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INVITATION TO AUTHORS

Essays, book reviews, syllabi, poetry, and letters are welcomed. Your essay should have a title, your name and address, e-mail address, and a brief summary of content. In addition, your telephone number (not for publication) would be helpful.

If possible, avoid footnotes; put references and bibliography at the end of the text, using a consistent style. Please put all figures on separate sheets of paper at the end of the text, with annotations as to where you would like them to fit within the text. These should be original photographs, high quality printouts, or drawn in dark ink. These figures can later be returned to you if you so desire.

Two copies of your submission, double-spaced and preferably laser-printed, should be sent to:

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Humanistic Mathematics Network Journal
Harvey Mudd College
Claremont, CA 91711

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NOTE TO LIBRARIANS

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COVER

Giving Math a Hand: Look closely and you can see that the hand is made of the names of different theories and areas of mathematics! Thanks to the brainstorming of her friends, production manager Fess Nelson was able to design this cover after being inspired by S. Robert Wilson's poetic essay (p. 7).

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Humanistic Mathematics Network Journal #25
August 2001

From Newsletter #1

Dear Colleague,

This newsletter follows a three-day **Conference to Examine Mathematics as a Humanistic Discipline** in Claremont 1986 supported by the Exxon Education Foundation, and a special session at the AMS-MAA meeting in San Antonio January 1987. A common response of the thirty-six mathematicians at the conference was, "I was startled to see so many who shared my feelings."

Two related themes that emerged from the conference were 1) teaching mathematics humanistically, and 2) teaching humanistic mathematics. The first theme sought to place the student more centrally in the position of inquirer than is generally the case, while at the same time acknowledging the emotional climate of the activity of learning mathematics. What students could learn from each other and how they might come to better understand mathematics as a meaningful rather than arbitrary discipline were among the ideas of the first theme.

The second theme focused less upon the nature of the teaching and learning environment and more upon the need to reconstruct the curriculum and the discipline of mathematics itself. The reconstruction would relate mathematical discoveries to personal courage, discovery to verification, mathematics to science, truth to utility, and in general, mathematics to the culture within which it is embedded.

Humanistic dimensions of mathematics discussed at the conference included:

a) An appreciation of the role of intuition, not only in understanding, but in creating concepts that appear in their finished versions to be "merely technical."
b) An appreciation for the human dimensions that motivate discovery: competition, cooperation, the urge for holistic pictures.
c) An understanding of the value judgments implied in the growth of any discipline. Logic alone never completely accounts for what is investigated, how it is investigated, and why it is investigated.
d) A need for new teaching/learning formats that will help discourage our students from a view of knowledge as certain or to-be-received.
e) The opportunity for students to think like mathematicians, including chances to work on tasks of low definition, generating new problems and participating in controversy over mathematical issues.
f) Opportunities for faculty to do research on issues relating to teaching and be respected for that area of research.

This newsletter, also supported by Exxon, is part of an effort to fulfill the hopes of the participants. Others who have heard about the conferences have enthusiastically joined the effort. The newsletter will help create a network of mathematicians and others who are interested in sharing their ideas and experiences related to the conference themes. The network will be a community of support extending over many campuses that will end the isolation that individuals may feel. There are lots of good ideas, lots of experimentation, and lots of frustration because of isolation and lack of support. In addition to informally sharing bibliographic references, syllabi, accounts of successes and failures. . . the network might formally support writing, team-teaching, exchanges, conferences. . . .

Alvin White
August 3, 1987

From the Editor

Four years ago Linley E. Hall came to Harvey Mudd College as a freshman. A short time later Linley became the Production Manager of HMNJ. It is clear to all that the journal flourished under her wise and steady hand. Her skills and high standards have been a boon to the readers and to me. I am grateful that she chose to give her energy and skills to the journal while studying and doing research in chemistry. Linley will go to graduate school at the University of California, Santa Cruz, where her concentration will be Science Writing. As you can see from the inside front cover, the production staff has added four students. I am grateful to and encouraged by the talented students who joined the staff and by their conscientious work in addition to their studies.

Of course, everyone is grateful to the ExxonMobil Foundation for their continuing support since 1987.

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A continuing source of anguish and frustration is the prominence of high stakes testing in the UnitedStates; the NEA Resolution reprinted below expresses that frustration.

NEA 2000-2001 Resolutions B-56. Standardized Testing of Students (http://www.nea.org/resolutions/00/00b-56.html):

The National Education Association believes that standardized tests should only be used to improve the quality of education and instruction for students. Standardized tests are most useful when selected by educational professionals closest to the classroom and integrated with assessment information specific to local programs. Affiliates should advocate the design and use of a variety of developmentally appropriate assessment techniques that allow necessary accommodations, modifications, and exemptions and are bias-free, reliable, and valid. When a test is mandated at the state or the national level, it should only be used to evaluate programs toward meeting state or national standards and/or goals.

The Association opposes the use of standardized tests when—

a. Used as the criterion for the reduction or withholding of any educational funding
b. Results are used to compare students, teachers, programs, schools, communities, and states
c. Used as a single criterion for high-stakes decision making
d. They do not match the developmental levels or language proficiency of the student
e. Student scores are used to evaluate teachers or to determine compensation or employment status
f. Programs are specifically designed to teach to the test
g. Testing programs or tests limit or supplant instructional time.

The administration of a standardized test includes the responsibility to educate the stakeholders in the purpose of the test, the meaning of test results, and the accurate interpretation of conclusions. (1978,2000)

Nuclear Magnetic Resonance and Humanistic Mathematics: A Farewell

Linley Erin Hall
Production Manager, Humanistic Mathematics Network Journal

One important tool in various chemical disciplines is nuclear magnetic resonance spectroscopy, commonly referred to as NMR. Because protons are positively charged, their spinning motion creates a magnetic field. When exposed to an external magnetic field, the proton can either align with (α)or against (β) the field. Exposing a proton to the right strength of magnetic field will cause it to flip from one orientation to the other, a condition called resonance. By exposing a molecule to a range of magnetic fields, the strength of field needed to achieve resonance of each NMR-active nucleus in the molecule can be seen as a peak on a spectrum (Figure 1). Nuclei with even numbers of both protons and neutrons are NMR-inactive; they do not appear on spectra. Also, chemically equivalent nuclei, such as the four hydrogens in methane, CH_4, will resonate at the same field strength and thus appear as one peak. Chemists commonly look at the resonance of hydrogen, ^1H. Figure 1 shows the NMR spectrum of 1,1-dichloroethane.

I was introduced to NMR in my sophomore organic chemistry course. Professor Phil Myhre explained the basics, in somewhat more detail than I have presented them here, then went on to talk about how you can use an NMR spectrum to figure out what an unknown molecule looks like.

One important aspect of NMR spectrum interpretation is coupling. In an organic (carbon-based) molecule, hydrogen atoms that are one carbon-carbon bond away can "see" one another. In the spectrum, this corresponds to a single peak for one hydrogen (or several chemically equivalent hydrogens) being split into many peaks. How many? The number of neighbors plus one, for the hydrogen could see its neighbors in, for the case of three, $\alpha\alpha\alpha$, $\alpha\alpha\beta$, $\alpha\beta\beta$, $\beta\beta\beta$. What are the intensities of these peaks? There's only one way to get each $\alpha\alpha\alpha$ and $\beta\beta\beta$, but three ways to get each $\alpha\alpha\beta$ ($\beta\alpha\alpha$, $\alpha\beta\alpha$) and $\alpha\beta\beta$ ($\beta\beta\alpha$, $\beta\alpha\beta$). Thus, in this case you get a quartet with peak intensities of 1:3:3:1. This can be seen in Figure 1; the hydrogen attached to the carbon with the two Cl atoms sees three ^1H neighbors and so appears as a 1:3:3:1 quartet around 330 Hz. The three hydrogens are chemically equivalent and see one ^1H neighbor, so they appear as a 1:1 doublet around 120 Hz.

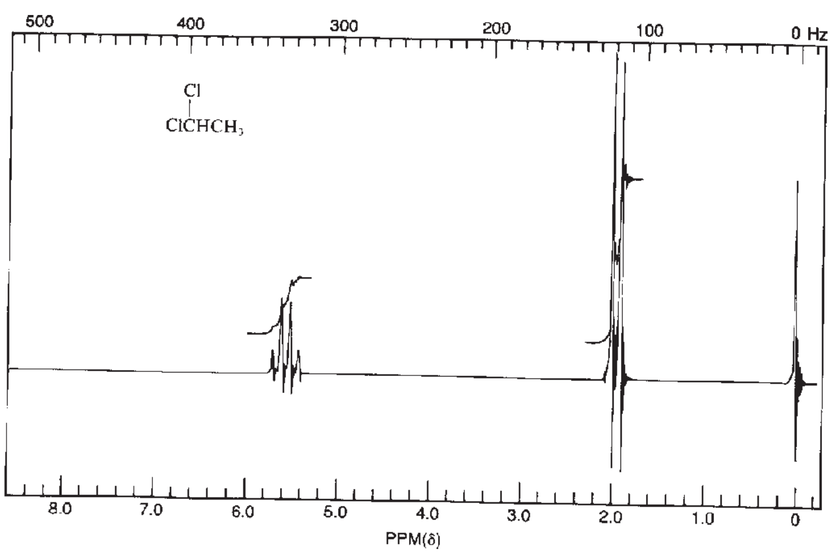

Figure 1
^1H NMR spectrum of 1,1-dichloroethane

Think about other splitting possiblities. To get a triplet a nucleus would see two neighbors. These neighbors could be in four different combinations: $\alpha\alpha$, $\alpha\beta$, $\beta\alpha$, and $\beta\beta$. Since $\alpha\beta$ and $\beta\alpha$ are the same, this works out to splitting with peak intensities of 1:2:1.

Seeing a pattern? Put everything together and you find Pascal's Triangle (Figure 2). The elegant pattern that gave me the binomial coefficients in algebra class also tells me ideal peak intensities in chemical spectra.

And so it was, sitting in chemistry class, looking at Pascal's Triangle on the overhead, that I really understood what humanistic mathematics is. It is finding the mathematics that is everywhere.

I have been the production manager of the HMNJ since March 1998 (Issue #17). I majored in chemistry and am now off to University of California, Santa Cruz to pursue a graduate degree in science writing. I leave the HMNJ in wonderful hands. Expect to see issues in your mailbox more frequently; Stephanie, Fess, Mary and Kathe are a great team, and I wish them well.

0	singlet (1)	1
1	doublet (2)	1 : 1
2	triplet (3)	1 : 2 : 1
3	quartet (4)	1 : 3 : 3 : 1
4	quintet (5)	1 : 4 : 6 : 4 : 1
5	sextet (6)	1 : 5 : 10 : 10 : 5 : 1
6	septet (7)	1 : 6 : 15 : 20 : 15 : 6 : 1

Figure 2
Pascal's Triangle.

"Stairway to Seven"

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May be sung to the tune of the 7 verse song "Stairway to Heaven" by Jimmy Page and Robert Plant.

There's a student who's sure if she rolls two fair dice,
The most likely sum is seven.
On the seventh day she knows many stores will be closed
'Cause lots of folks call it the day of rest.
Ooh, ooh, she's inquiring: "Where is there a seven?"

Well there's a seven on the wall, but she wants to be sure
'Cause you know uncrossed sevens can look like ones.
From seven notes in a scale, there's a songbird who sings—
It's the first sour note in the harmonic series.
Ooh, it's quite a number. Ooh, seven wonders.

There's a feeling I get from the seven continents
And shuffles needed to mix the cards:
Snow White's dwarves all could be a water polo team—
It's the limit of short-term mem'ry.
Ooh, it's quite a number. Ooh, telephone number.

It's the steps in ballet's art, it's the Big Dipper stars,
And it's how many times you can fold paper.
First polygon to elude the classical tools
And it's how many patterns for borders.

If a track meet takes a long time, don't be alarmed now—
It's just what's called a heptathlon.
Can you remember when the 7th month was September?
Then Caesar added August and July! And it makes me wonder...

Seven verses make this song maybe too long,
The piper fights for airplay!
First whole number whose reciprocal does use
Its maximum block of digits.

As we wind on down the road, with 7 chakras I am told,
And 7 colors of the rainbow make white light when they all show.
If you listen to this rhyme of this odd Mersenne prime,
May it make you want to find each number's special shine...
And she's buying a stairway to seventh heaven!

On the Preparation of High School Mathematics Teachers

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SUMMARY
In this paper I discuss some results got in 1997/98 with Brazilian mathematics school teachers. The research was done to investigate their mathematical language as related to the concept of function. A dichotomy was detected between "formal" and "practical" language they used to express their own conceptions of function, as well as to teach their students this subject. Also, I found teachers' conceptual images "shrinking" as soon as they were far from their colleges or universities programs.

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In the 17th issue of this Journal, Wenstrom, Martin & King (1998) wrote about the necessity of re-examining programs for college and university mathematics departments as concerns the preparation of mathematics school teachers.

Those authors emphasize that "high school mathematics teachers are the products of these programs. They not only teach what they learned to their students but also how they learned it" (Wenstrom, Martin & King (1998), p. 12).

This paper is intended to resume the subject above, reporting an investigation done in Brazil about the mathematical language used by high school teachers. Despite the differences one can observe in the educational policies in many countries, I have reasons to believe that the problem of preparation of mathematics teachers is essentially the same everywhere, and it is much more complex than one can suppose.

The results of my research show that these teachers (at least in Brazil) seem to use "models" of mathematical language in their classes unlike those they learned at their college or university. Instead, these "models" are much closer to those ones they had in their own experiences as students in high school.

Thus a new question takes place: what kind of influence do college or university programs have on the preparation of high school teachers today? If these programs seem not to interfere significantly in their mathematical language, to teach or to express their own conceptions about mathematical notions, what are their effective contributions to the professional development of these teachers?

As quoted in Wenstrom, Martin & King (1998, p. 12), "unfortunately, few university mathematics departments maintain meaningful links with mathematics in school or with the mathematical preparation of school teachers... Only when college faculty begin to recognize by deed as well as word that preparing school teachers is of vital national importance can we expect to see significant improvement in the continuity of learning between school and college" (Moving Beyond Myths, 1991, p. 28).

THE INVESTIGATION
In 1997/98 I conducted a qualitative study (Ande, 1995; Rockwell, 1985) of how secondary [or high] school mathematics teachers used mathematical language to treat ideas about the concept of function (Zuffi, 1999). My purpose was to investigate the ways these teachers—being mediators and 'catalysts' of the developing processes of their students (Vygotsky, 1962, 1989)—deal with their own conceptions about functions as well as how they explore them in their classrooms. Also, I was interested in knowing how conscious these teachers were about their use of mathematical language.

Seven high school math teachers were interviewed and answered a collection of twenty written questions related to the subject "function." These questions were proposed to give them the opportunity to express their own conceptions about that notion through mathematical language, and were freely answered by the teachers, in such a way that they could write *everything* they knew about functions, beyond the facts they teach in high school.

Even thus, it was very surprising to see, after the analyses of data, that the investigated teachers expressed themselves through mathematical language essentially in the same way they teach, and not in the way they had learned in their college or university courses. Even after telling them to go beyond the ideas they teach, they kept pointing to exactly the same topics and patterns they teach. None of those individuals went far from the language they used in the classrooms with their students.

However, the teachers' formal mathematical expressions tended to approach present day definitions for function (as the ones by Bourbaki or Dirichlet), although they had a very small formal repertoire to communicate it safely and correctly. Many mistakes were made, especially when they insisted on using symbolic notation.

I observe here that the preparation they received in their undergraduate mathematics courses seems to be insufficient to develop self-confidence and awareness of the use of mathematical language, mainly with respect to its formal aspects. On the other hand, my results reveal that there seems to exist a real dichotomy between the teachers' mathematical language dealing with theoretical frameworks and the expression of "practical" questions and situations.

For instance, while Dirichlet or Bourbaki are invoked in formal definitions, in dealing with examples and problem solving these teachers are restricted to classical conceptions for functions, such as Euler's definition. That is, they pointed out only "patterns" given by analytic formulas in very simple algebraic expressions, similar to the ones they often teach in their classrooms (e.g. $f(x)=3x+5$, or $f(x)=5x^2-7x+3$). In the "practical" situations, for the investigated teachers, the ordinary examples they present to their students seem to be enough to "encapsulate" (Dubinsky & Harel, 1992) all the meanings involved in the concept of function. This may be contributing to building narrower conceptual images (Vinner, 1991, 1992) in the high school teachers' expressions for the idea of function, and I don't believe they are really conscious of this fact.

Their conceptual images tend to be limited to the ones they use to teach in high school, and the images seem to "shrink" as these teachers become more and more distant from their undergraduate courses.

In a second part of my research, observing three high school teachers in their classrooms, I got similar results to those obtained with the questionaire and interviews. In their classrooms these teachers use formal mathematical language in such a way that definitions seem to be of much less importance than the "practice" for functions. What really should "count" for the students is the way the teacher deals with algorithms, examples, and techniques for solving mathematics problems. Definitions are in a second plane, which it is not necessary for students to reach.

The mathematical language pointed out in the observed classrooms was static, with purposes in itself, and syntactic aspects were much more emphasized than the meanings of the language. The concepts related to the notions of functions, as I saw in the high school classrooms, do not emerge from a context which has to do with the students' lives. Nor have they to do with the construction of a powerful way of communication, such as the ideal of mathematics. On the contrary, these notions are associated with abstract symbols and algorithms, and these symbols, in turn, become objects for themselves, in a fragmented and truncated language.

