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Invitation to Authors

Essays, book reviews, syllabi, poetry, and letters are welcomed. Your essay should have a title, your name and address, e-mail address, and a brief summary of content. In addition, your telephone number (not for publication) would be helpful.

If possible, avoid footnotes; put references and bibliography at the end of the text, using a consistent style. Please put all figures on separate sheets of paper at the end of the text, with annotations as to where you would like them to fit within the text; these should be original photographs, or drawn in dark ink. These figures can later be returned to you if you so desire.

Two copies of your submission, double-spaced and preferably laser-printed, should be sent to:

Prof. Alvin White
Humanistic Mathematics Network Journal
Harvey Mudd College
Claremont, CA 91711

Essays and other communications may also be transmitted by electronic mail to the editor at AWHITE@HMC.EDU, or faxed to (909) 621-8366. The editor may be contacted at (909) 621-8867 if you have further questions.

The Journal uses the programs Microsoft Word and PageMaker for the Macintosh. If you used either of these programs to prepare your paper, it would be very helpful to us if you sent a copy of your file on diskette along with the printed copies of the paper.

Note to Librarians

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Subscriptions

The Humanistic Mathematics Network Journal is distributed free to all libraries and individuals who request copies and subscriptions. Please address subscription requests to Prof. Alvin White.

Cover

A great mathematician: the cover features pictures of Karl Menger taken at different stages of his life. "A Tribute to Karl Menger" is the feature article in this issue of the Journal. Seymour Kass' account of Menger's life has been reprinted along with responses from Menger's daughter and colleagues.

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EXXON EDUCATIONAL FOUNDATION.

Humanistic Mathematics Network Journal #16
November 1997

The poems located throughout the journal are by David L Finn, Merrimack College 315 Turnpike Street, North Andover, MA 01845. They were inspired by last issue's interview with Reuben Hersh "What kind of thing is a number?"

From Newsletter #1

Dear Colleague,

This newsletter follows a three-day **Conference to Examine Mathematics as a Humanistic Discipline** in Claremont 1986 supported by the Exxon Education Foundation, and a special session at the AMS-MAA meeting in San Antonio January 1987. A common response of the thirty-six mathematicians at the conference was, "I was startled to see so many who shared my feelings."

Two related themes that emerged from the conference were 1) teaching mathematics humanistically, and 2) teaching humanistic mathematics. The first theme sought to place the student more centrally in the position of inquirer than is generally the case, while at the same time acknowledging the emotional climate of the activity of learning mathematics. What students could learn from each other and how they might come to better understand mathematics as a meaningful rather than arbitrary discipline were among the ideas of the first theme.

The second theme focused less upon the nature of the teaching and learning environment and more upon the need to reconstruct the curriculum and the discipline of mathematics itself. The reconstruction would relate mathematical discoveries to personal courage, discovery to verification, mathematics to science, truth to utility, and in general, mathematics to the culture within which it is embedded.

Humanistic dimensions of mathematics discussed at the conference included:

a) An appreciation of the role of intuition, not only in understanding, but in creating concepts that appear in their finished versions to be "merely technical."

b) An appreciation for the human dimensions that motivate discovery: competition, cooperation, the urge for holistic pictures.

c) An understanding of the value judgments implied in the growth of any discipline. Logic alone never completely accounts for what is investigated, how it is investigated, and why it is investigated.

d) A need for new teaching/learning formats that will help discourage our students from a view of knowledge as certain or to-be-received.

e) The opportunity for students to think like mathematicians, including chances to work on tasks of low definition, generating new problems and participating in controversy over mathematical issues.

f) Opportunities for faculty to do research on issues relating to teaching and be respected for that area of research.

This newsletter, also supported by Exxon, is part of an effort to fulfill the hopes of the participants. Others who have heard about the conferences have enthusiastically joined the effort. The newsletter will help create a network of mathematicians and others who are interested in sharing their ideas and experiences related to the conference themes. The network will be a community of support extending over many campuses that will end the isolation that individuals may feel. There are lots of good ideas, lots of experimentation, and lots of frustration because of isolation and lack of support. In addition to informally sharing bibliographic references, syllabi, accounts of successes and failures. . . the network might formally support writing, team-teaching, exchanges, conferences. . . .

Alvin White
August 3, 1987

From the Editor

In the statement attributed to George Polya on page 51 of issue #15, two words were transposed. The statement should have read: **"George Polya, when asked why he became a mathematician, said that he was too good to be a philosopher, and not good enough to be a physicist."** Thanks to those readers who pointed out the error.

In the July '97 issue of FOCUS, Peter Renz ends his editorial with the words of Roger Godement in the preface of his book *Algebra* which is worth repeating again:

> [I disagree] with the large number of public personalities at the present time who demand of scientists in general , and mathematicians in particular, that they should devote their energies to producing the legions of technologists whose existence, it appears, is urgently indispensable for our survival. Things being as they are, it seems to us that in the scientifically and technologically over-developed "great" nations in which we live, the first duty of a mathematician—and of many others—is to produce what is *not* demanded of him, namely men who are capable of thinking for themselves, of unmasking false arguments and ambiguous phrases, and to whom the dissemination of truth is infinitely more important than, for example, world-wide three-dimensional T.V.: free men, and not robots ruled by technocrats. It is sad but true that the best way of producing such men does not consist in teaching them mathematics and physical science; for these are the branches of knowledge which ignore the very existence of human problems, and it is a disturbing thought that our most civilized societies award them the first place. But even in the teaching of mathematics it is possible to attempt to impart a taste for freedom and reason, and to accustom the young to being treated as human beings endowed with the faculty of reason.

The (US) National Science Foundation report NSF96-139 "Shaping the Future" quotes Eli Noam from *Science*, 10/13/95, pages 247-249:

> "The scenario suggests a change of emphasis for universities. True teaching and learning are about more than information and its transmission...(are) based on mentoring, internalization, identification, role modeling, guidance, socialization, interaction, and group activity. In these processes, physical proximity plays an important role. Thus, the strength of the futrue physical university lies less in pure information and more in college as community; less in wholesale lecture, and more in individual tutorial; less in Cyber-U, and more in Goodbye-Mr.-Chips College. Technology would augment, not substitute."

Copies of that report are available from **pubs@nsf.gov** or **(703) 306-1130.**

At the January '98 meeting in Baltimore, the Humanistic Mathematics Network is sponsoring AN EVENING OF POETRY, Thursday, Jan. 8, 7-9 pm.

A Tribute Karl Menger

Article by Seymour Kass
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Originally printed as "Karl Menger" in May 1996 edition of Notices of the AMS :Volume 43, Number 5.

"His office was a showplace of chaos, the desktop covered with a turbulent sea of papers. He knew the exact position of each scrap. On the telephone he could instruct a secretary exactly how to locate what he needed. Once, in his absence, a new secretary undertook to 'make order', making little stacks on his desk. Upon his return, discovering the disaster, he nearly wept, because 'Now I don't know where anything is.'"

Karl Menger died on October 5, 1985, in Chicago. Except in his native Austria [5], no obituary notice seems to have appeared. This note marks ten years since his passing.

Menger's career spanned sixty years, during which he published 234 papers, 65 of them before the age of thirty. A partial bibliography appears in [15]. His early work in geometry and topology secured him an enduring place in mathematics, but his reach extended to logic, set theory, differential geometry, calculus of variations, graph theory, complex functions, algebra of functions, economics, philosophy, and pedagogy. Characteristic of Menger's work in geometry and topology is the reworking of fundamental concepts from intrinsic points of view (curve, dimension, curvature, statistical metric spaces, hazy sets). A few of these and some of his other accomplishments will be mentioned here.

Menger was born in Vienna on January 13, 1902, into a distinguished family. His mother, Hermione, was an author and musician; his father, Carl Menger, well known as a founder of the Austrian School of Economics, was tutor in economics to Crown Prince Rudolph (the illstarred Hapsburg heir-apparent, played by Charles Boyer in "Mayerling").

From 1913 to 1920 Menger attended the Doblinger Gymnasium in Vienna, where he was recognized as a prodigy. Two of his fellow students were Nobel Laureates Richard Kuhn and Wolfgang Pauli. He entered the University of Vienna in 1920 to study physics, attending the lectures of physicist Hans Thirring. Hans Hahn joined the mathematics faculty in March 1921, and Menger attended his seminar "News about the Concept of Curves". In the first lecture Hahn formulated the problem of making precise the idea of a curve, which no one had been able to articulate, mentioning the unsuccessful attempts of Cantor, Jordan, and Peano. The topology used in the lecture was new to Menger, but he "was completely enthralled and left the lecture room in a daze" [10, p. 41]. After a week of complete engrossment, he produced a definition of a curve and confided it to fellow student Otto Schreier, who could find no flaw but alerted Menger to recent commentary by Hausdorff and Bieberbach as to the problem's intractability, which Hahn hadn't mentioned. Before the seminar's second meeting Menger met with Hahn, who, unaccustomed to giving first-year students a serious hearing, nevertheless *listened* and after some thought agreed that Menger's was a promising attack on the problem.

Inspired, Menger went to work energetically, but having been diagnosed with tuberculosis, he left Vienna to recuperate in a sanatorium in the mountains of Styria, an Austrian counterpart of "The Magic Mountain". He wrote later that at the same time Kaika was dying in another sanatorium. In Styria he elaborated his theory of curves and dimension, submitting a paper to *Monatshefte fur Mathematik und Physikin* 1922 which contained a recursive definition of dimension in a separable metric space. P. S. Urysohn simultaneously and independently of Menger developed an equivalent definition. The Menger/Urysohn definition has become the cornerstone of the theory [6].

After receiving a PhD. in 1924, Menger, interested in L E. J. Brouwer's fundamental work in topology and wanting to clarify his thought about intuitionism, which he saw as the counterpart in mathematics to Ernst Mach's positivism in science, accepted Brouwer's invitation to Amsterdam, where he remained two years as docent and Brouwer's assistant. Though he found Brouwer testy, he retained warm feelings about his stay in Amsterdam and cited the good Brouwer did for some young mathematicians and "the beautiful experience of watching him listen to reports of new discoveries" [15].

In 1927, at age 25, Menger accepted Hahn's invitation to fill Kurt Reidemeister's chair in geometry at the University of Vienna. During the year 1927-28, in preparing a course on the foundations of projective geometry, Menger set out to construct an axiomatic foundation in terms of the principal features of the theory: the operations of joining and intersecting, which Garrett Birkhoff later called lattice operations. Menger was critical of Hilbert's and Veblen's formulations: Hilbert's requiring a different primitive for each dimension; Veblen's giving points a distinguished role not played in the theory itself, since hyperplanes play the same role as points. Von Neumann in [17] took a line parallel to Menger's, seeking "to complete the elimination of the notion of point (and line and plane) from geometry". He refers to Menger as "the first to replace distinct classes of 'undefined entities' by a unique class which consists of all linear subspaces of the given space, an essential part of his system being the axiomatic requirement of a linear dimensionality function." Menger subsequently gave a self-dual set of axioms for the system.

In 1928 he published *Dimensiontheorie* (Teubner). Fifty years later, J. Keesling wrote: "This book has historical value. It reveals at one and the same time the naivete of the early investigators by modern standards and yet their remarkable perception of what the important results were and the future direction of the theory" [7]. The authors of [10] illustrate the remark with Menger's theorem that every n-dimensional separable metric space is homeomorphic to part of a certain "universal" n-dimensional space, which can in turn be realized as a compact set in $(2n+1)$-dimensional Euclidean space. The universal 1-dimensional space, (the "Menger universal curve" or "Menger sponge"), appears in Mandelbrot's *The Fractal Geometry* [9] and in [2]. Menger's *Kurventheorie* appeared in 1932 (Teubner; Chelsea reprint, 1967). It contains Menger's n-Arc Theorem: Let G be a graph with A and B two disjoint n-tuples of vertices. Then either G contains n pairwise disjoint AB-paths (each connecting a point of A and a point of B), or there exists a set of fewer than n vertices that separates A and B. Menger's account of the theorem's origins appears in [11], an issue of the *Journal of Graph Theory* dedicated to his work. In an introductory note Frank Harary calls it "the fundamental theorem on connectivity of graphs" and "one of the most important results in graph theory". Variations of Menger's theorem and some of its applications are given in Harary's *Graph Theory* (Addison-Wesley, 1969).

In that same year Menger joined philosopher Moritz Schlick's fortnightly discussion group, which became known as the Vienna Circle [10], [18]. In 1931 in tandem with the Circle, Menger ran a mathematical colloquium in which Rudolph Carnap, Kurt Godel, Alfred Tarski, Olga Taussky, and others took part. Godel (also a Hahn student) first announced his epoch-making incompleteness results at the colloquium. Menger edited the series *Ergebnisse eines Mathematischen Kolloquiums* in the years 1931-37. Additionally, during this period a series of public lectures in science was held. Einstein and Schroedinger were among the speakers. Admission was charged, and the money generated was used to support students.

In the posthumously published *Reminiscences of the Vienna Circle and the Mathematical Colloquium* [10] Menger gives perceptive and lucid accounts of the philosophical and cultural atmosphere in Vienna, the personalities that streamed through the Circle and those who participated in the colloquium, and the topics discussed and issues that engrossed the participants. There are chapters on Wittgenstein and a moving account of Menger's long association with Godel. There is also an account of Menger's time at Harvard and The Rice Institute (1930-31), where he met most of the leading U.S. mathematicians of the day. Though he recognized E. H. Moore as the father of American mathematics, Percy Bridgman and Emil Post made the strongest impression on him. Menger regarded Bridgman as the successor to Mach. And he had great admiration for Post, who (with Haskell Curry) is a source for Menger's later work in the algebra of functions.

In the early 1930s Menger developed a notion of general curvature of an arc A in a compact convex metric space. Consider a triple of points of A, where A is an ordered continuum, not necessarily described by equations or functions. The triangle inequality implies the existence of three points in the Euclidean plane isometric to the given triple, and their Menger curvature is the reciprocal of the radius of the circumscribing circle. This curvature is zero if and only if one of the points is between the other two. Menger defined the curvature at a point of A to be the number (if it exists) from which the curvature of any three sufficiently close points in the Euclidean plane differs arbitrarily little. Numerous results and modifications of Menger's concept were obtained by his student Franz Alt and by Godel [4]. The extension to higher-dimensional manifolds was achieved by Menger's student Abraham Wald (later a distinguished statistician), who obtained a fundamentally new way of introducing Gaussian curvature. Menger's comment: "This result should make geometers realize that (contrary to the traditional view) the fundamental notion of curvature does not depend on coordinates, equations, parametrizations, or differentiability assumptions. The essence of curvature lies in the general notion of convex metric space and a quadruple of points in such a space" [12].

Menger had a lifelong interest in economics [15]. Oscar Morgenstern reports that Menger's paper "Das Unsicherherheitsmoment in der Wertlehre" (1934) played a primary role in persuading von Neumann to undertake a formal treatment of utility [8]. Two essays of Menger's appear in *Economic activity analysis*, the 1954 Princeton Economics Research Project's collection of essays edited by Morgenstern.

With Hitler's coming to power in 1933, Austrian agitation for unification with Germany intensified. With the resulting turmoil and street violence and Chancellor Dolfuss taking dictatorial powers in 1934, the vigorous intellectual life in Vienna atrophied. Extreme pro-German "nationalists" ruled the faculty and the student body of the university, and the Circle was disparaged and maligned. Hahn, a progressive force, had died in 1934, and in June 1936 Schlick, founder of the Circle, was shot dead by a deranged student. Still stunned by the tragedy, Menger attended the 1936 International Congress of Mathematicians in Oslo, becoming one of its vice-presidents. He described the deteriorating situation to friends and associates. Shortly thereafter he received a cable offering him a professorship at the University of Notre Dame. He and his family arrived in South Bend in 1937. In March 1938, the month of the *Anschluss*, Menger resigned his professorship in Vienna. He did not return until the 1960s.

European intellectuals who fled Europe for refuge in America were sometimes uncomfortable with American ways and unaccustomed to teaching elementary courses. Though central European in dress, manner, and style, Menger felt at home in America and enjoyed teaching undergraduates, believing that when properly done, it stimulated research During the 1960s he lectured to high school students on the subject "What is x?".

His fourth child was born in 1942, and the others were then reaching school age. Menger was drawn to Chicago, where Carnap and others were developing a sort of "Chicago Circle" and where his children's education would benefit from a cosmopolitan environment. L. R Ford, whom he had met at Rice in 1931, had become IIT mathematics department chairman. Thus in 1948 Menger came to IIT and remained in Chicago the rest of his life.

In the note "Statistical Metrics" [14] Menger made a contribution to resolving the interpretative issue of quantum mechanics. He proposed transferring the probabilistic notions of quantum mechanics from the physics to the underlying geometry. He showed how one could replace a numerical distance between points p and q by a distribution function Fpq whose value $Fpq(x)$ at the real number x is interpreted as the probability that the distance between p and q is less than x. Studies of such spaces by numerous authors followed, as did the book *Probabilistic metric spaces* (North Holland, 1983), by Berthold Schweizer and Abe Sklar.

In 1951 Menger introduced the concept of a "hazy set" [16], in which the element-set relation is replaced by the probability of an element belonging to a set. Hazy sets were rediscovered and renamed "fuzzy sets" in [1] and have become the subject of a broad research area.

In the war years Menger taught calculus to future navy officers in the V-12 program, which prompted him to rethink the foundations of the subject. It led to the monograph *Algebra of analysis* [13] and some papers which aimed to systematize and clarify the foundations of analysis, and it rekindled his interest in the algebra of functions. It also acquainted him with the calculus textbooks of the time, which he found made scant distinction between f and $f(x)$, and in which specific functions, used everywhere in calculus—like the identity function and selector functions for ordered n-tuples—were not identified. But Menger was operating in the framework of the Vienna Circle and the emphasis there on clear thought and expression. One might guess that he found an assault on the traditional calculus irresistible. Thus Menger's attempt to reform calculus: *Calculus, a modern approach* (Ginn, 1955). Written in characteristically vigorous style, it was a radical revision of textbooks of the period, scrapping some traditional notation. It received a lengthy, thoughtful, and cautiously favorable review in the *Monthly* by H. E. Bray 13] but was never accorded serious attention. He sent a copy to Einstein, who replied that he liked it and recognized the need for some clarity in notation, but advised against attempting too much "housecleaning".

That his book was ignored saddened Menger's later years. When Menger addressed the foundations of dimension theory, topology, projective n-space, or differential geometry, attention was paid by the best mathematical minds of his generation: Hahn, Brouwer, von Neumann, Godel. The failure of the calculus endeavor strained his relations with the mathematical community.

Menger has been described as a fiery personality. As a junior faculty member at IIT in the 1960s, I found him gracious, charming, and vivacious. Menger was solicitous of students. From his early days in Vienna onward he invited students and faculty to his home. In Chicago it included a tour of his decorative tile collection, which lined the walls of his living room. And he sometimes invited doctoral students for early morning mathematical walks along Lake Michigan.

His office was a showplace of chaos, the desktop covered with a turbulent sea of papers. He knew the exact position of each scrap. On the telephone he could instruct a secretary exactly how to locate what he needed. Once, in his absence, a new secretary undertook to "make order", making little stacks on his desk. Upon his return, discovering the disaster, he nearly wept, because "Now I don't know where anything is."

Menger liked America. He even liked the Marx Brothers. I once met him emerging from the Clark Theater in Chicago, where "A Night at the Opera" was playing. Still suffused with laughter, all he could say was, "Funny! Funny!"

Though born into a family with ties to the Austrian crown, Menger did not like establishments. His work shows that he could shed traditional ways as called for. He was a peerless mathematician and an independent and original spirit.

