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PERSONAL EXPERIENCES

My story is similar to that of many other mathematicians now approaching the last decade of their professional lives. We were educated in the '60s by a mathematics faculty feeling the mandate of the Sputnik era for training mathematicians and scientists and encouraged by considerable financial support. The search for Ph.D. candidates brought an increase in rigor in mathematics courses and an expansion in the number of graduate programs.

We were taught almost exclusively by the lecture method; the professor transferred his notes to the blackboard and the students dutifully copied them down, usually with little interaction on the spot, hoping to answer their questions on their own by studying the notes and whatever related material they could find. If they developed discussion groups with other students, they were lucky, for mathematics was a solitary activity, even a competitive activity, especially on the undergraduate level. The discussion groups developed more naturally in graduate school; at the Ph.D. level, mathematical research and personal interchange with the thesis advisor and other Ph.D. students enlightened the candidate as to how mathematics was really done by the professionals.

I had been attracted to mathematics in the eighth grade when I discovered that I liked solving story problems. Though my school courses emphasized story problems less and less, I continued to do story problems just for fun when I ran across them. It was during high school that I began collecting mathematical puzzles and problems.

When I was about fourteen, I became fascinated by the coconut problem [1] that I found in a desk encyclopedia at my grandfather's house. It was a story of five men on a tropical island who spent all day gathering coconuts. At the end of the day they had a large stack, but being too tired they decided to wait until morning to divide them up. During the night one of the men awoke and decided to take his share right then. He counted the coconuts, finding one more than a multiple of five, tossed the extra coconut to a monkey, and took one fifth of the rest. He hid them and then went back to sleep. Later, each of the other men awoke in turn; each decided to take his share then, found one more than a multiple of five, tossed the extra coconut to the monkey, and took a fifth of the remainder. In the morning the stack was greatly reduced, but no one said anything. They counted the coconuts and again found one more than a multiple of five, tossed the extra coconut to the monkey, and each took one fifth of the rest. The question was, what is the least number of coconuts that could have been gathered? I puzzled over the problem mightily, and waded enthusiastically but laboriously through the generalized solution presented. Though it involved algebra and number theory at the limit of my understanding, I was undaunted.

When I was a senior in high school, my cousin Bob was a freshman at Caltech. I had admired his intellectual prowess to an extent and wrote to him about my applying at Caltech, too. In his reply, he mentioned something his high school math teacher had told him the year before; why he mentioned it or what it involved I don't remember, but he used the expression $2n + 1$ to represent an odd integer. I do remember being completely amazed that such a simple thing could be so powerful and so general. From then on, mathematics was my major.

As I progressed through the study of mathematics, I liked it increasingly because it became more and more like solving story problems. In undergraduate topology, for example, the entire point of the course seemed to be discovering why a theorem was valid; we spent our time not only finding solutions (i. e., proofs), but explaining them to each other. Graduate mathematics was more of the same, and research for the Ph.D. was nothing but problem-solving. Not that it was applicable to anything in the "real world" sense, but tackling tough problems of any sort brought on the thrill of the chase, as it were, and solving tough problems resulted in a genuine "high" different from any other.

When I began to teach, I tried to share with my students the thrill of solving problems, but the way I had been taught (which was the source of most of my teaching strategies), the textbooks available, and the lack of time to deviate from the syllabus prevented me from really communicating that thrill to my students with any degree of success. In effect,

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teachers at the undergraduate level were constrained to leave out the real problem-solving aspects of mathematics; all we taught was prelude to the real mathematics to be done, and consisted of symbol manipulation rules and recipes for solving template problems. Our material and approach were still designed to bring potential Ph.D. candidates to the forefront; students not majoring in math became more and more of a "load" to whom we paid less and less attention.

