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Investigating the Development of Proof Comprehension: The Case of Proof by Contradiction

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INVESTIGATING THE DEVELOPMENT OF PROOF COMPREHENSION: THE CASE OF PROOF BY CONTRADICTION

by

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ABSTRACT

This dissertation reports on an investigation of transition-to-proof students' understanding of proof by contradiction. A plethora of research on the construction aspect of proof by contradiction is available and suggests that the method is one of the most difficult for students to construct and comprehend. However, there is little research on the students' comprehension of proofs and, in particular, proofs by contradiction. This study aims to fill this gap in the literature. Applying the cognitive lens of Action-Process-Object-Schema

(APOS) Theory to proof by contradiction, this study proposes a preliminary genetic decomposition for how a student might construct the concept 'proof by contradiction' and a series of five teaching interventions based on this preliminary genetic decomposition. Data was analyzed in two ways: (1) group analysis of the first two teaching interventions to consider students' initial conceptions of the proof method and (2) case study analysis of two individuals to consider how students' understanding developed over time. The genetic decomposition and teaching interventions were then revised based on the results of the data analysis. This study concludes with implications for teaching the concept of proof by contradiction and suggestions for further research on the topic.

INDEX WORDS: Proof by contradiction, Proof comprehension, APOS Theory, ACE teaching cycle, Indirect proof

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INTRODUCTION

1.1 Statement of the problem

Proof is central to the curriculum for undergraduate mathematics majors. Indeed, the *2015 CUPM Curriculum Guide to Majors in the Mathematical Sciences* contains the following recommendation: "Students majoring in the mathematical sciences should learn to read, understand, analyze, and produce proofs at increasing depth as they progress through a major" (Schumacher & Siegel, 2015, p. 11). Even though proof is essential to the mathematical curriculum, there exists a wide variety of definitions for mathematical proof (Balacheff, 2002; Weber, 2014). This study will use the following definition of proof, reproduced from Stylianides (2007):

> Proof is a *mathematical argument*, a connected sequence of assertions for or against a mathematical claim, with the following characteristics:
>
> 1. It uses statements accepted by the classroom community (*set of accepted statements*) that are true and available without further justification;
> 2. It employs forms of reasoning (*modes of argumentation*) that are valid and known to, or within the conceptual reach of, the classroom community; and
> 3. It is communicated with forms of expression (*modes of argument representation*) that are appropriate and known to, or within the conceptual reach of, the classroom community. (Stylianides, 2007, p. 291).

The focus on classroom community in the definition above allows for flexibility in describing students' increasing proficiency with proofs as they progress in a mathematics major. In particular, a flexible definition of proof is necessary to describe the vast differences in what students consider a proof at the beginning and at the end of a transition-to-proof course.

Transition-to-proof courses are normally situated between the Calculus sequence and proof-heavy courses (e.g., Linear Algebra, Abstract Algebra, and Analysis) to facilitate the transition from computation-based mathematics to proof-based mathematics (*Transitions to Proof*, 2015). To achieve this transition, the course normally introduces various proof methods that are applied to statements in Number Theory[1] (David & Zazkis, 2017; A. Selden, 2011). Despite this specially-designed transition course, students continue to struggle with mathematical proof (Samkoff & Weber, 2015). These persisting struggles illustrate a pressing need for research on students' interactions with mathematical proof.

Review of the current literature on mathematical proof at the undergraduate level point to four interconnected aspects of proof: construction, comprehension, validation, and evaluation. A short discussion of each aspect follows.

1.1.1 Four aspects of proof

Proof construction refers to "attempting to construct correct proofs at the level expected of university mathematics students (depending upon the year of their program of study)" (A. Selden & Selden, 2017, p. 339). Students' difficulties constructing proofs have been well documented in the literature (e.g., Andrew, 2007; Antonini, 2003; Antonini & Mariotti, 2006, 2009; D. Baker & Campbell, 2004; J. D. Baker, 1996; Barnard, 1995; Dubinsky, 1986, 1989; Dubinsky, Elterman, & Gong, 1988; Dubinsky & Lewin, 1986; Dubinsky & Yiparaki, 2000; Harel, 2002; Harel & Sowder, 1998, 2007; Moore, 1994; Piatek-Jimenez, 2010; Reid & Dobbin, 1998; Stavrou, 2014). These difficulties typically include difficulties with the underlying mathematical logic (e.g., Knipping, 2008; Moore, 1994; Stavrou, 2014), difficulties understanding the mathematical concepts within the proof (e.g., Mejía-Ramos, Weber, & Fuller, 2015; Moore, 1994; Stavrou, 2014), difficulties writing proofs by induction and contradiction (e.g., Andrew, 2007; Antonini & Mariotti, 2009; J. D. Baker, 1996; Barnard & Tall, 1997; Dubinsky, 1986, 1989; Dubinsky & Lewin, 1986; Harel, 2002; Harel & Sowder,

[1]While Number Theory is the most common content area used for transition-to-proof courses, other content areas may include Euclidean Geometry, Set Theory, and Algebra.

1998; Reid & Dobbin, 1998), difficulties knowing how to approach proving statements, i.e. lack of proof-writing strategies (e.g., Hanna, 2000; Hoyles, 1997; Moore, 1994; Weber, 2001, 2004), and difficulties with quantified[2] statements (e.g., Barnard, 1995; Dubinsky, 1989; Dubinsky & Yiparaki, 2000; Piatek-Jimenez, 2010). Of the four aspects of proof, construction has been given the most attention by the mathematics community.

Proof comprehension refers to "understanding a textbook or lecture proof" (A. Selden & Selden, 2017, p. 339). Understanding a proof can be thought of in two distinct ways: understanding the overarching structure of a proof and understanding a proof line-by-line (Leron, 1983; Mejía-Ramos, Fuller, Weber, Rhoads, & Samkoff, 2012; J. Selden & Selden, 1995). Research suggests that focusing on proof comprehension may aid students in understanding and constructing proofs (Hodds, Alcock, & Inglis, 2014; Samkoff & Weber, 2015). However, there is relatively little research on proof comprehension when compared to proof construction (Mejía-Ramos & Inglis, 2009; Samkoff & Weber, 2015), especially concerning empirical results on proof comprehension.

Proof validation refers to "the reading of, and reflection on, a proof attempt to determine their [the proof's] correctness" (A. Selden & Selden, 2017, p. 340). In other words, proof validation refers to a student's attempt to determine whether a sequence of assertions has the characteristics of a proof[3]. For example, a student could check any or all of the following: that all statements used in the proof are true and available without further justification, that the proof employs valid forms of reasoning, and that the proof is communicated appropriately. Research suggests that validation is separate from construction and comprehension (A. Selden & Selden, 2014) and that validation is a cognitively-demanding task (A. Selden & Selden, 2003) students approach in different ways based on their understanding of proof (Harel & Sowder, 1998). Little attention has been given to proof validation, especially when compared to the literature on proof construction and proof comprehension.

[2] A statement which contains quantifiers, such as "for all" and "there exists."
[3] Characteristics defined by their own definition of proof.

Proof evaluation refers to "making value judgments about a finished proof text" (A. Selden & Selden, 2017, p. 340). Research suggests there may be a need for basic competency in proof construction, comprehension, and validation before meaningful proof evaluation can be attempted (Inglis & Aberdein, 2015). Even then, these value judgments typically include aesthetic properties that make it difficult to quantify proof evaluation (Hanna & Mason, 2014). Of the four aspects of proof, evaluation has received the least attention.

These four interconnected aspects of proof can be used to study particular proof methods more thoroughly. In a transition-to-proof course, proof methods are classified as either direct or indirect. A brief description of direct and indirect proofs, along with examples to illustrate differences between the methods, follows.

1.1.2 Direct and indirect proof methods

The first direct proof method students are introduced to is based on *Modus Ponens*, or logical implication. By making an initial assumption and proceeding by implication in a sequence, one arrives at the conclusion. In symbolic notation, a direct implication proof is of the form $P \to P_1 \to P_2 \to P_3 \to \ldots \to P_n \to Q$, where P is the initial assumption, P_k, is an intermediate statement, and Q is the conclusion (Smith, Eggen, & Andre, 2015). For example, consider the following statement and its proof:

> *Statement*: Let x be an integer. If x is odd [**P**], then $x + 1$ is even [**Q**].
>
> *Proof:* Let x be an integer. Suppose x is odd [**P**]. Then $x = 2k + 1$ for some integer k [$\mathbf{P} \to \mathbf{P_1}$]. Then $x+1 = (2k+1)+1$ [$\mathbf{P_1} \to \mathbf{P_2}$]. Because $(2k+1)+1 = 2k + 2 = 2(k + 1)$, we see that $x + 1$ is the product of 2 and an the integer $k + 1$ [$\mathbf{P_2} \to \mathbf{P_3}$]. Thus, $x + 1$ is even [$\mathbf{P_3} \to \mathbf{Q}$]. (Smith et al., 2015, p. 33)

Note how each statement in the proof is a direct consequent of the previous statement. Due to the linear and straightforward nature, direct implication is considered the easiest proof method.

Indirect proofs are introduced immediately after direct implication proofs. An *indirect proof* replaces the statement to be proved by a tautologically equivalent statement (Smith et al., 2015). Two indirect proof methods are commonly taught in a transition-to-proof course: contraposition and contradiction.

A proof by contraposition[4] is based on the tautological equivalence of a conditional statement and its contrapositive. By directly proving the contrapositive of a conditional statement, one can conclude the statement itself. In symbolic notation, proof by contraposition is of the form $(\sim Q \rightarrow \sim P) \rightarrow (P \rightarrow Q)$, where P is the initial assumption and Q is the conclusion. Contraposition is useful when proving a conditional statement where the negations of the assumption and conclusion are easier to work with than the assumption and conclusion themselves (Smith et al., 2015). For example, consider the following statement and its proof:

> *Statement:* Let m be an integer. If m^2 is even [**P**], then m is even [**Q**].
>
> *Proof:* Let m be an integer. Suppose m is not even [$\sim$ **Q**]. Then m is odd, so $m = 2k + 1$ for some integer k. Then
>
> $$m^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1.$$
>
> Since m^2 is twice an integer, plus 1, m^2 is odd. Therefore, m^2 is not even [$\sim$ **P**]. Therefore if m is not even, then m^2 is not even [$\sim \mathbf{Q} \rightarrow \sim \mathbf{P}$]. By contraposition, if m^2 is even, then m is even [$(\sim \mathbf{Q} \rightarrow \sim \mathbf{P}) \rightarrow (\mathbf{P} \rightarrow \mathbf{Q})$]. (Smith et al., 2015, p. 42)

A proof by contraposition is preferred in this case as students can more easily square a number and track the preserved properties, such as evenness, than they could square root a number and track the same preserved properties.

As mentioned previously, another common type of indirect proof is proof by contradiction. A proof by contradiction is based on the *law of excluded middle*: either a statement

[4]Proof by contraposition is sometimes referred to as proof by contrapositive.

is true or the negation of the statement is true. By showing the negation of the statement cannot be true (i.e. leads to a contradiction), the statement must be true. In symbolic notation, proof by contradiction is of the form $(\sim P \rightarrow (Q \wedge \sim Q)) \rightarrow P$, where P is the statement to be proved and Q is any other statement (Smith et al., 2015). For example, consider the following statement and its proof:

> *Statement:* $\sqrt{2}$ is an irrational number [**P**].
>
> *Proof:* We show that the assumption that a rational $\frac{p}{q}$ exists such that $\frac{p^2}{q^2} = 2$ [$\sim$ **P**] leads to a contradiction [$\mathbf{Q} \wedge \sim \mathbf{Q}$]. Suppose that $\frac{p^2}{q^2} = 2$ [$\sim$ **P**] where p and q are integers with no factor in common [**Q**], then $p^2 = 2q^2$. Thus p^2 is even. But if p were odd, then p^2 would be odd, so this means that p must be even, $p = 2r$ where r is an integer. Substituting in $p^2 = 2q^2$, we obtain $4r^2 = 2q^2$, we obtain $4r^2 = 2q^2$, so $2r^2 = q^2$. The same argument shows that q must also be even. So p and q have a common factor 2 [$\sim$ **Q**], contradicting the fact that they have no common factor [**Q**]. Since assuming $\sqrt{2}$ is rational [$\sim$ **P**] leads to a contradiction [$\mathbf{Q} \wedge \sim \mathbf{Q}$], $\sqrt{2}$ is irrational [$(\sim \mathbf{P} \rightarrow (\mathbf{Q} \wedge \sim \mathbf{Q})) \rightarrow \mathbf{P}$]. (Tall, 1979, p. 2)

The statement above cannot be proven directly nor can it be proven by contraposition (Conway & Shipman, 2014), illustrating the necessity of proof by contradiction as a proof method. Furthermore, every statement that can be proven by contraposition can also be proven by contradiction[5], though the reverse is not true.

The direct and indirect proof methods described above are by no means an exhaustive list of methods taught in a transition-to-proof course. However, research suggests that proof by contradiction[6] is a difficult proof method for students to construct and comprehend (Antonini & Mariotti, 2008; Brown, 2017; Harel & Sowder, 1998) and thus will be the primary focus of this study. Utilizing the previously discussed four aspects of proof, a short summary

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[5] If contraposition can be used, then $\sim Q \rightarrow \sim P$. Combining this with the tautological equivalence $\sim (P \rightarrow Q) \leftrightarrow (P \wedge \sim Q)$, it is clear that $(\sim (P \rightarrow Q) \rightarrow (\sim P \ \wedge P)) \rightarrow (P \rightarrow Q)$.

[6] Proof by contradiction and proof by contraposition are considered cognitively similar proof method by some authors (e.g., Antonini & Mariotti, 2008) and thus proof by contraposition is relevant to this study.

of how each aspect has been studied with respect to proof by contradiction, and results from these studies, follow.

1.1.3 Four aspects of proof by contradiction

The construction aspect of students' proofs by contradiction, and in particular students' difficulties with writing proofs by contradiction, has received the most attention in the literature. Student difficulties constructing proofs by contradiction include: negating intuitive claims (e.g., Barnard, 1995; Dubinsky et al., 1988; Reid & Dobbin, 1998), constructing and manipulating impossible mathematical objects (e.g., Antonini & Mariotti, 2008; Leron, 1985), and identifying a contradiction (e.g. Antonini & Mariotti, 2009; Barnard & Tall, 1997).

The comprehension aspect of students' proofs by contradiction, and in particular models to describe students' understanding of proofs by contradiction, has received relatively little attention when compared to the two previous aspects of proof (Brown, 2014). Two models related to understanding proof by contradiction have been proposed: a model describing how students develop an understanding of proof by contradiction by Lin, Lee, and Wu Yu (2003) and a model to analyze student understanding of proof by contradiction by Antonini and Mariotti (2008). Unfortunately, neither of these models adequately describe how students develop an understanding of proof by contradiction in a transition-to-proof course[7].

The validation aspect of students' proofs by contradiction, and in particular students' responses and approaches to validating proofs by contradiction, has only recently been studied (Brown, 2016b). Brown (2016b) found that the approaches undergraduate students utilize to validate a proof by contradiction differ greatly from the approaches of advanced mathematics students and mathematicians.

The evaluation aspect of students' proofs by contradiction, and in particular students' preference for alternative proof methods, has been explored in conjunction with students' difficulties constructing proofs by contradiction. A preference for constructive proofs (Harel

[7]Section 2.2.2 will explain in detail why neither model adequately describes how students develop an understanding of proof by contradiction in a transition-to-proof course.

& Sowder, 1998), the historical view of necessary causality[8] (Harel & Sowder, 2007), difficulties dealing with impossible mathematical objects (Leron, 1985), and difficulties accepting the logical underpinning of proof by contradiction itself (Antonini & Mariotti, 2008) have been posited as explanations for students' avoidance of proof by contradiction. However, the view that students avoid proof by contradiction for any of these reasons has been recently challenged (Brown, 2011, 2016b, 2017).

Due to the plethora of research on the construction aspect of proof by contradiction, currently inadequate models of how students develop an understanding of proof by contradiction proposed to date, and recent considerations on the validation and evaluation aspects of proof by contradiction, this study will focus on the comprehension aspect of proof by contradiction. In particular, this study will focus on how students may develop an understanding of proof by contradiction and the implications of this understanding for a transition-to-proof course.

A cognitive framework is needed in order to describe possible ways in which students may develop understanding of mathematical concepts. The next section will outline a general cognitive theory that describes how individuals may develop understanding and introduce a related theoretical perspective, specific to mathematics, through which this study will investigate student understanding of proof by contradiction.

1.2 Theoretical perspective

Jean Piaget, a cognitive psychologist, theorized that individuals develop knowledge and understanding through reflective abstraction (Piaget, 1975). Reflective abstraction can be thought of in two repeating levels. The first level, *reflection*, is described as the process of contemplative thought about content and operations from a lower cognitive stage to a higher stage. The second level, *abstraction*, consists of reconstructing and reorganizing the content and operations on this higher stage, which results in the operations themselves becoming content to which new operations can be applied (Arnon et al., 2014). An example of reflective

[8]That proofs should provide a sense of cause, i.e. why a statement is true (Mancosu, 1991).

abstraction through the integers, synthesized from Piaget (1965) by Arnon et al. (2014), is included below:

> At one stage, an integer is an operation or process of forming units (objects that are identical to each other) into a set, counting these objects and ordering them. At a higher stage, integers become objects to which new operations, e.g., those of arithmetic, are applied. (Arnon et al., 2014, p. 6).

This example illustrates how the concept of integer develops: from a process of counting objects into objects themselves (that can be used in arithmetic). Reflective abstraction can again occur on these new integer objects to develop a single object: the set of integers $\mathbb{Z}$. Utilizing this new object, one can now use set multiplication to create a new multiplicative object: $\mathbb{Z} \times \mathbb{Z}$. As illustrated through the previous example, knowledge is continually constructed and reconstructed through reflective abstraction (Piaget, 1975).

By refining and applying reflective abstraction to mathematics, Ed Dubinsky developed Action-Process-Object-Schema (APOS) Theory[9] (Asiala et al., 1997). APOS Theory posits that "Individuals make sense of mathematical concepts by building and using certain mental *structures* (or *constructions*) which are considered in APOS Theory to be *stages* in the learning of mathematical concepts." (Arnon et al., 2014). A description of these mental structures follows. An *Action*[10] is a transformation of Objects by the individual requiring external, step-by-step instructions on how to perform the transformation. For example[11], a student may think of a function as a procedure for assigning elements from one set (i.e. domain) to another (i.e. range) through a specific procedure, such as $f(x) = 3x + 4$. In this case, the student cannot consider what would happen to an element, such as $x = 2$, without evaluating the function at that element. As an individual reflects on an Action, they can now perform and describe, in his or her mind, the transformation without the external, step-by-

[9] See Dubinsky and McDonald (2001) or Meel (2003) for validation of APOS Theory as a theoretical framework.

[10] Actions, Processes, Objects, and Schemas are capitalized to differentiate between the mental constructions of APOS Theory and the general use of these terms.

[11] APOS Theory is normally exemplified using functions. Examples in the context of proof by contradiction will be provided in Chapter 3.

step instructions of an Action; this is referred to as a *Process*. Returning to the example of a function, the student can now imagine the function $f(x) = 3x + 4$ as multiplying an element from the domain by 3 and adding 4 to get an element from the range without needing to refer to a specific element. As an individual reflects on a Process, he or she may think of the Process as a totality and can now perform transformations on this Process; this totality is referred to as an *Object*. Again returning to the example of a function, the student can consider translating a function $f(x) = 3x + 4$ up 4 by imagining the function in its entirety and moving up 4 rather than moving a series of elements in the range up 4. In other words, by imagining the function as a totality, the individual can now apply Actions to the function itself and not the individual elements of the domain. Finally, a *Schema* is an individual's collection of Actions, Processes, Objects, and other Schemas that are linked by some general principals to form a coherent framework in the individual's mind (Dubinsky & McDonald, 2001).

The mental structures introduced above are constructed through five types of reflective abstraction: interiorization, coordination, reversal, encapsulation, and generalization (Dubinsky, 1991). These types of abstraction are normally referred to as mental mechanisms and are used to describe how an individual develops an understanding of a concept. A short description of each mechanism follows, organized by the mental constructions (Processes, Objects, and Schemas[12]) they construct.

Processes may be constructed by one of three mental mechanisms: interiorization, coordination, or reversal. *Interiorization* occurs as an individual reflects on a series of Actions and constructs internal procedures, i.e. Processes, to make sense of the Actions. For an example with the linear function concept, a student can convert a set of explicit, step-by-step instructions for assigning elements from the domain to the range to general rules by considering the slope and the y-intercept of a line. *Coordination* occurs as an individual composes two or more Processes into a single Process. Again using the example of the lin-

[12]Actions are external and thus not created by the individual. Therefore, there is no mental mechanism that constructs Actions.

ear function concept, the student can coordinate a Process for finding the slope of a linear function with the Process of finding the y-intercept of a linear function to construction a Process for finding the equation of a linear function, thus creating a Process for defining a linear function. *Reversal* occurs as an individual reflects on a previously constructed Process in order to reverse the Process into a new, reversal Process. Returning to the example of the linear function concept, the student can reverse the Process of defining a linear function based on its slope and y-intercept into a Process of determining the slope and y-intercept of an already-defined function (Dubinsky, 1991).

Objects may be constructed through *encapsulation*, which occurs when an individual thinks of a Process as a totality by applying Actions on said Process. This encapsulated Object can be de-encapsulated back to the underlying Process when necessary. (Dubinsky, 1991). Continuing with the example of a linear function concept, a student can consider the Process of defining a linear function as a totality and apply the translation Action to this totality, thus creating a new linear function Object that other Actions can be preformed on.

Schemas may be constructed through *generalization*, which occurs when an individual can apply an existing Schema to a wider collection of phenomena (Dubinsky, 1991). Generalization may occur via two different mechanisms: assimilation and accommodation. *Assimilation* occurs when an individual can apply a cognitive structure with minimal change in order to deal with a new situation while *accommodation* occurs when an individual needs to reconstruct and modify a cognitive structure in order to deal with a new situation. Again returning to the example of a linear function concept, a student could apply the same translation Actions of linear functions to quadratic functions, resulting in a new Schema of functions. The student's need or lack of need to modify their previous Schema would determine whether the student accommodated or assimilated their linear function Schema.

Two aspects of APOS Theory distinguish itself from other theoretical perspectives that explain how students develop understanding of mathematical concepts: Schema development and a genetic decomposition. These two aspects will be introduced below and a brief description of their usefulness in the context of this study will be provided. An elaboration of each of these aspects, complete with examples and relevant literature on the two aspects, will be provided in Chapter 2.

Schema development occurs in three stages: Intra-, Inter-, and Trans-[13], collectively referred to as the *triad of Schema development* (B. Baker, Cooley, & Trigueros, 2000). The first stage, *Intra-*, is characterized by viewing Actions, Processes, or Objects within the Schema in isolation from one another. The second stage, *Inter-*, is characterized by some relationships being formed between the Actions, Processes, or Objects within the Schema. The third and final stage, *Trans-*, is characterized by an implicit or explicit underlying structure that provides coherence and understanding of the relationships developed in the Inter- stage of the Schema (B. Baker et al., 2000). The triad of Schema development will be essential in providing a structure for interpreting students' understanding of proof by contradiction.

A genetic decomposition outlines the hypothetical constructions students should[14] make in order to understand a concept (Arnon et al., 2014). This cognitive outline is constructed based on the following: an analysis of the historical development of the concept in question, a literature review, and the conception of the instructor or researcher. Once developed, a genetic decomposition is then used as a guide for developing all instructional material (Arnon et al., 2014). A genetic decomposition will be essential in modeling how students develop an understanding of proof by contradiction.

Applying the cognitive lens of APOS Theory to proof by contradiction, this study will focus on how students may develop an understanding of proof by contradiction and the implications of this understanding for a transition-to-proof course. In particular, this study will attempt to answer the following questions:

[13]The hyphen "-" is used to describe the stage of development for a particular Schema. For example, Trans-contradiction would refer to the Trans stage of proof by contradiction Schema development.

[14]There may be more than one distinct genetic decomposition for a concept as there is more than one way to develop understanding of a concept.

1.3 Research questions

1. How do transition-to-proof students initially conceptualize proof by contradiction?
   (a) What are students' initial conjectures about the proof method?
   (b) How does outlining the logical structure of a presented proof affect students' initial understanding of the proof method?
   (c) How do comprehension questions affect students' initial understanding of the proof method?
   (d) How does comparing logical outlines of proofs affect students' initial understanding of the proof method?
2. How do students develop an understanding of proof by contradiction over time?
   (a) How do students explain the underlying concept of a proof by contradiction as they develop an understanding of the method?
   (b) What types of tasks support transition-to-proof students' development of the proof method over time?
   (c) How do cognitive obstacles inhibit transition-to-proof students' understanding of the proof method over time?

1.4 Chapter summary

This chapter began with a definition of proof as a mathematical argument with the following characteristics: (1) it uses statements accepted by the classroom community, (2) it employs forms of reasoning that are valid and known to the classroom community, and (3) it is communicated with forms of expression that are appropriate and known to the classroom community. The classroom community of this study, transition-to-proof, was then briefly described.

After introducing the setting of this study, the chapter introduced literature on mathematical proof in a transition-to-proof course. This review was organized around four aspects of proof: construction, comprehension, validation, and evaluation. Then, common direct and indirect proof methods presented in transition-to-proof courses were then summarized, which lead to a focus on the most difficult of these: proof by contradiction.

After introducing the focal proof method of this study, the chapter introduced literature on this method. This review was organized around the previously described four aspects of proof and is summarized below.