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May 6, 1996

CORNING

Professor Seymour Kass
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University of Massachusetts
Boston, MA 02125

Dear Professor Kass:

Thank you so much for writing the wonderful tribute to my father. The whole family is grateful to you.

After my father passed away, the family established a fellowship through the American Mathematical Society for the mathematics winner of the International Science & Engineering Fair. For that purpose, we prepared a short biography of my father (enclosed) which is presented with the check to the awardee. However, since there are no mathematicians in this generation of Mengers, writing the intellectual biography was quite a chore and the outcome is not entirely satisfactory. With your permission, I would like to include the obituary you wrote with the pamphlet.

On a personal note, I will add that my father was also a great person. I miss him sorely.

Yours truly,

Ev. Menger

Dr. Eve L. Menger
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Professor Seymour Kass
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May 26, 1996

Dear Professor Kass,

I wish to thank you and to congratulate you the very well written article about our admirable former colleague and teacher, Professor Karl Menger in the May issue of the Notices. The last time I saw Prof. Menger was, when he waited for a Bus at a stop in Western Avenue. I passed him by car during my short stay in Chicago in the early 1980s.

I found it especially good that you did speak openly about the saddened later years of Prof. Menger. Also I recall that his desk looked unusable most the time. But he managed to survive.

As it happens often when Americans write foreign names: They, the Americans, often misspell foreign names: On page 558 of your article it should have been instead of "Hapsburg" correctly "Habsburg". On p. 560 instead of "Dolfuss" correctly "Dollfuss" (who by the way, was killed by the two Nazis Holzweber and Planetta - both were hanged).

Sincerely

Günther Goes

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

EPARTMENT OF ECONOMICS CAMBRIDGE, MASSACHUSETTS 02139-43

3 June 1996

Professor Seymour Kass
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Dear Professor Kass:

Through the good offices of Lionel McKenzie of Rochester University I have received a copy of your fine obituary article on Karl Menger (Menger the young: Carl Menger, the economist, was his father). We know Karl in economics because he showed that, with unbounded U(wealth) function, the St. Petersburgh Paradoxes reasserts itself even when $U''(W) < 0$. Also Karl Menger clarified some issues connected with the law of Diminishing Returns. And, of course, his 1931-37 Vienna *Symposia* were famous for works by Wald and by von Neumann. (Gödel, thought by many to be so odd as a person, in the Wald discussion said: "Why not let wage rates and other factor prices be determined by supply and demand so that incomes are endogenous variables and the relevant demand functions are $q_j = f^j(p_1,p_2,\ldots)$ instead of Wald's $p_j = F^j(q_1,q_2,\ldots)$. Out of the mouths of babes: that way the mathematicians would have caught up with Walras, Cassel, and other economists. We'll never know whether Wald left a lost manuscript (when leaving Hitler's Vienna) that pursued Gödel's hint.)

You say his Mother was an author and musician. I suppose that does not contradict the gossip of Gottfried Haberler my Harvard mentor: Carl married late, after conceiving a child with his housekeeper?

Here is more elevated gossip. Carl (like his brother Anton) achieved fame early. But he was thought to grow stale in his creativity also early. Some I knew feared that Karl might have inherited that pattern?

Appreciatively,

Paul A Samuelson/jm

Paul A. Samuelson

To: Seymour Kass <kass@umbsky.cc.umb.edu>

Dear Prof. Kass:

I have thoroughly enjoyed your appreciation of Karl Menger that appeared in the Notices of the AMS.

A couple of specific reactions.I recall buying during WW II a copy of Menger's preliminary version of his calculus book, entitled, "Algebra of Analysis" , Notre Dame Math Lectures, No.3. And I recall thinking: "Interesting, but it wont sell. When you reduce a subject to its skeleton all you find is bare bones."

What shocked me was to learn from your article that the non acceptance of this point of view soured Menger's later life. In the past century many calculus books promoting special points of view have been written. Most flop. What survives is determined by a Darwinian process. Of course the specialized training of professional mathematicians often leads them to think that there is a unique "right way" to present any subject (i.e., their own way).

I was also interested in the fact that Menger's father was tutor to Rudolf Hapsburg. What is known about this connection? I assume that Menger was Jewish. Rudolf had a number of Jewish connections including the owner/editor of the leading Vienna newspaper. He was a considerable liberal in an impossibly stuffy and reactionary court. The conjecture that Mayerling was murder and not suicide has a certain credibility, and the question is still raised from time to time in Austria.

Once again, many thanks for your informative article.

Phil Davis

Book Review: *GARBAGE PIZZA, PATCHWORK QUILTS, AND MATH MAGIC:Stories about Teachers Who Love to Teach and Children Who Love to Learn* by Susan Ohanian

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Garbage Pizza, Patchwork Quilts, and Math Magic: Stories About Teachers Who Love to Teach and Children Who Love to Learn. Susan Chanian. W. H. Freeman, 1992. 248pp. ISBN: 0-7167-2360-3.

How was mathematics transformed from the most hated to the most loved subject for some K-3 children?

The Exxon Education Foundation, disturbed that children turn-off mathematics by fourth grade, created "K-3 math specialists" and invited applications. Everyone who applied was given a planning grant. After a year each school whose planning grant showed any sort of promise was given an implementation grant to get both the math specialists and the new ideas into primary classrooms. This book describes the transformations of teachers and students after several years of the project.

The book is a celebration of happy, involved children and their parents. The teachers have a new sense of professionalism and confidence. The children have a sense of of ownership and excitement about their mathematical inventions and discoveries. Meaningless skill-drill is abandoned in favor of student involvement in creative opportunities for doing mathematics.

The book is filled with pictures the children have made of their activities. Patchwork quilts show symmetries and various geometric patterns. Bar graphs report on data collected. Sketches help children count the legs on farmyard animals. Garbage pizza illustrates the proportion of waste in the home. Paper gets the largest slice, followed by yard waste, food, etc. Students decorate their pizzas with samples of the appropriate trash.

Rote memorization is not present among these exuberant children who invent activities that use and extend mathematical concepts. Family Math Night when students bring their parents to school and "do" hands-on-math with them is a sell-out in projects all over the country. Parents who experience the thrill of understanding the real mathematics underlying a rote procedure they memorized years ago are not eager to join a petition drive to substitute skill-drill worksheets for the manipulative materials that help their children understand, say the geometric properties of multiplication. These parents remember their own agony in school and don't want to see it perpetuated in their children.

Institutional change occurs in districts where there is a framework for administrators, teachers and parents to work and learn new things together - with the children and for the children. Teachers find colleagues at the NCTM (National Council of Teachers of Mathematics) meetings and guidance from the NCTM Standards. Research shows that learning does not occur by passive absorption. "We torture our students with the teaching of too many 'facts' too soon...Inappropriate practice and memorization produce muddled thinking."

An experienced teacher in Orlando confessed, "I hated math. That's why I applied to the math specialist program. I hoped it would help me improve. I owed it to my students to improve...I knew I had to get better in math because I was killing these kids with my narrow computational view. I was the kill-drill queen."

As an experienced teacher the author is not an impartial observer. Her remarks enliven the book. "I began my study for the Exxon Education Foundation thinking that my job would be to record what I saw; I did not expect to be changed by it. I saw myself as merely an observer, never imagining that I would become a learner along the way. I was wrong."

The book is filled with vignettes of children and teachers doing and learning mathematics in creative and happy ways. I enthusiastically recommend it.

Reflections of Glenmount Reform Effort

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It was at this first NCTM conference that I became excited about math teaching.

REFLECTION BY GAIL R. BLACK

I was never very comfortable with math during my school years. In fact, I did all I could to avoid that dreaded topic.I was encouraged by teachers and family to pursue anything but math. Under the circumstances, it is not unusual that for about thirty five years of my life I was math-phobic. I graduated from high school and college and entered the teaching profession without a quality math experience. Despite little math knowledge, I became an accomplished teacher. However, I never really considered myself a mathematician until 5 years ago when a little towhead named Curtis entered my Kindergarten classroom. How could a little boy change my way of thinking? Well, this is where the story of my reform begins.

I had just received grant money to purchase a math manipulative package. (I knew it was about time I conquered my fear of teaching math.) Here I was with great things for children, but not much procedural knowledge with which to present it. This is the point when Ruth Miller, the mother of little Curtis, enters. Ruth was a stay-at-home mother, who was a math teacher before her child rearing days. It seemed that she longed to do more than just change diapers. When I asked for volunteers to package and organize the math games, Ruth being a frustrated, stuck-at-home mathematician answered my call--and then some.

Through classroom volunteerism and general "butting-in," Ruth championed my math education and reform. As soul "guru" of my efforts, she began to show me strategies, styles and methods of teaching mathematics to my kindergarten class. I was generally surprised at how much I was already doing, but calling it "thinking games." But, there was much more for me to learn. On a regular basis, Ruth presented math lessons to my class according to the NCTM standards. Soon I began to understand how the standards can make a difference in my teaching and my students' learning.

After a year, it was time for Curtis to leave my class. Sadly, I bid farewell to him, but gratefully, his mother continued to nurture my math education and reform efforts. During this second year at Glenmount, she was offered a part-time teaching position for selected primary and upper elementary classes, and continued to volunteer in my kindergarten classroom. She received a nominal salary for this job. However, Ruth did not keep her salary, but generously purchased much needed math manipulative for the primary grades. She also funded my way to the regional NCTM conference in New York. (She will never admit this happened.)

It was at this first NCTM conference that I became excited about math teaching. I was convinced that I was truly committed to bringing math reform to early elementary teachers at Glenmount. Through my contact with NCTM, I received a quantity of math information about standards, manipulatives, and curricula which helped me to become more confident in my math presentations. I was unable to keep my new mathematics strategies and methods to myself. From conversations with other teachers in my school, I also found that they too were somewhat uncomfortable with math teaching, but never really told anyone. I let them in on my "secret", but much to my chagrin, they were not as excited about knowing the standards as I was to talk about them. It was then, I knew I had an uphill climb in trying to bring reform to Glenmount.

I made efforts to continue to educate myself in all of the math techniques available. I attended several workshops and lectures about teaching math and math reform. I participated in the BCPS Math Leadership Cadre. (Imagine that!) I became a regular at the NCTM Conferences (now at my own expense), eagerly absorbing all I could at each session. I would return to school bursting with enthusiasm, and eager to try new ideas. I became very interested in the University of Chicago Mathematics Project. I began to use Everyday Mathematics. I was fortunate to receive a review copy from the sales rep, with whom I established a business relationship during my frequent visits to the "Chicago" booth at the NCTM conferences.The students responded favorably. When I would point out that they were real mathematicians, they grinned with pride. I would speak of the students successes in hope there was someone who would want to listen. Sure enough a new staff member,a second grade teacher, opened her ears; but to my surprise she was already an NCTM member. We connected, and began to seek out more teachers to align themselves with the standards. Still, most of the teachers were workbook oriented, or work bound and were unable to take on any more (so they said,)

Another year passed. Ruth obtained a full time job doing what she liked best--teaching math. (This time I think she kept her salary). I was on my own now, and feeling more confident with every new math lesson and strategy I could find. I continued to put out "feelers" for interested parties. Ruth as a parent volunteer and I promoted our first Math Madness Month.(I got the idea at an NCTM conference, of course.) The students were asked to produce a project that reflected their math awareness. The school wide project was not mandatory, but you can imagine our delight when we displayed 265 out of 375 projects. The students portrayed math in various ways. Some doubled a recipe. Others defined math theory, while others wrote about famous mathematicians. All in all, the total student body became involved. Either by participation with a project, or as a spectator, we all received heightened awareness in mathematics.

Because Math Madness was so successful with the students, a door finally opened to math reform at Glenmount. Teachers were now becoming interested in new pedagogy and asked for training.

Math Madness was a marked success and the following year we promoted the projects again. This time we added bridge building. Math Madness Month ended with a real bang. We tested the bridges for structural strength with weights. The students were so excited they asked to do it again the following year. Now, Math Madness is infectious with Glenmount students.

Because Math Madness was so successful with the students, a door finally opened to math reform at Glenmount. Teachers were now becoming interested in new pedagogy and asked for training. It was difficult or impossible to get any financial resources from our ever-in-debt system. So, we formed a group to seek creative ways to fund our development. We knew if we could attend the NCTM conferences, and return with great ideas, we would continue to be motivated to reforming math teaching at Glenmount.

In 5 years, our efforts to initiate math reform at Glenmount have met with some success. We have nine members now, and look forward to more teachers who are interested to join the ranks of the math informed. Our main objective is to have all Glenmount staff at least informed about the NCTM standards, if not committed. As co-organizer of the Math Discussion Group at Glenmount, it is my ultimate goal to seek financing to aid in the professional development of myself and the group. Being informed and knowledgeable about the NCTM standards enhances my / our teaching ability which ultimately produces "math-friendly" students.

I am not sure how much longer it would have taken me to find out about math reform or national standards. Our system is seeking to have us informed, but is very slow in doing so. I just know that my journey has been unique in that it was parent initiated. And for that, I am grateful I was the teacher who had that little towhead, Curtis.

REFLECTION BY RUTH MILLER

I am a mathematician, a teacher, and a parent of three Glenmount students, so I wear a lot of hats, and I have a lot at stake. I think of myself primarily as a mathematician, though, because the skills that I have developed as a result of my love of mathematics help me everyday to be a better teacher and a better parent. I think that the key to any successful appreciation of the subject of mathematics is the realization that it is not a discrete area to be studied, but a journey to be traveled which can enhance and enrich other areas of life. I have a poster of Einstein in my room and he is quoted, "Do not worry about your problems in mathematics, I assure you that mine are still greater." To me this means that whether you are seeking to understand calculus or addition or unified field theory you are still seeking to understand, and there are many paths to understanding.

My son, Curtis, entered Kindergarten at Glenmount five years ago. At the time I was a stay-at-home Mom, so I volunteered in my son's class, which was taught by Gail Black. I cut out letters, and did other things around the classroom. I organized math games, and I was really popular when Gail found out I could run a mimeograph machine! One day I told her that in a "previous life" I had been a math teacher and her response was along the lines of "You know, I'm just not as comfortable teaching math as I am teaching reading." And then she uttered my least favorite 6 words: "I'm just not a math person." When I hear that I always wonder what the reaction would be if I said, "You know, I just don't read, I never have been good at it."

I have a poster of Einstein in my room and he is quoted, "Do not worry about your problems in mathematics, I assure you that mine are still greater."

I didn't have much power back then; I'm a parent so I can force a teacher into a conference, but as a teacher, I know that unless that teacher is invested in what I have to say, no meaningful change can occur in a classroom. So I started bringing math into the things that Gail asked me to do. I organized a set of manipulative that she had and started to teach some lessons. At first she would use the time to get her paperwork done, and do housekeeping around the room; but eventually she started to watch and ultimately we began a sort of team teaching: I knew the math and she knew how to present ideas to small children (I learned some techniques that I now use teaching calculus). The next year I helped out by working on math enrichment with my son's first grade class. Gail and I continued our collaboration and we went to New York together for the National Council of Teachers of Mathematics Regional Conference. There Gail became a true born-again mathematician (teachers new to Glenmount can't believe that she was ever math-phobic; trust me, she was!). The following year we started Math Madness Month: A celebration of math culminating in an exhibition of projects in the hall. We knew that Gail's class would do projects and that Curtis's class would do projects, so we set up two tables and waited. Imagine our joy when there were so many projects that we eventually used every spare table in the school and a significant amount of wall space. By this time I was working full time at Roland Park Country School, which is generous enough to give me release time every week to continue working here at Glenmount. In 1994, we took Curtis's second grade teacher Sarah Pickett with us to NCTM's National Conference in Indianapolis. In the next year we added a Toothpick Bridge building Contest to Math Madness Month, and Gail hitched a ride with me in the Roland Park Country School Van to the NCTM Conference in Boston. During these years I have always offered to link math to any lesson and I have had many Glenmount teachers ask for help or insight into the theory behind a certain algorithm or method. This year nine teachers are involved in a reform oriented math discussion group and seven of us just returned from the NCTM regional conference in Philadelphia.

I think that, if any of Glenmount's success can devolve from my influence it is because, as a person with no real power, I have had to teach rather than mandate.

Whenever we came up against a wall we played a game called "That's all well and good but it won't happen here because..." and somehow we always found a way to make it work in spite of whatever limitations we faced.

I never told Gail Black how to teach math, I just exposed her to what math really is and trusted in her professionalism and intelligence. Her colleagues saw her enjoying a part of her job that she was known to dislike and wondered what had happened. She encouraged them as I had encouraged her. Whenever we came up against a wall we played a game called "That's all well and good but it won't happen here because..." and somehow we always found a way to make it work in spite of whatever limitations we faced. We have lived by the first and second commandments of mathematics: 1) Every problem has a solution somewhere, and, 2) The first time something works it's a trick—by the fourth time it's a method. Like Einstein,everyone must butt up against her own mathematical wall, and a wise teacher allows a student to learn and grow in her own time. My advice to anyone seeking to improve math scores is to improve math teaching, and that is best done by improving the teacher's exposure to what math really is and trusting her intelligence and professional desire to do the best for her students.

Research on Children's Learning as a Tool to Improve Math and Science Teaching: A Resource Review

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Math/Science Matters: Resource Booklets on Research in Math and Science Learning, Tina A. Grotzer. Project Zero, Harvard Graduate School of Education.

This set of three booklets is intended as a practical guide for teachers of math and science. The stated goal of this series is to summarize "the findings of current research applicable to math and science education." The author argues in her introduction that such summaries are necessary background knowledge for teachers, who often are presented with educational reforms that they are not able to fully evaluate. The information in these booklets is therefore provided as a tool that allows teachers to better understand the theoretical models and empirical findings that are the foundation for a host of educational innovations in math and science teaching. With increased understanding, teachers will presumably be more successful in implementing reforms because they will understand innovative techniques at a conceptual level rather than simply as formulas to be applied by rote and easily abandoned at the first sign of difficulty.

Each of the three booklets addresses a specific research literature that is deemed relevant to math and science learning. Within a booklet, the specific readings are bound separately, and it is not necessary to read them as a set or in any particular sequence.. Booklet #1 focuses on findings from the study of cognition. Four essays consider children's information processing and ways of knowing, theories of constructivism, and theories of achievement motivation. Booklet #2 addresses research on teaching and learning in five essays. These include discussions of curriculum theory, thinking skills, and problem based learning. Booklet #3 concentrates on the individual characteristics of learners, exploring ethnic diversity, gender differences, learning disabilities, and gifted education in four separate essays.

Overall these booklets are very basic and highly accessible introductions to specific bodies of research literature, suitable for those with limited knowledge in the areas discussed. Each essay is also accompanied by a reference list for those interested in furthering their knowledge of the topic. The author provides very broad overviews of highly complex issues and runs the risk of oversimplification and overgeneralization in a desire to make the knowledge available. The discussion of achievement motivation in Booklet #1 for example, is a clearly written introduction to cognitive theories of motivation. However, it ignores how children develop their understandings of ability and effort. Although these fairly brief essays can hardly be expected to fully cover the topic presented, a basic discussion of developmental change in motivation seems necessary to avert misunderstandings of children's behavior in the classroom. Similarly, an essay in Booklet #2 addresses the distinction between teaching for conceptual knowledge and rote memorization of mathematical algorithms. However, the discussion ignores the role of cognitive development in shaping children's "invented procedures." A developmental perspective would be helpful for the experienced teacher, because it would allow the teacher to assess the level of thinking represented by the invented procedure.

