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Music and Mathematics

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INTRODUCTION

As a child, music played a big role in my life. My father is a musician, and he tried to expose my brother, sister, and me to as much music as possible. Each of us was given the opportunity to play a musical instrument and encouraged to perform whenever given the chance. Although my brother and sister excelled with their instruments, I chose not to continue with lessons after the seventh grade. I enjoyed music immensely, but playing a musical instrument was really not my forte. In high school, I found something else that made me get excited: Mathematics. I enjoyed it so much that I decided to major in it when I went to college. Now I am here, and both music and mathematics continue to play a big role in my life. Instead of playing a musical instrument, I listen to music while doing my mathematics.

This semester I was given the opportunity to do an independent study in the mathematics department focusing on any topic that I desired. I now had a chance to combine two driving forces in my life, and to try to find some connection between them. I chose to investigate the relation of music and mathematics.

The focus for this paper is to find the commonalities between music and mathematics, with the hope that beauty will abound within this connection.

NOISE VS. MUSIC

First, we must establish that noise and music are two different entities. As defined in the tenth edition of *Merriam Webster's Collegiate Dictionary*, noise is a sound that "lacks agreeable musical quality or is noticeably unpleasant" [5]. The same dictionary defines music as "the science or art of ordering tones or sounds in succession, in combination, and in temporal relationships to produce a composition having unity and continuity" [4].

Sound waves are produced by vibrating matter. The sound waves produced by irregular vibrations in matter are called noise, whereas the sound waves produced by regular vibrations in matter are classified as musical sounds. These regular vibrations are the simple harmonic motion that can be represented graphically by adding a sufficient number of sine waves [1] (see Figure l). Jean-Baptiste Fourier is the man credited for this discovery. The frequency of the vibration determines the pitch of the musical sound, represented on the graph by the number of waves per unit time. The wave's amplitude, which indicates the intensity of the sound, is represented by the height of each crest.

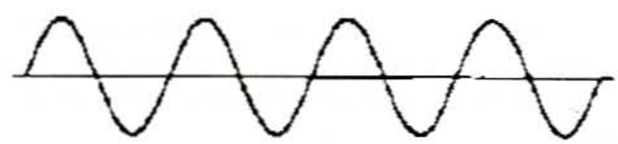

Figure 1
Sine Curve

A Fourier representation of a sound would consist of a series of simple, regular sine waves that, when added, represent the sound being analyzed. As the number of individual sine wave graphs increases, so does the complexity of the sound. Fourier analysis is useful for describing long, regular sounds in a very concise way [7].

Fourier, who studied mathematical vibration analysis circa 1800, knew that there was a flaw in his representation. He realized that a sound could not exist unchanged unless that sound was infinite in its duration. Because musical sound has a beginning and an end, the graphical representation of that sound must also be discrete [8]. Fourier analysis fails to reproduce accurately the timing of a sound when focusing on its pitch [7]. That is, there is a problem determining the time when a particular sound occurred.

It is now possible to represent both the pitch and the timing of musical sounds, thanks to Ingrid Daubechies. Daubechies uses a method that breaks down complex signals into what are called wavelets. The length of each wavelet represents the pitch of the sound -- the higher the pitch, the briefer the wavelet. Unlike Fourier representation, wavelets have no redundancy. With redundancy comes unnecessary information needed for reconstructing a sound. When using wavelets for analysis, "each wavelet is an essential component of the complex signal it represents" [7]. Wavelets are not only useful for representing sounds heard individually, but they are so precise that they can be used to single-out sounds in a graph of several simultaneous sounds.

Research in this area is very new. Because of this, the information regarding waveless is limited. Keep your eyes open; information on this topic is bound to explode!

HARMONY OF MUSIC

Its Frequency, Intensity, and Duration

As many of us may know, Pythagoras is the man credited with being the first to discover the relationship between musical harmony and mathematics [2]. It all happened one day, or so the story goes, when Pythagoras was considering whether it was possible to systematize musical sounds. He thought: sight is made precise with tools like the compass and ruler, as is touch by measures and balances. While thinking about this, he passed by a brazier's shop where he heard hammers beating on a piece of iron. Some sounds produced by hitting the same piece of iron were harmonious; others were not [3].

Later, after considering what he heard, Pythagoras went back to the brazier's shop to investigate how hammers beating on the same piece of iron could produce harmonious sounds. He discovered something astounding! When comparing the hammers, he found that they were of different weights. There was a six, eight, nine, and twelve pound hammer. When using the six and twelve pound hammers together, where the first hammer was half the weight of the second, the sound was harmonious. Harmony resulted when using the eight and twelve pound hammers together as well. But the hammers that were eight and nine pounds, when used together, produced a sound that did not harmonize [3].