- *Construction*: Student difficulties constructing proofs by contradiction include negating intuitive claims, constructing and manipulating impossible mathematical objects, and identifying a contradiction;

- *Comprehension*: Two models related to understanding proof by contradiction have been proposed, but neither model adequately describes how students develop an understanding of proof by contradiction in a transition-to-proof course;

- *Validation*: The approaches students utilize to validate a proof by contradiction differ greatly from the approaches of advanced mathematics students and mathematicians; and

- *Evaluation*: Explanations for students' avoidance of proof by contradiction, posited by the literature, included a preference for constructive proofs, the historical view of necessary causality, difficulties dealing with impossible mathematical objects, and difficulties accepting the logical underpinning of proof by contradiction itself. However, this preference for constructive proofs has been recently challenged.

After introducing the known literature on proof by contradiction and illustrating a need for research on how students understand proof by contradiction, student understanding of a general concept was discussed in the form of reflective abstraction. This led to a short description of a theoretical perspective grounded in reflective abstraction, APOS Theory, and the noteworthy aspects of the theory for this study: the triad of Schema development and genetic decomposition.

Finally, the research questions guiding this study were posed, which focused around two main questions: (1) How do transition-to-proof students initially conceptualize proof by contradiction and (2) How do students develop an understanding of proof by contradiction over time?

CHAPTER 2

LITERATURE REVIEW

The previous chapter situated student understanding of the contradiction proof method within the larger context of literature on proof and proof by contradiction. This chapter will provide details of the most relevant results in the literature on understanding proof and on proof by contradiction. First, Section 2.1 will outline results on the aspects of proof relevant to student understanding of proof. Then, Section 2.2 will provide a comprehensive literature review on proof by contradiction. Finally, Section 2.3 will discuss literature on the triad of Schema development and genetic decomposition.

2.1 Literature on the aspects of proof

Until recently, research on mathematical proof has been dominated by a focus on proof construction to the exclusion of other aspects. Indeed, Inglis and Mejia-Ramos (2009) presented an exhaustive bibliographical study conducted in 2008 of published articles discussing mathematical proof or argumentation. In their comprehensive review of 106 empirically-based research articles on mathematical proof, each article was sorted based on the tasks and goals of the study. The majority of studies, 82 out of 106, asked students to construct a proof to do one of the following: answer an open-ended question (44 of 82), estimate the truth of a conjecture (16 of 82), or justify a statement (22 of 82). A minority of the studies, 21 out of 106, asked students to read an argument and assess whether or not the proof was valid. The remaining studies, 3 out of 106, asked students to read an argument and assessed whether the student understood the argument read. In their discussion, the authors state:

> [...] we hypothesise that (i) the comprehension of given mathematical arguments and (ii) the presentation of these arguments to demonstrate one's understanding of them, are two of the key activities involved in the assessment of undergraduate

> students' proving skills [...] If this is indeed the case, our findings suggest that we, mathematics educators, know very little about students' behavior in some of the main types of activities involved in the assessment of their proving skills, which in turn may become the type of activities many students focus on, precisely because of their involvement in assessment. (Inglis & Mejia-Ramos, 2009, p. 92-93)

Note the authors describe (i) proof comprehension and (ii) demonstrating proof comprehension through proof construction as two of the key activities in the assessment of students' proving skills. Yet, only 3 of the 106 studies discussing mathematical proof focused on proof comprehension. It is therefore clear that more research is needed on the comprehension aspect of proof in general.

Given the previous lack of research on student understanding of proof, as exemplified in the study above, the rest of this section will be devoted to recent studies that provide insight into student understanding of proof along the four aspects of proof introduced in Chapter 1: construction, comprehension, validation, and evaluation. For each aspect, the known results and their implications on proof by contradiction will be presented, with greater details provided for any results that will be especially useful in describing student understanding of proof by contradiction or directly relevant to proof by contradiction in some other way.

2.1.1 Proof construction

Of the four aspects, proof construction has received the most attention (Inglis & Mejia-Ramos, 2009). Early work in proof construction research focused on students' difficulties while writing proofs. Some of the difficulties reported include:

- Difficulties with the underlying mathematical logic (e.g., Knipping, 2008; Moore, 1994; Stavrou, 2014);
- Difficulties understanding the mathematical concepts within the proof (e.g., Mejía-Ramos et al., 2015; Moore, 1994; Stavrou, 2014);
- Difficulties writing with particular proof methods:

  - Induction (e.g., Andrew, 2007; J. D. Baker, 1996; Dubinsky, 1986, 1989; Dubinsky & Lewin, 1986; Harel, 2002) and
  - Contradiction (e.g., Antonini & Mariotti, 2009; Barnard & Tall, 1997; Harel & Sowder, 1998; Reid & Dobbin, 1998);

- Difficulties knowing how to approach proving statements, i.e. lack of proof-writing strategies (e.g., Hanna, 2000; Hoyles, 1997; Moore, 1994; Weber, 2001, 2004); and
- Difficulties with quantified statements (e.g., Barnard, 1995; Dubinsky, 1989; Dubinsky & Yiparaki, 2000; Piatek-Jimenez, 2010).

Of note for proof by contradiction are the last two difficulties: knowing how to approach statements and quantified statements. For the former, students must realize they need to write a proof by contradiction and have some general strategies to complete the proof before they can successfully do so. For the latter, it is commonly required that students negate a quantified statement when writing a proof by contradiction. The rest of this subsection will describe prominent literature on these two difficulties.

Many researchers and educators have noted that in some situations, students may be aware of the theorems necessary to complete a proof and yet cannot do so. To uncover why this may occur, Weber (2001) conducted interviews with four undergraduate and four graduate students in which participants were asked to prove statements in abstract algebra. While the undergraduate students were aware of the facts necessary to complete the proof, as well as the proof methods to use, they were unable to complete the proofs. Unlike the undergraduates, the graduate students were able to complete the proofs using *strategic knowledge*, defined as: "knowledge of how to choose which facts and theorems to apply" (Weber, 2001, p. 101). The author hypothesized that there are four types of strategic knowledge: knowledge of the domain's proof techniques, knowledge of which theorems are important and when they will be useful, knowledge of when and when not to use syntactic strategies[1], and procedural[2] strategies.

Weber (2003) expanded on this research with a focus on six undergraduates' use of procedural strategies in understanding proof. From this study and his previous study, Weber (2004) presented a framework for describing undergraduate proof construction processes based on the observations of 176 undergraduate students' proofs over multiple studies. This framework classified the types of proofs produced as one of the following: procedural, syntactic, or semantic.

In a proof using *procedural* methods, "one attempts to construct a proof by applying a procedure, i.e., a prescribed set of specific steps, that he or she believes will yield a valid proof" (Weber, 2004, p. 426). The procedure can either be an algorithm or a process. *Algorithms* are characterized as external and highly mechanical to the student, whereas a *process* is internal and flexible. Note the close relation of algorithms in proof construction to Actions in APOS Theory; both are external and require step-by-step instruction. Also note the close relation of processes in proof construction to Processes in APOS Theory; both are described as internal and more flexible than the step-by-step nature of algorithms/Actions.

In a proof using *syntactic* methods, "one attempts to write a proof by manipulating correctly stated definitions and other relevant facts in a logically permissible way" (Weber, 2004, p. 428). Proofs of this form are no more than unpacking definitions and using tautologies to manipulate symbols in order to reach the desired conclusion. Students using this method do not need to consider the meaning of their syntactic statements. In terms of APOS Theory, syntactic methods may be either an Action (e.g. if the students have memorized which definitions to unpack for specific problems), or a Process (e.g. if students have a general rule to "unpack definitions" when necessary).

In a proof using *semantic* methods, "one first attempts to understand why a statement is true by examining representations (e.g., diagrams) of relevant mathematical objects and then uses this intuitive argument as a basis for constructing a formal proof" (Weber, 2004,

[1]Proof by simply manipulating symbols.

[2]Proof by external, explicit step-by-step instructions.

p. 429). Very few undergraduate research subjects, if any, attempted semantic proofs; 0 of 56 proofs in abstract algebra and 17 of 120 proofs in real analysis. In terms of APOS Theory, semantic methods seem to correlate with an Object level of understanding as both require the individual to understand the statement in totality. A lack of students attempting semantic proofs may indicate a lack of understanding with either the mathematical content or proof itself.

Another difficulty students may have when writing a proof is with the quantification of the statement (Dubinsky, 1989; Dubinsky & Yiparaki, 2000; Piatek-Jimenez, 2010). Student difficulties with quantification can be viewed in three categories: single-level quantification (e.g., for all ...), multiple-level quantification (e.g., for all ..., there exists ... such that ...), and the negation of any quantification. In particular, both Dubinsky and Yiparaki (2000) and Piatek-Jimenez (2010) focus on student difficulties with AE (for all ..., there exists ...) and EA (there exists ... for all...) statements, concluding that students struggle to understand the mathematical meaning behind multiple-level quantification. Dubinsky (1989) had difficulties isolating examples of students negating single-level quantification, though quoted an unpublished observation in which 49 of 52 sophomore mathematics majors could not negate the statement "Every member of my family is unemployed."

Quantification in proof by contradiction normally appears in one of three forms: nonexistence, uniqueness, and infinitely many. Nonexistence claims such as "There is no odd integer that can be expressed in the form $4j - 1$ and in the form $4k + 1$ for integers j and k" are commonly proved by assuming there does exist an object (in this example, an odd integer) that has the desired property and deriving a contradiction from this assumption. Uniqueness claims such as "For every non-zero rational number, there exists a unique multiplicative reciprocal" are commonly proved by assuming either there are no objects or there are two distinct objects with the desired quality (in this example, multiplicative reciprocals of a non-zero rational number), showing at least one object exists with the desired quality, and finally showing that two distinct objects are actually the same, i.e. deriving a contradiction. Infinity claims such as "The set of primes is infinite" are commonly proved by assuming the set is finite and constructing a new element not in the set, thus deriving a contradiction to the original assumption. In each of these three cases, the student must negate a single level of quantification before proceeding with the proof by contradiction. Therefore, students' understanding of quantification will be considered when studying their understanding of proof by contradiction.

2.1.2 Proof comprehension

There is relatively little research on proof comprehension when compared to proof construction (Inglis & Mejia-Ramos, 2009; Samkoff & Weber, 2015), especially concerning empirical research results on proof comprehension. The research results available suggest that focusing on proof comprehension may aid students in understanding and constructing proofs (Hodds et al., 2014; Samkoff & Weber, 2015). These studies relied on a proof comprehension model proposed by Mejía-Ramos et al. (2012). This model, designed based on the only previous proof comprehension model, proposed by Yang and Lin (2008), is currently in validation stages (Mejía-Ramos & Weber, 2016). In this way, the model proposed by Mejía-Ramos et al. (2012) could be considered the definitive assessment model on proof comprehension at the moment. This subsection will briefly describe both models and discuss the current results on proof comprehension based on these models.

In order to analyze and describe student understanding of geometry proofs, Yang and Lin (2008) considered *reading comprehension of proofs*, defined as "understanding proofs from the essential elements of knowing how a proof operates and why a proof is right, in addition to knowing what a proof can prove" (Yang & Lin, 2008, p. 60). A theoretical analysis of the existing literature on the contents of reading comprehension of geometric proof produced a hypothetical model of Reading Comprehension of Geometric Proof (RCGP). This model consisted of five facets of comprehension (basic knowledge, logical status and summarization, generality, and application) situated between four levels of understanding (Surface, Recognizing Elements, Chaining Elements, and Encapsulation). The meaning of each level of understanding, including the facets of comprehension, is reproduced in bulleted form below:

1. Surface - Epistemic understanding without analyzing elements of an argument in this proof;
    - Basic knowledge - Measure the understanding of mathematical terms, figures or symbols;
2. Recognizing Elements - Recognition of premises, conclusions or properties that may be implicity applied in this proof;
    - Logical Status - Measure the recognition of the status of an argument, which may be premises, conclusions or applied properties in proof;
    - Summarization - Measure the understanding of the given, the claim or a critical proof idea;
3. Chaining Elements - Logically to chain premises, properties and conclusions in this proof and to view figures as referential objects, which should be necessary for logically chaining arguments;
    - Generality - Measure the recognition of accuracy of a proposition or proof and what a proof proves;
    - Application - Measure the ability of knowing how to apply a proposition in another situation;
4. Encapsulation - Interiorize this proposition and its proof as a whole, where one can apply this proposition and its proof, and distinguish different premises from similar propositions. (Yang & Lin, 2008, p. 63, 65)

The last three levels of RCGP are consistent with the triad of Schema development. The second level, Recognizing Elements, aligns with the Intra-proof stage in which premises, conclusions, or properties are considered in isolation of each other. The third level, Chaining

Elements, aligns with the Inter-proof stage in which relations between the premises, conclusions, or properties are formed. The final level, Encapsulation, aligns with the Trans-proof stage in which the implicit or explicit underlying structure provides coherence to view the proposition and proof as a whole.

To test the hypothetical RCGP model, a 16 question assessment was distributed to 223 9th graders (who had not yet encountered geometry proof in school) and 378 10th graders (who had already encountered geometry proofs in school). Analysis of the results supported the hypothetical RCGP model, suggesting that proof comprehension can be developed and assessed in terms of reading comprehension of proofs.

Generalizing the RCGP model by Yang and Lin (2008), Mejía-Ramos et al. (2012) present a multidimensional model for assessing proof comprehension in undergraduate mathematics. This model contains seven different aspects of proof split into two categories: local and holistic. *Local* types of assessment focus on only one, or a small number, of statements within a proof, whereas *holistic* types of assessment focus on student understanding of a proof as a whole, much like the proof framework described by J. Selden and Selden (1995). These seven types of assessment are reproduced below from the study by Mejía-Ramos et al. (2012):

Local

1. *Meaning of terms and statements:* items of this type measure students understanding of key terms and statements in the proof;
2. *Logical status of statements and proof framework:* these questions assess students' knowledge of the logical status of statements in the proof and the logical relationship between these statements and the statement being proven;
3. *Justification of claims:* these items address students' comprehension of how each assertion in the proof follows from previous statements in the proof and other proven or assumed statements;

Holistic

4. *Summarizing via high-level ideas:* these items measure students' grasp of the main idea of the proof and its overarching approach;

5. *Identifying the modular structure:* items of this type address students' comprehension of the proof in terms of its main components/modules and the logical relationship between them;

6. *Identifying the general ideas or methods in another context:* these questions assess students' ability to adapt the ideas and procedures of the proof to solve other proving tasks;

7. *Illustrating with examples:* items of this type measure students' understanding of the proof in terms of its relationship to specific examples. (Mejía-Ramos et al., 2012, p. 15-16)

These comprehension questions focus on a presented proof of a statement and can be used to assess students' understanding of one or more lines of the particular proof.

Using the proof comprehension model above (but before publishing the model), Weber and Samkoff (2011) designed a study to identify strategies that successful undergraduates use to overcome difficulties in understanding mathematical proof in order to improve proof pedagogy. The authors suggested six proof-reading strategies: (1) rephrasing the theorem, (2) attempting to prove theorem, (3) identifying proof framework, (4) partitioning the proof, (5) checking inferences with examples, and (6) provide high-level summary. Samkoff and Weber (2015) developed an instructional experiment to teach students to apply these strategies and assess whether these proof-reading strategies would aid student understanding. The authors found that: (1) specific prescriptive guidance helped students implement them [the strategies] more effectively, (2) these strategies were beneficial to students, and (3) that there were impediments to proof comprehension that could not be addressed by these strategies. These results suggest that while the proof comprehension model by Mejía-Ramos et al. (2012) may assess student understanding of proof, it cannot, alone, be used as a pedagogical tool to develop instruction for a transition-to-proof course.

Utilizing the proof comprehension model by Mejía-Ramos et al. (2012), Hodds et al. (2014) developed a booklet containing self-explanation training focused on the logical relationships within a mathematical proof. Through a series of three experiments, the authors found that: (1) students who received the self-explanation training scored higher on a comprehension test, (2) self-explanation training increased cognitive engagement with a proof, and (3) a short self-explanation training session within a lecture improved students' proof comprehension and that this comprehension persisted over time. These results suggest that focusing on the logical relationships within a mathematical proof, and specifically for a proof by contradiction, may be key for developing student understanding for the proof method.

2.1.3 Proof validation

Proof validation is rarely taught explicitly in transition-to-proof courses and yet may be critical in developing a complete understanding of proof (Ko & Knuth, 2013; A. Selden & Selden, 1999). As mentioned in Chapter 1, reflection is necessary for developing an understanding of a concept. In this way, proof validation is key in developing a student's understanding of proof. Yet, very little research has focused on proof validation and so the extent to which validation and comprehension are linked is unknown (Ko & Knuth, 2013). In addition, there is no single set of validation techniques accepted by the mathematical community (Inglis, Mejia-Ramos, Weber, & Alcock, 2013), which may be a reason why proof validation is not explicitly taught in transition-to-proof courses. What is known, however, is that much of what constitutes convincing for undergraduate students (e.g., authority, appeals to physical objects, truth value of the statement) is not considered convincing for practicing mathematicians (Harel & Sowder, 1998; Weber & Alcock, 2005). The following subsection will thus be devoted to the most common ways in which undergraduate students validate proofs.

In an exploratory study, Harel and Sowder (1998) provided a complete classification of *proof schemes*[3], or that which convinces the individual a statement is true[4] (Harel & Sowder, 1998). This classification of consists of three categories: external conviction, empirical, and analytical, as well as many subcategories. A brief explanation of the overarching categories follows.

An *external conviction proof scheme* is characterized by students removing doubt by "the ritual of the argument presentation, the word of an authority, or the symbolic form of the argument" (Harel & Sowder, 1998, p. 246). The authors claimed that students are prone to accept false proof verifications on the basis of ritual and form when formality in mathematics is emphasized prematurely, emphasizing that "accepting false-proof verifications on the basis of their appearance is a severe deficiency in one's mathematical education" (Harel & Sowder, 1998, p. 246), though claimed that proofs by authority have some value. Indeed, Weber and Mejia-Ramos (2013) found that if a paper was published in a highly reputable journal or another highly reputable source, proofs were accepted as true without justification. This practical acceptance of proofs by authority show that external proof schemes are not always harmful. An *empirical proof scheme* is one in which "conjectures are validated, impugned, or subverted by appeals to physical facts or sensory experiences" (Harel & Sowder, 1998, p. 252). Research preceding Harel and Sowder (1998) validates the dominance of empirical proof schemes among novice and advanced students alike (e.g., Chazan, 1993; Goetting, 1995; Yerushalmy, 1986). An *analytical proof scheme* is "one that validates conjectures by means of logical deduction" (Harel & Sowder, 1998, p. 258). This scheme can be thought of in two parts: the transformation of mental images by deduction (transformational) and justification based on axioms (axiomatic). The authors state "Although the authoritarian and empirical schemes have value, we feel that mathematics majors in particular should also eventually show evidence of the analytical proof schemes" (Harel & Sowder, 1998, p. 277). In other words, mathematics majors should transition to an analytic proof scheme at some

[3]Note this is not the same as a proof Schema, which is the collection of Actions, Processes, Objects, and other Schemas linked by some general principal to form a framework of what proof is to an individual.

[4]Recall that the truth value of statements within a proof is connected to local comprehension of a proof and is thus of interest to proof comprehension.

point in their undergraduate studies.

Harel and Sowder's (1998) classification of proof schemes has been used in several studies (e.g., Hadas, Hershkowitz, & Schwarz, 2000; Housman & Porter, 2003) and has been useful in describing how students validate specific lines of proofs during proof construction. For example, both Hadas et al. (2000) and Housman and Porter (2003) found that generating examples aided students in validating specific statements (i.e., local comprehension). However, their classification of proof schemes is not useful when considering how students validate a proof (as opposed to a statement). What follows is a brief overview of two studies that analyzed how students validate a given proof of a statement.

A. Selden and Selden (2003) evaluated how eight undergraduate students read and validate purported proofs of a single theorem. While students professed they "check proofs step by step, follow arguments logically, generate examples, and make sure the ideas in a proof make sense" (A. Selden & Selden, 2003, p. 27) in order to validate the proof, the authors found little evidence that students actually do so based on their judgment results. That is, students purported to validate proofs by considering local and holistic aspects of a proof, yet their results suggested they did not actually do so. These results indicate that there is more to validating proofs than simply knowing useful validation strategies.

In a more narrow study on proof validation, Weber and Alcock (2005) conducted an in-depth theoretical analysis of students' use of warranted implications to understand and validate proofs, where *warranted implications* are understood as "good reason to believe that each statement follows from the preceding statements or from other accepted knowledge" (Weber & Alcock, 2005, p. 125). The authors conclude that the emphasis on the truth value of the precedent and antecedent over whether the implication was warranted lends to student difficulty validating proofs. This is especially relevant to proof by contradiction as students may focus on the truth value of the statement being negated over whether the implication is warranted.

2.1.4 Proof evaluation

While proof construction, comprehension, and validation work in concert to develop a totality of understanding with respect to proof, proof evaluation makes quality judgments that may not aid in understanding (A. Selden & Selden, 2017). Such quality judgments may include aesthetic properties such as "comprehensible, ingenious, explanatory, elegant, deep, beautiful, and insightful" (Hanna & Mason, 2014, p. 30). A few attempts have been made to quantify proof evaluation in order to more easily study the aspect, such as the concept of 'width of a proof' presented by Gowers (2007). These early attempts have not been well accepted by the mathematical community, as illustrated by the response of Hanna and Mason (2014). A short summary of Hanna and Mason's (2014) response will illustrate the early development of proof evaluation research in the mathematics community. While lacking objective qualities like logical correctness, these aesthetic judgments may account for students' preferences for or against particular proof methods and are thus useful to consider in the context of proof by contradiction.

Reflecting on Gowers' (2007) attempt to quantify aesthetic properties of a proof, Hanna and Mason (2014) discussed the concept of *width of a proof*, defined as the "number of distinct pieces of information, or ideas, needed to complete it" (Hanna & Mason, 2014, p. 34). Width of a proof developed out of the belief that a memorable proof is greatly preferred over an unmemorable proof and that memorability was directly related to other ascetic properties of a proof, such as elegance or explanatory power (Gowers, 2007). While width of a proof would be more easily quantified than elegance or explanatory power, Hanna argues against width of a proof in a number of ways: (1) that the width of a proof does not capture many of the ascetic properties it proposes to capture, like comprehensibility or insight; (2) that some ascetic properties, such as elegance and comprehensibility, are inversely related and thus one measure of ascetic is not appropriate; (3) that weights may be needed to account for especially sophisticated ideas when they are used to complete a proof; (4) that a lower width may not always be preferred; and (5) that there may not even be a need to quantify ascetic properties of a proof. Of note in this rebuttal is that comprehensibility is a part of evaluation and therefore students should develop a robust understanding of proof before considering evaluating proofs.

This section outlined results on the aspects of proof relevant to proof by contradiction specifically. The following section will provide a comprehensive literature review on proof by contradiction.

2.2 Literature on the aspects of proof by contradiction

Student understanding of proof by contradiction is the primary focus of this study as it is one of the most difficult proof method for transition-to-proof students to construct and understand (Antonini & Mariotti, 2006; Brown, 2013; Harel & Sowder, 1998). This section will be devoted to reviewing all of the literature on proof by contradiction, organized by aspects of proof. Subsection 2.2.1 will highlight various difficulties students encounter when constructing proofs by contradiction. Then, Subsection 2.2.2 will evaluate the current models on student comprehension of proof by contradiction. Next, Subsection 2.2.3 will describe the only study to date on validating proofs by contradiction. Finally, Subsection 2.2.4 will discuss the various reasons students avoid proof by contradiction.

2.2.1 Construction aspect of proof by contradiction

Proof construction and comprehension are intimately connected aspects of proof as they directly affect one another. Students gain understanding as they construct proofs, which in turn allows students to construct stronger proofs. Indeed, some of the difficulties student have when writing proofs by contradiction can be explained in terms of their understanding of proof by contradiction. This subsection will highlight various difficulties students encounter when constructing proofs by contradiction and provide details on the accompanying literature of these difficulties. In particular, this subsection will describe the following difficulties: (1) negating intuitive claims, (2) constructing and manipulating impossible mathematical objects, and (3) identifying a contradiction.

Every proof by contradiction begins with a negation of the claim to be proven and thus student difficulties negating statements are necessary to discuss with proof by contradiction. As already mentioned, negating any quantified statement can be difficult for students, with multi-level quantified statements such as 'for all, there exists...' (Dubinsky et al., 1988; Dubinsky & Yiparaki, 2000; Piatek-Jimenez, 2010) being especially difficult for students to negate. In order to study how university students negate various types of statements, Barnard (1995) chose 7 statements with varying levels of quantification, reproduced below:

1. x satisfies P, for all x in X;
2. x satisfies P, for some x in X;
3. x and y satsify P;
4. x satisfies P and Q, for all x in X;
5. A implies B;
6. There exists x in X such that $S(x, y)$ is true for all z in Z. (Barnard, 1995, p. 1)

Each statement was first set in everyday context and in mathematical context, after which they were given to 78 first year and 78 second/third year university students to negate. Beyond difficulties based on logical structure, transitioning between different representations of a statement (e.g. from a symbolic representation to a semantic representation), and degree of complexity (i.e. number of quantifications), students struggled to negate statements they believed to be true or intuitive. This difficulty, negating intuitive statements and using these negations to prove a statement the student already believes to be true, has been reproduced by authors such as Reid and Dobbin (1998). When presented with a statement such as "the $\sqrt{2}$ is irrational", the student is unable to divorce their knowledge that the statement is true from their proof of the statement and thus the assumption "the $\sqrt{2}$ is rational" is a difficulty the student cannot overcome (Reid & Dobbin, 1998).