All this can be supported by the following evidence:

i) The observed teachers used the term "dependency" as a synonym of "function," as if that word had clearly encapsulated all the mathematical subtleties the ultimate definition for function presents;

ii) The relation in a functional correspondence was always given by an explicit and very simple "rule" or "law" (algebraic expression);

iii) The symbolic notations "x," "y," "a," "b," "c" are always in straight association with the ideas of "independent variable," "dependent variable," and "constants," respectively. This leads the students (and very often, even teachers themselves) to think about these notations always in a limited meaning. (When the roles of "x" and "y" were interchanged, these teachers had difficulty identifying independent variables and constants);

iv) During observed classes, two of the teachers refered to “x” sometimes as “the variable,” other times as “the domain,” and finally, as one specific element of the domain which should be determined by the students. Since these teachers did not make clear the contexts in which they used each of the terms, I concluded that their own comprehension about these notions were limited. Even more, the sets of domain and image, in the teachers’ expressions, seemed to be determined only by the sequence in which they appeared—the first one is the domain, and the second one is the set where the image lies. Therefore, these teachers seemed not to realize that both sets don’t have symmetric roles (Sierpinska, 1992);

v) Although the observed teachers worked with real functions of real variables, the variation of elements they proposed for the domain—to build graphs, mainly—had “models” always in the set of integer numbers. They rarely “picked up” rational numbers, and never selected the irrationals to plot the graphs;

vi) The graphic forms were previously presented to students by the teachers, so that these same students only had to locate the graphs. To do that, three or five coordinates seemed to be enough. Hence, continuity was not discussed, and there were many difficulties (with high school students and teachers) dealing with discontinuous graphs of functions.

vii) The interviewed teachers’ conceptual images (Vinner, 1991, 1992) for functions are restricted to the facts they teach in high school, even when I asked for broader answers.

SOME REFLECTION

Of course most of the results of my research are not really new. The important fact revealed was that many of the problems we see with high school students are still the difficulties of their teachers. The distance between high school mathematics teachers’ conceptions about functions, and the knowledge they received in college seems to be wider and wider as they become more and more involved with their classrooms, and as they move further away from their undergraduate courses.

Here are some reasons I identify for this fact:

1. High school teachers depend almost exclusively on mathematics textbooks to prepare their classes and compose their mathematical language.

As Dancis (1999) reported:

> “It is standard for math textbooks and K-8th grade teachers to provide students with cookbook type directions of what to do in math. It is rare for students to be assigned problems that they have not been programmed to do. It is rare for textbooks and K-8th grade teachers to provide the students with understanding-based explanations which tell the whys and the wherefores of mathematics” (Dancis, 1999, p. 3).

I am sure the same is valid for high school textbooks and teachers in Brazil. And, since these textbooks very often propose a limited and static mathematical language, so is the teachers’ language. “Providing students with understanding-based explanations of mathematics is not a common teaching technique” (Dancis, 1999, p. 3). Therefore, syntactic aspects are emphasized, while the construction of meanings of mathematical language is still underestimated.

2. A social fact is involved in the question. There exists a school mathematical culture, at least in Brazil, where teachers must cover a great deal of content, even when the students are not able to reach comprehension for everything. In this case, the mathematical language proposed by these teachers becomes as reduced as possible, to promote very rapid memorization of technical procedures by students.

As Dancis (1999) asserted for middle school, and as we can also read for high school:

> “The natural result [of this situation] is that while the students may develop some proficiency in math skills, they do not gain any understanding of the mathematics. This results in students collecting all sort of misconceptions about mathematics and making a wide range of mistakes while doing calculations. This, in turn, results in less success in high school math classes. Remedying these misconceptions is difficult” (Dancis, 1999, p. 3).

> "The overemphasis on testing, skill development and fact content, etc. [in schools] seems to have inhibited [student] interest in learning, motivation, ability to work with and enjoy ideas, use creativity, and attain satisfaction from an educational experience" (Dancis, 1999, p. 4).

3. There is a great gap between pedagogical disciplines and those related to advanced mathematical content in college and university programs.

These programs generally have the conception that in the first terms the student must learn a lot of advanced math, to be able to apply this content to pedagogical situations. However, most of the program in those courses has nothing to do with the real situations of teaching. Specific math disciplines are isolated from high school programs, and the pedagogical ones are frequently too general to be connected to secondary school or to advanced math.

In the case of functions, many advanced disciplines, such as Algebra, Analysis, Topology, etc. deal with them. And there seems to exist a strong belief that, even isolated, those disciplines are enough to produce a full conceptual image of function for the future high school teachers. But our research revealed that they are not sufficient to produce such a result. Even the teachers who had a strong experience in those disciplines lost self-confidence in using symbolic notations for functions and had a limited conceptual image for them.

All this means that high school math teachers are not being properly prepared at college. And we can see that, even having been educated in the best of institutions, high school math teachers retain a strong influence from social scholar facts that are not foreseen by those undergraduate programs.

I believe that some questions, such as the choice and use of textbooks, the interface between advanced math disciplines and pedagogical ones, and continuity of studies for experienced high school teachers, raising them with future teachers, should concern everyone who cares about the preparation of school teachers, and those who are responsible for college and university programs.

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Fivefolded Asymmetrical Hand: A Poetic Essay

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An outline of one possible solution is given through an abstract characterization of the discipline as a whole.

At the January 1992 joint meeting in Baltimore, I read two rather obscure and cryptic poems, $\sqrt{2}$ and *GOD-137*. I have and continue to write poetry, though rather sporadically. The origin of these two poems was constructive in nature, each beginning with a simple concept and then being somewhat lifted into poetical form. The first of these, $\sqrt{2}$, was constructed using the symmetry of the 45° right triangle, two words of ten letters and one of fourteen. *GOD-137* began with the idea that when considering infinite quantities to represent sums and products we use the symbols Σ and Π. In finite elementary arithmetic we know product as "times" and sum becomes associated with "some" through a language deformation. Then, through a compound operation, the symbol $\Sigma\,\Pi$ is associated with the word "sometimes." For this construction the title is what needs explanation. After constructing the first "some" as one, continuing to construct the square and cube, the entire entity, the poem, contains this "three." The context of the "three" as referred to here is line, plane and space and it is through Cantor that these do become a one, in the sense of continuity. The name *GOD-137* becomes a representation where "1" is the poem, "3" this fundamental set of three, and "7" the number of lines in which the poem presents itself. The aspect "GOD" comes from the superstitious idea in Christian mythology that one is three. What makes the title more auspicious is that when I wrote it I had no knowledge of the approximation to 137 in the fine structure constant, which as I interpret, is a numerological equivalent to the word "god" from a physicist's point of view.

As my research in pure mathematics has been progressing I have often touched philosophical ground. After consistent achievement, during a restructuring and reckoning of research projects new concepts and ideas began to overflow. Through numerous discussions with a collaborator, a conception of ideal types emerged.

Philosophically, the problem being addressed is one of balance. The notion of transcendence takes on a religious flavor being framed in mythology for expression. However, ethically the call is for responsibility. Collectively this composes a current, beginning with balance, but incorporating the transcendental religious nature, in a secular ethical morality, most accurately described as spirituality.

Beginning with the most troublesome notion of religious differences, the opinion expressed is of religious belief not properly contained in the known domain of religious expressions, yet it is not devoid of any expression. The approximation this produces is simultaneously of a theist and an atheist. Further, recognizing this expression as not properly contained in its own transcendence makes this and anything said about it faulty.

Philosophically, a transcendental aspect is at work. The three traditional philosophies, Platonism, intuitionism, and formalism, at some level form a whole, an interdependent unity. In this light we may consider the abstract "some one is three." The question of balance enters at this point. The opinion is, that in light of the notion of transcendence in religious expression, this same transcendence property may be applied to philosophy of mathematics.

It is now my moral obligation as a mathematician, and my ethical responsibility as a member of the American Mathematical Society to make available results when consequences are apparent. It is then appropriate that I share these philosophical results of my pure research, in view of the fact that the mathematical community as a whole seems to be experiencing some internal difficulties.

The issues of recognition and understanding of mathematics by society as reflected in problems of employment, education, technology, and the importance of pure research, came to surface after a brief but inspirational conversation with a mathematics librarian. Touching on ideas of transcendence and foundations, I pointed out our current difficulties reasoning about "god" or the continuum hypothesis, two concepts I view with an amount of logical similarity. Through the subsequent evening the elaboration of the ideas based themselves in the two original "mathematical" poems of 1992 and took form in a not perfect but more aesthetically "natural" poem *Fivefolded Asymmetrical Hand*:

Why is the hand fivefolded asymmetrical
Apart from functionality?

Could our being created in our Father's
image
And our intuition have a connection?

Why of course

For before mathematicians formalize
We intuit
Intuition gives us the base
And the sight

All of mathematics, as the base of our
scientific study
Formed in initial intuition, something all men
possess
Defines a consequence that all men are at least
Minimally unconscious mathematicians

If being in the Image
Then He is Obvious, Eternal, and far from
minimal
That Master then is Self-Dual, Platonic in
Nature, by eternity

The blessed line
In Thales, Pythagoras, Archimedes...
Through Descartes, Newton, Euler, Gauss...
...Lobachevskii, Bolyai, Abel, Galois...
...Cantor, Brouwer, Hilbert, Ramanujan,
Gödel...

Defines a sequential web
Of institutions, legacies, branches, and
fields...the Mathematical Realm
With equations worth 10,000 pictures
The pictures worth 1000 words
The words of the language
The symbols of the words
Compactify
To one symbol
The Word

He spoke it
We hear it

But what do we do with it?

Why is the hand fivefolded yet asymmetrical
apart from functionality?

Knowing all of our philosophies
Shows us a clue as to why we are out of
balance

The chairs are extremely finite
The potential is rapidly growing
We are recognized as unimportant
But we know that is extremely inaccurate
That is our problem, our accuracy
But that is our greatest treasure, rigor
We are born, all of us with this paradox

How is it a problem?
Accuracy and details are the driving force

But for few, the princess and future kings
To tame this chaotic land
We must understand
Each of our roles

Each, but we are all created equal?

Not exactly so
In this group I see four

Few on top
Several forced to relate and interact
Many forced out, with nowhere to go
And a minimal, on the rise, with new light in
their eyes

That is to say the first specialize and innovate
The second specialize and integrate
The third specialize and disintegrate
The fourth appreciate and generalize

As it stands this core, the corps of modern life
Upon which the whole of Gaia depends
Which has brought so much good
So much prosperity
Is collapsing deep below the ground

Grasping for hope
We are sliding down a slope
We are caving in
And the Earth Herself is following

But on the surface it appears we have never
had a better time
Do not be fooled
Observation, Liberation, and Acceptance

Our number Three is hurting the most
If they will change we can rebuild, and
restructure
Reconstruct, and redefine

But how can we help them change?
We must change
Collectively, we must all change

If this is true
What must we do?

Rather than disintegrating, the Three must
reintegrate
Looking back we have seen this works
However, we would still be out of balance,
why?
Because they specialize
If they generalize, the more they will see
The easier it will be to reintegrate, rejuvenate
But that is not enough

Yet when they try it will happen that their
appreciation changes
A new angle of appreciation
Sparks the fire in the rejuvenation

This fire spreads like wildfire
This forest is parched
But this flame will not kill it
Beautiful parks need it from time to time
Lightning strikes, it is very natural
Especially in this, the most beautiful of parks
known to man

But why is the hand fivefolded yet
asymmetrical
apart from functionality?

Because the Four are nothing without one
more
The Custodian of the Knowledge
The Holder of the Flame
The Keeper of the Faith

It is he, the vital thumb of this hand
That is responsible for the reintegration
The rejuvenation
The rebuilding
The restructuring
The reconstructing
And the redefining

With an attitude of kindness, courteousness,
and helpfulness
He is the Pilot on this sea
He has the map
He has the control
He believes
He knows
Ultimately, he commands
But never demands

But what happens when the map is not
complete?
Where does he go on a sea of uncharted
waters?
The current
But when storms begin to swell, we are often
blown off course
But then again in uncharted territory, there is
No course

So there must exist yet another, a Sixth
Unseen servant to the entire hand
He can help by taking note
Together they both, the Fifth and the Sixth
Form an inseparable pair
So when the Pilot is at the helm
This Steward can draw the map
So that when
The storm
Has all but faded

The Pilot sails our ship

In peaceful waters
To discover the undiscoverable

But it takes all of us
Though we all are just one
A one that is Five
A complete unifying Whole
Our Sixth, hidden and unseen
Virtually forgotten, is absolutely crucial for our
construction

For the undiscovered to be discovered
We as the Collective Institution
Are crying for hope

The extension of hope is faith
Since the Pilot and the Steward have faith
And together they can complete the
incompletable
Delivering the message, they see the need for
change
Therefore we know that the Collective
Institution needs faith
This we already have

Building faith, the experience of the scientific
enterprise
Only comes after hard won discovery
So to build the faith stronger
Our need is justified in wanting
Rejuvenation
Rebuilding
Reconstructing
And redefining
To achieve this, our precursor is reintegration

If we so choose
We can and will discover the undiscoverable
And complete the incompletable
All, only if we decide

Knowing all philosophies
We have therefore
Stumbled upon a new ideal type
A type classification
A solution to our problems
A rectification of our differences
A healthier Institution

That type is the radically asymmetrical
pentagon
The radical, so discovered
Was an unfortunate occurrence for one
But a tree of infinite fruit for the multitude

He, floating on this ship
Expelled from school
And further sentenced to death at sea
Had a vision

He saw $\sqrt{2}$

I have claimed this poetic interpretation
Continuous
Associated
Transformation
+.014213562...

His school had as badge
The symbol of Health
To restore the Health that radical must be
reintegrated
Into that badge

Yes that is a divine plan
A radically asymmetrical pentagon

Now the members of our school must
recognize
As we do
The types of apprentices, the students that
have enrolled
Being good teachers we recognize who is who
And steer them if we can
But only upon the acceptance of all types
Can they, those drowning at sea be helped

Therefore we define our type class:

Custodian of the Knowledge
The Laboring Weavers
The Laboring Administrators
The Inspirational Movers
The Inspirational Laborers

As respectively, the Fifth, Third, Second, First,
and Fourth
Of the Four,

Individually, each is dependent on the other

three
And stylistically they split into pairs
All owing the deepest of gratitude to the
Custodian

For this five folded asymmetrical hand
To those that know it as an image of the
Father
Has actually something very unique
A metasymmetry

For those with this knowledge
They have no doubt
There exists God
In fact there exists,

God-137:
sometimes one is some
unless some is none
for one to be none
just can't be done
yet sometimes some is square
and sometimes some is cube
and someone is three

The paradox we are amidst is most easily expressed by the acknowledgement that the mathematical discipline is making great internal strides in development and it is facing serious difficulty structurally as an entity. Thus, in the model of real world interpretation with respect to society at large, these two aspects are precisely opposite. So we understand this as $M \equiv m \wedge s$ where $s \equiv -m$. Hence, this is a fundamental contradiction. Metaphorically we may think of us as just stretching a muscle.

Our first reconciliation is to view the three philosophies, Platonism, intuitionism, and formalism, as defining a metaphilosphy, under balance. As mathematicians at this point in history, we have accumulated our Body of Mathematics. We see the beauty and necessity, but the unconscious mathematicians, be it student or layman, may not see the beauty we do. Yet they may be literary scholars, physicians, or musicians having some sense of a Beauty. Is this Beauty the same? Is it unique in origin but manifest in expression? For mathematicians and physicists alike, it does exist, platonic and real. This is the core, a self-dual Nature, with expression in number, in mathematics.

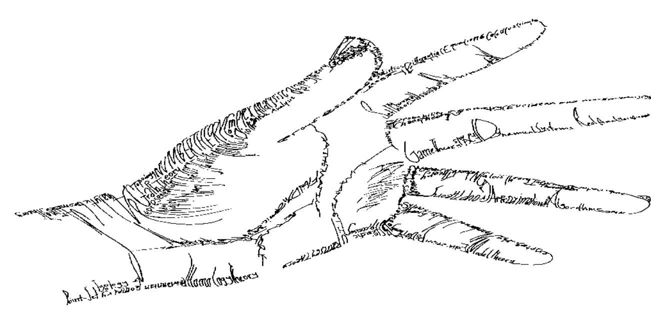
Our language is then nothing more than a sentence, a word, a symbol, finite in representation. We have used it to talk of the continuous and the discrete. Our physicists with the gift of sight have been leading in one direction and our mathematicians with the gift of sight in another; they are our dedicated professors. Our community interacts with society at large by administering, by repetition of ideas, application. The brilliant students following sound advice develop and drive home existing results, speciality is the notion. Our results become documented in literature that quickly becomes obscure. The traditional types, pure and applied, are failing to describe our deeper structure, and the students of today are faced with an innate binary option of intra- or extra-applied mathematics.

The structure that we see developing is simply one of pure academia, industrial application, or unemployment, the equivalent of nonmathematics. This is not a new observation. We have been asking for a solution.

The conclusion that we may immediately draw is that if we have more of our specialized mathematicians that are nontraditional, by generalizing and collectively working, an additional type is defined. From a practical standpoint, this type needs its own internal structure. Clearly, its definition is one of augmentation.

It is this rejuvenating type of mathematician that defines itself as an integral part of a whole, a whole that is under construction. Collectively, with an attitude of redefinition but a negation of that stance, a support for our contradictory base is obtained. We proceed in this direction with the aim of communication.

Because of the inherent contradiction that is at the center of the communication we are consequently seeking to define new ways to effect the communication, that is we are seeking and defining new mathematics, and this is the conclusion so sought.



The Natural Role of Mathematics in the Sciences: How Maharishi's Vedic Science Answers the Question of the Unreasonable Effectiveness of Mathematics in the Sciences

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ABSTRACT
Mathematicians and scientists have for a long time tried to understand why mathematics, a subjective creation of the human intellect, is so effective in the sciences, which study the objective, physical world. Satisfactory reasons have not been found because there has not been a comprehensive understanding of the relationship between the subjective and the objective aspects of life. In this paper we will see that Maharishi's Vedic Science, by explaining the link between the subjective realm where mathematics is located and the objective world that science examines, can resolve this problem in a natural way.

INTRODUCTION
Mathematics is fundamental to all areas of science and technology. The language of mathematics has been used since antiquity to express our knowledge of the physical world, to derive new knowledge from old, and to predict the behavior of physical systems. For example, Newton's Second Law of Motion says that a force exerted on an object is the product of its mass times the resulting acceleration, $F = ma$. Newton used this law, together with his newly developed calculus and the law of gravitation, to derive the elliptical shape of the planetary orbits. Today, mathematical analysis similar to Newton's has placed a man on the moon.