The relationship between the weights of the hammers and harmonious sounds can be represented by using any musical instrument. For ease of explanation, I will discuss the representation in reference to a stringed instrument. The procedure is as follows:

1. A single stretched string vibrating as a whole produces a ground note. The frequency of the vibration determines the pitch of the musical sound.
2. Allow only half the string to vibrate, and the pitch will rise an octave above the ground note.
3. Allow 2/3 of the string to vibrate, and the pitch will rise a fifth above the one produced by the total length.
4. 3/4 - tone is a fourth higher.
5. 8/9 - tone is a whole step higher.

etc.

If the still point on the string, called the node, is not at one of these exact divisions, the sound is discordant. As we continue to divide the string, the fractions become more complex, and the two notes represented by the resulting intervals become more dissonant, or unpleasant, when they are sounded together. The smaller the whole numbers in the fractions, the more consonant, or pleasing, the sound is [2]. This is the reason Pythagoras felt that the six and twelve pound hammers sounded harmonious together, but the eight and nine pound hammers did not. Eventually, the fractions of the vibrating portions of the string became expressed as ratios. For example, the octave was expressed as a ration of $1:2$.

The frequencies of intervals between the tones of a musical scale can also be represented as a ratio. The frequency of middle C is 261 cycles per second. The ratio of $1:2$ describes the interval of an octave, so by doubling that frequency, we obtain a note defined by 522 cycles per second, or C one octave above middle C.

The chromatic scale, used in western music, consists of twelve intervals. Because of this, each tone in the scale has a frequency ratio of $\sqrt[12]{2} \approx 1.0595$ to the next tone (where the two comes from the ratio of an octave). It is with this ratio ($1:\sqrt[12]{2}$) that frequency intervals are spread equally over the twelve tone intervals of the octave. The break down of one octave is shown in Table 1. Because all twelve tones are necessary to construct musical scales, we can now find the frequency of any note in any octave [1]. The intensity of a tone is determined by the rate at which sound energy flows through a unit area. Intensity can simply be thought of as the loudness of a tone. The duration of a tone refers to how long a tone exists. With these three properties specifically stated, a musical sound can be duplicated.

ANALYSIS OF A COMPOSITION

When writing a piece of music, composers usually do not write a mathematical function and then compose the piece around the function. Instead, the composer might hear music in her head and then record that thought on paper. Whatever the process, I believe it is safe to say that mathematics is generally not the motivation for a composition. What is amazing is the fact that music is very organized. We have seen how harmony is made. We understand the idea of consonance and dissonance. Now let us investigate the mathematics of a composition.

First, let us look at a single, generic sound. Our sound will be an event that is considered as a whole and will be considered neither pleasant nor unpleasant. We can consider the abstract relations within the event or between several events, and the logical operations that may be imposed on them. Our event will be denoted as a.

Note	Frequency Approximation
middle C	261
C#/D♭	276.5199
D	292.9626
D#/E♭	310.3831
E	328.8394
F	348.3932
F#/G♭	369.1097
G	391.0581
G#/A♭	414.3117
A	438.9479
A#/B♭	465.0491
B	492.7024
C	522

Table 1
Notes and Frequency Approximations of an Octave

Properties :

1. If the sound is emitted once, all we have is its single existence that appears and then disappears. Here, we only have a.
2. If the sound is emitted several times in succession and compared, all that we can conclude is that they are identical.
   Now we can say that repetition implies the notion of identity, or tautology:
   $$a \vee a \vee a \vee \ldots \vee a = a$$
   where $\vee$ is the logical operator "or", disregarding time.
3. Modulation of time imposed on the sound.

When the element of time is considered, our sound takes on new meaning. Instead of just a sound, we now have potential for a code. For example, the Morse Code is an emission of a single sound that varies in duration. It is the duration of the sound, rather than the sound itself, which gives meaning to the code. For this reason, we will disregard the modulation of time and consider the case of two or more generic sound.

Let a, b, and c be distinct, easily recognizable sounds $(a \neq b \neq c)$.

Properties :

1. $a \vee b = b \vee a$
   Since time is not considered, our events are commutative.
2. $(a \vee b) \vee c = a \vee (b \vee c)$
   If we combine two elements, the combination can be considered as forming another element, or an entity, in relation to the third. This combination will allow our events to be associative.

When we exclude the time factor in composition, we end up with the commutative and associative laws of composition outside-time [9]. If we do consider the element of time (denoted with the logical operator T), then the sonic events, when played in succession, have a new meaning.

$$a\mathrm{T}b \neq b\mathrm{T}c$$

The comutative law no longer holds. Because our events are distinct and easily recognizable, it follows that a played before b sounds different from b played before a.