The strain on knowing what is true and not true in a proof by contradiction also leads to problems for students when they construct and manipulate impossible mathematical objects (Antonini & Mariotti, 2008; Leron, 1985). For example, consider the following statement and its proof, with the constructed false object marked [**C**], below:

Statement: The set of primes is infinite [**P**].

Proof: Suppose the set of primes is finite [$\sim$ **P**]. Let $p_1, p_2, p_3, \ldots, p_k$ be all those primes. Let n be one more than the product of all of them: $n = (p_1 p_2 p_3 \ldots p_k)+1$ [**C**]. Then n is a natural number, so n has a prime divisor q. Since q is prime, $q > 1$ [**Q**]. Since q is prime and $p_1, p_2, p_3, \ldots, p_k$ are *all* of the primes, q is one of the p_i in the list. Thus, q divides the product $p_1 p_2 p_3 \ldots p_k$. Since q divides n, q divides the difference $n - (p_1 p_2 p_3 \ldots p_k)$. But this difference is 1, so $q = 1$ [$\sim$ **Q**]. From the contradiction, $q > 1$ and $q = 1$ [$\mathbf{Q} \wedge \sim \mathbf{Q}$], we conclude that the assumption that the set of primes is finite is false [$\sim\sim$ **P**]. Therefore, the set of primes is infinite [**P**]. (Smith et al., 2015, p. 44)

In the proof above, the constructed false object, $n = (p_1 p_2 p_3 \ldots p_k)+1$, seemingly appears out of nothing. While constructions in any proof can seem arbitrary, some researchers believe constructions within a proof by contradiction are even more troublesome (Leron, 1985). (e.g., Leron, 1985) explained this difficulty as the strain of knowing what is and is not true in a proof by contradiction. Returning to the proof that the set of primes is infinite, the mathematical object $P = \{p_1, p_2, p_3, \ldots, p_k\}$, the set of 'all' prime numbers, can be viewed as an impossible mathematical object as a finite list of all prime numbers is not possible. The difficulty in constructing a new prime number that is not in P could then be explained by a student being unable to deal with the impossible set P rather than the construction being arbitrary. For another example, consider the following statement and its proof:

Statement: Let a and b be two real numbers. If $ab = 0$, then either $a = 0$ or $b = 0$.

Proof: Assume $ab = 0, a \neq 0$, and $b \neq 0$ [$\sim$ **P**]. Since $a \neq 0$ and $b \neq 0$, both sides of the equality $ab = 0$ can be divided by a and by b, obtaining $1 = 0$ [**Q**]. However, $1 \neq 0$ [$\sim$ **Q**], a contradiction [$\mathbf{Q} \wedge \sim \mathbf{Q}$]. Therefore if $ab = 0$, then either $a = 0$ or $b = 0$ [**P**]. (Antonini & Mariotti, 2008, p. 404)

In this example, $ab = 0$ is an impossible mathematical object when $a \neq 0$ and $b \neq 0$.

Antonini and Mariotti (2008) presented a student, Maria, who expressed the following issue with her proof of the statement above:

> M: Moreover, so as $ab = 0$ with a different from 0 and b different from 0, that is against my common beliefs [Italian: "*contro le mie normali vedute*"] and I must pretend to be true, I do not know if I can consider that $0/b = 0$. I mean, I do not know what is true and what I pretend is true. [...] It is as if a, b and ab move from the real world to the absurd world, but the rules do not function on them, consequently they have to come back ... But my problem is to understand which are the rules in the absurd world, are they the rules of the absurd world or those of the real world. (Antonini & Mariotti, 2008, p. 406)

In order for Maria to deal with the impossible mathematical object $ab = 0$, she suspended all her mathematical beliefs and no longer knew if algebraic operations, such as division, on the impossible mathematical object $ab = 0$ were valid. Leron (1985) postulated that the separation of construction and the negative assumption in a proof by contradiction would alleviate this difficulty as "the moment the negative assumption is declared, along with the intention of falsifying it by means of a future contradiction, a cognitive strain is set up in the mind of the learner" (Leron, 1985, p. 324). While this may alleviate the construction difficulty, it may prevent the student from developing a robust understanding of proof by contradiction (Tall et al., 2012). Antonini and Mariotti (2008) do not offer any pedagogical strategies to alleviate this construction difficulty.

The final difficulty a student could have constructing a proof by contradiction would be an inability to identify a contradiction, thus leaving the student unable to complete the proof. Students have even more difficulty identifying a contradiction when it does not directly relate to the primary statement they are trying to prove (Antonini & Mariotti, 2009; Barnard & Tall, 1997). Again consider the statement "Let a and b be two real numbers. If $ab = 0$, then either $a = 0$ or $b = 0$." There are essentially[5] two contradictions that can be reached:

[5]The same could be said for $b \neq 0 \ \wedge b = 0$ without the loss of generality.

$a \neq 0 \wedge a = 0$ or $1 = 0 \wedge 1 \neq 0$. Yet, it is not clear to students at the onset of a proof which of these contradictions they will eventually encounter. It is especially unclear why $1 = 0 \wedge 1 \neq 0$ would be a contradiction the student could conclude, given the statement to be proved. Again, Antonini and Mariotti (2008) do not offer any pedagogical strategies to alleviate this construction difficulty.

2.2.2 Comprehension aspect of proof by contradiction

Two models related to student understanding of proof by contradiction are available in the literature: one model describing how students develop understanding and another model for analyzing how students understand proof by contradiction. For each model, I will provide a detailed description of the model followed by its merits and limitations.

Lin et al. (2003) are the only researchers to present a model of how students develop an understanding of proof by contradiction. This model is based on a few assumptions, quoted below:

> Understanding proof by contradiction shall mean to have both the procedural and conceptual knowledge of proof by contradiction. The procedure knowledge is: negating the conclusion q, and then inferring a mathematical fact or assertion that is contradicted to p. The conceptual knowledge is: "if $\sim q$ then $\sim p$ implies "if p then q". This step, which is the principle of proof by contradiction, is based on the law of contrapositive. (Lin et al., 2003, p. 443)

Under these assumptions, the authors developed a questionnaire to address three categories: negating a statement, recognizing the procedure, and recognizing the law of contrapositive. This questionnaire was distributed to 140 high school students and 64 college students, whose responses were analyzed for correctness. Statistical methods were then employed to find significant factors that affect the understanding of proof by contradiction, defined as having both procedural and conceptual knowledge of proof by contradiction. Five such factors emerged from the analysis: (1) the ability to negate statements without quantifiers or with the quantifier "some", (2) the ability to recognize the law of contrapositive, (3) the ability to negate statements with the quantifier "only one", (4) the ability to describe the procedural knowledge of proof by contradiction, and (5) the ability to negate statements with the quantifier "all". Using this data, the following model of how students develop an understanding of proof by contradiction was proposed:

> The first step of proof by contradiction is to negate the conclusion. After a student is able to negate a basic statement, he/she can begin to learn the procedure knowledge of proof by contradiction. However, only until a student understands the law of contrapositive, he/she will know why the procedure is finished. The ability of negating a statement might be developed unrelated to the understanding of the procedural knowledge of proof by contradiction. (Lin et al., 2003, p. 448)

In other words, the ability to negate a statement, while necessary to constructing a proof by contradiction, is not necessary to understanding proof by contradiction. Understanding proof by contradiction thus develops first from knowing the process of writing a proof by contradiction to then knowing both the process of writing a proof by contradiction and understanding the law of contrapositive.

As the first model on student understanding of proof by contradiction, Lin et al. (2003) drew the attention of the research community to the need for a model of student understanding of proof by contradiction. However, limitations of the authors' study may mitigate the usefulness of this model. First, the authors assumed "the conceptual knowledge of proof by contradiction is the law of contrapositive" (Lin et al., 2003, p. 446). As argued in Chapter 1, proof by contradiction is sufficiently different than proof by contraposition. The distinction between these two methods is most apparent when the statement "Q" is not explicitly stated, as exemplified in the proof of the irrationality of $\sqrt{2}$. Therefore, it is not appropriate to use one model to describe a student's understanding of both proof methods. The student composition of the authors' study also limits the usefulness of the model. The subjects for this study are mixed, with 140 high school students and 62 college students participating. The expected understanding of high school and college students is distinct enough that a single model for student understanding of proof by contradiction may not be appropriate. Indeed, the authors stated "this study also found that the difficulty levels of students' negating a statement can be ordered decreasingly as negating statements without quantifier, negating "some", negating "all", and negating "only one" (Lin et al., 2003, p. 448). Note this means no multiple-level quantification was tested - likely due to the inclusion of high school students in the sample. This limits the usefulness of the model for describing student understanding in a university-level transition-to-proof course. Finally, this model does not address how to develop student understanding in order to avoid the various difficulties with constructing proof by contradiction described in the previous subsection. Due to these limitations, the model proposed by Lin et al. (2003) is not a very informative model for describing how student understanding of proof by contradiction develops in a transition-to-proof course.

Antonini and Mariotti (2008) presented a model through which student understanding of proof by contradiction can be identified, analyzed, and interpreted. This model is the culmination of numerous studies (Antonini, 2003, 2004; Antonini & Mariotti, 2006, 2007). First, it should be noted that this model is for indirect proof and not solely for proof by contradiction, though Antonini and Mariotti (2008) stated:

> [...] what seems to be psychologically meaningful in proof by contradiction is the starting point that is the negation of the thesis. This characteristic is also shared by proof by contraposition. In spite of significant differences, we can point out some important commonalities of these types of proof. Therefore, in this paper, we deal with both proof by contradiction and proof by contraposition, referring to them through the term *indirect proof*. (Antonini & Mariotti, 2008, p. 402)

Antonini and Mariotti (2008) also presented the theoretical construct Cognitve Unity as a lens through which to examine indirect proof. *Cognitive Unity* is based on the notion that "a mathematical theorem consists in the system of relations between a statement, its proof, and the theory within which the proof makes sense" (Antonini & Mariotti, 2008, p. 404). This triplet is referred to as (S, P, T), where S is the statement, P is the proof, and T is the theory.

For indirect proofs, students move from a principal statement S to a secondary statement S^*, which is tautologically equivalent to S. It is through proving the secondary statement S^* that the primary statement S is proved, i.e. $S^* \to S$. This secondary statement will have its own proof and theory and so has its own triplet (S^*, P^*, T). In this view, students should now prove two statements: S^* and the meta-statement $S^* \to S$. Consider the following examples of principal and secondary statements presented in Antonini and Mariotti (2008):

> Principal Statement S: Let a and b be two real numbers. If $ab = 0$, then $a = 0$ or $b = 0$.
> Secondary Statement S^*: Let a and b be two real numbers. If $ab = 0, a \neq 0, b \neq 0$, then $1 = 0$. (Antonini & Mariotti, 2008, p. 404)

Note the secondary statement can now be proved directly. Using Cognitive Unity, the authors can identify nearly all difficulties students have with proof by contradiction as student difficulties with the proof of the meta-statement $S^* \to S$.

This analysis model is useful in a few ways. By separating the proof of a secondary statement and of the meta-statement $S^* \to S$, students may become aware of the underlying cognitive processes of a proof by contradiction. Cognitive Unity can also explain why students have issues dealing with impossible mathematical objects; when transitioning to the secondary statement S^*, a student may decide that the previous theory T is no longer adequate for proving S^*. This was the case for Maria, an example student from the study by Antonini and Mariotti (2008) presented in the previous subsection on student difficulties constructing proofs by contradiction, who was not sure the number theory rules she was applying were still valid when dealing with the impossible mathematical object $ab = 0$ (with $a \neq 0$ and $b \neq 0$).

There are two mitigating factors that limit the usefulness of this model. First, while the model easily handles how students can transition from the principal statement S to the secondary statement S^* when working with a proof by contraposition, it is less clear how students can transition from S to S^* when working with a proof by contradiction.

For example, it is not clear how a student would know to transition 'Let a and b be two real numbers. If $ab = 0$, then $a = 0$ or $b = 0$' to 'Let a and b be two real numbers. If $ab = 0$, $a \neq 0, b \neq 0$, then $1 = 0$'. The authors did not address how students would make this transition without already completing the proof. Secondly, the model is only useful in analyzing understanding and does not describe how student understanding develops over time[6]. It is not clear from the model how a student should develop their understanding of proof by contradiction, only that it should involve understanding both the secondary statement S^* and the meta-statement $S^* \to S$. Due to these limitations, the model proposed in Antonini and Mariotti (2008) is not a very informative model for describing how student understanding of proof by contradiction develops in a transition-to-proof course.

2.2.3 Validation aspect of proof by contradiction

Proof validation tasks present a student with the opportunity to reflect on a proof by contradiction and develop an understanding that extends beyond a specific proof. A recent study by Brown (2016b) considers how students and mathematicians validate the implication of the meta-statement $S^* \to S$, described in Antonini and Mariotti (2008) as necessary to the understanding of proof by contradiction. This subsection will provide a short summary of the methodology and results of the study by Brown (2016b) as no other study has been conducted that examined how students validate proofs by contradiction.

Brown (2016b) used three stages of data collection to investigate students' and mathematicians' reasoning on the meta-statement $S^* \to S$ using the following statements:

> *Theorem 5.* For all positive integers n, if $n \bmod (3) \equiv 2$, then n is not a perfect square.
>
> *Statement A.* There exists no positive integer n such that $n \bmod (3) \equiv 2$ and n is a perfect square. (Brown, 2016b, p. 4)

[6]Indeed, Antonini and Mariotti (2008) claim "[...] the Cognitive Unity approach can also be an efficient didactical tool for designing teaching/learning situations aimed to introduce indirect proofs." (Antonini & Mariotti, 2008, p. 411). However, the authors do not describe a specific lesson plan nor do they provide a general guide to developing a lesson plan using the Cognitive Unity approach.

In the first stage of data collection, electronic surveys were distributed to students who had recently completed a transition-to-proof course that asked respondents two questions: (1) whether the statement "You can prove Theorem 5 by proving Statement A" was true or false and (2) which statement, Theorem 5 or Statement A, the student would pursue first if they were asked to prove Theorem 5. In the second stage of data collection, 21 students who had recently completed a transition-to-proof course were interviewed for their thinking behind the survey questions. In the third stage of data collection, 6 mathematicians were asked the same questions provided in the electronic survey and were interviewed for their thinking behind the survey questions. Note that in terms of Antonini and Mariotti (2008), Theorem 5 is the principal statement S and Statement A is the secondary statement S^*.

The author found that while advance students and mathematicians were successful in validating the claim 'You can prove Theorem 5 by proving Statement A', novice proof writers were not as successful. The author concluded by stating:

> [...] it appears that, at least for novices, the metatheoretical issues described by Antonini and Mariotti (2008) may play a role in students' interpretations and sense of certainty in the context of the results of a proof by contradiction. [...] a potential source of these difficulties may be validating the relationships between S^* and S. (Brown, 2016b, p. 8)

In other words, student difficulties validating the relationship between S^* and S may affect students' understanding of the results of a proof by contradiction, which would in turn affect students' understanding of proof by contradiction as a proof method.

2.2.4 Evaluation aspect of proof by contradiction

Student avoidance of proof by contradiction may be due to a lack of understanding proof by contradiction as a proof method. In order to help determine alternative reasons students may avoid proof by contradiction, this subsection will discuss the various reasons students avoid proof by contradiction, as identified in the literature.

While proof by contradiction is currently accepted as an equally valid proof method as direct proofs, this has not always been the case[7]. Originally, the acceptance of a proof was based on the amount of evidence a proof gave for the truth of the statement (Mancosu, 1991). In this way, constructive proofs were accepted more readily than a proof by contradiction that provided no evidence a statement was true beyond its logical truth. This historical perspective on proof by contradiction as a less accepted proof method may help explain why students avoid this particular proof method. Leron (1985) questioned what insight a proof by contradiction gives to the student and supported the historical perspective for the need of evidence in a proof, stating:

> At its best, mathematical learning of a proof is based on the learner's construction of a corresponding *mental* entity, an image perhaps, that can then be manipulated in the mind in place of the mathematical object or its symbol on the paper . . . This is also what gives us the Platonic sense that we are working on a mathematical 'reality', manipulating real objects. (Leron, 1985, p. 323)

As noted before, Leron (1985) does not provide empirical evidence to fortify his position. However, Harel and Sowder (1998) provided evidence that students prefer constructive proofs over proofs by contradiction through students' comments such as "I have never really understood proofs by contradiction, they never made sense" (Harel & Sowder, 1998, p. 272). Antonini (2003) provided more evidence that students prefer construction by observing that constructions of non-examples, i.e. examples that do not verify some statement, can lead to spontaneous productions of proof by contradiction. In the author's study, 7 pairs of secondary school students and 2 pairs of college students were given open-ended problems and asked to formulate a conjecture that they then prove. Spontaneous proofs by contradiction arose after students constructed non-examples that led to a specific argumentation 'if it were not so, then . . .' and used the non-example as the contradiction.

Not all literature believes students avoid proof by contradiction due to the lack of

[7]For a more through discussion on the historical acceptance of proof by contradiction, see Mancosu (1991).

evidence a proof by contradiction provides. Reid and Dobbin (1998) presented small groups of 7 year old children reasoning with contradiction while playing a card game. Due to some of the children spontaneously reasoning using contradictions, the authors concluded that student difficulties with proof by contradiction arise from emotioning, i.e. reasoning as a result of emotions, and not from the lack of evidence a proof by contradiction provides nor the logical structure of a proof by contradiction. In this view, claims such as 'the $\sqrt{2}$ is irrational' need not be verified by the student and thus attempts to prove this statement, whether they be by contradiction or not, will be difficult for the student.

Questioning whether students always avoid proof by contradiction, Brown (2011) analysed 20 advanced mathematics students' proof preferences. Students completed a survey of three types of proof comparison tasks: (I) compare direct to indirect, (II) compare proof by construction to an existence proof, and (III) compare proof by contradiction and proof by contrapositive. Within task II, "students overwhelmingly selected an existence argument, with an implicit proof by contradiction, when compared with a constructive proof" (Brown, 2011, p. 4), which seems to contradict the earlier findings of Harel and Sowder (1998). Brown (2016a) follows up with a comparative selection task in which a theorem is stated and two proofs, one direct and one by contradiction, are presented to the student. The proofs were controlled for length, familiarity to the student, and complexity so that the choice between proofs was based solely on the proof method employed. The student selected a proof along 6 rationales: (1) simplicity of the proof chosen, (2) error in the alternative proof, (3) directness or straightforwardness of the proof chosen, (4) the proof chosen matched their own thinking, (5) familiarity with the proof chosen, and (6) that the proof chosen was a stronger argument. In this task, students overwhelming choose the direct proof, citing simplicity (21:7), directness (23:4), and familiarity (12:1) with this proof. The author cautions against concluding this implies that students lack a preference for proof by contradiction as "[...] it may be the case that students are more prone to comprehension difficulties with contradiction proofs rather than lack a preference for this form of proof." (Brown, 2016a, p. 7). The author concludes that "[...] while far from proving a definitive conclusion, this research raises multiple questions regarding students' preferences for or against proof by contradiction" (Brown, 2016a, p. 7). Note the author suggests that student understanding of proof by contradiction may affect their proof evaluation - further evidence that students should develop a robust understanding of proof methods before evaluating proofs.

This section provided a comprehensive literature review on proof by contradiction organized along four aspects of proof. The following section will discuss literature on the triad of Schema development and genetic decomposition

2.3 Literature on APOS Theory

APOS Theory has been used as a theoretical framework to analyze how students develop an understanding of a variety of concepts, including mathematical induction (Dubinsky, 1986, 1989), quantification (Dubinsky et al., 1988), the chain rule (Clark et al., 1997), graphing in calculus (B. Baker, Cooley, & Trigueros, 2000; Cooley, Trigueros, & Baker, 2007), spanning set and span (Kú, Oktaç, & Trigueros, 2011), linear transformation (Roa-Fuentes & Oktaç, 2012), and parametric function (Stalvey & Vidakovic, 2015), to list a few. In each, a hypothetical model was constructed to describe how students come to understand a certain mathematical topic. This model, known as a genetic decomposition, was described in Chapter 1 as an outline of the mental constructions a student should make in order to understand a concept (Arnon et al., 2014). A genetic decomposition may "include a description of prerequisite structures an individual needs to have constructed previously, and might explain differences in students' development that may account for variations in mathematical performance" (Arnon et al., 2014, p. 28). Given a genetic decomposition's use as a tool in analyzing differences in students' development of a concept as well as its role in providing a guide to teach a concept, this study will construct and test a genetic decomposition of proof by contradiction.

A genetic decomposition may not be sufficient to describe how a student develops understanding of a particular mathematical concept (Clark et al., 1997). Clark et al. (1997) conducted 41 individual interviews with students designed to search for issues related to understanding concepts of calculus. The concept of the chain rule arose from an initial analysis of the data, after which the authors constructed a preliminary[8] genetic decomposition of the chain rule. The data collected relevant to the chain rule was then analyzed based on this genetic decomposition. During this data analysis, the authors “realized that Actions, Processes, and Objects alone were insufficient to describe student understanding of the chain rule” (Clark et al., 1997, p. 353). In order to sufficiently describe student understanding of the chain rule, the authors considered the stages through which a concept develops, introduced in Chapter 1 as the triad of Schema development. While the triad was useful in describing student understanding of the chain rule, the authors lamented “we could not determine this [Trans level of development] with our data because the interview questions were developed before we knew to look for these Triad stages” (Clark et al., 1997, p. 359). Therefore, this study will construct a preliminary triad of Schema development for proof by contradiction before collecting any data.

In order to develop both a triad of Schema development and genetic decomposition for proof by contradiction, examples of a triad and/or genetic decomposition for similar concepts to proof by contradiction were sought out in the literature. Three such examples emerged: (1) a triad of Schema development and genetic decomposition of a calculus graphing Schema, (2) a genetic decomposition for quantification, and (3) a genetic decomposition for proof by induction. The rest of this section will be devoted to describing how each triad of Schema development or genetic decomposition relates to proof by contradiction, providing an overview of the methodology of each study, and detailing the triad of Schema development or genetic decomposition of the study.

2.3.1 Triad of Schema development for a calculus graphing Schema

The triad of Schema development can be used to describe Schema thematization, which is “a mental construction of a Schema so that it may be dissected, broken down, examined by its parts, reassembled, acted upon as an Object, and brought to bear in appropriate

[8]A genetic decomposition before empirical testing is referred to as preliminary (Arnon et al., 2014).

situations" (Cooley et al., 2007, p. 371). In this way, Schema thematization can be thought of as a highly desirable and robust conceptual understanding of a particular idea. The triad of Schema development is then a description of the levels of understanding a student passes through as they develop understanding of a particular concept. Schema thematization of proof by contradiction would allow students to understand the proof method and know when to apply the method. However, only Cooley et al. (2007) have studied Schema thematization of any mathematical concept. Therefore, the rest of this subsection will be devoted to describing the methodology of Cooley et al.'s (2007) study and presenting the resulting triad of Schema development.

B. Baker et al. (2000) aimed to describe how students understand and solve non-routine calculus graphing problems. Forty-one students were interviewed, during which they were given a list of conditions and asked to graph a function that satisfied these conditions. After completing the interviews, the authors constructed a preliminary genetic decomposition for a calculus graphing Schema in order to establish specific criteria to categorize student thinking. The calculus graphing Schema that emerged was unique in that it involved the coordination of two Schemas: the condition-property Schema and domain-interval Schema. The condition-property Schema, shortened to property Schema, involves "understanding each analytic condition as it relates to a graphical property of the function and coordinating these conditions" (B. Baker et al., 2000, p. 565). The domain-interval Schema, shortened to interval Schema, involves "understanding the interval notation, connecting contiguous intervals, and coordinating the overlap of the intervals" (B. Baker et al., 2000, p. 566). The data was then analyzed in terms of the triad of Schema development to provide a framework for categorizing student understanding. As the calculus graphing Schema involved the coordination of two Schemas, nine categories were developed[9]. The progression from intra-property to trans-property and from intra-interval to trans-interval have been separated and are presented below:

[9]Each stage of the property Schema, such as intra-property, was coordinated to each stage of the interval Schema, such as trans-interval, for a total of nine categories.

- Property Schema
  - Intra-property: Student interprets only one analytical condition at a time in terms of its graphical feature. There is awareness of other properties but student cannot coordinate them to produce a graph.
  - Inter-property: The student begins to coordinate two or more conditions simultaneously. This coordination is not applied throughout all connected intervals or across the entire domain.
  - Trans-property: Student can demonstrate coordination of all the analytic conditions to the graphical properties of the function on an interval.
- Interval Schema
  - Intra-interval: Work is on isolated intervals. The overlap of intervals or connection of continuous intervals causes confusion.
  - Inter-interval: There is coordination of two or more contiguous intervals simultaneously. This coordination, however, was not applied throughout all connected intervals or across the entire domain.
  - Trans-interval: Describes the coordination of the intervals across the whole domain. He or she is able to overlap intervals and connect contiguous intervals. (Cooley et al., 2007, p. 375)

Note how connected the mental constructions are at each stage. For the property Schema, the student begins by analyzing each condition separately at the intra-property level. Then, the student coordinates some of these conditions together at the inter-property level. Finally, the student can coordinate all of the properties together at the trans-property level. Similarly for the interval Schema, the student works on a single interval at a time at the intra-interval level. Then, the student can work on two or more intervals at a time at the inter-interval level. Finally, the student can work on all intervals at a time at the trans-interval level.