The author describes these works as accessible for parents, although intended primarily for teachers. The reference list that accompanies each essay is a valuable tool for anyone interested in the topic of discussion. Current teachers of math and science who lack this basic level of knowledge and information, should consider these booklets required reading. However, the booklets' content may indeed be more suitable for parents, and other noneducators, as well as for new and preservice teachers, than for experienced professionals working in the field. The author has done a great service in providing a point of entry into the research literature for those who lack basic information. Whether experienced teachers of math and science are the appropriate audience is not completely clear.

A Situational Pedagogy for Elementary Mathematics

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It is now clear that the NCTM publication in 1989, *Curriculum and Evaluation Standards for School Mathematics*, was successful beyond expectation in focusing attention on the directions American mathematics education should take. But how does one design a complete, self-contained, K-9 curriculum that wholly incorporates and meets these laudable standards?

A reading of the *Standards* reveals an underlying call for "humanism" in the teaching of children, not only in mathematics but in all disciplines. To this writer that reading suggests a mathematics program (and an implicit definition of the word "humanism") in which . . .

• children are continually involved in the process of learning mathematics as participants, not as spectators;

• mathematical ideas grow out of reactions of children to intriguing situations posed by the program (such as pretend situations in early years, situations from elementary science and/or history of mathematics in later years);

• learning of routine arithmetic skills occurs in the process of reacting to the situations posed and not only in rote drill;

• the many facets of mathematics reinforce each other (for example, in the elementary grades) with arithmetic, geometry, and probability each interacting with and clarifying the others;

• having accepted the proposition that no one can master an idea completely on first, or even second, encounter, the main mathematical ideas continue to reoccur in a spiral curriculum in which the ideas are viewed many times from different perspectives and in deeper and more revealing form;

• the teacher relinquishes the role of mathematical authority, becoming a facilitator as the children discuss and investigate ways of dealing with the situations posed; the authority then belongs to logical thinking and the mathematics as it is developed;

• reliance on language is reduced in the early years to allow development of simple mathematical concepts that only seem complex when "described" in words;

• intuitive logic and thinking skills are developed indirectly through games and story situations that involve cataloging attributes of arithmetic and geometric relations;

• students write their own textbooks in the form of journals that record the high points and discoveries of the lessons.

An example of such a K-9 program now exists (more of which later), but the nagging questions are: How to "sell" such a program to the textbook publishers? To the test makers? To the "mastery" advocates? To the education establishment?

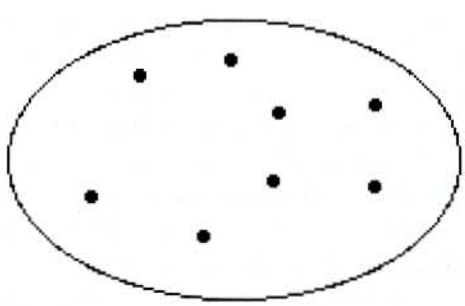
There is a long list of problems in the '90's that thwart wide implementation of such a curriculum, a list to which everyone interested in education can add a favorite. On the list is the *problem of teacher training and retraining*. Some would argue that the need is for a revamping of the teacher training programs in the universities. But in what directions and under what standards? Others would argue for broad programs of *local teacher retraining*. But by whom and with what agendas? Others would prefer to put political pressure on the *test makers* to revise the content and format of the standardized tests, as, for example, is now being done with some success in Japan. But it is clear to many observers that in spite of the *Standards*, we already have a deeply entrenched set of national standards, which are set by the textbook publishers and the test makers, as well as by the inertial carryover of traditional and comfortably familiar curricula and pedagogy in the schools.

Thus it would appear that before any serious reform effort can be mounted in teacher training, in test making and consequently in commercial textbooks, there must first be a national sense of the *way* children should be taught and *how* they should be involved in the learning process; i.e., we should confront the issues and make a decision about pedagogy.

It is easy simply to say, "We need a pedagogy that provides for continued intellectual involvement of children in the learning process." But to describe fully such a pedagogy, along with its philosophical basis, would require volumes and years of debate. Instead, it might be possible to understand what is meant by "a pedagogy of situations", to coin a label, by looking at some examples of situations and typical reactions of children to them. These examples, taken from an existing complete K-9 program, show how children of various ages can be engaged in intellectual experiences, some of which might not seem to be mathematical by traditional standards. We argue that such situations have strong implications for mathematics, and for other disciplines as well.

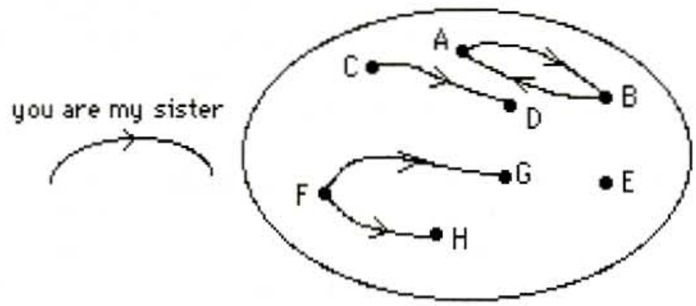
Pedagogically rich situations similar to the following examples can be used successfully to teach computational skills and at the same time give students enjoyable experiences with numerical relationships. We will also see non-numerical examples that are more dramatic in their continued stimulation of intellectual involvement at all levels of mathematics.

CHILDREN OF AGES 5 OR 6:



Tell a story about some children in a playground. While telling the story, we draw the above diagram and say, "The children in our story are playing a game." Explain that each child in the playground is represented by exactly one dot, and each dot is for exactly one child. It is not necessary to draw pictures of children; in fact, such pictures might even distract the children from the purpose of the lesson. On other days, for example, dots might represent ducks in a pond, or marbles in a bag, or numbers, or lines in a plane, etc.

In the game the children are playing each child points to his or her sister(s) and says, "You are my sister." Draw red arrows as indicated below to show how the children responded.



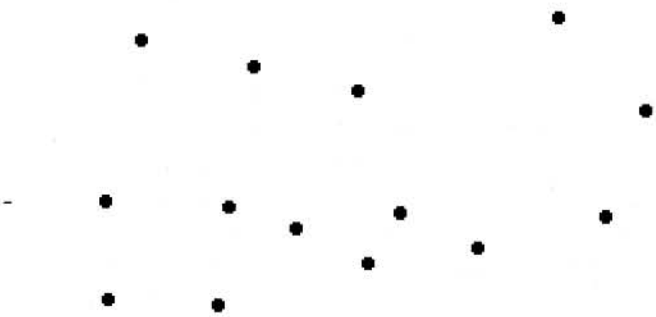
What sort of comments will five or six year-olds make when asked, "What does this picture tell us about the children in the playground?"

1. There are eight children in the playground.
2. D is C's sister.
3. D is a girl. (Why? Because D is someone's sister.)
4. E has no sisters in the playground.
5. A and B are sisters, so that both are girls.
6. G and H did not listen to the instructions. (Why?) They should have pointed to each other because they are both sisters of F. (Let children draw these arrows. Then tell them that now all the children that can say, "You are my sister" have done so.)
7. At this point some child in every group will probably say, "C is a boy." If we ask, "How do you know that?" we get an answer that in adult language goes like this: Suppose C were a girl. Then C would be D's sister. But there is no arrow from D to C. So we see that C is not D's sister, and hence C is not a girl. So C is a boy.
8. There are children from four different families in the playground.
9. C is D's brother. (Now ask students to draw blue arrows for "You are my brother".) They will draw three blue arrows: from D to C, from G to F and from H to F.

In this type of situation young children grapple with problems of symbolization and the relation of a symbol to the object it represents. They also give beautiful arguments quite like those that are at the heart of mathematics.

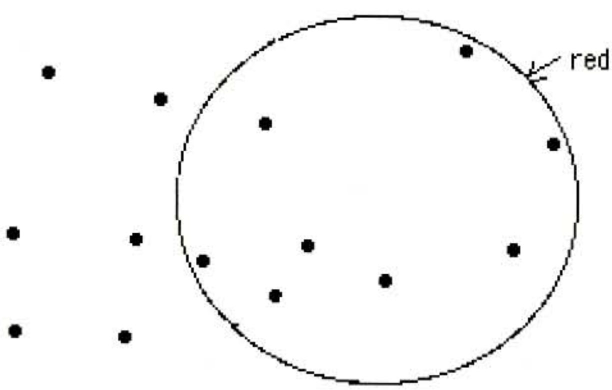
CHILDREN OF AGES 6 OR 7:

Tell a story about a boy, Nick, who wanted to be a detective when he grew up. His grandmother makes up a game for Nick when he visits her. She draws this diagram

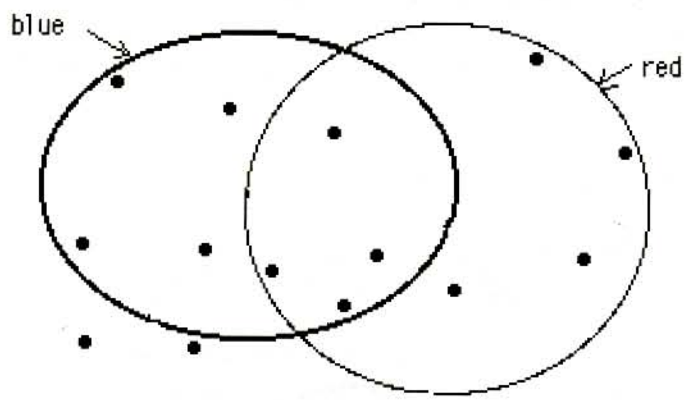


and tells him the dots are for three rabbits, her poodle, her three cats, three bugs, a squirrel, herself, Nick, and Nick's nose.

She asks, "Can you tell which dot is for which object?" Nick was puzzled, but she assured him that his talents as a detective would help him. "Here are some clues," she says as she draws a red string. "The red string is for animals with four legs. Is the dot for you inside the red string?"

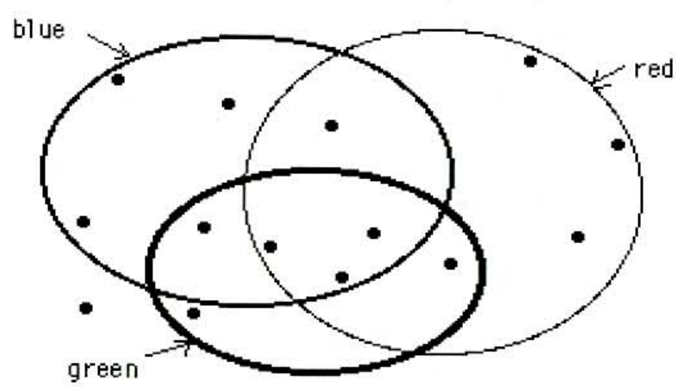


Nick immediately knew the answer; what was it? He also knew that the rabbits, the poodle, the cats and the squirrel were inside the red string, but he didn't know which were which. "What about the dots outside the red string?" One is for Nick, one is for Nick's nose, one is for grandmother, and the other three are for the bugs." (Why?) "Can you now tell me what dots are for which objects?" Nick was discouraged and his grandmother gave him another clue by drawing a blue string. "The blue string is for all living things that can climb trees." Nick is a great tree climber, and so he knew his own dot is inside the blue string. "But what about those two dots outside both strings?" Nick knew. Do you? (They must be for Nick's nose and grandmother, neither of which climbs trees.)



"Where is my poodle?" (Inside the red string but outside the blue string.) "Where are the three bugs and the three cats?" Nick needs another clue, because he can't tell which dot is for which thing.

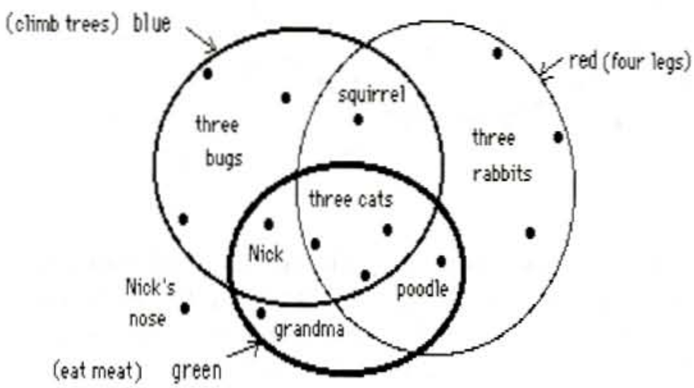
"Your last clue, a green string, is for all living things that like to eat meat."



Nick became very excited, because now he knew exactly which dot was for him, since he loved to eat meat.

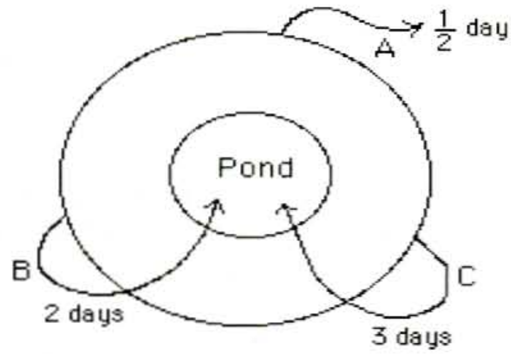
Which dot is for Nick? Children hearing the story tend to reason somewhat like this: Nick climbs trees and eats meat but doesn't have four legs. So the dot for Nick is the only one that is inside both the blue and green strings but outside the red string. Then Nick thought about the strings and soon was able to identify all the dots. Can you, for example, find the dot for his grandmother? Nick's grandmother likes to eat meat, but can't climb trees and doesn't have four legs. So the dot for her is the only one inside the green string but outside both the red and blue strings. Similar reasoning identifies all the other dots. Why is the dot outside all three strings for Nick's nose? Nick's nose, as a separate object, has no legs, cannot climb trees and does not eat meat. So its dot is outside all three strings.

See the diagram below.



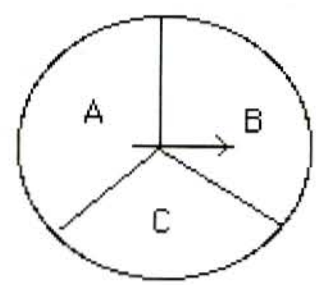
CHILDREN OF AGES 8 OR 9:
Tell a story about a prisoner in a dungeon in a mythical land. The dungeon has three tunnels leading from it. Unknown to the prisoner are these facts:

> TUNNEL A LEADS TO FREEDOM AFTER ONE- HALF DAY OF CRAWLING.
>
> TUNNEL B WINDS BACK AFTER TWO DAYS OF CRAWLING TO A CHUTE THAT DUMPS HIM INTO THE POND IN THE MIDDLE OF THE DUNGEON.
>
> TUNNEL C WINDS BACK TO THE SAME POND AFTER THREE DAYS OF CRAWLING.



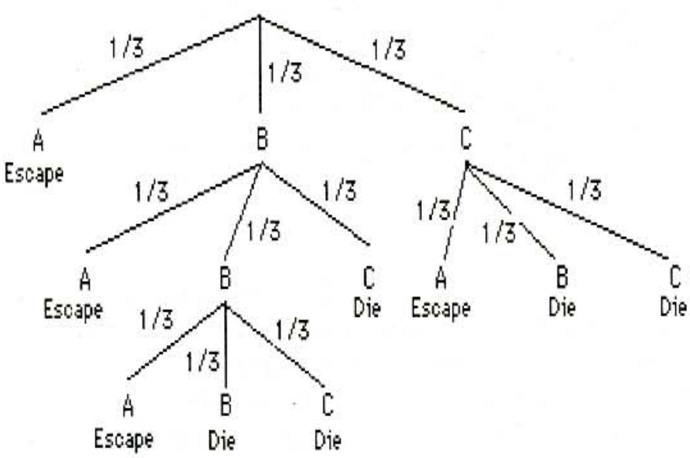
The king is a sporting fellow; so he gives his prisoners a chance to gain freedom. Each prisoner is given enough food to stay alive for 9/2 days and are allowed to select any of the three tunnels at random and try to crawl to freedom. What do you think the chances are of this prisoner escaping? (By the way, what the prisoner doesn't know is that if he lands in the pond he will be so confused that he won't know which tunnel he chose before. So he will again have to choose a tunnel randomly, and possibly again, and again — if he has enough food left.)

Children of this age will have met similar situations and will accept a suggestion to simulate this situation with a random device. Suppose they select a spinner such as the one shown below, with equal regions, because all the tunnels are equally likely to be chosen randomly.



Each child then puts himself in the place of the prisoner and selects tunnels by using his spinner. For example, one child might spin and get B (two days elapsed) and then spin again and get C(three days elapsed). So this prisoner does not escape because his food ran out. Another child might get C on the first spin, then A on the second; thus, this prisoner escaped. Still another might get B on the first spin, B on the second and A on the third; this prisoner also escaped. Putting together the total experiences of the class it might be decided that it is slightly more likely that the prisoner escapes than not, but it also might be too close to tell.

Later, at ages of 11 or 12, the children will have had enough experiences with random experiments to analyze this situation with probability tree diagrams. Using the knowledge that the three outcomes of the spinner are equally likely, that is, that P(A) = P(B) = P(C) = 1/3, the children draw a tree such as this,

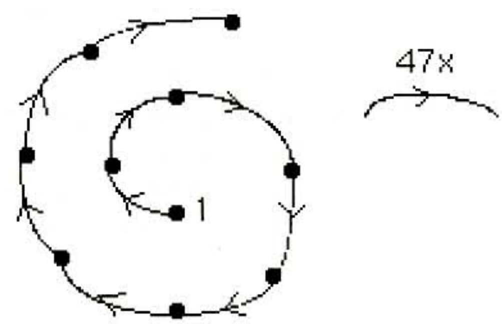


which records the possible adventures of the prisoner. Why does the sequence of spins B, then B, then B lead to death while the sequence B, then B, then A leads to escape? (The first sequence takes 6 days, while the second takes 4+1/2 days.) Adding the probabilities of escape, P(A) + P(B,A) + P(B,B,A) + P(C,A) = 1/3 + 1/9 + 1/27 + 1/9 = 16/27, which is greater than 1/2. Why is P(A) = 1/3? Why is P(B,A) = 1/9? Why is P(B,B,A) = 1/27? Why is P(C,A) = 1/9? (Imagine that 27 marbles are poured into the mouth of a toy in which one-third of them flow into each of three tubes that dump into other tubes, etc., where the same process continues as in the tree diagram. How many of the marbles will follow the path A? (9) The path B,A? (3) The path B, B, A? (1) The path C,A? (3) So 16 of the 27 marbles will end in the escape positions, i.e., P(escape) = 16/27)

CHILDREN OF AGES 9 OR 10:

There has been much argument about the role of hand calculators in elementary mathematics education. Too often these cheap, little machines are thought of as merely tools for calculation, but in reality they also can have a significant role in thinking about and obtaining insights about number relationships. Here is a dialogue between teacher (T) and students (S) in a class of 9-10 year-old children who have 8-digit displays on their hand calculators.

T: (*Draws on board*) Who can label this dot (*pointing to the first unlabeled dot in the spiral*)?



S: 47, because 47 times 1 is 47.
T: The next one? You many use your calculator if you wish.
S: 2209, because 47 times 47 is 2209.
T: Go on.
S: The next one is 103,823. And then comes 4,879,681. And then—what's that funny thing on my screen?
S: On my calculator at home that's the overload symbol.
T: What do you mean by "overload symbol?"
S: That means the number is too big for the machine? It can only work with numbers up to eight digits..
T: Does this mean that we can't do this last calculation on your calculators?
S: Yes.
T: What should we do?
S: Do it by hand.
T: You mean we must use our brains instead of the calculators. Is the calculator of no help with this fifth calculation?
S: I think I know how we can use the calculator to do this problem. First multiply 4,879,681 by 7 and then multiply 4,879,681 by 40 and add the two numbers.
S: That's right. That would be the same as multiplying by 47.
T: Good. Let's try that. What's 4,879,681 times 7?
S: 34,157,767.
T: What should we do next?
S: You'd better write that on the board so we won't forget it, because the calculator will lose it when we do the next part of the problem.