With these properties of sound, we can now investigate the concept of the interval. As defined in the *Norton/Grove Concise Encyclopedia of Music*, an interval is simply "the distance between two pitches" [6]. An interval is described according to the number of steps between notes, inclusive. For example, from C up to D, the interval is a major second. From G down to C, the interval is a perfect fifth.

With this in mind, let us consider a set of pitch intervals, $P = \left(p_a, p_{b,\ldots}\right)$, and the binary relation $\geq$ meaning greater than or equal to.

Then:

1. $p \geq p, \forall p \in P$
   - reflexive
2. $p_a \geq p_b \neq p_b \geq p_a$ *except for* $p_a = p_b$
   - antisymmetric
3. $p_a \geq p_b \wedge p_b \geq p_c \rightarrow p_a \geq p_c$
   - transitive

So, the set of pitch intervals, P, with the binary relation $\geq$ (P, $\geq$), forms a partially ordered set.

The ultimate goal of composers, let us assume, is likely to be the ability to share their musical inclinations with others. To do this, a composer must tell the musician exactly what she is thinking or hearing in her head. In order for a musical sound to be duplicated, all aspects of that sound must be considered. These aspects include frequency (pitch), intensity, and duration. With these three elements correctly combined, any musical sound can be constructed and repeated. In this case, the number 3 is irreducible.

Structure

When considering the set of pitch intervals, we are forced to consider the structure within that set. If p_a is a pitch interval going from C up to D (a major second), and p_b is a pitch interval going from D up to F (a minor third), then a third element, p_c, can be made to correspond when combining p_a and p_b. The element p_c would then be a pitch interval going from C up to F (a perfect fourth). Xenakis refers to this as the "law of internal composition" (consecutive pitch intervals, $p_a, p_b \in P$, can be made to correspond to a third pitch interval, $p_c \in P$, by the composite of p_a by p_b and is denoted as $p_a + p_b = p_c$) [9]. With this in mind, and once again disregarding time, we can say:

1. The law of internal composition for conjuncted intervals is addition.
2. The law is associative:
$$\left(p_a + p_b\right) + p_c = p_a + \left(p_b + p_c\right)$$
3. $\forall p_a \in P$, $\exists p_0 \in P$, a neutral element, such that:
$$p_0 + p_a = p_a + p_0 = p_a$$
4. $\forall p_a \in P$, $\exists p'_a \in P$, called the inverse of p_a, such that:
$$p_a + p'_a = p_a + p'_a = p_0$$
5. The law is commutative:
$$p_a + p_b = p_b + p_a$$

These five axioms hold for pitch *outside-time*. This example of pitch intervals can be extended to intensity intervals and durations, the other two fundamental factors of musical sound. It should be noted that the sets form an Abelian additive group structure.

So far, it has been established that the idea of sound possesses a structure outside-time. The element of time forms a temporal structure. When we combine these two structures, the result is a structure in-time, or an actual composition.

Before considering a musical composition, let us first consider the notes that a composer uses. The only limitation imposed on what notes and in which octaves are usable is with the instruments that the composer chooses to use. If the piece is written for a bassoon, then only the notes in the available octaves can be used. The composition would not be written in the same octave as, say, the upper register of a piccolo.

APPLICATION

For a composition with one instrument

Let

$R = \{\textit{all the notes of a particular instrument}\}$

$A = \{\textit{a certain choice of notes of the instrument}\}$

$B = \{another\ choice\ of\ notes\ of\ the\ instrument\}$
Where A and B are subsets of the universal set R.

If we first hear A, and then B, and then compare the two sets, we can establish some relationships between them.

1. If certain notes are common to both sets A and B, the sets intersect (see Figure 2a).
2. If no elements are common between the chosen sets, they are disjoint (see Figure 2b).
3. If all the elements of B are common to one part of B, then our set B is included in A (see Figure 2c).
4. If all the elements of A are found in B and all the elements of B are found in A, then the two sets are indistinguishable, or equal (see Figure 2d).

Now that we understand the basic relationships between sets, we can investigate a method of creating new sets given existing sets. When we choose A and B so that they have some elements in common, we can then establish those new sets.

1. If we hear the notes in common between A and B, we are using the operation of intersection (conjunction) to form a new set consisting only of those common elements:

$$A \cdot B \text{ or } B \cdot A$$

2. If we hear the notes of both sets and interpret them as a mixture of the elements of A and B, we have a new set formed using the operation of union (disjunction):

$$A + B \text{ or } B + A$$

This set consists of all the elements of set A and set B.