With such dramatic successes, it is not surprising that many people, particularly those who have been at the forefront of developing new applications of mathematics, have wondered why mathematics has proven to be so practical and why the laws of nature are so effectively expressed by mathematical formulas. Mathematics is theoretical and completely abstract, created in moments of inspiration and afterward verified by the intellect. Science, on the other hand, seeks to accurately and objectively describe and predict how the physical world around us behaves. Nevertheless, these two approaches to knowledge have been intimately linked since we began observing and thinking about the world. The basis for understanding the role of mathematics in science must depend on an understanding of how the subjective world of the mind and intellect, the source of mathematics, is connected to the world of matter, forces, and energy studied by science. This connection can be understood through Maharishi's Vedic Science, which gives a comprehensive explanation of the nature of consciousness and its manifestations in the physical world and how the subjective world of consciousness and the mind is connected to the objective physical world around us. Because Maharishi's Vedic Science is so comprehensive, an analysis of the nature of mathematics according to its principles can provide the link between the subjective and the objective aspects of knowledge necessary to properly explain the role mathematics plays in the sciences.

According to Maharishi's Vedic Science, the mind and the physical world are not two separate entities, but two different aspects of one reality. The mind is subtler, more abstract, and more intimate than the physical world, but both exist simultaneously and inseparably. As we will discuss in later sections, Maharishi sees both the mind and the physical world as having their source in the self-interacting dynamics of pure consciousness, which he identifies as the total potential of natural law. Both mathematics and science are studying those aspects of natural law which are quantifiable and exact, although using different methodologies. Thus, the effectiveness of mathematics in the sciences is no surprise but is, in fact, natural and expected.

Moreover, this understanding of Maharishi's Vedic Science shows us that to make mathematics even more powerful, effective, and complete, mathematicians must go even deeper into their subjective nature and connect themselves to their source in consciousness.

The same can be said for scientists, who can make science more productive by linking the objective, natural world that they study to the same source in consciousness.

In this paper, we will first look at mathematics and the question of its effectiveness in the sciences as it has been posed by the physicist Eugene Wigner and the mathematician Richard Hamming. This will be followed by a discussion of points from Maharishi's Vedic Science relevant to this question, a resolution of the question based on these ideas, and a look at the implications of this resolution.

THE QUESTION: THE ROLE OF MATHEMATICS IN THE SCIENCES

Throughout time, mathematics has always been associated with its applications, and from these applications mathematicians have derived new impetus and new directions. For example, the Sulba Sutras, one of the earliest records of mathematics from the Vedic civilization, includes geometric constructions that were used to describe the procedure for the construction of ceremonial platforms (see Henderson, to appear, and Price, to appear). The Rhind Papyrus of the Egyptians gives computational techniques alongside sample problems for applying the techniques to everyday situations such as computing the size of a barn used to store grain. Babylonian clay tablets give mathematical tables for astronomical predictions as well as for business transactions (van der Waerden, 1971).

With the Greeks, however, the discipline of pure mathematics was separated from its applications. As seen in Euclid's Elements, mathematicians had become concerned not with applied problems, but rather with the logical foundations (or postulates) of geometry and the rigorous, systematic derivation of new results from the postulates and previously established results. Mathematical proof became the central feature of the research, communication, and exposition of mathematics.

As mathematics progressed from the classical study of geometry and calculus to the more abstract areas of group theory, non-Euclidean geometry, and topology, its ancient connection to applications weakened still further. In the nineteenth and twentieth centuries, mathematics became replete with concepts that, on the surface, appear to be unrelated to science and the physical world. For example, in certain abstract algebraic systems, the equation $2 + 4 = 1$ can be correct. In hyperbolic geometry, one can draw many different lines through a point parallel to another line, something strictly forbidden in Euclidean geometry. Topologists and analysts regularly study infinite-dimensional spaces, even though the space around us is only three-dimensional. As mathematicians pursued these and other more abstract ideas for their own intrinsic interest and without regard for possible applications, a large body of seemingly "useless" mathematics was developed. This mathematics nevertheless proved to be beautiful and profound. It provided new insights into applied mathematics and became the core of mathematical research. Some purely theoretical mathematicians, notably G. H. Hardy, even expressed disdain for concerns with applications and were proud that their work could have no applications. For Hardy (1976), the value of mathematics is purely subjective, purely in the realm of ideas:

> A mathematician, like a painter or a poet, is a maker of patterns. If his patterns are more permanent than theirs, it is because they are made with ideas...A mathematician...has no material to work with but ideas, and so his patterns are likely to last longer, since ideas wear less with time than words...The mathematician's patterns, like the painter's or the poet's, must be beautiful; the ideas, like the colours or the words, must fit together in a harmonious way. Beauty is the first test: there is no permanent place in the world for ugly mathematics (pp. 84-85).

Hardy sees mathematics as essentially disconnected from the world of applications. In discussing the mathematical significance of the proofs of the infinitude of the number of primes and the irrationality of $\sqrt{2}$, Hardy (1976) says,

> There is no doubt at all, then, of the 'seriousness' of either theorem. It is therefore the better worth remarking that neither theorem has the slightest 'practical' importance. In practical applications we are concerned only with comparatively small numbers; only stellar astronomy and atomic physics deal with 'large' numbers, and they have very little more prac-

tical importance, as yet, than the most abstract pure mathematics. I do not know what is the highest degree of accuracy which is ever useful to an engineer—we shall be very generous if we say ten significant figures. Then 3.14159265 (the value of π to eight places of decimals) is the ratio

$$\frac{314159265}{1000000000}$$

of two numbers of ten digits. The number of primes less than 1,000,000,000 is 50,847,478: that is enough for an engineer, and he can be perfectly happy without the rest (pp. 101-102).

He goes on to claim that what he considers "real mathematics," the purest, most abstract mathematics, is without applications (Hardy, 1976):

> There is one comforting conclusion which is easy for a real mathematician. Real mathematics has no effects on war. No one has yet discovered any warlike purpose to be served by the theory of numbers or relativity, and it seems very unlikely that anyone will do so for many years (p. 140)... I have never done anything 'useful'. No discovery of mine has made, or is likely to make, directly or indirectly, for good or ill, the least difference to the amenity of the world (p. 150).

These deep-seated ideas notwithstanding, history had a surprising twist in store for mathematicians. At the beginning of the twentieth century, developments in quantum physics and relativity theory required the most abstract theories of algebra, analysis, and geometry. Furthermore, computer technology has required precisely the mathematics that Hardy felt to be impractical. In fact, one multi-million dollar company, RSA Cryptosystems, specializes in finding for its customers prime numbers 100 to 200 digits long, primes which far exceed the numbers considered by Hardy to be "enough." This mathematics has even proven to be crucial to the military; for instance, extremely large prime numbers are used daily in securing military communications.

As the abstract mathematics that had seemed so irrelevant to the pragmatic world began to have exciting and unexpected applications, it was inevitable that scientists would search for an explanation. One such individual was Eugene Wigner. Noted for his deep insights into mathematical physics, he gave fresh insight into the usefulness of mathematics in his now classic paper, "The Unreasonable Effectiveness of Mathematics in the Natural Sciences," first published in 1960 (Wigner, 1967).

Wigner begins his paper with the belief, common to all those familiar with mathematics, that mathematical concepts have applicability far beyond the context in which they were originally developed. Based on his experience, he says "it is important to point out that the mathematical formulation of the physicist's often crude experience leads in an uncanny number of cases to an amazingly accurate description of a large class of phenomena" (Wigner, 1967, p. 230). He uses the law of gravitation, originally used to model freely falling bodies on the surface of the earth, as an example. This fundamental law was extended on the basis of what Wigner terms "very scanty observations" (Wigner, 1967, p. 231) to describe the motion of the planets and "has proved accurate beyond all reasonable expectations." Another oft-cited example is Maxwell's equations, derived to model familiar electrical phenomena; additional roots of the equations describe radio waves, which were later found to exist. Wigner sums up his argument by saying that "the enormous usefulness of mathematics in the natural sciences is something bordering on the mysterious and that there is no rational explanation for it" (Wigner, 1967, p. 233). He concludes his paper with the same question he began with:

> The miracle of the appropriateness of the language of mathematics for the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve. We should be grateful for it and hope that it will remain valid in future research and that it will extend, for better or for worse, to our pleasure, even though perhaps also to our bafflement, to wide branches of learning (p. 237).

Wigner has drawn many others into this discussion on the applicability of mathematics. R. W. Hamming repeats Wigner's observation about its usefulness: "constantly what we predict from the manipulation of mathematical symbols is realized in the real world...The enormous usefulness of the same pieces of mathematics in widely different situations has no rational explanation (as yet)" (Hamming, 1980, p. 82). Hamming carefully examines his own experiences of using mathematics, his understanding of the origins and history of mathematics, the nature of mathematics, mathematical discovery and proof, the foundational crisis of mathematics, and the nature of science and scientific laws, and then finally proposes some explanations. Nevertheless, he is unsatisfied with his reasoning and must, like Wigner, leave the question of the role of mathematics unanswered:

> From all this I am forced to conclude both that mathematics is unreasonably effective and that all of the explanations I have given when added together simply are not enough to explain what I set out to account for. I think that we—meaning you, mainly—must continue to try to explain why the logical side of science—meaning mathematics, mainly—is the proper tool for exploring the universe as we perceive it at present. I suspect that my explanations are hardly as good as those of the early Greeks, who said for the material side of the question that the nature of the universe is earth, fire, water, and air. The logical side of the nature of the universe requires further exploration (p. 90).

Thus, we are left with the question of why mathematics, which is developed and verified by mathematicians according to human logic and reasoning, is so perfect a tool for investigating the physical world around us.

MAHARISHI'S VEDIC SCIENCE

This question of the effectiveness of mathematics can be answered by considering the Vedic knowledge brought to light by Maharishi Mahesh Yogi in his Vedic Science. Everywhere we look in nature, whether as a scientist or not, we see orderliness and growth. Natural laws, still not yet understood by scientists, govern the universe of billions and billions of stars moving throughout space in perfect harmony. The delicate balance of the environment on earth is the result of thousands of species living together in an intricately organized way. Maharishi points out that observations such as these lead us to recognize that intelligence is inseparable from life.

> We see things around us exist. We also see that things around us change and evolve. We also see that there is order in evolution—an apple seed will only grow into an apple tree, etc. Thus it is obvious that existence is endowed with the quality of intelligence—existence breathes life by virtue of intelligence (Maharishi Mahesh Yogi, 1994, pp. 57-58).

Maharishi (1994) goes on to locate consciousness at the basis of life, as fundamental as existence and intelligence, "Consciousness is the existence of everything, and consciousness is the intelligence of everything" (p. 58). Science and mathematics are intimately linked to questions of existence and intelligence, so knowledge of the field of consciousness is important for the question of the role of mathematics in science. To give experiential knowledge of the total range of consciousness, Maharishi has made available the Transcendental Meditation technique, a simple, natural, effortless technique:

> "
> ***Science and mathematics are intimately linked to questions of existence and intelligence...***

> During this technique, the individual's awareness settles down and experiences a unique state of restful alertness: as the body becomes deeply relaxed, the mind transcends all mental activity to experience the simplest form of human awareness—Transcendental Consciousness—where consciousness is open to itself. This is the self-referral state of consciousness (Maharishi Mahesh Yogi, 1994, p. 260).

In the pure self-referral state of transcendental consciousness, consciousness is conscious of itself, and the subject of knowledge is the same as the object of knowledge. Since consciousness is the link between itself as subject and as object, it is also the process of knowing. Maharishi (1986) describes the importance of this fundamental relationship, "This state of pure knowledge, where knower, known, and knowledge are in the self-referral state, is that all powerful, immortal, infinite dynamism at the unmanifest basis of creation" (p. 27). In particular, this dynamism of consciousness is the source of subjective experience: "When consciousness is flowing out into the field of thoughts and activity, it identifies itself with many things, and this is how experience takes place" (Maharishi, 1986, p. 25). Furthermore, since knowledge has organizing power, Maharishi concludes that the field of pure consciousness is also a field of absolute organizing power and from there the laws of nature emerge (Maharishi Mahesh Yogi, 1980):

> Knowledge has organizing power and therefore in the absolute structure of knowledge, in the state of the absolute observer-observed relationship, we have absolute organizing power. Once we have the field of absolute organizing power in this state of pure transcendental awareness, the seat of absolute knowledge, we have the source of all the streams of organizing power in nature. All the laws governing different fields of excitation in nature, all the innumerable laws known to all the sciences, have their common source in this field of absolute organizing power (pp. 74-75).

In this way, we see that the self-interacting dynamics of pure consciousness are at once the source of subjective experience and of the laws of nature governing all aspects of the world around us. The principles of intelligence and orderliness inherent within consciousness therefore govern all the expressions of consciousness—and, as Maharishi explains, that is all that there is.

> All speech, action, and behaviour are fluctuations of consciousness. All life emerges from and is sustained in consciousness. The whole universe is the expression of consciousness. The reality of the universe is one unbounded ocean of consciousness in motion (Maharishi Mahesh Yogi, 1994, pp. 67-68).

Since every part of life is sequentially unfolded from its source in consciousness, the full range of life is from the unbounded field of pure consciousness, the home of all the laws of nature, to the subjective realm of the mind where mathematics is located to the objective physical world around us. As Maharishi (1980) puts it:

> All the relationships and activity in the different parts and structures in the universe are nothing other than expressions of natural laws, and we have discussed that the natural laws themselves are the expressions of consciousness. The expressions of consciousness in their turn are the expressions of the non-expressed, non-changing value of pure consciousness (p. 78).

Furthermore, Maharishi goes on to explain that one whose consciousness is fully developed is able to perceive at an extremely refined level the sound of the eternal process of the transformation of the singularity of consciousness into the diversity of the physical world and the transformation of the diversity of the world into the singularity of consciousness. And it is this sequence of sound and silence in consciousness that is the Veda:

> All the material and non-material expressions of creation have specific frequencies (sounds). These fundamental frequencies, non-material values, are the sounds of the Vedic Literature: the intellect, the hum of the intellect, and with the hum, the flow and stop of it in sequence. The expression of melody, forming the whole Vedic Literature, gives us the entire process of the basic mechanics of transformation within the self-referral state of consciousness (Maharishi, 1994, p. 66).

Since all the fundamental frequencies of creation are lively in the Veda, Maharishi refers to the Veda as the Constitution of the Universe, "The structure of this level of self-referral pure intelligence is the structure of Veda, which is the very well structured Constitution of the Universe" (Maharishi, 1994, pp. 208-209). Thus, the laws that govern all manifest and unmanifest aspects of creation are structured within the consciousness of each individual.

With this explanation of the fundamental role of consciousness and the intimate connection of consciousness and the physical universe, we are ready to answer the question about the connection of mathematics, a subjective creation of the human mind, with the structure of the objective physical universe around us.

RESOLUTION OF THE QUESTION

We now consider how the description of consciousness as the source of life in Maharishi's Vedic Science resolves the question that Wigner and Hamming have set before us. The question is, simply put: why is mathematics, which is developed as a subjective discipline, so effective in its applied forms in the natural sciences, which describe nature in a purely objective manner. First, we clarify what is meant by mathematics so that we can more easily put it into the framework of Maharishi's Vedic Science.

Mathematics is the search for and study of abstract and precise patterns of orderliness in number, shape, and form. The objects studied by mathematics—numbers, shapes, sets, patterns, relationships, and so on—do not have any real physical existence. Rather, as pointed out by Hardy, they exist as ideas in the awareness of the mathematician, and they are therefore part of the subjective realm of life. Accordingly, new mathematical ideas are discovered on the subjective level by intuition, insight, and creativity, and mathematics is considered to be an art by those who practice it. The results of mathematics are expressed in very precise language as formulas and theorems and are verified and proved according to strict standards of logic, so mathematics has the reliability and objectivity associated with science, but it is nevertheless a subjective study.

> **"**
> ***...in Maharishi's Vedic Science, we understand that these two worlds are both the expressions of the same underlying field of consciousness and are both governed by the same natural laws.***

Mathematics investigates the structure of the laws governing the subjective values and functioning of intelligence and consciousness; it quantifies subjective and abstract patterns in a precise way, and it offers an exact and systematic description of purely subjective phenomena. Science, on the other hand, investigates the underlying structure of objective phenomena. Wigner and Hamming made the seemingly obvious assumption that mathematics and science were therefore studying two completely separate worlds. However, in Maharishi's Vedic Science, we understand that these two worlds are both the expressions of the same underlying field of consciousness and are both governed by the same natural laws.

Thus, mathematicians and scientists are both studying the same laws of nature. Furthermore, they are both looking for those properties of natural law that are general enough to capture the underlying structure of many different situations, as for example in the way the law of gravity applies to objects on earth, planets orbiting the sun, and galaxies in the heavens or in the way the quadratic formula can solve all possible quadratic equations. Mathematicians and scientists are both looking for exact, concise, and systematic representations of their discoveries. Both demand that knowledge be nonvariable and verifiable.

There are differences between mathematics and science, however, and these differences have given rise to the question of the effectiveness of mathematics in science. Mathematicians, by going deep into the structure of their own intellect, are studying how the laws of nature govern subjective aspects of creation, and they verify their discoveries by the intellect. Scientists, by looking out at the world around them, are studying how the laws of nature govern objective aspects of creation, and they verify their discoveries by experimentation. The understanding given by Maharishi's Vedic Science allows us to reconcile these differences. Although from two different vantage points, mathematicians and scientists are both looking at the same phenomena, the same "unbounded ocean of consciousness in motion," so the patterns and structures which the mathematician sees on an abstract level are exactly those that the scientist studies on the physical level. There must be not only parallels in what they find, there must be perfect coincidence—and this is exactly what so puzzled Wigner and Hamming. Maharishi (1996) explains this as follows:

> This universality of applications can be traced back to the fact that all aspects of Nature and areas of life are governed by the same principles of order and intelligence that have been discovered subjectively by mathematicians by referring back to the principles of intelligence in their own consciousness. Great scientists like Einstein have marveled in the past about this profound relation between the subjective and objective aspects in creation, a relation which has its foundation in the identity of the Unified Field of Natural Law and the field of pure self-referral consciousness displaying the

universal principles of intelligence and order (pp. 304-305).

Working on the level of the intellect where understanding about natural law can be expressed in concise and exact mathematical formulations, the mathematician is able to provide powerful and comprehensive tools for the scientist. Abstract mathematical formulations are able to capture in a simple way the understanding of the scientist, and scientific laws are generally expressed as mathematical equations. Since the principles of order and intelligence expressed in the mathematical model of a physical system are the same as the principles governing the behavior of the system, we see that the computational consequences of a mathematical model of a physical system can exactly describe or predict the evolving conditions of that system. The great speed and efficiency with which the mind can derive predictions from a mathematical model give science great power. For example, in a few minutes, one can set up and solve the equations describing a trajectory that can take a comet months or years to traverse.