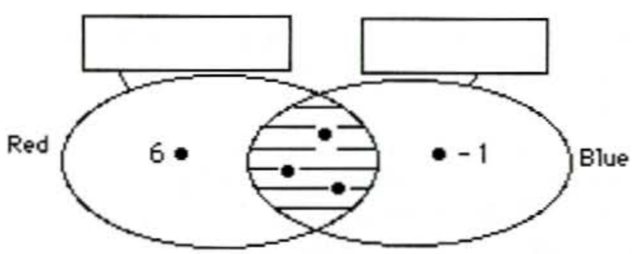
T: Good thinking. I'll record that number while you calculate 40 times 4,879,681.
S: But it overloaded again.
S: That's OK. We can multiply 4 times 4,879,681 on the calculator and then multiply that by 10 by putting a "0" at the end to get 195,187,240.
T: Very clever thinking! Now can we add these two on the calculator to get the answer?
S: No. We already know that the answer has more than eight digits. So we must add the numbers by hand.
S: I did it. the number is 229,345,007.
T: Good. I wonder what the largest number is that can be put on the display of the calculator. Think about that. And if you press the "3" button and then the "x" button successively, what is the largest number that will show before overload?

The above examples suggest that children can be turned on just as much by intellectual challenges as they are often turned off by performing prescribed tasks to acquire computational speed and skills. These examples exploit what we could call the "languages of strings and arrows", which have wide potential to deal meaningfully with mathematical situations such as those in the following examples.

Students of ages 11 or 12:

An example of the "language of strings":

T: Let's play another numerical string game. (*Draws the string diagram as shown below, and writes the list of possible attributes [unknown to the students] of the two strings as shown below.*



In the many versions of the game, the object is to identify the labels of the strings based on evidence given by dots for numbers placed in the various regions of the diagram.) I've given you some clues as a starter. One clue is the hatching of the region common to the red and blue strings. What does that tell us?

	Red	Blue
multiples of 2		
multiples of 3		
multiples of 4		
multiples of 5		
multiples of 10		
odd numbers		
positive prime numbers		
greater than 50		
less than 50		
greater than -10		
less than -10		
positive divisors of 12		
positive divisors of 18		
positive divisors of 20		
positive divisors of 24		
positive divisors of 27		

S: If a region of the string diagram is hatched, that means that the only numbers in that region are the ones shown. If no dots for numbers are in the a hatched region then there are no numbers there.
T: Good. So that means there are exactly three numbers in both the red and blue strings. Do we know what numbers they are?
S: No. All we know is that there are exactly three of them, because the dots are not labeled.
T: I wonder whether we have enough clues already to tell what labels are on the strings, or whether we need to make some plays to get more clues.
(*Sometimes the game is played between two teams, each having available a set of cards labeled with numbers from which to play alternatively, with the teacher saying "yes" if the card is placed in the proper region and "no" otherwise.*) You can use your analysis sheets to help you decide whether these clues will be enough to identify the strings. (*Each student has a copy of the list of attributes*)
S: We know that "6" is not a multiple of 4, or 5, or 10; so for the Red string we can cross out "multiples of 4, 5, and 10". (*Students do this in the Red column of their analysis sheets with "x"s.*)
S: And Red can't be "odd numbers" or "greater than 50" or "less than -10" or "is a positive divisor of 20 or 27". (*All agree after giving reasons.*)
T: Very good. We crossed out 8 of the labels for Red. Are there others we can eliminate?
S: I have another. Red can't be "positive prime num-

bers".

S: I don't see any more for Red. But for the Blue string we can cross out all the "multiples" because -1 is not a multiple of 2 or 3 or 4 or 5 or 10. And we can cross out "positive prime numbers", "greater than 50", "less than -10" and all the "positive divisor" labels. (*Again the class checks these suggestions and agrees.*) But we still don't know the labels of the strings.

(*At this point the students' analysis sheets look like this.*)

	Red	Blue
multiples of 2		×
multiples of 3		×
multiples of 4	×	×
multiples of 5	×	×
multiples of 10	×	×
odd numbers	×	
positive prime numbers	×	×
greater than 50	×	×
less than 50		
greater than -10		
less than -10	×	×
positive divisors of 12		×
positive divisors of 18		×
positive divisors of 20	×	×
positive divisors of 24		×
positive divisors of 27	×	×

T: Have you thought about the fact that both strings have exactly three numbers in common?

S: OK. Let's suppose Red is for "multiples of 2 or "multiples of 3". Then those three dots in the middle are for "multiples of 2" or "multiples of 3". But those three dots are also in Blue, and the only possibilities for Blue are "odd", "less than 50" and "greater than -10". But -1 is not a multiple of 2 or 3; so Blue could not be for "odd". And blue could not be for "less than 50" or "greater than -10" because there are more than exactly three multiples of 2 and of 3 that are less than 50 and greater than -10; so blue can't be for "multiples of 2" or "multiples of 3". These are the only possibilities for Blue if Red were for "multiples of 2" or "multiples of 3". That means we can cross these two off the Red.

S: I have some more. Suppose both strings are for "less than 50". Then all numbers less than 50 are in the middle region. But there are only three numbers there. So we can't have *both* strings for "less than 50". Now suppose Red is for "less than 50" and Blue is for "greater than -10". Then all the numbers between -10 and 50 are in the middle region. But since there are only three numbers in the middle, this can't be true. So suppose Red is for "is less than 50" and Blue is for "odd numbers". But there are more than three odd numbers less than 50. So we must conclude that Red can't be for "less than 50".

S: The same argument shows that Red can't be for "greater than -10". (The students will repeat the argument and then agree.)

T: Now all that are left for Red are "divisors of 12, or 18, or 24", and for Blue are "odd numbers", "less than 50" or "greater than -10".

Now the students' analysis sheets look like this:

	Red	Blue
multiples of 2	×	×
multiples of 3	×	×
multiples of 4	×	×
multiples of 5	×	×
multiples of 10	×	×
odd numbers	×	
positive prime numbers	×	×
greater than 50	×	×
less than 50	×	
greater than -10	×	
less than -10	×	×
positive divisors of 12		×
positive divisors of 18		×
positive divisors of 20	×	×
positive divisors of 24		×
positive divisors of 27	×	×

S: We can use the same argument to show that Blue can't be "less than 50" or "greater than -10". Suppose Red is for "positive divisors of 12, 18, or 24", and Blue is for "less than 50". Then there are more than four positive divisors of each of 12, 18, or 24 that are less than 50. So Blue can't be "less than 50", and the same argument can be used to show that Blue can't be "greater than -10". Hey, we found the Blue string is "odd numbers".

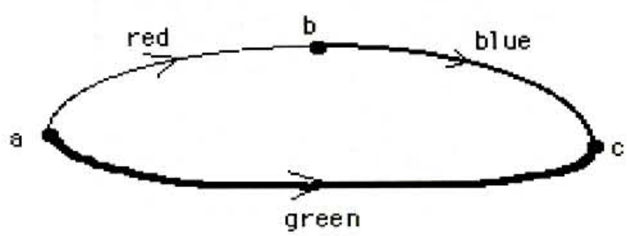
T: Who can finish the game?

S: We know that Blue is "odd numbers". Then the three numbers in the middle are all positive odd divisors of either 12 or 18 or 24. The positive divisors of 12 are 1, 2, 3, 4, 6,12, with only two odd divisors. The positive divisors of 18 are 1, 2, 3, 6, 9, 18, with three odd divisors. The positive divisors of 24 are 1, 2, 3, 4, 6, 8, 12, 24, with only two odd divisors. So the Red string is for "positive divisors of 18". (*Loud cheers from the class.*)

Although the depth of thinking involved in the above analysis seems remarkable for students of ages 11 or 12, it must be understood that these students began thinking in this way at ages 6 or 7 by playing simpler string games with attribute blocks. Then they moved on at ages 8 or 9 to string games which extended to three strings and then to string games whose attributes are mathematical relations or mathematical operations. What is most interesting about these games and the other mathematical situations in the program is the enthusiasm and enjoyment that children experience when their intelligence is engaged—and it turns out that very few children are immune to the joys of good thinking.

SOME EXAMPLES OF THE "LANGUAGE OF ARROWS"

Much emphasis is placed on exploiting the power of relational thinking. From the beginning children use arrows from dot to dot to describe the relation of the second object to the first, without relying on the natural language. For example, consider this arrow diagram:



Ages 6 or 7: Suppose the dots are for numbers, the red arrow is for "+5" and the blue arrow is for "-9". What is the relation of the green arrow? (That is, number "a" says to number "b", "you are 5 more than me" and the number "b" says to number "c", "you are 9 less than me"). Then what does the number "a" say to number "c"? (*Ans: Green arrow is for "-4"*)

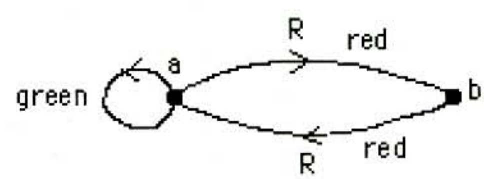
Ages 7 or 8: Suppose the dots are for people, the red arrow is for "you are my father" and the blue arrow is for "you are my sister". What is the relation of the green arrow? In other words, what would person "a" say to person "c"? (*Ans: you are my aunt, or more precisely, my paternal aunt.*)

Ages 11 or 12: Suppose the dots are for numbers, the red arrow is for "4x" and the blue arrow is for "÷5". What is the relation of the green arrow? (For example, suppose "a" is 20; then what is "c"(16). If "a" is 35, what is "c"? (28) If "a" is 1, what is "c"?, (4/5). Etc.)

Comment: The relation denoted by the green arrow in each case is the composite of the relations of the red and blue arrows, in that order. In most situations the composite of the blue followed by the red gives an entirely different relation from the composite of the red followed by the blue, although in this case each order of the composition yields the same relation, namely "4 x/5". In fact, this composite relation precisely defines a fraction. For example. the fraction 4/5 is the result of applying the composite relation 4x/5 to the number 1.

Ages 12 or 13: Suppose the dots are for lines in a plane, the red arrow is for "you are parallel to me" and the blue arrow is for "you are perpendicular to me". What is the relation of the green arrow? (In this case, we would have to change the color of the green arrow to blue, since line "a" would say to line "c", "You are perpendicular to me".

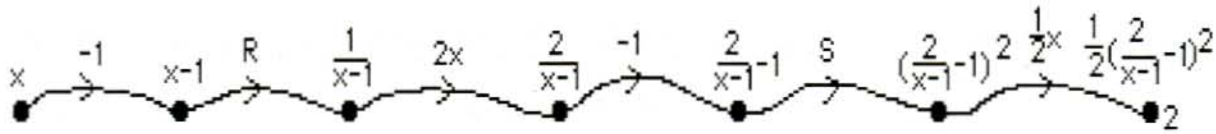
Ages 13 or 14: Suppose the dots are for rational numbers other than 0, the red arrow is for "R: you are my reciprocal" and the blue arrow is also a red R arrow. What is the relation of the green arrow? (Here we are forced to change the diagram, because of the convention that each dot is for one and only one object. So the green arrow will be a loop from "a" back to "a", as below.)



Ages 14 or 15: Suppose the students are faced with a situation that requires the solution (s), if any, of the equation

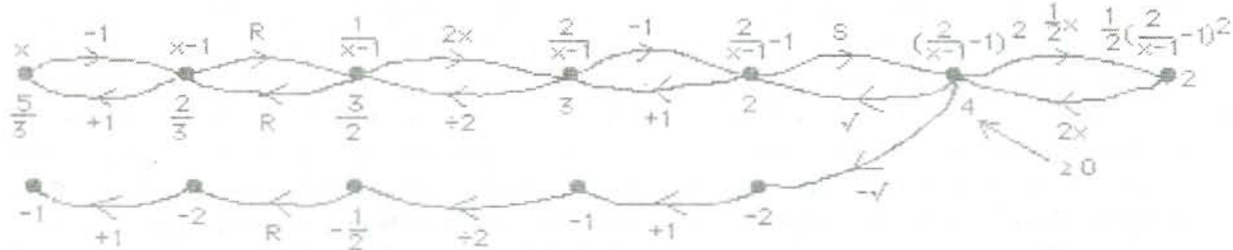
$$1/2(2/(x-1) -1)^2) = 2, \ x \neq 1$$

From past experience in the program students ordinarily would tackle this equation with an arrow diagram (shown below) but some would need only *visualize* an arrow diagram to guide them as they apply relations, all of which are functions (i.e., mappings), as follows.



The mapping "-1" takes x to x-1; "R" takes x-1 to 1/(x-1); "2x" takes 1/(x-1) to 2/(x-1); "-1" takes 2/(x-1) to 2/(x-1) -1 ; the square mapping "S" takes 2/(x-1) -1 to (2/(x-1) -1)^2; finally, the mapping "(1/2)x" takes (2/(x-1) -1)^2 to (1/2)(2/(x-1) -1)^2. Each dot in the diagram denotes a unique object and reminds students of its role: "one object—many names". In this case the last dot in the diagram is for one number with two names: " (1/2)(2(x-1) -1)^2" and "2".

Then the return arrows (the converse relations) are drawn, as below. All of these converse relations also are functions, except for the converse of the square function S, which is two-valued ("√" and "-√") only if operating on a number greater than 0, and single-valued ("√") if operating on 0, a fact which the students have discovered in the past. Notice that the converse of "(1/2)x" is "2x", which maps 2 to 4, which is greater than 0; then the converse of S is two-valued, and maps 4 to 2 and to -2; etc. The process ends when it is found that *if* there is a number x satisfying the equation, then possible names for x are 5/3 or -1, neither of which is 1. Finally, it can be checked that if x = 5/3 or -1, then the equation is satisfied.



Notice that here there is no need for any confusing "rules", such as "doing the same thing to both 'sides' of an equation", etc. Much of mathematics consists of the study and application of the composition of functions; in fact, arrow diagrams are often used by mathematicians when they "doodle with diagrams" as they think about abstract theories, where the dots might even represent functions.

The above examples implicitly define a *pedagogy that assures respect for children by providing for their continued intellectual involvement in the learning process.* It is a pedagogy that emphasizes logical and relational thinking of a kind that applies not only to mathematics but to a whole spectrum of disciplines. Such a pedagogy can improve not only the intellectual climate but also the social climate of schools that employ it. There is another strength of such a program that applies particularly in school populations with special cultural characteristics; namely, it is essentially culture-neutral. There is no inherent advantage in coming to it from the "right side of the tracks" nor from the other side. In regard to its diagrammatic and intuitive approach, no child enters such a program with special "deprivation" baggage.

As stated at the outset, there does exist a complete K-9 curriculum that meets the *Standards* and from which the above examples of pedagogically rich situations were taken. The K-6 portion is the elementary component of the Comprehensive School Mathematics Program (CSMP), which was developed, tested nationally , evaluated extensively and published from 1969 to 1982 by CEMREL, the regional education laboratory in St. Louis, with NIE funding . From 1982 to the present time the CSMP K-6 materials have been revised and extended to CSMP-MS, the 7th to 9th grade program, by McREL, the regional education laboratory in Aurora, CO. The K-6 portion is available for use and the 7-9 portion is in the process of national pilot testing and available for trial by contacting:

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Helping Students with Attention Deficit Disorder Succeed in a College Mathematics Class

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Section 504 of the Rehabilitation Act of 1973 directs postsecondary institutions to provide educational opportunities to qualified students with disabilities. Increasingly, individuals with disabilities are attending college and voluntarily disclosing their disorder in order to take advantage of the services available to them mandated by law. (Goldstein, 1996)

Attention Deficit Disorder (ADD) is a neurobiological condition that causes difficulties with impulsiveness, concentration, retention of information and simple facts, organization, staying on task, and paying attention to details. So many of us have had students in our mathematics classes who have exhibited one or more of these behaviors that we can inadvertently overlook the ADD student. It is tempting to conclude students with ADD are unable to master sophisticated mathematical concepts.

In spite of the special student services available to them on most campuses, some ADD students fail to disclose their disability to their instructors or to the college, because they fear they will not be believed, understood, and/or that the instructor won't understand the condition. Unfortunately, some students resort to behaviors to hide their disability only to exacerbate their problem of performing inadequately in class.

Highly intelligent non-hyperactive ADD students can do quite well in elementary ana secondary school. This is true particularly if both the home and school environment were well structured. For some, schoolwork was not very challenging and they could depend on the teacher to repeat important information many times. ADD symptoms tend to intensify when the demands on performance exceed the student's ability to function. For some people this debilitating stage may not occur until they are faced with a more demanding but less structured academic environment such as a college mathematics class. (Nadeau, 1996)

Helping the adult student learn mathematics becomes particularly challenging when the individual suffers from ADD coupled with low self-esteem. Many non-hyperactive children with ADD go undetected until the disorder is diagnosed in adulthood. Over the years, these students can develop self-doubt in their academic abilities because throughout their schooling they were told they were not trying hard enough or they lacked motivation.

So what can we do as instructors when ADD students enroll in our college classes? Foremost, instructors must believe ADD students can be successful in their courses. Since these students have made it to college, they have managed their problems with procrastination, distractibility and disorganization to some degree. If I have not been informed that a student has ADD, I simply ask the student, if I suspect something is not right. This is only after I feel I have established a rapport with the student and the student feels safe to communicate openly with me. After the law was enacted, I was fortunate to be teaching at a college that trained faculty on how to recognize learning disabilities.

An effective strategy I found that worked for me is to provide the ADD student with a structured framework in which to achieve the course objectives. In spite of their constant struggle to stay focused, the ADD student can develop a cognitive processing system that enables them to move information stored in short term memory to long term memory. The challenge for them is to get the information in their long term memory within the course deadlines. Often it is suggested to allow ADD students to take untimed tests. This accommodation assumes the student is retaining all the information he needs for the test by the time the test is given. Having to meet these deadlines may cause a high degree of anxiety in the student. This level of stress interferes with learning and delays even longer what he is expected to know.

The structured framework provided the ADD student should detail specific measurable expectations the student must demonstrate in order to pass the course. I also discuss with students ways in which they feel they can best demonstrate what they must learn. An untimed test is frequently requested because it is most commonly recommended by Special Student Services. However, this option may be inadequate. When and where the ADD students take a test are as important as how much time they have to complete it. The environment must be conducive to concentrating. Even taking an exam, timed or untimed, in class with other students can be distracting to some ADD students. Another strategy to assess knowledge is to have the student tape record his responses to test or homework questions.

He submits you both the tape and written work that illustrates what he has recorded. Giving the student an oral exam where your direction keeps her focused is also effective.

Permitting ADD students to leam at their own rates within the framework enhances their opportunity to pass the course. This means the time lines for completion of the course work for your ADD student can be very different from deadlines set for each assignment or test you have given the rest of the class. Students should be expected to attend class even if they are learning or retaining information at a slower rate than you are teaching it, as it facilitates them staying on task. Strategies such as allowing and encouraging ADD students to tape your presentation enables them to replay it until they are able to process the information as described above. Accommodating them with a note taker is quite helpful because the student can then focus all of his or her attention on visualizing and listening to what you are saying and writing. Ideally, important information, like due dates and expectations, should always be in writing. Sending an email to relay a message to an ADD student has its advantages over making a telephone call.

