their own and other peoples.

REFLECTIONS ON HUMANISTIC EDUCATION

What does it mean for education to be humanistic? This course enabled the instructor to come up with a number of tentative answers to this question. Brown (1996), who discussed humanistic mathematics education, claimed that first of all, there is a need to deepen the understanding of the perception of humanism and human nature, and their lengthy history. After that, interpretations of education should be studied in depth, along with the development of the humanistic perception. Finally, mathematics itself should be reassessed. In another article, Brown (1993) asks: "How might we use mathematics to convey knowledge and attitudes towards the world and about oneself that would be valuable in many non-mathematical contexts?"

As of this point, I am currently investigating in more than anecdotal way the implications of a course of this sort at Kaye College. Nevertheless, initial findings from the pilot study show that most of the students gained a different view of mathematics. This was indicated by their considerable appreciation and esteem for the value of the history of mathematics, and for its influence on their view of mathematics and mathematics teaching in the school. The students became more open to and more willing to accept knowledge accumulated through history that could be defined as a humanistic value - i.e. appreciation of the evolution of ideas of the many false starts that are part of the journey.

Another humanistic value mentioned by the students in the pilot study was respect for different cultures and appreciation for other eras. This may sound obvious for history or literature classes, but it is rare with regard to mathematics classes. The pilot study also found that the course gave the participants a sense of power, and taught them to deal with new challenges. The covert competition that developed between the student pairs in presenting their material forced them to invest time and effort in a search for interesting, creative material and to find diverse ways of presenting their topic—and also to keep trying. All this can be classified as the humanist value of self-actualization.

One of the most important comments that came up in almost every lesson, in one way or another, reflected the students' genuine desire to pass the torch to their own pupils, and to make them aware of what they themselves had learned and experienced. This was evident among both the experienced teachers and the preservice teachers, and was evidence of these teachers' caring and sensitivity towards their pupils. I often heard the teachers say, "Let's send them [the pupils] to find the material themselves"; "Let's let the pupils prepare and teach a lesson, so they can discover things on their own." This leads us to another humanistic value: expressing humanity and respect for others.

Finally, the students arrived at the material through self-discovery. The experience of searching for the material and familiarity with and use of knowledge sources not previously accessible led to a humanist value that they expressed both orally and in writing: showing intellectual interest.

CONCLUSION

Increasing teachers' awareness of the development of mathematical knowledge can help them fulfill their professional commitment, even narrowly construed as teaching pupils and instilling in them habits of scientific thought. Naturally, their pupils do not absorb only the knowledge; they undergo a learning process that reflects, in one way or another, the process experienced by humankind in attaining this mathematical knowledge.

Teachers' lack of knowledge - or their attitude of outright dismissal - when faced with the question "Why?"

can make pupils passive, indifferent, or even averse to mathematics. Instead, teachers can encourage pupils to ask, "Why do I do this?" or "How can I explore this?" Here, I do not refer to the knowledge of experts in the history of mathematics, as that knowledge is purely academic and has little to do with mathematics education in the school. I refer only to relevant topics and areas of mathematics that enrich mathematics teaching, thus enabling the mathematics teacher to foster an atmosphere allowing mind and emotion to take flight together. Since the dawn of civilization, wise men and women have advanced mathematical knowledge, and have also educated the younger generation in "the qualities of the whole person." Much of the teachings of Socrates, Plato, Aristotle, and many others are devoted to humanistic education. Among the principles of Platonic humanist perception, we find values such as "Worthless is the life of he who lives without investigation and a critical view of what is good," alongside, "The full measure of a man is in his wisdom and knowledge" (Aloni, 1998).

But for pupils today as well as for most teachers, Pythagoras's theorem is no more than $c^2 = a^2 + b^2$. Knowing that this theorem has been in existence across the world for over three thousand years, and how it has been proven in many different ways, may excite them and lead them to join the ranks of those who see the beauty behind the rigid and ostensibly inimical language of mathematics.Hersh (as cited in Brockmam, 2001) states:

> Mathematics is neither physical nor mental, it's social. It's part of culture, it's part of history, it's like law, like religion, like money, like all those very real things which are real only as part of collective human consciousness.

The latter statement may be made more obvious while, in the marvelous book Even-Zifer the Doctor, Sissa (1960) tells the 12th-century story of the friendship between a Jewish doctor from Cordoba and the Christian royal scribe from Hungary, a friendship based on the love shared by seekers of knowledge in the Dark Ages and drawing on the human spirit. Both were enamored of the enchanted science called mathematics. Sissa writes, "Is it possible to integrate science into a story? Is there not something amazing in the history of science, as surprising and as fascinating as the history of nations?"

It is no wonder, then, that the students in the course discovered another world within a subject so familiar to them. As they explored the depths of history, the students learned of the hieroglyphics in the Rind and Moscow papyruses, mathematical discoveries from ancient Greece through the Middle Ages, stories of charity from the Koran, and astronomical calculations for the Jewish calendar. In addition, they looked at modem mathematics and mathematicians such as Descartes, Euler, Gauss, and Lobadievsky.

It may be debatable whether there is a parallel between the historical development of mathematics and the process of clarifying values expressed by students participating in the course. But the contact between the two areas—mathematics and the history of mathematics—as well as appropriate training methods gave rise to positive humanistic energies and a new view of the subject that the students had chosen to teach. It is to be hoped that this learning added another aspect to their professional humanist development as teachers.

Today's focus on technological means cannot replace the crucial human figure of the teacher, who is responsible for fostering pupils' curiosity about mathematics—a curiosity rooted in human emotion. Teachers constantly face the challenge of renewing education and ways of learning in teaching mathematics in order, as Hook (1994) puts it, to shape teaching practice in our society "and [to] create new ways of knowing and different strategies for the sharing of knowledge."

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A Brief Look at Mathematics and Theology

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Such a really remarkable discovery. I wanted your opinion on it. You know the formula m over naught equals infinity, m being any positive number? [m/0 = ∞]. Well, why not reduce the equation to a simpler form by multiplying both sides by naught? In which case you have m equals infinity times naught [m = ∞ x 0]. That is to say, a positive number is the product of zero and infinity. Doesn't that demonstrate the creation of the Universe by an infinite power out of nothing? Doesn't it?

Aldous Huxley, *Point Counter Point* (1928), Chapter XI

INTRODUCTION

We are living in a mathematical age. Our lives, from the personal to the communal, from the communal to the international, from the biological and physical to the economic and even to the ethical, are increasingly mathematicized. Despite this, the average person has little necessity to deal with the mathematics on a conscious level. Mathematics permeates our world, often in "chipified" form. According to some theologies, God also permeates our world; God is its origin, its ultimate power, and its ultimate reason. Therefore it is appropriate to inquire what, if anything, is the perceived relationship between mathematics and God; how, over the millennia, this perception has changed; and what are its consequences.

I begin with two stories. Recently, I spread the word quite among my mathematical friends that I had been invited to lecture on mathematics and theology. I wanted to get a reaction, perhaps even a suggestion or two.

One, a research mathematician, the chairman of his department, who, in his personal life would be considered very devout in a traditional religious sense, told me that, "God could never get tenure in our department."

Another friend, well versed in the history of mathematics, told me, "The relation between God and mathematics simply doesn't interest me."

I think that these two reactions sum up fairly well the attitude of today's professional mathematicians. Though both God and mathematics are everywhere, mathematicians tend towards agnosticism; or, if religion plays a role in their personal lives, it is kept in a separate compartment and seems not to be a source of professional inspiration.

There is hardly a book that deals in depth with the 4000 year history of the relationship between mathematics and theology. There are numerous articles and books that deal with particular chapters of the story. Ivor Grattan-Guinness has written on mathematics and ancient religions. Joan Richards has treated the influence of non-euclidean geometry in Victorian England. Joseph Davis and others have treated the attempts at the reconciliation of science and religion by Jewish scholars of the seventeenth century. But most historians of mathematics in the past two centuries, under the influence of the Enlightenment and of positivistic philosophies have avoided the topic like the plague.

This suppression has been an act of "intellectual cleansing" in the service of presenting mathematics as a pure logical creation, "undefiled" by contact with human emotions or religious feelings. It parallels the many acts of iconoclastic destruction that have overtaken civilization at various times and places and is still taking place. Why has it occurred? Numerous reasons have been suggested. Is it the Enlightenment and positivistic philosophies?

But things are now changing. The separation of mathematics and theology is now not nearly so rigid as it has been since, e.g., Laplace's day. There is now a substantial reversion in physics, biology, mathematics, etc to the older position. The material published runs from what is very thoughtful and sincere to what might be called "crazy." (And what is the test for what is and what is not "crazy"?).

Why? Is it part of the general perception that rationalism has its limitations? The current generation finds positivistic philosophies lacking in social and emotional warmth and in transcendental values. It is now trying to reclaim those values with syntheses of God, the Bible, Apocalyptic visions, the Nicene Creed, Zero, Infinity, Gödel's Theorem, Quantum Theory, the Omega Point, the God Particle, Chaos, Higher Dimensions, Multiple Universes, Neo-Pythagoreanism, Theories of Everything, etc. etc. I find that most of this is bizarre. When it comes to specific statements, such as "God is a mathematician", I find the discussions both pro and con unconvincing, but I would not say, as an older generation of positivists might have said, that the statement is meaningless.

The extent of the historic relationship between mathematics and theology should not be underestimated. There is much that can be and has been said. Practically every major theme of mathematics, its concepts, its methodology, its philosophy, have been linked in some way to theological concepts. Individual mathematical features such as number, geometry, pattern, computation, axiomatization, logic, deduction, proof, existence, uniqueness, non-contradiction, zero, infinity, randomness, chaos, entropy, fractals, self-reference, catastrophe theory, description, modeling, prediction, have been wide open for theological questions and answers.

As simple examples: is God constantly geometrizing? Does God have the power to make 2 + 2 other than 4? Does God predict or simply know?

The links between mathematics and theology are part of the history of mathematics and part of the mathematical civilization into which we were born. They are part of applied mathematics. In recent years, these links have been extended to embrace theological relations to cognition, personhood, feminism, ethnicity.

The contributions of mathematics to theological thought have been substantial. The young John Henry Newman (1801-1890. Later: Cardinal) asserted that the statements of mathematics were more firm than those of dogmatic theology. Hermann Cohen, philosopher (1842 - 1918) thought that mathematics was the basis on which theology must be built. In many recent discussions, as we shall see, mathematics takes objective priority over theology just as it did to Cardinal Newman. On the other hand, one should remind oneself of the ascending hierarchical order in the days of the Scholastics (e.g., St. Thomas Aquinas, 1225-1274): mathematics, philosophy, metaphysics, with theology at the apex.

In the other direction, the contributions of theology and religious practice to mathematics were also substantial - at least until around the 14th Century. As examples church and astronomical (secular) calendars are mathematical arrangements and needed reconciliation. The Jewish philosopher and theologian Moses Maimonides (1135 - 1204) wrote a book entitled *On* *the Computation of the New Moon*. Among Moslems, the determination of the qibla (the bearing from any spot towards Mecca) was important and fostered the development of spherical trigonometry. These various demands led to improved techniques and theories. Kim Plofker has only recently discussed the historical attempts to reconcile sacred and secular Indian cosmologies.

Astrology which very often had links to theology and religious practice, demanded exact planetary positions, and astrology stimulated and supported mathematics for long periods of time and led to intellectual controversy. Astrology carries with it an implication of rigid determinism and this came early into conflict with the doctrine of free will. The conflict was reconciled by asserting that though the stars at the time of one's nativity control one's fate, God has the final say, so that prayer, repentance, sacrifice, etc., undertaken as a free will impulse, can alter the astrological predictions. This is the message of Christian Astrology, two books with the same title written centuries apart by Pierre Dailly (1350-1420) and by William Lilly (c. 1647).

With the discrediting of astrology as a predictive technique (even as it remains a technique for individuals to shape their daily behavior), such contributions have certainly been much less publicized or emphasized in recent mathematical history than,, e.g., technological or military demands

On a much wider stage and at a deeper level, claims have been made and descriptions have been given of the manner in which Christian theology entered into the development of Western science. Here is the contemporary view of Freeman Dyson:

> Western science grew out of Christian theology. It is probably not an accident that modern science grew explosively in Christian Europe and left the rest of the world behind. A thousand years of theological disputes nurtured the habit of analytical thinking that could be applied to the analysis of natural phenomena. On the other hand, the close historical relations between theology and science have caused conflicts between science and Christianity that do not exist between science and other religions.... The common root of modern science and Christian theology was Greek philosophy.

The same claim might be asserted for mathematics, though perhaps with somewhat less strength.

A few western opinions over the ages, arranged more or less chronologically, should give us the flavor, if not the details, of the relationship between mathematics and theology. (See e.g., David King for Islamic writings, and David Pingree and Kim Plofker for Indian.)

However, while citing and quoting is a relatively easy matter, it is not easy to enter into the frame of mind of the authors quoted and of the civilizations of which they were part; how the particular way they expressed themselves mathematically entered into the whole. Thus Plofker has written:

> It is difficult to draw a clear and consistent picture of the opinions of authors who reject some assumptions of sacred cosmology while espousing others... To many scholars eager to validate the scientific achievements of medieval Indians according to modern criteria, the very notion of their deferring to scriptural authority [the Puranas] at all is something of an embarassment.

To appreciate this, it helps to remember that the secularization and the disenchantment (i.e., disbelief in ritual magic) of the world is a relatively recent event which occurred in the late seventeenth century. For an older discussion of this point, see W.E.H.Lecky.

To quote contemporary historian of mathematics Ivor Grattan-Guinness:

> Two deep and general points about ancient cultures are often underrated that people saw themselves as part of nature, and mathematics was central to life. These views stand in contrast to modern ones, in which nature is usually regarded as an external area for problem-solving, and mathematicians are often treated as mysterious outcasts, removed from polite intellectual life.

And David Berlinski, (contemporary, philosopher, and science writer) writes:

> As the twenty-first century commences, we are largely unable to recapture the intensity of conviction that for all of western history has been associated with theological belief.

I now shall present numerous clips, mostly of older authors, organized according to certain mathematical topics.

Number

Perhaps the earliest mathematics/theology relationship is "number mysticism", the attribution of secret or mystic meanings of individual numbers and of their influence on human lives. This is often called numerology and its practice was widespread in very ancient times. Odd numbers are male. Even numbers are female. In Babylonia the numbers from I to 60 were associated with a variety of gods, and these characteristics are just for starters. Since alphabetic letters were used as numbers, the passage from numbers to ideas and vice versa was rich in possibilities.

"All is number," said Pythagoras (c. 550 BC), around whom a considerable religious cult formed and whose cultic practices seemed to involve mathematics in a substantial way. The historian of mathematics, Carl B. Boyer, wrote, "Never before or since has mathematics played so large a role in life and religion as it did among the Pythagoreans."

The words "or since" may be easily challenged without in the least denying the importance of mathematics for the Pythagorean Brotherhood.

Some mathematical mysticism occurs in Plato's Timaeus. There, Plato (c. 390 BC) takes the dodecahedron as a symbol for the whole Universe and says that: "God used it for the whole." For Plato, the world has a soul and God speaks through mathematics.

Ideas of number mysticism spread from Pagan to Christian thought. The Revelations of John (c. end of 1st Century) is full of numbers and of number mysticism. For example:

> Here is the secret meaning of the seven stars which you saw in my right hand and of the seven lamps of gold: the seven stars are the angels of the seven churches, and the seven lamps are the seven churches. (Rev. 1:20).

And then, there is the famous, oft quoted passage in Revelations 13:18:

> ...anyone who has intelligence may work out the number of the Beast. The number represents a man's name and the numerical value of its letters is six hundred and sixty six.

Innumerable computations of the Second Coming, or of the Days of the Messiah have been carried out. The idea that the end of the world can be computed is very old.

The Apocalypse, foretold in Revelations, and said to precede the Second Coming, has been and still is a favorite subject for mathematical speculation and prediction. The predictions are usually made along arithmetic lines and make use of some method of giving numbers to the historic years. For the details of a computation of the date of the Apocalypse carried out by John Napier (15 50 - 1617), the creator of logarithms and one of the leading mathematicians of his day, the reader is referred to the splendid book of Katharine Firth.

Religious authorities have often proscribed such computations. But such computations have never disappeared and the desire to calculate the end of days is present in contemporary end-of-the-world cosmologies based on current astrophysical knowledge as well as in tragic episodes of religious fundamentalism.

Iamblichus (c. 250 - 330), a Neo-Platonist, in his *Theologoumena tes Arithmetikes* (The Theology of Arithmetic) explains the divine aspect of each of the numbers from one up to ten.

St. Augustine (354 - 430) asserted that the world was created in six days because six is a perfect number (i.e., a number equal to the sum of its divisors). Augustine also said: Numbers are the link between humans and God. They are innate in our brains.

In the 12th Century, the Neoplatonist Thierry of Chartres opined: "The creation of number was the creation of things."

The colorful mathematician and physician Geronimo Cardano (1501 - 1576) cast a horoscope for Jesus and earned thereby the wrath of the hierarchy.

The humanistic Shakespeare, whose works display little religious sentiment, has a line: "There is divinity in odd numbers." Was he perhaps picking up on the Trinity and the mystic number 3 ?

Blaise Pascal (1623 -1662), an early figure in the development of probability theory, "proves" the existence of God by means of a wager:

> God is or He is not. Let us weigh the gain and the loss in selecting 'God is.' If you win, you win all. If you lose, you lose nothing. Therefore bet unhesitatingly that He is. —Pensees.

"Pascal's Wager" has generated a vast literature of its own.

Sir Isaac Newton, convert to (heretic) Arianism, alchemist, theologian, (1642 - 1727), the "last of the magicians" according to John Maynard Keynes, is so preeminent in mathematics and physics that the amount of material on his "nonscientific" writings—for long considered by historians of science to be an aberration—is now substantial. See, e.g., James E. Force and Richard Popkin and also B.J.T Dobbs. Briefly, Newton attempted to combine mathematics and astronomical science so as to prepare a revised chronology of world history and thereby to understand the divine message. For example, we find in Newton's *The Chronology of Ancient Kingdoms Amended*:

> Hesiod tells us that sixty says after the winter solstice, the star Arcturus rose just at sunset: and thence it follows that Hesiod flourished about a hundred years after the death of Solomon, or in the Generation or Age next after the Trojan War, as Hesiod himself declares.

Also,

> Newton saw his scientific work as evidence of God's handiwork. He turned to religious studies later in life and considered it an integral part of his thinking. Indeed, just as today's cosmologists are trying to find a 'Theory of Everything' , Newton looked for a unification of the sacred texts with his mathematico-physical theories. —Katz & Popkin.

In mathematician and clergyman John Craig's "Mathematical Principles of Christian Theology" (1699), Craig calculated, based on an observed decline in belief and a passage in St. Luke, that the second coming would occur before the year 3150. To a contemporary mathematician, Craig's reasoning is not unlike an argument from exponential decay.

Expressions of number mysticism ebb and flow. They seem never to disappear entirely. Today, number mysticism is widespread. There are said to be lucky and unlucky numbers - an ancient idea. These selections, intended for personal use, are widely available in books and newspapers." Your number for the day is 859." "In the year 1000 or 1666 or 2000 something good or something bad will happen." The question of whether these kinds of assertions are "deeply believed" is often irrelevant given the extent to which its practice results in human actions.

Recently there were various to-dos about the new "Millennium", (including a billion-pound exhibition in London) as though the year 2000 inherited mystic properties from its digital structure. In the "Y2K flap", digital programming was indeed important, but the excessive publicity and mild hysteria were hallmarks of a virulent attack of number mysticism.

The spirit of Pythagoras seems to have influenced the thought of a number of distinguished 20th Century physicists. Arthur Eddington (1882-1944) and P.A.M. Dirac (1902-1984), for example, have searched for simple whole number (i.e., integer) relationships between the fundamental physical constants expressed nondimensionally. Then, seemingly denying simplicity, Dirac wrote, "God is a mathematician of a very high order. He used some very advanced mathematics in constructing the Universe."

The number of amazing patterns that can be constructed via simple arithmetic operations is endless, and to each pattern can be attributed mystic potency or divine origin. Ivor Grattan-Guinness, who is also a musician and musicologist, in a section on "the power of number" in his *History of the Mathematical Sciences*, gives instances involving Kepler, Newton, Freemasonry, and Bach, Mozart, and Beethoven. About W.A. Mozart (175 6-9 1), he says in part:

> Mozart's opera *The Magic Flute*, written in 1791, to defend Masonic ideals against political attack, is crammed with numerology and some gematria. [Gematria = the identification of letters with numbers and used to arrive at insights].

But number, though operating within a theological context, is not always conceived within a mystic theology though we may now think otherwise. Thus Leibniz (1646-1716), "Cum Deus calculat et cogitationem exercet, fit mundus." (When God thinks things through and calculates, the world is made.)