Finally, then, in Maharishi's Vedic Science, we are able to find a resolution to the question of the role of mathematics in the sciences. The same laws of nature, with their source in consciousness, are responsible for both the subjective and objective aspects of creation. The mathematician intellectually studies the subjective side of creation; the intimacy of the intellect with the subjective side of creation gives mathematics its profundity, elegance, and naturalness. The scientist intellectually studies the objective side of creation. The subjective language and tools of the mathematician provide the precise and appropriate intellectual structures for the scientist to comprehend the physical world.

CONCLUSION

This explanation of the role of mathematics based on the principles of Maharishi's Vedic Science allows us to come to a number of conclusions and to suggest some new directions. Firstly, because mathematicians are studying the same principles of order and intelligence that are studied by science, but in a subjective and abstract way, mathematics is the natural language for scientists to record their understanding of the physical world, and the methodology of mathematics provides the natural means for predicting the behavior of the physical world. On the other hand, new discoveries and problems arising in the sciences are naturally a resource for the mathematician looking for new ideas and directions.

Next, we see the value for mathematicians to pursue pure mathematics without consideration of its applications. There has been concern in the discipline that by following their individual aesthetics and judgments, mathematicians might go off in directions that are unproductive. But we see here that it is precisely by following their own tastes and preferences that mathematicians are able to uncover deeper and deeper principles governing the structure of subtler and subtler values of natural law. According to Maharishi (1996), "These principles describe the dynamics of Cosmic Intelligence—the Unified Field of Natural Law—as it functions within itself, and are directly cognized on the level of the consciousness of the mathematician" (p. 302). Since these principles are also responsible for the physical world, they must have some reflection in the physical world, and whether they have been located now or not, eventually they will be. As Lobachevsky, a founder of non-Euclidean geometry, said, "There is no branch of mathematics, however abstract, that will not eventually be applied to the phenomena of the real world" (Lobachevsky, 1984).

Finally, this understanding of the role mathematics plays in the sciences shows us that in order to have a complete science, we must have complete mathematical knowledge, and in order to have complete mathematical knowledge, we must have complete knowledge of all levels of life. This means that mathematicians must have complete knowledge of the structure of pure knowledge and complete knowledge of the structure and functioning of consciousness. To be a good mathematician, one must develop one's consciousness fully—from the finest level to the grossest level. Maharishi Mahesh Yogi has provided theoretical knowledge and practical techniques, including the Transcendental Meditation and TM-Sidhi program, for this purpose. In his Vedic Mathematics, Maharishi has gone on to show how this knowledge of consciousness can be applied to fulfill the goals of modern mathematics. Maharishi's Vedic Mathematics is the mathematics of consciousness itself.

Vedic Mathematics is the mathematics of the

> absolute, eternal, unbounded, which deals with the absolute reality, self-referral singularity—the total potential of infinite diversity at the unmanifest basis of creation, the transcendental level of consciousness (Maharishi Mahesh Yogi, 1996, pp. 366-367).

With the comprehensive knowledge of Maharishi's Vedic Mathematics (see also Price, 1997), mathematics will be able to rise to its full potential and guide life in a more holistic, mistake-free, and evolutionary way.

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To Myself

Abba Kovner
(1918-1987)
one of Israel's leading poets

Mathematicians take a huge area like a whole world
and divide it into smaller areas, identical,
smaller than the eye can see.
Parts so exact don't need
an empty space between them.
Mathematicians
do it with only three forms:
isosceles triangle, square,
and hexagon, reliable instruments,
of course. My fear taught me
to try something else: when I could no longer bear
the space surrounding me, I wanted to manage
something smaller
like a cell, dividing itself
without fission. Not looking for answers
to every question. Only to discover what is
nagging me. Still trying: forty years
and more. Why did I want to get rid
of that hidden fear?
After all, if I fall dead in the empty space
it's not the mathematicians who'll be surprised.

Kovner, Abba. *Against Forgetting: Twentieth Century Poetry of Witness*. Edited and with an introduction by Carolyn Forché. New York: W. W. Norton, 1993. 542.

What Does It Mean to Understand Mathematics?

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INTRODUCTION

There is unanimous agreement among mathematics educators (e.g., Davis 1986; Lambdin 1993; NCTM 1989) including teachers that a prime objective of mathematics teaching is to promote understanding of the subject. According to the National Council of Teachers of Mathematics [NCTM] , Teachers must help every student develop conceptual and procedural understandings... "in various aspects of mathematics (1991, 21). Unfortunately, however, there does not exist the same degree of unanimity with respect to what is meant by 'understanding mathematics.'" According to Tom Romberg (Kieran 1994, 590), "There isn't a common definition of understanding." Perceptions of mathematical understanding vary greatly and could range from a simple recall of mathematical facts to an evaluation of a mathematical masterpiece (for example, an original proof).

DEFINITIONS

In an in-depth discussion of mathematical learning and understanding, Kieran pointed out that perceptions of mathematical understanding have changed over the years. For example, in the '70s understanding was included in learning and was equated with "knowing, applying, and analyzing" (1994, 593). She concluded that understanding is ongoing, not an "all or nothing" phenomenon, but that "some level of understanding is involved in all of mathematical learning" (1994, 598).

Bruner says, "To understand something well is to sense wherein it is simple, wherein it is an instance of a simpler, general case...In the main, however, to understand something is to sense the simpler structure that underlies a range of instances, and this is notably true in mathematics" (1995, 333). Skemp (1978) discusses "relational" and "instrumental" understanding and their differences. He describes the former as "knowing what to do and why," and the latter as "rules without reasons."

According to NCTM, understanding concepts "involves more than mere recall of definitions and recognition of common examples..." (1989, 223). It goes on to say that evidence of students' understanding of a concept is their ability to apply that concept to novel situations. NCTM agrees with Kieran (1994) that understanding is an ongoing process. It says "The development of conceptual understanding is a long-term process; understanding is developed, elaborated, deepened, and made more nearly complete over time" (1989, 69).

Cangelosi differentiates between "literal" and "interpretive" understanding. Students demonstrate literal understanding if "they can accurately translate" the "implicit meaning" of a statement. They demonstrate interpretive understanding if "they can infer implicit meaning" of a statement and can give illustrations to elucidate what is contained in the statement (1992, 98).

Cramer and Karnowski define understanding in mathematics as "the ability to represent a mathematical idea in multiple ways and to make connections among different representations" (1995, 333).

INDICATORS OF UNDERSTANDING

In this paper I illustrate with examples some indicators of mathematical understanding and then suggest how teachers can facilitate understanding in mathematics among their students.

To understand mathematics means that the learner is able to:

1. Recognize relationships among concepts and within a concept (NCTM 1989); for example, the relationship between addition and subtraction, or between the logarithmic and exponential func-

tions; so that if the learner is given *a-c=b*, he/she must be able to conclude that *a=b+c*; and vice versa. Similarly, the learner should recognize that $\log_b a = c \Leftrightarrow a = b^c$; and that $\log (ab) \Leftrightarrow \log a + \log b$.

2. Represent a concept in different ways, identify the connections among these representations, and transform and translate easily from one representation to another (NCTM 1989; Huinker 1993). For example, the learner must be able to transform $3x + 2y = 7$ into $y = -\frac{3}{2}x + \frac{7}{2}$. The learner must also be able to translate from a concrete representation to a symbolic or other representation, and vice versa; and recognize that the slope of a line, for example, can be represented trigonometrically as the tangent of an angle, geometrically as the ratio "rise over run," and as a rate of change, all of which are related.

"Recognize that a result is meaningful and makes sense; the learner must realize that an answer such as "eight and five-sixths buses" borders on absurdity.

3. Recognize the underlying structure of the mathematics embedded in a situation. For example, the learner must recognize that $\log y = \log a + n \log x$ is of the same form as $Y = A + nx$; $2\left(\frac{1}{h}\right) + 3\left(\frac{r}{5}\right) = 7$ is the same form as $2x + 3y = 7$. Also, if $m \times n = 1$, then *m* is the multiplicative inverse of *n*, and *n* is the multiplicative inverse of *m*. More generally, the learner must recognize that $a \otimes b = e \Leftrightarrow a^{-1} = b$ and $b^{-1} = a$ where e is the identity element with respect to the operation $\otimes$.

4. Communicate mathematics orally and in writing; for example, students must be able to explain their solutions to problems to the class or to the teacher. Talking about mathematics helps students to clarify their thoughts and improve their understanding (Buschman 1995; Garofalo and Mtetwa 1990; Helton 1995; NCTM 1989; Owen 1995).

5. Apply mathematics to real-life and other situations (NCTM 1989); it does not make much sense to be able to enunciate the Pythagorean theorem, for example, but not be able to use it to answer a question in geometry.

6. Generate examples and nonexamples of concepts (NCTM 1989); for example, the learner must be able to recognize that a square is a rectangle but a rectangle is not a square; a rectangle is a parallelogram but a parallelogram is not a rectangle; and so on.

7. Monitor and control his/her thought processes so that he/she can recognize when something is not correct and take the appropriate steps.

8. Recognize that a result is meaningful and makes sense; for example, the learner must realize that an answer such as "eight and five-sixths buses" borders on absurdity. In other words, the learner must interpret the answer to a problem within its context.

RECOMMENDATIONS

1. Use multiple representations in teaching, including physical models and manipulatives. The learner must be provided with experiences to recognize concepts in different situations and contexts and from different perspectives. For example, a triangle should be represented in different sizes and orientations.

2. Teach relationships, for example rules, in both directions; the learner must become aware of the reversibilty of relationships; for example, given $\log a + \log b$, the learner must be able to state that this is equal to $\log (ab)$.

3. Provide opportunities for students to write and talk about mathematics. Some activities could be journal writing (students may include what they learned in a lesson), interviews, and peer tutoring.

4. Provide opportunities for students to solve a wide range of problems individually and in groups so that they can apply their knowledge, skills, and concepts to familiar and unfamiliar situations.

5. Emphasize relational understanding rather than

instrumental understanding.

6. Provide students with experiences in self-assessment to help them develop self-correcting and self-monitoring abilities.

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Galactic Hippodrome

Arnold L. Trindade
Glen Cove, N.Y.

The sun whirls round
The Milky Center–eleven years

The earth mobile whirls around the sun
Three hundred and sixty five days

The satellites and space stations
Around earth: ninety minutes

Congressmen and Senators
Around the amphitheater floor
Two or six years

The wife and hubby
Around the cradle of baby
Two, five or seven years

The sum total of all mobile rounds
Square roots of distance
Speed over time?
On the Galactic Hippodrome!

Using Guided Inquiry in Teaching Mathematical Subjects

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The guided inquiry method of teaching promotes students' active participation in the learning process. It increases students' ability to analyze, synthesize, evaluate, and relate the intended learning concepts to multiple disciplines and everyday life, thereby making the material more relevant to students. In this paper, we introduce the guided inquiry method in teaching mathematical concepts. This method is used here to teach the golden ratio and the golden rectangle concepts.

INTRODUCTION AND RATIONALE

In order to capture the students' attention and interest, a teacher must actively engage the students in discovery activities that demonstrate the mathematical concept. After students have had opportunities to explore real life phenomena surrounding the concept and to understand the concept's correlation with other disciplines, the teacher provides students with a formal presentation of the concept. Finally, discussion of its application in multiple environments, including professional and non-professional settings, reinforces the understanding of the concept.

Although there are many proposed forms of inquiry development (e.g. Dewey, 1910; Schwab, 1962), all contain basic similarities that pertain to teachers and students alike. In Suchnian's method (1966), which parallels other proposed methods, the steps in the process of inquiry are to (1) present discrepant events or specific problematic situations, (2) encourage observation for developing a statement of research objectives, (3) ask students for observations and explanations, (4) encourage the testing of those hypotheses, (5) develop tentative conclusions and generalizations, and (6) debrief the process. In order for this process to work, the teacher must create an appropriate classroom climate where asking questions and hypothesizing about the given problem are encouraged. Teachers must also create an environment where the students do not just passively take notes and/or regurgitate factual information, but where they actively participate in the learning process.

There are many mathematical concepts that lend themselves to the guided inquiry method. First, we will explain what we mean by "mathematics," then we will introduce the general technique for using the guided inquiry method in the teaching of mathematics, and finally we will demonstrate the use of this method in teaching one specific mathematics lesson.

WHAT IS MATHEMATICS?

There is a common belief that mathematics is the study of numbers. In this oversimplified perspective, mathematics seems straightforward enough not to need an approach to teaching like guided inquiry. Sometimes, more comprehensively, it is believed that mathematics is the science of numbers. Still this perspective is not quite accurate: it is both simpler and more complicated than that. If we consider mathematics historically, we can trace an evolution of the understanding of this subject, and begin to understand why a more comprehensive approach to the subject is required.

Up to 500 B.C. (Egyptian-Babylonian understanding of mathematics), mathematics was indeed the study of numbers.

From 500 B.C. to 300 A.D. (in Greece), geometry became the foundation for mathematics. The study at this time involved forging a relationship between shapes and numbers. Mathematics was now regarded as an intellectual pursuit, having both aesthetic and religious elements. It was at this time that axioms and theorems were born. Because of this development, precisely stated assertions of mathematics could be logically proven by a formal argument.

There was no major change in this conception of mathematics until the middle of the 17th century, with the conception of calculus. Newton (English) and Leibniz

(German) studied the relationships among previously disparate elements.

At the end of the 19th century, the study of numbers became augmented by its more complex relationship with motion, change, space, and, most importantly to the issues raised in this essay, the utilization of specific mathematical tools.

Today, mathematics is not just the science of numbers, but of patterns. These patterns can be real or imaginary, visual or mental, static or dynamic, qualitative or quantitative. They can arise from the world around us, from the depths of space and time, or from the inner workings of the human mind. They are, then, both more complicated than the first historical understandings of mathematics, for they involve much more than just numbers, and more simple, for they are the foundations of much of what we already observe in the world and in our own schema of thought and discovery (Gialamas, 1997).

To introduce the mathematical concept of concern to this study, then, let us initially consider a more elementary mathematical topic: proportion. A proportion is a relationship between two ratios and is expressed as $a : b :: c : d$, or as $\frac{a}{b} = \frac{c}{d}$. A ratio, in turn, is a comparison of two different sizes, quantities, qualities, or ideas, and is expressed by the formula $a : b$ or $\frac{a}{b}$.

With this in mind, there is a unique geometric proportion of terms that has been called the golden ratio. It is this ratio which is the focus of this mathematical investigation. To make it truly an investigation, students must be able to discover, understand, and relate the learning concept to real life situations. Using the guided inquiry method with the steps outlined below, they learn to do so.

USING THE GUIDED INQUIRY TECHNIQUE TO TEACH MATHEMATICAL CONCEPTS FOR CONCEPTUAL CHANGE

For this investigation seven main questions have been modified from Cherif's (1988) proposed guided inquiry method for teaching science, to be used as general guided inquiry questions for the teaching of mathematical concepts (Appendix I).

With these questions, the teacher engages the students in discovery activities that will eventually demonstrate the mathematical concept for the students. The questions have been grouped into "Before the Activity" and "After the Activity":

Before the Activity

1) *What do you think will happen, given an initial set of conditions and a specific set of procedures to follow?* (Here, students should make conjectures for what they believe will happen.)
2) *Which conjectures seem the most mathematically viable?*

After the Activity

3) *What is the result of completing the procedures?*
4) *Which initial conjectures were most reasonable?*
5) *To arrive at a conclusion, what steps were needed in order to complete these procedures?*
6) *Where can you identify correlations between introduced mathematical concepts and activities in daily lives?*
7) *Can you generalize the results by completing these procedures?*

Using these steps drawn from the principles of guided inquiry, mathematics may be taught in a manner that engages students' intellectual curiosity.

USING GUIDED INQUIRY QUESTIONS TO TEACH THE GOLDEN RATIO

One might define that two quantities a and b satisfy the golden ratio property if the ratio $\frac{a}{b}$ is approximately 1.618. The questions that follow are derived from the general guided inquiry approach and suited for use in a lesson on the golden ratio. These questions were used in a seventh grade classroom, and those students' conjectures and answers are given below.

Before the Activity

1) *What do you think will happen when you compare the length of your hand and the length of your arm?*

According to Cherif (1988), a question such as this belongs to the Synthesis Level of Bloom's educational objectives. Its aim is to arouse interest, to stimulate thinking, and to produce educated conjectures. It deals with expectations. When we ask "What will happen if...?" we set the stage for the students to recognize that there is a problem, and therefore capture their immediate interest. Furthermore, this question promotes the ability to use mathematical tools in order to express a hypothesis, an assumption, or possible conclusion clearly. Here are some examples of students' initial hypotheses:

1. The ratio of measurement among male students will be twice as great as the ratio of measurement among the female students.
2. The ratio of measurement among female students will be 1 and 1/2 times greater than among the male students.
3. The ratio of measurement among taller students will be greater than the ratio of measurement among the shorter students.
4. The ratio of measurement among taller girls and shorter boys will be almost the same.
5. The ratio of measurement among male and female students most of the time will be the same.

2) *Which of the above conjectures seems like the best answer?*

To promote educated conjectures, learners must have enough time to discuss their conjectures amongst themselves. Moreover, they must be able to justify their conjectures and also to change their conjectures in the case that someone else has a better point of view. In this situation, the student might discuss the idea that two sets of numbers might be different but might have the same ratio (e.g. 8/16 and 4/8). The educated conjectures then go up on the blackboard for further use.

At this point, students complete the activity and record the results. Students are given the opportunity to test their own conjectures by performing the measurements and calculating the corresponding ratio of the length of arms to hands among male and female students in the classroom. Therefore it promotes the integration of students' understanding and the manifestation of their understanding on the investigated problem. In addition, the students must observe carefully, measure and calculate accurately, and describe their findings in writing (the actual final result) in a concise manner.

After the Activity

3) *What is the result of completing the procedures?*

This question offers students the opportunity to actually plan and carry out experiments on their own to determine whether their conjecture is reasonasable. As a result, they will have the opportunity to gain the skills of designing experiments, testing hypotheses, reasoning and debating results, etc.