At first some of the suggestions can seem time consuming. However, if your college is receiving federal funding of any sort, it should have in place services that can assist you in accommodating the ADD student's needs. Another strategy is to call the ADD student's major advisor who can be quite helpful with suggestions on how to structure the course work.

If we take seriously the needs of students with ADD, we are challenged to rethink our whole class structure, how we test, and deadlines we impose. Becoming sensitized to differences between our teaching style and how students learn is another possible outcome. Our response to this new awareness can benefit all of our students, whether they have ADD or not.

REFERENCES

1. McDonald-Richard, M. (1996). College Programs and Services. In S. Goldstein, *Managing Attention and Leaming Disorders in Late Adolescence and Adulthood* (pp. 266-288). New York: John Wiley & Sons.
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NUMBERS - 1

Numbers are such simple little things
Surely known before time begins.

We start by counting fingers and stones
In answer to how many of those.

Then, we number things in measuring
Answering how big and in comparing.

Eventually numbers become symbolic
And finally ensconced and mystic.

This lead us to philosophize
Pondering numbers --- what are they?
Concepts, things, or social entities.

The Reform Calculus Debate and the Psychology of Learning Mathematics

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It is a basic truth for the vast majority who learn mathematics, that they come to understand and appreciate the foundations of a subject only when they have some maturity in that subject.

This essay is prompted by a recent debate in the American Mathematical Monthly between Professor Tucker[1] and Professor Swann[2]. Their debate is a good example of the national, sometimes acrimonious debate on "reform calculus," with Professor Tucker on the reform side and Professor Swann on the side of tradition. I claim that the debate is misdirected, that at issue are long standing problems in mathematics education.

THE NEED FOR FORMAL PROOFS IN BEGINNING CALCULUS (THERE ISN'T ANY)

Professor Tucker says at the beginning of his article (p. 231): "I have serious doubts about the need for formal theorems and proofs in a standard calculus course." Likewise, Professor Swann says near the beginning of his article (p. 241): "I agree that proofs of the extreme value theorem and other global results from basic principles do not belong in today's beginning calculus texts in the present educational climate." This is amazing in that the debate then focuses on the foundations of calculus and like so many such discussions centers on the mean value theorem (for derivatives). Hence the exchange is basically like a knife fight between two pactfists. To be fair, Professor Swann has hedged his philosophy with the key phrase *in the present educational climate*. Later (p. 242) he adds "It surely is imperative to introduce the notion of proof in beginning multisemester calculus and keep it alive though actual proofs are few." I understand his misgivings about today's educational environment, and I think that during the basic calculus sequence the students should be exposed to δ–ε proofs perhaps twice in the perhaps vain hope of priming the pump: that maybe some idea may germinate or at least the gist of what a precise statement from calculus looks like will make some lasting (positive) impression.

Whereas I see periodically in print that δ–ε proofs are inappropriate for the basic calculus sequence, the fact is, that many teachers employ them and this is true in junior college classrooms on up. This is symptomatic of more widespread problems in mathematics pedagogy and will bring us back to the reform calculus debate. The use of δ–ε proofs in the basic calculus sequence is simply a bad idea. Too few students are ready for it. Students who endure it often learn to do various δ–ε exercises by rote. It adds a great deal of stress to a course that has enough stress and it diverts attention from basic concepts that have to be learned. The problem is simple: δ–ε proofs are foundational in nature and the students are not ready for foundations.

FOUNDATIONS COME LAST

It is a basic truth for the vast majority who learn mathematics, that they come to understand and appreciate the foundations of a subject only when they have some maturity in that subject. Most learning in elementary mathematics starts with some rote. Rote learning seems to be universally condemned, yet if one starts with theory, the students usually have no idea what one is talking about. Those education experts who condemn rote learning so often wind up teaching nothing at all. It is important to recognize that rote learning in itself isn't necessarily bad; it is bad only when it is the main object of learning rather than a foundation for higher learning. If you want to teach a child arithmetic, you teach her to count largely by rote. Later she must learn addition and multiplication tables and again this is largely rote. Perhaps you would rather try to explain Peano's axioms to her? Then, of course, define addition and multiplication recursively and prove their basic properties. Then explain to her how important associativity is and then, of course, how Hindu-Arabic notation is a product of genius.

THE CRISIS OF SHALLOWNESS

Part of the crisis in calculus education stems from the lament that students who have completed the basic sequence have a rather shallow understanding of what they are doing. It is said that they do problems mechanically without appreciation of the substance at hand. The reformers want to use geometry and calculation to instill greater insight into the fundamentals of calculus. However, this occurs at the expense of symbolic manipulation. I don't think it makes much difference: one type of lesson is going to be replaced with another and in any case the student is going to wind up with a shallow understanding of the subject.

I would like to know what are the math areas where the students come out of an introductory course with deep understanding. Sure most students come out of the calculus sequence having learned to solve problems by rote. During their courses they ignore the proofs and often the theorem statements and they study the text examples to learn to do the homework problems and if the instructor uses similar problems on the tests then they may graduate. What they get out of lecture is inversely proportional to the time spent on proofs and theory. The fact is, these are the good students. Many students are unable to learn to do the examples by rote and they must go to other areas. As for those who do get past the sequence through learning by rote, give them time; some of them will acquire depth and will be good prospects for graduate school. The real problem in calculus is that too many of the students capable of rote learning and later deeper learning are filtered out with the others. They change majors not because they didn't have the potential and the work ethic to make it in calculus, but because they had a bad instructor, often a totally useless instructor at a critical time. They get the instructor who thinks his job is to present the material. And he thinks that first semester calculus is a fine time to introduce the students to full rigor. This teacher is untroubled by the fact that he flunks eighty per cent of his students. These students just don't have the right stuff. Again and again this happens and the teacher never seems to realize that the students who pass are students who have had the material before, along with one or two students who are simply brilliant. By teaching the course the way he understands the material himself, he is in effect teaching a review course, and the lecture is stimulating for the students who already know the material. I claim that a significant portion of mathematics teachers perform in a review mode most of the time they teach, and this drives away students who might otherwise do quite well in mathematics.

It is not an accident that the foundations of calculus followed the discovery of calculus by two-hundred years.

ONE SEMESTER EQUALS FIFTY YEARS

Although a historical approach to teaching a subject can be over done, in learning it is safe to say that ontogeny should follow phylogeny (or should I say ontopedagogy should follow phylopedagogy?). It is not an accident that the foundations of calculus followed the discovery of calculus by two-hundred years. A good rule of thumb is: when students first reach calculus, each semester is equivalent to fifty years of history. First semester calculus is the late seventeenth century: the definition of derivative and the fundamental theorem of calculus. Second and third semester calculus are the eighteenth century: loads of applications and much problem solving. The semester following the calculus sequence is the time of Cauchy: it is time to ponder what has been learned and to start thinking more precisely about fundamentals. The semester after that (the fifth semester) brings us to the time of Weierstrass: now is the time to look at δ–ε proofs. Those students who simply cannot handle it should not do graduate work in mathematics but may have opportunities elsewhere. Interestingly, there are students who two years earlier thought δ–ε definitions were completely unintelligible and now find them rather simple, almost obvious, perhaps elegant. These students have promise in the mathematical sciences. Their promise as teachers is another matter. Because, it is rather easy for the student to convince himself that he could have understood this material two years earlier but it was not explained properly. However, it is really quite clear and he knows that when it is his time to teach calculus he will show the students δ–ε definitions and proofs because there is no reason that the students can't understand them right off the bat. It is quite simple really, we only need to be precise, we must have rigor and so on, and the students are stuck with another mathematics teacher from hell.

Consider the (promising) student just out of the calculus sequence. At that point calculus is a large basket of definitions, theorems, formulas, techniques, and a huge variety of problems. Although it was covered in one of his tests, he doesn't remember what the mean value theorem says. The fundamental theorem of calculus says that to evaluate an integral you find the anti-derivative and plug in the end points: that is all of it he remembers and with just that amount of precision. Following the sequence, some of his calculus is reinforced by a course in differential equations. The material starts to gel and he starts to think and work more precisely. He takes an analysis sequence which makes him think more rigorously and to finally contemplate and start to appreciate the foundations of analysis. In graduate school he gets a great deal more analysis, and also he teaches calculus recitation classes. On leaving graduate school he gets a job as a teacher and it so happens that virtually every year he teaches some calculus course or analysis course. Now the interesting thing is that virtually every time he faces the material of calculus, he learns something new. To be sure there may be no great revelations, and perhaps he is not always conscious of the new insight. But after twenty years of this he can get extremely involved in discussions on calculus and, say, the mean value theorem. More important is that as he has acquired knowledge over the last thirty years, the calculus has become simpler and simpler to him. What once was horrendously complex seems simple; as far as our professor is concerned calculus boils down to maybe four things: the completeness of the reals, the definition of derivative, the fundamental theorem of calculus and the mean value theorem. Having great insights, he endeavors to pass it on to his students. Each lecture is filled with insights gathered over the years, and he is handing this hard-won knowledge over to the students so that they can get it immediately without trudging through the swamps as he did. However, the damn students appreciate none of this. They nod their heads politely during lecture and concentrate on learning to do the exercises in the text. This, of course, is a crisis. We need to change the way we teach calculus because the current students who complete the sequence seem to have a shallow understanding of the topic.

If you really want to increase the understanding of your students on leaving the course, you might try the following trick: cover less material! !

SLOW DOWN, DAMN IT

If you really want to increase the understanding of your students on leaving the course, you might try the following trick: cover less material! ! Consider the undergraduate abstract algebra sequence. Often it is taught out of a text like Fraleigh[3]. At the second half of the second semester the professor covers Galois theory and at the very end he proves the non-existence of a quintic formula. Out of his fifteen students, ten have no idea what he is talking about most of the second semester. Four others are picking up fragments. The fifteenth student, the young woman who actually has comprehended most of the course, is the student who will be offered scholarships to do graduate work at Harvard, Berkeley, and Cal Tech. At least four students and probably more, could have gotten much more out of the course. That the course went too fast is because it was taught as the teacher sees the subject; the lectures were appropriate for a review course.

Surprisingly, it can be more work to teach less material. It requires effort to focus on the students and what they actually comprehend. A course that is taught in a theorem-proof format, much like the way Edmund Landau wrote books, is relatively easy to teach, and that is one reason that many teachers use that style. It is, of course, enjoyable to teach the one brilliant student; she comprehends so much more than the others do. But she is the one student who does not need much detailed attention. She might benefit more from reading a book, say the text on Galois theory by John Stillwell[4]. It is a lousy choice for a text, but it is an enlightening read, and could be inspirational to our brilliant student. One other point: covering less material does not mean giving less homework. Not only that, but the students might do more homework because they are not lost all of the time.

A RELEVANT NOTE

In 1995 the Mathematical Association of America published a book about Ralph P. Boas, Jr. who died in 1992[5]. On page 98 of that book, Professor Boas says: I once heard Wiener admit that, although he had used the ergodic theorem, he had never gone through a proof of it. Later, of course, he did prove (and improve) it. It so happens that I once expressed surprise at this to Professor Boas. Here is his reply[6]:

> I do not think my story about Wiener is very surprising One can 't always be going back to first principles.
>
> I quite agree that—at least for some people (I am one of them) calculation precedes understanding I have probably said before that I knew how to calculate with logarithms long before I knew how they worked. The idea that proofs come first is, I think, a modern

> fallacy. Certainly—even in this calculator age—a child learns that 2 x 2 = 4 before understanding why. The trouble with "new math " was (in part) the fallacy of thinking that understand needs to come first.

At the risk of overstating the point, let me repeat the key phrase: The idea that proofs come first is, I think, a modern fallacy.

THE REFORM CALCULUS QUESTION

There are reasons why the reform calculus is attractive. There are also good arguments for some of the more traditional approaches. I have not given any of these arguments here because they are irrelevant to my point. Whichever approach we choose will trade off one superficial knowledge for another. Any approach, if it is at all a feasible approach, and if it is done well, will provide a foundation for the student to go on to a deeper knowledge of calculus and its applications. If the student is not going to go further, then his taking the sequence in the first place was probably a waste of his time and the teacher's, and the approach used is still irrelevant. Perhaps in twenty years, something of a consensus will emerge about what approach to teaching elementary calculus is most fruitful. In the mean time, the controversy does not deserve the present rancor. It is true that today's students are less well prepared than thirty years ago, and are less inclined to work. Nonetheless, were this not so, I would still suggest: try teaching less material, try to save the theoretical overview for later, and lighten up.

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NUMBERS - 2

Numbers, delight and entrance
Necessary for music and dance

Patterns abound with numbers
Some are found rhyming this verse

Such pure simplicity
Hides vast complexity

Eternal truths found by addition
Show questions without solution

The most basic being
One, two, three
What art thee?

TEACHING MATHEMATICS APPRECIATION TO NONSCIENCE MAJORS

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I. WHAT DO I MEAN BY MATHEMATICS APPRECIATION?

As H. O. Pollak (35) very accurately observed, "the perception most people have of mathematics has been molded by their educational experience, and neither the experience, nor its recollection tends to be happy." Ironically, many people dislike and fear a mislabeled enemy, because they have only seen what David Fowler (18) calls schoolmath, a quite different subject with its own terminology, methods, and beliefs. Although it is true that most nonscience majors may have forgotten a good part of their schoolmath, the bad feelings about their experience remain.

On the other hand, in spite of the indisputable applicability of mathematics, it takes some planning to communicate the effectiveness of mathematics to a nonspecialized audience. "How can it be—Albert Einstein asked in 1920—that mathematics, being after all a product of human thought independent of experience, is so admirably adapted to the objects of reality?"

As a result of all these factors, the dealings of mathematics seem too often closed off as by a high wall. How do we breach this wall, how do we present mathematics in a way that a passerby may enjoy it? Better yet, how do we lure a reluctant spectator into becoming, to some extent, a performer? After teaching mathematics appreciation for several years to numerous nonscience majors, I am still struggling with these questions, although it seems that by the end of each semester I am able to come up with better answers.

I do not think that there is a universal recipe for what constitutes appreciation of mathematics. However, it is my experience that any successful approach should recognize the special characteristics and the wealth of nonscientific knowledge that the students have. To this purpose, I find it very effective to present mathematics in several combined ways: as a powerful tool in the students' own business, as a means to develop effective thinking and communication skills, as a phenomenon of cultural history, and as a collection of fundamental thoughts and ideas. I think that one of my tasks is to convincingly present the central role that mathematics has played in people's lives throughout history, and to emphasize the continuity and connectedness in the way mathematics develops. Another task is to convey what the physicist Eugene Wigner called "the unreasonable effectiveness of mathematics"

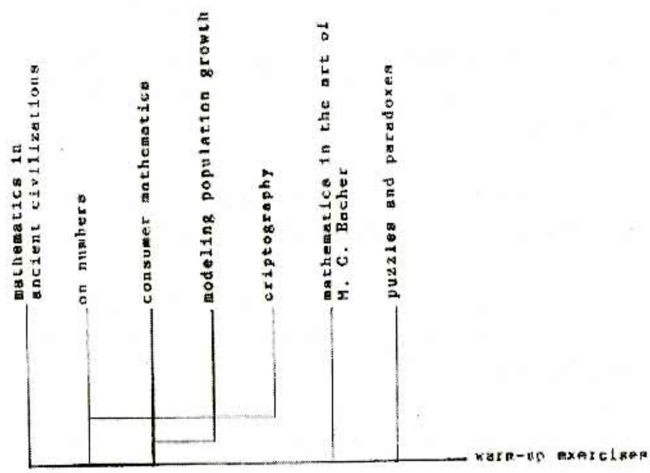
I do not think that a course on mathematics appreciation should insist on technical skills. Rather, it should make the best out of whatever skills the students have, typically a vague recollection of high school algebra. Still, I view this course as a mathematics course, where the students are expected to see and do mathematics. Many of the most beautiful and enduring mathematical ideas have a common sense quality that makes them fairly natural to grasp.

II. HOW DO I TEACH A BASIC COURSE ON MATHEMATICS APPRECIATION?

I do not require a textbook. Most of the books appropriate for this basic level, seem to be collections of methods and examples that fail to show how and why these methods arose. Instead, I provide the students with an extensive and eclectic list of materials gathered from scholarly books and journals, newspapers, magazines, etc. I am including a sample list at the end of this article.

What I have seen working best in my classes is to choose paths that go through a collection of topics that have some common thread whether in the subject itself or in the approach by which they can be presented. My purpose is to emphasize the continuity and connectedness in the way mathematics develops and progresses. For this purpose, it is crucial to emphasize credible beginnings, links to previous experiences or human needs. I keep reminding myself that, probably, this is the last chance to expose these students to the beauty and power of mathematics, that their feelings towards mathematics will undoubtedly influence their children's.

I start the semester with a set of simple problems that I call "Warm-up Exercises" and use them to preview what we will cover in detail later. These problems are the trunk from which the branches of the course will stem. Typically, I cover topics on mathematics in ancient civilizations, numbers, modeling population growth, consumer mathematics, cryptography, puzzles and paradoxes, and mathematics in the art of M. C. Escher. Here is a picture of such a course:



I show this picture to my students quite often along the semester, to emphasize the flow of mathematical events.

The class format is highly interactive, with students doing a lot of group work. For each topic, I prepare extensive handouts, including many historical references and hands-on activities. An overhead projector is frequently used to illustrate a topic. I require the students to have and use a scientific calculator. I prove and discuss on the blackboard some of the formulas to be used. Even very simple formulas, such as compound interest or annuities, provide powerful examples of mathematics at work. For most students, this is the first time they see how a formula is obtained. It also gives some credibility to the algebraic manipulations they may have seen before.

I limit my lecturing to an absolute minimum. My role in the classroom is more of a moderator and watcher of the students' discussions. I am fortunate to have a classroom with movable furniture that encourages and facilitates group work.

As part of the course requirements, the students have to prepare two group papers, and work individually on some in-class written essays and problems.

III. WHAT DID I LEARN FROM TEACHING MATHEMATICS APPRECIATION?

I have asked the students to answer brief questionnaires at the beginning and at the end of each semester. Their answers show a tremendous, positive change in their attitude towards mathematics. I quote here one of my favorites answers: "I expected drudgery of trying to complete copious amounts of stupid problems such as if two pipes fill a pool that is unplugged..." I am grateful to my students for many eye opening responses.

After teaching my first mathematics appreciation course in 1994, I wrote in my annual report: "This course opened up a completely new, challenging, and very rewarding world in my teaching career." I have taught since then several versions of the course and I still feel the same way.

Prior to teaching this course, I had frequently used group work and writing assignments in other courses. Teaching mathematics appreciation gave me the opportunity to improve my overall understanding of the historical development of mathematics. This newly acquired ability to seek and appreciate a bigger picture has made me a better teacher at the undergraduate and graduate level, as well as a better researcher, more able to look into the rich history of my own field of interest. I now make a special effort to look for supplemental reading materials for my students and myself that stress the historical development of a subject. And I am very pleased to find that more mathematical authors are trying to offer a more balanced picture between techniques and their historical development.

IV. SAMPLE LIST OF RESOURCE MATERIALS

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NUMBERS - 3

Numbers originate with who?
Did God or man define two?

Maybe Plato was right
Possibly a divine insight

Truths existing before time
In an ideal plane they reside

Numbers thus exist there
Not here or anywhere

However, its just possible
Numbers were definable

To appear mysterious timeless
That's speculation, I guess

Nevertheless
Numbers are what they are
Nothingless

John Dewey, The Math and Science Standards and the Workplace

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Are there lessons to be learned by today's educational reformers from the fate of Dewey's pedagogical principles over the past century?