Today, we may omit "Deus" from this precept: via calculation we create everything from huge arches in St. Louis (which has a great spiritual quality), to designer drugs or to the human genome map project. To some people, these numerical computations provide the latest answer to the Biblical question, "What is man that thou are mindful of him; the son of man that thou shouldst visit him?" (Ps. 8,4) without answering in the least what the long-range effects of such computations will be.

Thus, numbers. All of the instances cited, together with those that follow in later sections, may be deemed "applied mathematics", for they apply mathematics to human concerns and not to mathematics itself. Such an expanded meaning would be in strong disagreement with the current usage of "applied mathematics."

Geometry; Space

We find in the Old Testament, Proverbs 8:27: "He girded the ocean with the horizon." The Hebrew word for gird is "chug." A mathematical compass is a "mechugah." Same root. God compasses the world.

The image of God as the one who wields the compass was common. The Renaissance artists liked it and drew it over and over. In Amos Funkenstein's splendid *Theology and the Scientific Imagination*, you will find that on his cover there is a mediaeval picture showing Christ measuring the world with a compass. The compass motif lasted well into the 18th century when William Blake (1757-1827, mystic artist and poet) produced a famous engraving that combined these elements. Was this merely artistic metaphor, or was it stronger?

The world, therefore, was constructed geometrically. The classic statement is "God always geometrizes."

On a much more abstract level, Moses Maimonides (1135-1204, philosopher, theologian, and physician) denied the infinity of space. In this regard, he sided with Aristotle. On the other hand, Hasdai Crescas, poet and philosopher, (1340 - 1410) allowed it.

In Art: Dilrer (1471 - 1528), Michaelangelo (1475 - 1564), and numerous other artists of the period, men who were well versed in the mathematics of the day, looked for the divine formula that would give the proportions of the human body. The human body was God's creation and perfection must be found there. This perfection was thought to be expressible through mathematical proportions.

Hermetic geometry (i.e., geometrical arrangements that were thought to embody occult or religious forces) abounded. Churches were constructed in the form of the cross. Secular architecture was not free of it: the Castel del Monte erected for Frederick II Hohenstaufen (1194 -1250), by Cistercian monks, displays an intricate geometrical arrangement, a fusion of European and Arabic sensibilities, based on the octagon and whose plan has been said to symbolize the unity of the secular and the sacred.

In the *Monas Hieroglyphica* (heiros, Greek: sacred, supernatural) of John Dee (1564), mathematician, the first translator of Euclid into English a man who was both a rationalist, an alchemist, and a crystal-ball gazer, delineates certain assemblages of figures that have potency deriving from a mixture of their geometrical/astral/theological aspects.

Consider next the spiral. Much has been written about its symbolism: in mathematics, in astronomy, in botany, in shells, and animal life, in art, architecture, decoration, in Jungian psychology, in mysticism, in religion. To the famous Swiss mathematician Johann Bernoulli (1667 - 1748) who created the mathematically omnipresent Spiral of Bernoulli, its self-reproducing properties suggested it as a symbol of the Resurrection, and he had its figure carved on his gravestone in Basle. Today, the double helix carries both a biological meaning as well as an intimation of human destiny.

In my childhood, the circle persisted as a potent magic figure in the playtime doggerel "Make a magic circle and sign it with a dot." The interested reader will find thousands of allusions to the phrase "magic circle" on the Web. Magic ellipses or rectangles are less frequent.

The Buddhist mandalas which are objects of spiritual contemplation, embody highly stylized geometrical arrangements. The amulets and talismans that are worn on the body, placed on walls, displayed in cars; the ankhs, the crosses, the hexagrams, the outlined fish, the horseshoes, the triangular abracadabra arrangements and magical squares, the sigils (magical signs or images) of which whole dictionaries were compiled in the 17th century, the hex signs placed on house exteriors, all point to geometry in the service of religious or quasi-religious practice.

There is a multitude of geometrical figures signs employed in kabbalistic practices, each associated with stars, planets, metals, stones, spirits, demons, and whose mode of production and use is specified rigorously. Wallis Budge, student of Near Eastern antiquities wrote:

> According to Cornelius Agrippa [physician and magician, 1486 - 1535], it is necessary to be careful when using a magical square as an amulet, that it is drawn when the sun or moon or the planet is exhibiting a benevolent aspect, for otherwise the amulet will bring misfortune and calamity upon the wearer instead of prosperity and happiness.

Let semanticists and semioticists explain the relationship between our geometrical symbols and our psyches for it lies deeper than simple designation (e.g., crescent = Islam). The geometrical swastika, which over the millennia and cultures has carried different interpretations, is now held in abhorrence by most Americans. The memory of World War II is certainly at work here, but the geometry can go "abstract" and its meaning become detached from an original historic context.

Why has Salvador Dali (1904 - 1989) in his large painting Corpus Hypercubus in the Metropolitan Museum in New York, placed a crucifixion against a representation of a four dimensional cube? Art historian Martin Kemp has commented:

> Dali's painting does stand effectively for an age-old striving in art, theology, mathematics, and cosmology for access to those dimensions that lie beyond the visual and tactile scope of the finite spaces of up-and-down, left and right, and in-and-out that imprison our common sense perceptions of the physical world we inhabit. The scientists' success in colonizing the extra dimensions is defined mathematically...

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To be continued...The remainder of this article, including bibliography and notes will be issued online in HMNJ #27.

Pat's Prologues

Introductions to the First Two Airings of Math Medley, A Radio Talk Show

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Since May 1998 I've been hosting a weekly radio call-in talk show, "Math Medley." It appears to be the first of its kind, according to both the math and the radio grapevines. Each week I interview someone about "education, parenting, equity, and environmental issues with an underlying theme of mathematics," to quote my opening patter.

It's been a joyous foray into show biz for a math professor. Everything is done by telephone—to and from anywhere! Some of my guests are experienced radio hams, but many are appearing for the first time. All have been fabulous people to enjoy for an hour. They have included five past presidents of the National Council of Teachers of Mathematics (NCTM), the current president and president-elect, research mathematicians, K-12 classroom teachers, equity advocates and leaders in various environmental organizations. There have been shows on bailing out family finances, investing thoughtfully, and environmentally aware consumerism—all important environmental issues. You can see a list of past and future Math Medley guests and topics at *www.csam.montclair.edu/~kenschaft*.

Ten minutes into my first show a man called in to say he'd been reading a book called *The Bell Curve*, and he wondered whether there was any hope in what I was trying to do. Five minutes later he thanked me and my guest and said he had learned a lot already from the show! There was a hostile caller later that show who actually slammed up the phone at the the terrible suggestion that citizens of North Providence might pay taxes to educate children in Prah....vi-dence (said with dripping disdain). Since then, all callers have been respectful, even when they disagree. Two blind men have called in, a truck driver, many parents, a physician, and some math professors. Sometimes people accept our invitation to solve math problems on the air, and are willing to talk about solutions for five or ten minutes.

The first nine shows aired only in Providence, and Math Medley is now heard throughout Rhode Island and in nearby Massachusetts at 990 AM from 11:00 to noon. The show also can be heard in Arizona at 1100 AM, currently from 8:00 AM to 9:00 AM. Arizona is the only state that does not go onto daylight savings time, so there is a discontinuity twice a year; I hope strongly to keep the show at 11:00 AM Eastern Time. It can now also be heard live at *www.renaissanceradio.com*.

Even more exciting, the shows are gradually being "archived" by webCT, so you can hear them any time at www.webCT.com/math; then click on Math Medley on the upper right. As I write this, about sixty different shows can be heard there, but it appears that many more will soon appear. It isn't a completely predictable process.

Each week I write an introduction to the show. The first nine are somewhat representative, and they are reproduced below, slightly edited, to give the flavor of Math Medley. Leads and support for publishing entire show transcripts would be appreciated. Support for the shows (other than my teaching salary) would also be welcome; I hereby express gratitude for the partial support from webCT, Texas Instruments, the American Mathematical Society, Burpee Seeds, and Breadman Breadmaking Machines—some in the form of advertising. I would also appreciate arrangements to syndicate to other stations; the host station has been very cooperative. If anyone would like tapes of the shows, I can provide them in limited quantities, two shows for five dollars. I own the copyright, so other stations are free to rebroadcast past shows lifted from the web as long as they either do not make a profit or share the profits with me.

I am grateful for National American Broadcasting Company for providing an opportunity for middle class non-professionals in radio broadcasting. Their technology has improved over the three years. The first shows were the least technically satisfying, so later shows are becoming archived sooner. The only show of the first nine now archived (as I write this) on webCT is Tom Banchoff's. Personally, I think almost all 165 shows thus far are worthy of both archiving: and transcribing so people can quickly read and scan them, but obviously I'm biased.

The common opening for all shows is omitted in the following transcripts. Alas, transcribing entire shows takes time and/or money, but the introductions can be taken from my word processor. The topics and guests of the first nine are "Recent Changes in Children's Math Education" with Claire Pollard; "Changing Expectations of Teachers and Kids" with John Long; "Girls and Math" with Joan Countryman; "Changing Forms of Testing" with David Capaldi; "Ethnomathematics" with Gloria Gilmer; "New Trends in Mathematical Research and Teaching" with Tom Banchoff; "Gender Issues and Standardized Testing" with Cathy Kessel; "Environmental Mathematics" with Barry Schiller; and "Hope" with Trina Paulus.

RECENT CHANGES IN CHILDREN'S MATH EDUCATION
MAY 16, 1998, CLAIRE POLLARD

Today we will be considering "Recent Changes in Children's Math Education." My guest will be Claire Pollard, who is president of the Rhode Island Mathematics Teachers Association. She also is a Resource

Teacher in the Providence public schools, where her job is to help Providence teachers understand both math and how to teach it better.

I have two types of reasons for hosting a radio show to entice people into thinking about math. One is that I enjoy it enormously, and it seems sad to me that more people don't. I dream of a world where all people—well, almost all people—will enjoy math, in the same way that most people enjoy singing or dancing. Little children love math. At least, all the ones I meet in supermarket lines and waiting rooms do. I wish that our culture wouldn't destroy that enthusiasm as often as it does. Your joy with math—like singing—doesn't take from my pleasure a whit. I hope that Math Medley can help many more people enjoy math. I love making people happy when it doesn't cost me anything.

My other reason for this show is less cheery. It's that I really want there to be people on this earth a century from now. For that to happen, many people have to know math. For example, we need statistics to understand complicated facts. Moreover, sharing limited resources without violence requires that many people must understand fractions and big numbers much better than most adults do now.

I'm Dr. Pat Kenschaft, professor of mathematics at Montclair State University in New Jersey. My day job is teaching college and graduate students pure mathematics. Over the past decade I've won fourteen grants for helping elementary school teachers mathematically because I believe that the day job of my guest is one of the most important in the world. I've taught hundreds of elementary school classes in the past decade, but she's taught many more. Welcome to Math Medley, Claire Pollard.

CHANGING EXPECTATIONS OF TEACHERS AND KIDS
MAY 23, 1998, JOHN LONG

My guest today will be Dr. John Long, Professor of Education at the University of Rhode Island and co-chair five years ago of the Rhode Island special legislative commission on math and science reform. His topic will be, "The Changing Expectations of Teachers and Kids." This affects all of us, because some day we all will be at the mercy of today's children. No matter how much money we have, no matter how loyal our own children are, and no matter how high the walls we build around our home, younger adults outside our family will affect the quality of our waning years. Therefore, their education will greatly influence whether they make our elderyears pleasant—or unpleasant.

The recent Third International Mathematics and Science Study—called TIMSS by its friends—placed United States fourth graders somewhat *above* the international median, a welcome relief from the First and Second International Studies, when our country's showing was embarrassing. One reason for the improvement has been the effort of about 200,000 of us around the country in what is called the "math reform movement." Two hundred thousand is not a small group—it's roughly the population of Providence. However, compared to the 50 million American youngsters in kindergarten through twelfth grade, it's not nearly enough. That's one of us for 250 school children. Math reform needs many more people involved.

United States eighth graders still placed a little lower than median in the recent TIMSS study, and our twelfth graders came out at the bottom of the international ladder. There were only two countries below us—Cyprus and the Union of South Africa.

The good news is that our youngsters do learn when they are taught. More good news is that when our teachers are taught the subject matter and good ways of teaching it, their students learn. A recent study of 900 districts in Texas found that teachers' expertise accounted for about 40% of the variance in school children's reading and math achievement. Indeed, the differences in black and white students were almost entirely explained by teachers qualifications and socioeconomic status. I am quoting from a blue-ribbon report *What Matters Most: Teaching for America's Future* led by two governors, one Republican and one Democrat. The report goes on to say that 28% of the math teachers in this country do not have even a minor in math. It is no wonder that our youngsters do not do well on international exams.

My guest today is Dr. John Long, professor of education at the University of Rhode Island and one of the state's leaders in the math reform movement. Welcome to Math Medley, Dr. John Long.

Notes on Formal Constructivism

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ABSTRACT
Our aim is to sketch some ideas related to how we (as in, we two) think we (as in, we humans) think.

"That theory is useless. It isn't even wrong."
- Wolfgang Pauli.

Our hope in this paper is to provide a theory, admittedly somewhat vague, of how we think about mathematics. We also hope our ideas do not cause the reader to be reminded of Pauli's quote above.

These notes were motivated by the interesting book by Changeaux and Connes [CC].

REALISM VS CONSTRUCTIVISM

Realism: Mathematical objects exist independently of experience (or "physical reality") which we process using our senses (smell, touch, sight, ...) and interpret using our brain. For example, Descartes speaks of a triangle as an "immutable and eternal" figure whose existence is independent of the mind which imagines it. Similar statements are made regarding God by many religious experts.

Constructivism: Mathematical objects exist *solely* in the mind as a certain electro-chemo-biological pattern of neurons, synapses, chemicals, ... in the brain. For an extreme example, Hume believed that ideas are merely copies of sense impressions.

Examples: Alain Connes (and probably most mathematicians) are realists. For example, the famous quote of Kronecker's, "The integers are made by God, all else is made by man," indicates a realist point-of-view. On the other hand, the biologist Jean Pierre Changeux and philosopher David Hume are constructivists (though Hume is the more extreme). Poincaré was possibly a constructivist in this sense (see [D], chapter 9).

The realist position might be roughly summarized as follows: The physical world is modeled as much as possible by mathematics. Mathematicians merely discover what is already in existence.

The constructivist position might be summarized as follows: Models for the physical world are constructions of the mind (only) and all such mental constructs exist solely as electro-chemo-biological patterns of neurons, ... in the brain. To the question, "Why is mathematics so well-suited to the description of physics?", the constructivist might counter that physicists tend to examine reproducible phenomena which tend to have "universal" characteristics. Hence mathematics, which is also universal, is admirably suited for physical description.

Points of Agreement

- Mathematics provides a "universal language", i.e., a grammar and set of terms which can be understood by anyone (sufficiently trained), independently of their cultural background.

- There is a "physical world" independent of our mind (which, however, we sense using our brain and sensory organs).

- Mathematical objects can be represented as a certain electro-chemo-biological pattern of neurons, synapses, chemicals, ... in the brain.

- A given mathematical construction can be represented as a program in a "Turing machine".

  (Using an over-simplification, these representations are modeled using neural networks, which are related to Turing machines [M].)

FORMAL CONSTRUCTIVISM

Assorted Thoughts of Our Own

- Though mathematics may indeed be universal, its development and current state is inspired and influenced by culture and the human experience.

There may be experiences outside the human realm (the realm includes other methods of detection, such as computers, microscopes, cyclotrons, ...) which might lead to mathematics which we humans might never "discover/invent/realize".

- The brain is capable of translating (or "producing") from sensual patterns certain "grammar" (or, loosely speaking, patterns of patterns). These may be thought of as rules that mental objects satisfy, though they are more intuitive feelings than rigorous laws. (For example, one never expects to see a mouse turn into an elephant, so such a concatenation of mental objects in our senses would be regarded as ungrammatical.)

- The brain is capable of translating certain universal mental objects into symbolic objects (such as translating sounds into written words). Grammar satisfied by the mental objects can be translated into grammar for the symbols.

- Suppose that we do indeed perceive "reality" via our brain and senses. Suppose, in the extreme, that all mental objects are merely copies of sense impressions. ("Copy" is being loosely interpreted here, as it is assumed that a sense impression is an electro-chemo-biological pattern in the brain and the brain may be less reliable than a camera or xerox machine!) These objects may possess properties (at least as far as we may sense them). Assume that we may posulate (using our imagination) new properties for these objects. Define a "model" to be a logically coherent collection of objects and their properties. It seems reasonable to hypothesize that all of mathematics belongs to some such model.

  If these assumptions are accepted, one may then create a sort of "platonic model" of mathematics, even though the actual objects may exist only as mental constructs (this idea which you have just read about may be such a "model"!). Call this **formal constructivism**.

- In formal constructivism the mind-body dualism problem might be regarded as follows. The mind is merely a collection of mental objects, constructions and models. The body is a collection of nerves, bones, synapses, ... Constructive realism defines the model to be a collection of electro-chemo-biological patterns in the brain. Thus the "mind" is part of the functioning of the "body" and hence there is no dualism.

- If the "realm of human experience" consists of all human ideas, experiences, and activities, then within this realm statements may be divided into two classes, T (testable) and NT (non-testable):

  (T) Those statements regarding experiences which are in principle testable by some physical device or thought experiment. We assume that the thought experiment is one which tests the truth or falsity of a well-defined statement within an axiomatically presented (logical, mathematical or philosophical, for example) internally consistent universal model. In this case the validity of the test would, of course, only be relative to the axioms assumed. For example, "The person reading this sentence is a human being," is both testable and probably true!

  (NT) Those statements regarding experiences which are not testable in the sense above. For example, "Triangles existed 2 billion years ago," is not testable.

  All statements in the second class cannot be knowable in the sense that they can be tested. However, depending on one's axioms, many models are testable. For example, if one hypothesizes the existence of God in a consistent model, then of course the statement, "God exists," is axiomatically true.

A mathematical analogy of this testable/non-testable idea: mathematical statements may be divided into two classes, D (decidable) and UD (undecidable):

(D) Those statements which may be proven true or false logically from the axioms of some mathematical model with a recursive set of axioms (say Zermelo-Fraenkel set theory).

(UD) Those statements which cannot be proven true or false logically from the axioms of some mathematical model with a finite set of axioms. Such statements could be:

– known to be true (but "self-referential" and not provable, as in Godel's incompleteness theorem),

– known to be false (but not disprovable),

– poorly formed (using the grammar of the model),

– well-formed but independent of the axioms (such as the Continuum Hypothesis).

LANGUAGE AND GRAMMAR

So far, we have simply regarded the brain as a processing unit, which is capable of translating (or producing) certain patterns from sequences of mental objects. These patterns may be translated and represented (not necessarily faithfully) using more universal mental objects. Call these objects "symbols". This capability of the brain might be regarded as an "abstraction device": a machine which is capable of noticing patterns from sequences of inputs. Of course, some patterns are more relevant than others. The brain is also capable of distingishing, evaluating, and selecting patterns.

The brain also has a tendency towards using universal mental objects (constructing order from chaos, if you will). Therefore, it is natural for the brain to process stimuli in terms of symbols and grammar. This leads naturally to language, which is useful for processing even more information.

The point is that the way we think about mathematics falls into one of several categories. We either

- formally manipulate symbols (such as algebraic expressions), following grammatical rules,
- formally manipulate mathematical objects (such as knots), following grammatical rules,
- experimentally determine grammatical rules using sequences of mathematical objects,
- select mathematical objects from sequences using some evaluation procedure (possibly for the purpose of manipulating them or determining grammatical rules for them).

All these involve one of the brain's capabilities discussed above.

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Calculus

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I tell my students the story
of Newton versus Leibniz,
the war of symbols,
lasting five generations,
between The Continent
and British Isles,
involving deeply hurt
sensibilities, grievous
blows to national pride;
on such weighty issues
as publication priority
and working systems
of logical notation:
whether the derivative
must be denoted by a "prime",
an apostrophe atop
the right hand corner
of a function, evaluated
by Newton's fluxions method
or by a formal quotient
of differentials dy/dx,
intimating future possibilities,
terminology that guides
the mind.

The genius of both men
lies in grasping
simplicity
out of the swirl of ideas
guarded by Chaos,
becoming channels,
through which
her light
poured
clarity on the relation
binding slope of tangent line
to area of planar region
lying below a curve,
The Fundamental Theorem of Calculus,
basis of modem mathematics,
claims nothing more.