Cooley et al. (2007) built upon their previous study by focusing on Schema thematization of the calculus graphing Schema described previously. In order to examine whether students demonstrated a thematized Schema while solving questions related to the calculus graphing Schema, the authors interviewed 28 strong[10] mathematics students. These interviews involved participants completing a set of eight problems as well as explaining their thought processes and methods they used to assemble graphs.

After categorizing the student stages of Schema development, established by B. Baker et al. (2000), only one of the 6 students at the trans-property, trans-interval level of Schema development were determined to have thematized her Schema due to the student's ability to "clearly explain what would change and what would remain invariant when the conditions of the problem were changed in the final question" (Cooley et al., 2007, p. 379). The authors concluded it is was thus possible for students to thematize their Schema, though the authors suggested that more research is needed to address *how* students thematize their Schema. This study will attempt to address how students thematize their Schema by conducting multiple sessions to capture students' thinking over time.

2.3.2 Genetic decomposition of quantification

Quantified statements are regularly encountered by students in a transition-to-proof course in order to slowly increase the difficulty of statements that must be proved by the student (especially after the majority of proof methods have been taught). Proof by contradiction is the preferred method to prove statements with certain quantifications, such as nonexistence, uniqueness, and infinitely many. Therefore, quantification is a prerequisite concept for proof by contradiction. Examining the genetic decomposition for quantification, proposed by Dubinsky, Elterman, and Gong (1988), will provide details on how students can develop understanding of this prerequisite concept for proof by contradiction as well as serve as an example genetic decomposition for this study.

In order to examine how student understanding of quantified statements develops in

[10]Considered strong by their professors and had completed at least differential and integral calculus.

undergraduate students, Dubinsky et al. (1988) conducted an informal study of a Discrete Mathematics class in the spring of 1986. After quantification was covered in this Discrete Mathematics course, each student participated in an interview during which the student was asked to solve problems connected with quantification and to explain their reasoning while doing so. Observations from these interviews were used to illustrate the cognitive development of quantification proposed by the genetic decomposition of quantification, reproduced below:

1. The construction begins with simple declarations that are made more complicated in two ways:

    (a) Two or more declarations are coordinated by linking them with the standard logical connectors and

    (b) Variables are introduced and the learner interiorizes the Process of iterating this variable through its domain, checking the truth or falsity of the proposition valued function for each value of the variable;

2. The learner interiorizes a Process of iterating a variable through its domain to obtain a set of propositions and applying a quantifier to obtain a single proposition, resulting in a transition from the preliminary stage 1 to the stage of single-level quantification by coordinating the two extensions 1a and 1b;

3. The learner constructs and interiorizes a Process which is a nested iteration over two variables. This Process is again encapsulated to a single proposition so it is possible to proceed to higher-level quantifications, resulting in a transition to two-level quantifications by encapsulating Process 2 to obtain a single proposition;

4. The learner groups the two inner quantifications and applies the two-level Schema 3 to again obtain a proposition which depends on the outermost variable. This proposition valued function is then quantified as before to obtain a single quantification, resulting in a transition to three-level quantifications;

5. The entire procedure can now be repeated indefinitely and with an awareness of this possibility, the learner has constructed a Schema which can handle quantifications which are nested to any level. (Dubinsky et al., 1988, p. 20-21)

It should be noted that a student does not need to have progressed through the entire genetic decomposition above in order to prove a statement by contradiction. Indeed, a student should understand single-level quantification before attempting to prove a single-level quantified statement by contradiction, yet a student need not understand multi-level quantified statements at this same point. In this way, a student's quantification Schema may develop as a student's proof by contradiction Schema develops.

2.3.3 Genetic decomposition of proof by induction

Proof by induction is another proof method that students have difficulties utilizing and understanding (Dubinsky, 1986; Harel, 2002). While proof by induction is not an indirect proof, and thus does not share a structural similarity to proof by contradiction, it is the only proof method in the literature with a genetic decomposition. Therefore, a review of the genetic decomposition for proof by induction, constructed by Dubinsky and Lewin (1986) and expanded upon by Dubinsky (1986, 1989), will serve as the closest example in the literature to model a genetic decomposition of proof by contradiction as well as validate the usefulness of a genetic decomposition in understanding a proof method.

Dubinsky and Lewin (1986) constructed a genetic decomposition of induction to explain some students' difficulties with the proof method after interviewing 22 university students on their understanding of proof by induction. An outlined version of this genetic decomposition is reproduced below[11]:

1. Generalize the function Schema to include a proposition-valued function of the positive integers, $N \to P(N)$;

[11]The mental mechanisms by which Actions, Processes, Objects, and Schemas are constructed (i.e. interiorization, coordination, reversal, encapsulation, and generalization) were introduced in Chapter 1.

2. Logical necessity is encapsulated into the implication $A \to B$ which becomes, cognitively, an Object which can serve as the value of a function;

3. Coordinate step 1 and step 2 to obtain the Schema of implication-valued function of positive integers, $N \to (P(N) \to P(N+1))$;

4. Interiorize logical necessity into a Process, *Modus Ponens*, whereby the student is conscious of receiving the inputs $A, A \to B$, and then concluding B;

5. Implication-valued function is coordinated with *Modus Ponens* to obtain a Process that begins with $P(1)$ and then successively alternates evaluation of the implication-valued function $N \to (P(N) \to P(N+1))$ with modus ponens ad infinitum;

6. Encapsulate the explain induction Schema from a cognitive Process into an Object that can serve as an aliment to the Schema method of proof so that the student is ready to apply induction; and

7. Generalize Actions on a multitude of problems to tricks that are coordinated with the Schema method of proof so that students can apply induction. (Dubinsky, 1986, p. 307-308)

Dubinsky (1986, 1989) conducted a pair of studies designed to describe and evaluate the genetic decomposition above. The author conducted a teaching experiment through which he taught a Finite Mathematics course utilizing SETL, which is a "very high level procedural programming language with standard constructs of assignments, procedures, for-loops, while loops, and so on" (Dubinsky, 1986, p. 308). SETL was used to aid students in understanding the mathematical syntax and assisting students in applying Actions on specific mathematical concepts, such as mathematical induction. Throughout the course, activities designed using the genetic decomposition for induction were incorporated into the regular course material. Student understanding of induction was assessed in two ways: (1) a take-home assignment consisting of two induction proofs and (2) 20-minute individual interviews on induction.

The author concluded that the students in this course made progress toward understanding mathematical induction for the following reasons: "They were able to describe the process and discuss it meaningfully. Almost all of them were able to deal with problems in which they were asked to construct an induction proof. They set up the arguments properly and most of them produced correct proofs" (Dubinsky, 1986, p. 315-316). In addition, all of the Schemas in the genetic decomposition for induction appeared in the interviews and thus the genetic decomposition proposed in Dubinsky and Lewin (1986) was effective at describing how students develop an understanding of mathematical induction. As a genetic decomposition was able to aid students' understanding of proof by induction, it is reasonable to believe a genetic decomposition will aid students' understanding of proof by contradiction.

2.4 Chapter summary

This chapter began with a presentation of the recent studies of proof in general that provide insight into student understanding of proof along the four aspects of proof introduced in Chapter 1: construction, comprehension, validation, and evaluation. Two difficulties with proof construction, a lack of proof-writing strategies and difficulties with quantified statements, were examined in detail for their relation to student understanding of proof by contradiction. A proof comprehension assessment model by Mejía-Ramos et al. (2012) that describes two aspects of comprehension, local and holistic, was then detailed and described in terms of APOS Theory. A presentation of common ways in which undergraduate students validate proofs (e.g., authority, appeals to physical objects, truth value of the statement) as well as they purport to validate proofs (e.g., step-by-step, logical status of statements, and generating examples) followed as these closely resemble the local and holistic aspects of proof comprehension. This section ended with a short summary of Hanna's (2014) response to Gowers' (2007) attempt to quantify aesthetic properties of a proof as these aesthetic judgments may account for students' preferences for or against the contradiction proof method.

The second section of this chapter began with a discussion of how comprehension of the contradiction proof method may explain difficulties students encounter when constructing proofs by contradiction. Two models related to student comprehension of proofs by contradiction, proposed by Lin et al. (2003) and Antonini and Mariotti (2008), were then evaluated and determined to not adequately describe how students develop an understanding of proof by contradiction in a transition-to-proof course due to various limitations. Limitations of Lin et al.'s (2003) model include: (1) assuming the conceptual knowledge of proof by contradiction is the law of contrapositive, (2) the student composition of 140 high school students and 62 college students, and (3) the model does not address how to develop student understanding in order to avoid the various difficulties students have with constructing proofs by contradiction. Limitations of Antonini and Mariotti's (2008) model include: (1) it is not clear how students can transition from the principal to the secondary statement when working with a proof by contradiction and (2) the model does not describe how student understanding of proof by contradiction develops over time. Brown's (2016b) study on student difficulties validating the relationship between the two tautological statements, $\sim P \rightarrow (Q \wedge \sim Q)$ and P, of a proof by contradiction was then discussed as a potential obstacle in student understanding of proof by contradiction. This section ended with a discussion of whether students always prefer construction proofs to contradiction proofs, with Brown (2016a) suggesting that comprehension of the contradiction proof method may play a role in this preference.

The third section of this chapter began by describing how a genetic decomposition may be used to analyze and teach a concept, followed by how the triad of Schema development may describe student understanding in ways that a genetic decomposition cannot (i.e., the stages through which a concept develops). The section ended with a discussion of three research projects that developed either a triad of Schema development or genetic decomposition for concepts related to proof by contradiction: (1) Cooley et al.'s (2007) proposed triad of Schema development and genetic decomposition for a calculus graphing Schema (2) Dubinsky et al.'s (1989) proposed genetic decomposition for quantification, and (3) Dubinsky and Lewin's (1986) proposed genetic decomposition for proof by induction.

CHAPTER 3

METHODOLOGY

The previous chapter provided details of the most relevant results in the literature on student understanding of proof and on proof by contradiction in particular. This chapter will provide details on how the theoretical perspective guiding this study, APOS Theory, can be applied to proof by contradiction as well as describe the methods employed leading up to and by this study. First, Section 3.1 will discuss the various roles of APOS Theory in this study. Then, Section 3.2 will provide an overview of the pilot studies that contributed to this study. Next, Section 3.3 will detail the research design for this study. Finally, Section 3.4 will describe how the data was collected while Section 3.5 will detail how the data was analyzed.

3.1 Role of APOS Theory

Mental processes, such as understanding, are by their very nature impossible to observe directly. Instead, we rely on the observable actions individuals make, such as the words they use or movements they make when responding to a particular question. Cognitive frameworks inform researchers of how individuals should be observed and how to develop models of a mental process from these observations (Weisberg & Reeves, 2013). APOS Theory, a cognitive framework described in Chapter 1, will inform this study on how to observe and model student understanding of proof by contradiction. This section will provide details on how APOS Theory will be used to investigate and analyze student understanding of proof by contradiction. In particular, this section will (1) exemplify various levels of understanding of proof by contradiction in terms of APOS constructs (Actions, Process, Object, and Schema), (2) present a preliminary triad of Schema development for proof by contradiction, and (3) present a preliminary genetic decomposition for proof by contradiction.

3.1.1 Proof by contradiction through the lens of APOS Theory

As mentioned in Chapter 1, APOS Theory posits that individuals develop an understanding of mathematical concepts through the construction of mental structures: Actions, Processes, Objects, and Schemas. Building these mental structures is considered to be levels in the learning of a mathematical concept (Arnon et al., 2014). That is, we can use students' exhibited construction of these mental structures to describe a hierarchy of understanding for a particular concept. What follows is a description of this hierarchy for proof by contradiction in a transition-to-proof course.

A student with an *Action* conception of proof by contradiction would have to work with a concrete example and perform memorized or specifically given instructions for steps one needs to perform when using this proof method. At this level, a student would not be able to prove a statement that required a deviation from these steps. For example, a student may have memorized the following steps that need to be done to prove an implication ($P \to Q$) statement by contradiction:

1. Convert the statement into symbolic notation;
2. Identify the assumption (P), the conclusion (Q), and write the statement in the 'If P, then Q' form;
3. Write the negation of the conclusion;
4. Assume the negation of the conclusion and the assumption are both true;
5. Symbolically manipulate the conclusion to get a statement that contradicts the assumption; and
6. State that a contradiction has been made and thus the proof is done.

In this example, the student would not be able to prove a statement that required a deviation from these steps, such as if the statement needs more than symbolic manipulation to reach a contradiction or if the initial statement was not an implication. The student would also need to deal with an explicit proof to apply these memorized steps to and could not talk about these steps in general.

A student with a *Process* conception of proof by contradiction would have written many different particular proofs, repeating the above steps. When the student can reflect on these Actions and no longer needs to work with concrete examples or external cues in order to explain how a proof by contradiction works, the student is said to have interiorized the Action into a Process. At this level, a student is able to describe a general procedure to prove a statement by contradiction without referring to a particular example and without a need to perform the procedure. For example, consider how a student may interiorize the example step-by-step procedure presented above. After reflecting on the specific procedure of an implication statement over some particular proofs, the student would then be able to state that the following general steps could be used to prove any[1] statement by contradiction:

1. Assume the premise and the negation of the conclusion are true;
2. Show that step 1 leads to a mathematical absurdity, i.e. a contradiction; and
3. Conclude the statement to be proved is true.

Note the generality of steps 1 and 2. In step 1, the student no longer relies on the symbolic representations of statements to begin the proof. In step 2, the contradiction can be of any form, rather than being restrained to arising from symbolic manipulation (as in the example Action conception). The generality of these two steps encompasses steps 1 through 5 in the Action conception example. Moreover, the student would be able to describe *why* they follow the general steps above in order to complete a proof by contradiction. For example, the student could describe that the assumption leading to a contradiction means the assumption is not true and so the statement to be proved is "not not" true, and thus is true. The ability to describe *why* the student follows the general procedure is critical to a Process conception.

[1]It is important to note that the student may believe this Process encompasses all proofs by contradiction until they are presented with a statement that their Process does not work for (e.g. to prove a uniqueness statement). At this point, a student may rephrase the first step and thus refine their Process conception of proof by contradiction.

A student with an *Object* conception of proof by contradiction would have encapsulated the Process described above into a static Object by performing some Action on the contradiction proof method itself. This Action could be proving the validity of the proof method itself by utilizing the *law of excluded middle*. Alternatively, this Action could be comparing the contradiction proof method to other proof methods, such as a proof by contraposition or a proof by cases. Either way, the proof method itself is transformed from a dynamic procedure into a static structure that can be acted on. When necessary, the student can de-encapsulate their Object conception back into a Process it came from in order to prove a particular statement by contradiction.

While these levels are hierarchical, they are not necessarily linear. That is, an individual may construct and reconstruct many mental structures before being considered to have developed a conception at that level. For example, a student may construct multiple Processes for specific types of statements (e.g., implications, nonexistence, uniqueness) that they must then restructure and coordinate together in order to develop a single Process, at which point the student would be considered to have developed a Process conception of proof by contradiction. Indeed, Piaget and Garcia (1989) argued "that cognitive development is never linear, but generally requires the reconstruction of what had been acquired during the preceding stages in order to advance to a higher level" (Piaget & Garcia, 1989, p. 109). The authors stated these reconstructions occur as three distinct stages: intra-, inter-, and trans-. In other words, these stages of development, known as the triad of Schema development, allows us to describe how a student's conception may develop between the levels in learning a mathematical concept (Arnon et al., 2014). The next subsection will introduce a preliminary triad of Schema development for proof by contradiction.

3.1.2 Preliminary triad of proof by contradiction Schema development

Cooley et al. (2007) described how a triad of Schema development can be used in general to analyze students' understanding of a mathematical concept:

> [. . .] using the triad as a tool with which to analyze the knowledge composition

> of students, researchers are able to take into account the richness of problem situations by focusing attention on relationships among ideas. Because a student may proceed through levels in a unique and nonlinear fashion, the triad, as a tool, allows researchers to use a detailed approach in a flexible manner. It utilizes complexities of the problem, relationships among ideas, newly formed structures, and coherence all important aspects of the development of logico-mathematical structures. (Cooley et al., 2007, p. 560)

In other words, the triad of Schema development focuses on the relationship between mental structures while the levels of conception (Action, Process, and Object) focus on the construction of these mental structures themselves. Taken together, we can describe how a student develops an understanding of a mathematical concept. A preliminary triad of Schema development for proof by contradiction is thus a useful analysis tool as well as a way to provide a more detailed description of how a student may develop their proof by contradiction Schema. What follows is a description of a preliminary genetic decomposition for proof by contradiction.

As noted when describing the different conceptions of proof by contradiction, the primary mental structures are procedures (either external or internal). The triad of Schema development thus focuses on the relationship between these procedures in the individual's mind. At the *intra-contradiction* stage, there are no relations between these procedures and thus each procedure may be called by a different name. For example, a student may determine what the structure of the statement to be proved is so that they may "choose the correct" procedure. The student would not be able to compare these procedures in a meaningful way and so each would be given a different name, such as nonexistence proof, existential proof, and infinity proof. At the *inter-contradiction* stage, the student begins to develop relationships between these procedures by partitioning the procedures into a series of roles or purposes. By comparing the series of roles in two or more procedures, procedures can be grouped together and called by the same name. For example, any proofs that contain a construction step to derive a contradiction may be grouped together regardless of the logical structure of the initial statement. In addition, new procedures may now be included as proofs by contradiction, such as a proof by cases that proceeds by contradiction. As another example, students could consider a proof by contradiction on a uniqueness statement as a proof by cases where one arrives at a contradiction for each case. This would allow students to group together any procedure where at least one case arrives at a contradiction and consider them all as a proof by contradiction with cases. Finally, at the *trans-contradiction* stage, the student would have constructed a single procedure for proof by contradiction. This single procedure provides a relation through which all procedures are grouped together and called the same name. The single procedure also provides coherence to the Schema in that can be used to determine whether a presented proof is a proof by contradiction or not.

Utilizing the levels of conception and the triad of Schema development for proof by contradiction, we can provide a detailed description of the mental structures and relations between these mental structures as they develop. We can then use this understanding to construct an outline of the hypothetical constructions a student should make in order to understand proof by contradiction, known in APOS Theory as a preliminary genetic decomposition. The next section will present this outline.

3.1.3 Preliminary genetic decomposition of proof by contradiction

As mentioned in Chapter 1, a genetic decomposition is constructed based on the following: the conception of the instructor or researcher, a literature review of the concept, and an analysis of the historical development of the concept. In addition to outlining the mental constructions a student should make in order to understand a concept, a genetic decomposition may include a description of prerequisite knowledge a student should possess in order to begin developing a concept[2]. A description of the prerequisite knowledge for proof by contradiction follows.

First, a student should have a general understanding of the propositional and predicate

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[2]Begin developing a concept at the appropriate level of the individual. In this case, for an undergraduate in a transition-to-proof course

logic that underscores all proofs. In particular, a student should be able to perform and understand negations of the following types of statements: statements without quantification, implications, and single-level quantified statements. This is necessary as every proof by contradiction requires a negation of some statement. Secondly, a student should be able to move between semantic, symbolic, and algebraic representations of mathematical statements. For example, a student should be able to represent the statement 'If a is even, then $a+1$ is odd' as the propositional implication '$P \to Q$', as the predicate statement $(\forall a \in \mathbb{Z})(2 \mid a \to 2 \nmid (a+1))$, and as the algebraic representation 'If $a = 2n$ for $n \in \mathbb{Z}$, then $a+1 = 2m+1$ for $m \in \mathbb{Z}$. Moving between representations of mathematical statements (either consciously or subconsciously) is necessary to make sense of mathematical statements and thus is necessary before developing a notion of proof by contradiction. Finally, a student should be familiar with the direct proof method in a mathematical context as this forms the basis of understanding for a proof by contradiction.

With this prerequisite knowledge in mind, a preliminary genetic decomposition for proof by contradiction is presented below:

1. Action conception of propositional or predicate logic statements consists of following a given or memorized specific step-by-step instructions to construct proofs by contradiction for the following types of statements: (I) implication, (II) single-level quantification, and (III) property claim.

2. Interiorization of each Action in Step 1 individually as general steps to writing a proof by contradiction for statements of the form (I), (II), and (III).

3. Coordination of the Processes from Step 2 as general steps to writing a proof by contradiction.

4. Encapsulate the Process in Step 3 as an Object, for example, by utilizing the law of excluded middle to show proof by contradiction is a valid proof method. Alternatively, encapsulate the Process in Step 3 as an Object by comparing the contradiction proof method to other proof methods.

5. De-encapsulate the Object in Step 4 into a Process similar to Step 3 that then coordinates with a Process conception of quantification to prove multi-level quantified statements.

Note that the first constructions a student should make are with the underlying logic of proof by contradiction. This allows the student to construct a specific step-by-step procedure for proof by contradiction in symbolic form for each type of statement encountered [Step 1]. For example, the student could have memorized procedures for proof by contradiction based on the structure of the statement, such as the procedures included in Table 3.1.

Table 3.1 Possible memorized procedures for implication statements (left) and uniqueness statements (right).

Statement: $P \to Q$	**Statement:** $(\nexists x)(P(x))$
1. Assume $\sim (P \to Q)$	1. Assume $\sim (\nexists x)(P(x))$
2. $P \wedge \sim Q$	2. $(\exists x)(P(x))$
3. $\sim Q$	3. $P(n)$
4. $(\sim Q \wedge P) \to Q$	4. Using $P(n)$, get to a contradiction.
5. Q	
6. $Q \wedge \sim Q$	5. $\sim (\sim (\nexists x)(P(x)))$
7. $\sim (\sim (P \to Q))$	6. $(\nexists x)(P(x))$
8. $P \to Q$	

At this level, a student relies on translating mathematical statements into their symbolic logic form in order to use the step-by-step procedure for each specific type of statement. In the example above, a student with an Action conception would not be able to describe the logical reasoning behind the procedure but would have memorized the steps and applied them in close to identical situations.

As a student reflects on a particular step-by-step procedure, the student interiorizes the memorized procedure into general steps to write a proof. At this stage, a student no longer needs to translate statements into their underlying symbolic logic in order to prove a mathematical statement or could think of the statements in their head without the need to write them step-by-step.

As a student reflects on these different general steps to prove different types of statements, the student can coordinate these general steps into a single series of general steps, which can be used to prove a variety of statements by contradiction. The following is an example of general steps for writing a proof by contradiction: (1) negate the statement, (2) find a contradiction, and (3) conclude the proof. At this stage, a student has a unifying structure to how a proof by contradiction is written, which allows the student to determine whether a presented proof is by contradiction or not.

A student may encapsulate the Process in Step 3 of the genetic decomposition as a proof method by utilizing the *law of excluded middle* to show proof by contradiction is a valid proof method [Step 4]. By encapsulating either Process, a student can think of proof by contradiction as a totality, or entity that he or she can compare to other proof techniques, rather than a dynamic procedure. In this case, we say that a student has constructed an Object level of understanding of proof by contradiction. Furthermore, such a constructed Object can subsequently be de-encapsulated into a Process that then can be coordinated with a Process conception of quantification to prove multi-level quantified statements.

This section described how APOS Theory would be used to observe, analyze, and model student understanding of proof by contradiction. The following section will describe three pilot studies that were implemented to test the instruments based on the previously presented cognitive models.

3.2 Pilot studies

Three pilot studies were performed to develop and test some instruments for the primary study. While data for these pilot studies were collected from students taking the same transition-to-proof course, the semester in which they were performed, methods employed, participants recruited, and, most importantly, the goals of each pilot study were all different. First, I will provide a brief outline of each pilot study, after which I will describe how these pilots influenced the primary study.

In the first pilot study, two transition-to-proof students (one undergraduate and one graduate student) volunteered to be interviewed during Spring 2015. The undergraduate, James, was a double major, in Computer Science and Mathematics, while the graduate, Frank, was an Economics major. Despite the difference in degree program, both James and Frank had completed similar mathematics courses and could be considered to have similar mathematical backgrounds. Data for this pilot study consisted of students' written attempts to prove three number theory statements[3] as well as individual interviews on their thought process as they constructed the proofs. The goals of this pilot were to (1) determine whether students in a transition-to-proof course could successfully construct and comprehend proofs by contradiction by the end of the course, (2) examine their understanding of proof by contradiction, and (3) determine whether comprehension of proof by contradiction could predict the types of proof strategies a student would employ. A report by Chamberlain Jr. and Vidakovic (2016) presents a more detailed description of this pilot study as well as some preliminary result.

In the second pilot study, one transition-to-proof student volunteered to be interviewed during Spring 2016. Data for this pilot study consisted of a series of proof comprehension questions on a presented proof of the statement "There are infinitely many primes," some of which were reproduced from Mejía-Ramos and Weber (2016) while others were developed for the pilot. The goals of this pilot study were to (1) determine whether transition-to-proof students could answer proof comprehension questions without explicit instruction focused on proof comprehension, (2) determine the level of students' understanding of a particular proof that appeared on both a test and the final exam of the course, and (3) test proof comprehension questions on a particular proof. Results from this pilot are unpublished as the participating student did not provide a significant amount of data.

In the third pilot study, one transition-to-proof student, Chandler, volunteered to participate during Summer 2016. Data for this pilot study consisted of three 1-hour teaching sessions that each had three phases: (i) a problem statement with the intention for Chandler to think about a proof by contradiction and use his previous knowledge (as well as provided

[3]Each statement could be proved by contradiction, though contradiction was not necessary.

information) to make certain observations and/or conjectures about proof by contradiction; (ii) discussion to elaborate on Chandler's responses in part 1 and provide an opportunity for the teacher/researcher to guide Chandler in properly formulating (with proper notation and terminology) proof by contradiction; and (iii) a series of comprehension questions that reinforced the idea of a proof by contradiction. The goal of this pilot study was to test and refine the teaching sessions that would be used during Fall 2016. A report by Chamberlain Jr. and Vidakovic (2017) presents a more detailed description of this pilot study as well as some preliminary results.