1. INTRODUCTION

One of the main points of this paper is to demonstrate that John Dewey's ideas on education are alive and well today, certainly in major efforts to reform school mathematics and science. A century ago he emerged as America's most influential philosopher. A powerful critic of the *status quo* in education, he was revered and reviled. For about the last fifty years his ideas have been denigrated, sometimes distorted, in broad educational circles (see Section 2). In Section 3 there is a brief discussion of the background of recent reform efforts in school mathematics and science. I show in Section 4 the strong similarities between Deweyan principles and major themes of the "standards" for school mathematics and science (which I shall call here the "math/sci standards"), formulated in recent years by leading professional organizations.[1-8] Then in Sections 5 and 6 I attempt to answer two questions raised by the renewed vitality of Dewey's ideas: Are there lessons to be learned by today's educational reformers from the fate of Dewey's pedagogical principles over the past century? Are the prospects better today than earlier in the century for the implementation ot those ideas in educational practice? My optimism about these prospects derives from the relevance of major themes of the math/sci standards to the needs of the changing industrial environment.

Dewey's educational ideas had radical implications for school organization and practices. In 1894, when he went to the University of Chicago as Head Professor of Philosophy, Psychology and Pedagogy, he established his so-called Laboratory School as an instrument to experiment with and refine his pedagogical principles. Although he insisted that it was not meant to be a "model" school for others to emulate, his influence and the power of his ideas were such that numerous schools were established, not only in the United States but abroad, which tried to carry out his ideas in varying degrees.

This "progressive education" movement flourished for several decades, particularly in private schools, but barely penetrated the great majority of American schools, those in the public school systems of the nation. Most of the resistance came from supporters of traditional educational practices, some from critics of certain aspects of his philosophy. Dewey himself acknowledged that many adherents of his philosophy had failed to appreciate the hard, detailed work required to implement new pedagogical principles (see Section 5):

> I think that only slight acquaintance with the history of education is needed to prove that educational reformers and innovators alone have felt the need for a philosophy of education. Those who adhered to the established system needed merely a few fine sounding words to justify existing practices. The real work was done by habits which were so fixed as to be institutional....It is, accordingly, a much more difficult task to work out the kinds of materials, of methods, and of social relationships that are appropriate to the new education than is the case with traditional education. I think many of the difficulties experienced in the conduct of progressive schools and many of the criticisms leveled against them arise from this source. [9, p. 29].

Having majored neither in philosophy nor education, I had not had occasion to read Dewey. In the late 50s (when my children began attending the local public school system) and in the 60s (when I served on its board of education), I would hear an occasional reference to progressive education, invariably derogatory. Thus, before this year it was my impression, when I thought about it at all, that hardly a trace remained of John Dewey's influence on American education. In spring 1995, however, I read the announcement of a conference on Dewey, [10] and one of the topics listed for discussion was "experiential education in and across the disciplines." In view of the attention given to experiential education in the math/sci standards, I decided to investigate how similar the major themes of these standards might be to Dewey's ideas.

In the course of this inquiry the only reference I found to such similarity is in a paper by Ratner: "John Dewey, E.H. Moore and the Philosophy of Mathematics Education In the Twentleth Century" [11]. The paucity of such references is consistent with Ratner's observation that since the 1920's, "Dewey and Moore have not been cited often in the reports of successive committees and commissions on mathematical education. Their seminal insights, however, have either been used or rediscovered and developed in new settings and situations..." [11, p. 114]. Here, besides comparing Dewey's ideas and the math/sci standards in some detail, I examine implications of their similarity.

2. CONFUSION ABOUT DEWEY'S IDEAS

At a recent mathematics conference where I spoke briefly on this subject, a questioner criticized Dewey's ideas, citing some "facts" which were simply wrong. Therefore I wish first to describe a few of the misunderstandings which are part of the Dewey legend in many people's minds. For example, he was criticized for advocating an adjustment ethic, the need to adapt oneself to existing social conditions. On the contrary, he emphasized repeatedly the importance of reconstructing or reshaping the social environment into a more desirable and ideal form.

Dewey himself contributed to this confusion, giving aid and comfort to his critics, for repeatedly he used the term "adjustment"—not in the usual sense, but in the sense of mutual accomodation. For example, he wrote early in his career that "every living form is dynamically not simply statically adapted to its environment. I mean by this, it subjects conditions about it to its own needs. This is the very meaning of 'adjustment'; it does not mean that the life-form passively accepts or submits to the conditions just as they are, but that it functionally subordinates these natural circumstances to its own food needs" [12, p. 51]. Later he remarked that, "We are also given to playing loose with the conception of adjustment, as if that meant something fixed—a kind of accommodation once for all (ideally at least) of the organism to an environment. But as life requires the fitness of the environment to the organic functions, adjustment to the environment means not passive acceptance of the latter, but acting so that the environing changes take a certain turn" [13, p. 8]. As one editor of his works has put it, "It is an irony of history that an adjustment ethic should be attributed to Dewey, for one of the concerns constantly manifest in his writings is that modern technological society is creating a more docile, passive individual" [14, p. xii].

More unjustly, he was attacked for suggesting that "education ought simply to cater to the needs and whims of the child" [14, p.xi]. The truth is quite otherwise. According to Dewey, "The fundamental factors in the educative process are an immature, undeveloped being, and certain social aims, meanings, values incarnate in the matured experience of the adult. The educative process is the due interaction of these forces" [15, p.182]. Conceding that "the kind of external imposition which was so common in the traditional school limited rather than promoted the intellectual and moral development of the young," he insisted nevertheless that the real problems of education "are not even recognized, to say nothing of being solved, when it is assumed that it suffices to reject the ideas and practices ot the old education and then go to the opposite extreme" of dispensing with the teacher's responsibility for planning and guiding the students' educational experiences [9, p. 22].

He stressed that "the greater maturity of experience which should belong to the adult as educator puts him in a position to evaluate each experience of the young in a way in which the one having the less mature experience cannot do....There is no point in his being more mature if, instead of using his greater insight to help organize the conditions of the experience of the immature, he throws away his insight" [9, p. 38].

3. THE MATH AND SCIENCE STANDARDS AND PREVIOUS REFORM MOVEMENTS

In 1989 the National Council of Teachers of Mathematics (NCTM) published "Curriculum and Evaluation Standardson School Mathematics," the first of several volumes expressing the consensus of professionals in the mathematical sciences "to guide reform in school mathematics in the next decade...in terms of content priority end emphasis" [1]. This was followed by the volumes "Professional Standards for Teaching Mathematics" [2], "Assessment Standards for School Mathematics," [3] and a number of booklets offering more detailed examples and guidance for various grade levels.

In a similar vein the National Research Council, following years of preparation and consultation with national science teachers organizations, has recently published "National Science Education Standards" [4], to offer guidance for school science. Taking their cue from these national trends (and often drawing from the "standards" documents), other organizations have formulated quite similar sets of principles; e.g., see [5, 6, 7, 8].

Since World War II there have been several "crisis" in mathematics and science education, starting with the American reaction to the launching of Sputnik. In school mathematics, for instance, the major reform thrusts were the "new math" (in the 1960's) and "back-to-basics" (in the 1970's). It is significant that these earlier reform efforts reached nowhere near the level of broad political and social recognition accorded to the educational concerns of the last decade. In the "education summits" of 1989 and 1996, convened by the nation's governors and attended by the sitting President and top business executives, particular attention was paid to the need to strengthen math and science education. (The leading figures at the 1989 conference were Governors Lamar Alexander of Tennessee and Bill Clinton of Arkansas.)

Willoughby has remarked that, "In 1894 the first national commisslon on mathematics education, known as the Committee of Ten," issued a report recommending, among other things, that mathematics be taught in a more integrated way rather than as isolated subjects, that more attention be given to realistic problem solving, and that there be "more emphasis on intuition and thinking" [16, p. 8].

"In the intervening hundred years," he goes on to say, "reports of committees, commissions, and others have regularly told us what is wrong with mathematics education and what we should do to fix it. Today the teaching of mathematics in most American classrooms resembles the teaching of 1894 more closely than it resembles the recommendations of the Committee of Ten or its many successors. What has gone wrong? Will this time be different?" [16, p. 8]. In my opinion it may well be different this time, and I give my reasons in Sections 6 and 7.

4. SIMILARITIES BETWEEN DEWEY'S IDEAS AND ELEMENTS OF THE STANDARDS PROPOSED FOR SCHOOL MATHEMATICS AND SCIENCE

The high degree of compatibility between Dewey's educational philosophy and major ideas of the math/sci standards may be noted by comparing statements on several aspects of pedagogy, as follows.

A major principle of current reform literature is that teachers should build on students' prior understandings. For example, it is urged that "in determining the specific science content and activities that make up a curriculum, teachers consider the students who will be learning the science. Whether working with mandated content and activities, selecting from extant activities, or creating original activities, teachers plan to meet the particular interests, knowledge, and skills of their students and build on their questions and ideas....Teachers are aware of and understand common naive concepts in science for given grade levels, as well as the cultural and experiential background of students and the effects these have on learning" [4, p. 30]. According to Dewey, "It is a cardinal precept of the newer school of education that the beginning of instruction shall be made with the experience learners already have; that this experience and the capacities already developed during its course provide the starting point for all further learning" [9, p. 74].

A broad theme of the standards is constructivism, a philosophy of learning in which "the focus is on 'allowing students to make meaning for themselves' rather than just barraging them with information" [8, p. 3]. According to the New York State Education Department's "Learning Standards for Mathematics, Science and Technology": "Students formulate questions independently...construct explanations independently for natural phenomena...seek to clarify, to assess critically, and to reconcile with their own thinking the ideas presented by others, including peers, teachers, authors, and scientists" [5, p. 4]. In Dewey's words, "The final problem of instruction is the reconstruction of [the student's] experience" [17, p. 74]. In this regard it is worth noting that the work of the Swiss psychologist and educator Jean Piaget on learning and cognition has profoundly influenced current thinking about cognitive science and intellectual development, and Piaget was a major contributor to the constructivist philosophy. "One of the most enduring and influential of Piaget's beliefs about cognition is that individuals actively construct their world...individuals operate with and on the environment, constructing their own perceptions as they assimilate new experiences into existing schemes and adapt the schemes to accommodate the constraints of the experiences" [18, p. 15].

In the 1989 NCTM "Standards" it is asserted that "'knowing' mathematics is 'doing' mathematics. A person gathers, discovers or creates knowledge in the course of some activity having a purpose" [1, p. 7]. A similar sentiment about knowledge in general was expressed by Dewey: "Knowledge is not something separate and self-sufficing, but is involved in the process by which life is sustained and evolved" [19, p. 87]. The editor of a volume of Dewey's writings has paraphrased Dewey's ideas as follows: "Thought is not theoretical, but a doing; for the solutions it proposes for the elimination of obstacles are not mere hypotheses, devised for intellectual or aesthetic satisfaction, but hypotheses to be tested in action, so that if they are successful, experience may move on to a further stage" [20, p. 15].

A core concept of Dewey's pedagogical principles was the transactional character of experience. "The nature of experience can be understood only by noting that it includes an active and a passive element peculiarly combined. On the active hand, experience is trying—a meaning which is made explicit in the connected term experiment. On the passive, it is undergoing. When we experience something, we act upon it, we do something with it; then we suffer or undergo the consequences. We do something to the thing and then it does something to us in return" [21, p. 139]. A similar notion, expressed somewhat differently, is found In the 1989 NCTM "Standards": "These goals imply that students...should be encouraged to explore, to guess, and even to make and correct errors...." [1, p. 5].

The most common buzzword of reform efforts in the past two decades has been "problem-solving". Dewey remarked that, "problems are the stimulus to thinking...growth depends on the presence of difficulty to be overcome by the exercise of intelligence" [9, p. 79]. The most effective learning, he thought, is that based on inquiry, on the application of intelligence to resolve a problematic situation. One of the New York State "Learning Standards" is "Interdisciplinary Problem Solving: Students will apply the knowledge and thinking skills of mathematics, science, and technology to address real-life problems and make informed decisions" [5, p. 61].

There is agreement not only on conceptual or philosophical aspects of education, but also on desirable modes of classroom behavior. Both Dewey and the math/sci standards emphasize the importance of communication between student and teachers and among students, and the value of students working together for common purposes.

Dewey criticized severely the traditional classroom scenario in which children were expected to sit quietly and learn by listening, for "The language instinct is the simplest form of the social expression of the child. Hence it is a great, perhaps the greatest of all, educational resources" [15, p. 43]. Similarly, in the 1989 NCTM "Standards" one ot the five listed "New Goals for Students" is "that they learn to communicate mathematically" [1, p. 5]. "This is best accomplished in problem situations in which students have an opportunity to read, write and discuss ideas in which the use of the language of mathematics becomes natural. As students communicate their ideas, they learn to clarify, refine, and consolidate their thinking" [1, p. 6].

Where the school work consists in simply learning lessons, mutual assistance, instead of being the most natural form of cooperation and association, becomes a clandestine effort to relieve one's neighbor of his proper duties. Where active work is going on, all this is changed.

Finally, Dewey has extolled the educational and social values of student collaboration. In the traditional classroom, he wrote, "The mere absorbing of facts and truths is so exclusively individual an affair that it tends very naturally into selfishness....Indeed, almost the only measure for success is a competitive one, in the bad sense of that term—a comparison of results in the recitation or in the examination to see which child has succeeded in getting ahead of others in storing up, in accumulating, the maximum of information. So thoroughly is this the prevailing atmosphere that for one child to help another in his task has become a school crime. Where the school work consists in simply learning lessons, mutual assistance, instead of being the most natural form of cooperation and association, becomes a clandestine effort to relieve one's neighbor of his proper duties. Where active work is going on, all this is changed. Helping others, instead of being a form of charity which impoverishes the recipient, is simply an aid in setting free the powers and furthering the impulse of the one helped. A spirit of free communication, of interchange of ideas, suggestions, results, both successes and failures of previous experiences, becomes the dominating note of the recitation" [15, p. 15 - 16].

Similarly various formulations of the standards advocate "cooperative learning" and elaborate for teachers various ways of guiding such efforts. For example, the "Science Teaching Standards" suggest that "using a collaborative group structure, teachers encourage...students to work together in small groups so that all participate in sharing data and in developing group reports. Teachers also give groups opportunities to make presentations of their work and to engage with their classmates in explaining, clarifying, and justifying what they have learned. The teacher's role in these small and larger group interactions is to listen, encourage broad participation, and judge how to guide discussion—determining ideas to follow, ideas to question, information to provide, and connections to make. In the hands of a skilled teacher, such group work leads students to recognize the expertise that different members of the group bring to each endeavor and the greater value of evidence and argument over personality and style" [4, p. 36].

5. LESSONS TO BE LEARNED

Certainly it is of interest to Dewey scholars that the math/sci standards contain major elements of Dewey's educational philosophy. This fact has practical implications, however, for a much wider constituency in the educational community. It is reasonable, for example, to ask whether we can learn any lessons about the prospects for current reform efforts from the problems of progressive education.

The best critique of progressive education that I have found is by Dewey himself. In 1938, forty years after he had begun articulating his objections to traditional education and his principles for a new educational philosophy, he reflected at length on how progressive education was being carried out [9]. In general he cautioned against a new orthodoxy: "It is not too much to say that an educational philosophy which professes to be based on the idea of freedom may become as dogmatic as ever was the traditional education which is reacted against" [9, p. 22].

Criticizing a common tendency of progressive teachers to define their practice "negatively or by reaction against what has been current in education," he cited three specific points of concern, namely, that "many of the newer schools tend to make little or nothing of organized subject-matter of study, to proceed as if any form of direction or guidance by adults were an invasion of individual freedom, and as if the idea that education should be concerned with the present and future meant that acquaintance with the past has little or no role to play in education" [9, p. 22].

"It is not too much to say that an educational philosophy which professes to be based on the idea of freedom may become as dogmatic as ever was the traditional education which is reacted against"

The concern about neglect of the past does not appear to be relevant to mathematics and science education. For example, discussion of the Pythagorean theorem with a class of middle school children that I meet weekly immediately spans the millennia. In other words discussion of many topics in mathematics and science revolve around seminal results and insights bearing the name of a major figure in the history of the subject. The other issues, however, require further comment.

Dewey had not rejected the need for organized subject matter, but rather the way in which it was used in traditional education; similarly he had criticized the manner in which adult guidance was exercised in traditional education, not the need for such guidance. It is instructive to recall his view of the proper roles of these ingredients in his pedagogy, in what he called "the organic connection between education and experience" [9, p. 26].

An essential characteristic of Dewey's "newer education"—in fact, of any educational philosophy—is "continuity or the experiential continuum. This principle is involved in every attempt to discriminate between experiences that are worthwhile educationally and those that are not" [9, p. 33]. While the teacher must try to harness the student's attention and involvement by an interesting experience or activity—an enjoyable one, if possible—this is by no means sufficient. There has to be an end in view for the experience.

In Dewey's view an effective program requires the design (by the teacher) of a sequence of experiences, each building on the previous one, each preparing for the next. The program begins where the student is at, with the knowledge and conceptions (or misconceptions, for that matter) based on prior experiences; then the teacher has the responsibility of guiding the student through the process of developing and grasping new concepts to the point where the student has constructed (or reconstructed) the subject-matter for him/herself in an objective, organized form. Dewey observed sadly that many teachers in progressive schools focused on the design of individual stimulating experiences without giving adequate attention to the follow-up.

The lesson is clear. There must be an awareness of the need for continuity and planning on the part of those who want to teach mathematics or science according to the Standards' (or Dewey's) principles, especially on the part of elementary teachers, who are not in general math or science specialists. In the late 70's and early 80's, when the use of manipulatives in elementary mathematics was becoming increasingly popular, I remember seeing some teachers get excited about the fun to be had by teachers and pupils from working with manipulatives, without giving thought to the purpose, the mathematical insight to be achieved by the hands-on activity.

Happily those who developed and formulated the standards are aware of the teacher's critical role as educational guide and coach and the need for organization and continuity in guiding students' experiences. In the "Science Teaching Standards", under "TEACHING STANDARD A: Teachers of science plan an inquiry-based science program for their students," the first injunction for the teacher is to "develop a framework of yearlong and short-term goals for students" [4, p. 30]. Likewise, it is noted in the 1989 "Standards" that "it takes careful planning to create a curriculum!" [1, p. 11] and in the 1991 "Standards" that among "the important decisions that a teacher makes in teaching" are "setting goals and selecting or creating mathematical tasks to help students achieve these goals" [2, p. 5].

6. IS THE GROUND MORE FERTILE TODAY FOR IDEAS LIKE DEWEY'S?

Since education in the U.S. is a state responsibility, there is no official or uniform national curriculum; but the math/sci standards are being embraced broadly by state educational establishments. According to the National Science Foundation, "after the release of the NCTM standards in 1989, states began modifying their curriculum frameworks for science and mathematics. By 1994, twenty-four states had published revisions, and by 1995 still more states were in the process of publishing new or revised guidelines—thirty-seven in science and thirty-three in mathematics" [22, p. 34].

Official endorsement is one thing; implementation in the classroom is quite another (as was the case with the "new math"). The NSF reports that "based on the indicators presented here, the learning environment is becoming more like the one envisioned in the standards," but admittedly at a slow and uneven pace. "Teachers are beginning to implement many of the recommendations in the science and mathematics standards. In general, high school teachers are the group most resistant to reform. Despite recommendations to increase the use of hands-on activities and approach subjects in more depth [they] continue to rely heavily on lectures, and less than one-half assign long-term projects. In addition, most are not using computers for science and mathematics instruction. Generally, science and mathematics classes are poorly supported in terms of facilities and supplies" [22, p. 68 - 69].