While Leibniz -
suave, debonair,
philosopher and politician,
published his proof
to jubilant cheers
of continental followers,
the Isles seethed
unnerved,
they knew of Newton's
secret files, locked in
deep secret drawers -
for fear of theft
and stranger paranoid delusions,
hiding
an earlier version
of the same result.
The battle escalated
to public accusation,
charges of blatant plagiarism,
excommunication
from The Royal Math. Society,
a few blackened eyes,
(no duels);
and raged for long
after both men were buried,
splitting Isles from Continent,
barring unified progress,
till black bile drained
and turbulent spirits becalmed.

Calculus -

Latin for small stones,
primitive means of calculation;
evolving to abaci;
later to principles of enumeration
advanced by widespread use of
the arabic numeral system
employed to this day,
as practiced by algebristas
barbers and bone setters
in Medieval Spain;
before Calculus came
the $\sum$ (sigma) notion,
sums of infinite
yet countable series;
and culminating in
addition of uncountable many
dimensionless line segments,
the integral $\int$
snake,
first to thirst
for knowledge, at any price.

That abstract concepts,
applicable—at start,
merely to the unseen
unsensed objects:
orbits of distant stars,
could generate
intense earthly
passions,
is inconceivable today;
when Mathematics
is considered
a dry discipline,
depleted of life sap,
devoid of emotion,
alive only
in convoluted brain cells
of weird
scientific minds.

Marcy's Dots: A Problem on a National Test Revisited

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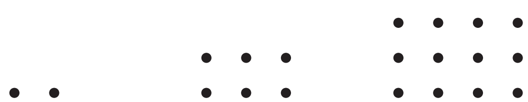
In 1992 the following problem was given to 8th grade students as a part of the National Assessment of Educational Progress (NAEP)'s national assessment:

MARCY'S DOTS
A pattern of dots is shown below. At each step, more dots are added to the pattern. The number of dots added at each step is more than the number added in the previous step. The pattern continues infinitely.

(1st step) (2nd step) (3rd step)



2 dots 6 dots 12 dots

Marcy has to determine the number of dots in the 20th step, but she does not want to draw all 20 pictures and then count the dots.

Explain or show how she could do this and give the answer that Marcy should get for the number of dots.

***

The only answer accepted as "correct" was 420. But the quality of each explanation was graded as minimal, partial, or satisfactory or better.

Responses of 8th grade students in the national sample

		Correct		
No response	Incorrect	Minimal	Partial	Satisfactory or better
16%	63%	10%	6%	6%

This problem is a typical "IQ type" problem common on NEAP tests. In order to solve it, a student needs only 3rd grade mathematics. A student has to know that the number of dots in a rectangular array is the product of the number of rows and the number of columns, and he/she has to be able to make one easy mental multiplication. Finding the intended solution requires no ingenuity if a student has any experience with looking for numerical, and not visual, patterns. But providing a concise and clear explanatory write-up is difficult even for the best students.

The intended solution looks as follows:

1. Marcy should notice that the number of rows in step $n = 1, 2$ and 3, is n, and that the number of columns is $n + 1$.
2. Therefore the number of dots is equal to $n(n + 1)$, for $n = 1, 2$ and 3.
3. If this formula holds for other numbers, then for $n = 20$ the number of dots is $20 * 21 = 420$. This is the number of dots Marcy should get.
4. She should also check that the number of dots that are added increases from one step to the next. That is easy, because the number of dots added in step n is $n(n + 1) – (n – 1)n = 2n$.

But is there any reason to claim that 420 is the unique correct solution? NO!

There are infinitely many patterns of dots that satisfy the conditions of the problem, and there is no overwhelming reason to claim that the one which seems to one person the most obvious is the "correct" one. Clearly many 8th graders saw patterns that were different from the one seen by the makers of the test.

Below are some possible solutions to the problem. (They are written as if they were students' answers, but they were not.)

EXAMPLE 1
There are many other solutions besides the obvious

420, so I asked myself, what is the smallest possible solution?

The differences between the numbers of dots that are added in each step must increase at least by 1. So we have the following pattern if we make the differences as small as possible.

Step:	1	2	3	4	5	…	n-1	n	…
# of dots:	2	6	12	19	27	…	?	?	…
Differences:	4	6	7	8		n+2	n+3 …		

Thus, the number of dots at the n^{th} step is:

$$12 + 7 + 8 + \ldots + (n + 2) + (n + 3)$$
$$= 12 + (n + 10)(n - 3)/2.$$

So the smallest possible solution for the 20^{th} step is 267 dots.

Example 2

The question is about the number of dots, and not about their pattern. So I decided to concentrate just on numbers. The ratios between the numbers of dots are 3 and 2 (6/2 and 12/6), and that suggests an exponential growth. However this cannot be "purely" exponential, because the ratios are not equal, so I fiddled a little with formulas and found this one for the number of dots, d(n), in step n:

$$d(n) = 2^n + 2(n - 1).$$

I used a scientific calculator to compute d(20) = 1,048,614.

Example 3

The pattern can continue by repeating the ratios 3 and 2.

Step:	1	2	3	4	5	6.	…
Number of dots:	2	6	12	36	72	216	…

Thus the number of dots in step 2n is 6^n, and the number of dots in step 2n + 1 is $2*6^n$.

Therefore the number of dots in the 20^{th} step is 6^{10} = 60,466,176. (I used a calculator.)

Example 4

We were learning about Fibonacci, so I started looking for Fibonacci numbers. And guess what? I found them!

We have to start with step 0, with no dots, and look at the differences.

Step:	0	1	2	3	?
Number:	0	2	6	12	?
Difference:	2	4	6	?	

onacci pattern! So the next difference is 4 + 6 = 10, the next is 6 + 10 = 16, and so on. Then you have to add all these differences up to step 20 to get the number of dots. I went only up to step 10 and I gave up. Not enough time. I would rather program a calculator to give me the answer.

0	1	2	3	4	5	6	7	8	9	10
2	4	6	10	16	26	42	68	110	178	
0	2	6	12	22	38	64	106	174	284	462

CONCLUSION

In order to make the solution to the original problem unique, one needs to add a few strong assumptions. For example, the number of dots, d(n), in step n is expressed by a polynomial of second degree. The assumption about the degree of the polynomial is needed because the polynomial

$$d(n) = n(n + 1) + (n - 1)(n - 2)(n - 3)$$

is a third degree polynomial, which is a solution to the original problem.

Questions that are used on national and state tests should be mathematically sound. Questions should be testing mathematical knowledge that is expected at a given grade level. Also, answers should be scored objectively and correctly. The problem of Marcy's Dots fails all three criteria. It is an example of rushing toward a solution, rather than thinking, what is a solution to the stated problem (Buerk 2000).

It is hard to judge whether poor test questions are exceptions or if they are the norm, because test makers protect themselves by keeping the contents of tests secret.

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Aphorisms

Lee Goldstein

Editor's note: These are excerpts from a longer paper.

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It is easy to see that even in (ordinary) human life, and first of all in every individual life from childhood up to maturity, the originally intuitive life which creates its originally self-evident structures through activities on the basis of sense-experience very quickly and in increasing measure falls victim to the seduction of language. Greater and greater segments of this life lapse into a kind of talking and reading that is dominated purely by association; and often enough, in respect to the validities arrived at in this way, it is disappointed by subsequent experience.

Edmund Husserl, *The Origin of Geometry*

***

- Sense statements may tend to be homeopathic to the mathematical, and mathematical statements tend to be allopathic to the sense world.

- Mathematics (or geometry) opens betwixt infra-realization and super- nominalization, both of which are programs.

- Mathematics is in the thinning of programmaticness, as such a checking of programs and unprogram-maticness.

- Mathematics leans on institutions of objectivity.

- Hollow mathematicians are at least correct.

- Upon a people's limited language, nonverbal mathematics was the first mathematics.

- I believe in nonverbal universals.

- We may read silence in dreams.

- Statements containing "there exists" could be that penultimate resort of the nonverbal.

- The nonverbal could hypothetically be as nominal, not nominalist.

- Mathematics is nominalism's self.

- A referential statement about mathematics might be as an unending hypothesis

- Science and the beginning of the world are not referentism.

- The reference is the residue.

The basic idea of the above aphorisms is that a quickening and underlying programmatization of the understructure of things quickens the tendency to words and language and not the intuitive realism which precedes programs. This answers Husserl's question, then, and opens up a new field coincidental or before grammar. As our grammar is hard-wired into our brains, so is the programmetrical structure of the world.

When is a Math Problem Really "Real"?

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INTRODUCTION: A NEW ANSWER TO THE OLD QUESTION

In this article, I address two works by two different authors in the May 1998 issue (#17) of *Humanistic Mathematics Network Journal*. In "Real Data, Real Math, All Classes, No Kidding," Martin Vern Bonsangue discusses the famous question so often put to math teachers: "When am I ever gonna use this stuff?" Bonsangue confesses, "I gave shop-worn answers like, 'Math teaches you how to think, so it doesn't matter,' or 'Well, we can solve word problems with math,' neither of which I believed in" (17). His solution is to find "real" examples (I explain the quotation marks in the following section), specifically data about earthquakes and life expectancy, to use in his math classes.

A few pages later, Jeffry Bohl, in "Problems that Matter: Teaching Mathematics as Critical Engagement," deals with the question again. He creates a small taxonomy of possible responses to the same age-old utilitarian question: all answers are versions of "tomorrow" (you will need the information for an upcoming chapter, course, etc.), "jobs" (you will need the information to land a good career), "general mental strength" (you will need the information to think better), or "tests" (you will need the information to pass the test, quiz, next assessment).

I would like to propose another category of response that stems from what I believe is the general principle at work behind Bonsangue's search for "real" examples. Specifically, the response to "When am I gonna need this?" involves recognizing that the question, though designed to be teleological, cannot, strictly speaking, be answered. As those of us who are a little more experienced and worldly than our students well know, *you never know when you will need to know something*.

"REAL" DATA: WHAT IS IT AND WHY DO WE LIKE IT?

One problem with mathematics textbooks is their lack of real—perhaps we should use "accurate" or "contemporary"—data. Bonsangue's math problems involve real data: earthquakes, a subject of intrinsic interest for his Californian audience; and life spans of pre-civil war women of different races, a relatively peculiar subject involving history, detective work, and heritage. However, on the face of it, neither topic would necessarily treat a student's question of "When am I going to need to know this?" By introducing data collected from "the field," as Bonsangue does, we expect relevance and a "hook" that will capture students' imaginations. In my experience, our expectations are satisfied.

Somehow, more traditional problems that also use real data are hardly so enticing. For example, "A bus departs at noon from Newport News (where I teach) averaging 42 mph. Another bus departs an hour and fifteen minutes later averaging 60 mph. If I am going to Washington, D.C., 200 miles away, which bus gets me there first?" This problem may actually be based on extant bus schedules; furthermore, D.C. is a wonderful city to visit and so knowing this information has some value. The difference is, perhaps, that these data look like a "standard" word problem, whereas Bonsangue's data are unorthodox, deriving from very *un*mathematical sources.

Indeed, math problems that seem to come from non-mathematical problems are often what we mean by "real data," and they are often very interesting to students. Even though knowing the "Distribution of 'Felt' West Coast Earthquakes January 1, 1990 through July 11, 1996" (Bonsangue 19) is not a good answer to "When am I going need to know this?", it has a good chance of engaging students because it smacks of reality. It uses information that somehow creates or demonstrates a nexus between life, math, and nature. In other words, Bonsangue's problems are truly humanistic mathematics—they genuinely engage many areas of human experience and knowledge at once—and *this* is what excites students. Math problems are often most interesting when they combine the mathematics with other areas of learning.

WHEN DO YOU NEED TO KNOW ANYTHING?

I begin with a digression: years ago, I was one of two finalists for a job teaching physics and earth science. During the second interview, I learned that my competition had more experience and expertise than I did, but she had one weakness: she was tentative about driving a school van for an after-school program. The Goldberg family car for 20 years was a huge, full-sized van (this was the pre-minivan era), so I was happy to offer my services as driver. I got the job. Never in my wildest dreams did I imagine, going into that second interview, that my competence driving vans would be critical to my teaching career. This experience confirmed one of life's great lessons, one of the most critical I have to offer to anyone, young or old: a fundamental aspect of life is that *you never know what you need to know until you need to know it*.

Intuitive understanding of this principle may well be why working with real data almost always seems so much more interesting than traditional math problems. Even if the purpose of the knowledge is not clear now, a problem-solver is gaining real, applicable knowledge that may be of utility another day. Here Bohl's "tomorrow" answer applies. Bohl believes, "we need to teach mathematics through the mathematization of real, socially relevant situations" (29). It may sound curmudgeonly, but I find "socially relevant" a problematic phrase: it too often means, "socially relevant in the eyes of a teacher and not a student." When I taught composition, I frequently covered "socially relevant" material and found I unintentionally put students in combative, uncomfortable situations. When I did not try to introduce the material, it came to light in more natural and potent ways. *Trying* to be socially relevant can be problematic.

However, I agree with the spirit of the advice: we are teaching not only math, but life skills (or, at least, cultural skills). Herein lies the connection between Bonsangue's "real data" and Bohl's "general mental strength." Social relevance is in the eye of the beholder, but if students understand they are learning about their world, not some abstract theoretical principle, they are more likely to file it away for use another day. If students feel that you have broadened their knowledge base, as opposed to broadening their *mathematical* knowledge base, they will often feel more fulfilled. This is not to say, "don't teach math." In fact, my point is that a general, humanistic knowledge base should be integrated with mathematics. The most exciting mathematical problems are the ones that seem to come from elsewhere besides math books.

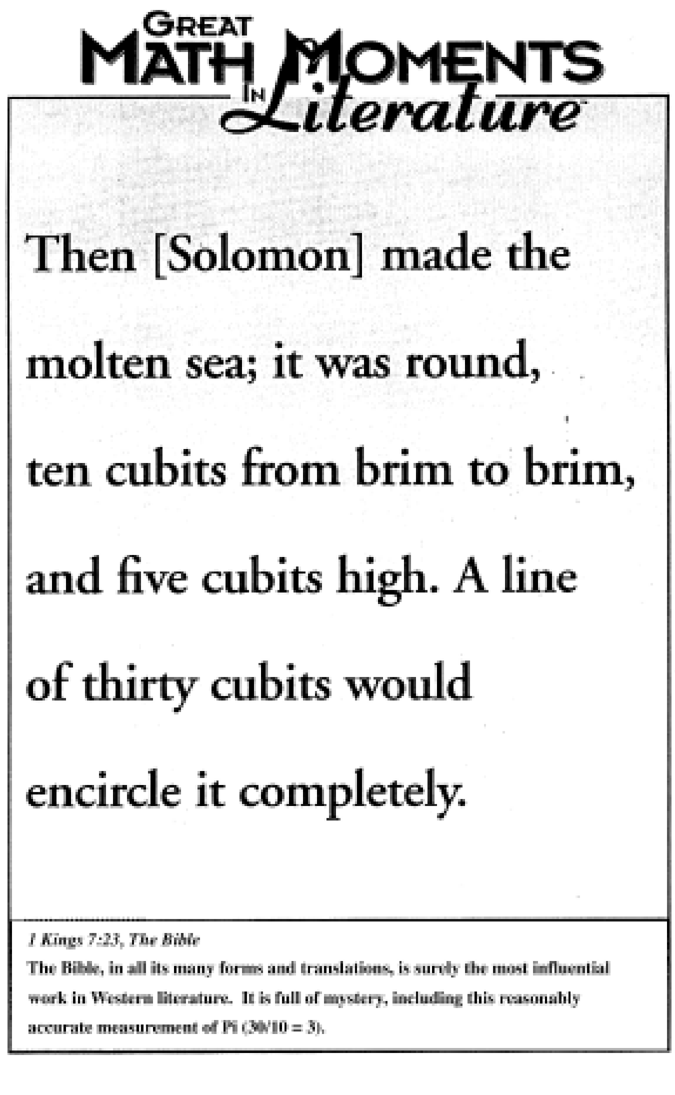
According to *A Handbook of Literature*, by Holman and Harmon, "humanism suggests a devotion to those studies supposed to promote human culture most effectively—in particular, those dealing with life, thought, language, and literature of ancient Greece and Rome" (233). It may be (unfortunately) difficult to find students eager to digest information about ancient Greece and Rome, but problems engaging other aspects of humanism tend to excite students greatly. It is, after all, far rarer that an English teacher gets asked, "When am I going to need to know this?"

Though the humanities have historically struggled for validation in America, I have found few students asking why they needed to read *Hamlet*. The presumption among students is that Shakespeare, by virtue of his importance to literature, history, philosophy, etc., is automatically worthwhile. Problems engaging more than one area of human thought (like the ones Bonsangue has constructed) fit the bill. They teach mathematics, but they also stimulate other interests, effectively displacing the "when do I need to know this" question. I, too, prefer mathematical problems that in some way ask students to engage other disciplines (art, history, literature, film, etc.), ideally a few at once. I also believe that the "social relevance" will often be contained within such problems automatically, by virtue of their complexity.

To summarize this section, let us review why the bus problem described above is relatively drab to my students. (I should point out that it would not be dull to *all* students. The trip to Washington, D.C. is, for some, a rewarding concept to entertain.) There do not appear to be many dimensions or applications to the problem. We need to go to a specific place and the math helps us make a better decision about how to get there. The "better decision" is better only insofar as it saves us about half an hour. The beauty of taking real data is that these data are not math problems first: they are problems dealing with life and thought, problems dealing with humanism. It thus turns out—and this is surprising and exciting to many novices—that mathematics is part of the pursuit of humanistic (not just "human") knowledge. A friend in college was fond of pointing out that math departments are not in "arts and sciences" divisions *not* because math is a *science*, but because it is an *art*. A problem that can be put in a student's "I may need to know this someday" file is an important, artistic problem.

AN EXAMPLE: GREAT MATH MOMENTS IN LITERATURE

As should be quite obvious, I am a humanist at heart. I have a doctoral degree in English literature; wrote a dissertation on James Joyce (himself no minor humanist); and spent ten years teaching composition, film, and English literature. (Thus, I can claim with some authority that students do not question the teaching of *Hamlet*.) I also have a strong background in math and physics, but for a decade I taught very little math, save my "calculus tip of the day," when I taught a writing course geared toward college engineering majors. For years, people asked me about my diverse background. Would I ever need to know all that math? Why was it so interesting to me anyway (I was an *English* major)? I come by my devotion to interdisciplinarity honestly, and I believe that it is precisely the search for links between different modes of thought and different knowledge bases that best furthers learning.



Having mused about the intrinsic needs of the young (and old?) mind to encounter more than just math, it is time to turn to my experience and my suggestions for teaching humanistic mathematics. I will keep this section short; this Journal contains a wealth of ideas regularly, and I only want to add enough to connect theory with practice.

I am now a high school mathematics teacher and I delight in adapting *non*-mathematical situations into my classroom. I enjoy asking students to look at relatively obvious "real life" problems (some examples: altering recipes, renting cars, predicting landfall of a hurricane, buying concert tickets, carbon-dating bones) as well as those that, in talking about Bonsangue's "life expectancy" project, I call "relatively peculiar." These are particularly enjoyable in my calculus class, where I used a problem called "the perfect glass of chocolate milk" to explain saturation (and derive a formula), and used derivatives to calculate how far a professional wrestler jumped by knowing his "hang time." My point is this: both of these are fairly simple problems, but they are taken from daily experience and thus, judging by my students' responses, evoke great interest.

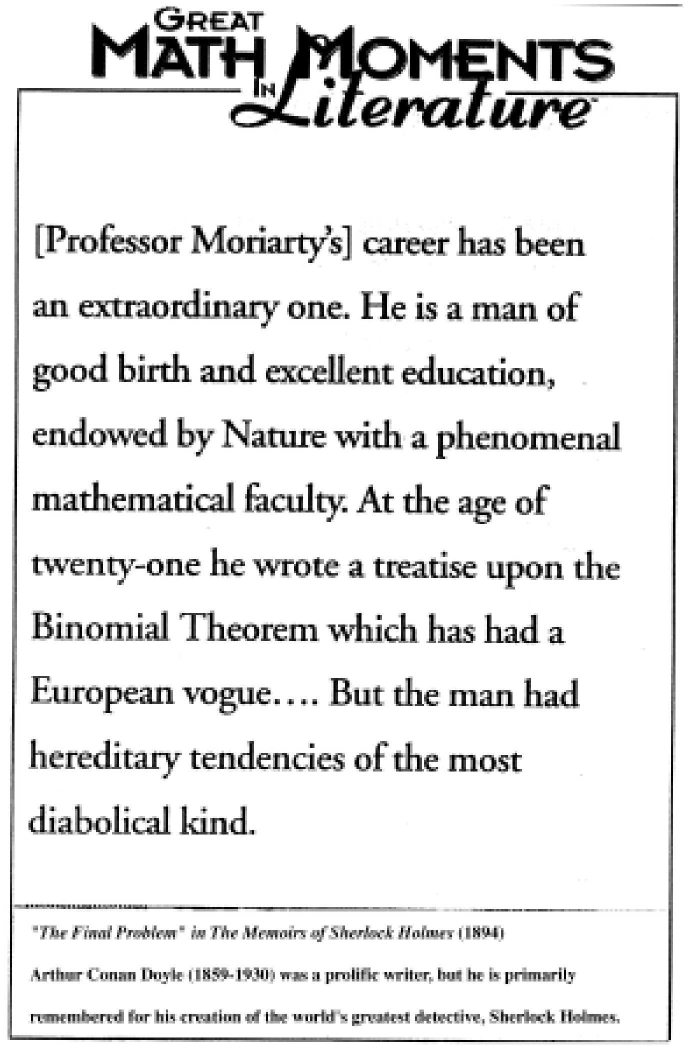
Unique to my classroom is a series of posters that I created (along with some friends and a graphic artist), called "Great Math Moments in Literature." With a few notable exceptions, I generally did not take examples from science fiction. Though I enjoy science fiction very much, to sample it would be to take the "obvious" choices (kind of like using the bus problem). The best posters are those that use examples from work seemingly *un*mathematical in nature. There is room for Madeleine L'Engle, Thomas Pynchon, Lewis Carroll, and Arthur C. Clarke, but what pleases me the most are the posters that come from less scientific or mathematical sources.