4) *After completing the measurements and finding the ratios, what do you think about your initial assumption? Which of the initial assumptions were the most reasonable?*

This is a descriptive-discovery question based on the careful observation that characterizes any scientific process. It is aimed toward building an awareness of what actually happened and encouraging students to willingly change their thinking (conjectures) based on the results of the experiments (Cherif, 1988).

In this case, an example of an accepted answer is: "There is no significant difference in the ratio of the length of hands to arms on male and female students." When all the students have agreed about the actual findings and the conclusions pertaining to the compared ratio, they are asked to compare their own initial conjectures with the actual finding. Then, they are asked to come forward and erase from the blackboard any matched predictions.

This is an exciting stage of self-correcting where the students, while engaged in the whole process independently, are actually learning by thinking and doing. Since students are devoted to conducting experiments to test them, the analysis of experimental results will allow for some hypotheses to be rejected and some to retained (Cherif, 1988).
At this stage, Cherif has warned teachers from failing into the "right answer syndrome," where many teachers feel they must give the right answers to students' questions. In the spirit of inquiry, the students should be allowed to make discoveries for themselves. To use Popp's words (1981), teachers should help students develop or enhance a frame of mind "which can allow familiar and perhaps pet beliefs to be released in favor of alternative better supported ones."

The following are examples of how seventh grade students tested their hypotheses that were listed in question number two:

1. They measured the ratios for their sisters and brothers and used those results to justify their con-

jectures pertaining to hypothesis (3).

2. They measured the ratios for their pets and drew conclusions in terms of their conjectures in hypothesis (5).
3. One student used charts from his father's medical office in an attempt to determine patterns of growth, as they relate to hypotheses (1) and (2).
4. A student used her footprints from the hospital certificate created on her date of birth to compare with the current ratio of segments of her foot length, to test hypothesis (1).

$$\frac{\text{The length of the foot}}{\text{The length of the ankle to toe}} : \frac{\text{The length of the ankle to toe}}{\text{The length of the middle toe}}$$

5) What steps have you taken in order to conclude that there is no significant difference in the ratio of the hands to the arms of different students?

Students need to describe precisely and in detail all the previous steps they and/or the teacher have taken before reaching the final conclusion. In other words, with this question, students need to be able to describe the experimental pattern that led to the final results. Cherif (1988, 1993) has stated that the objectives of asking this question are:

(a) to keep students up-to-date with the inquiry processes,
(b) to establish in their minds the cause and effect relationship and that the final results could not have been determined without all the previous steps, and
(c) to encourage students to think of everything that took place not as a separate or isolated event, but as a total and integrated whole.

Most teachers go directly to the question "Why?" after they ask the question "What happened?" Teachers should be cautioned not to pass over the process too lightly, simply because the students have gained some skills and information and have developed an awareness of the problem. It is necessary for the students to reflect on the experience of having discovered the final result, in order to help them deepen their understanding and appreciation of the gained knowledge and processes (Cherif, 1988 and 1998).

In answering this question, a seventh grade student wrote:

1. We measured the lengths of several parts of our bodies. We calculated the ratios of two of the measurements.
2. We compared all of the ratios.
3. We drew conclusions about the ratios and our conjectures.
4. We discovered that not all of our conjectures were wrong.
5. After taking all the measurements, we compared our findings with our conjectures.

6) Can you identify a correlation between the demonstrated mathematical concept and real life?

Cherif (1988) calls this question an idea-application or testing-understanding. He argues that its aim is to help students generalize from the ideas at hand and to encourage them to think of the investigated concept as a part of their lives. This question is asked in order to confirm the following:

(a) to make sure that students understand the idea or the concept under investigation,
(b) to make sure that they master the inquiry processes,
(c) to help them develop the ability to apply the reasoning pattern in other situations,
(d) to see mathematics as a part not only of society, but also of themselves, and
(e) to accept mathematics as a way of knowing and understanding. Once the students have undergone the process of guided inquiry in order to understand a specific mathematical concept, it is important to reinforce their understanding with applications in other disciplines and in daily life.

7) Can you generalize the results of completing these procedures? How can you show mathematically that there is no significant difference in the ratio of the length of hands to arms on a variety of students?

Here, students must provide enough evidence in their attempt to prove their conjectures in general. This is the causal question or the reasoning explanation. The point of this question is that students are asked to generate a reasoned and testable hypothesis. At this stage, it is the generation of a hypothesis and not the testing of the hypothesis that is of concern. Teachers must remember that it is "the theory and not the experiment [that] opens up the way to new knowledge"

(Karl Popper; cited in Hurd, 1969, p 17).

Furthermore, in this stage of the inquiry, Cherif has argued, the tentative explanations (testable hypotheses) offered by students should reflect their ideas, experiences, and understanding, and thus present teachers with the opportunity to find out how and what their students think about the given instance. Based on such findings, teachers should make the decision to continue the session of inquiry without further assistance, with more guided assistance, or to give the students more time to look for related information needed for generating testable hypotheses related to the investigated problem(s). Teachers should have a set of follow-up questions ready for use to stimulate the students should there still persist many ill-founded and unsettled hypotheses.

The following are examples of students' testable hypotheses in seventh grade:

1. Humans grow symmetrically.
2. Human body parts grow proportionally.
3. The growth pattern is the same for the human body in males and females.
4. The growth pattern is the same within all living organisms (plants and animals).
5. The growth pattern is constant within each species in mammals.

As Cherif has argued, only those conjectures that have provided enough evidence of how they might be proven must be considered. The students who generate conjectures, but fail to provide enough evidence of how they can prove them, should have their conjectures rejected by the teacher for consideration.

THE FORMAL INTRODUCTION OF THE MATHEMATICAL CONCEPT

At this stage in the guided inquiry method, the teacher formally introduces the mathematical concept that was previously intuitively presented to the students. The first mathematics concept under investigation is the golden ratio.

The Golden Ratio

Given a line segment AB and a point C between A and B:

the ratio $\frac{AB}{AC} = \frac{AC}{BC}$ is denoted by ϕ and is called the golden ratio. One can compute the value of ϕ as follows. Let $AB = x$ and $AC = m$. Then $BC = x - m$.

The ratio becomes

$$\phi = \frac{x}{m} = \frac{x}{x-m}$$
$$\Rightarrow x(x-m) = m^2$$
$$\Rightarrow x^2 - mx = m^2$$
$$\Rightarrow x^2 - mx - m^2 = 0$$

Let us consider x as the variable and m as the constant. Then we have a quadratic equation with the solution as follows:

$$x_1 = \frac{m + \sqrt{m^2 + 4(1)(-m^2)}}{2(1)} = \frac{m + \sqrt{5m^2}}{2}$$
$$x_2 = \frac{m - \sqrt{m^2 - 4(1)(-m^2)}}{2(1)} = \frac{m - \sqrt{5m^2}}{2}$$

Then $x_1 = \frac{m + m\sqrt{5}}{2} \Rightarrow x_1 = m\frac{1+\sqrt{5}}{2}$

or $x_2 = \frac{m - m\sqrt{5}}{2} \Rightarrow x_2 = m\frac{1-\sqrt{5}}{2}$,

a negative number.

Therefore, there is only one positive solution, and the ratio that is accepted is

$$\frac{x}{m} = \frac{1+\sqrt{5}}{2} \cong 1.618033989\ldots$$

or $\phi = \frac{1+\sqrt{5}}{2} \cong 1.618033989\ldots,$

which is an irrational number.

For our computations, we will be using a 3-digit approximation for the value of ϕ, which will be 1.618.

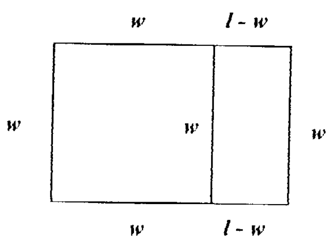
The Golden Rectangle

A rectangle with length l and width w is a golden rectangle if the ratio, $\frac{l}{w} = \frac{w}{l-w} = \phi$.

Alternatively, a rectangle with length *l* and width *w* is a golden rectangle if, when we remove a square with side *w* from the original rectangle, the remaining rectangle is similar to the original–that is to say, two rectangles ABCD and KLMN are similar if the following condition is satisfied:

$$\frac{AB}{KL} = \frac{DC}{NM} = \frac{AD}{KN} = \frac{BC}{LM}$$

So that golden rectangle would look like this:



where $\frac{l}{w} = \frac{w}{l-w}$

and the original rectangle with sides *l* and *w* is similar to the remaining rectangle with sides *l-w* and *w*.

The Fibonacci Sequence of Numbers

The golden ratio is related to a special sequence of numbers, discovered by Leonardo Fibonacci of Pisa, an Italian mathematician in the 11th century. A sequence of numbers $a_0, a_1, \ldots, a_{m-1}, a_m, a_{m+1}, \ldots$ is called a Fibonacci sequence when the following relationship between its tems is satisfied

$$a_{n+1} = a_n + a_{n-1}, \text{ for } n \geq 1, \text{and } a_0 = 0, a_1 = 1$$

By replacing *n* with 1, 2, 3,..., 21, we obtain the first 21 terms of the sequence. Therefore

$a_0 = 0,$ $a_1 = 1,$
$a_2 = 1,$ $a_3 = 2,$
$a_4 = 3,$ $a_5 = 5,$
$a_6 = 8,$ $a_7 = 13,$
$a_8 = 21,$ $a_9 = 34,$
$a_{10} = 55,$ $a_{11} = 89,$
$a_{12} = 144,$ $a_{13} = 233,$
$a_{14} = 377,$ $a_{15} = 610,$
$a_{16} = 987,$ $a_{17} = 1597,$
$a_{18} = 2,584,$ $a_{19} = 4,181,$
$a_{20} = 6,765,$ $a_{21} = 10,946.$

Sequence of "Almost" Golden Rectangles

Let us apply the principle of the golden rectangle using certain selected numbers as sides of a rectangle.

We create a sequence of "almost-golden" rectangles as follows:

1. Choose as the first rectangle the one which has sides $a = 10,946$ and $b = 6,765$ which are consecutive terms in the Fibonacci sequence). We see that the ratio a/b is approximately 1.618.

2. Removing a square with side *b* (6,765) from the first rectangle we create the second rectangle. The length and width of the new rectangle are respectively 6,765 and 4,181, and their ratio is approximately 1.618.

If we continue this process of removing squares with a side length equal to the shorter sides' length of the rectangle, we obtain a sequence of rectangles with corresponding lengths and widths as indicated in the following table.

COMPARISON	STAGES	RECTANGLE LENGTH	RECTANGLE WIDTH	RATIO
Ratio < ϕ	1	10,946	6,765	1.618033963
Ratio < ϕ	2	6,765	4,181	1.618033963
Ratio > ϕ	3	4,181	2,584	1.618034056
Ratio < ϕ	4	2,584	1,597	1.618033813
Ratio > ϕ	5	1,597	987	1.618034448
Ratio < ϕ	6	987	610	1.618032787
Ratio > ϕ	7	610	377	1.618037135
Ratio < ϕ	8	377	233	1.618025751
Ratio > ϕ	9	233	144	1.618055556
Ratio < ϕ	10	144	89	1.617977528
Ratio > ϕ	11	89	55	1.618181818
Ratio < ϕ	12	55	34	1.617647059
Ratio > ϕ	13	34	21	1.619047619
Ratio < ϕ	14	21	13	1.615384615
Ratio > ϕ	15	13	8	1.625
Ratio < ϕ	16	8	5	1.6
Ratio > ϕ	17	5	3	1.666666667
Ratio < ϕ	18	3	2	1.5
Ratio > ϕ	19	2	1	2
Ratio < ϕ	20	1	1	1

One realizes that there is a pattern involving the ratios of the dimensions of the rectangle at each stage. In particular when we compare these ratios with ϕ at each stage we observe that these ratios alternating from being less than ϕ to being greater than ϕ. Finally it is clear that after Stage 12, the differences are increasingly divergent from the golden ratio.

One might conclude that the Fibonacci number sequence, which appears in many cases in nature, is closely related to the golden ratio. The visual presentation of the first six stages of the table and the curve associated with the sequence of the rectangles is presented in figure 1.

The Spiral Curve Associated with the Sequence of "Almost" Golden Rectangles

To begin, we take the first "removed" square (from the golden rectangle), with its center as one of the vertices of the square and radius the *l* of one side, and draw an arc from one adjacent vertex to the other. We continue the same process for each removed square at each stage in the sequence of "almost" golden rectangles. The resulting continuous curve is called the equiangular spiral curve. In looking at the chambered nautilus seashell in relation to this curve, one can see that the spiral curve appears on the boundary of the shell.

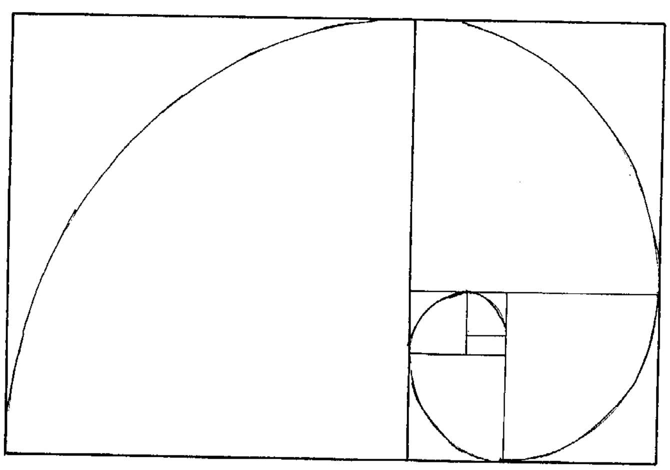


Figure 1
A visual presentation of the first six golden rectangles from the sequence of the rectangles represented by the table and the spiral curve associated with them.

ACTIVITIES TO REINFORCE THE MATHEMATICAL CONCEPT

After a student has studied the mathematical concept through the process of guided inquiry, there are several ways in which that concept can be reinforced to ensure a more complete understanding of the mathematics. First, as the guided inquiry approach is meant to be a process-based form of learning, for both the instructor and the student, it would make sense for the instructor to embark on the teaching of related concepts in subsequent lessons, so that the student might build upon the information he or she has recently learned through guided inquiry.

Also, in the ending stages of the guided inquiry approach, an instructor might also use other disciplines to reinforce the mathematical concept. The students might draw a representation of the process by which the concept was learned, or the student might write a creative piece demonstrating his/her understanding of the concept in new terms altogether. Both methods would meliorate the students' initial interaction with the mathematical concept. In addition, these forays into other disciplines provide the instructor of the class an opportunity to assess students' understanding of the concept in alternative ways. If a student excels, for example, in the arts and has had a general disdain for mathematics before this lesson, he can demonstrate his understanding of the topic in his own terms, according to his strengths, and his grade would be decided based on a broader range of activities. Any sort of project relating this mathematical concept to other disciplines (science, art, history, etc.) is a fine way to continue the active learning process initiated by the guided inquiry approach.

CONCLUSION

The guided inquiry approach promotes active learning: not just hands-on learning, but minds-on learning. Activities in any discipline that capitalize on the guided inquiry approach will help students and teachers alike make academic material more meaningful, for guided inquiry inspires intellectual curiosity rather than defensiveness. For students who ask,"why do I need mathematics, again?" and for insouciant students who'd rather stare out the window than engage in listening to a teacher lecture on fundamental mathematical principles, the guided inquiry approach offers a reason to become participants in the learning process.

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APPENDIX 1

No.	Guided Inquiy Question in Teaching Science Concepts	Nature of the Question	Aim and Objective of the Question
1.	What do you think will happen given this set of conditions? (If, for example, X is added to Y?)	Predicted Question	To arouse interest, stimulate thinking, and provide predictions.
2.	What actually happened?	Descriptive-Discovery Question	To build an awareness of what actually happened.
3.	How did it happen?	Holistic-Descriptive Question	To estabilsh in students' minds the cause and effect relationship; to think of all the processes that took place as a total integrated whole; to provide general understanding of the process that took place and resulted in what actually happened.
4.	Why did this happen?	Casual Question or Reasoning Explanation	To develop and apply some kind of mental analysis that enables students to generate a reasoned and testable hypothesis (tentative explanations) using their ideas, experiences, and understanding.
5.	How can we find out which of these hypotheses is the most reasonable?	Experimental Question	To provide the opportunity to actually plan and carry out experiements of their own; to gain skills of designing experiments, testing hypotheses, reasoning, and debating results.
6.	How can you relate the investigated ideas, concept, or principle to your daily lives?	Idea-Application or Understanding-Testing Question	To understand the idea or the concept under investigation; to master the inquiry processes; to apply reasoning patterns in other situations; to accept science as a way of knowing and understanding.

A Sabbatical Experience: Nurturing a Partnership

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One might ask why any professor of mathematics education would return to the elementary school classroom after earning a doctoral degree. The appropriate question should really be why would she or he not return to where the action in mathematics education really is!

It has been almost twenty-five years since I have taught elementary school children concepts and skills in mathematics for any extended period of time. During my career as a mathematics educator, I have conducted numerous workshops involving students of all ages (grades K-11), but not at the intense level as a classroom teacher. The NCTM *Professional Standards for Teaching Mathematics* document advocates that mathematics educators at the university and college levels should spend time in schools working with teachers and children (NCTM 1991). Not only should these educators spend time in the schools, but they should also cultivate and nurture partnerships between LEA's (local educational agencies) and universities (Curcio, Perez, and Stewart 1994). I chose to take this proactive challenge to heart and applied for and was awarded a sabbatical leave at a local elementary school for the 1998 spring semester.

Prior to my sabbatical, I enjoyed a strong relationship with the faculty and administration at a local elementary school (grades K-5). This relationship was primarily due to establishing and conducting a mathematics teaching practicum at this school for my preservice elementary school teachers. The relationship evolved so well that I wanted to establish more of a partnership with the school than just mentoring my preservice teachers; hence, the sabbatical leave.

During the sabbatical, I decided (after consulting with the teachers and principal of the school) to establish a mathematics club for a select number of gifted-and-talented students (sixteen students in grades K-2 and sixteen students in grades 3-5, per semester), to act as a mathematics consultant to the faculty, tutor students (individually and in small groups), teach students in a classroom setting, conduct some qualitative research, and perform other school-related duties and functions. The two most demanding, yet most rewarding, of these activities were administering the mathematics club and teaching mathematics concepts and skills to the children.