As I taught mathematics, I gradually became aware of some of its history, something that had not been particularly fashionable at the times or places of my formal education. When I was assigned to teach the math history class out of Eves' book [2], I was amazed at the quantity of rich information of which I previously had been totally unaware. The history class led me to use a collection of articles reprinted from Scientific American, edited by Morris Kline [3], as text for a sophomore seminar. Statements in Kline's introductions to the sections led me to explore the nature of mathematics. As I discussed it with others, we came to the conclusion that we didn't really know what mathematics was, beyond the fact that it was what mathematicians did.

We knew mathematics was not a description of the "real world"; that had been settled in the middle of the nineteenth century by the development of non-Euclidean geometry. When Cayley and Klein showed that hyperbolic and elliptic geometries were just as consistent as Euclidean geometry, the question arose as to which was a description of the real universe. The profession as a whole gradually came to the conclusion that none of the three need be "true" of reality; soon mathematics became independent of the physical universe in the minds of mathematicians.

On the other hand, mathematics was not just an elaborate logical game that existed only in the mind, for how then could it attract the attention and enthusiasm of serious scholars? While some claimed that Russell and Whitehead had shown that all of mathematics could be derived from the clear blue of pure logic, it also had an "unreasonable effectiveness" [4] in predicting real-world phenomena. Each working mathematician felt that mathematics was somehow "out there," external to himself, but he was never quite sure whether his mathematics was discovered or invented. Someone suggested that mathematics was "composed," but that notion failed to gain any currency.

During the decades of the '70s and '80s, enrollments in mathematics classes, particularly in calculus courses, increased dramatically. At our institution, the growth rate was about eight percent, compounded annually, and that under fixed-ceiling enrollments overall. Most of that increase consisted of non-majors and therefore expanded the service load. Burgeoning classes but constant resources forced creative arrangements to meet the demand—large classes, laboratory-based courses, and cheap labor (TA's) were used widely. During the academic year 1986—1987, of the approximately 300,000 students who began the study of mainstream calculus in American colleges and universities, only 140,000 completed the year-long sequence with grades of D or better [5].

During much of this time, I was working on my own calculus text. I became convinced in the mid '70s that I could write a better book than the ones I had to teach from; I finally succeeded in producing a "good book" in 1988. By that time, dissatisfaction with the results of current strategies had produced the Tulane conference of 1986 which spawned the calculus reform movement. Participants cited unacceptably high failure rates as a waste of human potential, tradition-bound text materials, and concerns with the way students learn as reasons for paying attention to the way mathematics, particularly calculus, was taught.

In 1988 the first symbol-manipulating calculator hit the scene; from the centennial banquet of the American Mathematical Society, 1500 mathematicians took home a new toy that would not only do arithmetic but would manipulate algebraic expressions, draw graphs, and differentiate and integrate as well. Many professors began to see that the new technology would have a profound effect on the way they taught. Several professors reported that the new technology could easily pass the previous semester's calculus final.

I first began experimenting with computers in my math classes in 1983, but lack of resources prohibited any large-scale or permanent effort. I began to see what technology in the hands of students would do to the way mathematics was taught; my vision, limited though it was, became possible when students could arm themselves with the HP28S. By 1989 I was using the calculator freely in my classes and allowing my students the same privilege.

Much experimentation showed that there were ways to use the technology that greatly enhanced the acquisition of concepts. For example, the calculator could produce a dozen good graphs in the time it used to take for the student to produce a single decent graph; consequently, graphical properties became intuitive and were much more easily applied to the analysis of functions. The graph itself was no longer the point. The same could be said for many algorithms; by turning over to the technology the drudgery it could do well, the student was freed to think about what it all meant and how it applied to solving problems. The technology could also compress time; in a single class period, second-semester calculus students could start with the Riemann sum definition of the definite integral and, by observing what was happening to errors, could guess for themselves the trapezoidal and Simpson's rules.