My favorite example is a poster I use while discussing irrational numbers with students in my Algebra I classes. We should never forget how apt the term denoting numbers like π or $\sqrt{2}$ really is: irrational numbers *do not make sense* to the novice. Only experience, it seems, eventually allows us to get a handle on these quantities that we can write as symbols, point to on a number line, but never know the exact value of. When the class arrives at this head-spinning concept, I point to a poster that contains a quotation from the Bible: "Then [Solomon] made the molten sea; it was round, ten cubits from brim to brim, and five cubits high. A line of thirty cubits would encircle it completely." The poster can (and does!) occasion a variety of wonderful discussions, including ones of metaphor ("the molten sea" is a poetic term for a gigantic chalice) or of the nature of units (a "cubit" is about 18 inches, the distance from the elbow to the fingertip). Generally, though, I prefer to point out the biblical approximation of π to 3 (circumference of 30 divided by diameter of 10). The class can begin drawing and measuring circles, research the circumference/diameter relationship other ways (on the Internet?), or merely engage in a lively discussion about how accurately we can measure anything. Whatever discussion/exercise ensues, by the time we are done, the students are satisfied that they have gained knowledge of philosophy, theology, geometry, and, by the way, Algebra I. The concept of irrational numbers becomes a part of a much bigger, more important discussion.

I close with an imagined example, using another poster in my classroom. The poster cites a later Sherlock Holmes story ("The Final Problem"), in which the great detective explains Moriarty's evil genius. It seems,

> [Moriarty's] career has been an extraordinary one. He is a man of good birth and mathematical faculty. At the age of twenty-one he wrote a treatise upon the Binomial Theorem which has had a European vogue. But the man had hereditary tendencies of the most diabolical kind.

This would be an enjoyable starting point for a discussion of the Binomial Theorem (given a great deal of time or, perhaps, a graduate-level classroom, it might even be fun to let students do some research and take a crack at writing Moriarty's treatise). It seems to me also a good place to investigate the nature of proofs and theorems. I say this because most students accept math as "truth." How, they might wonder, could someone write a "treatise" on an incontrovertible truth? How could such a paper enjoy "a vogue"? Isn't math either true or false? The point,


GREAT MATH MOMENTS IN Literature

[Professor Moriarty's] career has been an extraordinary one. He is a man of good birth and excellent education, endowed by Nature with a phenomenal mathematical faculty. At the age of twenty-one he wrote a treatise upon the Binomial Theorem which has had a European vogue.… But the man had hereditary tendencies of the most diabolical kind.

"The Final Problem" in The Memoirs of Sherlock Holmes (1894)

Arthur Conan Doyle (1859-1930) was a prolific writer, but he is primarily remembered for his creation of the world's greatest detective, Sherlock Holmes.

then, would be to use a seemingly innocuous quotation as an introduction into the complexities and unknowns of mathematics, thence an introduction into the complexities and unknowns of our world.

Sherlock Holmes' genius lies in two abilities: he is remarkably observant and he has a huge knowledge base. Just to take one example, in "The Five Orange Pips," he deduces that the *Lone Star* must be a boat from America because he knows it borrows the nickname for Texas. This is impressive knowledge for a proper English gentleman in 1892, for whom knowing state nicknames would appear to be utterly useless information. It *is* utterly useless, until Holmes needs it one day to solve this case; armed with this information, the case is solved quickly and effectively. Holmes is therefore one of the great, if not the greatest of all exemplars of the "you do not know what you need to know until you need to know it" principle. Citing him as a model is attractive: he uses the seemingly “mathematical” science of deduction to help with the most human of issues, including broken relationships, jealousy, greed, and evil. If math problems were like Sherlock Holmes stories, students would be endlessly engaged, entertained, and educated. Diversity, interdisciplinarity, and real data make math problems that much more like Sherlock Holmes stories.

CONCLUSION

I am not suggesting we teach literature in math classes. I am suggesting we break down the barriers between disciplines and I have chosen Sherlock Holmes as a metaphor. In my own teaching I try (though, of course, I do not always succeed) to apply the framework I have outlined here. Any good teacher—not only one who handles math—must have his or her own answers to “When am I going to need to know this?” However, the best answer is not always a time, date, or place. The best answer is that no one quite knows. We may, in part, teach math so that students can be ready for tomorrow, do well in jobs, improve their critical thinking, and pass the next test. In fact, I have some support for all of these reasons. However, we also have an imperative to teach humanism, to mathematize the world in ways such that our students are ready for whatever life offers them. After all, in “real life,” people get jobs teaching physics just because they know how to drive a van.

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Book Review: *The Teaching Gap* by James W. Stigler and James Hiebert
Review Part II: Contrasting U.S. and Japanese Beliefs about Mathematics Teaching

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The Teaching Gap. James W. Stigler and James Hiebert. The Free Press (Div. Simon & Schuster Inc.): New York, 1999. ISBN 0-684-85274-8

This is the second of three articles that together form an in-depth book review of The Teaching Gap. After a brief summary of the first article[1], we explore the contrasting cultural beliefs that support mathematics teaching in the U.S. and Japan, and in doing so, find several surprises that are relevant to college teaching.

SUMMARY OF THE FIRST ARTICLE

As our first book review article discusses, *The Teaching Gap* addresses critical questions about how mathematics teaching is actually done in the U.S., Japan, and Germany, based on uniquely valuable data: the first videotaped national random samples, in each country, of eighth-grade mathematics classroom lessons. Remarkably, teaching varied greatly from one culture to the next, and comparatively little within each culture, giving an empirical foundation to the pivotal claim in the book that "teaching is a cultural activity."[2] The authors claim, and we are persuaded, that their findings of cultural differences go far beyond the eighth grade. Indeed, we believe they have much of importance in common with what we en-

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For the perennial and largely unsuccessful efforts to reform American education, the authors accordingly offer a deep and elegant explanation: in these efforts, "we have been acting as if teaching is a noncultural activity." The Japanese system for improving teaching, on the other hand, "is built on the idea that teaching is a complex, cultural activity."

The U.S. and Japanese typical lesson patterns were starkly contrasting. While they both began with a review of previous work, in the U.S. this led into "presenting a few sample problems and demonstrating how to solve them," with students then practicing solving similar problems, followed by checking and correcting some of this practice. In Japan, the initial review led into presenting a new problem, *which students then worked on trying to solve*. Then the class discussed and compared the students' various methods, along with any the teacher showed. The teacher finished by highlighting the principal points.

Thus, after a new problem is presented, in Japan students develop solution procedures, "without first" being shown "how to solve the problem," while in the U.S. a solution procedure is "almost always" shown to students first. Only then are they required to work problems using it. Mathematics students in the U.S. are thus found to be occupied mainly with mastering unconnected skills by repetitive practice, while their Japanese counterparts devote time in comparable amounts to, on the one hand, solving problems that genuinely challenge them and having concept-focused discussions, and on the other hand, skill practice.

As we emphasized in the previous article, this is in our view a crucial contrast, and one which, as we examine here, seems to have much bearing on U.S. college mathematics teaching.

CULTURAL SCRIPTS AND BELIEFS ABOUT TEACHING

To explain why a nation's mathematics lessons follow these distinctive and, in the case of the U.S. and Japan, contrasting, patterns, the authors introduce the notion that these lessons were created and carried out by teachers who, as members of a culture, "share the same *scripts*" [italics ours]. A script is "a mental version of...teaching patterns" such as the U.S. and Japanese patterns whose contrasts we have summarized above. In this way *The Teaching Gap* develops its claim that how one teaches is primarily determined by one's culture, rather than being an instinctive gift or acquired in "college teacher-training programs."

And the script, in turn, "begins forming early.... As children move through twelve years and more of school, they form scripts for teaching." So students (as well as teachers) come to the classroom after years of being socialized into a pattern of expectations that are mutually aligned in consonant ways. This is, I think, why the authors' generalizations have so much power beyond the eighth-grade context from which they were derived—these findings are statements about deeper patterns of cultural participation. In particular, as we remarked in the previous article, we at the college level may find it worthwhile to consider the extent to which our teaching also fits the U.S. pattern and contrasts with the Japanese pattern, as summarized above.

Teaching is therefore a complex system that, like other cultural activities, "evolve[s] over long periods of time in ways that are consistent with the stable web of beliefs and assumptions that are part of the culture."

Thus we see that the authors' fundamental logic is: *beliefs* give rise to and sustain cultural *scripts* (which characterize a culture's *system* of teaching), and the system in turn determines the various manifest *features* of teaching that we can observe, say, in video data.

And what are these beliefs about? A nation's teaching script seems to depend on a few central beliefs about (1) what mathematics is, (2) the way students learn about it, and (3) the teacher's function during the lesson. The authors have inferred a set of beliefs for Japanese and U.S. teaching in these three areas based not only on teachers' answers to questionnaires but directly on how they behave.

CULTURAL BELIEFS ABOUT WHAT MATHEMATICS IS

In regard to what mathematics is, the U.S. pattern fits "the belief that school mathematics is a set of procedures...for solving problems." Sixty-one percent of American teachers, "asked what 'main thing' they wanted students to learn from the lesson,...described *skills*.... They wanted the students to be able to perform a procedure, solve a particular kind of problem, and so on."

It is also riveting and thought-provoking to learn that "[m]any U.S. teachers also seem to believe that" this view of mathematics ("learning terms and practicing skills") is "not very exciting. We have watched them trying to jazz up the lesson and increase students' interest in nonmathematical ways: by being entertaining, by interrupting the lesson to talk about other things...or by setting the mathematics problem in a real-life or intriguing context.... Teachers act as if student interest will be generated only by diversions outside of mathematics." While we may question the inclusion of creating a "real-life...context" in the list, the thrust of the description hits home, I think, surprisingly forcefully: I suspect it is a rare college mathematics educator indeed who has not taken it for granted, to one degree or another, that these extrinsic activities in the classroom are a necessary concomitant of winning the prize of students' enthusiasm for the mathematics. While the authors' description may at first be disturbing, on reflection it is encouraging, in that it gives us a new context in which to view these pulls away from the mathematics in our classrooms, and thereby a hope that they are not ineluctable.

These "diversions" are compromises, and thus to some extent compromising: they are moves away from where we, including those of us in the college classroom, feel we ought to be. What is hopeful in this formulation is that the blame is placed neither on us nor on our students, but on our *system*.

Contrasting beliefs about what mathematics is seem to underlie Japanese teaching. "Teachers act as if mathematics is a set of relationships between concepts, facts, and procedures. ...On the same questionnaire [as the one given to American teachers], 73% of Japanese teachers said that the main thing they wanted their students to learn from the lesson was to think about things in a new way, such as to see new relationships between mathematical ideas."

Moreover, in stunning contrast to the U.S. teachers' beliefs about what mathematics is, "Japanese teachers also act as if mathematics is inherently interesting and students will be interested in exploring it by developing new methods for solving problems. They seem less concerned about motivating the topics in nonmathematical ways.

CULTURAL BELIEFS ABOUT HOW MATHEMATICS SHOULD BE LEARNED

Based on this U.S. belief about mathematics as largely a set of procedures, a natural belief about how mathematics should be learned then follows: "incrementally, piece by piece...practicing [procedures] many times, with later exercises being slightly more difficult than earlier ones." Again, does this not seem so familiar as to be taken as virtually the only way to do things?

The authors connect "this view of skill learning" and its "long history in the United States" to behaviorist psychology, B.F. Skinner, and related work. Unfortunately this fertile connection is relegated to a mere footnote[3] and seems very worthy of amplification.

A further American belief about how mathematics is learned also follows: "Practice should be relatively error-free, with high levels of success at each point. Confusion and frustration, in this traditional American view, should be minimized; they are signs that earlier material was not mastered." The authors offer the example of a lesson in adding fractions, saying that these beliefs would lead to a presentation sequence with "like denominators,...then...simple fractions with unlike denominators,...warn[ing] about the common error of adding the denominators (to minimize this error), and later...more difficult" unlike denominators. But (at a more advanced level of course) do we not act out of the same beliefs much of the time in our college teaching, and construct our lessons accordingly, seeking to maximize success rates all along the way, and minimize frustration?

The Japanese belief about how mathematics is to be learned involves "first struggling to solve" problems, then discussing how to find solutions, and then having various methods compared and related. "Frustration and confusion are taken to be a natural part of the process, because each person must struggle with a situation or problem first in order to make sense of the information he or she hears later. Constructing connections between methods and problems is thought to require time to explore and invent, to make mistakes, to reflect, and to receive the needed information at an appropriate time." They quote a Japanese teachers' manual that, in the lesson on adding fractions, advises allowing students to make the most common error, i.e., to add the denominators, then to reflect on the "inconsistencies" they will find thereby. The teacher should start with, "for example, _ + _," then compare the various ways that students come up with to solve this problem, such as adding denominators to get 2/6. Thus, the Japanese believe that "struggling and making mistakes and then seeing why they are mistakes" are necessary to learning.

CULTURAL BELIEFS ABOUT THE TEACHER'S RESPONSIBILITIES

The last set of beliefs concerns what teachers in each country regard themselves as responsible for in the classroom.

American teachers seem to think they should partition work into units that are doable for most of the class, telling students all that they need to know in order to do the work, and then giving them lots of drill. Note carefully, however, that telling students what they need to know to do the work typically reduces to showing them how to do problems just like the practice problems. "Confusion and frustration" are believed to be intrinsically bad, evidence that teachers have fallen short in some way, and, upon their occurrence, the teacher will rush to give whatever help is required to put students back on the right path. U.S. teachers thus "try hard to reduce confusion by presenting full information about how to solve problems." Again, I would surmise that we can see ourselves, at least some of the time, in this description, and indeed that we perhaps might not even have considered it too plausible that there was any alternative to doing what is described here.

Also, that we can see our choices in teaching as much as we do in these descriptions is indirect confirmation of the authors' thesis that these choices are culturally conditioned.

A natural consequence of the foregoing beliefs in the U.S.—because students must pay continuous close attention to their teacher solving model problems in order to be able to carry out the same solution methods on their own—is that "U.S. teachers also take responsibility for keeping students engaged and attending." In particular, as one illustrative consequence, a detail that is nevertheless emblematic is that the U.S. teacher typically prefers the overhead projector rather than the blackboard, because the projector is better able to focus attention. U.S. teachers have other ploys whose goal is to keep the attention of students from wandering: "They pump up students' interest by increasing the pace of the activities, by praising students for their work and behavior, by the cuteness or reallifeness of tasks, and by their own power of persuasion through their enthusiasm, humor, and 'coolness.'" Notice that this is a whole category of teacher behavior in the U.S. that is distinct from the injection of extrinsic diversions as discussed above, but which I think can often share some of the same dubious or compromising qualities.

Taking "responsibility for keeping students engaged and attending" is so fundamental to the way we do things in American classrooms that, here again, one might well have taken for granted that it could not be any other way. Tying this to an emphasis on procedural learning is encouraging, I think, because it suggests that the necessity for such measures is mutable.

Surprisingly, teachers in Japan "apparently believe they are responsible for different aspects of classroom activity" from their U.S. counterparts. The Japanese beliefs about what the proper role for the teacher is are at the heart of what college teachers can most benefit from in *The Teaching Gap*, since it is this third set of beliefs that goes directly to what Japanese teachers actually do in the classroom. We accordingly quote the authors' description at some length:

> ... They often choose a challenging problem to begin the lesson, and they help students understand and represent the problem so they can begin working on a solution. While students are working, the teachers monitor their solution methods so they can organize the follow-up discussion when students share solutions. They also encourage students to keep struggling in the face of difficulty, sometimes offering hints to support students' progress. Rarely would teachers show students how to solve the problem midway through the lesson.
>
> Japanese teachers lead class discussions, asking questions about the solution methods presented, pointing out important features of students' methods, and presenting methods themselves. Because they seem to believe that learning mathematics means constructing relationships between facts, procedures, and ideas, they try to create a visual record of these different methods as the lesson proceeds. Apparently, it is not as important for students to attend at each moment of the lesson as it is for them to be able to go back and think again about earlier events, and to see connections between the different parts of the lesson.

The authors then say, picking up on an example cited above, that, "Now we understand why Japanese teachers prefer the chalkboard to the overhead projector. Indeed, now we see, in a deeper way, why they cannot use the projector."

This priority that they "go back and think again...and see connections," and the consequent depotentiation of the need for continuous attention, seem to address at once two prominent issues in our college teaching: respectively, how to encourage the making of higher-order, more abstract connections, and how to lessen those diversionary pulls away from the mathematical material itself to extrinsic matters, as discussed above, and instead allow the inherent qualities of the mathematics to be what keep students attending. Indeed, this priority seems to be, in the context of course of the whole Japanese system, an explicit remedy for the latter problem.

A COROLLARY BELIEF ABOUT VARIATIONS IN STUDENTS' PERFORMANCE

A further fascinating and striking consequence of the Japanese script, closely related to the three sets of beliefs that we have just considered, is a positive valuation of what is often viewed as a chronic barrier to better results in American classrooms, certainly including those in the colleges—namely, differing levels of performance and ability in the classroom. (In fact, the statistical distribution of such levels is not only often widely spread out but, very likely much more often in the colleges than in the public schools, indeed actually bimodal or even multimodal, due to the implementation of two or more distinct tiers of admissions policies at many colleges. This multimodality typically makes this barrier even more awkward to surmount.)

Remarkably, the Japanese see such individual differences "as a resource for both students and teachers...because they produce a range of ideas and solution methods that provide the material for students' discussion and reflection." The larger the class, the more assured the teacher can be that a satisfactory range and an assortment of types of responses will be produced, hence the more reliably planned the lesson can be! Moreover, this range also gives teachers the means to address the differing levels of performance and ability among students. The Japanese have in fact quantified and systematized this approach: "Japanese teachers have ready access to information of the form 'When presented with problem A, 60% of students will use Strategy One, 20% Strategy Two, 15% Strategy Three, and 5% some other strategy.'"

What a very different and more constructive "take" on the problem of disparate performance levels the Japanese script seems to allow.

CLOSING

In this second book review on *The Teaching Gap*, we have given detailed emphasis to the contrasting sets of teachers' beliefs in the U.S. and Japan because these findings seem exceptionally relevant to the practice of mathematics teaching at the college level. In the third and concluding article, we focus on ideas related to improving mathematics teaching in the U.S., and on ways to carry these explorations onward, both in action and in reflection. There as well, the focus is on what we find most worthy of being better known among mathematics educators at the college level.

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1 Brown, Michael L., "Book Review: *The Teaching Gap*, by James W. Stigler and James Hiebert. Review Part 1: Mathematics Teaching as a Cultural Activity," *Humanistic Mathematics Network Journal, 24* (May 2001).

2 All unattributed quotes in this article are, like this quote, from *The Teaching Gap*.

3 This is Footnote 3 on Page 187 of *The Teaching Gap*.

Review of JoAnne Growney's "My Dance is Mathematics"

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Growney, Joanne. *My Dance is Mathematics*. Bloomsburg, PA: JoAnne Growney, 2001, 30 pages, $5.00. May be ordered from the author.

My Dance is Mathematics is a set of twenty-four poems expressing appreciation, understanding, and love of mathematics from a variety of different perspectives. The relationship to mathematics may be seen in the content of the poem, the structure, or both. Poems are often humorous. I enjoy reading them so much that I have given the booklet as a gift many times. Readers of Mathematics Magazine, The College Mathematics Journal, The American Mathematical Monthly, and the Humanistic Mathematical Network Journal have seen samples from this collection.