The need for more and better computers is receiving much attention now from government officials and the media. This problem should be eased considerably in the next few years. But high school teachers seem to be more wary of change. As specialists in their subjects, they tend to be more comfortable with what they have been teaching in the past and how they have been teaching it. The greater receptiveness of elementary teachers to new approaches to math and science instruction may well be due to a felt need for better understanding in these areas, for a non-trivial percentage of elementary teachers are uncomfortable about their grasp of math and/or science and their ability to teach them.

Let me describe some of my own experience in this regard. Three years in a row, from 1979 to 1982, the Department of Mathematical Sciences at Rensselaer Polytechnic Institute offered an in-service course for elementary teachers, "Add Intuition to Math, Subtract Anxiety" [23], which embraced principles and pedagogical approaches similar to those formulated later in the mathematics standards [1, 2]. Not all the course participants were "math-anxious"; some were primarily interested in helping students so afflicted. But tests of the participants' mathematics skills, questionnaires about their feelings and attitudes toward math, and pre- and post-course assessments of their degree of math anxiety demonstrated that our course improved substantially, sometimes dramatically, the participants' understanding of mathematics and their level of comfort in teaching it.

The difficulty of carrying out any educational reform should not be underestimated. Apart from issues of technology and attitudes, many teachers, if not most, will need in-service instruction in the content and pedagogy of the standards; this will incur increased public expenditure, always an argument against change. For example, a news story about New Jersey updating its curriculum notes that, "The new standards brought an outcry from school districts that feel caught between new programs and demands to cut their budgets....While praising the idea of pushing students to learn more, critics said teachers would need new training" [24].

Yet there are substantial reasons for believing that ideas like Dewey's will in the next decade send deep roots into our educational soil. In answer to Willoughby's [16] question as to whether this reform movement will be any more successful than previous ones, I suggest that what is different this time is that the demands of today's workplace are very compatible with the math/sci standards.

A major role of the public schools is to equip young people with the skills they will need to make a living. In the early decades of this century mass production was increasingly characteristic of manufacturing. An historic development in this regard, in 1913, was the Ford Motor Company's first use of the assembly line, where workers had to perform repetitive tasks prescribed by designers and production engineers. One historian has remarked that, "Even more than previous manufacturing technologies, the assembly line implied that men, too, could be mechanized" [25, pp.10 - 11].

Industrialists viewing the assembly line as the new paradigm for production would scarcely seek critical thinkers or problem solvers for the factory floor.

'Look, it isn't a matter of doing a better job teaching what people used to need. We expect our workers to tackle problems they have never seen before, to work together and to communicate their ideas to others'

Dewey's ideas, however, were aimed precisely at encouraging children to maintain and exercise their curiosity and imagination. "Plato somewhere speaks of the slave as one who in his actions does not express his own ideas, but those of some other man. It is our social problem now, even more urgent than in the time of Plato, that method, purpose, understanding, shall exist in the consciousness of the one who does the work, that his activity shall have meaning to himself....How many of the employed are today mere appendages to the machines which they operate!" [15, pp. 23 - 24]. Would businessmen and other community leaders view such ideas as contributing properly to the preparation of young people for the world of work?

Today, however, cutting-edge technology has quite different expectations of production workers. Discussing production and innovation in the semiconductor industry, Kenney and Florida have written that, "In Japanese corporations...once the technology is designed and implemented, factory workers make continuous suggestions on how to upgrade and improve both the quality of the technology and the manufacturing process. . ." [26, p. 67]. In reference to the steel industry they use more dramatic language to describe how production workers have contributed to improving the efficiency of the "cold-rolling" process: "Nippon Steel has turned the cold-rolling of steel into a continuous process that takes less than one hour from start to finish. It achieved this by unleashing the collective intelligence of its workers" [26, p. 67].

What a conceptual change: from "men...could be mechanized" to "the collective intelligence of...workers"!

American corporations also envisage an altered role for production workers. "In the mid-1980's," according to Craig R. Barrett, chief operating officer of the Intel Corporation, "it became brutally apparent" to the U.S. semiconductor industry "that all the smart technologists in the world would not make this industry a success. We had to get down and vastly improve our manufacturing efficiency" [27]. An essential part in this task is now being played by a new breed of workers, usually with two-year degrees from technical schools or electrical engineering degrees; they "work with one thousand personal computers in the factory, searching for...'a plethora of small continuous improvements' intended to hasten production and improve chip yields" [27].

Finally, consider again the auto industry, where in 1913 the assembly line worker was to be an unthinking cog in the production machinery. In a *New York Times* article of April 21, 1996, discussing a huge new hiring spree by America's Big Three auto manufacturers, it was estimated that 170,000 new factory workers will be hired by Ford, General Motors, and Chrysler in the next seven years:

> The Big Three have...borrowed Japanese management practices, which emphasize teamwork and job flexibility on the factory floor...now that they are hiring again, they are putting quick minds ahead of strong bodies. [Applicants go] through a grueling selection process that emphasizes mental acuity and communication skills. All three companies have contracted with...a human-resources consulting firm...to screen candidates. The firm checks their reading and math abilities, manual dexterity and understanding of spatial relations....Those who jump that first hurdle are tested for drug use.Then, for about three hours, applicants are put in groups of four to six and given a task to complete while...consultants assess their ability to work together [28].

7. CONCLUSION

In trying to relate the demands of industry to the changes in mathematics content and pedagogy called for in the NCTM standards, Robert J. Kansky, associate executive director of the Mathematical Sciences Education Board, has suggested that the search for a new way of teaching math began because business and industry leaders complained that students were not learning what they needed to know on the job. "They were saying, 'Look, it isn't a matter of doing a better job teaching what people used to need. We expect our workers to tackle problems they have never seen before, to work together and to communicate their ideas to others'" [29].

Kansky's suggestion that it was pressure from industry that drove math reform is simplistic. As noted above, already in the late 1970's the widespread dissatisfaction of educators with traditional math curricula and teaching practices had given rise to many experimental programs around the country. Yet there is significant agreement between the emphases in the math/science standards and the capabilities being sought increasingly in new employees by major corporations.

A century ago, when John Dewey urged that American education stress the cultivation of such qualities, the message did not appear relevant to the needs and goals of the practical people, the leaders of business and industry. Now, however, we can expect that the new "standards" for school math and science, which incorporate so many of Dewey's ideas, will grow in favor with the educational community and the larger society because their time has come, because they are designed to encourage students' curiosity and imagination and other qualities which industry is finding increasingly valuable in its workers.

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The OBJECT and the STUDY of Mathematics

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This paper consists of my reflections on the object and the study of mathematics. One of the main issues that we tried to deal with in the 'Philosophy of Math' meetings with professor Alvin White of Harvey Mudd College was the question: "What is the object of mathematics?" This question raises many relevant questions such as: What is mathematical truth? What is the foundation of mathematics, if there exists one? How do we obtain mathematical knowledge? Is the method of mathematical proof the only conceivable method in mathematics? The questions above are all linked to one another, and let us tackle them as a whole.

As Evert W. Beth argues: "According to the current view, mathematics is concerned with immaterial objects: points without dimensions, lines with no thickness and so on."[1]

In other words, this common view argues that mathematics is concerned with abstractions.

"Mathematics abstracts, idealizes, schematizes, constructs, simulates."[2] According to Putnam, mathematical truth comes from the fact that mathematics is an objective science, and as he states:

> Mathematics should be interpreted realistically -that is, that mathematics makes assertions that are objectively true or false, independently of the human mind, and that something answers to such mathematical notions as 'set' and 'function'. This is not to say that reality is somehow bifurcated -that is, there is one reality of material things, and then, over and above it, a second reality of 'mathematical things'.[3]

If we accept Putnam's view on mathematical truth, how can we, human beings, witness and fully conceive the foundations of mathematics? We, human beings, have access only to the material world. If mathematics has a second reality of 'mathematical things' which we cannot observe and envision, can we still believe that mathematics has its foundations in our real world?

Putnam argues that mathematics does not have a crisis in its foundations. He does not believe that mathematics either has or needs foundations. In our traditional thinking, we ascribe properties such as length, width, thickness to material objects, and therefore, when a human being thinks of a material object, she/he directly links it with the subjective object, not a real external object. If such objects ,which we call external objects, exist, we cannot visualize them, but can conceive their powers. For example, we cannot visualize a 'set' as we can visualize an 'apple', but we can use the notion of a 'set' in constructing powerful mathematical theories, and conceive the power of this construction, although it is not a subjective object.

The first step to obtaining mathematical truth is by obtaining mathematical knowledge. Because mathematical objects are non-physical realities, the common view, as Putnam points out, is that the kind of knowledge we have in mathematics is strictly a priori. However, Putnam also argues that mathematical knowledge ,in fact, resembles empirical knowledge - " that is, that the criterion of truth in mathematics just as much as in physics is success of our ideas in practice, and that mathematical knowledge is corrigible and not absolute."[4] Therefore, what matters in mathematical truth is the power of its non-physical realities in making a coherent link that is understandable by the physical world. In this sense, mathematical knowledge plays an important role in mediating between the physical world and the non-physical realities of mathematical truth. In other words, mathematical knowledge makes mathematical truth understandable by the physical world.

Mathematical knowledge starts with precise definitions and auxiliary assumptions. Then, these definitions are linked together under the assumptions, and theories are formed. Thomas Hobbes's definitions of science,in his book Leviathan, is that science is the knowledge of consequences. In the light of his definition, mathematical objects are the knowledge of the consequences of each link between given definitions, conclusions and auxiliary assumptions.

What are the criteria for a good foundation of a mathematical theory?

1) Assumptions must be realistic: Assumptions are made to simplify situations, but they must have a large scope. They should be broad enough to be applied to more complicated situations.

2) Definitions must be simple, clear and precise.

3) Assumptions and definitions must be consistent: The theoretician must use them consistently in each link of the axiomatic structure, and newly built definitions must not contradict the older ones.

4) Definitions and each step of the axiomatic argument must be fruitful: They must enable the construction of new definitions and new steps from them.

If these Kuhnian inspired conditions are satisfied, the mathematical objects which are the basic ingredients of mathematical knowledge will be based on a strong foundation. Theorems, propositions and so on which will arise from this foundation will be coherent since their build up will consist of consistent links between the knowledge of consequences of each step we take.

As Putnam argues, generally, in empirical sciences, for each theory, there exists other altemative theories, or those which are struggling to be born. He notes:

> As long as the major parts of classical logic and number theory and analysis have no alternatives in the field—alternatives which require a change in the axioms and which effect the simplicity of total science, including empirical science, so that a choice has to be made—the situation will be what it has always been. [5]

This argument suggests that once a theory which is more powerful than the already existing one appears, it is justified to accept the new theory. Putnam believes that the mathematicians can be wrong, not in the sense that their proofs are misleading, but that the auxiliary assumptions they use might be wrong. He believes that this flexible character of mathematics which allows alternatives into the field makes mathematics *'empirical'*.

His understanding of mathematics resembles the Kuhnian notion of paradigms. As Kuhn suggests, many different paradigms can exist in science. He argues that if an already existing paradigm has anomalies, and contradictions, a new paradigm is formed. According to Kuhn, as long as both of the paradigms are able to generate and solve puzzles, they are equally adequate. It does not have to be that the supporters of the different paradigms are in disagreement. Kuhn's paradigms are incommensurable. Similarly, Putnam allows the existence of different paradigms in mathematics. An example to this notion in mathematics is the non-rival existence of both Euclidean and non-Euclidean geometry.

What happens if a newly formed theory is in contradiction with the already existing one? According to Popper, we would then have to test the two theories and falsify the less-satisfactory one. However, this contradicts the notion of mathematical truth that once a mathematical problem is solved, it is solved forever.[6] If a mathematical theory is formed on the basis of the criteria I have suggested, it is impossible that a proof would be wrong since each knowledge we acquire at each step of our theory is a consequence of the conclusions we acquire in the preceding step. Therefore, if it happens that a theory contradicts the other, we might want to follow Duhem's suggestion and look at the auxiliary assumptions we make. It might be that our assumptions are false, or that the assumptions of the two contradicting theories are incommensurable.

Having discussed the foundations of mathematical knowledge, next we ask the question: " Once we have all the ingredients, how can we cook our recipe to obtain mathematical knowledge?" As Putnam argues: "It does seem at first blush as if the sole method that mathematicians do use or can use is the method of mathematical proof, and as if that method consists simply in deriving conclusions from axioms which have been fixed once and for all by rules of derivation which have been fixed once and for all." [7] Putnam creates an interesting story about Martian mathematics where Martians, in testing theories, use 'quasi-empirical' methods which consist of statements generalized by induction. This method of acquiring mathematical knowledge creates a new concept which we can call: 'mathematical confirmation'. Putnam argues that our refusal to use this 'Martian' method limits the range of our proofs to only analytical ones. If we were to use quasi-empirical methods, we could also enjoy the discovery of synthetic truths in mathematics. Actually, when we look back at our history, we encounter the use of quasi-empirical methods. As Putnam suggests, the Greeks lacked the mathematical experience and mathematical sophistication, and therefore, they used generalizations in their mathematical conjectures. The simplest example is that the real numbers were not introduced through a rigorous mathematical justification. However, the use of real numbers in mathematics now enables us to construct more complicated theories.

To conclude, mathematics is a very interesting branch with lots of questions concerning its origin, study, methodology and its direction. It is, though, a unique branch because it is precise, objective and universal. It does not directly conjecture on nature as do the natural sciences, but it provides a language and foundation on which natural sciences can base their studies securely. The object of mathematics is not a subjective object, but one can feel its power through its preciseness, consistency and fruitfulness.

NOTES

1. Beth, pg: 25

2.ibid pg 24

3.Hilary Putnam, Mathematics. Matter and Method, Cambridge, 1975, pg: 60

4. ibid. pg 61

5. ibid. pg: 51

6. Rota, Mathematics and Philosophy

7. Putnam, pg: 61

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Inspiration in England

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Classes greatly reduce the amount of unstructured time available for study. They also take place during our brains' most productive hours.

In my Philosophy of Math class, we learned much about the role inspiration plays in any creative process. Inspiration is the vital element of any successful enterprise. Inspiration, however, cannot strike a "burned out" mind [1]. While feeling "burned out," Richard Feynman writes that his attempts to do research are fruitless. Similarly, I rely heavily upon inspiration to do math problems. A few weeks into the semester, however, I already feel "burned out." While college life in America burns me out, I found while studying abroad that college life in England inspires me.

The difference is rooted in Pomona College's academic structure. American college practices place much pressure and limits on students. For example, Pomona students take four classes per semester, often in a wide variety of topics geared towards fulfilling general education requirements. Managing four very different courses is stressful. Each course demands attention, so we are restrained from focusing all of our time on something we find interesting. Another confining convention is the large amount of time we spend in class. Classes greatly reduce the amount of unstructured time available for study. They also take place during our brains' most productive hours.

The grading system creates a stress that dulls inspiration. Of course, in both England and America students receive final grades for their course of study. However, at Pomona, every piece of work I turn in is graded and represents a fraction of my extremely important GPA. Knowing that a specific project is worth exactly X% of a course grade often heightens the stress associated with the work. This quantification also allows students to calculate the bare minimum effort needed for a particular grade. For example, after a classmate calculated that a 0% math paper grade would not hurt her average, she decided to not write it.

The scheduling and grading practices develop a negative attitude towards schoolwork. College becomes a function of balancing and minimizing input to maximize grades. The simple objective of learning is not important. For example, in some of my math classes, we are given seven homework problems, five of which we have to complete. When I start the assignment I often concentrate on choosing which 5 of the 7 are the easiest. My fellow classmates and I do not discuss which problems are the most interesting, but which problems are the easiest or take the shortest amount of time to finish. I would never go beyond what was required and complete an extra problem.

My college schedule, like other Pomona students, includes classes, an internship, plus other activities, which means most work must be done in the evenings. By this time, however, we are no longer as energetic as we were earlier in the day. In this state of mind, I completely dread doing anything that might overexert my brain. I have to force myself to start work, often after much procrastination. One glance at a math problem set almost makes me physically ill. The surprising thing is that once I get started on the set, I remember how much I enjoy it. My state of mind at the end of my day just prevents me from remembering how rewarding work can be.

This enjoyment, however, is completely confined to the academic compartment of my life. At Pomona, we experience a strong boundary between our academics and recreation. Therefore I am not allowed to let any excitement I feel about math to leak into the "fun" sector of my life. Once, I shared my excitement about an abelian group theorem with a friend. After teasing me for "getting abelian" on him, he remarked that my enthusiasm was "cute" and subsequently changed the subject. On another occasion, a friend commented that "we're such nerds" because we started discussing philosophy during a social occasion. Our negative view towards academic work keeps us from letting it enter other parts of our life.

The structure of college life at Oxford University greatly affected my attitude towards academics. One helpful change was that we did not receive grades during the term. We received feedback in terms of comments, but not quantification. In addition, we did not take four demanding classes, but experienced the tutorial system. We took one main tutorial and two less demanding tutorials. We met with our tutors only once a week, leaving the bulk of our time unstructured. This arrangement called for much more work to be done independently.

The difference in scheduling revolutionized our work habits. We still had sports and activities to lend some structure to our days, but compared to Pomona, we were very rich in unstructured time. For the first time in years, I had free days. This time took the pressure off and allowed us to have fun with our studies. Although I was not in class, I still spent this time on academics. Preparing for each tutorial meant spending many hours in the college library, so we had to work during the day. For some reason, working in the morning and afternoon was much better than working at night. Our minds were fresh and not tired from a long day of classes. Learning independently also required active participation rather than passive listening. Consequently our work stimulated our brains, so our time spent in the library preparing for tutorial was much more fruitful than the time spent in class. I also did not feel spread thin between my classes. Having a main tutorial saved me from juggling my attention between courses. The tutorial structure also offered us a more relaxed structure regarding curriculum. In some cases, our own curiosity played a major role in determining the content of our tutorials. Our tutors took advantage of the fact that we would develop interests and directions of our own. We could immerse ourselves in our studies without performing a balancing act.

When I shared the seemingly odd fact that a valid conclusion could be drawn from false premises, my friends actually listened and thought of examples.

Although we knew we would have grades at the end of the term, the fact that we were not graded along the way helped us to have fun with our studies. The lack of immediate grades removed the immediate concern with quantified achievement, refocusing our goals on learning. At first I was completely disoriented because I did not know where I stood in my classes and had no idea what standards a good grade required. As a result, I could not just do the bare minimum. This changed the way I approached my assignments. For example, my tutor gave me a Logic assignment that included a few extra challenging proofs that I was not required to do. Surprisingly, I worked very hard to figure out the challenging problems. At Pomona, I would not have done them.

Our different outlook toward work also changed our attitudes toward play. The line between work and fun blurred. Although much of our work was done in the library, our brains did not turn off when we closed our books. During tea time in the Junior Common Room, dinner, evenings at the college bar, or late nights over kebab van fare, we did not cease to share our thoughts and new ideas from our tutorials. Our sharing sessions often developed into passionate debates. These discussions inspired us to learn more about our interests and go beyond the minimum for our tutorials. Sharing my enthusiasm for logic wasn't regarded as "cute," but was taken seriously. When I shared the seemingly odd fact that a valid conclusion could be drawn from false premises, my friends actually listened and thought of examples. I have fond memories of discussing how it could be so that A implies B is true when A is false and B is true. Not only did our discussions help my work, but they affected my friends. My friend Sierra remarked that our Logic discussions helped her to understand LSAT questions. My friend Devon decided to take a Logic course when we returned to Pomona. Instead of feeling burned out on the subject of Logic, I felt excited about it.