In 1990, my publisher told me to start thinking about a second edition of my calculus book. When colleagues invited me to attend a workshop at Harvard in May of 1991 on teaching calculus, I consented to go along to see what I could pick up for my book. Just before we went, my publisher informed me they had changed their minds about a second edition; previous sales didn't warrant it. When I got to the workshop and saw the prepublication version of the "Harvard calculus," I was forced to admit to myself that I had come upon a better way to teach calculus. Here was a whole calculus book based on the idea of problem-solving the way I had approached it and loved it as a student but had failed to pass on to the students in my classes or to incorporate well into my textbook. I quit using my own book that Fall and began class-testing the Harvard materials. I also required my students to obtain and use HP48S calculators.

My experiences in teaching that year are almost indescribable. I was totally unprepared for the enthusiasm with which students attacked the new materials. Their love of the technology was astounding. But the thing that surprised me most was the sense of community that developed and the amazing amount of mathematics that the students did as I joined them in learning the calculus from a new approach. I pretty much quit lecturing and used a great deal of collaborative learning in small groups; as I moved around among the groups, I found myself gaining insights right along with them. I saw more mathematics being done by far than when I was the only performer.

Feedback was immediately positive. Students reported feeling much less anxiety and much more self-confidence than was reported the year before by very similar students. One young woman reported being in a chemistry class when the instructor started putting up a problem of a type that she recognized from calculus. She whipped out her calculator and had the problem finished long before the instructor finished presenting it. She said that what pleased her most was the incredulous looks on the faces of the young men sitting around her; her self-confidence grew by leaps and bounds.

I later taped a conversation among several of the students about their experiences in the class. Concerning their work in groups, they said:

Teresa: "Working in a group was a different experience for me because you're getting different people's opinions on ideas and you realize that mathematics is not just a set, defined pattem—that there are different ways to look at things. It was hard for me to get used to that setting—that everyone looks at math in a different light."

Kari: "Sometimes when we'd be working on our problems, you'd come to a point where you couldn't figure it out—you're stuck and you can't see any way out of it. Someone may say something and it triggers something in your mind and you can go from there and figure out the rest of it. You need that little help that somebody else can give to you."

Kristin: "The thing I enjoyed about the group work even beyond the concepts was the people that we worked with, because that created a foundation for a study group so that outside of class we could get together and work on assignments. The group work was especially fun, with [the instructor's] help to keep our ideas going...."

Kari: "Your ideas get a little bit more in depth when you're working with a group, too, because everyone sees different details...and it all comes together and you see the detailed, whole picture."

Chad: "I think that group work was very essential in the whole process of learning what we learned last year in calculus."

Monica: "It wasn't individual learning at all...but it was just the class learning together. Everybody worked together and if one person didn't understand, three or four people would help until they did. It was a community, I guess.... We all got to be really good friends. I think most of us were freshmen and most of the best friendships we made were from that class."

Concerning the use of the calculators:

Monica: "I was scared to death of that calculator when we first got it. I don't like computers, I don't want to like them, and I was really not happy to have to get the calculator."

Chad: "All I could think of was the price, and it was a different way of using a calculator also because it uses reverse Polish logic and so it was difficult to adapt..., but I learned. I had so much fun using that calculator after getting over the initial shock.... I realized that this thing could do so much more and it was so much easier to do my homework with ...."

Monica: "Even though it's a calculator and it does rote manipulations and calculations, I thought more because the calculator was there. As I was using it, my mind would be clicking just as fast, or more so, than if I'd been doing it on paper. Using the calculator made me think about problems a lot more."

Teresa: "In any sort of problem the HP would basically analyze it and do the work for you so you could take it one step higher and say, 'OK, what is actually going on here?' You could look at the graphs and say, 'OK, I've got this graph now; what is taking place?' and you didn't have to sit there and graph it out all by yourself..."

During the ensuing summer, four of the students let me know that they had changed their majors to mathematics; such a thing had never happened to me before.

Not everything went smoothly, but I was happy to see that most of my worries about changing my teaching habits were unnecessary. One prominent worry had been giving up control in the classroom. (Perhaps I had only imagined I had control before, and the students had been merely passive.) I had already been aware that when students have technology in their hands, they aren't listening to you talk, but are off on their own, doing things you never thought of. I discovered that the best way to get them back was to use interesting material that they perceived as relevant and for which they felt responsible. I turned out to be quite happy to relinquish control, turning it over to the material.