The mathematics club met before school twice per month, once for the K-2 group and once for the 3-5 group. Presenters consisted of teachers from the school (including both the instrumental music and the physical education teachers), my preservice elementary school teachers (from the practicum course), one of my secondary mathematics education majors, and mathematics and science education professors from Towson University. According to research, the use of cooperative groups (Thornton and Wilson 1993) and the appropriate use of physical materials (Dougherty and Scott 1993) can help students develop a stronger foundation for acquiring mathematical concepts and skills. Therefore, the focus of these sessions was the effective use of cooperative groups and manipulative materials to enhance the learning of interesting and challenging mathematics and science concepts and skills. The activities that were presented challenged students to construct their own understanding of the mathematics or science concepts and/or skills. In this way, the children would acquire and "own" the knowledge on their own terms and be more able to apply the concepts and skills in different contexts.

As a consultant, teachers would approach me during the school day and solicit my opinion about teaching a particular mathematical concept or skill. However, after about two weeks, the teachers not only wanted my advice, they wanted me to model my teaching style with their students. After agreeing to do this, I was extremely busy for the remainder of my time at the school! Teaching children mathematics on a daily basis has been one of the most rewarding experiences of my professional life as a mathematics educator.

Regardless of their level of mathematical sophistication or age, the students were almost always enthusiastic, attentive and willing to work hard at the activities that were presented. These students enjoyed working in cooperative groups, using physical materials and technology, and applying their knowledge to novel situations. Using activities which apply mathematics to other content areas is advocated by many professional education organizations, such as the National Council of Teachers of Mathematics and the National Research Council (NCTM 1989, NRC 1996). With the type of activities I used and still use with children, I always try to empower them mathematically by acting as a "facilitator of active learning," not just a "dispenser of knowledge," and by connecting mathematics to other disciplines. For example, in a graphing activity which analyzes the forces affecting a falling parachute, I used plastic grocery bags, small plastic soldier figures, a ladder, a graphing calculator and a CBL (calculator-based laboratory) unit with a regular class of fourth grade students. The students were asked to explain the forces that might act upon the parachute as it descends to the ground and to select which of three graphs might best illustrate the relationship between height and time. After a lively discussion of these issues (led by the students), the students actually made the parachutes and conducted the experiment. Most students were amazed that the linear graph was the one that best reflected the data from the experiment. After a few minutes of deliberation (without any input from me), the students were able to accurately describe why the graph of this data should be linear in nature.

With respect to my preservice elementary school teachers and other professional colleagues, the sabbatical experience lends credibility to my career as a professor of mathematics education. It provides evidence to my peers and students that I am cognizant of current issues in elementary school mathematics education from a first-hand perspective and that I am a believer in and a practitioner of learning as a lifelong pursuit.

Based on the research conducted during my sabbatical, it is my opinion that a strong and long-lasting partnership, with an LEA, can be established and fostered by university-based mathematics educators by performing a few, if not all, of the following activities:

- providing a cohort of preservice elementary school teachers as mathematics teaching interns (establishing a field-based practicum),
- conducting mathematics content staff development workshops for teachers,
- consulting with teachers and administrators concerning mathematics content, pedagogy, and assessment,
- establishing a mathematics club for gifted-and-talented children,
- providing tutoring sessions for all children (sessions conducted by the professor and/or preservice teachers),
- modeling effective mathematics teaching strategies for classroom teachers (in front of actual children!), and
- collaborating with classroom teachers and/or administrators in writing staff development and/or technology grants.

Most importantly, it is my opinion that a carefully planned sabbatical experience can make an enormous contribution to an increased level of a mutually beneficial partnership between a local public elementary school and a university department of mathematics, which is a need shared by both school-based and university-based educators.

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The Pythagorean Theorem and Area: Postulates into Theorems

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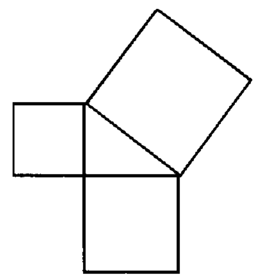
Considerable time is spent in high school geometry building an axiomatic system that allows students to understand and prove interesting theorems. In traditional geometry classrooms, the theorems were treated in isolation with some of the more interesting and powerful theorems posed as only postulates. NCTM's *Curriculum and Evaluation Standards* (1989) called for a rethinking of the structure of geometry.

In particular, students should be given an opportunity to discover the ideas of geometry through concrete experiences and direct measurement so that they can build intuition for the central elements in the axiomatic structure. The more recent *Principles and Standards 2000* (NCTM, 2000) calls for a return to reasoning and proof through the K-12 curriculum. The learning of geometry is an inductive/deductive process. Students should experience specific instances that allow them to generalize the postulates, theorems and definitions of geometry. Many of the ideas of geometry can be easily introduced in a discovery setting in which students explore the ideas of measure, congruence, inequality, parallelism and similarity. Once students have inductively acquired an understanding of the ideas of the axiomatic system through these concrete experiences, they can *then* deductively explore short sequences of interesting theorems that demonstrate the elegance of the axiomatic system.

This article deals with the deductive process, highlighting some central theorems in geometry which are too frequently bypassed as postulates in the standard geometry texts. It is curious, for example, that the familiar similar-triangle proof of the Pythagorean theorem is based on something called the Angle-Angle Similarity *Postulate*. When one takes this circuitous route to the Pythagorean Theorem the notion of area never appears. Yet, Euclid's proof depends largely on the notion of area, as shown below. He simply shows that the sum of the area of the two smaller squares is equal to the area of the square on the hypotenuse.



Lightner (1991) speculates on the method of the Pythagoreans when he describes an algebraic/geometric approach that involves dissecting squares and using the idea of combining areas. It seems that area is an essential component in the various proofs of the Pythagorean Theorem.

The following two sequences of theorems include some of the standard "postulates" and culminates with an interesting "area" proof of the Angle-Angle Similarity Theorem which would then allow us to prove the Pythagorean Theorem by the usual similar triangle approach. The theorems are found in a variety of texts, but rarely are they found in high school geometry texts. When the synthetic approach to geometry *is* emphasized, it is important that theorems be arranged in meaningful sequences and that they are connected so that students can understand and connect the various elements of the axiomatic system In much of what follows we use the important idea of one-to-one correspondence given by the Ruler and Protractor Postulates.

TRIANGLE CONGRUENCE

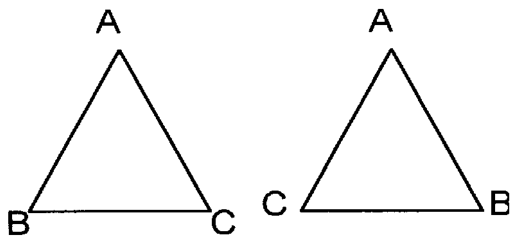
High school geometry texts typically pose SAS, SSS and ASA as postulates. Here we postulate SAS and develop proofs for the other two.

Postulate (SAS): If two sides and the included angle of one triangle are congruent to the corresponding parts of a second triangle, then the triangles are congruent.

The Isosceles Triangle Theorem is an immediate consequence of the SAS Postulate if we take the following transformational perspective. The proof is elegant and simple.

Theorem (Isosceles Triangle): The base angles of an isosceles triangle are congruent.

Proof: Consider isosceles ABC with AB=AC from two perspectives.



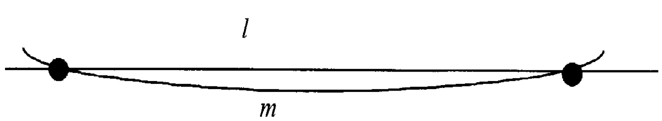
From left to right, since AB=AC and AC=AB with $<A \cong <A$, we have $\Delta ABC \cong \Delta ACB$ by SAS. The corresponding angles B and C are then congruent by definition of congruent triangles.

We are almost ready to investigate the proofs of SSS and ASA. It is important that students spend some time with proof by contradiction. We digress momentarily to develop a simple example using one of the first postulates in the axiomatic system.

Postulate: Two distinct points determine exactly one line.

Theorem: When two distinct lines intersect, they intersect in exactly one point.

Proof: Suppose not. Suppose, given distinct lines *l* and *m*, they intersect in two points.

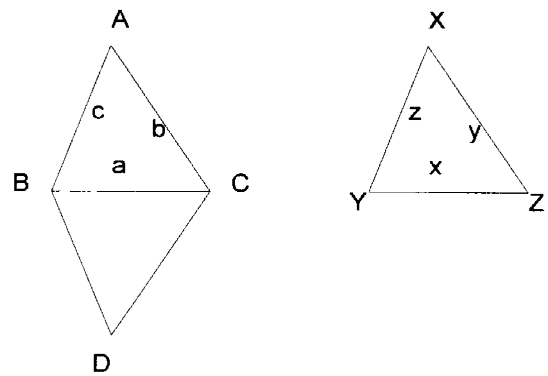


Since two points determine exactly one line, we contradict the hypothesis that *l* and *m* are distinct. Conclude that two distinct lines can intersect in only one point.

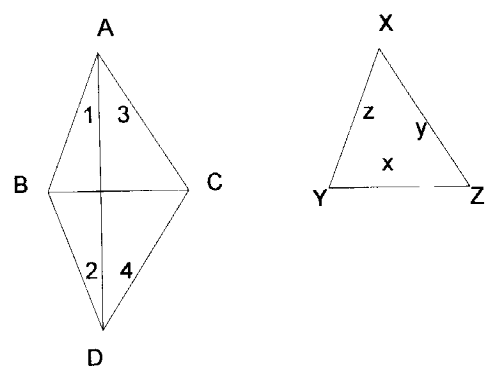
In proof by contradiction, we assume the hypothesis (distinct lines) and the negation of the conclusion (not one point) and reach a contradiction forcing us to accept the conclusion. We use this idea later in the ASA Theorem.

Theorem (SSS): If three sides of one triangle are congruent to the corresponding sides of a second triangle, then the triangles are congruent

Proof: Here we will use the SAS Postulate twice. Consider the two triangles shown with a = x, b = y and c = z. The Protractor and Ruler Postulates allow us to consider $<CBD \cong <Y$ with BD = z as shown. Now with a=x we have $\Delta XYZ \cong \Delta DBC$ by SAS.

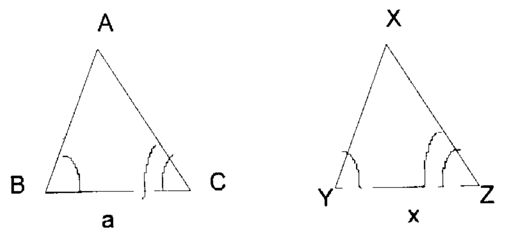


Now we will show that $\Delta ABC \cong \Delta DBC$. Consider segment AD. By the Isosceles Triangle Theorem we have $<1 \cong <2$ and $<3 \cong <4$. Applying the Angle Addition Postulate, we have $<BAC \cong <BDC$ and $\Delta ABC \cong \Delta DBC$ by SAS.



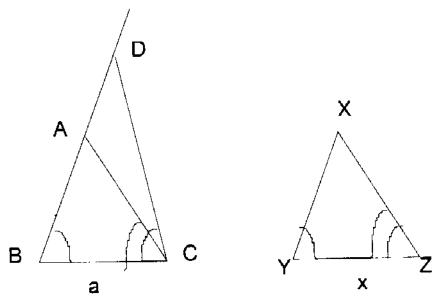
Now $\Delta ABC \cong \Delta XYZ$ by transitivity as desired.

Theorem (ASA): If two angles and the included side of one triangle are congruent to the corresponding parts of a second triangle, then the triangles are congruent.



Proof: Consider the two triangles shown with a = x, <B = <Y and <C = <Z.

We proceed by supposing the two triangles are not congruent. In particular, suppose AB≠YX. On ray BA consider BD = YX and consider CD.



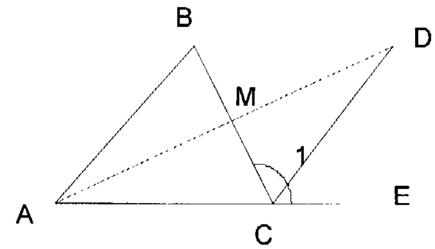
Now we have ΔDBC ≅ ΔXYZ by SAS and corresponding angles BCD and Z congruent. By hypothesis <C ≅ <Z. This contradicts the Protractor Postulate which only allows one ray to determine an angle. Thus ΔABC = ΔXYZ.

GETTING TO THE PYTHAGOREAN THEOREM

The familiar Exterior Angle Equality Theorem, which states that an exterior angle is equal icon measure to the sum of its remote interior angles, is preceded by what we will call the Greater Exterior Angle Theorem, which is often rnisplaced in the high school texts. In fact, this theorem is usually proved *after* the Exterior Angle Equality theorem because of unnecessary assumptions. Yet, the Greater Exterior Angle Theorem allows us to establish the Equality Theorem and, more importantly, sets the stage for the Pythagorean Theorem proof by similar triangles as evidenced in the following discussion.

Theorem (Greater Exterior Angle): An exterior angle of a triangle is greater in measure than either of its remote interior angles.

Proof: Consider triangle ABC with exterior angle 1. Through the midpoint M of segment BC consider segment AD such that AD is twice AM. Considering the two vertical congruent angles and the bisected segments AD and BC we have ΔABM ≅ΔDCM by SAS.



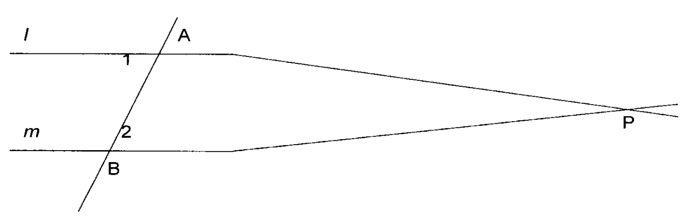
Corresponding angles DCM and B are congruent. Now m<1-m<DCM = m<DCE and since m<DCE>0 we have m<1-m<DCM >0. Then m<1>m<DCM and by substitution m<1 > m<B as desired. A similar construction will show that m<1>m<A.

With the Greater Exterior Angle Theorem behind us, we can now turn the Alternate Interior Angle Postulate into the Alternate Interior Angle Theorem. But, first, we need a very important, controversial postulate.

Postulate (Parallel): In a plane, through a point outside a line, there is exactly one parallel to the line.

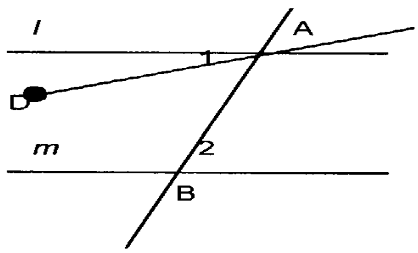
Theorem: Alternate interior angles formed by two lines and a transversal are congruent if and only if the lines are parallel.

Proof: We begin by showing that congruent alternate interior angles imply parallel lines. This part is usually posed as a postulate. Suppose alternate interior angles 1 and 2 are congruent but the lines *l* and *m* are not parallel. Suppose they intersect in some point P as shown.



By the Exterior Angle Theorem, <1 must be larger than <2, but by hypothesis we know that they are congruent. Thus we have reached a contradiction and conclude that *l* and *m* are parallel.

Conversely, suppose we know that *l* and *m* are parallel. Suppose angles 1 and 2 are not congruent. At A consider <BAD ≅<2 as shown.



The previous result tells us that line AD must be parallel to *m* since the alternate interior angles BAD and 2 are congruent. Now we have two lines parallel to *m* through A. This contradicts the Parallel Postulate, so we conclude that <1 = <2. The following theorems follow immediately from these results and will be of use later.

Theorem: The sum of the measures of the angles of a triangle is 180.

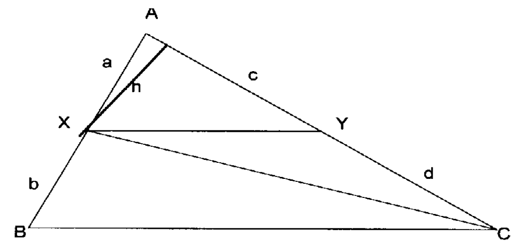
Theorem (*Exterior Angle Equality*): An exterior angle of a triangle is equal in measure to the sum of its two remote interior angles. (Ironically, in many texts, the more powerful Greater Exterior Angle Theorem follows here.)

Theorem: Corresponding angles formed by two lines and a transversal are congruent if and only if the lines are parallel.

We state the above without proof, so that we can turn our attention to proportionality and similarity, ideas that allow us to connect these theorems and postulates as the foundation for the Pythagorean Theorem. We are almost ready to prove the AA Similarity Theorem. We begin by proving the following important theorem which is often proved *after* the AA Similarity Postulate. It can, however, be proved first using the notion of area and turns out to be a necessary condition for the AA Theorem. It is necessary to assume area of a square and the resultant area of a triangle theorem for the following.

Theorem (Proportional Segments): A line parallel to one side of a triangle that intersects the other two sides in distinct points divides those two sides into proportional segments.

Proof: Consider triangle ABC with XY parallel to BC as shown. We would like to show that a:b=c:d. Consider segments XC and the altitude from X to AY with length *h*. Observe that this is the altitude for both triangles AXY and XCY to bases with lengths *c* and *d* respectively.



Now using α to denote area,

$$\alpha(AXY) = \frac{1}{2}hc$$

$$\alpha(CXY) = \frac{1}{2}hd$$

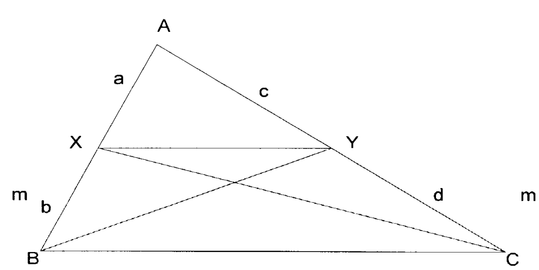
and if we consider the ratio of the areas of the two triangles we have

$$\frac{\alpha(AXY)}{\alpha(CXY)} = \frac{\frac{1}{2}hc}{\frac{1}{2}hd} = \frac{c}{d}.$$

Similarly by considering the segment from B to Y and the altitude with length *k* from Y to AX we can show:

$$\frac{\alpha(AXY)}{\alpha(BXY)} = \frac{\frac{1}{2}ka}{\frac{1}{2}kb} = \frac{a}{b}.$$

Now, if we can show the areas of CXY and BXY equal, we are done. Since parallel lines can easily be shown to be equidistant, the altitude of both triangles to the common base, segment XY, have the same length, *m*, as shown below.