All results of all three pilot studies were used in developing and refining the instruments for this study. The first pilot showed that asking students to construct proofs was not sufficient to accurately and completely assess their understanding of proof by contradiction. To remedy this, a series of proof comprehension questions were developed for five different statements that could be proved by contradiction. The second pilot provided a time estimate for completing the series of comprehension questions, as well as provided an opportunity to test the clarity and formulation of the questions. In addition, the number of comprehension questions was reduced from 10 to 8 to ensure the students have enough time to consider and discuss all of the comprehension questions within 45 minutes. The third pilot study provided an opportunity to test and refine the teaching interventions of the instrument to be used during Fall 2016. The following section will detail the culmination of these three pilot studies: the research design of this study.

3.3 Research design

As mentioned in the previous section, mental processes such as understanding are by their very nature impossible to observe directly. Indeed, researchers must rely on a variety of observable actions (e.g., words, gestures, pauses, and tone) in order to determine a student's understanding. These types of observable actions take place during a problem solving process rather as opposed to being observable in the final product. My focus on the *process* of understanding over the *product* of understanding necessitates the use of qualitative methods, or methods that gather descriptive data that focuses on how individuals make meaning during the process of a task (Bogdan & Biklen, 2007). First, I will describe the setting for this study. Then, I will detail the specific type of qualitative methods that were used in this setting.

3.3.1 Setting

This study was conducted with students enrolled in *Bridge to Higher Mathematics* at Georgia State University during Fall 2016. *Bridge to Higher Mathematics* has been chosen as it is the first time students are formally introduced to proof by contradiction in a mathematical context. Students taking this course come from a variety of majors, including but not limited to: mathematics, physics, chemistry, computer science, economics, and secondary education. What follows is a description of *Bridge to Higher Mathematics* at Georgia State University, the goals of the course, and a brief description of how proof by contradiction is usually taught within the course.

Prerequisites for *Bridge to Higher Mathematics* include a grade of C or higher in *Calculus of One Variable II* and *Discrete Mathematics*. *Calculus of One Variable II* covers: applications and techniques of integration, transcendental and trigonometric functions, polar coordinates, infinite sequences and series, indeterminate forms, and improper integrals. *Discrete Mathematics* covers: number bases, logic, proofs, sets, and Boolean algebra. It is important to note that the proofs taught in *Discrete Mathematics* are Propositional Logic proofs, i.e. logical proofs without any mathematical content, which are then reinforced in the first weeks of *Bridge to Higher Mathematics*. Therefore, *Bridge to Higher Mathematics* is the first course in which students are formally introduced to mathematical proofs and, in particular, proof by contradiction.

The primary focus of *Bridge to Higher Mathematics* is to develop students' understanding of proof methods and certain mathematical concepts. The learning outcomes for this course can be organized into four general topics:

1. Basic Logic (e.g., truth tables, negation, quantification),
2. Proof Methods (e.g., direct, contradiction, induction),

3. Introductory Set Theory (e.g., union, intersection, power set, cardinality), and
4. Introductory Analysis (e.g., least upper bound and greatest lower bound, open/closed sets, limit points). (*MATH 3000 - Learning Outcomes*, 2014)

Mathematical concepts in Set Theory and Analysis are used as context to practice and refine the basic logic and proof methods introduced earlier in the semester.

The following is a description of how proof by contradiction is generally taught within *Bridge to Higher Mathematics*[4]. Proof methods, and specifically proof by contradiction, are covered in the first three weeks of the course. Proof by contradiction is typically covered in two to three 50-minute class periods simultaneously with proof by contraposition. First, the propositional logic of a proof by contraposition, i.e. $(P \to Q) \leftrightarrow (\sim Q \to \sim P)$, is presented and explained through an example proof. Next, the propositional logic of a proof by contradiction, i.e. $(\sim P \to (Q \wedge \sim Q)) \to P$ is presented and explained through an example proof. The differences between the two proof methods, and when it is best to use each method, are then described. Students are then expected to work through exercises in the textbook, Smith et al. (2015), on both proof methods. After this instructional treatment, proof by contradiction and other proof methods are reinforced throughout the remainder of the course through their use in proving mathematical statements in the context of Set Theory and Introductory Analysis.

3.3.2 Methods

This study will utilize a teaching experiment as its primary methodology. Steffe and Thompson (2000) defined a *teaching experiment* as a sequence of teaching episodes that are recorded and retrospectively analyzed for how students develop understanding of a particular concept over time. Consider the following quote by Steffe and Thompson (2000) on the contrast between a teaching experiment and a clinical interview:

[4]Based on observations of multiple instructors of the course and discussions on how the course is taught with professors in the department who have previously taught the course.

> Whereas the clinical interview is aimed at understanding students' current knowledge, the teaching experiment is directed toward understanding the progress students make over extended periods. It is a dynamic way of operating, serving a functional role in the lives of researchers as they strive to organize their activity to achieve their purposes and goals. In this, it is a living methodology designed initially for the exploration and explanation of students mathematical activity. (Steffe & Thompson, 2000, p. 273)

Note the clinical interview is focused on a static "snapshot" of student understanding at a particular point while a teaching experiment is focused on the dynamic evolution of student understanding over time. A teaching experiment can simultaneously investigate students' initial conceptions and how these conceptions develop over time - the primary focus of this study. Therefore, a teaching experiment is the most suited methodology for this study.

To develop a teaching experiment, one must first decide on an instructional approach. One such approach, aligned with APOS Theory, is the ACE teaching cycle; an instructional approach that consists of three phases: Activities, Classroom discussion, and Exercises[5]. In the *Activities* phase, students work in groups to complete tasks designed to promote reflective abstraction about new mathematical concepts introduced in the phase. These tasks should assist students in making the mental constructions suggested by the genetic decomposition for a particular mathematical concept that is a focus of learning. In the *Classroom discussion* phase, the teacher leads a discussion about the mathematical concepts the Activities focused on. For example, during this phase the teacher may formally state the theorem that was central in the Action phase and, together with students, write its complete proof. In the *Exercises* phase, students work on standard problems designed to reinforce the Classroom discussion and support the continued development of the mental constructions about mathematical concept(s) as suggested by the genetic decomposition. These problems also provide students with the opportunity to apply what they have learned

[5]The Activities, Classroom discussion notes, and Exercises used for this study are included in Appendix A.

in the Activities and Classroom discussion phases to related mathematical concepts (Arnon et al., 2014). By utilizing the ACE teaching cycle in this teaching experiment, it is a hope that designed activities and tasks will induce a development of students' understanding of proof by contradiction as indicated in the genetic decomposition presented in the previous section.

Unlike a typical instructional sequence of ACE teaching cycle that, in a regular classroom, usually lasts for a week, this teaching experiment consisted of 5 shorter, consecutive teaching sessions over a period of 5 weeks that each mimicked the ACE teaching cycle. That is, each session consisted of: students working on the Activity worksheet focusing on a particular component of the genetic decomposition for proof by contradiction **(A)**; a discussion about the concepts from the worksheet **(C)**; and a typical series of proof comprehension questions related to the content of the worksheet **(E)**. A more detailed description for each session, including the role of the teacher/researcher[6] during each phase, is described below.

- **Activity** - Sessions began with a presented statement and proof. Students would work in groups to talk out how this statement and proof can be converted to propositional or predicate logic. If the episode was conducted with only one student, the teacher/researcher acted as another student with incomplete knowledge. Otherwise, the teacher/researcher acted as a knowledgeable agent who disseminated each group's understanding throughout the classroom.

- **Classroom Discussion** - After the Activity, the teacher/researcher would ask students to summarize the structure of the proof. At this point, students would be guided to develop a formal structure of the proof by the teacher/researcher, who would make sure definitions and mathematical properties (theorems) were accurately stated. Then, the teacher/researcher would discuss how this structure compared to the one written during the Activity. During this phase, the teacher/researcher acted as a knowledgeable agent who guided the students to make comparisons and appropriate conclusions.

[6]The researcher acted as the teacher during the teaching episodes.

- **Exercise** - After the Classroom Discussion phase, the students answered comprehension questions on their own or in groups (their preference), after which the teacher/researcher prompted students for their answers and thinking behind the answers provided. Note that as textbooks do not normally provide comprehension questions on proofs, the proof comprehension assessment model by Mejía-Ramos et al. (2012) was used to develop standard proof comprehension questions for the Exercises. During this phase, the teacher/researcher acted as a knowledgeable agent to gain insight into the students thinking.

Five shorter, consecutive ACE teaching cycles (as opposed to a single cycle) were necessary for a number of reasons. First, each course instructor alloted the teacher/researcher a different amount of time to conduct part of the teaching experiment during class meeting time. The first episode was thus developed to provide an adequate introduction to proof by contradiction without the need for further episodes[7]. Secondly, students could not dedicate more than an hour at a time to meet outside of class. Each episode was thus developed to be conducted entirely within an hour. Thirdly, two distinct concepts were being developed during the teaching experiment: proof comprehension and proof by contradiction. None of the instructors explicitly taught proof comprehension in class and thus, out of necessity, more than one concept needed to be covered during the teaching experiment. Finally, reading a proof for comprehension is a time-intensive task. The pilot studies suggested students could read and answer comprehension questions on a single proof in about 30 minutes. As there are a variety of statements that can be proved by contradiction, each statement type required a separate episode to be introduced and subsequently compared. These reasons, together, necessitated a sequence of five ACE teaching cycles (the details of each cycle can be found in Appendix A.1) to assist students in developing a robust understanding of proof by contradiction. Each of the five teaching episodes were developed based on the preliminary

[7]The first episode provided an adequate introduction to *constructing* proofs by contradiction for the most commonly encountered statement types in the course. This satisfied the expectation that students know how to reproduce and construct familiar proofs by contradiction of implication statements.

genetic decomposition presented in Section 3.1. What follows is a description of the goals for each episode.

Episode 1 was designed to introduce a set of step-by-step instructions for students to utilize in order to construct proofs by contradiction for implication statements, which aligns with step 1 for developing an Action conception for implication statements (type I statements) in the preliminary genetic decomposition. In addition, focusing students' attention on the roles of specific lines in the proof and having students identify the key steps of the proof provided students with the opportunity to interiorize the step-by-step instructions for implication into a Process, which aligns with step 2 in the preliminary genetic decomposition.

Episode 2 was designed to introduce a set of step-by-step instructions for students to use to construct proofs by contradiction for nonexistence statements, which aligns with step 1 for developing an Action conception for single-level quantified statements (type II statements) in the preliminary genetic decomposition. In addition, focusing students' attention on the roles of specific lines in the proof and having students identify the key steps of the proof provided students with the opportunity to interiorize the step-by-step instructions for nonexistence statements into a Process, which aligns with step 2 in the preliminary genetic decomposition. Encouraging students to compare the Processes for implication and nonexistence statements would allow students to coordinate these Processes into a single Process for constructing a proof by contradiction, which aligns with step 3 in the preliminary genetic decomposition.

Episode 3 was designed to introduce a set of step-by-step instructions for students to use to construct proofs by contradiction for uniqueness statements, which aligns with step 1 for developing an Action conception for single-level quantified statements (type II statements) in the preliminary genetic decomposition. In addition, focusing students' attention on the roles of specific lines in the proof and having students identify the key steps of the proof provided students with the opportunity to interiorize the step-by-step instructions for uniqueness statements into a Process, which aligns with step 2 in the preliminary genetic decomposition. Encouraging students to compare the Processes for implication, nonexistence, and uniqueness statements would allow students to coordinate these Processes into a single Process for constructing a proof by contradiction, which aligns with step 3 in the preliminary genetic decomposition[8].

Episode 4 was designed to introduce a set of step-by-step instructions for students to use to construct proofs by contradiction for infinity statements, which aligns with step 1 for developing an Action conception for single-level quantified statements (type II statements) in the preliminary genetic decomposition. In addition, focusing students' attention on the roles of specific lines in the proof and having students identify the key steps of the proof provided students with the opportunity to interiorize the step-by-step instructions for infinity statements into a Process, which aligns with step 2 in the preliminary genetic decomposition. Encouraging students to compare the Processes for implication, nonexistence, uniqueness, and infinity statements would allow students to coordinate these Processes into a single Process for proof by contradiction, which aligns with step 3 in the preliminary genetic decomposition[9].

Episode 5 was designed to introduce a set of step-by-step instructions for students to use to construct proofs by contradiction for property claim statements, which aligns with step 1 for developing an Action conception for property claim statements (type III statements) in the preliminary genetic decomposition. Similar to the previous episodes, students were encouraged to interiorize this Action into a Process for the specific type of statement and coordinate this Process to construct a Process for proof by contradiction in general. However, it is also possible students would recognize the presented proof as proving the contradiction proof method and thus the episode encourages students to encapsulate their Process for proof by contradiction into an Object, which aligns with step 4 in the preliminary genetic decomposition.

[8] Alternatively, the student could coordinate the single Process for both implication and nonexistence statements with the Process for uniqueness statements into a single Process.

[9] Alternatively, the student could coordinate the single Process for implication, nonexistence, and uniqueness statements with the Process for infinity statements into a single Process.

Four primary types of tasks were assigned to achieve the goals described above. For each type of task, I will describe what it required students to do, its goals, and the phases in which it generally appeared.

Outlining tasks asked students to logically outline a presented proof by contradiction. These tasks were included to prompt students to identify the logical argument within a presented proof by contradiction. According to the proof comprehension assessment model by Mejía-Ramos et al. (2012), both the logical status of statements and the modular structure of a proof play a role in students' proof comprehension. That is, examining the logical relation between one or more lines in a presented proof aids students in understanding the particular proof. Authors have substantiated this claim, such as Hodds et al. (2014) who developed a booklet containing self-explanation training focused on the logical relationships within a mathematical proof that improved students' proof comprehension. For the purposes of this study, it was conjectured that improving students' proof comprehension of particular proofs by contradiction would aid students' comprehension of the proof method in general.

Defining tasks asked students to define and explain proof by contradiction. These tasks focused on verbalizing students' conceptions of proof by contradiction. The goals of these tasks were to make students aware of their proof by contradiction Schema and ensure students were provided a formal definition of proof by contradiction during each episode. These tasks occurred during the Activity and Classroom Discussion phase. That is, students were asked to complete these tasks during the Activity phase and were subsequently discussed during the Classroom Discussion phase.

Comparison tasks required students to compare two or more logical outlines of presented proofs. These tasks were used as a reflection tool for students to consider the necessary logical lines of a proof by contradiction and how these lines logically relate. The goal of these tasks was to encourage students to compare the purpose of lines in multiple outlines and develop a series of general steps to prove any proof by contradiction. These tasks occurred during the Classroom Discussion phase.

Comprehension tasks assessed students' understanding of a presented proof. These tasks focused on a presented proof or the student producing their own proof of a statement and included: (1) asking for the key steps of a presented proof, (2) stating the purpose of a statement related to their procedure, and (3) writing a complete proof by contradiction. The goal of these tasks was to reinforce students' conceptions of proof by contradiction after they were formalized. Therefore, these tasks occurred primarily during the Exercise phase.

3.4 Data collection

Subjects for this study were students enrolled in a *Bridge to Higher Mathematics* course with instructors who were not associated with this study. No limits were placed on the number of participants. Twenty-seven students (out of twenty-nine enrolled students) participated during Fall 2016: nine were enrolled in a section taught by Dr. Chi and eighteen were enrolled in a section taught by Dr. Khan. As the teaching experiment consisted of five episodes and students could drop out at any time, participation varied greatly between volunteers. This subsection will describe the volunteers and their amount of participation in the teaching experiment, organized by instructor.

Dr. Chi allowed the researcher to complete one teaching episode in-class. All students who were present[10] agreed to participate in teaching episode 1. Five of the nine in-class participants (Jean, Nick, Ren, Nora, and Yara) agreed to participate in additional, out-of-class teaching episodes. What follows is a description of each student and any reason they stopped participating in out-of-class episodes (if any).

Jean was a sophomore Mathematics major. After the first out-of-class teaching episode, he declined to participate in future teaching episodes as he was "trying to find the best method for me to memorize a proof" and felt that the set of steps developed in the second teaching episode would be enough to memorize any proof by contradiction he encountered. Nick was a junior Mathematics major. Halfway through the first out-of-class teaching episode, Nick mentioned that he had a meeting with an advisor to determine whether he met the course prerequisites for *Bridge to Higher Mathematics*. After a short discussion

[10]Two students were absent and thus did not participate

with the teacher/researcher, the teaching episode was ended early as Nick did not meet the prerequisites and would need to drop the course. Ren was a junior Computer Science major. While he initially expressed interest in completing all teaching episodes, he dropped the course after completing two out-of-class teaching episodes with Nora. Nora was a part-time sophomore Mathematics major. She completed two out-of-class teaching episodes with Ren and, once Ren dropped the course, did not return for additional teaching episodes. She gave no reason for withdrawing when contacted and thus it can be assumed she stopped participating because Ren stopped participating. Finally, Yara was a senior Mathematics major with a minor in Educational Psychology. Beyond the required prerequisite courses for *Bridge to Higher Mathematics*, she had already taken Mathematical Statistics, Methods of Regression and Analysis of Variance, Foundations of Numbers and Operations, and Applied Combinatorics. However, none of these courses required proof writing and thus *Bridge to Higher Mathematics* was her first experience with formal proofs. She completed all five teaching episodes.

Dr. Khan allowed the teacher/researcher time to complete two teaching episodes in-class. All students who were present[11] agreed to participate in the teaching episode. Two students, Beth and Wesley, completed out-of-class teaching episodes. What follows is a description of each student and any reason they stopped participating in out-of-class episodes (if any).

Beth was an undergraduate Pre-Middle Level Education major. While Beth expressed interest in completing all five teaching episodes, she did not attend the scheduled, final episode. As she was contacted multiple times and did not reply, it is unknown why she declined to participate in the final teaching episode. Wesley was a non-traditional sophomore Mathematics major returning to the classroom after many years in the work force. He completed all five teaching episodes.

In summary, a total of sixteen independent teaching episodes were conducted with the following groups: Dr. Chi's section (1), Jean (1), Nick (1), Ren and Nora (2), Yara (4), Dr.

[11]All students registered for Dr. Khan's class were present.

Khan's section (2), Beth (2), and Wesley (3). The next section will detail how the researcher analyzed the data collected during these sessions.

3.5 Data analysis

Multiple forms of data were collected during the study. The primary source consists of two types of data collected during the teaching episodes: audio/video recordings and written work. Video recordings were primarily used to transcribe each teaching episode and include any gestures or other inaudible cues students may have performed during the teaching episodes. In addition, video recordings were used to identify students that were working together in groups. These groups were then used to organize the transcript in cases when multiple students were talking at the same time. Audio recordings were then used to verify that transcriptions from the video recordings were accurate. Secondary sources of data include notes the teacher/researcher took directly after each teaching episode as well as assignments (homework, quizzes, and tests) developed and administered by the instructor as part of the course. What follows is a description of how the primary forms of data were analyzed.

In total, sixteen separate teaching episodes were conducted and recorded. Each video recording was first transcribed with special attention to any gestures or extended pauses the students made while speaking. I then used the audio recording to validate the transcription of the teaching episode. Once all sixteen transcriptions were completed and verified for accuracy, they were uploaded to MAXQDA - a qualitative analysis software[12]. Analysis of the data was motivated by the overarching research question: How do students develop an understanding of proof by contradiction? The focus of analyzing the primary data (transcripts and written responses during the teaching experiment) was thus to interpret and categorize how students' understanding evolved during each of the five teaching episodes. A description of the multiple passes of data analysis follows.

[12]To be clear, MAXQDA was used to organize the data analysis I performed and was not used to analyze the data.

During the first pass of data analysis, I organized the data based on students' groups during the teaching episode. Then, I identified each group's understanding of proof by contradiction, according to APOS Theory, during each teaching episode. This included levels of conception and, if evident, the stage of participants' proof by contradiction Schema. For example, after analyzing and interpreting all of group G13's responses to tasks in teaching episode 2, the group was identified as having exhibited a Process conception of proof by contradiction during the episode. Groups' levels of conception per teaching episode were then used to form categories of exhibited students' understanding of proof by contradiction.

During the second pass of data analysis, groups' responses to each phase of teaching episodes 1 and 2 were categorized and then interpreted via APOS Theory. This analysis provided a tool to identify which task or tasks aided students in developing an understanding of proof by contradiction. For example, consider group G13's conception of proof by contradiction during teaching episode 2. As previously mentioned, G13 was identified as having exhibited a Process conception of proof by contradiction by the end of the teaching episode. However, after an analysis and interpretation of the group's responses during each phase of teaching episode 2, it was identified that G13 exhibited an Action conception of proof by contradiction during the first phase and a Process conception during the second phase. This detailed analysis suggested the tasks during the second phase of teaching episode 2 may have aided G13 in developing a Process conception of proof by contradiction and therefore should be analyzed further.

During the third pass of data analysis, two students, Wesley and Yara, were chosen as *case studies* - detailed examinations of a single subject's understanding (Bogdan & Biklen, 2007). Wesley and Yara were chosen for a variety of reasons, including: (i) they were enrolled in different sections of *Bridge to Higher Mathematics* (ii) they completed all five teaching episodes, (iii) each student's evolution of understanding was distinct and representative of participants' understanding during teaching episodes 3-5, and (iv) these students provided the most data through which to analyze and support how their understanding of proof by contradiction evolved throughout the teaching experiment. As the second pass of data analysis provided a detailed description of groups' understanding of proof by contradiction for teaching episodes 1 and 2 (including their own), the third pass of data analysis focused on these particular students' understanding during teaching episodes 3, 4, and 5.

3.6 Chapter summary

This chapter began by detailing how APOS Theory would be used to analyze student understanding of proof by contradiction in two ways: levels of conception and stages of their contradiction proof Schema. These levels and stages are summarized in Tables 3.2 and 3.3, respectively.

Table 3.2 Levels of conception for proof by contradiction.

Action:	Students work with a concrete example and perform memorized or explicitly provided steps when using this proof method. At this level, students would not be able to prove a statement that required a deviation from these steps.
Process:	Students no longer need to work with concrete examples or external cues in order to explain how a proof by contradiction proceeds. At this level, students are able to describe a general procedure without referring to a particular example and without a need to perform the procedure.
Object:	Students can act on the method itself. At this level, students could prove the validity of the method or could compare the method to other proof methods.

Table 3.3 Stages of development for proof by contradiction.

Intra-contra:	Students describe proof by contradiction as a procedure (either externally or internally) based on the structure of the statement to be proved. They have separate, unrelated procedures for each type of statement and there is no reasoning behind why these procedures are valid.
Inter-contra:	Students begin to develop reasoning behind why these procedures are valid. In particular, students develop roles for parts of their procedures. By comparing roles, procedures are grouped together and called by the same name.
Trans-contra:	Students construct a single procedure based on the roles of the proof method. In particular, students relate the key steps to describe the general reasoning of the proof method. Students demonstrate coherence of the Schema by determining whether a presented proof is a proof by contradiction or not.

Then, I presented an outline of the hypothetical mental constructions students should make in order to understand proof by contradiction, referred to as a preliminary genetic decomposition. This outline, along with results of three pilot studies, was used to develop a teaching experiment consisting of five teaching episodes. Each of these episodes included the following three general phases:

1. Students worked cooperatively on tasks designed to help them to make the mental constructions suggested by the genetic decomposition. The focus of these tasks was to help students to make abstractions about certain ideas related to proof by contradiction rather than obtain correct answers. If students were alone, the teacher/researcher acted as another student with incomplete knowledge of proof by contradiction.

2. Students worked on tasks that built on tasks and activities completed during the first phase. During this phase, the teacher/researcher guided the discussion, provided definitions when needed, offered explanations, and/or presented an overview to tie together the particular ideas about proof by contradiction that the students had been thinking about and worked on.

3. The teacher/researcher assigned some standard problems to reinforce the development of particular mental constructions about proof by contradiction suggested by the genetic decomposition that were the focus of study during the first two phases. Often, these tasks required students to apply what they had learned and to consider the concept they were studying in relation to other mathematical concepts.

After providing details on the context of this teaching experiment, the goals of each teaching episode, in relation to the preliminary genetic decomposition, were described. A summary of these goals is provided in Table 3.4, where steps in the genetic decomposition are abbreviated GD#, with I, II, or III provided to indicate the type of statement: I for implication, II for single-level quantification, and III for property claims. Then, the primary types of tasks to achieve theses goals were described and are summarized in Table 3.5.

Table 3.4 Goals of each teaching episode in relation to the preliminary genetic decomposition.

Episode 1:	Action conception for implication *(GD1-I)* and opportunity to interiorize the Action into a Process for implication *(GD2-I)*.
Episode 2:	Action conception for nonexistence *(GD1-II)*, opportunity to interiorize the Action into a Process for nonexistence *(GD2-II)*, and opportunity to coordinate the Processes into a single Process for any statement *(GD3)*.
Episode 3:	Action conception for uniqueness *(GD1-II)*, opportunity to interiorize the Action into a Process for nonexistence *(GD2-II)*, and opportunity to coordinate the Processes into a single Process for any statement *(GD3)*.
Episode 4:	Action conception for infinity *(GD1-II)*, opportunity to interiorize the Action into a Process for infinity *(GD2-II)*, and opportunity to coordinate the Processes into a single Process for any statement *(GD3)*.
Episode 5:	Action conception for property claims *(GD1-III)*, opportunity to interiorize the Action into a Process for property claims *(GD2-III)*, opportunity to coordinate the Processes into a single Process for any statement *(GD3)*, and opportunity to encapsulate the Process into an Object *(GD4)*.