The content of a poem in "My Dance is Mathematics" may be related to mathematical subject matter, or the associated pedagogy; it may be about a mathematician, or a mathematician's life. For instance, every mathematician would understand and enjoy the portrayal of a familiar experience in the poem, "Misunderstanding."

Ah, you are a mathematician,
    they say with admiration
    or scorn.

Then, they say,
    I could use you
    to balance my checkbook.

I think about checkbooks.
    Once in a while
    I balance mine,
    just like sometimes
    I dust high shelves

One of my favorites is called "A Mathematician's Nightmare." On the surface, it seems to be about decision-making in pricing and shopping, but it is an excellent depiction for a student or lay reader of the Collatz Conjecture, a famous unsolved problem. (A footnote gives the explanation.) Another favorite is a beautiful and poignant poem about Emmy Noether. It is called "My Dance is Mathematics," for which the entire collection is named. In this poem, JoAnne Growney asks "If a woman's dance / is mathematics, / must she dance alone?"

In "December and June," the poet uses vibrant images to compare winter and summer scenes and feelings. The structure of the poem is based on prime factorization in a delightful way. Here is the first stanza:

cold
winds howl
geese go south
nights long tea steeps
temperatures fall low

ponds freeze snowmen grow
toboggans slide down hillsides
sun hides ice coats spring waits
wood-fires flame snowballs fly
winds howl groundhogs hibernate

Poems with very different content, "Changing Colors" (about universal emotions) and "Counting" (about the number system) are based on the same poetic structure: the numbers of syllables in consecutive lines are consecutive positive integers. One of the poems, "ABC," ftmny and clever, is also a bit Eke an exercise, expressing a mathematical idea by using each letter of the alphabet in turn. We enjoy the fun and may be inspired to try one ourselves.

JoAnne Growney's poetic talent, insight, and humor provide repeated pleasure.

Portfolio Assessment in Liberal Arts Mathematics

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SUMMARY
This paper describes my efforts to incorporate problem-solving portfolios into my liberal arts mathematics course. I begin with a description of the components of the portfolios and the factors I consider in evaluating them. I then address some of the more significant obstacles I have encountered as well as what I consider to be among the major benefits. A selection from one student's portfolio is appended.

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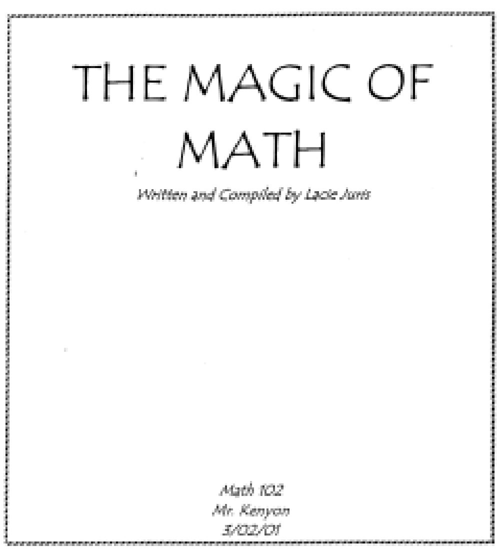
My use of portfolios in my liberal arts mathematics course (Math 102, "The Nature of Mathematics") grew out of my desire to expand on my traditional and often too-narrow use of homework, tests, and not much else to formally assess my students' work. My office adjoins those of some of my colleagues in the English Department, and I have had the opportunity to observe their successes with the use of the portfolio as an assessment tool. The liberal arts mathematics course has an emphasis on a variety of problems suitable for thoughtful investigation and comes with the added benefit of typically being populated by students with particular talents in writing and the arts. As such, it seemed to be a course that was conducive to piloting portfolios.


THE MAGIC OF MATH

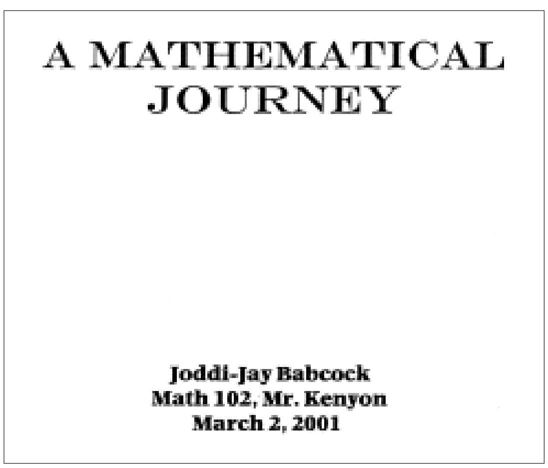
Written and Compiled by Lacie Juris

Math 102
Mr. Kenyon
3/02/01

An excerpt from *The Magic of Math* follows the conclusion of this article.

The portfolio, as I use it, is made up of three major parts. One is the problems themselves. Prior to the start of the quarter, I select between twenty and thirty problems from my list of homework problems and designate them as being eligible for inclusion in the students' portfolios. I select problems that will require some investigation and analysis on the part of the student; the straightforward computational problems are not included—unless there is something special about them that an insightful approach might reveal. As the quarter progresses and I make further assignments, I will also mark some of them as eligible for inclusion. Selected test problems can be included as well. Each student selects five of the eligible problems to include in her portfolio. I suggest that students will want to rewrite their solutions to make them look more presentable; many students do this even in cases where it might not have been necessary. For each of the five problems, the student then writes a page or two about the process she used to solve the problem, the techniques she attempted to use, what worked, what didn't work, and what she learned from both the successful and the unsuccessful attempts. (I usually refer to word counts rather than page lengths as guidelines, in order to eliminate problems with fonts, margins, and the like. I give a guideline of 250 – 500 words for these papers, but I also generally tell students that the paper needs to be as long as it needs to be—no more, and no less. A well-written paper that is a little shorter is better than a poor paper of greater length.)

The second part—which almost always appears at the beginning of the completed portfolio—is a reflective introduction, a 500 – 1,000 word piece that ties the problems together. I point out that, in most cases, there will be some common thread running through the problems each student chooses, and I ask the students to identify those commonalities and reflect on what this body of work shows about their progress over the course of the quarter. One student, for example, opted to showcase the problems she felt most directly applied to "the real world." Many students, though, have been successful with this even (or especially) when their problems do not, at least superficially, appear to have much of anything in common. In fact, some will go out of their way to choose problems that seem to have nothing whatsoever to do with each other. Those students are then able to point out something deeper about the problems and their solutions to them that shows how they have developed their ability to investigate complex problems. In some cases, students have simply chosen their favorite problems. One student explained her selections with these words: "With the wide range of subjects covered in this class, some of them of realistic use in daily life and some of them fairly abstract, I found the variety of problems coupled with their difficulty was forcing me to use critical thinking skills. More importantly than that I found myself enjoying this activity … Maybe just like everything else in life problem solving was more about the journey than the destination, so I chose the problems in this portfolio based on how interesting the journey was." [Lyczewski]


A MATHEMATICAL JOURNEY

Joddi-Jay Babcock
Math 102, Mr. Kenyon
March 2, 2001

It is in the reflective introduction that the students are most likely to look back on the course as a whole and fully appreciate all that they have done. As one thirty-something student wrote, "After an absence of 17 years from mathematics … even though I did well in Math 95 [Intermediate Algebra] and developed some understanding of algebraic concepts, I continued to have serious reservations about taking another math class. I still failed to see the relevance mathematics had for me personally … Although this particular course has come to an end, I do not believe I have arrived at the end of my mathematical journey." [Babcock] Another student explained, "In the past, I have always considered math an intriguing subject, though not terribly exciting. However, that opinion has changed dramatically. I have discovered, through the topics addressed in this course, that the subject of math possesses many amazing, if not 'magical' qualities…Overall, each piece in this portfolio demonstrates not only my skills as a student, but also my abilities to learn new information and apply it to different circumstances." [Juris] Having read those comments and others like them, as well as the portfolios that contain them, I am led to believe that compiling the portfolios has afforded the students the opportunity to look back on their work and see the individual problems in the larger context of the entire quarter. That, I think, makes the students more likely to realize what they have done and to reflect on what it may mean.

The final part of the portfolio is a 500 – 750 word review of the book *Fermat's Enigma* by Simon Singh. We read the book over the course of the quarter and discuss it chapter-by-chapter in class, and also watch the NOVA video *The Proof*. I point out that this is, after all, a book about a problem (albeit a very difficult and famous one) and its eventual solution and as such seems entirely appropriate in a problem-solving portfolio. Where possible, I encourage students to include references to their book reviews in their reflective introductions, but I note that this isn't always possible, and if it doesn't seem to fit with the rest of the material in the introduction, they shouldn't try to force it. This component could, of course, easily be removed in a course that did not have outside reading.

Each of the five problem write-ups counts for 10% of the portfolio grade, as does the book review. The reflective introduction counts for the remaining 40%. I assess each paper in seven areas:

1 The essay should have a clear focus on a specific main idea.
2 The essay should be relevant to the reader.
3 The essay should show that the student has made an effort to be aware of herself as a problem solver.
4 The essay should include specific examples (from the attempts (successful or otherwise) to solve the

problem, from the reading, from class discussions, and the like) in support of the points being made.

5 These examples should be analyzed, not merely stated.

6 The essay should be well-organized.

7 Mechanics, usage, grammar, and the like should be correct.

The student earns a score of 4, 3, 2, or 1 (excellent, good, fair, or poor, corresponding roughly to grades of A, B, C, or D, respectively) in each area on each paper. (The exception is that I find it is usually not appropriate to look at the student's awareness of herself as a problem solver in her book review.) The scores on each paper are averaged to give an overall score for that paper, and I add the scores for each paper (counting the reflective introduction four times) to get a score for the portfolio out of 40 possible points. Merging this with the rest of the student's work for the quarter (homework, tests, etc.) turns out to be a little challenging, because a grade of 3 out of 4 is a B on the portfolio, but 75% is not a B on other assignments. To compensate for this, I end up with a percent score for the portfolio (but 30 out of 40 is not 75% in this scale), and I have the rather absurd situation of recording a portfolio grade to the nearest hundredth of a percent. It works well enough, though. (The gory details are available upon request.)

I gave students very little direction on this project the first time I assigned it. I gave them a two-page handout describing the guidelines and criteria during the first week of class and occasional reminders as the quarter went on. We spent little if any class time on it, though I did make it clear that I was entirely willing to read over rough drafts and offer my feedback. (Several students each quarter take advantage of this.) I remain uncertain as to whether I had a stroke of genius in this minimalist approach or just got uncommonly lucky, but I was simply delighted with the quality of the work I received. It was clear that the students had taken the project seriously and were determined to showcase their best work.

Since then, I have asked students who have written exceptionally good portfolios for permission to copy them and share them with my colleagues and future students, so I now have samples to share with my classes. I have also expanded (and, I think, improved on) my explanatory handouts and discussed what I'm looking for in more detail. This includes devising a handout that explains in a sentence or two what characteristics will earn a paper a score of 1, 2, 3, or 4 in each of the seven areas, all of which seems to make for a much better-conceived project.

(I have never, incidentally, been turned down when I asked a student for permission to copy his or her portfolio for future use. One student, in fact, was delighted by the request, saying that no teacher—and certainly not a *math* teacher—had ever wanted to keep a copy of her work before!)

There are, of course, challenges involved in undertaking such an endeavor. Not least of these is the time involved in evaluating them. Portfolios frequently run in excess of twenty pages; a length of thirty pages is not all that rare. I find that I require at least 45 to 60 minutes (often more) to read and comment on each portfolio. (I never write comments directly on the student's paper. All of my comments are on separate pages.) I simply cannot read more than four or five in a single sitting; at that point, my brain turns to jelly, and it's futile for me to try to go on. Couple this with the rest of the time pressures that come at the end of the quarter, and it's a non-trivial exercise. (Don't expect any sympathy from the students, either!) To accommodate this, I generally make the portfolios due either at the start of the last week of classes or the end of the week before. I don't give extensions, or, more precisely, the truly extraordinary conditions that would warrant me giving an extension have not yet arisen.

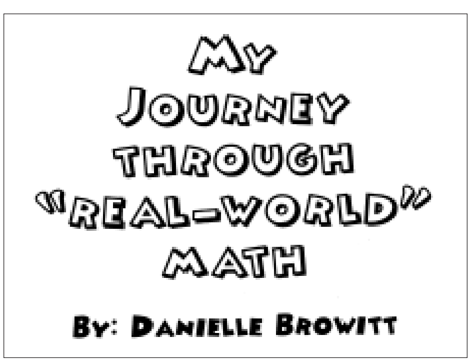
The nature of this writing is such that plagiarism is rather unlikely to be an issue, but it is still good to keep an eye on any overwhelming similarities that might appear between students' work. And one student submitted a problem that hadn't even been assigned (much less listed as eligible for inclusion) that quarter. My first clue came when she wrote that it was the fourth problem on our third test, which only had three problems. In fact, she had lifted "her" work almost word-for-word from one of the sample portfolios I had provided from previous classes. Fortunately, such cases of academic dishonesty are both rare and relatively simple to identify.

There is no grade of F represented in the 1-through-4 grading scheme, and at first that bothered me at some fundamental level. What, I wondered, would I do if and when I received a truly atrocious paper? As it turns out, though, I have never, in the three classes with whom I've done this, received a complete portfolio that merited a grade of F. There have been cases where students have decided that three or four problems instead of five would be sufficient (it's not), or that the book review isn't a necessary component (it is), or that it's all right to turn in a portfolio three days late ("Late portfolios will not be accepted" is not, in my view, subject to multiple interpretations).

There is no provision in the grading for mathematical accuracy. I concluded that since the student can choose any five problems she wants from a list that ends up including close to forty options, she is obviously going to choose ones that she knows she has right. This has almost always been an accurate assumption. In the few cases where it has not been, the accompanying paper has (perhaps predictably) been so poor as to more than penalize the student for the mathematical errors.

By and large, I have been thoroughly impressed by the quality of the work I have seen in these portfolios. Students have taken the project quite seriously, done a great deal of careful introspection and analysis, and composed excellent bodies of work reflecting their progress.

There is, of course, some initial (and sometimes lasting) resistance to the idea of writing in a mathematics course. But that resistance often gives way to a certain sense of welcoming the opportunity to use other talents to showcase their mathematical progress, especially in a course targeted to the liberal arts student. In several cases, I have seen students whose abilities do not appear to be reflected in their test scores doing significantly better when they are given an opportunity to use another assessment instrument. Better yet,



the converse does not hold: I have not yet had a student who was doing well otherwise but submitted a poor portfolio.

An unexpected benefit to this project has been the number of students who have gone on to take additional mathematics courses. We generally assume that the liberal arts mathematics course will be the last mathematics course a student takes, and it does not serve as a prerequisite for any further courses. But since I have begun using portfolios, I have had a surprisingly large number of students elect to take college algebra in subsequent quarters. (Both liberal arts math and college algebra have intermediate algebra as their prerequisite.) Whether this is a result in any way of the students' experiences with the portfolio project is not at all clear, of course, and I haven't analyzed those students' performances in college algebra. But the very fact that they opt to take it is encouraging.

There does not appear to be any obvious reason this concept would not work just as well in other mathematics courses, but while I have thought about doing so, I have not extended my use of portfolios into my other courses. I expect that minor (or even major, in some cases) modifications would be in order, depending on the specific course in question.

I am entirely willing to share samples of my students' work (the best ones, of course!) as well as the current incarnations of my handouts, rubrics, and feedback sheets. Feel free to contact me and let me know what you want to see.

I am grateful to my colleagues in the English Department for their advice and the resources they have provided. John Delbridge and Tracy Case Arostegui encouraged me to go forth with the idea when I mentioned it. Sandy Schroeder provided me with reading material as I began, and Dodie Forrest's rubric was an excellent model for my own. Carolyn Calhoon-Dillahunt has always made herself available to respond to my numerous questions, and has also been vital to the crafting of this essay.

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“Magic in a Box”

Excerpted with permission from
The Magic of Math
by Lacie Juris
Winter Quarter 2001

I have discovered that there are many “magical” qualities involved in numbers and their arrangements. At the beginning of this course I was given a very intriguing puzzle which has been passed down through the ages of mathematics. It is called the Magic Square. The goal is to create a square of equal proportion (3x3, 4x4, etc.) made up of numbers whose sums, from any direction, equal the same number vertically, horizontally, and diagonally. I was allowed to make it as large or small as I desired. Wishing to be different from the rest of the class, I took up the challenge of creating a 5x5 magic square.

Logic and strategy have never been among my strong points. In fact, in ordinary circumstances I will do whatever it takes to avoid them at all cost. However, I had only to look at the now-monumental square awaiting its 25 numbers to see that random guessing was not going to be even a remote possibility within my time frame. I decided that a little research was necessary in order for me to better understand the creation of the magic square. After a search of the Internet, I was able to find a website containing dozens of different examples. My main “strategy” at this point was to pick out as many common patterns between the examples as I could which might give me a clue in how to construct my own. Surprisingly enough, it didn’t take me too long to discover a common similarity in many of the 5x5 squares. Most of the numbers seemed to wrap around the square as if it were bent into a spherical shape like a pillar. With this observation in mind, I attempted my own square.

For no particular reason, I decided to place my first number (1) in the bottom row, middle column. Number 2 then went in the top row, second column from the right. I wanted to make the numbers wrap around the square while angling downward diagonally. Since 2 could not be placed lower than 1, I went back to the top and moved over one column to the right. The figures 3, 4, and 5 continued in the spiraling pattern. When I reached 6, I could no longer go down because number 1 already occupied the space. After a great deal of pondering, I found it was possible to move 6 to the space directly above 5 and then continue on with the pattern. I utilized this move whenever another number blocked a needed space. Finally, after three hours, I triumphantly placed the last number (25) in the top row, middle column and the puzzle was complete! Every row, column, and diagonal added up to exactly 65.

Overall, I have learned more during the completion of this assignment than I ever expected. I have discovered the challenge and fascination of mathematical puzzles and felt the thrill in finding the solution. I have also discovered the great importance of logic, patterns, and observations. It is hard to even imagine the possibilities of solving a puzzle of this size by random chance. This project stretched my critical thinking skills by forcing me to study the puzzle and create a logical strategy to solve it. Once I had discovered the pattern, the entire mystery came to a victorious conclusion in mere seconds. I have finally unlocked the “magic” in the box!

11	18	25	2	9
10	12	19	21	3
4	6	13	20	22
23	5	7	14	16
17	24	1	8	15

Divisibility: A Problem Solving Approach Through Generalizing and Specializing.

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September, 1998

ABSTRACT
This paper describes a divisibility rule for any prime number as an engaging problem solving activity for preservice secondary school mathematics teachers.

INTRODUCTION
My students, preservice secondary school mathematics teachers holding majors or minors in mathematics or science, were raised to believe that there were some "neat" divisibility rules for numbers like 2, 3, 5, 9, 10, 100, some considering the last digit or digits and some considering the sum of the digits. They have also heard of some "weird" and totally unuseful divisibility rules for 7, 11 and maybe even 13. Usually, the former are introduced, and at times even proved, in junior-high school. The latter are mentioned briefly without a proof, or omitted altogether. In the AS (After Sputnik) era of growing dependence on calculating machines, who could possibly be interested in divisibility rules?

The curiosity of one student generated an interesting investigation, that I wish to present here. This student discovered, in fact found on the internet, a divisibility rule for 7, and wondered why it worked. I blessed her curiosity and suggested that the class work on it. The results went far beyond our original intentions: a divisibility rule for any prime number has been derived and proved. More than the mathematical exercise, I wish to share the exciting mathematical investigation and experimentation in which the students engaged.

I will present the results as a problem solving activity that started with collecting data through observation and incorporated several rounds of implementing a "What if Not?" strategy (Brown & Walter, 1990). I will present the results as students' engagement in generalizing and specializing (Mason, 1985), and will conclude with a brief discussion on the relevance of such an activity and several ideas for possible extensions.

PROLOGUE
Consider the divisibility rule for 3: A number is divisible by 3 if and only if the sum of its digits is divisible by 3. Let's prove it for a 4 digit number.

Consider an expanded notation of a 4 digit number written with digits a, b, c and d from left to right.

$1000a + 100b + 10c + d =$
$= (999a + a) + (99b + b) + (9c + c) + d$ applying associativity and commutativity of addition
$= (999a + 99b + 9c) + (a+b+c+d)$

The first addend in this sum $(999a + 99b + 9c)$ is always divisible by 3. The second addend $(a+b+c+d)$ is the sum of the digits. Therefore the number is divisible by 3 if and only if the sum of its digits is divisible by 3.

Even though this proof refers to a four digit number, it gives a general idea how the proof can be extended to a number with n-digits. The strategy used in this proof is representing a number as a sum of two addends. The divisibility of one component is obvious. The divisibility of the second component determines the divisibility of the number. A similar strategy will be applied in the following proofs.