3. If we are allowed to hear all the notes in our universal set R except those of A, then we have a new set defined by the negation A with respect to R:

$$\overline{A}$$

4. In music, there is another set which is represented by silence. This set is equivalent to the empty set, and is called a rest.

With a proper choice of notes for each set, and a proper grouping of these sets, we can write a mathematical function to represent a composition. When given three sets, A, B, and C we can write a Boolean function in the form called disjunctive cannonic:

$$\sum_{i=1}^{8} \sigma_i k_i$$

where,

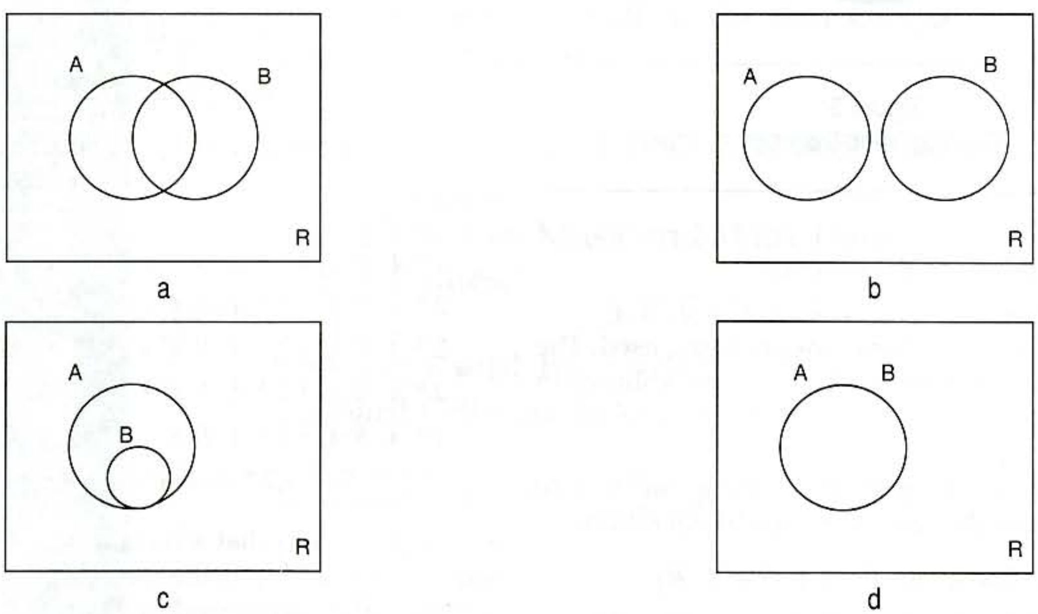

Figure 2
Relationships Between Sets

$$\sigma_i = 0, 1$$

and

$$k_i = A \cdot B \cdot C, A \cdot B \cdot \overline{C}, A \cdot \overline{B} \cdot C, \overline{A} \cdot B \cdot C, A \cdot \overline{B} \cdot \overline{C}, \overline{A} \cdot B \cdot \overline{C}, \overline{A} \cdot \overline{B} \cdot C, \overline{A} \cdot \overline{B} \cdot \overline{C}$$

A Boolean function can always be written in a way that brings a maximum of operations using (+), (.), and (¯), equal to $3n \cdot 2^{n-2} - 1$, where n is the number of sets being used. In this case, $3 \cdot 3 \cdot 2^{3-2} - 1 = 9 \cdot 2 - 1 = 17$ [9].

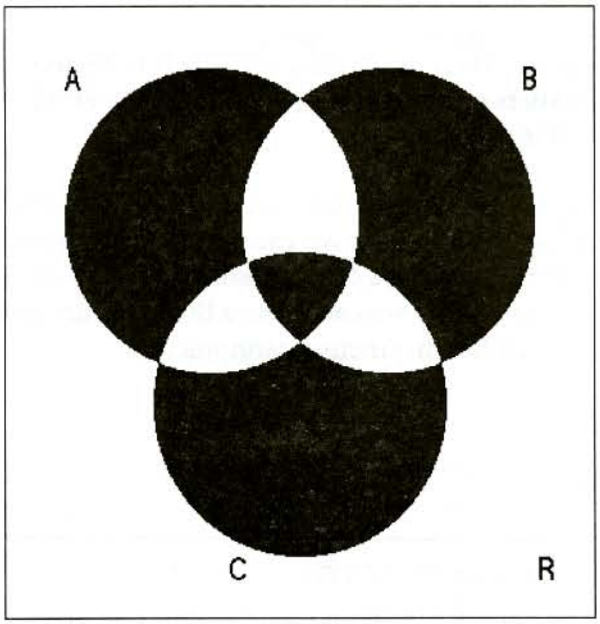

Figure 3
Example Venn Diagram

For example, if we use the function:

$$F = A \cdot B \cdot C + A \cdot \overline{B} \cdot \overline{C} + \overline{A} \cdot B \cdot \overline{C} + \overline{A} \cdot \overline{B} \cdot C$$

we will notice that 17 operations are being used. The Venn diagram representing this function is shown in Figure 3.