Table 3.5 Primary types of tasks used during the teaching experiment.

Outlining:	Asked students to logically outline a presented proof by contradiction. These tasks were included to prompt students to identify the logical argument within a presented proof by contradiction.
Defining:	Asked students to define and explain the method. These tasks focused on verbalizing students' conceptions of proof by contradiction to make students conscious of their contradiction proof Schema.
Comparison:	*Comparison tasks* required students to compare two or more logical outlines of presented proofs. These tasks were used as a reflection tool for students to consider the necessary logical lines of a proof by contradiction and how these lines logically relate.
Comprehension:	Assessed students' understanding of a presented proof. These tasks focused on a presented proof or the student producing their own proof of a statement. These tasks were used to reinforce the students' current conception of the proof method.

After describing the design of the teaching experiment, the participants of the study were described. A summary of participation in the teaching experiment is provided in Figure 3.1, where each block represents an individual teaching episode.

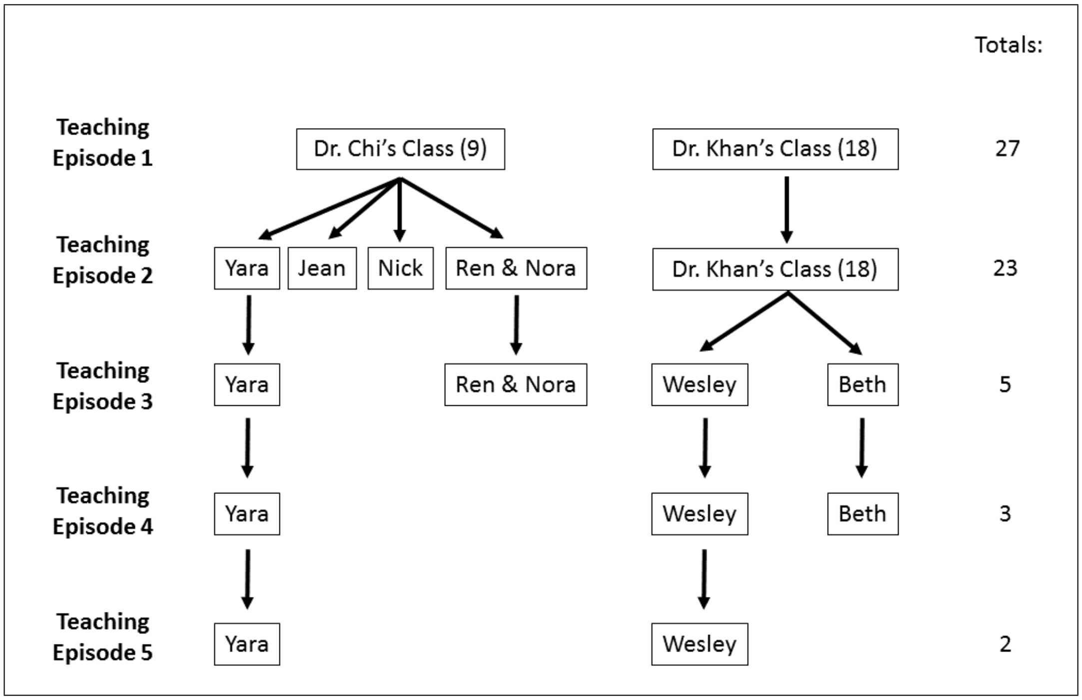


Figure 3.1 Summary of student participation by teaching episode.

The chapter concluded with a description of the primary and secondary data collected during the teaching experiment as well as detail how this data was analyzed. First, I organized students into groups based on who they worked with during the teaching episode, after which I identified each group's understanding of proof by contradiction for each teaching episode. Secondly, groups' responses to each phase of teaching episodes 1 and 2 were categorized and then interpreted via APOS Theory. Finally, two students (Wesley and Yara) were chosen for case study analysis that focused on their understanding of proof by contradiction during teaching episodes 3, 4, and 5.

CHAPTER 4

RESULTS

The previous chapter introduced the design and participants of the teaching experiment as well as described how the data was analyzed. This chapter will present the analysis and interpretation of the data collected throughout the course of the teaching experiment.

Results of the teaching experiment will be organized into sections by teaching episode and each section into subsections by phases of the ACE teaching cycle (see Section 3.3.2 for details on the ACE teaching cycle). Each phase will present participants' responses to the tasks of the phase and describe how these responses indicate the participants' understanding of proof by contradiction and related concepts, according to APOS Theory. The chapter will then conclude with a summary of results from teaching episodes 1 through 5.

4.1 Teaching Episode 1: Introducing a procedure for proof by contradiction

Teaching episode 1 was designed to achieve two goals. First, groups were guided to construct a set of step-by-step instructions for proofs by contradiction of an implication statement (i.e., $P \to Q$). Groups could then use this set of step-by-step instructions to prove similar statements by contradiction and thus possibly develop an Action conception of proof by contradiction for this type of statement. Secondly, groups were encouraged to focus on the roles of collections of lines in a proof and on the key steps of the proof method. Groups could then relate these roles to interiorize the previous step-by-step instructions into general steps for an implication statement and thus possibly develop a Process conception of proof by contradiction for this type of statement.

A total of 11 self-selected, temporary[1] groups of one to four students were formed during this teaching episode and will be denoted as G1 through G11. While these groups

[1]By temporary, I mean that students worked in these groups only during one class period.

were formed in two separate sessions, the sessions were conducted in a similar way - following the same lesson plan and asking the same or very similar questions. Therefore, this section will focus on how students' understanding emerged from an analysis of groups' responses to the tasks and teacher/researcher prompts during teaching episode 1.

The following subsections will be organized by the three phases of the teaching episode. Subsection 4.1.1 will focus on groups' initial conjectures about proof by contradiction based on a logical outline of a presented proof on an implication statement. Subsection 4.1.2 will focus on how the teacher/researcher guided groups to formalize the initial conjectures from the previous phase. Subsection 4.1.3 will focus on responses to proof comprehension questions designed to reinforce the formalized conjectures in the previous phase. For each subsection, I will begin by describing the lesson plan of the phase and then proceed by presenting the analysis and interpretation of data collected during the phase. The tasks used for this each phase of the teaching episode can be found in Appendix A and will be reproduced as needed to describe the lesson plan.

4.1.1 Initial conjectures about proofs by contradiction

This subsection is a report about groups' work during Activity 1 (see Appendix A for the complete set of tasks). The goals of Activity 1 were twofold: (1) guide groups to construct a set of step-by-step instructions for proofs by contradiction of an implication statement and (2) prompt groups to make initial conjectures about proof by contradiction based on this set of step-by-step instructions. Thus, the phase consisted of two tasks: outlining a given proof by contradiction for an implication statement and defining proof by contradiction. For each task, I will first detail the goal and reasoning behind the task. Then, I will present an analysis and interpretation of groups' responses to the task.

4.1.1.1 Outline of an implication proof by contradiction Groups were asked to read a presented proof of the statement "If every even natural number greater than 2 is the sum of two primes, then every odd natural number greater than 5 is the sum of three primes $[\boldsymbol{P} \rightarrow \boldsymbol{Q}]$" and subsequently outline this proof (see Figure 4.1) utilizing propositional logic.

Statement: If every even natural number greater than 2 is the sum of two primes, then every odd natural number greater than 5 is the sum of three primes $[\boldsymbol{P} \rightarrow \boldsymbol{Q}]$

Proof: Assume it is not true that if every even natural number greater than 2 is the sum of two primes, then every odd natural number greater than 5 is the sum of three primes $[\sim (\boldsymbol{P} \rightarrow \boldsymbol{Q})]$. Then every even natural number greater than 2 is the sum of two primes and it is not the case that every odd natural number greater than 5 is the sum of three primes $[\boldsymbol{P} \wedge \sim \boldsymbol{Q}]$. Then there exists an odd natural number greater than 5 that is not the sum of three primes, call it k $[\sim \boldsymbol{Q}]$. Then $k = 2n+1$. Since $k > 5$, $k-3 > 2$. Thus $k-3 = 2n-2 > 2$ and $k-3$ is even. By our assumption, $k-3$ is then the sum of two primes: p and q. Thus $k-3 = p+q$. Solving for k, we get $k = p+q+3$ $[\boldsymbol{Q}]$. This is a contradiction, as we assumed k was not the sum of three primes $[\boldsymbol{Q} \wedge \sim \boldsymbol{Q}]$. Therefore it is not the case that it is not true that if every even natural number greater than 2 is the sum of two primes, then every odd natural number greater than 5 is the sum of three primes $[\sim (\sim (\boldsymbol{P} \rightarrow \boldsymbol{Q}))]$. In other words, if every even natural number greater than 2 is the sum of two primes, then every odd natural number greater than 5 is the sum of three primes $[\boldsymbol{P} \rightarrow \boldsymbol{Q}]$.

Figure 4.1 Presented proof for Activity 1.

This particular statement was chosen as both the premise and conclusion have an unknown truth-value. For example, the premise "Every even natural number greater than 2 is the sum of two primes" (known as the *Strong Goldbach Conjecture*) is assumed to be true as part of the proof but has neither been proven nor disproven in general. That is, this statement is logically justified in the proof and is not mathematically justified. This would encourage groups to consider logical justifications for all lines in the proof (i.e., generalize the need for a logical justification of this particular line to the possible need to logically justify a general line in a proof) rather than focus solely on mathematical justification. Reasoning behind the exact presentation of the proof in Figure 4.1 is provided below.

The purpose of outlining the logical structure of this presented proof was to encourage groups to generalize the specific proof by contradiction into a series of steps that could be used to write proofs by contradiction for implication statements. In this way, groups were encouraged to think about proof by contradiction as a series of logical steps and thus possibly develop an Action conception of proof by contradiction for this type of statement.

Symbolic representations for the statement and lines of the proof were provided at the end of nearly all lines in the proof (bolded in Figure 4.1). These representations were provided to prompt groups to recall the propositional logic they were expected to have learned in the prerequisite course and apply this propositional logic to a mathematical proof. In addition, providing the symbolic representations allowed groups to focus on the logical relation between statements rather than focus on symbolically representing statements, as it has been shown in the literature that students may exhibit difficulty representing mathematical statements using symbols (e.g., Inglis & Alcock, 2012).

A desired logical outline of the proof is illustrated in Figure 4.2.

Statement: $P \to Q$

1. Assume $\sim (P \to Q)$
2. $P \wedge \sim Q$
3. $\sim Q_k$
4. $(\sim Q_k \wedge P) \to Q_k$
5. Q_k
6. $Q_k \wedge \sim Q_k$
7. $\sim (\sim (P \to Q))$
8. $P \to Q$

Figure 4.2 Desired logical outline of presented proof during Activity 1.

In the outline above, Q_k represents a specific odd natural number with the properties described by the representation Q. That is, k is an odd natural number greater than 5 that is the sum of three primes. While predicate logic may be more appropriate to describe statements P and Q (and thus provide clarity on what Q_k represents) propositional logic was provided as the literature suggested students exhibit difficulty understanding quantified statements (e.g., Dubinsky & Yiparaki, 2000).

The outline in Figure 4.2 is desired for three main reasons. First, the outline described the two key steps of a proof by contradiction: assuming the statement is false and arriving at a contradiction. Secondly, statements with different meaning are assigned different symbols (e.g., the difference between Q and Q_k). Finally, the outline describes how the proof arrives at Q_k from $\sim Q_k$. Line 4 noted that $\sim Q_k$ and P were necessary to arrive at Q_k. The lack of this step would create an undesirable "logical leap" as $\sim Q_k$ cannot imply Q_k alone. A discussion of groups' outlines of the presented proof follows.

Groups needed some initial prompting and additional explanation from the teacher/researcher of what was meant by an outline of the proof before writing their responses. Two categories emerged from the analysis of groups' responses[2]: (1) attending to the algebra of the proof and (2) an incomplete logical outline of the presented proof. In particular, no group provided a complete logical outline of the presented proof. A detailed description and interpretation of each category of response, accompanied by an example to illustrate the category, is provided below.

The first category of responses was an attention to the algebra of the presented proof. In particular, the group attended to the specific algebra of the proof over the logical argument of the proof. For example, the proof in Figure 4.1 used algebra to show that k could be written as a sum of three primes. Groups that attended to the algebra of the presented proof described this algebra in detail and did not describe the logical purpose of this algebra in the overall argument. This focus on algebraic manipulation over the logical argument of a proof has been identified in the literature (e.g., Inglis & Alcock, 2012; A. Selden & Selden, 2003) as a proof comprehension difficulty and indicates the group did not attend to the logical argument of the proof. These groups did not provide a logical outline even though they were given the necessary symbolic representations to logically outline the proof, which is indicative of a Pre-Action conception of mathematical logic. Groups G1 and G4 responded in this way. An example of this category of response follows.

[2]Groups G2, G8, and G11 did not respond.

Group G1 provided the most complete description of the algebraic manipulation within the proof (see Figure 4.3).

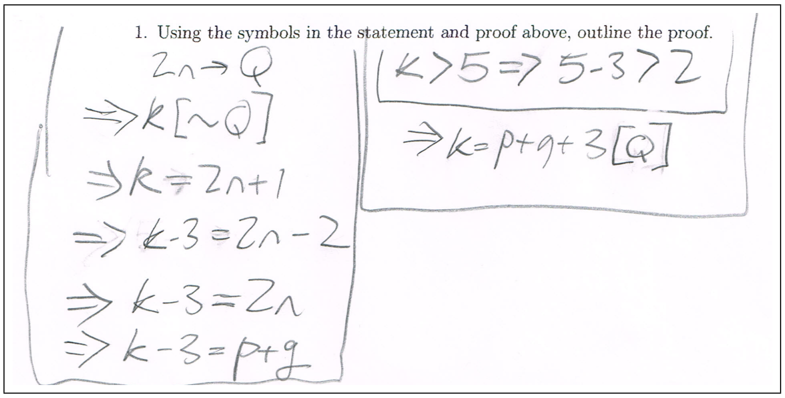


Figure 4.3 G1's outline of the proof presented during Activity 1.

The line "$2n \to Q$" is an interpretation of the premise "Every even natural number greater than 2 is the sum of two primes" (represented as $2n$) and the conclusion "every odd natural number greater than 5 is the sum of three primes" (represented as Q). The outline then skipped the first key step of a proof by contradiction (assuming the statement is false) and directly represented a consequence of this assumption: that a k exists to make the representation $\sim Q$ true. The outline then proceeded, by a series of implications justified by algebraic manipulation, to show that this k is the sum of three primes and made the representation Q true. No other lines of the proof are represented, including the second key step of a proof by contradiction (arriving at a contradiction). Therefore, G1 exhibited a Pre-Action conception of mathematical logic as the group did not include either of the two key steps of a proof by contradiction in their outline of the presented proof.

The second category of responses was an incomplete logical outline of the presented proof. This type of response focused to the logical argument of the proof but contained one or more unexplained lines of the proof. For example, removing line 4 in Figure 4.2 would leave line 5 unexplained as $\sim Q_k$ cannot logically imply Q_k. Responding with an incomplete logical outline suggests the group represented lines of the proof individually and did not consider how these lines logically related to one another, which is indicative of a Pre-Action conception of mathematical logic. Groups G3, G5, G6, G7, G9, and G10 responded in this way. An example of this category of response follows.

Group G10 provided the most common incomplete outline of the proof (see Figure 4.4).

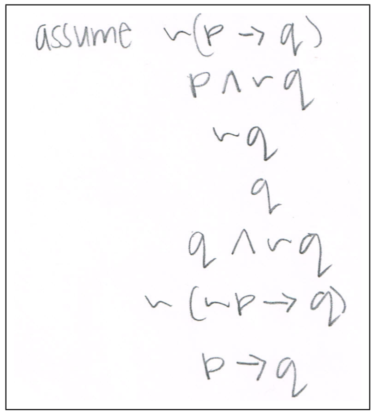


Figure 4.4 G10's outline of the proof presented during Activity 1.

This outline listed every representation included in the presented proof as a separate line. However, G10 did not provide any logical reasoning between $\sim Q$ and Q. Since Q cannot logically follow from $\sim Q$, there is an undesirable "logical leap" between these lines that is not addressed by G10. This suggests G10 copied the symbols provided at the end of each sentence in the presented proof and organized these symbols as a series of steps, however did not describe the logical relations between these steps. Therefore, G10 exhibited a Pre-Action conception of mathematical logic as the outline did not provide a complete logical justification for the proof.

4.1.1.2 Initial definition of proof by contradiction After reading the presented proof and writing an outline of the proof, groups were asked to write a definition for proof by contradiction. At this point in *Bridge to Higher Mathematics*, groups had not been formally introduced to proof by contradiction in a mathematical context. However, groups had been introduced to proof by contradiction in a logical context in their prerequisite course *Discrete Mathematics*. The goal of this question was to encourage groups to reflect on their knowledge of proof by contradiction in a logical context, along with a specific example of the proof method in a mathematical context, to consider the necessary steps to construct a proof by contradiction. While it was not included in the written task, the teacher/researcher prompted groups to reflect on the presented proof by contradiction to write a definition of the proof method in a mathematical context.

A desired definition for proof by contradiction would include the two key steps of the proof method (assume the statement is not true and arrive at a contradiction) as well as describe how these two key steps logically prove the statement. For example, the definition "Start by assuming the statement is not true and then arrive at a contradiction, which means your assumption was wrong and so the statement is true" is desirable as it includes both key steps of the proof method and describes how these steps relate to prove the statement is true. Results of the analysis of groups' definitions for proof by contradiction follow.

Three categories of definition for proof by contradiction emerged from the analysis of all groups' responses[3]. These categories are:

1. Generic indirect proofs;
2. Proofs that start with the negation of a statement; or
3. Proofs that contradict themselves (i.e., to prove statement P, show $P \wedge \sim P$).

No group provided a definition that could be considered as complete and close to a formal definition. A detailed description and interpretation of each category of response, along with an example, is provided below.

The first category of definitions represents groups that conceptualized proof by contradiction as a generic indirect proof. In other words, these groups did not differentiate between a proof by contraposition and a proof by contradiction. This type of definition did not describe either of the key steps of a proof by contradiction, which is indicative of a Pre-Action conception of the proof method. Groups G2, G6, G9, and G10 responded in this way. An example of this type of definition follows.

[3]Groups G4, G5, and G8 did not respond

When asked for a definition of proof by contradiction, group G2 responded "Proof by contradiction is stated to prove opposite proposition is true." The phrase "opposite proposition is true" is indicative of how an indirect proof does not prove the original statement and thus this definition did not recognize a distinction between proofs by contraposition and proofs by contradiction. Indeed, one could replace 'contradiction' by 'contraposition' in G2's definition and the definition would still be valid. The definition also does not include either of the key steps of a proof by contradiction. Therefore, G2 exhibited a Pre-Action conception of proof by contradiction.

The second category of definitions represents groups who conjectured that beginning with the negation of the statement is indicative of a proof by contradiction. This type of response is consistent with authors such as Antonini and Mariotti (2008) who claim that students' conceptions of proof by contradiction and contraposition can be analyzed together as both methods begin with the negation of a statement. This category of definitions is different than considering both methods as an indirect proof as this type of definition specified what is negated for a proof by contradiction: the statement to be proved. That is, this type of definition includes only one of the two key steps of a proof by contradiction, which is indicative of a Pre-Action conception of proof by contradiction. Groups G3 and G7 responded in this way. An example of this type of definition follows.

When asked for a definition of proof by contradiction, group G3 responded "Assuming something is not true, proving that it is." This definition is distinct from a proof by contraposition as it describes assuming the statement is not true (as opposed to proving the opposite of the statement is true). In addition, this definition described to the first step of a proof by contradiction (assuming the statement to be proved is false) and does not describe a contradiction. Therefore, G3 exhibited a Pre-Action conception of proof by contradiction.

The third category of definitions represents groups that focused on the specific contradiction of the presented proof, $Q \wedge \sim Q$, and viewed it as directly related to the initial statement, $P \rightarrow Q$. These groups then overgeneralized this direct contradiction to be with the statement to be proved itself. That is, if P is the statement to be proved, the contradiction would be $P \wedge \sim P$. This overgeneralization is related to *circular reasoning* - a proof construction difficulty recognized in the literature (e.g., Stavrou, 2014) where the student first assumes the statement is true in order to conclude that the statement is true. This type of response suggests the group recognized that negating the initial statement and arriving at a contradiction are necessary steps for the proof method but could describe the reasoning behind these steps, which is indicative of an Action conception of proof by contradiction. Groups G1 and G11 responded in this way. An example of this type of definition follows.

When asked for a definition of proof by contradiction, group G1 stated “Proof by contradiction is showing that a statement or equation contradicts itself, or is false. So therefore the *[does not finish statement]*.” This definition suggests that given a statement, S, a proof by contradiction is to show this statement contradicts itself (i.e., $S \wedge \sim S$). The phrase “or is false” clarifies this interpretation by suggesting to prove S, one shows $\sim S$. G1 overgeneralized the contradiction as relating directly to the statement itself and thus could not describe the reasoning behind the method, which explains why G1 was not able to complete their statement of the definition. Therefore, G1 exhibited an Action conception of proof by contradiction.

4.1.2 Formalization of initial conjectures about proof by contradiction

This subsection is a report about groups’ responses during Classroom Discussion 1 (see Appendix A for general questions that guided this discussion). The purpose of this phase was to guide students into formalizing the initial conjectures from Activity 1. In particular, the goals of this phase were to formalize a set of step-by-step instructions for proofs by contradiction of an implication statement and formalize groups’ initial definitions of proof by contradiction. Thus the phase consisted of two parts during which the teacher/researcher provided formalizations of the outline and definitions from Activity 1. For each part, I will begin by presenting the topic of discussion and how the topic would be formalized. Then, I will present an analysis and interpretation of groups’ responses to the formalization of the topic.

4.1.2.1 Discussion of an outline of the presented proof The teacher/researcher began by asking groups for a line-by-line outline of the presented proof using the propositional logic. Collectively, all groups provided an incomplete outline of the presented proof, illustrated in Figure 4.5.

Statement: $P \to Q$

1. Assume $\sim (P \to Q)$
2. $P \wedge \sim Q$
3. $\sim Q$
4. Q
5. $Q \wedge \sim Q$
6. $\sim (\sim (P \to Q))$
7. $P \to Q$

Figure 4.5 Groups' outline of the presented proof from Activity 1.

As a group suggested a particular line, the teacher/researcher prompted the group for the logical justification of the line. For example, when he asked for the logical justification of the line $(P \wedge \sim Q)$, he was prompting groups to recognize the line was logically equivalent to the preceding line, $\sim (P \to Q)$. The goal of prompting groups for the logical justification of each line was to formalize the outline as a list of step-by-step instructions the groups could use to prove similar statements.

Both sections were able to provide a logical justification for some lines in the presented proof, but could not agree on a logical justification for others. In particular, no groups could provide the logical justification of two lines in the outline in Figure 4.5: lines 4 and 6. A detailed description of these lines, along with groups' responses to the logical justification of these lines, follows.

Line 4 from the outline in Figure 4.5 represents the statement "k is an even, natural number greater than 5 that is the sum of three primes." This line is not the same Q from the statement as it refers to a specific natural number. Line 4 logically follows from showing $k-3$ is a specific case of P: that every even natural number greater than 2 is the sum of two primes. An ideal response to the logical justification for line 4 would both note that P is true (based on the initial assumption) and that $k-3$ is a specific case of P. When asked for the logical justification for line 4, groups responded in one of three ways: (1) there is no logical justification for the line, (2) it logically follows from line 3, or (3) it logically follows from the algebra of the previous statements in the proof.

The first category of responses was that there was no logical justification for the line. This type of response would suggest the group's conception of mathematical proof in general does not require that every line have a logical justification. Thus, groups that responded in this way exhibited a Pre-Action conception of mathematical logic as they did not properly provide a logical justification for a line in the outline.

The second category of responses was that line 4 followed from the previous line. This type of reasoning, that one line logically follows from the previous line, has been identified in the literature (e.g., Weber & Alcock, 2005) and suggests an inattention to the logical structure of the argument. This inability to appropriately describe the logical justification for a statement in a proof is indicative of a Pre-Action conception of mathematical logic. For example, G3 responded that the justification of line 4 was "Not not Q?" and thus suggested that line 4 logically followed from line 3.

The third category of responses was that line 4 followed from the algebra of the previous statements in the proof. This type of response suggested a focus on the algebraic manipulation over the logical argument of the proof. In this case, the algebraic manipulation is part of the logical implication but cannot, alone, imply line 4. Therefore, this type of response is indicative of a Pre-Action conception of mathematical logic. For example, G1 responded that the justification of line 4 was "Algebra?" and continued to describe their outline in Figure 4.3 (see page 83). This suggested G1 to the algebra over the logical justification of line 4 and thus exhibited a Pre-Action conception of mathematical logic.

Line 6 from the outline in Figure 4.5 represents the statement "Therefore it is not the case that it is not true that if every even natural number greater than 2 is the sum of two primes, then every odd natural number greater than 5 is the sum of three primes." This line required groups to understand the underlying concept of proof by contradiction: the law of excluded middle. An ideal response to the logical justification of line 6 would describe how line 1 (the assumption that the statement is not true) and line 5 (that this assumption lead to a contradiction) jointly imply that line 6 is true. When asked for the logical justification for line 6, groups responded in one of two ways: (1) recognized the lines but did not explain justification and (2) with a desired justification.