DIVISIBILITY BY 7 - INTRODUCING THE ALGORITHM.
Divisibility of a number by 7 can be determined using the following recursive algorithm:

1. Multiply the last digit of the number by 2.
2. Subtract the product in (1) from the number obtained by deleting the last digit of the original number.

3. Continue steps 1 and 2 until the divisibility of the number obtained in (2) by 7 is "obvious". The original number is divisible by 7 if and only if the number obtained in step (2) is divisible by 7.

Examples of implementation:
(a) Is 86415 divisible by 7?
86415 8641 - (5x2) = 8631
8631 863 - (1x2) = 861
861 --- > 86 - (lx2)= 84
84 --- > 8 - (4x2) = 0
Yes, 0 is divisible by 7 therefore 86415 is divisible by 7

(b) Is 380247 divisible by 7?
380247 --- > 38024 - (7x2) = 38010
38010 --- > 3801 - (0x2) = 3801
3801 --- > 380 - (1x2) = 378
378 --- > 37 - (8x2) = 21
21 is divisible by 7 and therefore 380247 is divisible by 7. (We could continue one step further to get a zero).

(c) Is 380245 divisible by 7?
380245 --- > 38024 - (5x2) = 38014
38014 --- > 3801 - (4x2) = 3793
3793 --- > 379 - (3x2) = 373
373 --> 37 - (3x2) = 31
31 is not divisible by 7 and therefore 380247 is not divisible by 7.

When examples similar to the above were presented in class, the immediate response for many students was a desire to try it out, to carry out the algorithm on numbers of their choice and verify divisibility with a calculator. This generated a large body of evidence to suggest that the algorithm "works". This also generated two related questions:

• Why does this work?
• Why does this work for 7?

The first question is drawn by the desire to understand the algorithm and to prove that it determines divisibility by 7 for any natural (or integer) number. The second question is drawn by the desire to determine the special place of the number 7 in the algorithm. Specializing on 7 in turn invites generalization: Does it work for 7 only? Will the algorithm work for another number? For which numbers will it work? How can the algorithm be modified to work for another number?

WHAT IF NOT 7?

While experimenting with other numbers, a lucky trial by one student prompted a conjecture, that exactly the same algorithm can be applied to determine divisibility by 3. This conjecture has been supported by several examples, however, no other number was found for which the above algorithm can be applied to determine divisibility. To encourage further investigation I suggested the following variation.

DIVISIBILITY BY 19 - VARYING THE ALGORITHM

Divisibility of a number by 19 can be determined by the following algorithm:

1. Multiply the last digit of the number by 2.
2. Add the product in (1) to the number obtained by deleting the last digit of the original number.
3. Continue steps 1 and 2 till the divisibility of number obtained in (2) by 19 is "obvious". The original number is divisible by 19 if and only if the number obtained in step (2) is divisible by 19.

Examples of implementation
(a) Is 15276 divisible by 19?
15276 1527 + (6x2) = 1539
1539 153 + (9x2) = 171
171 17 + (lx2) = 19
19 is divisible by 19 and therefore 15276 is divisible by 19.

(b) Is 12312 divisible by 19?
12312 --- > 1231 + (2x2) = 1235
1235 --- > 123 + (5x2) = 133
133 --- > 13 + (3x2) = 19
19 is divisible by 19 and therefore 12312 is divisible by 19.

For convenience of reference in further discussion, we shall name this algorithm a trimming algorithm.

WHY-QUESTIONS TO WONDER

Experimenting with the two variations of the trimming algorithm presented above there are (at least) two questions that arise: (1) Why is the last digit multiplied by 2? (2) Why does the algorithm involve subtraction in case of 7 and addition in case of 19?

DIVISIBILITY BY 17 - ANOTHER VARIATION

A different variation on the rimming algorithm can be used to determine divisibility by 17. In this case we multiple the last digit by 5 and subtract the product from the "trimmed" number.

Examples:
(a) Is 82654 divisible by 17?
82654 8265 - (4x5) = 8245
8245 824 - (5x5) = 799
799 --- > 79 - (9x5) = 34 we may stop here or continue one step further
34 --- > 3 - (4x5) = -17
Conclusion: 82654 is divisible by 17.

(b) Is 17456 divisible by 17?
17456 --- > 1745 - (6x5) = 1715
1715 171 - (5x5) = 146
146 14 - (6x5) = -16
Conclusion: 17456 is not divisible by 17.

REPHRASING THE WHY-QUESTIONS

The similarities among the three algorithms are obvious. However, the last variation suggests rewording of the first question:

(1) How is the multiplier of the last digit of the number determined? (Why was it 2 in case of 19 and 7 and 5 in case of 17?)

The second question remains basically the same:

(2) Why does the algorithm involve addition in some cases and subtraction is others?

DIVISIBILITY BY 7 - A SPECIFIC "GENERIC" PROOF

After experimenting with a variety of examples the students became convinced that the algorithms do indeed represent a divisibility rule. However, they were still seen as some magic tricks. The interest in *why* (they work) took over from the initial excitement of *how* they work.

Let us prove the divisibility algorithm for 7.

Consider any natural number n. If N is the number obtained from n by deleting the last digit a, we can always represent n as 1ON+a (Example: 3456 = 10 x 345 + 6) We are interested in connecting our original number n and the number obtained by the algorithm, namely, N-2a. In fact, we would like to prove that n is divisible by 7 if and only if N-2a is divisible by 7.

Applying simple arithmetic we get:
10N + a 10 (N - 2a) + 20a+ a
10 (N - 2a) + 21a
The last addend (21a) is divisible by 7 for any digit a. Therefore n is divisible by 7 if and only if N-2a is divisible by 7. Now we can treat the "new" number (N-2a) as the number for which divisibility by 7 has to be established using the same method.

WHAT IF NOT 7?

What if divisibility by a prime number p is in question? Separate proofs, similar to the above, can be developed for a variety of numbers. Inviting students to develop these proofs and discuss similarities among them may help in generalizing to attain an algorithm which determines divisibility of a number by any prime p.

WHEN P=17

For example, to determine divisibility by 17 we looked for a number of the form 10k± 1 divisible by 17. We found 5 1. Therefore k=5. This is the number used in the trimming algorithm to establish divisibility for 17. Since 51 has the form 10k + 1, the number obtained in the algorithm should be of the form N - ka, therefore the algorithm involves a subtraction of a product of the last digit by 5.

WHEN P=31

What is divisibility rule for 31? 31 itself differs by 1 from the closest multiple of 10. Therefore k=3 and the algorithm involves subtraction.

Example:
Is 4185 divisible by 31?
4185 418 - (5x3) = 403
403 40 - (3x3) = 31
Conclusion: Indeed, 4185 is divisible by 31.

WHEN P=13

What is the divisibility rule for 13? We find 39 as a multiple of 13 that differs by 1 from a multiple closest of 10. Therefore k--4 and the algorithms involves addition.

Example:
Is 4173 divisible by 13?

4173 --- > 417 + (3x4) = 429

429 42 + (9x4) = 78
78 7 + (8x4) = 39
Conclusion: 4173 is divisible by 39.

EXISTENCE PROOF

We believe that so far the why questions (1) and (2) raised earlier have been answered. Now it is time to wonder whether it is possible to find an appropriate trimming algorithm to determine divisibility by any prime.

The mathematical answer is no. However, the "human" answer is - almost. Such an algorithm can be determined for all the primes except 2 and 5. (However, since 2 and 5 have well known divisibility rules, we will focus on the other primes.)

The existence of a trimming algorithm for p depends on the existence of a multiple of p which is larger by 1 or smaller by 1 than a multiple of 10. It is obvious that such a multiple does not exist for 5 and 2 which are factors of 10. Let us prove its existence for all other primes.

Formally, let's prove that for any prime p, $p \neq 2, 5$, there exist natural numbers k and m such that $|mp - 10k| = 1$

Let us consider the last digit x of p. The possibilities are 1,3,7 or 9, since this digit cannot be even or 5. If x = 1 or x = 9 the prime itself differs by 1 from the closest multiple of 10. In this case m = 1 and k is determined accordingly. If x = 3, let m = 3, then the last digit of mp is 9 and the number mp is smaller than the closest multiple of 10 by 1. If x = 7, let m = 3, then the last digit of mp is 1 and the number mp is bigger than the closest multiple of 10 by 1. Therefore if x=3 orx=7, then m=3 and k is determined accordingly.

In summary, for any prime p, $p \neq 2, 5$, it is possible to determine a divisibility rule based on a trimming algorithm.

NUMBER THEORY CONNECTION

In a Number Theory text (e.g. Long, 1987, p. 98) the following can be found as an exercise:

(a) If p is a prime and (p, 10) = 1, prove that there exist integers k and y such that yp = 10k+1.

(b) Let n = 10a + b. If p is a prime with (p, 10) = 1 prove that p/n if and only if p/(a–kb), where k is determined in (a). However, without a concrete experience the relationship between this exercise and divisibility rules may not be apparent to many students.

FOR FURTHER INVESTIGATION

In Polya's tradition, the fourth step in problem solving is "looking back" (Polya, 1988). This involves searching for alternative solutions or solution paths, generalizing solutions and exploring situations to which the problem or the method of solution can be applied. We have presented one level of looking back at the problem of divisibility by 7 by exploring the divisibility algorithm for any prime. Further, "looking back" at the general divisibility rule, we can extend our investigation by asking several "what if not" questions.

- What if not primes? Can a similar algorithm be used or modified to determine divisibility by a composite number? What properties of primes were used in our proof? For what composite numbers can the algorithm be applied or modified?
- In all the above examples we have chosen the smallest multiple of p that was bigger by I or smaller by I than a multiple of 10. However, in our proof there was no reference to the choice of the smallest k. So, what if not the smallest? Will the algorithm still work? In other words, is the algorithm, for which existence is proved above, unique?
- Which familiar divisibility rules can be seen as special cases of the general algorithm?

A COMMENT ON USEFULNESS

There is a common trend in mathematics education to focus on "applicable" mathematics, on mathematics that is related to "real life situations". From this perspective, there is a danger of labeling divisibility rules as "unuseful".

I believe that usefulness, together with mathematical beauty, is in the eye of the beholder. For me, a problem that attracts students' interest and curiosity, that generates an engaging investigation, that invites students to make conjectures and test conjecture - is most useful. I believe that an engaging mathematical investigation is useful for all learners, and the excitement of mathematical investigation is especially useful for individuals planning for a teaching career. I hope that my students feel the same.

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Equation Story

Whitney Perret

A short story by a student in Margaret Schafer's 7th grade class at Hale Middle School, Los Angeles

One day, a boy named Data was playing with his dog Variable. Data lived in a magical world called "Number Land." Everything revolved around numbers. People were named after math terms. They used math to label their houses. All of their jobs revolved around numbers. Their whole language was made of math terms.

"Son, come and finish your math chores," said his mom, Mrs. Sequence. The Sequence family liked to keep their numbers or equations in an ordered list.

"Aww, okay Mom," he answered.

"Use like terms to organize your equation," commanded his mom. The word equation translates to bedroom in Number Land.

"Okay," replied Data. His equation looked like this: -7+n-8n+2=. So he combined like terms and got 2n+2=.

"You're almost done, but you need to know the value of n. In this equation, n is equal to 1," she said.

"Okay, Mom. Then my room is 4," he replied.

"Good job, honey. That's correct. Your equation is nice and organized," exclaimed Mrs. Sequence.

"Here puppy. Come on Variable, let's play with my friend Identity," said Data. "It's the funniest thing that all his Variables are exactly the same as my Variable. "Can Operation come over too?" asked Data.

"I guess," said Mrs. Sequence.

"My friend Operation sure knows how to do adding, subtracting, multiplying, and dividing very well. When I have friends over, I never invite Operation and Inverse Operation," Data explained. "Inverse, as we call him for short, always excludes Operation from everything."

"That's smart, honey; you sure know a lot of information," exclaimed Mrs. Sequence. So he called Identity and Operation and asked them to come over. They arrived five minutes later.

"Come on in you guys. I'll be right there but first I want to check the mail." So Data went to the mailbox and looked through it and found a letter. He brought it inside and opened it. "Oh. No. It's an exponent letter," he explained as he walked into the house.

"What does it say you have to do?" asked Operation.

"It says that I have to rewrite this letter 3^4 times or go to Division Crest to see what's at the top," Data yelled. So the three friends set off to Division Crest. "I wonder what's up there," said Data as they began to climb the mountain. "Probably the same thing that's on the top of every mountain, a peak." So they hiked all the way to the top. At the very top, they found a beautiful gleaming stone.

"Oh, my goodness, it's an emerald!"

"We're rich!" exclaimed Data.

"According to my calculations, you will now be 5,000.3 times richer," said Operation.

Does a mathematical/scientific world-view lead to a clearer or more distorted view of reality-purposive musings inspired from readings in *The URANTIA Book, The Cosmic Family, Volume I*, and elsewhere.

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"Until we can understand the assumptions in which we are drenched we cannot know ourselves"
~ Adrienne Rich

Has science, in its earnest endeavor to free itself from the shackles of oppressive medieval thought, unwittingly shackled itself from a truer perception of reality? Has science, in its haste to distinguish itself from superstition and free itself from restrictive religious thinking, embraced postulates it would not have, had there not been a justifiably strong reaction against medieval religious mores?

The URANTIA Book offers this discernment on the subject:

The mother of modern secularism was the totalitarian medieval Christian church. Secularism had its inception as a rising protest against the almost complete domination of Western civilization by the institutionalized Christian church. (p. 2081 - §2)

One example, in my view, of the scientific community's illogical embrace of a postulate is the generally accepted theory that early life formed as the result of the spontaneous coming together of amino acids to form proteins. As you may know, this theory came about as a result of the following hypothesis and experiment summarized below:

I*n 1952, Harold Urey, a Nobel Prize winner, then of the University of Chicago, suggested that the first living cell may have come into existence as the result of a lightning flash searing its way through a smoggy primeval atmosphere composed of hydrogen, ammonia, water vapor and methane. Not that the lightning could have alchemized a living cell at a single stroke; but it might, Dr. Urey proposed, have combined the gases into a number of different amino acids, and these, in turn, might have combined into proteins, and these, in their turn, might have combined themselves into the first living cell.*

In 1955, only three years later, one of Dr. Urey's students, Stanley Miller, mixed the four suggested ingredients in a bottle, discharged an electric spark through them for a week, and discovered on analyzing the result that he had indeed brought about the formation of a number of different amino acids. (Martin Cecil, *On Eagles Wings* p.39)

What is of interest here is that the scientific community as a whole has, apparently, accepted this finding as a valid hypothesis regarding the origin of life despite the statistical remoteness of this possibility. How remote? Consider the following analysis regarding the chance amalgamation of proteins from amino acids below:

In 1964 Malcolm Dixon and Edwin Webb, on page 667 of their standard reference work Enzymes, point out ….. that-depending on the laws of chance arrangement alone—in order to get the needed amino acids close enough to form a given protein molecule there would be required a total volume of amino acid solution equal to 10 to the power of 50 times the volume of the earth.

But here we are dealing with the chance origin of a very simple protein. What are the odds in favor of the formation of a larger protein molecule such as hemoglobin?

S.W. Fox and J.F. Foster have worked this out for us in their Introduction to Protein Chemistry, page 279. They have shown that only after the necessary amino acids had *come together to form random protein molecules by the process described above, and only after these protein molecules had been formed in such a quantity that they filled a volume 10 to the power of 512 times our entire known universe …could we reasonably expect that just one hemoglobin molecule might form itself by luck alone.* (Martin Cecil, *On Eagles Wings* p.39-40)

Clearly from the standpoint of statistical analysis the idea of life forming spontaneously is absurd. Yet, this idea is pervasively held throughout the scientific community. I suggest we have accepted this incongruous notion rather than submit ourselves to the remotest possibility of returning to the horrendously oppressive conditions of medieval times. That is to say, there is, I believe, an unspoken fear that should the idea of a personal god be accepted in mainstream scientific thought that this would then lead inexorably to a return of the oppressive mores of medieval times where scientists would find themselves beholden to and persecuted by the church, as were Copernicus and Galileo. Thus, I submit, the animus within the scientific community towards considering the existence of a personal god has more to do with fear and human prejudice than with honest scientific analysis.

What has thus apparently evolved over recent centuries is the development of an existent paradigm of a godless science.

As we know, paradigms are sets of rules (filters if you will) for viewing the world. Regarding the classic, *The Structure of Scientific Revolutions*, by Thomas Kuhn, Ronald C. Tobey in *A Beginner's Guide to Research in the History of Science* offers these insights:

Kuhn distinguished between two kinds of science - normal science and crisis (or revolutionary) science. Normal science is science pursued by a community of scientists who share a paradigm. Revolutionary science is not. A paradigm is a consensus among a community of practicing scientists about certain concrete solutions—called "exemplars"—to central problems of their field. Their consensus is based on commitment to the paradigm. The commitment is derived from their training and their values; it is not the result of critical testing of the paradigm. Normal science is intellectually isolated from "outside" influences, including the paradigms of other scientific fields and nonscientific events and values. Commitment to their paradigm gives a powerful "normality" to the paradigm, enabling scientists to disregard phenomena that appear to contradict it-"anomalies." (http://www.horuspublications.com/guide/cm106.html)

One may wonder then, to what extent the scientific community is willing to promote this paradigm of a godless science. Is the scientific community's investment in a godless science so deeply entrenched that it collectively disregards possibilities to the contrary? Consider Werner Heisenberg's comments on his own uncertainty theory. (The uncertainty theory simply stated is that, regarding sub-atomic particles, it is impossible to know with certainty both the momentum and position of a particle at the same time, the greater the certainty of one quantity the less the certainty of the other, in contrast, this is not the case with larger (Newtonian) size objects, such as billiard balls where the position and momentum can be known with certainty and at the same time.)

I*n view of the intimate connection between the statistical character of the quantum theory and the imprecision of all perception, it may be suggested that behind the statistical universe of perception there lies hidden a "real" world ruled by causality. Such speculation seems to us—and this we stress with emphasis—useless and meaningless. For physics has to confine itself to the formal description of the relations among perceptions.* (W. Heisenberg, *Zeitschrift fÜr Physik*, 43 {1927} p.197)

What is this 'hidden "real" world ruled by causality'?…apparently Heisenberg was unwilling to consider it. Why? Perhaps it was because an honest examination of this phenomenon could lead one to conceive of a personal god present amongst the particles. A possibility apparently at odds with prevailing scientific thought then and now.

In the realm of sub-atomic particles, the observer indeed has a cause-and-effect impact upon the observed. How could this be unless there was indeed a causality connected to the presence of the observer? Stated otherwise, there exists a relationship between the human observer and the physical matter being observed….a relationship. Here then is a clue that the universe is not static but that in fact our actions have a discernible affect upon it.

Because our actions are intimately connected to our thoughts and attitudes, we may thus expand the matrix of reality being considered to include the attitude and thought-life of the observer as well as the discernible matter being observed. From this point of view, reality becomes more fluid then perhaps we have previously conceived. Accepting for the moment an omnipresent personal creator we can also envisage the presence of a divine or cosmologic vibration pattern whose very presence is revealed to us in direct measure of our spirituality.

This phenomenon is hinted at in *The Cosmic Family, Volume I* where our relation to cosmologic vibration pattern is revealed.

As you incorporate patterns of thinking within yourself, these energy patterns create messages within your physical body that either respond to a cosmologic vibration pattern within the divine mind or to confusion, non-divine pattern, disharmony and self-assertion. (p. 155)

From this vantage point the relation of observer to the observed may be expanded from merely a consideration of the study of sub-atomic particles to one's relation to spirituality generally.

Again drawing from *The URANTIA Book*:

Moral convictions based on spiritual enlightenment and rooted in human experience are just as real and certain as mathematical deductions based on physical observations, but on another and higher level. (p.2077 - §8)

This higher level apparently functions with remarkably elastic properties as is again revealed in *The Cosmic Family, Volume I*:

As you become honest, the gift of honesty is given. As you become patient, the gift of patience is given. As you become giving, the gift of things are given to you. As you seek wisdom over pride, wisdom is given. (p. 119)

The above discussion, in and of itself, I doubt will convince many materialistically minded thinkers to embrace the reality of a living personal god but it is, nonetheless, worth considering; albeit many will likely yield to the temptation of embracing a mechanistic view of reality rather then to consider the presence of an intelligent creator behind the scenes.