THOUGHTS ABOUT MATHEMATICS

These experiences have led me to think deeply about how students meet mathematics and how it ought to be presented to them. They have caused me to question the very nature of mathematics and have enabled me at long last to see how it is that I approach mathematics.

Historically, developments in what we now regard as elementary mathematics came about through the efforts of non-mathematicians to understand some aspect of the world around them. The developers of algebra were just playing around with numbers, trying to outdo each other with clever puzzles; Fibonacci was one of the foremost. Trigonometry was just a tool developed by astronomers. Newton was really a physicist who developed the calculus into a usable tool in order to understand motion and gravity. Maxwell developed the calculus of vector fields in an attempt to understand electric fields.

In each case, a "real world" problem presented itself and the tools of logical analysis were applied to it. Assumptions were made about the problem to make it more tractable, and order arose out of the assumptions. Techniques were developed for handling the order and drawing from it a prediction about the situation. The entire process was called mathematics.

Gradually, it was noticed that the same process of logical analysis could be applied to the perceived order itself, independent of the real situation. Modern abstract mathematics thus came into being. As the mathematics was refined, it drifted ever further in the minds of its practitioners from the real situations which had first given rise to it. Thus by the middle of the nineteenth century, mathematics had come to be defined as the abstract study of order or pattern, taught in a manner progressively axiomatic and devoid of physical content.

As a result, elementary mathematics has been taught for more than a century as a purely logical discipline, consisting of rules for manipulating the symbols that came to represent ideas. Because it is thus divorced from "reality," many students of mathematics regard their experiences as stultifying at best and mystifying more often than not. Most students do not survive in mathematics long enough to discover that the way mathematics is taught is not the way mathematics is done.

Mathematicians know that when they do their work, they are using logical analysis to understand the world around them, even if it is just the artificial and specialized world of mathematics. When they refer to mathematics, they include the thought processes they use in solving research problems, every bit as much as the body of knowledge consisting of all the manipulation rules, identities, and techniques that they wish their students knew. But in teaching elementary mathematics to beginning students, they never invite the students to use those same reasoning processes. It is not because, for

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beginning students, there is nothing appropriate to which to apply such reasoning processes, but because it has been forgotten that mathematics is every bit as much a process as it is a body of knowledge.

This leads me to a point of view of mathematics that seems to be valid. Both historically and as research mathematics is done today, mathematics is a means of dealing with the order that we see in the world around us.

Some remarks about this point of view are in order. I say "a means of dealing with the order" because thought processes are so varied as to defy any more specific categorization when taken in the aggregate. When one is wrestling with a problem, there are no holds barred and one catches as one can. The only criterion is that there should be some convincing, logical explanation afterward, even though most insights come from highly illogical combinations.

I say "the order that we see" because it is our perceptions to which we apply reasoning, not what is actually there. The traditional language is that a mathematical model is constructed and reasoning is applied to the model; in this language, mathematics is first of all modeling. Moreover, the "order" that arises from a situation is often the result of our assumptions, conditioned by previous experience. When shown a series of pictures of a cat in varying poses, some see only many pictures of a cat while others see the cat in motion and can even ascribe velocity and acceleration to it; those who see only many poses tend to lose interest quickly, while those who see motion find a myriad of things to analyze.

Human beings seem to need their perceptions of a situation to "make sense" if the situation is to be regarded at all. They are even willing to make unrealistic assumptions in an effort to understand. Thus we analyze a situation according to the way we construe it; it may or may not be an accurate or useful representation of reality. This basic uncertainty about our understanding of reality is what keeps most of us interested in learning about the universe.