The first category of responses was to identify the justification included both lines 1 and 5, but not describe the logical justification further. The inability to explain, in their own words, why lines 1 and 5 together justified line 6 indicates the group referred to some external rule for proof by contradiction that links the three lines. In other words, the response indicated the correct lines to logically justify line 6 but does not describe the reasoning behind the justification, which is indicative of an Action conception of proof by contradiction. An example of this type of response follows.

When asked to provide the logical justification for line 6, G3 first stated the contradiction (line 5) was related to "the negation" (line 6), after which another group, G1, interjected that the contradiction was related to the assumption (line 1). When asked to explain why lines 1 and 5 together justified line 6, neither group could do so. Therefore, these groups linked lines 1 and 5 together to logically imply line 6, but did not articulate exactly how the lines were related and thus exhibited an Action conception of proof by contradiction.

The second category of responses was to identify lines 1 and 5 to explain why these two lines, together, justified line 6. A response that explained, in the group's own words, why lines 1 and 5 logically implied line 6 suggests the group could explain the main idea behind a proof by contradiction without describing each step of the proof. This would indicate a Process conception of proof by contradiction. An example of this type of response follows.

When asked to provide the logical justification for line 6, G6 first stated "So *[line]* 5 is a contradiction and thus, then *[line]* 6, we are just showing that *[line]* 1 is not true." Group G10 then immediately interjected "In order for that to work, then P implies Q has to be a statement or, more precisely, Q has to be a statement." G10 further clarified that an inequality such as $2x > 16$ is not a statement as it could be both true and false. In this way, G10 questioned whether $Q \wedge \sim Q$, the representation of the contradiction the groups agreed upon (see Figure 4.5 on page 4.5) was truly a contradiction and thus if, in this particular presented proof, lines 1 and 5 logically implied line 6. In other words, G10 recognized the logical relation between lines 1, 5, and 6 but questioned whether the presented proof arrived at a contradiction and could justify line 6. Therefore both groups could describe, in their own words, how lines 1 and 5 logically implied line 6 and thus exhibited a Process conception of proof by contradiction.

4.1.2.2 Discussion of initial definitions for proof by contradiction After the outline of the presented proof was discussed, groups were prompted to develop a definition for proof by contradiction. The teacher/researcher then used these suggestions to state a formal definition for the method. Figure 4.6 illustrates how these definitions were agreed upon by multiple groups.

G10:	Prove that the statement is not false.
G9:	Prove a statement is true by proving the false statement is false.
G7:	Proving that a statement is false by assuming first that it is false.
G11:	Prove that the statement is true by assuming the negation and proving the negation is false.
Teacher:	Prove that the statement is true by assuming the negation and proving this assumption is false with a contradiction.

Figure 4.6 Successive refinements of the definition during Classroom Discussion 1.

Note that all three categories of definitions observed in the first phase appeared in these refinements: G10 and G9 first suggested a generic indirect proof, then G7 suggested a proof that starts with an assumption, and finally G11 suggested a proof with a direct contradiction (assumption is both true and false). In addition, each definition built off of and refined the previous definition. These definitions, once sufficiently refined by the students into a single definition, were then formalized by the teacher/researcher to include the two indicative steps of a proof by contradiction (1) assume the statement to be proved is not true and (2) using this assumption and other mathematical knowledge, arrive at a contradiction.

Therefore, by the end of Classroom Discussion 1, all groups were provided a formal outline of proof by contradiction for implication statements (see Figure 4.2 on page 81) and a formal definition of the proof method (see Figure 4.6). In other words, all groups were provided the external steps to follow in order to construct an Action conception of proof by contradiction, as desired by the lesson plan.

4.1.3 Reinforcing initial conjectures about proof by contradiction

This subsection is a report about groups' responses to tasks from Exercise 1 (see Appendix A for the complete set of tasks) - the last phase of teaching episode 1. The goal of this phase was to reinforce groups' conceptions of mathematical logic and proof by contradiction through a series of eight tasks aligned with the proof comprehension assessment model by Mejía-Ramos et al. (2012). The comprehension questions were written for the presented proof (Figure 4.1 on page 80) and are provided in Figure 4.7.

Groups were first given approximately 20 minutes to respond to the comprehension questions above. Then, the teacher/researcher asked groups to share their answers and describe the reasoning behind their response to each question. After eliciting the groups' response and reasoning behind their response, the teacher/researcher provided a desired response to the question (if necessary).

The remainder of this subsection will be organized by comprehension questions one through eight. For each question, I will first describe what it was meant to assess in terms of the proof comprehension assessment model by Mejía-Ramos et al. (2012) (described on page 23). Then, I will describe the goal of the assessment in terms of the students' understanding of the presented proof, mathematical logic, or proof by contradiction. Finally, I will present an analysis and interpretation of groups' responses to the question.

1. Please give an example of a prime number and explain why it is prime.
2. Why is every even natural number greater than 2 the sum of two primes?
3. Why exactly can one conclude that $k - 3$ is the sum of two primes?
4. What is the purpose of the statement "Since $k > 5, 2n + 1 > 5$ and so $n > 2$"?
5. Summarize in your own words the main idea of this proof.
6. What do you think are the key steps of this proof?
7. In this proof, we subtracted 3 and worked with $k-3$. Would the proof still work if we instead subtracted 5 and worked with $k - 5$? Why or why not?
8. Using the method of this proof, show that: if every odd natural number greater than 5 is the sum of three primes and one of those primes is 3, then every even natural number greater than 2 is the sum of two primes.

Figure 4.7 Comprehension questions for the presented proof during Exercise 1.

Question one assessed whether groups could identify the meaning of terms and statements in the presented proof. In particular, the question asked for groups to provide an example of a prime number and explain why this number is prime. The goal of this question was to determine whether groups understood one of the mathematical terms in the proof (prime) that was necessary to understand the logical argument of the proof, as the literature suggests students struggle with proof comprehension due to a lack of mathematical knowledge (e.g., Moore, 1994). In other words, the goal of this question was to assess prerequisite knowledge necessary to understand the particular proof.

Nearly all groups provided an example of an odd prime number and a variation of the definition "A number that is divisible by 1 and itself" without any further clarification (i.e., did not clarify that the numbers are positive integers greater than 1). For example, group G1 responded "3. Prime numbers are numbers that are only divisible by either 1 or the number itself." This definition is sufficient for the purposes of the proof, as the proof only requires the knowledge that 3 is a prime number. One group, G4, quoted the proof statement to improperly justify that 7 was a prime number and stated "7, according to the proof 7 is the sum of 3 primes, since it's greater than 5. So $2 + 2 + 3 = 7$." Therefore nearly all groups possessed the prerequisite knowledge of prime numbers in order to understand the particular proof.

Question two assessed whether groups could identify the logical status of a statement in the proof. That is, the question assessed whether groups could identify why one statement in the proof logically followed from another. The question focused on part of the initial assumption, "every even natural number greater than 2 is the sum of two primes", represented as P. The goal of this question was to assess whether groups could provide the logical justification for this statement, as the statement was required to logically justify line 5 of the outline (Figure 4.2 on page 4.2). Responses to this question would indicate groups' conception of mathematical logic as it required groups to justify a particular line in the proof.

As the teacher/researcher walked around and noted that groups did not provide any response to the question as formulated, it was rephrased as "Why can the proof use that every even natural number greater than 2 is the sum of two primes?" to emphasize that this may be particular to the presented proof. Groups responded in one of two ways: it is true (by definition or theorem unknown to them) or it appeared to be true (and provided examples). In other words, the groups justified the statement by theorem or example rather than noted it logically followed from the initial assumption and thus exhibited a Pre-Action conception of mathematical logic.

Question three assessed whether groups could justify a particular claim in the proof. This question focused on justifying the statement that led to the contradiction: $k - 3$ is the sum of two primes. The goal of this question was to assess whether groups could explain why $k - 3$ was an even natural number greater than 2 and why this implied that $k - 3$ was the sum of two primes. In other words, the goal of this question was to assess whether groups could justify line 4 from the desired logical outline of the proof (see Figure 4.2 on page 81), which was described during the previous phase of the teaching episode. Responses to this question would indicate groups' conception of mathematical logic as it required groups to justify a particular line in the proof. A desired response would indicate that $k-3$ is both even and greater than 2 and thus by the initial assumption P, $k-3$ was the sum of two primes.

More than half of the groups (G2, G3, G5, G9, G10, G11) stated that $k-3$ was both even and greater than 2. These responses implicitly used the claim that all even natural numbers greater than 2 are the sum of two primes and, based on responses to question two, did not rely on P as logically true for this proof. Therefore, these responses illustrated an Action conception of mathematical logic.

Group G2 provided the most complete explanation of why $k-3$ is both even and greater than 2 and stated "k is odd natural number greater than 5. k is equal to $2n+1$. Then, $k-3$ is equal to $2n-2$, and it should greater than 2. Therefore $k-3$ is even." The phrase "it should greater than 2" was interpreted as "it showed greater than 2" and referred to the line in the proof directly after k was defined as $2n+1$, which stated "Since $k>5, k-3>2$." The sentence "Therefore $k-3$ is even." seems to refer to the phrase $k-3=2n-2$, though this was not made explicit. Regardless, this response attended to the two conditions to show $k-3$ was the sum of two primes: greater than 2 and even.

Some groups (G1, G8) only presented one of the two conditions (even and greater than 2) in order to invoke P. For example, G1 responded "$k=2n+1 \rightarrow k-3=2n-2 \rightarrow k-3=2n \rightarrow k-3=p+q$." The last implication, $k-3=2n \rightarrow k-3=p+q$, suggests G1 only attended to $k-3$ as a even number and did not consider whether or not $k-3$ was greater than 2. The remaining groups' responses to the question were either too vague to be considered valid (e.g., G6 stated "By proving it directly with a proof") or did not provide a valid explanation (e.g., G7 stated "If the primes are different, the first two primes are 1 and 2, their addition is $1+2=3$ so k has to be $k-3$). Each of these three types of responses indicate a Pre-Action conception of mathematical logic as they do not provide an adequate justification of the claim, even though this justification was discussed during the previous phase of the teaching episode.

Question four assessed whether groups could identify the logical status of a statement in the proof, similar to question 2. This time, groups were asked for the purpose of the statement "Since $k > 5, k - 3 > 2$." which provided one of two conditions necessary to invoke part of the initial assumption, P. The goal of this question was to assess whether groups could describe the logical purpose of the statement (as part of the justification of another line) as opposed to justify the statement itself is true. Responses to this question would indicate groups' conception of mathematical logic as it required groups to consider the logical purpose of a particular sentence in the larger argument of the proof.

Groups responded[4] to question four in one of two ways: the purpose is to show $k-3 > 2$ (G2, G4, G5, G6, G11) or the purpose is to show that $k - 3$ is the sum of two primes (G1, G7, G9, G10). Both of these responses do not adequately describe the logical status of the statement. The first response described the truth value of the statement rather than the logical status of the statement (e.g., G5 responded "Because $k - 3 > 2$ is the same as $k > 5$ if you add 3 to both sides."). The second response does not acknowledge that another condition, $k - 3$ is even, is required to invoke the initial assumption and arrive at the contradiction (e.g., G7 responded "The purpose of the statement is to show that $k - 3 = p + q \rightarrow k = 3 + p + q$."). Both types of responses suggest groups did not attend to the purpose of the statement in the larger argument of the proof, even though this purpose was discussed during the previous phase of the teaching episode. That is, groups did not attend to the logical purpose of a statement in the proof, which is indicative of a Pre-Action conception of mathematical logic.

Question five assessed whether groups could summarize the proof via high-level ideas. The purpose of this question was to assess whether groups could summarize the proof in their own words and, if they could, how they summarized the proof. For example, a summary for the presented proof could be similar to the following:

> It's a proof by contradiction, where you assume the statement isn't true. So then

[4] Groups G3 and G8 did not respond.

> there is an odd number that is not the sum of three primes. But if we subtract 3 and look at $k-3$, it is even and greater than 2, so $k-3$ is the sum of two primes. That means k is the sum of three primes, but that's a contradiction. Therefore the statement is true.

This summary describes the specific procedure of the presented proof and would illustrate an Action conception of proof by contradiction. In contrast, a summary could also be similar to the following:

> If the statement were false, then there would be a number that is not the sum of three primes, but we can show that it is the sum of three primes with the premise of the statement, which is a contradiction. Therefore, the statement must be true.

This response describes the main idea of the proof (e.g., the premise of the statement necessarily implies the conclusion of the statement) and provides a generalization of the two key steps of a proof by contradiction, which is indicative of a Process conception of the proof method. Therefore, responses to this question would indicate groups' conception of proof by contradiction.

Groups responded to question five in one of two ways. The first type of response provided some relevant information but was incomplete (G2, G5). For example, G2 stated "The main idea of this proof is to show that what if it's not true, but it's actually true." The phrase "what if it's not true, but it's actually true" alludes to the procedure of the proof (proof by contradiction). In other words, G2 responded that the main idea of the proof is that it is a proof by contradiction. While this is valid and relevant to summarizing the proof, it did not provide a complete summary of the *particular* proof.

The second type of response provided some particular ideas and statements from the proof rather than a summary of the proof (G1, G4, G9). For example, G4 responded "All non-prime even numbers are the sum of two odd primes and all odd natural numbers are the sum of 3 primes." The phrase "non-prime even numbers are the sum of two odd primes" is an improper interpretation of the statement "even natural numbers greater than 2 are the sum of two primes" while the phrase "odd natural numbers are the sum of 3 primes" is an improper interpretation of the statement "odd natural numbers greater than 5 are the sum of 3 primes." In addition to improperly interpreting the statements individually, the phrase as a whole is an improper representation of the statement to be proved. In other words, G4's response described the statement to be proved rather than provide a summary of the particular proof.

Both types of responses indicated groups could not describe the proof, neither by reciting the procedure of the proof nor by citing the general ideas of the proof. In addition, the majority of groups (G3, G6, G7, G8, G10, G11) left this question blank and did not respond. Therefore, all groups exhibited a Pre-Action conception of proof by contradiction as they did not summarize the proof either by the procedure or the main ideas of the proof.

Question six assessed whether groups could identify the modular structure of the proof. In other words, whether groups could collect sentences in the proof together and provide a purpose for each of these collections of sentences (in contrast to an outline that provides representations for individual lines in a proof). By phrasing the question as 'key steps', the teacher/researcher attempted to illicit the purpose of particular lines (e.g., showing that $k-3$ is the sum of two primes) and how these purposes provided a complete proof rather than citing lines in the proof. A desired response would indicate both key steps of a proof by contradiction (assuming the negation of the statement is true and arriving at a contradiction) as well as synthesis how this particular proof arrived at a contradiction (showing $k-3$ is the sum of two primes). These three steps, together, would provide a complete modular structure to the presented proof.

Some groups (G5, G7, G9, and G10) responded with at least two key lines from their outline: 1 and 5. These responses referred to the two key steps of a proof by contradiction and relied on the outline to describe the key steps of the proof, which is indicative of an Action conception of proof by contradiction. In addition, these steps referred to a proof by contradiction in general and did not describe the key steps of the particular proof. Therefore, these types of responses did not provide a complete modular structure to the presented proof. For example, G9 rewrote the outline from Classroom Discussion 1 and circled[5] lines 1 and 5 without any further explanation (see Figure 4.8).

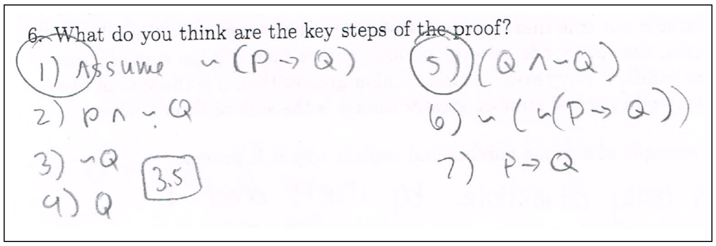


Figure 4.8 G9's response for question 6 in Exercise 1.

The remaining groups either responded with one (G8) or none (G1, G2, G3, G4, and G6) of the key steps for the proof method, which is indicative of a Pre-Action conception of proof by contradiction. In addition, these responses did not synthesize the intermediate steps that arrived at a contradiction and thus did not provide a complete modular structure to the presented proof.

Question seven assessed whether groups could justify a statement in the proof. This question focused on why the proof considered $k-3$ and not the difference of k and another odd number, 5. The goal of this question was to assess whether groups recognized that the condition $k-3>2$ was necessary to prove the statement. A desired response to this question would state that $k-5$ is not necessarily larger than 2 and thus the proof, as written, would not be valid.

Some groups (G5, G6, G7, and G10) responded that $k-5$ would not work unless the initial statement was changed to odd natural numbers greater than 7. These groups recognized that $k-5$ would not necessarily be greater than 2 and even provided a modification to part of the initial statement that would make $k-5$ necessarily greater than 2. These responses were valid and desired as they recognized the proof, as written, would not be valid

[5]'3.5' was added after the class presented their responses to the question and was boxed to indicate it was not part of their original response.

with the change to $k-3$ and identified specific condition that would not hold (i.e., $k-5$ would not necessarily be larger than 2). Therefore, a response of this type illustrated a Process conception of mathematical logic as it addressed the specific condition affected by this change and suggested a modification to the statement to reconcile this change.

Group G7 provided the most complete explanation of this type of response and stated "No, because it is greater than 0 but not 2 which we need it to be. If we switch it to $k > 7$, then it will be true." This response recognized that, based on the initial statement, $k-5$ would necessarily be larger than 0 whereas the proof relied on $k-3 > 2$. The phrase "then it will be true" in reference to the question "Would the proof still work" refers to the proof as valid (not true) given the change to the initial statement.

All other groups did not describe the condition that $k-5$ is not necessarily larger than 2 and thus did not recognize the condition that $k-3$ was necessarily greater than 2 as necessary for the proof, which is indicative of a Pre-Action conception of mathematical logic. Groups G1, G3, G4, and G11 did not provide any response while groups G2, G8, G9 attended to other conditions that were not affected. For example, G2 responded "Yes, because k is still *[an]* odd natural number." This response attended to k and does not recognize that $k-3 > 2$ is a necessary condition for the proof to be valid.

Question eight assessed whether groups could transfer the general idea or method to another context, this time in the form of writing a similar proof. The purpose of this question was to assess whether groups could utilize the logical outline provided for a proof by contradiction (see Figure 4.2 on page 81) and modify this outline to write a proof by contradiction. Responses to this question would thus indicate groups' understanding of proof by contradiction.

No group provided a response to this question, even though they were provided with a logical outline they could have followed step-by-step. Therefore, all groups exhibited a Pre-Action conception of proof by contradiction as they did not follow the step-by-step outline to write a similar proof by contradiction.

4.1.4 Summary of results

Teaching episode 1 was designed to achieve two goals. First, groups were guided to construct a set of step-by-step instructions for proofs by contradiction of an implication statement (i.e., $P \to Q$). Groups could then use this set of step-by-step instructions to prove similar statements by contradiction and thus possibly develop an Action conception of proof by contradiction for this type of statement. Secondly, groups were encouraged to focus on the roles of collections of lines in a proof and on the key steps of the proof method. Groups could then relate these roles to interiorize the previous step-by-step instructions into general steps for an implication statement and thus possibly develop a Process conception of proof by contradiction for this type of statement.

Subsection 4.1.1 reported on groups' work during Activity 1. The goals of this phase were twofold: (1) guide groups to construct a set of step-by-step instructions for proofs by contradiction of an implication statement and (2) prompt groups to make initial conjectures about proof by contradiction based on this set of step-by-step instructions. Thus, this phase was made up of two tasks: (4.1.1.1) outlining a presented proof of an implication statement and (4.1.1.2) defining proof by contradiction based on this outline. A brief summary of the results from each task follows.

Two categories of responses emerged from an analysis of groups' outlines. The first type of response focused on the algebra of the proof over the logical argument of the proof, which is indicative of a Pre-Action conception of proof by contradiction. The second type of response provided an incomplete logical outline of the presented proof, which is also indicative of a Pre-Action conception of proof by contradiction. No group provided a complete logical outline of the presented proof.

Three categories of responses emerged from an analysis of groups' definitions for proof by contradiction. The first definition explained the proof method as a generic indirect proof, which is indicative of a Pre-Action conception of proof by contradiction. The second definition explained the proof method as one that start with the negation of a statement, which is indicative of a Pre-Action conception of proof by contradiction. Finally, the third definition explained the proof method as one that contradict itself, which is indicative of an Action conception of proof by contradiction.

Subsection 4.1.2 reported on groups' responses during Classroom Discussion 1. The purpose of this phase was to formalize groups' conceptions of proof by contradiction based on their work and conjectures during Activity 1. Thus, the tasks for this phase were based on the two tasks from Activity 1. A brief summary of the results from each task follows.

When asked to provide and agree upon an outline of the presented proof, all groups initially agreed on an incomplete outline of the presented proof as it did not provide the logical justification necessary for two lines: that k is the sum of three primes (line 4) and that the initial assumption was not true (line 6). When asked for the logical justification for line 4, groups responded in one of three undesirable ways. Some stated there is no logical justification for the line, which is indicative of a Pre-Action conception of mathematical logic. Others stated the line logically follows from the previous line, which is also indicative of a Pre-Action conception of mathematical logic. Finally, other groups stated the line logically followed from the algebra of the previous statements, which is also indicative of a Pre-Action conception of mathematical logic. When asked for the logical justification for line 6, groups responded in one of two ways. Some groups identified the correct lines that justified the statement but could not describe the reasoning behind the justification, which is indicative of an Action conception of proof by contradiction. Other groups identified the lines that justified the statement and provided a desirable description of the reasoning behind the justification, which is indicative of a Process conception of proof by contradiction.

After groups presented their initial conjectures of a definition for proof by contradiction, the teacher/researcher presented the following formal definition: prove that the statement is true by assuming the negation and proving this assumption is false with a contradiction.

Subsection 4.1.3 reported on groups' work during Exercise 1. The goal of this phase was to reinforce groups' conceptions of mathematical logic and proof by contradiction through a series of eight tasks aligned with the proof comprehension assessment model by Mejía-Ramos et al. (2012). A short summary of results for each question follow.

Question one: When asked to provide an example of a prime number and explain why it is prime, nearly all groups provided an example of an odd prime number and a variation of the definition "A number that is divisible by 1 and itself." These groups possessed the necessary prerequisite knowledge of primes numbers in order to understand the presented proof. One group quoted the proof statement to improperly justify why 7 was prime and thus did not possess the necessary prerequisite knowledge of primes numbers in order to understand the presented proof.

Question two: When asked why the proof could use a mathematical conjecture (i.e., the *Strong Goldbach Conjecture*) as true in the proof, groups responded the conjecture was true or it appeared to be true. Both of these responses are indicative of a Pre-Action conception of mathematical logic.

Question three: When asked to justify the statement that directly led to the contradiction, the majority of groups indicated the two conditions necessary for the statement to be true and implicitly used another statement in the proof, which is indicative of an Action conception of mathematical logic. The other groups responded with either one of the two conditions or did not provide a valid explanation, which is indicative of a Pre-Action conception of mathematical logic.

Question four: When asked for the purpose to the line in the presented proof that provided one of two conditions necessary to justify the statement from question 3, groups responded in one of two ways. The first was to illustrate that the statement was true. The second was to show it directly justified the statement from question 3. Both of these purposes are not valid and are thus indicative of a Pre-Action conception of mathematical logic.

Question five: When asked to summarize the main idea of the proof, all groups either provided some relevant information but ultimately an incomplete summary or some particular ideas and statements from the proof. Both responses did not provide a desired summary of the proof and thus are indicative of a Pre-Action conception of proof by contradiction.

Question six: When asked to provide the key steps of the proof, groups responded in two ways. Some groups used the outline to identify the key steps in a proof by contradiction in order to identify the key steps of the particular proof, which is indicative of an Action conception of proof by contradiction. Other groups responded with either one key step or no key steps, which is indicative of a Pre-Action conception of proof by contradiction.

Question seven: When asked if the proof would still work if it used $k-5$ instead of $k-3$, groups responded in two ways. Some groups identified and modified a condition in the initial statement to reconcile a change in the procedure of the proof, which is indicative of a Process conception of mathematical logic. Other groups did not describe the conditions that made $k-3$ necessary, which is indicative of a Pre-Action conception of mathematical logic.

Question eight: When asked to write a similar proof using the method of the presented proof, no group provided an response, which is indicative of a Pre-Action conception of proof by contradiction.

4.2 Teaching Episode 2: Generalizing a procedure for proof by contradiction

Teaching episode 2 was designed to achieve three goals. First, groups were guided to construct a set of step-by-step instructions for proofs by contradiction for a nonexistence statement (i.e., $(\nexists x)(P(x))$). Groups could then use this set of step-by-step instructions to prove similar statements by contradiction and thus possibly develop an Action conception of proof by contradiction for this type of statement. Secondly, groups were encouraged to focus on the roles of groups of lines in a proof and on the key steps of the proof method. They could then relate the roles of lines or collections of lines to interiorize the previous step-by-step instructions into general steps for a nonexistence statement and thus possibly develop a Process conception of proof by contradiction for this type of statement. Thirdly, groups were encouraged to compare their general steps for implication statements (from teaching episode 1) to their general steps for a nonexistence statement. Groups could then coordinate these general steps for specific structures of statements to construct general steps for any proof by contradiction and thus possibly develop a Process conception of proof by contradiction.

A total of 10 self-selected, temporary groups of one to four students were formed during this teaching episode. Groups G5, G6, G7, G8, G9, G10, and G11 decided to work together again and remained the same groups from teaching episode 1. In addition, three new groups were formed: G12, G13, and G14. This section will focus on how students' understanding emerged from an analysis of these groups' responses to the tasks and teacher/researcher prompts during teaching episode 2.