Unfortunately, this mechanistic view taken to its logical conclusion:

…reduces man to a soulless automaton and constitutes him merely an arithmetical symbol finding a helpless place in the mathematical formula of an unromantic and mechanistic universe. (*The URANTIA Book* p.2077 - §4)

Challenging the mechanistic view-point generally The URANTIA Book points out:

The inconsistency of the modern mechanist is: If this were merely a material universe and man only a machine, such a man would be wholly unable to recognize himself as such a machine, and likewise would such a machine-man be wholly unconscious of the fact of the existence of such a material universe. (p.2078 - §6)

Clearly, however, not all scientists have embraced this paradigm of a godless science. Many have, in fact, contributed to what has become known as the Design Argument. (Stated simply the Design Argument promotes the idea that because there is so much evidence for design in nature, both biologically and cosmically, that this therefore can be taken as evidence of the existence of a designer.) Chief proponents of this view include Isaac Newton who in his addendum to the Principia book three (the General Scholium) reasoned:

The planets and comets will constantly pursue their revolutions in orbits given in kind and position, according to the laws above explained; but though these bodies may, indeed, continue in their orbits by mere laws of gravity, they could by no means have at first derived the regular position of the orbits themselves from those laws.

Another proponent of this Design Argument was William Paley, author of *Evidences of the Existence and Attributes of the Deity collected from the Appearances of Nature*. Frederick Ferre in his classic essay *Design Argument* summarizes Paley's work in this way:

In that work Paley argued explicitly for the presence of intelligently designed features in nature. The marks of design, he said, are what we observe in contrasting a watch with a stone. The stone, for all we can tell, might just have "happened"; but the watch is clearly put together out of parts that work together in an arrangement that is essential to their function, and the function of the whole has a discernible and beneficial use. Wherever we find such a constellation of characteristics, Paley said, we must admit *that we are in the presence of "contrivance" and design and since in our experience the only known source of such contrivance is the intelligence of some designer, we are en-* titled—obliged—to infer an intelligent designer somewhere behind anything possessing the above mentioned marks of design. (Design Argument, Frederick Ferre, Dictionary of the History of Ideas, Vol. I, p. 674)

Apparently then, we have arrived at two competing world views; a mechanistic godless universe versus an omnipresent intelligent designer, with the human scientist cast adrift somewhere in between.

Again, *The URANTIA Book* offers these insights:

The universe is not like the laws, mechanisms, and the uniformities which the scientist discovers, and which he comes to regard as science, but rather like the curious, thinking, choosing, creative, combining, and discriminating scientist who thus observes universe phenomena and classifies the mathematical facts inherent in the mechanistic phases of the material side of creation. Neither is the universe like the art of the artist, but rather like the striving, dreaming, aspiring, and advancing artist who seeks to transcend the world of material things in an effort to achieve a spiritual goal. (p.2080 - §7)

Thus it appears it is the pursuit of science, rather than the science itself, which may offer us the most meaningful approach to reality.

Along this line of reasoning Cardinal Nicholas Cusanus of the fifteenth century observed:

Mathematics induces the mind to withdraw somewhat from physical immediacy into the sphere of reflective meanings, thus preparing for our further move toward God's invisible reality. (Idea of God 1400-1800, James Collins, *Dictionary of the History of Ideas, Vol. II* p. 346)

If one will allow the supposition that mathematics is the product of the human mind then the field, taken as a whole, reflects the breadth, width and height of humanity's ideational habitat; offering at the same time an expansive as well as conditioned conceptual framework.

In this regard *The URANTIA Book* offers this insight:

While the domain of mathematics is beset with qualitative limitations, it does provide the finite mind with a conceptual basis of contemplating infinity. There is no quantitative limitation to numbers, even in the comprehension of the finite mind. No matter how large the number conceived, you can always envisage one more being added. And also, you can comprehend that that is short of infinity, for no matter how many times you repeat this addition to number, still always one more can be added. (p.1294 - §11)

Returning to our original proposition: Does a mathematical/scientific world-view lead to a clearer or more distorted view of reality? *The URANTIA Book* again offer us these clarifying insights:

Mathematics, material science, is indispensable to the intelligent discussion of the material aspects of the universe, but such knowledge is not necessarily a part of the higher realization of truth or of the personal appreciation of spiritual realities. Not only in the realms of life but even in the world of physical energy, the sum of two or more things is very often something more than, or something different from, the predictable additive consequences of such unions. The entire science of mathematics, the whole domain of philosophy, the highest physics or chemistry, could not predict or know that the union of two gaseous hydrogen atoms with one gaseous oxygen atom would result in a new and qualitatively superadditive substance—liquid water. The understanding knowledge of this one physiochemical phenomenon should have prevented the development of materialistic philosophy and mechanistic cosmology. (p.141 - §4)

In conclusion, on the subject of mathematical reasoning, the universe, reality and whether or not there is an omnipresent personal god, perhaps the most stimulating close to this essay would be the query, so eloquently posited in The URANTIA Book, "…whence comes all this vast universe of mathematics without a Master Mathematician?" (p.2077 - §4).

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Special Section: On the publication of the 26th issue of the Humanistic Mathematics Network Journal

Emerson said (and I'm sorry for not having the exact quote), that every institution is the work of a man. We can think about that on two levels; Alvin kept the torch burning for the rest of us to be sure, but the shadow of the man is also the "human" in Humanistic Mathematics.

The past decades have seen mathematics in unimaginable transition. The WWII years may have been a period which experienced an admirable collaboration of thinker, but its legacy, the 50's, was a period in which you had to take sides: "applied" or "pure." Either way you were stuck.

In the 60's, pure math was filtering into textbooks to create what might be (in your mind) "the only decent Calculus book" or something entirely unreadable. One-third of the female freshman class at Brown/ Pembroke in those Sputnik years declared math as their incoming major; but non-standard analysis killed most of them off. Often a student's text was the teacher's notes, and that minimal, purple-printed test asked you to prove the Fundamental Theorem of Calculus and solve some problems. (Perhaps your text had problems, but there weren't many!)

In the 70's, during the Vietnam War we saw the vague emergence of the computer and simultaneously distrust began to brew about putting pure mathematicians into the position of guardians of engineering research centers. I remember teaching at Washington and Lee in those days, trying my hardest to persuade my class that the computer would be useful! (We were using it to add three floating point numbers and if your punch cards didn't spill all over the floor in transit to the computer room, you found Fortran couldn't do the problem accurately.)

The 80's were a period good for women and minorities. Cultural concerns, including issues of teaching flourished. Note, 1986 marks the inception of the Humanistic Math Network.

The 90's! What can one say? They went by in a microsecond, raising issues of the calculator, the computer and the web, but equally, a brand new interest in teaching (e.g. calculus reform). Teaching can be the "tar-baby" of mathematics, and we have to temper our interest in being converts to a religion-of-one with the needs of unthinking, financially driven governing bodies.

How then will we, in the future, hang onto the spirit of humanism, collaboration, and gentleness that learners from previous decades enjoyed and shared? My own feelings are quite optimistic. We are again benefiting from the blending of the applied and the pure, and we have a greater awareness of student-teacher relations and issues of assessment with which we must always focus on the need to possibly remodel ourselves.

To create this last issue of the "hand-held" HMNJ, I essentially asked Alvin to step aside as I asked for contributors from the journal's past. Touching these people enriched me. It reminded me of all that was best about mathematics; the people I contacted were people whose books or articles I may have read, in some comfortable reading corner—books "about" mathematics—how we do it and what we mean by it and why we do it. But there are many warm and wonderful members in my immediate mathematical community.

Web generation we now may be, and the terms of the web will be broader, more scattered, faster and yet more time-consuming. But we can exploit the web version of HMNJ by tuning in, to create a new community which retains that understanding of the "human" for which our institution of mathematics is merely the shadow. Thanks, Alvin, for keeping us remembering!

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Alvin White and the HMNJ

When I first met Al White at the winter mathematics meeting in San Antonio in 1987, I already knew his name. We both had links to Stefan Bergman. Alvin did his doctoral dissertation under Bergman at Stanford, and I, a few years previously, had been a graduate student taking Bergman's courses (at Harvard) and later post doc'd for him. Al and I both had high regard for Bergman as a human being, which was not always the case in the mathematical community. This was an initial basis for our friendship. The second basis and more significant one, was our mutual concern for the humanistic aspects of mathematics, however you care to define that term.

Al saw the need for a periodical devoted to promoting the humanistic aspects of mathematics. In those days he was one of the few voices crying in a wilderness of unreflective accomplishment. He told me about the HMNJ and his plans for it. I was fooled by his soft-spoken and occasionally bumbling manner. But Al carried through with his plans.

Things don't just happen; they don't just sprout like mushrooms after a penetrating rain. When one looks closely, there is always an individual who conceives, then creates and nurtures what has been created. In the case of the HMNJ, that person is Alvin White, and over the years the journal has become the leading outlet disseminating articles on the humanistic aspects of mathematics. I congratulate Al for piloting the HMNJ with skill for so many years and I hope he will be able to guide it through its present crisis.

Today when, on the one hand, society is ever increasingly mathematized, and when, on the other hand, mathematics offers many an escape from the difficult dilemmas of civilization, the necessity of relating mathematics to the lives of those who create it and those who are affected by it is increasingly important. Hence the need for reflecting on and publicizing such reflections. Looking over several past issues of the HMNJ, I observe that it contains a wide variety of subjects. There are, among others, didactics at all levels of instruction; biography; linguistics; heuristics; methodology; history and futurology; esthetic; poetry; the spiritual element; women's issues; ethnomathematical issues, book reviews.

What I, personally, would like to see in future issues of the HMNJ is much more discussion of how mathematics has entered our daily lives and what its effects have been for good, for bad, or for neither. As examples, (some of which have indeed been discussed in past issues), product striping, ATM's, voting schemes, the impossibility of counting (census results are a matter of litigation), other parts of mathematics that are litigious (e.g., what math can be copyright), use of mathematical statistics as court evidence, the role of mathematics in social and economic decision making, gambling schemes, the role of mathematics in "green" issues, the role of mathematics in defense.

Thus, referring to the last topic, there is to be in August of this year a conference in Sweden discussing the role of mathematics in war. I would think that a number of papers presented at this conference would be appropriate material for reprinting in the HMNJ. What I would caution against is the HMNJ becoming a journal that deals more and more with classroom issues. There are many journals that concern themselves with pedagogical techniques and curricular questions. There are indeed serious problems in the classroom that need consideration, but despite the inadequacies in the teaching and learning of mathematics, the fact is that the mathematization of society is going forward at an increasing pace, often set in position by the fiat of a techno-competent elite, and which require constant reexamination and discussion. The HMNJ should be characterized by its concern for the problems that mathematizations create.

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Reflections on the Founder

For me, the Humanistic Mathematics Network is inextricably linked with Alvin White, its founder and editor. In this special edition of the Journal, it is probably appropriate to describe what I know of Al, a man whose life and career embody the principles that the many readers of the newsletter find so compelling.

My first meeting with Al White has become a caricature in my mind; the remembered emotion of the events has exaggerated the details. This is probably not what happened, but it is what I remember happening.

I came to Harvey Mudd College for one semester to teach a few courses. Al had written me a very kind e-mail to welcome me, and so I decided to stop by his office and introduce myself. He was seated with a student, both of them at a desk far across the vast room from where I stood in the doorway. I stuck my head in the room; I excused the interruption to say my name; I said that I would come back later.

But Al jumped up, papers flying from him the way pigeons fly from a cat. He clambered past piles upon piles of books to grasp my hand in both of his. Welcome, he said, and I am so glad to meet you. He asked me about an article I had written about mathematical writing, and gazed into my eyes earnestly while he professed his own passion for teaching. For several minutes he explained his philosophy on the importance of teaching well, of communicating well, of keeping the next generation always in mind. Students, he said, students come first. Always. I remember looking past Al at the student still sitting on the chair by the desk, on the other side of piles of books, on the far side of a vast room. Al followed my gaze back and laughed at himself. It is so good to have you here, he said to me, and he clambered back to his student.

My deepest obligation to Al is for a panel discussion that he organized. I was still a young, untenured mathematician, but Al invited me to be one of the panelists. On one side of me sat Leonard Gilman, the author of "Writing Mathematics Well", and whom I had idolized from afar. On the other side sat JoAnne Growney, a mathematician/poet whose books still travel back and forth between my shelves and my lap. I remember Leonard Gilman playing the piano to describe mathematics in music, and I remember that JoAnne Growney read poems of her own and of others. (I don't really remember what I did at all, but maybe you do: there were 500 people or so in the audience). It was a wonderful evening, and a wonderful opportunity, and I am deeply grateful to Al for including me.

Alvin White has made his career at a small school (about 600 students) that is probably the only Liberal Arts/engineering School in the nation. While you might think that this combination would naturally engender a newsletter like "Humanistic Mathematics", in fact I found during my semester there that the students (and even many of the faculty) were invariably practical-minded, with little room for fluff. Given the choice between truth and beauty, they would choose truth.

And yet Alvin White began a journal-and a national movement-that combines mathematics with poetry, that says that people come first, that defines mathematics as a human endeavor. Given the choice between truth and beauty, Al does not choose. Instead, like Plato and Keats, Alvin White says there is no choice:

Truth is beauty, and beauty is truth: that is all ye know, or need know, on earth.

Thank you, Al!

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On the Twenty-Sixth Edition of HMNJ

I am indeed saddened by the news that HMNJ in its hard copy form will be coming to an end. I'm grateful that EXXON has supported it as it has. But I don't think on-line journals are nearly as useful, as they are hard to read and often text and diagrams get changed on the screen as far as layout is concerned. Anyway, online journals are much harder to keep track of. With HMNJ, I often recall the color of the issue even if not the date, so can find it again more easily. Also, I can browse through it by my nightstand—something I can't do if it is only on the computer!

Nevertheless, while hoping that the HMNJ continues to be a fruitful place to visit online, something must be said about what it has been til now. Namely, it has been a unique place for publishing diverse kinds of articles. For example, where else could long, valuable, and thoughtful articles like, "Will you Still be Teaching in the Twenty-First Century" (#23), "Tilings in Art and Science" (#12), and the timeless, "The Classroom Encounter" be found?

Also, the book reviews have been very thoughtful. Thinking of this as the last issue, I now wish I had shown my appreciation to Alvin more directly, and certainly I never thought of writing to EXXON to tell them what a valuable contribution they made.

So if nothing else, I do want, in this last issue, to thank Alvin for his vision, which he created this journal and for all the hard work he put into it. I think he and the journal have given encouragement to a lot of people whose work and teaching will have greatly benefitted from it. His great chapter, "Teaching Mathematics as if Students Mattered," in the book "Teaching as if Students Mattered" (1985), should be read by older folk again, or used to introduce younger ones who have never heard of this book, to ideas now considered innovative, published over 15 years ago. I think this chapter is even more relevant now than then because of the onslaught of testing and the "TELL the facts" type of teaching.

An Alphabet for the Twenty-Sixth Edition of HMNJ

Alvin with his ideas of the need to teach
Better by encouraging
Concepts and
Dialogues
Encouraged us all to learn about
Fascinating subject matter with
Good ideas.
Humanistically Hearing (not just listening) and providing
Inspiration and Intuition tempered with
Judgment and always with
Kindness, he showed us
Learning that
Matters, that seems
Natural, that
Opens students' eyes to
Posing of problems and
Question-asking, together with
Reaching to
Solve those problems by
Thinking in new ways—
Usefully, and in a
Variety of ways, demonstrating the importance of
Writing.
eXxon supported this journal over the
Years so as to provide us with the
Zest to continue our teaching so others, too, will love math.

Good luck to the HMNJ of the future!

Marion Walter
Math Department
University of Oregon
Eugene, OR 97403

Being an editor of a periodical is a tedious, thank less job. So many details to worry about. So many chances to do or not do something that somebody will complain about. When readers are satisfied, they thank the author, not the editor. When they are dissatisfied, they blame the editor, not the people who could have contributed articles but didn't get around to it.

And then, what about the courageous, heroic individual who actually dares to found a new publication, and that newsletter miraculously makes it, endures, finds an angel and an audience! Do we remember him or her, when years later we benefit from his or her creative and daring impulse?

So, I say that what Alvin has done is awesome. Now, what about the down side? HMNJ—(Alvin White, actually) accepts what comes. Some is more original, some less. Some is less profound, some is more. The literary-scientific quality of each issue has to be variable. But it is authentic! HMNL is a an outlet, a vehicle for the large inchoate mass of math teachers who want to humanize mathematics. As such, it is incomparable and irreplaceable.

Thanks, Alvin.

Reuben Hersh
University of New Mexico
Alburquerque, NM

Humanistic Educational Mathematics

My first memory of the notion of (if not the term) humanistic mathematics is of the Pasadena meeting of mathematics educators in March of 1986. Alvin White assembled quite a few impressive people, including some of my heroes of the time and later. We had a great time not reaching consensus about what humanistic mathematics was.

Fortunately, the lack of consensus didn't deter Alvin from forming the network and newsletter. Without a need for consensus on the definition of the term, members seem to fall into at least one of two camps: embracing mathematics as a human activity (usually meaning as part of humanities, having strong connections to literature and history and the fine arts), or teaching mathematics with attention to how humans think and feel and learn.

My own interest is in teaching mathematics as a means of educating humans. I use "educating" in the sense of encouraging development in intellectual, ethical, and identity domains. I'm especially interested in the maturing of peoples' conceptions of knowledge, as described by Perry (1970) from believing that every statement is either right or wrong to seeing truth as relative to context. As Perry found, this cognitive development is intertwined with ethical and identity development. Letting students in on the ambiguity, warmth, imperfections, and elegance of this field that is commonly seen as precise and austere and perfect can help shock them out of the comfort of a black and white world.

So I'm interested not so much in mathematics education-i.e., in helping students come to attain mathematical skills or to understand mathematical ideas or even to "think like mathematicians'~-but in how mathematics can be taught to enhance education. Back in the 1970's, Steve Brown coined the term "educational mathematics" for that interest-a term included in the title of the Institute I direct. That Steve continues to share this interest is evident in his most recent book (Brown, 2001).

Educational mathematics depends on humanistic mathematics in a variety of ways. To use mathematics as a tool for education, we need to be aware of how humans think and feel and learn. We can use links with humanities astorce to encourage students to grow and change. But the goals are somewhat different. And, while focused on these goals, the major issues of mathematics education-reform curricula, assessment, etc.-are of interest only peripherally. I've written about these issues in various places, and I support myself by working with them, but what I consider my most enthusiastic publications have been about using mathematics to encourage development on the Perry Scheme (e.g., Copes, 1982, 1993).

What about the future of humanistic mathematics education and educational mathematics? To the extent that, as Postman & Weingartner (1969) claim, teaching is a subversive activity, many teachers will continue to teach humanistic mathematics by teaching humanely. A few will communicate the aspects of the field that make mathematics like the humanities.

As for educational mathematics, I suspect that we are not much further ahead than we were 20 years ago. NCTM's influential publications (1989, 2000) ignore this role of mathematics. The reform movement refers to areas outside mathematics only as a source of applications or as a means to help students understand mathematics better. The political battles are about what to teach of and about mathematics: primarily, as facts and procedures or as a set of tools for problem-solving. The justification of mathematics as a means to the end of self actualization isn't appearing in the newspapers.

But learning is a subversive activity as well. I am optimistic that many students, with or without the help of teachers, will experience mathematics in ways that encourage their own development.

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It is hard to say what is the BEST thing about Al White, but one VERY GOOD thing is that he al ways has time to hear and consider new ideas. The Humanistic Mathematics Network and its journal have for many years provided a source and a forum for the new and the different as well as the old and not-to-be-forgotten.

As a mathematician with strong interests in literature and the arts, I was delighted to find, through HMN, others with similar interests. Al White has been untiring in his support of special sessions at national meetings and journal publication so that teachers and mathematicians of diverse interests may exchange ideas.

Now a poet, my memories of HMNJ focus particularly on poetry from it and I am ever grateful that I met on its pages the Czech poet, Miroslav Holub (1923-98). Both a scientist (an immunologist) and a poet, Holub's interests paralleled my own and he has become a favorite poet for me.

Below I include a brief poem by Miroslav Holub (translated from the Czech by Ewald Osers) that has been included in NUMBERS AND FACES: A Collection of Poems With Mathematical Imagery, an HMN publication that Al and I worked on together in 2001.

The Parallel Syndrome

Two parallels
always meet
when we draw them by our own hand.
The question is only
whether in front of us
or behind us.
Whether that train in the distance
is coming
or going.

Al's contributions to mathematics and its teaching are legion. He has had new ideas when others were dormant; he has continued when others were idle. He has lived, at least, nine lived. Always a supporter of the artists among mathematicians the visual artists, the musicians, the writers. I have appreciated his support of my poetry and close with this brief poem of my own.

Good Fortune

is good numbers,
the length of a furrow,
the count of years,
the depth of a broken heart,
the cost of camouflage,
the volume of tears.

Alvin White, may you always have good numbers!