Of course, we can simplify the original function to obtain a function that only requires 10 operations:

$$F = \left(A \cdot B + \overline{A} \cdot \overline{B}\right) \cdot C + \overline{\left(A \cdot B + \overline{A} \cdot \overline{B}\right)} \cdot \overline{C}$$

but by doing this, we will change the procedure in the composition.

I must stress that this mathematical model deals only with which notes in a composition are played. It does not deal with other variables such as intensity or duration.

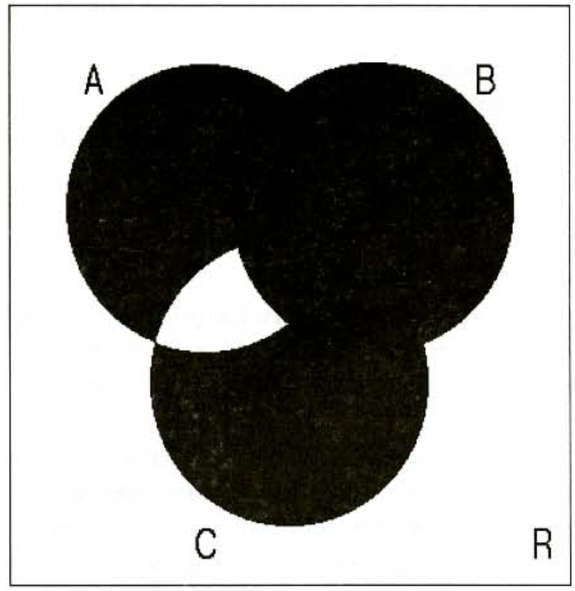

Figure 4
Challenge Venn Diagram

Finally, I leave you with the following.
Let

$$A = \{\mathrm{A, B, C^{\#}, G}\}$$
$$B = \{\mathrm{B, C^{\#}, D, E}\}$$
$$C = \{\mathrm{B, E, F^{\#}, G}\}$$

and let

$$F = \begin{array}{l} 2 * A \cdot \overline{B} \cdot \overline{C} + 2 * \overline{A} \cdot B \cdot C + 2 * \overline{A} \cdot \overline{B} \cdot C + 2 * \overline{A} \cdot B \cdot C + \\ 2 * \overline{A} \cdot B \cdot \overline{C} + 2 * A \cdot B \cdot \overline{C} + 2 * A \cdot B \cdot C + 2 * A \cdot \overline{B} \cdot \overline{C} + \\ 2 * \overline{A} \cdot B \cdot C + 2 * \overline{A} \cdot B \cdot \overline{C} + 2 * A \cdot B \cdot \overline{C} + 2 * A \cdot B \cdot C + \\ 2 * \overline{A} \cdot B \cdot C + 2 * \overline{A} \cdot B \cdot \overline{C} + 2 * A \cdot B \cdot \overline{C} + 2 * A \cdot B \cdot C + \\ 2 * A \cdot \overline{B} \cdot \overline{C} + 2 * \overline{A} \cdot B \cdot C + 2 * \overline{A} \cdot \overline{B} \cdot C + 2 * \overline{A} \cdot B \cdot C + \\ 2 * \overline{A} \cdot B \cdot \overline{C} + 2 * A \cdot B \cdot \overline{C} + 2 * A \cdot B \cdot C + 2 * A \cdot \overline{B} \cdot \overline{C} \end{array}$$

where $2*$ means that a certain note is played twice, sequentially, and $+$ is the transition from one note to another. The corresponding Venn diagram is shown in Figure 4.

Here is the challenge: Interpret the function (determine the sequence of notes), and give the interpretation to a musician. Ask her to play it, and try to *name that tune*!

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San Diego Joint Math Meetings

The Humanistic Mathematics Network has organized a panel at the San Diego Joint Math Meetings.

Saturday, January 11, 1997, 2:30 - 3:50 PM.

"Art, Literature, Music and Math: Degrees of Similarities."

Speakers will be Annalisa Crannell, Leonard Gillman, JoAnne Growney.

Moderated by Alvin White.