When I refer to "the world around us," I mean whatever attracts our attention. The process of mathematical analysis can be applied to any subject whatever, concrete or abstract. These days, the "scientist" tends to focus on some aspect of reality while the "mathematician" typically focuses on some aspect of an abstraction. In actuality, the scientist is also dealing with an abstraction; the main difference is the frequency with which the researcher checks with reality.

THOUGHTS ABOUT TEACHING MATHEMATICS

This point of view of the nature of mathematics has what I think are profound implications for the teaching of mathematics at least through calculus. If we want a catch phrase for it, I think we could say, "Mathematics is a process; to introduce students to mathematics, we must engage them in the process." The process, of course, is dealing with the order that we see in the world around us.

People are scientists at heart, in that they seek to understand the events that go on around them so as to predict and control (or at least be prepared for) future events [6]. To assist themselves in the process, they construct theories into which they seek to organize and understand the mass of information impinging on them. The information comes not as facts but as perceptions; thus people deal with the world as they construe it or as they believe it to be. Insofar as their theories involve quantity, order, and pattern, they can deal with their perceptions mathematically.

In teaching mathematics, I believe we should capitalize upon the natural scientific tendencies of each student. We should begin with the process of logical analysis of problems, not with the body of manipulation rules and recipes. Mathematics is first the process; the rules come later, both historically and in the solving of research problems. If we begin with the process, it will be much more clear to the student that reasoning and analysis are what mathematics is all about, not merely memorizing formulas.

The problems to which the beginning student applies logical reasoning must be in the world of the student's interest, not in some artificial world someone else creates. If not so, there is no motivation; we know well that telling a student to be motivated does not make it happen in most cases.

Mathematics is at base a social activity; work can proceed individually, but never in a vacuum, and it is never complete until shared.

This means that problems at first must come from what the student perceives as the real world; as the student gains success in analyzing situations, the process of abstraction becomes clearer as we point it out and eventually the student's attention can be turned to the abstraction itself. This applies to the beginning student at any level, as much to the beginning student of calculus as to the beginning student of counting or arithmetic.

Moreover, much of the process is in communication of ideas. Forcing students to work in isolation is not only contrary to the way in which mathematics is created but often insures that the student will fail to learn. Allowing, indeed requiring, the student to communicate with peers helps to correct, refine, and solidify concepts and introduces the student to many more ideas than he or she is able to imagine alone. Mathematics is at base a social activity; work can proceed individually, but never in a vacuum, and it is never complete until shared.

If the student develops the ability to solve problems by thinking deeply and productively about certain key problems, it is not necessary for the student to see a recipe for the solution of every problem that was ever solved. Remember the adage, "Teach a man to fish...." Each mathematical subject has its key problems; in fact, each discipline to which mathematical reasoning can be applied has its key problems illustrating that application. A student who has thought deeply about some key ideas and is armed with logical reasoning will always outperform the student with a book of recipes.

The wise use of technology can be a great aid to the learning of mathematics. Current graphing calculator technology, for example, allows for multiple representations of concepts, powerful visualization, the compression of time, ease with experimentation, and the elimination of much drudgery. We should turn over to the technology the rules and recipes, things computers do very well, and get on with the thinking process. After all, if a calculator can do it, is it really thinking?

Unwise use of technology would include using a computer as a "black box." The student should never be programmed simply to push the right keys; only after an algorithm is completely understood is it appropriate to rely on the computer to perform it. On the other hand, once an algorithm is understood, we can save a lot of time and get on to the higher-level thinking we value by using the technology freely; the fact that the teacher or the student's parents did it "by hand" for years implies no particular virtue in the student doing so .

The biologists have a heuristic point of view that "ontogeny recapitulates phylogeny," meaning that it is helpful to view the developing embryo as progressing through the stages of evolution of that species. The same idea, applied to the individual student, would be "education recapitulates civilization." I believe that students of mathematics should re-create for themselves the development of elementary mathematics, time-compressed by the appropriate use of technology and by the wise choice of problems to analyze. The challenge to mathematics educators is now to select those problems and promote their analysis so as to engage the student fruitfully in the mathematical process.

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