Basically, the Oxford structure allowed us to enjoy studying. It removed stress (such as constant grading and time constraints) where stress was counterproductive. We had much more flexibility in terms of time and curriculum. Without the rigidity of Pomona College, I found inspiration instead of burn out.

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A COURSE IN MATHEMATICAL ETHICS

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Summary: *This paper is a report of a course in mathematical ethics that I have developed and taught. The course is designed for math and computer science majors in their junior and senior years. A course outline is given and various projects involving mathematical ethics are discussed. Arguments are presented that ethics is relevant and crucial to mathematicians.*

This paper is based on a talk presented at the national meeting of the American Mathematical Society in San Diego on January 10, 1997.

Longwood College, as part of its general education requirements, specifies that every student must take an ethics course, preferable in the student's major. When the requirement was instituted, I was given the job of developing an ethics course for the Mathematics and Computer Science Department.

Several problems were apparent. First, I had no formal training in ethics. Second, I did not know where to look for material and resources. Third, I had no existing course to use as a model.

Having no formal training in ethics proved not to be much of a problem. I quickly found several books in computer science ethics, and most of them had an introductory chapter or two giving an overview of ethical theory. After reading them I used a member of the Philosophy Department as a resource. I did not become an expert, but today I can hold my own in a discussion of ethical theory.

The second problem, the lack of resources, proved to be more difficult. There is a good deal written about ethics in computer science and the natural sciences, but I wanted something pertinent to mathematics as well. It appears that very little work has been done on mathematical ethics. Indeed, most mathematicians appear not to have thought much about the subject at all. None of the major mathematical societies have codes of ethics for their members. (The American Mathematical Society recently adopted an ethics code, but it is a code for the society, not for individual mathematicians.) Even the National Council of Teachers of Mathematics has no code of ethics. This is in marked contrast to other disciplines.

I was able to develop and teach the course. It has the following assumptions and objectives:

*MATHEMATICIANS NEED TO THINK ABOUT THE CONSEQUENCES TO SOCIETY OF APPLYING THE KNOWLEDGE THEY POSSESS.

*MANY ETHICAL AND MORAL DILEMMAS ARISE IN THE NORMAL COURSE OF A MATHEMATICIAN'S CAREER, AND THE PROFESSIONAL MUST BE ABLE TO RECOGNIZE AND RESOLVE THEM.

*MATHEMATICIANS NEED TO UNDERSTAND THE IMPACT OF ACADEMIC MATERIAL ON THE REST OF SOCIETY.

*MATHEMATICIANS NEED TO HAVE A BETTER APPRECIATION OF THE RELATIONSHIP BETWEEN TECHNICAL ISSUES AND HUMAN VALUES.

Mathematical ethics textbooks do not exist, so I have had to resort to using those written for computer science. [1], [2], and [3] are good ones. While these books are intended for a computer science audience, much of the material they contain applies to mathematicians. Of these, [1] and [3] have excellent introductory chapters on ethical theory. [1] has more material than [3], and both are more directly focused on computer science than is [2]. Unfortunately, some of the readings in [2], while more general, are dated.

The course description from the college catalog is as follows. "Consideration of ethical implications of mathematics and computer science in society. Overview of ethical theory; case studies of situations illustrating ethical dilemmas. A knowledge of calculus and algorithms will be assumed. 1 credit."

The course begins with an overview of ethical theory. This is the first course in ethics for most of the students, and they need some foundation on which to base later discussions. I cover three of the theories which appear to be the most widespread and influential today: utilitarianism, Kantian (deontological), and relativism. I require the students to decide with which of these three they most agree and write a short paper defending their choice. This is not the same as requiring the students actually to believe in one of these three, of course. I regard requiring belief, or imposing my beliefs, to be unethical.

The rest of the course consists largely of readings, discussion, and case studies, with students expected to apply the theories to the cases. Lecture is kept to a minimum. Class discussion and writing are emphasized. The grade is based on class participation, a test, two short (two to three page) papers, and a longer (five to six page) paper requiring some research, in which the student is expected to analyze an ethical dilemma pertaining to his or her intended career involving mathematics. The calculus and algorithms prerequisites for the course are mainly for maturity. Most of the topics can be understood even by those who are mathematically weak.

A big challenge is to make sure the course contains material of interest to mathematicians; the temptation is to take the easy way out and use the readily accessible material that applies primarily to computer science. Fortunately, many topics naturally overlap the two disciplines. Professional obligations, including "whistle blowing" and complete disclosure; intellectual property rights, including copyrighting issues; and matters involving improper use of data, whether to invade privacy or to draw misleading and unsubstantiated conclusions, are examples of issues involving ethical dilemmas that are likely to be faced by mathematicians as well as computer scientists.

It is difficult to find good case studies pertaining solely to mathematical ethics. I have found two topics that may be used successfully in a course. The first concerns the ethical implications of patenting algorithms. Is it ethical to patent a problem solving method and to make others pay for use of one's intellectual products? Suppose, for instance, that you find a new, faster way of solving the Traveling Salesman Problem, a discovery that has great economic implications. Are you ethically justified in patenting your method and charging others for its use?

The second topic has to do with censorship. Two decades ago the National Security Agency (NSA) tried to require mathematicians working on cryptography to submit their work to NSA prior to publication for censorship, arguing that allowing America's enemies access to powerful techniques for making and breaking codes could be devastating to national security. Today there is great controversy over the export of American products involving sophisticated encryption schemes to foreigners. If you are doing research in cryptography, do you have an ethical obligation to submit your work for censorship - or are you ethically obliged not to do so?

I believe that this course is one of the most important mathematics courses that the majors take. Mathematicians possess powerful knowledge, knowledge that can have tremendous effects on other people and on society, and I believe we have a moral obligation to use it wisely. I am very concerned that few mathematicians think about the consequences of what they do, and I believe we must provide a framework for our students to use as they make professional decisions in their careers.

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Book Review: Ethnomathematics--Challenging Eurocentrism in Mathematics Education, Arthur B. Powell and Marilyn Frankenstein, Editors

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Ethnomathematics: Challenging Eurocentricism in Mathematics Education. Powell, Arthur B. and Marilyn Frankenstein, eds. State University of New York Press: Albany, 1997. 440p. ISBN 0-7914-3351-X.

In recent years, more mathematicians and mathematics educators have been introduced to the field of ethnomathematics. While many believe the field embraces the study of the mathematics of different ethnic groups or "race" mathematics, ethnomathematicians hold the view that the field is much broader. Professor Ubiratan D'Ambrosio, perhaps the most eminent scholar in the field, tells us, "...ethnomathematics invites us to look into how knowledge was built throughout history in different cultural environments." He goes on to describe ethnomathematics as "a comparative study of the techniques, modes, arts, and styles of explaining, understanding, learning about, and coping with the reality in different natural and cultural environments."

Ethnomathematics - Challenging Eurocentrism in Mathematics Education is a book which joins, in one volume, articles containing the diverse ideas of a group of respected ethnomathematics scholars. Each article is a classic in the field having made a significant contribution to the field of ethnomathematics. The authors attempt to challenge the Eurocentric opinion that "all significant historical development in science and technology developed in Europe." Those who hold an Eurocentric bias perhaps do not know that G.G. Joseph tells us that "recent studies of India, China, and parts of Africa would suggest the existence of scientific creativity and technological achievements long before the incursions of Europe into these areas."

The book is divided into six sections. Each section has an in depth introductory article by the editors of the book followed by articles by authors such as M. Ascher and R. Ascher, G.G. Joseph, U. D'Ambrosio, D.J. Struik, P. Gerdes, and C. Zaslavsky, to name a few. The sections are directed toward ethnomathematical knowledge, uncovering distorted and hidden history of mathematical knowledge, interactions of culture and mathematical knowledge, what counts as mathematical knowledge, ethnomathematical practices in the curriculum, and ethnomathematical research. The intent of each section is to focus on certain challenges to Eurocentrism in mathematics education.

The reader not acquainted with ethnomathematics must approach this book with an open mind. Among some in the mathematics community, there has already appeared concern that the editors, Powell and Frankenstein, advocate an approach to mathematics education that would further erode the pathetic background in mathematics that many of our students presently exhibit. These people believe that introducing "primitive" mathematics into the curriculum would further reduce the time spent on "worthwhile" mathematics. The United States would thus fall further behind other nations on studies such as the TIMMS Report. I have found in discussion groups that most criticism of the book is based not on having read the book, but on assumptions that are made about academic mathematics verses ethnomathematics. There is no denying the reality that Eurocentric mathematics is the mathematics of academia. However, the authors in the book are not suggesting that Eurocentric mathematics is without value. Rather, they want people worldwide to see Eurocentric mathematics as one important piece in the large picture of mathematics.

The editors write that according to the "Eurocentric account, Europe...has always been and currently is the superior Center from which knowledge, creativity, technology, culture, and so forth flow forth to the inferior Periphery, the so-called underdeveloped countries." I believe this to be an excellent definition of Eurocentrism. What the book and its editors try to do is challenge this notion and define mathematics in a much broader context. There are those in the academic community who view the emergence of such vocabulary as Eurocentrism as threatening to the integrity of mathematics. An example of the general attitude of many mathematicians toward ethnomathematics can be revealed in an incident that happened to me. Several years ago at a major mathematics conference, I was presenting a paper entitled "Celebrating the Mathematics of Native Americans." In the audience, I was pleased to observe an internationally respected Princeton University professor. Following the presentation, the gentleman was asked what he thought of the paper. He said that he found it interesting but containing very little mathematics. The mathematics of traditional peoples is not Eurocentric mathematics and, thus, was not destined to be included in academic mathematics. Fortunately, the editors of the book address the question "What counts as Mathematical Knowledge?"

The beauty of the book is that it offers much nourishment for thought in different areas. For me, a mathematics educator, the most important part of the book was the implications for mathematics education. This was found in Section V entitled "Ethnomathematical praxis in the curriculum." The first author in this section, Marcelo C. Borba, looks at relationships between ethnomathematics and academic mathematics. He proposes a system of education based on starting with the ethnomathematics of the group. This leads to his suggestion that the "problems to be solved would be chosen by both students and teachers in a dialogical relationship which fosters a critical consciousness." Knowledge is the result of such a dialogical relationship. Borba goes on to remind the readers that the relationship of teacher and students does not mean that the role played by both is the same. An equal relationship does not mean uniform. "In the classroom dialogue, the teacher can learn from the ethnomathematics 'spoken' by the students, just as the students are learning from the academic ethnomathematics of the teacher."

In Section V, Munil Fasheh encourages using culture to make mathematics learning more effective and meaningful. He describes mathematics education in Third World countries as being "taught as a set of rules and formulas that students have to memorize and a set of problems - usually nonsensical to students - that they must solve." I suggest that this also may describe the mathematics education in a good portion of the United States today, especially our impoverished areas. Fasheh adds that the "objective of teaching mathematics should be to discover new "facts" about one's self, society, and culture, to be able to make better judgments and decisions; and to build the links again between mathematical concepts and concrete situations and personal experiences." He believes that when math is taught without a cultural context and when it is said to be absolute, abstract, and universal, the result is the "alienation and failure of the vast majority of students in the subject."

A third article in Section V discuses the curriculum developed by its author, S. E. Anderson. He relates that after twenty years of developing a non-Eurocentric approach in his teaching of mathematics, he found his students to have a more positive, self assured attitude toward their ability to do mathematics. He found his students were interested enough in mathematics to try courses that they had not originally considered.

Claudia Zaslavsky finishes the section by encouraging the introduction of multicultural, interdisciplinary perspectives into the mathematics curriculum. Such a curriculum helps students realize that mathematics practices arose out of people's real needs and interests. It helps children take pride in their own heritage and helps develop their own self-esteem. Linking mathematics to the other disciplines gives the study of mathematics more meaning.

I highly recommend reading Ethnomathematics - Challenging Eurocentrism in Mathematics Education for several reasons. First, it brings together the views of a highly regarded group of ethnomathematicians. There is integrity in their scholarship. Secondly, it helps one realize how complex the notion of Eurocentrism is. As a caveat, I will remind the reader that some of the book may be discomforting to read. I was not comfortable with all the articles. However, recalling Piaget, remember "disequilibrium" is necessary for one to move to a different level of thought and understanding.

If the reader is further interested in ethnomathematics perspectives, I recommend looking at *Multicultural Mathematics—Teaching Mathematics from a Global Perspective*, Nelson, Joseph, and Williams, Oxford University Press, 1993.

Book Review: *Nexus: Architecture and Mathematics*, Kim Williams, ed.

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Nexus: Architecture and Mathematics. Kim Williams, ed. Edizioni Dell'Erba via Castruccio 1-50054 Fucecchio (Firenze), Italy, 1996. PHONE AND FAX (0571) 242093. ISBN 88-86888-04-X.

One way to judge the success of the mathematics curriculum in grades K-12 is to see if it results in large enough numbers of successful practitioners of mathematics (and in a broader sense scientists and engineers), if it creates a climate in which mathematics is appreciated and valued for its own sake and for its role in development of applications and new technologies, and if it delivers high school graduates who have basic skills as problem solvers and are flexible thinkers. There is much evidence that the high school mathematics curriculum of the past has failed on the last two points. Some might argue it has failed to meet all these tests.

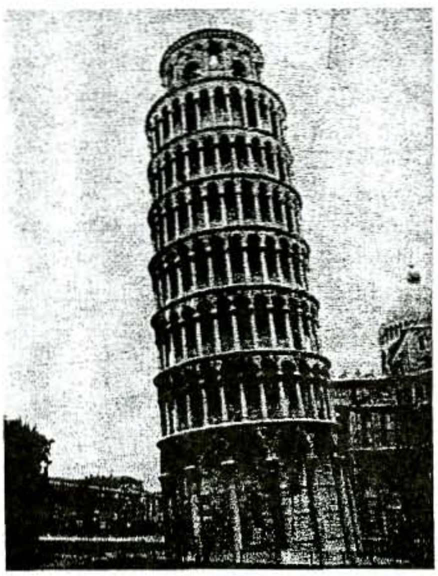

Fig. 1. The Leaning Tower of Pisa

In light of the malaise that the discussion above engenders, whenever a book appears that shows in a clear way the value of mathematical technique and the way that mathematics supports applications, poses interesting questions, and shows connections between different branches of knowledge, it is most welcome. Such a book is *Nexus: Architecture and Mathematics* edited by Kim Williams, published by Edizioni dell'Erba. The book grew out of a conference dealing with the relationships between architecture and mathematics held in June, 1996 near Florence, Italy which was organized by Kim Williams. Ms. Williams is an American-trained architect who now practices in Italy and who has a keen interest in the role that mathematics plays in architecture, and more generally, in the arts and engineering.

The full list of articles in this handsome volume appears below. Physically, the book is set in a easy-to-read, spacious type and profusely illustrated by line drawings and by crisply reproduced photographs. More importantly, the articles which make up the book provide a thoughtful attempt to deal with the support that mathematics provides for architecture, structural engineering and symmetry. The fact that the articles have a rich historical flavor and cover a broad range of times and cultures makes them excellent reading for a varied audience that should include teachers of mathematics at all levels and the general public as well. A knowledge of high school mathematics suffices for reading the book; those with a rusty knowledge will, I believe, find a reason to brush up on what they may have forgotten.

Fig. 3. The facade of the Cathedral of Pisa.

The articles, except for the introductory piece by Mario Salvadori (the very recently deceased author of the seminal book *Why Buildings Stand Up*) are arranged in alphabetical order. However, I did not read them in this order, nor do I suspect that most readers will, although a variety of choices will work. Personally, I started with **The Universality of the Symmetry Concept** (HARGITTAI AND HARGITTAI, figure 7 in the book), which makes the case for a broad approach to symmetry and followed up by reading **Architecture and Mathematics: Soap Bubbles and Soap Films** (EMMER), which provides a very nice history of work on minimal surfaces. From there I turned to the other pieces, each valuable in its own way.

I was intrigued to read **The Symmetries of the Baptistery and the Leaning Tower of Pisa** (SPEISER, figures 1 and 3 from the book) for personal reasons. I suppose everyone is familiar with the famous Leaning Tower of Pisa through photographs but I was unprepared for the emotions I felt when I got to see the Leaning Tower, Baptistery, and Cathedral in person for the first (and I hope not only!) time. The sheer beauty of these buildings with their exquisite proportions and delicate symmetry can not be done justice in photographs alone! Thus, it was a pleasure to read David Speiser's detailed analysis of the symmetry of these buildings, with his reservations about some aesthetic considerations involving the Baptistery.

One area that the book does not cover that I would liked to have seen treated is the way that relatively recent support for structural engineering, via the emerging mathematical insights being obtained into rigidity, is inspiring both artists, bridge designers, and architects.

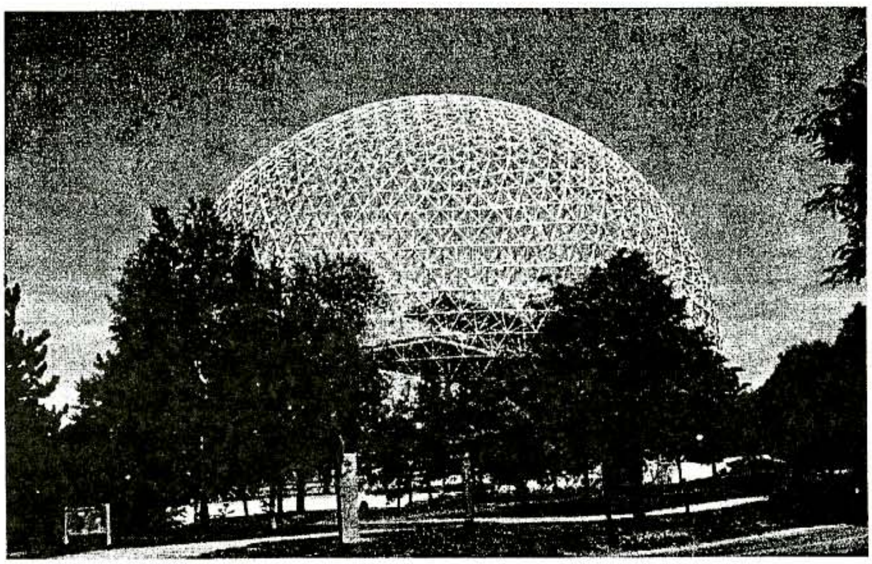

Fig. 7. Buckminster Fuller's Geodesic Dome at the Montreal Expo.

What makes this book exciting is the love of art, architecture, engineering, symmetry, history, and the role that mathematics plays in supporting these areas that perfuses all the articles. I hope this enjoyable and valuable book finds its way to the variety of readers who will enjoy and profit from reading it. Kim Williams deserves our thanks for bringing it to us.

CONTENTS:

Comments and Letters

I read with interest the review of Ethnomathematics [by Marcia Ascher] in the #14 Humanistic Mathematics Network Journal. However, when I went to find the book I discovered it is no longer published by Brooks/ Cole. In fact the representative of the company said "It is out of print."

But alas it is NOT out of print—just not printed by them! Your readers might like to learn that it was published April, 1994 by Chapman and Hall, ISBN 0-412-98941-7 ($36.50).

Thank you for your work on this interesting publication.

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In Future Issues...

Ein-ya Gura
Changing Ways of Thinking About Mathematics by Teaching Game Theory

Emamuddin Hoosain
Interviews in the Math Classroom

Julian F. Floren
Book Review: A Tour of the Calculus by David Berlinski

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Ralph A. Raimi
On Soving Equations, Negative Numbers, and Other Absurdities

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