Thus both triangles have area $\frac{1}{2}m < XY$.

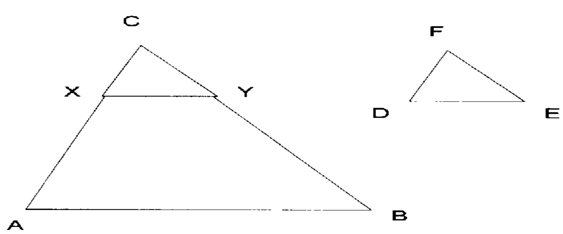
And then

$$\frac{a}{b} = \frac{\alpha(AXY)}{\alpha(BXY)} = \frac{\alpha(AXY)}{\alpha(CXY)} = \frac{c}{d}.$$

We are now ready to prove the Angle-Angle (AA) similarity theorem. It is easy to show that if two angles of one triangle are congruent to two angles of a second triangle, then the third angles from each are likewise congruent. We will need this in the proof.

Theorem (AA Similarity): If two angles of one triangle are congruent to two angles of a second triangle, then the triangles are similar.

Proof: In order to establish similarity, we must show that the sides are proportional. Consider triangles ABC and DEF with <A = <D and <C = <F. On segments CA and CB locate points X and Y such that CX=FD and CY = FE.



Now ΔDEF = ΔXYC by SAS and <A = < CXY by transitivity imply that XY | | AB. Applying the Proportional Segments Theorem, we just proved we know that

$$\frac{CX}{XA} = \frac{CY}{YB}$$

Using the definition of between and the fact that if a:b =c:d then a:a+b = c:c+d, we have

$$\frac{CX}{CA} = \frac{CY}{CB}$$

and finally by substitution

$$\frac{FD}{CA} = \frac{FE}{CB}.$$

By a similar method we can show the other sides proportional.

It is important for students to see that the idea of area can be used to prove the AA Similarity Theorem which, in turn, allows for the similar triangle proof of the Pythagorean Theorem. In fact, many of the so-called "postulates" in the secondary texts can be proved as theorems without much difficulty. When AA similarity is postulated we lose sight of the importance of our axiomatic system. Students can see how the SAS postulate, the Greater Exterior Angle Theorem and the Parallel Postulate combine forces to lay the groundwork for perhaps the most important theorem in Euclidean Geometry. One might be led to believe that the Pythagorean Theorem cannot be proved directly without area. Interestingly, Moise (1990) provides an elegant proof of the AA Similarity Theorem without area. While the proof may be beyond the scope of high school geometry, it is worth noting that one can, in fact, arrive at the Pythagorean Theorem without area.

Much of the richness of geometry is lost when theorems are treated in isolation and when key theorems are bypassed as postulates. Part of "problem posing" in geometry should include an investigation of the way the axiomatic system fits together. How do some key theorems like the Greater Exterior Angle Theorem allow us to generate important geometric ideas? For what later theorems is the Parallel Postulate a necessary condition? NCTM's *Principles and Standards 2000* challenges us to reevaluate both the content and methodology of geometry instruction. When the synthetic or analytic approach is taken, posing short sequences of interesting connected theorems encourages problem solving that engenders deeper understanding and appreciation of geometry.

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A Hypnotist Teaches Math: The Effect of Person Centered Math Support Classes on At-Risk Community College Students

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ABSTRACT
The same elements which make Ericksonian hypnosis a highly effective therapeutic tool are found to be at work in an algebra support class called Math Success Orientation. Interpersonal dynamics of Carl Rogers' person centered approach to counseling are also part of the philosophy of the class which has a four semester track record of improving both grades and attitude in at risk, math avoidant community college students. The article relates the history of the course with illustrations of how the elements of Person Centered Mathematics operate in the classroom. An appendix demonstrates the effect of the class on student grades...

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Four semesters ago, in the spring of 1996, I was asked to supervise four tutors who would be working with math avoidant college students as part of the Achieving a College Education Plus (ACE Plus) assistance program for first generation college students. To entice these math avoidant, math anxious, math hating—though otherwise ordinary—college students to give math another try, the experience was termed a college course with transferable credit, to be taught by a counselor, not a math instructor. Grading for *CPD 150 Math Success Orientation (MSO)* would be based solely on attendance.

Feeling a responsibility to provide a legitimate course of instruction, I developed a curriculum which cut into the available time for tutoring while providing study skills, logic training and strategies for dealing with math anxiety. Four semesters later, as I reflect on how our students are outperforming the rest of the student body, how MSO has been picked up by the math department, expanded from one to eight sections and now includes a second course in tutoring methodology, it is clear to me that what makes the course work is not the curriculum or the amount of tutoring but an underlying philosophy which combines the person centered therapy of Carl Rogers with a framework for hypnotherapy developed out of the work of Milton Erickson. In this article, I will describe several Ericksonian concepts and just a few Rogerian ways of being with people which have combined to form what I call *Person Centered Mathematics (PCM)*.

So far, I've identified 14 distinct Ericksonian elements at work in my teaching. They include *utilization, chunking down, priming, paradox, pacing, absorption, reframing, response sets, shock, confusion, indirection, metaphor, ordeal of choices,* and *linking*.

I need to clarify that while my methods reflect the practices of Ericksonian hypnotherapy, I do not hypnotize my students. What I've come to realize is that the same use of communication and relationship building which allows a hypnotic subject to relax, focus and enter an enhanced psychological state called trance, can be employed to facilitate students to relax in the face of math anxiety, focus on mathematical concepts and enter an enhanced psychological state called learning. Additionally, the same unconditional positive regard which Carl Roger used so effectively to promote personal growth in his clients is vital to the educational growth of students who have established a failure identity in mathematics.

Milton Erickson emphasized the importance of the relationship between hypnotist and subject, between teacher and student. He respected the learner's ability to make choices and made it the instructor's job to uncover more choices, not to decide for the learner. Erickson saw each student as a distinct individual with a unique internal logic responding to one's own positive intentions. He sought to develop rapport, create an atmosphere in which students internal resources could be utilized to move them toward their goals. Significantly, he did not believe in *resistance*.

CLASS BEGINS

The first day of MSO was marked by student resistance to trying anything new involving mathematics. I'd taught non-math courses to students from ACE Plus previously and always found them open to experimentation and risk taking. So it surprised me when I could not find a volunteer to come to the chalkboard to work a problem.

I decided to view their behavior as positively motivated. Erickson believed that what we call resistance is actually an individual's own internally logical method of meeting personal needs. I searched for a way to utilize their reluctance to expose themselves to chalkboard failure. On a hunch, I asked them all to go to the chalkboard and work the problem in pairs. Within moments they were all scribbling away; "resistance" had disappeared.

In retrospect, it appears that their fear had been of being singled out. My instruction reframed the situation to one in which the feared condition—exposure—could only be avoided by doing what I asked. Mirroring the mechanism of a hypnotic suggestion, my instruction could easily have been ignored had it been inappropriate but was just as easily followed, having become a comparatively attractive option to be singled out back at one's desk as the sole non-participant.

UTILIZATION

Erickson was uncanny in his ability to take what the client was doing to sustain the problem and use the same mechanism to resolve it. He called this *utilization*. In the example above, I utilized students' strong motivation to avoid being singled out as impetus to "join the crowd" at the chalkboard. Another example involved an older woman, newly divorced, feeling out of place in a class full of eighteen year olds; she was hesitant to accept peer tutoring. Out of class she proudly described raising her three children who are now in successful careers. Utilizing the importance she placed on her pride and mothering, I got her talking about how she developed her own kids confidence by letting them gradually accept greater challenges and increased responsibility. She then readily accepted the idea that by allowing these youngsters to tutor her, she was giving them similar nurturing, validating experiences—a nice trade off. She now proudly refers to herself not as a helpless tutee but as "co-trainer of tutors."

CHUNKING DOWN, PRIMING AND PARADOX

I later discovered that a gradual building up of small successes worked even better. First trips to the chalkboard would always be in groups. The task would move imperceptibly from simple to complex, from drawing to writing, from silly to serious. A typical student reaction was to remark later in the semester how he or she had always hated going to the chalkboard and could not remember when it had become so easy. The Ericksonian concept at work here was that of *chunking down*, breaking a seemingly impossible task into its component parts, "the idea that Custer could have won if the Indians had come over the hill one at a time" (Lankton & Lankton).

Seeing how quickly the atmosphere in the room had changed, I began to think in terms of *priming*, an Ericksonian concept having to do with building a foundation of ideas which encourage the client to start thinking in a particular direction so that later ideas will have a place to nest...

In MSO, priming involved not only keeping students informed of what was to come, but guiding and shaping their reactions. I seeded the idea that success involved being open to new experience, that success would sneak up on students, that success did not require perfection, that allowing math to be fun would dissipate anxiety and allow comprehension to flourish, that math was an interpersonal experience, that tutoring a peer was the best way for the tutoring peer to learn.

A paradoxical form of priming involves "predicting a relapse." After a new group of students begins to settle into the course, I make the following statement, "While many of you will experience failure early on, as your attitude gradually changes, you'll begin to see impressive improvement." This statement involves several Ericksonian concepts which will be examined in more detail.

The statement begins with a truism; these students have traditionally failed at math. By saying so, I am only validating their current experience. The wording is purposely vague. Who is and who isn't part of *many of you*? In what context will they *experience failure*? On a quiz? A test? When does *early on* begin and end? Vague wording improves the chance they will accept what I'm saying without challenge.

The second part of the statement links *failure* to *improvement* through the unspecified idea of *attitude change*. Which *attitude* will *change*? How fast or slow is *gradual*? After being validated by the first premise, *you will experience failure early on*, it becomes easier to accept the last part, *you'll begin to experience impressive improvement*. This imprecise wording allows each student to imagine what would constitute impressive improvement.

The result is that students are put in the unique position of seeing failure as a harbinger of good things to come. They are less likely to panic or become discouraged by a poor test result. When spoken congruently and sincerely, these kinds of statements result in sustained motivation which ultimately translates into better grades.

NO SUCH THING AS RESISTANCE

The clients of Carl Rogers drew strength from his unwavering belief in them. My tutors and I were continually challenged to stay future focused, believing in each student's potential. In a meeting with the instructor, a tutor referred to a student as a slacker who didn't do his homework and didn't care. After this student was switched to a tutor who believed in the student, he went from F's to B's. Whereas the first tutor knew her math, was creative and energetic, her replacement refused to view him as resistant or not trying. Her belief in him translated into his own success.

PCM joins Erickson in rejecting the idea of resistance because if students can resist, so can instructors. It is too easy for an instructor to think, "They obviously don't care, so why should I go the extra mile?" Instructors who view their students as resistant can become resentful or feel hopeless of reaching them. Students pick up on their instructors' attitudes, their "vibes." Whether or not these attitudes are verbalized, students get the message. They feel blamed, alienated, and unheard. They then view their instructors as "resistant," and the cycle continues with both sides putting in less and less effort.

PACING AND ABSORPTION

Rather than think in terms of reducing resistance, Erickson focused on *pacing* his clients. We want our students to be "on the same page" with us. Erickson recognized that before clients would turn to the therapeutic page, he first had to visit theirs. He sought to get into their rhythms. In trance induction, this meant speaking in rhythm to the subject's breathing. Rather than repeat some stock patter, Erickson absorbed attention by talking on topics known to be of interest to the individual. As rapport was being gained, he would ease into therapy. In MSO, this means that in every class period I present a meaningful transition to thinking and doing math.

A few students are ready from the get go. But for most, social needs must be met and interest gained. Tutors are encouraged to identify those students who have greater social needs, requiring a few moments to chat before getting down to business. Rather than see this as a waste, i.e., "Yak on your own time; I'm here to teach," we view it as an opportunity to take the pulse of the class, check out who needs what. Ideally, the transition to instruction occurs unnoticed. I'll present a brainteaser or tell a math joke designed to illustrate a way of breaking out of old patterns and discovering some alternative mode of thinking about math.

As an example, in math notation, shortcuts tend to confuse some students. Although they've been exposed to the same notation for years, under pressure they see the term 5t, and think "five t," not "five times t." They read 3/y as "three over y," not "three divided by y." They read 7(q) as "seven parenthesis q," not "seven times q." I call these unwritten operations *ghosts*. To create a memory trace which will always remind them to see the ghosts, I ask them the following riddle. As time permits, I dress up the story into a real drama. But in brief,

> A man and his wife are rushing back to town when they crash. He's thrown from the car but she's trapped inside. He rushes for help, but when he returns, she's dead and a stranger is in the car. The doors and windows are still intact and unopenable. Firemen have to break in to get them out. How did she die, and who

is the stranger? (*The answer is at the end of this article.*)

Usually after a few minutes of wrong guesses and clues, someone comes up with the right answer. I'll then remark on the resemblance of parenthesis to the woman's condition. We'll generate a list of mathematical ghosts. In future tutoring situations, I need only ask, "Do you see a ghost?"

In Ericksonian terms, the puzzle has playfully absorbed their attention. The answer is a surprise which heightens their arousal, creating a psychological space in which interest, attention and motivation are increased so that my next suggestion is accepted and comprehended without interference, i.e., they forget for a moment that math is hard and instead simply enjoy learning about mathematical ghosts—and they're now ready for an hour of algebra.

These kinds of lessons gradually change students' attitudes towards math from disdain to enjoyment, from terror to confidence. On the very first day of MSO, I had the students describe their individual math histories. While this gave us a lot of important information and let them know they were all in the same boat, the general impression was that ship was sinking; they had neither oars nor life jackets. They hated math and didn't believe they could ever succeed. Our challenge was to convince them that they could, without dismissing their feelings.

ATTITUDE AND VALIDATION

There is an art to validating a student's experience without getting mired in it. Elizabeth Ely, director of the Field School where I cut my eye teeth on teaching, was fond of quoting an aphorism of Pythagoras, "Help a man to take up his burden, but never help him put it down." She cares deeply for her students' struggles but never lets them quit. Carl Rogers would listen with complete absorption as his clients bared their deepest pains. He didn't insult them with quick fixes. Clients felt his empathy but also his belief in them. Milton Erickson kept the client focused on solutions, not problems; on future possibilities, not the injustices of the past.

After twelve or more years of struggle, my students were spoiling for a fight, tensed for the next blow. Our objective was to make sure that blow never came from our camp. Like a tennis player who leaps the net and takes a doubles stance, we refused to see our students as the opponent, no matter how many volleys they sent our way.

In practice, this meant no blaming, no cajoling, no threatening, no "I told you so's." From time to time, I would remind a student of where he or she stood in relation to the attendance/grading policy, then step back and respect whatever choice the student made. Occasionally a student would pile up enough absences to assure a failing grade. Typically that student would return the next semester both acknowledging that some personal issues had taken precedence over math, and thanking us for the space given.

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If we buy into the paradigm that they MUST PASS MATH NOW!, we've lost before we've begun.

A LEAP OF FAITH

In the third installment of the Indiana Jones saga, in order to save his father's life, our hero must step off a ledge above a deep canyon and walk through the air to a cave on the far side, hoping against hope that he won't fall into the chasm below. He makes the leap of faith, drops onto a camouflaged bridge which he crosses easily, and ultimately saves both his father and the day.

Developing Person Centered Mathematics required a similar leap of faith. We let go of the idea of grades. Not only do we not grade on math proficiency, we don't worry about what grades students get on their early math tests. Simply put, if we panic, they panic. If we buy into the paradigm that they MUST PASS MATH NOW!, we've lost before we've begun. In that case we'd be giving them exactly what they've had before, what they're used to, what they've continually failed with. To that end, and to quote from Monty Python, our motto has become, "And now for something completely different."

Students must perceive that they're not doing the same old, same old. They know the end of that story. To believe that a new result is possible, they must perceive new ways of doing things. To that end, we make a conscious effort to do things that just don't happen in traditional classrooms, like solve murder mysteries, memorize to rap, hand out awards, eat donuts, give back rubs, crumple paper and play catch.

REFRAMES AND RESPONSE SETS

I learned to toss paper from Bill Hammers, an amazing mathematician who has an outstanding rate of students completing and passing his courses. After everyone has caught and tossed our makeshift football we ask, "How did you know at what velocity to throw the ball? How did you detennine the angle of arc which would correctly counterbalance the effect of gravity, making a successful catch a greater statistical probability?" As the students stare dumbfounded, Bill just smiles reassuringly.

For Bill, the greatest mathematician of the modem age was Joe Montana. "Imagine," Bill explains, "the trigonometry involved in hurling a football just past the outstretched arms of an all-pro cornerback into the hands of a moving receiver at a distance of sixty yards. But that's the same geometry and calculus involved in tossing a crumpled sheet of scratch paper to a peer, or judging whether you have enough room to pass on a two lane highway." Bill concludes, "You're performing operations of geometry, trigonometry and calculus all the time. All we're asking you to do in this class is algebra!"

In Ericksonian terms, Bill is setting up a *response set*. Each of his examples represents a verifiable truism. Students can't deny that to toss the paper, they had to have an intuitive sense of velocity, gravity, angles, mass and wind resistance. After agreeing with each of those ideas, they're ready to agree with the next plausible idea Bill presents, namely that algebra is relatively easy.

PARADOX, CONFUSION AND PSYCHOLOGICAL SHOCK

Erickson was fond of using *paradox*, *confusion* and *psychological shock* to open students to new ideas, to shake them free of rigid and limiting mind sets. I use paradoxical instruction to change the way students look at their math books. These texts are filled with useful resources, yet students typically open them only to copy down assigned problems. From a local used book store I purchase a dozen assorted, out-of-print texts for about $2 each. After putting students in small groups, I give the following instruction: "Find a glossary of terms and bring it to me. Do not bring up the book." After a few confused looks, the question is inevitably asked, "How can we bring it to you without the book?" I give them a reassuring look and repeat the instruction. Inevitably, yet tentatively, a tearing sound is heard. Soon the room is buzzing with gasps, giggles and finally a great collective sigh as tension dissolves. Students eagerly litter my desk with every resource the text has to offer. By the end of class, they are left with little more than book covers.