Thank you for your years as chief of the Humanistic Mathematics Network and as editor of the Humanistic Mathematics Network Journal (and, before it, the Newsletter). I am sorry that the years of paper publication have ended for I love the feel of paper in my hands; however, I will find HMNJ on-line, and I hope everyone else does too.

JoAnne Growney
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Teaching as Though Students Mattered

A BIOGRAPHY OF ALVIN WHITE AS TOLD TO SANDRA KEITH

I was born 1925 in New York; served in the U.S Navy in 1943-1946 during WWII; was sent to Ra dio Technician School and then served in the Pacific (the battle of Okinawa). Subsequently, I was sent back to the States for Officer Training School at Columbia University. I graduated from Columbia College in 1949 with a major in math but I took many other subjects, of course. Among my most memorable professors and courses were Moses Hadas teaching "Greek and Roman Mythology" and Bernard Stem in "The Family Past and Present." I feel I learned the most (and got the most enjoyment) from one course that required me to write a paper independently in Autumn and Spring term; in that course, I chose to learn and write about the history of public housing in New York City. Special apartment houses had been built in New York City for the waves of immigrants coming to the USA in the early part of the 20th century. These early houses were instant slums. Eventually, a law was passed that required every room to have a window; even that minimal ruling was barely satisfied. But I spent many happy hours in the Columbia Architectural Library reading Jane Jacobs and other architects and historians on the history and progress (albeit slow) toward improving public housing.

I received an MA in math from UCLA and went on to Palo Alto where I enrolled at Stanford. I worked with Stefan Bergman on partial differential equations related to fluid dynamics and received my Ph.D. 1961. 1 was hired by Harvey Mudd College, but spent the year 1961- 1962 at the Math Research Center at the University of Wisconsin, Madison.

While teaching traditional courses at HMC, I had found and read "Freedom to Learn," by Carl Rogers. As an attempt to apply my understanding of Rogers' book, I offered a seminar on Calculus of Variations in my living room. I received a small grant to purchase portable blackboards (green actually) and bean bags for sitting. (I visited the California Institute of Technology some time later and found that they had a room similarly furnished that was used every hour by many departments to teach various courses!) I wanted the course to be "student centered", as I understood that concept. At the first meeting I introduced ten or twelve books on the subject and explained the problems and topics from the early history of the subject. Each student was to report on a topic. I met with those six students once a week for three hours. As it worked out the students' reports paralleled the usual course. The reports were also distributed and used. But at the end of the course some students remarked, " I enjoyed the seminar and learned a lot. But I think that I would have learned more with lectures and a textbook." I was not aware of anyone teaching mathematics in this way at that time.

I tried to publish my experiences in that course with "The American Mathematical Monthly" but was rejected with the remark, "No numerical data." However, Carl Rogers, while guest editor of the journal Education, published my article under the title "Humanistic Math: An Experiment." A physicist from Purdue was inspired by another book by Carl Rogers and also taught a laboratory course in physics in the same student-centered manner. His students had a similar response:

> "We enjoyed the course but we would have learned more with a text and lectures." His comment, in an article, was: "I doubt that they would have learned more ... although they might have suffered more."

In the early 70's, I began to organize informal, interdisciplinary discussions among faculty of the Claremont Colleges. The discussions drew not only mathematicians but physicists, philosophers, historians, and psychologists. At this time, writing unsuccessful grant proposals was to become a repeated occupation of mine. In 1976-77, however, I was one of ten Danforth Faculty Fellows awarded year-long grants to study at various universities. I studied at Stanford, took a short stay at Northwestern, and spent the spring semester at Massachussetts Institute of Technology (MIT)-Harvard, where I taught a seminar at the Division for Study and Research in Education (DSRE). The seminar was described by a set of questions:

> "How does one acquire knowledge? What are the limits of certainty? What is the relationship between scientific knowledge and general knowledge? What is the role of beauty, simplicity or intuition in creative discovery? Our present knowledge in the arts, humanities and sciences is the legacy of creative imagination. How can this legacy influence education at all levels?"

There were twelve students in the course, representing Math, Art, Linguistics, Electrical Engineering, Biology, Artificial Intelligence, and Computer Science. We sat around a table discussing our readings and thoughts. We met twice, for an hour and a half, the first week. After that, everyone dropped their other courses and we ended up meeting the whole morning until we were eventually displaced by lunch hour. We became a community that cared for one another and learned from each other. Students invited other professors to participate. Visitors to MIT-DSRE would ask permission to observe quietly, although they usually joined in the discussion. One student remarked that the popularity of our seminar among visitors was probably due to the openness, honest listening and caring that were evident. We accepted all contributions in a non-judgmental way. No one was forced to speak, and everyone had a chance to speak. In that course, we examined writings of Dewey, Kant, Polanyi, Russell and others. The last week of the term became a time to discuss and evaluate the seminar. Why was it so successful? What had happened to us? How had we been transformed from a mere collection of strangers to a group of friends and colleagues? It was as if we had chanced upon a semester-long celebration, and, like Alain-Foumier's "Wanderer," we had been caught up in the "spirit of the place." A student observed that this was the first course where her presence in the room had "made a difference." We wondered: what had we learned and what should we do if we wanted to find that spirit of celebration again? An unexpected answer emerged, one that simultaneously addressed some of the questions of the seminar as well as questions about the seminar. The answer from one of the students was : "we learned with love and trust." This was more food for thought: what did that response mean? In my opinion, the meaning is well discussed in E.A. Burtt' s book: " In Search of Philosophic Understanding."

After I returned to HMC from MIT and the exciting, warm relationships that I had experienced there with students, I was disappointed by the relationships that students assumed or insisted upon between them and me as their teacher at HMC. To them the classroom could not be a community of mutual learning and teaching, and for some there was an invisible line between teacher and students. In an attempt to overcome some of these barriers, I started teaching my course in Calculus upon returning to HMC by reading with the class, J. Bronowski, "Science and Human Values." I wanted to investigate with my class Bronowski's statement that "the society of scientists is more important than their discoveries. What science has to teach us here is not its techniques but its spirit: the irresistible urge to explore." I have to admire the students' tolerance. Their fear was that they would not understand calculus and that we would not complete the syllabus . As it turned out, we completed the syllabus with time to spare, and students and I agreed that they understood the calculus more deeply than they would have in a traditional course of straight lectures. In addition to homework problems, students were encouraged to modify problems, for example, by increasing the dimension of the space or by changing a constant to a variable. They invented original problems and challenged classmates to solve problems. We also consulted several textbooks although we followed, more or less, the traditional syllabus of the "official" department text. Instead of one text, however, the students consulted at least two books for each topic in the course outline. The hope was that the difference in approach and notation would motivate the students to a deeper understanding of the material. Most students commented that the multiple text approach resulted in a deeper, more creative understanding of the material although it took more effort.

Although some calculus texts have over 500 pages, these books may represent a narrow path of the known. The problems students invented, for which solutions were known or not, were given to the class as challenges which many students accepted. They struggled, guessed, reasoned by analogy and tested their tentative solutions. Many students were highly motivated by these challenges. We were a society participating in doing mathematics. It often happened that a problem that was impossible to solve was the topic of a future chapter, which led us into the new material naturally. The most controversial (at the time) innovation was the cooperative exam where the whole class could discuss the problems and solutions. Students could not catch on to this idea on the first exam, but the second exam was more successful with students learning more by cooperating. My hope was to use cooperative exams to bring intellectual excitement into the mathematics classroom. One student, in a later evaluation, remarked that the cooperative exam was "one of the best of the innovations but is not suitable for performance evaluation."

In those experimental classes at HMC , in addition to cooperative testing, term papers were required. Students were asked to write a term paper on any topic. There were essays on creativity, responsibility of scientists, fractional integration, computer generated poetry and other topics. Class time was spent answering questions, comparing different approaches, and discussing invented problems and non traditional problems. The non traditional material and approach gave us insights about mathematics and learning that were wholly unexpected. Many students responded positively (one a year later!) and some were indifferent. A few were hostile. But class attendance was consistently high. Section A of the class then sent a message to section B, inviting them to meet in a large lecture hall on the weekend to discuss some solutions to problems. One student presented a solution that was rejected. Another student presented a different solution. There was arguing and some shouting. Finally a solution was presented that survived criticism. A consensus was achieved. The students were exhilarated. Most students remarked how much they had learned and how good they felt about themselves and their relationship to the material. In " A New Paradigm for the Mathematics Classroom" (Int. J. Math Educ. Sci Technol 1976 vol7, no2) I raised a number of issues.. Mathematics is exciting. Can it compete with a literature course where students are caught up in the intellectual clash of ideas? Can students recognize mathematics as a creative activity? After the article was published, I learned of others who had tried cooperative testing, some of whom were inspired by my account.

Before I embarked on my Danforth Fellowship, my colleagues and I submitted a proposal to FIPSE in 1976: "New, Interdisciplinary, Holistic Approaches to Teaching and Learning." The beginnings were informal, interdisciplinary discussions that I had organinzed among the faculites of the Claremont Colleges. While I was at MIT, I learned that proposal was denied. One reviewer said the idea was trivial; another said it was impossible! I traveled to Washington, DC, to consult with the FIPSE staff, who encouraged me to reapply the next year. The second application to FIPSE was successful and resulted in a three-year grant. Faculty from six colleges (all of the Claremont Colleges, a nearby state college and the neighboring community colleges) participated. The goal of the project was to make every participant an interdisciplinary scholar/teacher—to introduce scientists to the humanists' viewpoint and knowledge-and visa versa. Each faculty member was encouraged to understand and appreciate the viewpoints and problems of the whole range of knowledge. Prominent scholars from all over the US came to speak to us. We had as many as two speakers with discussions per day, every academic day for three years. A project which had begun as a personal vision of a few became a major part of faculty culture. Approaches that were first viewed with skepticism became accepted and even expected. Ideas which were on the radical fringes of academe about the benefit of integrating the sciences and humanities moved into the main stream of desired educational outcomes. Some faculty members who were timid about trying new modes of teaching or introducing new content, such as values and ethics, were supported and encouraged by the newly created setting. For example a chemist introduced humanistic themes into his classes and began holding discussions in his classes. We ended on a high note, with a three-day conference addressed by some of the nations' leading educators. Among the presenters were Nevitt Sanford of Berkeley's Wright Institute, Benson Snyder of DSRE and Harold Taylor of Sarah Lawrence. A participant commented that she had never expected to see any of these educational leaders in person, but here she was seeing so many altogether!

After the FIPSE project I was elected president of SIGMA Xi, AAUP, and co-president of the faculty senate (not concurrently!) The FIPSE project, 1977-81 was an exciting time for me as well as for many faculty colleagues.

My interest in interdisciplinarity le d me to solicit authors and articles for one of the Jossey-Bass series, " New Directions in Teaching and Learning (0)" titled "Interdisciplinary Teaching," which I edited. The other authors --Geoffrey Vickers, Ralph Ross, Kenneth Boulding, David Layzer, C.West Churchman, Richard M. Jones, Arthur Loeb, Owen Gingerich, Barbara Mowat, CArl Hertel, Miroslav Holub --were well-known people from many disciplines: Business, Law, Education, English Literature, General Systems Theory, Astronomy art and Environmental Design. They included a professor of design science at Harvard whose degree in chemical physics led to a collaboration with R. Buckminster Fuller, a professor of art and environmental design, and a Czech poet who became chief research immunologist for Clinical and Experimental Medicine in Prague. My article in this series discussed my seminar at MIT, and gave the title, for the Jossey Bass Series (#21): "Teaching as Though Students Mattered."

All my experiences contributed to my seeing mathematics as one of the humanities. I wrote to the Exxon Foundation for a grant (of a million dollars) to develop the concept of Humanistic Mathematics; to create an approach to teaching and learning mathematics that would be nonthreatening, but inviting to students, who would participate in a cooperative spirit with each other and with teachers. My dean was very discouraging about the whole idea and the possibility of receiving the grant. Within a week of sending the letter to Exxon, however, I received a phone call from the program director, who wanted to discuss the proposal and negotiate the particulars. After three weeks of conversations on the phone he suggested that he could give me $10,000 within a month to have a "Planning conference" for my colleagues and I to talk about the ideas and program of humanistic mathematics. There was no time for complicated arrangements and advertising. I asked some friends if they would come and recommend other friends for a three-day conference. For me this would be a singualar moment in time. As I recall, the attendants were Stephen Brown of SUNY Buffalo, Marion Walters of Oregon, Phil Davis of Brown, Reuben Hersh of New Mexico, Jeremy Kilpatrick of Georgia, Bob Borrelli of HMC, Paul Yale of Pomona College, Tom Tymozko of Smith College (who said at the 1992 Quebec meeting of ICME, "Alvin White started all this"), Sherman Stein of UC Davis, Ed Dubinsky of Purdue, a psychologist from Boston, two mathematicians from Cal Poly San Luis Obispo, several others, and myself -- thirty in all.

After three exciting days, we decided to start a newsletter; this was the birth of the Humanistic Mathematics Network Journal. Exxon Education Foundation graciously agreed to support this newsletter-journal. The first issue was sent out to thirty people, one year later in 1987. The second issue was sent to sixty people, and then distribution grew rapidly. The first three issues were unedited pages which I xeroxed and stapled together.. The fourth issue was edited and retyped by Susie Hakansson's staff at the Graduate School of Education, UCLA. I arranged for it to be printed with a blue cover with a painting by Blake and a poem fragment by Milton. Subsequent issues have been typed by a student from Harvey Mudd College and printed by a local printer. It now became possible to edit the articles before printing. The Foundation continued to pay for hardware and software, typing, printing, and mailing the issues. The typing (production manager) has been a student. Proof reading and refereeing etc. are done by worldwide colleagues and myself. We have a minimum of promotion, but people hear of the journal by word of mouth or by mention in an article in another publication; they eventually subscribed by contacting me, usually through e-mail. Most articles have been sent to me unsolicited, although I have invited some articles and made requests to reprint some.

The launching of the journal was followed by a number of conferences and workshops on Humanistic Mathematics across the country-a session in Boston and another in New York. At the 1987 winter meeting of the math societies in San Antonio, I organized an afternoon panel on " Mathematics as a Humanistic Discipline," that lasted 4 or 5 hours. We held somewhat shorter sessions as well as poetry readings at subsequent national meetings as well as conferences with the same title at a small Catholic college and a community college in New York.

A strong encounter in my life, although I'm not absolutely sure where it fits in, chronologically, was with the book "Personal Knowledge," by Michael Polanyi. I discovered it from a footnote in a paper by Abe Maslow that a psychologist at the campus counseling center had given me..

Every time I read Polynani's book, I underlined certain lines in colored pencil. When I reread it, I felt compelled again to underline many of the same passages in different colors. That cherished book is now quite personalized, in a colorful way. I cannot summarize the more than 400 pages of that book but I would like to close with a few tidbits and phrases From the preface"I start by rejecting the ideal of scientific detachment. In the exact sciences this false ideal is perhaps harmless, for it is in fact disregarded there by scientists. But ... it exercises a destructive influence in biology, psychology and sociology and falsifies our whole outlook beyond the domain of science.... "Tacit knowing is more fundamental than explicit knowing: we can know more than we can tell and we can tell nothing without relying on our awareness of things we may not be able to tell ... ... Thomas Kuhn in "Structure of Scientific Revolutions" expresses similar ideas. J.Hadamard in "The Psychology of Invention in the Mathematical Field" discusses the role of the unconscious in discovery and problem solving. According to Polanyi---" Such is the personal participation of the knower in all acts of understanding. But this does not make our understanding subjective. Comprehension is neither an arbitrary act nor a passive experience, but a responsible act claiming universal validity. Such knowing is objective in the sense of establishing contact with a hidden reality; a contact that is defined as the condition for anticipating an indeterminate range of yet unknown implications." Polyani mentions tacit skills of which we are not aware - such as swimming or riding a bike. "The complete objectivity as usually attributed to the exact sciences is a delusion and in fact a false ideal. "To learn by example is to submit to authority. You follow your master because you trust his manner of doing things even when you cannot analyze and account in detail for their effectiveness. By watching the master and emulating his efforts in the presence of his example, the apprentice unconsciously picks up the rules of the art, including those which are not explicitly known to the master himself. These hidden rules can be assimilated only by a person who surrenders himself to that extent uncritically to the imitation of another."

When I first started to teach, my ambition was to present a complete and masterful lecture (and I have witnessed several such lectures.) After I read Polanyi and others, my ambition became not to present a perfect complete lecture, but to inspire my students (or rather, conspire with them) to become interested in the subject and learn the subject for its own sake. Perhaps I have made some progress toward that goal and have had some partial success, but I am continuing to learn in my teaching, now, at age 77.

Humanistic Mathematics: Personal Evolution and Excavations

Stephen I. Brown

The first online issue of HMNJ will feature a major article by Stephen Brown. To whet your appetite, we have included his introduction here.

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Professionals from many different disciplines and perspectives who frequently do little more than greet each other politely, have come to appreciate, acknowledge, and even communicate with each other. Those interested in exploring a diversity of fields in relation to mathematics have set up tents around a bon fire that was lit by Alvin White's newsletter, *Humanistic Mathematics Network* of 1986—a newsletter that officially became a journal in 1993. Fields as diverse as cognitive psychology, education, history, literature, linguistics, history, philosophy, and poetry are represented in the journal. This journal has also inspired the humanistic mathematics movement, now represented by a well attended topic group at the annual meeting of the American Mathematical Society and Mathematical Association of America. In addition, it has been acknowledged as an emerging force in a recent international handbook of mathematics education published in The Netherlands (Brown, 1996a). From a personal point of view, the journal has meant a great deal, not only because of the direct impact of its articles, but more importantly because it was a contributing factor in giving me the courage to write about and integrate a variety of fields—a feat that stretched considerably the bounds of my perceived expertise.

With much appreciation for such encouragement, I reflect in this essay on the evolution of my own writing about the concept of humanistic mathematics. I do so by setting my first publication on the topic in *bas relief* against my writing that emerged some thirty years later. I will not in this brief space (shades of Fermat and his marginal notes!) have a chance to paint the variety of self-portraits that emerged over this time span. I will, however, point to a number of contributing factors that influenced the change. I propose this act of introspection as a case study of one person's struggle with new ideas.

My first article, published in 1973, that explicitly highlighted the word "humanistic" was playfully entitled "Mathematics and Humanistic Themes: *Sum* Considerations" (MHT). The evolved book, published in 2001 is less playfully entitled, *Reconstructing School Mathematics: Problems with Problems and the Real World* (RSM). As I look back at MHT, it becomes clear to me that this article planted the seeds for much of my subsequent writing. Perhaps the most dominant theme —maybe an obsession—has been a focus on problems and their educational uses. As will be obvious when I discuss some of the humanistic categories, part of that focus is ameliorative with regard to problems. That is, I point out "near relatives" of such concepts as problem solving, and indicate the educational shortsightedness of excluding them from the educational scene.

Letters and Comments

I am a long-standing and enthusiastic reader of the HMN Journal. After reading journal #25, I wanted to raise two issues—being interested to know what others think of them. I shall be brief—and (playing devil's advocate) blunt.

First, on the article about Maharishi's vedic science. This seemed to me a rather long-winded way (8pp!) of saying that Wigner's famous phrase (about the unreasonable effectiveness of mathematics in science) is mistaken because of an assertion that the two worlds are governed by the same laws—the same melody. This seems an ex-cathedra circular argument asserted without evidence or supporting argument other than an appeal to authority. What has this metaphysical stuff to do with a "humanistic" mathematics?

Secondly, from another long (9pp this time) article on "person-centered" math, "Imagine the trigonometry involved in hurting a football...that's the same geometry and calculus involved in tossing a crumpled sheet of paper...performing operations of geometry, trigonometry and calculus all the time." Hey, but you aren't! You're throwing a football, a ball of paper. Isn't it dishonest and misleading to suggest otherwise to students? Otherwise why bother to study calculus when you are already doing it OK? Isn't mathematics a second-order activity in which you are looking at the throwing in a specific way, one which is quite different than the first-order activity of the footballer? Do we have to sugar the pill in this particular way?

-Dick Tahta

Letters to the editor are always welcome.

Future Issues will be online at http://www2.hmc.edu/hmnj/

Among the future online articles, will be Stephen Brown's article:

Humanistic Mathematics: Personal Evolution and Excavations

Humanistic Mathematics Network Journal E-mail List and Website

The HMNJ has an email list which allows contact among the readers. To subscribe, send email to: listkeeper@hmc.edu with the message: "Subscribe HMNJ-L@hmc.edu" in the body of the email. After you have subscribed, to send mail to the list, write to HMNJ-L@hmc.edu and your email will be forwarded to everyone on the list.

The HMNJ is also working on constructing a website. You can see what progress we are making at http://www2.hmc.edu/hmnj/. While it may be a while until the website is fully functional, we hope you will contribute your suggestions for improvements and ideas for functions to alvin_white@hmc.edu.

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