As this example illustrates three Ericksonian concepts, it deserves a closer look. First there was the *paradoxical suggestion*, roughly, *bring it to me but don't bring it to me*. This instruction acts to replace the students' state of complacency with one of *confusion*. Yet my demeanor remains reassuring, not mocking or competitive. When one of them finally tears a page from the text, the rest are thrown into a state of mild *psychological shock*. Sacrilege has been committed, yet the sky does not fall. This creates a space of doubt in their belief systems. The pleasure of tearing up old math books fills that space with the new ideas like *math can be fun* and *exploring a text can be fun and informative*.

Laughter in a mathematics classroom is not an event to be taken lightly. For students whose ingrained emotional response to math is anxiety, anger and embarrassment, it is a powerful experience to laugh and think math in the same breath. On a visceral level, Ericksonians recognize that such pairing can result in a lessening of math anxiety. On a metaphoric level, tearing up math books provided a release for anger so closely associated with past math problems which, not incidentally, came from similar math books. The use of used texts is not completely a monetary decision. We are destroying the *old* ways, releasing ourselves of the power of the past—not the new.

I could just as easily have directly assigned them to look up a list of textbook resources and write down the page number to verify they'd done so. Not only would that be a dry, utilitarian exercise, but little or no memory trace would have attached.

INDIRECTION AND METAPHOR

Obviously there are times when an instructor needs to be direct and to the point. But when complacency, frustration and math anxiety inhibit the learning process, it's time to employ *indirection*. Erickson is most fondly remembered for the stories he told, each one indirectly illustrating a particular idea he wanted a client to grasp. Some were motivational, others instructional, and others rapport building.

Metaphors provide a conveniently indirect way to bring someone to an understanding without engaging confrontation or inviting rejection. Struggling math students have experienced so many failures that many will reject any idea they recognize consciously as a new intervention. But when a student deciphers the metaphor on one's own, there is a sense of ownership which allows the student to accept the new idea as a valid option.

Erickson liked to layer his metaphors one inside another. My preference is to slip a metaphor into a logic puzzle.

> Billy and Maria were out playing when they spotted a train coming towards them. Billy ran one way and Maria ran another. Billy ran directly away from the train while Maria ran towards the speeding locomotive. Billy got run over. Maria escaped to play another day. How might one explain this curious occurrence? (*The answer is at the end of this article.*)

Before checking your answer, count how many ideas the story illustrates. First, it describes the danger of meeting a challenge head on. Second, it demonstrates how it is worse to run away. Third, it suggests a new direction to move. The beauty of the puzzle is that when they hear the answer, students must admit that it was better not to avoid the challenge. A fourth idea is that being stronger, whiter, or maler is no advantage.

Fifth, should we criticize Billy? No. Both kids acted to prolong their lives—Billy by a few seconds, Maria by 70 years. Still, Billy's intention was positive—and he ran as fast as he could—faster than Maria. Struggling math students try even harder. They just need to be pointed in the right direction. Then their hard work pays double.

Sixth, notice the phrase, *curious occurrence* at the end of the puzzle. With two little words, a gory story is reframed as an enticing enigma, i.e., what used to cause math anxiety can alternatively lead to satisfaction.

Seventh, beginning the question, *How might one...* provides an invitation rather than a command. Use of the third person singular *one* further distances the student from any threat that solving puzzles might present. It can be an enlightening experience for an instructor to tape one's instructions to students and examine the many meanings that struggling students might attach. To misquote Thoreau, "The unexamined instruction is not worth giving."

ORDEAL OF CHOICE

Erickson would never suggest that teachers sugarcoat their instructions. On the contrary, he often motivated his clients by presenting a narrowed list of choices in what became known as an *ordeal of choices*. In MSO, this consists of determining a learning objective which requires active student participation such as completing homework, practicing new skills or doing in class presentations. In my experience, the majority of struggling math students also loathe presenting for their peers. These same students are usually too passive or insecure to participate in their required math courses, where they fear looking foolish if they risk asking questions.

The ordeal of choices I present to them is to either conduct an interview with one's math instructor or do a brief presentation for the class. The crucial factor is how the options are presented. First I describe the many benefits to be gained by doing an oral presentation. Although the potential benefits are real and my manner compassionate, the description taps into a considerable amount of anxiety. By the end, students are turning several shades of green. Only then do I let them know there is an alternative, i.e., to interview one's instructor. Their collective sigh can be heard three classrooms away. I then give them easy step-by-step instructions on how to conduct the interviews which most then opt for and complete on time. After the interviews, they report back that they now ask many more questions in class. Had I presented the assignments in reverse order, many would have chosen oral presentations in order to avoid the interviews. Either way, motivated to avoid an ordeal, they are more likely to complete one of the tasks.

RELATING TO STUDENTS

Person Centered Mathematics is about relating to students in ways that provide validation and motivation. Students often view math instructors as possessing knowledge far out of their reach. Traditional math instruction reinforces this chasm between instructor and student. At the other extreme, Carl Rogers renamed his therapy from "Client Centered" to "Person Centered" to eliminate all hierarchical distinctions. While my students are confident that I know the subject matter, they don't see me as that different from themselves. When I arrive on Halloween as a math magician, "The Wizard of Odds," I am lampooning my own position as hallowed font of all knowledge mathematical. Each semester I register for some impossible math course at a nearby university so my students can watch me struggle.

Back in MSO, I make it my goal to visit briefly with as many students as possible each class. I also try to make at least one mistake each class period. It's hard for students to get down on themselves for "stupid mistakes" when they see me making them. In the tutoring portion of each class, every available board is in use. I watch from the middle of the room although my presence is intentionally peripheral. I pitch in as needed, but whenever possible, I maneuver students into peer tutoring situations. There comes a point midsemester when the class has passed into students' hands. I can usually identify this as the day I arrive ten minutes late and no one notices.

CONCLUSION

Most colleges offer some sort of math assessment test to determine the appropriate level of study for new students. These tests do not take into account human factors that lead to math anxiety and lack of confidence. PCM bridges that gap, dissolving anxiety and increasing confidence so that students' true math potential can be realized.

PCM is not just a fun way to do algebra. All the strange and different experiences have legitimate educational objectives. They also serve to distract students from their own worst fears. No hypnotic subject has ever been able to identify the exact moment he or she entered trance. It happens precisely because the subject stops worrying whether it will. Distracted from debilitating self-doubt, students are pleasantly surprised to discover that they are actually doing and understanding mathematics. Once that attitude shift occurs, they no longer need our course. In succeeding semesters, they can create their own support systems and learning networks to complement their improved math self concepts.

ANSWERS TO RIDDLES

First Answer: His wife had delivered a baby and died of complications.

Second Answer: Billy and Maria were in a tunnel. By running away, Billy delayed by a few seconds being run over. Maria on the other hand, ran out the entrance of the tunnel, then jumped out of the way, just before the train entereed and ran down her fleeing friend.

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ADDENDUM: STATISTICAL COMPARISON OF PRE VS. POST INTERVENTION AND INTERVENTION VS. CONTROL GROUP

The Math Success Orientation class (MSO) at Glendale Community College (GCC) was designed to serve students who performed at the college level in all areas but math. Because the course proved so successful, MSO students were compared to a control group who had graduated from the same high schools in the same years and had comparable math assessment test scores. While the control group could be matched for age, gender, ethnic group, and prior math history, the students who took Math Success began as a far greater challenged math group. Whereas the control group's prior history included 52.7% already succeeding with at least C's in math courses, only 21.6% of the pre-Math Success students could say the same. The control group exceeded the pre-Math Success group in percent of A's, B's and C's.

The first chart compiles grades received by each group in all required Algebra courses taken prior to the introduction of Math Success to the campus. These courses included Introductory, Intermediate and College Algebra. The Algebra courses were taught by a variety of instructors on campus who had no connection to the MSO course. MSO students chose their Algebra courses independently and in most cases were the only MSO student in that particular class. *Grades received in the one credit MSO class are not included in these comparisons*. The great majority of students in both the control and Math Success group had been full-time college students for no more than two semesters prior to the intervention semester. Since the total number of classes taken is not the same for bothgroups, the percentages are the most important statistics to note.

PRE-INTERVENTION

Math Grades prior to Semester of intervention	A	B	C	D	F	Y	W
Control Group	2	10	7	3	2	3	9
	5.6%	**27.7%**	**19.4%**	8.3%	5.6%	**8.3%**	25%
Math Success group	1	1	3	5	6	0	7
	4.3%	4.3%	13%	**21.7%**	**26.1%**	0%	**30.4%**

(numbers in bold indicate highest percentage with grade)

The next chart compares the math grades received by students during a semester in which they also took Math Success, to the grades of the control group in the same semester. Note that 58.9% of the Math Success group passed with A's, B's and C's compared to only 36.3% of controls.

POST-INTERVENTION

Math Grades	A	B	C	D	F	Y	W
Control Group	1	4	7	4	9	0	8
	3%	12.1%	21.2%	**12.1%**	**27.3%**	0%	24.2%
Math Success	7	13	13	4	2	0	17
	12.5%	**23.2%**	**23.2%**	7.1%	3.6%	0%	**30.4%**

(numbers in bold indicate highest percentage with grade)

The third chart demonstrates that when the percentage of change is calculated for both groups, the control group grades dip while MSO students' grades rise significantly.

PERCENTAGE OF CHANGE

Math Grades	A	B	C	D	F	Y	W
Math Success Group	**+8.2%**	**+18.9%**	**+13.2%**	-14.6%	-22.5%	0%	0%
Control Group	-2.6%	-15.6%	+1.8%	**+3.8%**	**+21.7%**	-8.3%	-.8%

(numbers in bold indicate highest percentage of increase)

Since the control group's grades dipped in the post-intervention semester we had to consider whether they might represent some aberration and not be representative of their typical grades. Therefore, the fourth chart lumps together all math classes taken by the control group before and after intervention, and compares them to the Math Success group's grades received after the intervention. Yet again, the MSO students fared significantly better.

Post-Intervention MSO vs. Control Group in All Semesters

Math Grades	A	B	C	D	F	Y	W
Control Through Spring 1997	5	16	6	8	13	3	25
	6.6%	21.1%	7.9%	**10.5%**	**17.1%**	**3.9%**	**32.9%**
During Math Success	7	13	13	4	2	0	17
	12.5%	**23.2%**	**23.2%**	7.1%	3.6%	0%	30.4%

(numbers in bold indicate highest percentage of increase)

Finally, we had to consider whether the entire control group was somehow aberrant and not representative of the pool from which it was taken. The final chart lumps together all classes taken by all students in the pool which consisted of the 182 non-ACE Plus, non-Honor students who matriculated to GCC from the same high schools in the same years, 1994 and 1995. Honor students were excluded because ACE Plus does not accept them into its programs, meaning that the MSO group contained none.

Pre-Intervention Grades

	A	B	C	D	F	Y	W
MSO (39 students)	4.3%	4.3%	13%	**21.7%**	**26.1%**	0%	**30.4%**
Pool (182 students)	**10.4%**	**22.0%**	**18.1%**	10.4%	15.4%	**2.2%**	21.4%

(numbers in bold indicate highest percentage with grade)

Post-Intervention Grades

	A	B	C	D	F	Y	W
MSO	**12.5%**	**23.2%**	**23.2%**	7.1%	3.6%	0%	30.4%
Pool	9.8%	11.1%	16.5%	12.0%	17.3%	**1.3%**	**32.3%**

(numbers in bold indicate greatest improvement)

Summary of Results

1. MSO markedly improved grades in required algebra courses.
2. Previously at-risk, low performing math students improved to a level significantly higher than their peer group.
3. Students in MSO classes improved their grades from first to second semester while their peers showed a marked drop in grades.

A FINAL NOTE

Ron Bell and colleague Dr. John Coles of Truckee Meadows Community College presented Person Centered Mathematics to the American Counseling Association at the 2000 national conference.

In the time since Bell wrote this article in 1997, he has left the community college at which the program was begun to become Counseling Faculty at Southwestern Oregon Community College. Even without Bell's constant guidance, the strong foundation of the course has enabled it to have its availability expanded to ten course sections under the name "MAT 108 Tutored Math." It is remarkable that the course has achieved transfer level, considering that the courses it is designed to support are not.

Mathematics and Cultural Diversity in the Curriculum

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Author's Note: The following remarks were made to a university-wide audience at the second workshop in a series of workshops on cultural diversity at the university and in the curriculum. The comments were made in May 1993, but they are as relevant today as they were four years ago.

I would like to take a few minutes to talk about how I see mathematics fitting into the general discussion of multicultural and international diversity. When I attended this workshop last year I was unsure of the role of mathematics in these efforts. So were many other people. But in the year since then I have come to a very clear understanding of its role.

Mathematics has been and is part of every human culture. From the empirical geometries of the Egyptians and Babylonians to the formal geometry of Greece and the dynamic geometries of the Navaho and Inuit Indians; from the stylized algebra of the Chinese to the manipulative algebra of the Arabs and the calculus of the Europeans, every culture has a mathematical heritage. Every culture does mathematics.

Mathematics is a culturally based, human endeavor, but it has an importance that transcends its mere existence. In some sense mathematics forms a commonality across all cultures. Indeed, modern "western" mathematics is a truly international effort. It has a communication scheme transcending language barriers.

I currently am looking at the proceedings of a conference held in Germany to honor a German mathematician. There were 69 participants from 17 countries. The book, published in Germany with an introduction in German, is entirely in English. Viable and dynamic practitioners are found in every part of the world: China and Japan, the former SU, the Western Industrial Nations, Vietnam, Iran, Israel, Egypt, Brazil and Pakistan, to name a few locations.

Why should this be so? To quote Lynn Steen from *On the Shoulders of Giants*:

> What humans do with the language of Mathematics is to describe patterns. Mathematics is an exploratory science that seeks to understand every kind of pattern—patterns that occur in nature, patterns invented by the human mind, and even patterns created by other patterns.

Even more than being a language or being a science, mathematics is a way of thinking. People are thinking mathematically whenever they do the following list of activities, taken from the document *Everybody Counts* by the National Research Council.

Modeling: Representing worldly phenomena by mental constructs, often visual or symbolic, that capture important and useful features.

Optimization: Finding the best solution (least expensive or most efficient) by asking "what if" and exploring all possibilities.

Symbolism: Extending natural language to symbolic representation of abstract concepts in an economical form that makes possible both communication and computation.

Inference: Reasoning from data, from premises, from graphs, from incomplete and inconsistent sources.

Logical Analysis: Seeking implications of premises and searching for first principles to explain observed phenomena.

Abstraction: Singling out for special study of certain properties common to many different phenomena.

So where does this get us? To quote the character of Mr. Escalante in the movie *Stand and Deliver*: "Mathematics is the great equalizer."

The best indicator of salary ten years after high school graduation is the amount of mathematics studied. The ten best jobs in the United States—rated on salary, stability, stress, and future—all need a mathematics background.

To paraphrase Dr. Marable, who spoke here on Martin Luther King Day:

> In the 1990s a new segregation characterizes American society: unequal access to education in mathematics, technologies, and sciences results in people being excluded from full status as functioning, contributing members of society. This segregation is seen particularly among individuals outside the mainstream of American society—individuals from ethnic minorities, from rural areas, and among women.

I would like to argue here that the issue is even greater than having individuals fit in the American society at large. I see mathematics as a great equalizer—a form of empowerment—even within the ethnic and racial cultures of the nation. In writing a paper for my Mathematical Ideas course, a student came to this same realization. He wrote:

> ...I was born in the inner-city called Gary, IN, the product of a broken marriage and a father that I never knew. I've seen all of the males on my fathers side go to jail, run away, or get killed...In August I will become the first person in the history of my family (on either side) to graduate from college. I will also become the first male on my father's side to do something legit (non-illegal), and no thanks to math... By the time of my senior year in high school I was the top history student in my school. I soon went on to star in that category on our school's academic superbowl team. Subsequently, I was accepted into Valparaiso University, with no thanks to math.
>
> In high school I took algebra one, algebra two, and geometry... For one thing the fact that I'm black basically excludes me from the "normal" American culture in the first place, so I don't need math to exclude me from a culture that I don't exist in. Secondly, math is not needed to be a part of the black community. And lastly, I don't need math to fulfill my obligation as an American citizen, because in my opinion blacks are not American citizens.

Yet later in the paper he wrote:

> ...When I first began writing this paper I was in a terrible funk, when concerning mathematics. But as I started writing my paper it became increasingly difficult to think of new points against math, and this is when I realized how immense my task was. In an attempt to gain further insight I asked one of my friends, Mr. Elliott Fourté, his opinion on this paper. Elliott told me that he couldn't give me any ideas on this paper for he thought that mathematics and life success were synonymous. He also told me that though I didn't feel that I've had a lot of math, that three years of math is a lot of math when looked at from a global and national standpoint.
>
> After thinking about his I've realized that I've failed to give math enough credit... I've also come to realize that the statement "blacks don't need math" could be a cop out, and part of the reason that we as blacks are in our current situation could be attributed to a lack of science and math skills. Blacks in America's economic situation is very similar to that of third world countries, and the technology that comes from math and science seems to be the thing that separate the industrialized nations from the non-industrial.

What then is the role of mathematics in a world of diversity? This role is obvious: it is to provide individuals with the tools needed to be empowered in their own culture, and to provide them with a common language that can, and does, transcend cultural barriers.

How can mathematicians fulfill that demanded of them by their discipline? By the time an individual is old enough to attend college, he or she has acquired these tools, or it is frequently too late to provide them with the tools they need. It is therefore imperative that practicing mathematicians take a positive role in K-12 mathematics education by participating in professional development and curriculum projects, as well as by working with young students themselves.

In Future Issues...

Michael E. Goldberg
When Is a Math Problem Really "Real"?

George W. Hart
Loopy

David Joyner
Notes on Formal Constructivism

Pat Kenschaft
Pat's Prologues: Introductions to the First Nine Airings of Math Medley, A Radio Talk Show

Lawrence Neff Stout
Aesthetic Analysis of Proofs of the Binomial Theorem

Humanistic Mathematics Network Journal E-mail List and Website

The HMNJ has an email list which allows contact among the readers. To subscribe, send email to: listkeeper@hmc.edu with the message: "Subscribe HMNJ-L@hmc.edu" in the body of the email. After you have subscribed, to send mail to the list, write to HMNJ-L@hmc.edu and your email will be forwarded to everyone on the list.

The HMNJ is also working on constructing a website. You can see what progress we are making at http://www3.hmc.edu/~jnelson. While it may be a while until the website is fully functional, we hope you will contribute your suggestions for improvements and ideas for functions to jnelson@hmc.edu.

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