The following subsections will be organized by the three phases of the teaching episode. Subsection 4.2.1 will focus on groups' initial conjectures about proof by contradiction based on a logical outline of a presented proof on a nonexistence statement. Subsection 4.2.2 will focus on how the teacher/researcher formalized the initial conjectures from the previous phase. Subsection 4.2.3 will focus on responses to proof comprehension questions designed to reinforce the formalized conjectures from the previous phase. For each subsection, I will begin by describing the lesson plan of the phase and then proceed by presenting the analysis and interpretation of data collected during the phase. The tasks used for this teaching episode can be found in Appendix A and will be reproduced as needed to describe the lesson plan for each phase of teaching episode 2.

4.2.1 Initial conjectures on nonexistence proofs by contradiction

This subsection is a report about groups' work during Activity 2 (see Appendix A for the complete set of tasks). The goals of this Activity reflect the three goals of this teaching episode. Thus, the phase consisted of three tasks: outlining a given proof by contradiction for a nonexistence statement, defining proof by contradiction, and comparing the presented proofs from Activity 1 and the previous task. For each task, I will first detail the goal and reasoning behind the task. Then, I will present an analysis and interpretation of groups' responses to the task.

4.2.1.1 Outline of a nonexistence proof by contradiction The first task of Activity 2 asked groups to read a presented proof of the statement "There is no odd integer that can be expressed in the form $4j-1$ and in the form $4k+1$ for integers j and k.

$(\nexists \boldsymbol{x})(\boldsymbol{P}(\boldsymbol{x}))$" and subsequently outline this proof (see Figure 4.9) utilizing predicate logic.

Statement: There is no odd integer that can be expressed in the form $4j-1$ and in the form $4k+1$ for integers j and k. $(\nexists \boldsymbol{x})(\boldsymbol{P}(\boldsymbol{x}))$

Proof: Assume it is not true that there is no odd integer that can be expressed in the form $4j-1$ and in the form $4k+1$ for integers j and k. Then there is an odd integer that can be expressed in the form $4j-1$ and in the form $4k+1$ for integers j and k. Let n be that integer; that is, $n \in \mathbb{Z}$ such that $n = 4j-1$ and $n = 4k+1$ for $j, k \in \mathbb{Z}$. Then $4j-1 = 4k+1$ and so $2j = 2k+1$. Note that $2j$ is an even number and, since $2j = 2k+1$, $2j$ is an odd number. A number cannot be both even and odd and thus this is a contradiction. Therefore, it is not true that it is not true that there is no odd integer that can be expressed in the form $4j-1$ and in the form $4k+1$ for integers j and k. In other words, there is no odd integer that can be expressed in the form $4j-1$ and in the form $4k+1$ for integers j and k.

Figure 4.9 Presented proof for Activity 2.

The purpose of outlining the logical structure of this presented proof was to encourage groups to generalize the specific proof by contradiction into a series of steps that could be used to write proofs by contradiction for nonexistence statements. In this way, groups were encouraged to think about proof by contradiction as a series of logical steps and thus possibly develop an Action conception of proof by contradiction. In addition, this particular proof relied on a well-known mathematical partition between even and odd integers (i.e., that an integer cannot be both even and odd) and thus groups were expected to have this particular mathematical knowledge in order to understand the contradiction of the particular proof. Reasoning behind the exact presentation of the proof in Figure 4.9 is provided below.

Unlike the presented proof in Activity 1, symbolic representation was provided only for the statement and not for the body of the proof. A representation of the original statement was provided to guide groups to make the same or similar representations for the outline. For example, by representing the initial statement as $(\nexists x)(P(x))$, groups were encouraged to represent the first line of the proof as $\sim (\nexists x)(P(x))$ and to continue representing lines in the presented proof utilizing the representation $P(x)$. With similar outlines, groups could then more easily compare and contrast each other's outlines to agree upon a formal outline. Representations were not provided for lines of the proof as, at this point in the course, groups were expected to represent quantified statements using predicate logic. Not providing symbolic representations for the quantified statements also encouraged groups to reflect on the original statement and compare the meaning of lines in the proof to this statement.

A desired logical outline of the presented proof would describe the two key steps of a proof by contradiction (assuming the statement is not true and arriving at a contradiction) and, in particular, would describe that this contradiction is generic. By 'generic', I mean that it can relate directly to the statement proved (e.g., in the presented proof in Figure 4.1 on page 80, the conclusion of the statement is part of the contradiction) or be a contradiction to a know mathematical theorem or axiom (e.g., in the presented proof in Figure 4.9, the mathematical theorem contradicted is that an integer cannot be both even and odd). In addition to including the two key steps of a proof by contradiction, the outline should completely describe the logical argument of the proof. That is, each line should follow either from the previous line or from the general procedure of a proof by contradiction (e.g., negating the assumption after arriving at a contradiction). One such logical outline for the proof presented in Figure 4.9 is provided in Figure 4.10 below.

Statement: $(\nexists x)(P(x))$

1. Assume $\sim((\nexists x)(P(x)))$
2. $(\exists x)(P(x))$
3. $P(n)$
4. Using $P(n)$, get a contradiction.
5. $\sim(\sim(\nexists x)(P(x)))$
6. $(\nexists x)(P(x))$

Figure 4.10 Desired logical outline of presented proof during Activity 2.

The outline above is desired for two main reasons. First, it includes the two key steps of a proof by contradiction (line 1 and line 4). Similar to the outline in Figure 4.2 (see page 81), line 4 in Figure 4.10 synthesizes multiple lines of the presented proof to one purpose: arriving at a contradiction. Note that in this outline, neither the structure nor the relation of the contradiction to the initial statement is specified. Secondly, the outline provides a complete logical argument for the proof. In particular, the outline provides additional clarification for how to arrive at a contradiction (Using $P(n)$). The justification for this line is general enough to encompass any nonexistence proof - the goal of the task. However, an outline that provided the specific algebraic justification (as opposed to a general justification) would still be appropriate. A discussion of groups' outlines of the presented proof follows.

Nearly all groups were able to represent lines 1, 2, 5, and 6 of the outline in Figure 4.10 using predicate logic. Groups' representations of lines 3 and 4 were categorized in one of three ways: (1) an undesired and/or unclear representation that results in a contradiction, (2) reliance on the outline from Classroom Discussion 1, and (3) a desired representation of lines 3 and 4 in algebraic symbols. A detailed description and interpretation of each category of response, accompanied by an example to illustrate the category, is provided below.

The first type of outline response for lines 3 and 4 described an undesired and unclear representation to justify the contradiction. An undesired representation included using symbols in an improper way, such as using an equal sign to signify an element is in a set. An unclear representation included not defining a represented symbol (e.g. writing Q and not defining what Q represents) or using the same symbol for multiple representations (e.g. P representing that all numbers can be written in a certain form and P representing a specific number can be written in the same form). An undesired and unclear representation suggests the group did not understand how to write arguments following mathematical logic and thus indicated a Pre-Action conception of mathematical logic. Groups G6 and G11 responded in this way. An example of this category of response follows.

Group G6 provided a desired representation of lines 1 and 2, but did not provide a desired representation for any other lines (see Figure 4.11).

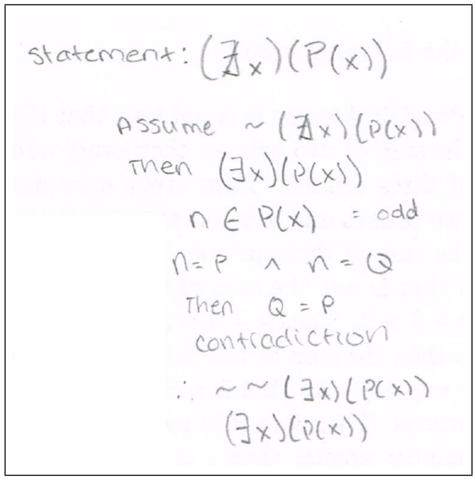


Figure 4.11 G6's outline for the presented proof in Activity 2.

For line 3, G6 wrote "$n \in P(x) =$ odd" and seemed to focus on the meaning of the third statement in the presented proof, though it was an undesired use of symbols to represent the statement. In addition, it is not clear what G6 meant by "$n = P \wedge n = Q$, Then $Q = P$" or how this equality resulted in a contradiction. Therefore, G6 exhibited a Pre-Action conception of mathematical logic through the use of undesired and unclear representations.

The second type of outline response for lines 3 and 4 relied on the outline from Classroom Discussion 1. In particular, this type of response represented line 3 in Figure 4.9 as $\exists x \wedge \sim P(x)$ and line 4 in Figure 4.9 as $\sim P(x)$. These representations include an improper use a logical operator and do not properly represent statements in the proof. However, these representations emulated lines 3 ($P \wedge \sim Q$) and 4 ($\sim Q$) of the outline from Classroom Discussion 1. This emulation suggests groups improperly represented statements in the presented proof in order to follow the step-by-step procedure from Classroom Discussion 1 (see page 88). The improper use of symbolic representations is indicative of a Pre-Action conception of mathematical logic while emulation of the external step-by-step procedure is indicative of an Action conception of proof by contradiction. Groups G10 and G12 responded in this way. An example of this category of response follows.

Group G10 provided the most clear emulation of the outline from Classroom Discussion 1 (see Figure 4.12).

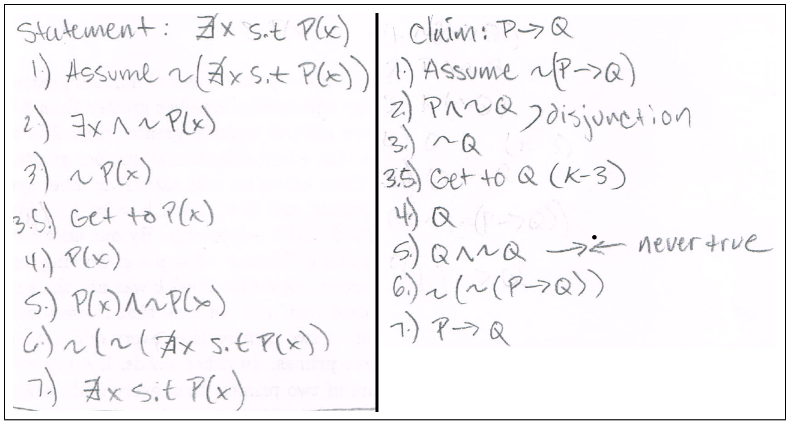


Figure 4.12 G10's outline for the presented proof in Activity 2 (left) and Classroom Discussion 1 (right).

Line 1 in G10's outline is a desired representation of the first statement. Then, line 2 provided an undesired and unclear representation of the meaning of line 1 as $\exists x$ is not a complete statement and thus cannot be joined by the logical operator $\wedge$, which is indicative of a Pre-Action conception of mathematical logic. However, by comparing line 2 for Activity 2 to line 2 in Classroom Discussion 1 (provided on the right in Figure 4.12) it is evident that these lines were written in the same form. This suggests G10 represented the line "Then there is an odd integer that can be expressed in the form $4j-1$ and in the form $4k+1$ for integers j and k" by focusing on the step-by-step form of the outline in Classroom Discussion 1, as an 'and' statement, rather than as $(\exists x)(P(x))$, which is indicative of an Action conception of proof by contradiction. Indeed, a side-by-side comparison shows lines 2 through 5 in G10's outline were improperly represented and were written in the same form as the corresponding line in the previous outline. This resulted in an outline that contained the key steps of a proof by contradiction (indicative of an Action conception of proof by contradiction) but did use proper representations to outline the current proof (indicative of a Pre-Action conception of mathematical logic).

The third type of outline response for lines 3 and 4 relied on algebraic representation. That is, the outline focused on the specifics of the proof to describe how the presented proof arrived at a contradiction, which is indicative of an Action conception of proof by contradiction. While this type of outline would not be a general outline for the type of statement (nonexistence) for the presented proof, it would be a desired outline of the presented proof. In addition, modifications of the outline in Figure 4.10 would suggest the group focused on the logical argument of the proof. Therefore, outlines that relied on algebraic representations to describe how the proof arrived at a contradiction are indicative of a Process conception of mathematical logic. Groups G5, G7, G8, G9, G13, and G14 responded in this way. An example of this category of response follows.

Group G8 provided the most complete outline of the presented proof while utilizing algebraic representations to describe how the proof arrived at a contradiction (see Figure 4.13).

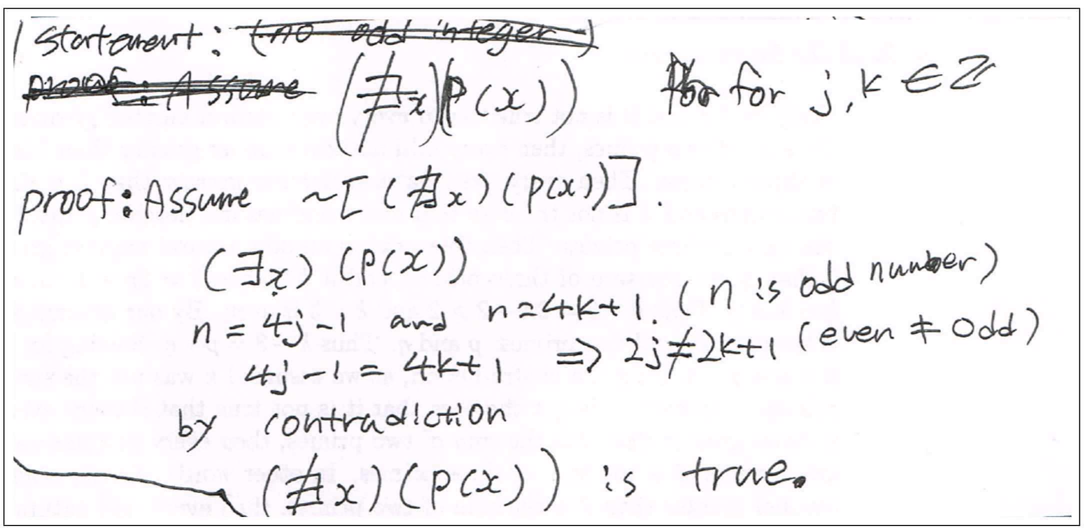


Figure 4.13 G8's outline for the presented proof in Activity 2.

G8's representation utilized predicate logic for line 1 (Assume $\sim [(\nexists x)(P(x))]$) and line 2 ($(\exists x)(P(x))$). These lines addressed the first key step of a proof by contradiction: assuming the statement is true. Then, the group represented $P(n)$ algebraically as $n = 4j - 1$ and $n = 4k+1$, after which their representation described how setting these values equal produced a number that is both even and odd, which is a contradiction. The equation $2j \neq 2k + 1$ was interpreted as representing that a number cannot be both even and odd, which was further clarified in the parentheses. These lines, all represented algebraically, addressed the second key step of a proof by contradiction: using the initial assumption and mathematical knowledge, arrive at a contradiction. G8 then concluded that "by contradiction $(\nexists x)(P(x))$ is true." The phrase "by contradiction" suggests the previous lines of the proof satisfied the requirements of the proof method. This line addressed the relation between the two key steps: since assuming the negation of the statement arrived at a contradiction, the statement must be true. In other words, G8's representation described the two key steps of a proof by contradiction without relying on an external, step-by-step procedure and described how these two lines prove the statement. Therefore, G8 exhibited a Process conception of mathematical logic.

4.2.1.2 Definition of proof by contradiction with quantification The second task of Activity 2 asked groups to write a definition for proof by contradiction. As in teaching episode 1, groups were prompted to reflect on the presented proof to consider the necessary steps to construct any proof by contradiction. The goal of this task was to encourage groups to reflect on their knowledge of proof by contradiction from teaching episode 1 and a new outline for a particular proof by contradiction to establish the necessary steps to construct a proof by contradiction in general. While it was not explicitly included in the written task, the teacher/researcher prompted groups to reflect on the presented proof by contradiction to write a definition for the proof method.

An ideal definition for proof by contradiction would include the two key steps for the proof method: (1) assume the statement to be proved is not true and (2) arrive at a contradiction. This definition is the same as the ideal definition from Activity 1 and was used again to reinforce the development of the formal definition of proof by contradiction introduced previously. What follows are groups' responses to defining proof by contradiction.

Groups' definitions for proof by contradiction were classified into four categories: generic indirect proofs (G7 and G11), proofs that start with the negation of a statement (G12 and G14), proofs that contradict themselves (G5 and G6), and formal definitions presented during

Classroom Discussion 1 (G8, G9, G10, and G13). The first three categories were the same as the classification of definitions in Activity 1 and were discussed in Section 4.1.1.2. Therefore, these explanations will not be repeated here. Instead, I will present a detailed description and interpretation of the fourth category of responses as well as provide an example to illustrate this category of response.

The fourth category of definitions represents groups who stated the formal definition. This definition included the key steps of a proof by contradiction: assuming the statement is not true and arriving at a contradiction. In addition, the definition was described in almost the exact same way as the formal definition stated by the teacher/researcher in Classroom Discussion 1: Prove that the statement is true by assuming the negation and proving this assumption is false with a contradiction. Providing a memorized definition is indicative of an Action conception of proof by contradiction. Groups G8, G9, G10, and G13 responded in this way. An example of this category of definition follows.

Group G10 provided the most clear formal definition of proof by contradiction: "Prove the statement, by arriving at a contradiction when assuming the negation of the statement is true." This definition included both key steps of a proof by contradiction and provides how these two key steps logically relate to prove the statement. This definition also uses nearly the same phrases as the definition given by the teacher/researcher, though the definition switches the order of the negation and contradiction. Therefore, G10 exhibited an Action conception of proof by contradiction.

4.2.1.3 Comparing procedures for proof by contradiction The third and final task of Activity 2 asked groups to compare the presented statements and proofs from Activity 1 and the first task of Activity 2. The goal of this task was to modify or reinforce groups' conception of the necessary steps for any proof by contradiction. This task prompted groups to reflect on the similarities and differences between two proofs by contradiction. Steps that are similar for each proofs, such as the initial assumption that the statement is false, could then be considered necessary for any proof by contradiction. Steps that are different for each proofs, such as the contradiction's relation to the initial statement, could then be unnecessary for every proof by contradiction. The identification of necessary and unnecessary steps in any proof by contradiction would either modify or reinforce groups' conception of the necessary steps for any proof by contradiction.

While it was not written on the task, the teacher/researcher encouraged groups to also compare their outlines for the presented proofs. In this way, groups were encouraged to coordinate these general steps for specific structures of statements to construct general steps for any proof by contradiction and thus possibly develop a Process conception of the proof method. What follows are groups' responses to the task of comparing and contrasting the presented proofs between the two teaching episodes.

Groups needed some initial prompting and additional explanation from the teacher/researcher of what was meant by comparing and contrasting the presented statements and proofs. Even after additional clarification, most groups either superficially compared and contrasted the two statements (e.g., describing the mathematical content of each proof) or did not respond at all. An example of this type of response follows.

Group G10 (consisting of K2, K10, and K15) initially could not compare the two presented proofs. An excerpt of their discussion is provided below.

K10: I don't know this. I don't know how to do this. [...] I don't know how they are similar.

K15: I wrote how they are different right now, first. [...] No I don't know what these questions are about *[K10 laughs]* because I don't know exactly what they are asking you to do.

K2: Yeah.

K15: I can prove it on my own, but... they give the proof in the set up and ask us weird questions about it. I don't like that.

The group voiced uncertainty about what the question asked them to do. K15 suggested starting with the differences, which he wrote were "Claim 2 is a conditional statement and

Claim 1 is a 'not-and" kind of proposition." That is, he identified the structure of the statements as the only difference between the two presented statements and proofs. For similarities, he wrote that they were both proofs by contradiction. These types of differences and similarities are valid but do not provide any insight into the proof method and thus are considered superficial comparisons.

After additional prompting to focus on the outlines and ignore the presented proofs, three groups (G10, G13, and G14) were then able to compare the form of lines in the two presented outlines. In particular, these groups identified the first line of each proof as assuming the statement is not true, identified that each proof contained a contradiction, and identified the second-to-last line of each proof as negating the initial assumption. However, these groups were not able to synthesize the lines between these as "using the assumption and mathematical knowledge, arriving at a contradiction" which describes, in general, how the proofs arrived at a contradiction. In other words, these groups focused on the outlines of the presented proofs in order to compare the forms of specific lines, which is indicative of an Action conception of proof by contradiction. An example of this type of comparison is provided below.

After prompting to compare the outlines of the presented proofs, one member of group G10, K10, explained similarities between the two outlines to the rest of the group. An excerpt of this discussion is provided below.

K2: So what are we supposed to do with that *[the two outlines]*?

K10: So every contradiction is going to do this *[pulls up outlines]* like where they assume to not be true.

K2: Okay.

K10: And use the negation. But, I don't know. It's like the same thing except for this part *[points to algebra on outline]*. Because you can't get to one answer the same way you get to another.

K10 first compared the initial line in both outlines and generalized the step as assuming the statement is not true. K10 then stated "And use the negation" which referred to the line after reaching a contradiction in which one takes the negation of the initial assumption. The phrase "you can't get to one answer the same way you get to another" referred to how the presented proof arrived at a contradiction and suggests K10 could not describe, in general, how the proofs arrived at a contradiction. No other group members added to the similarities and differences between the outlines of the proofs. Thus, the group focused on the form of specific lines in the outlines of each proof in order to compare and contrast the presented proofs. Therefore, G10 exhibited an Action conception of proof by contradiction.

4.2.2 Formalization of proof by contradiction with quantification

This subsection is a report about groups' responses during Classroom Discussion 2 (see Appendix A for general questions that guided this discussion). The purpose of this phase was to guide students into formalizing the initial conjectures from Activity 2. Therefore, the tasks for this phase are based on the three tasks of Activity 2. For each task, I will first present the goal and reasoning behind the task. Then, I will present an analysis and interpretation of groups' responses to the task.

4.2.2.1 Discussion of outline of a nonexistence proof by contradiction The teacher/researcher began the second phase of teaching episode 2 by asking groups to share their outlines of the presented proof using predicate logic. During this part, groups suggested an individual line that the teacher/researcher then wrote on the board. If all groups agreed on the line, they were prompted to share the next line.

All groups initially agreed on an incomplete outline of the proof (see Figure 4.14).

Statement: $(\nexists x)(P(x))$

1. Assume $\sim ((\nexists x)(P(x)))$
2. $(\exists x)(P(x))$
3. $P(n)$
4. Contradiction
5. $\sim (\sim (\nexists x)(P(x)))$
6. $(\nexists x)(P(x))$

Figure 4.14 Agreed-upon logical outline of presented proof during Classroom Discussion 2.

Similar to the second phase of teaching episode 1, the teacher/researcher then prompted groups for the logical justification of the each line. The goal of this task was to formalize and logically complete the agreed-upon outline. In particular, this outline is the first to provide a general contradiction (i.e, a contradiction that does not directly relate to the statement) and thus a significant focus of discussing the outline was focused on the contradiction.

Groups were able to provide a logical justification for all lines except for line 4: contradiction. Line 4 is a synthesis of the algebraic manipulation that results in a number, $2j$, that is both even and odd. In other words, line 4 of the outline provides the contradiction to the proof. The exact lines that correspond to line 4 from the outline are:

> Then $4j - 1 = 4k + 1$ and so $2j = 2k + 1$. Note that $2j$ is an even number and, since $2j = 2k + 1$, $2j$ is an odd number. A number cannot be both even and odd and thus this is a contradiction.

As noted when reviewing the design of this presented proof, the contradiction is that the number $2j$ is both even and odd. The number $2j$ is not n - the specific value we initially assumed existed. In other words, the contradiction is not part of the initial assumption but is a consequence of this assumption. A desired justification for the contradiction would describe how assuming that if such a number n existed, there would exist another number that was both even and odd, which is a contradiction as it is a well known theorem that an integer cannot be both even and odd. That is, $P(n)$ implies a contradiction that does not directly involve the assumed number n. The two categories of justification for line 4 that emerged from an analysis of groups' responses follow.

The first type of response was to state the contradiction was directly related to the initial assumption. As previously stated, the presented proof was selected as the contradiction is not directly related to the initial assumption (i.e., the contradiction does not directly involve the initial statement nor the assumed element n). This would suggest the group did not describe the meaning of the contradiction statement. Instead, the response suggests the group relied on something other than mathematical logic to respond. As relating the contradiction directly to the initial assumption was a category of response for the definition of proof by contradiction, it is likely the group leveraged their understanding of proof by contradiction in order to justify the contradiction step of the presented proof and thus exhibited an Action conception of proof by contradiction. An example of this type of response follows.

When prompted to state the contradiction of the presented proof in Figure 4.14 (The number $2j$ is both even and odd), group G10 stated "There exists $P(x)$ and there does not exist $P(x)$?" where $P(x)$ referred to the representation "x is an integer that can be expressed in the form $4j-1$ and $4k+1$." Note that $(\exists x)(P(x))$ represents the initial assumption of the presented proof and thus G10 stated that the contradiction related directly to the statement to be proved. G10 appeared to utilize what they understood to be a contradiction in a proof by contradiction (a direct contradiction with the assumption) rather than interpret and logically justify the statement in the presented proof. Therefore, G10 exhibited an Action conception of proof by contradiction when asked to justify the contradiction step.

The second type of response was to justify the contradiction using mathematical knowledge. Algebraically manipulating the assumption $(\exists x)(P(x))$ (i.e., assuming such an n exists and setting the two forms equal to each other) provided a contradiction in the form of another number, $2j$, being both even and odd. That is, the presented proof does not call on another line of the proof to logically justify the contradiction but instead relies on a commonly known theorem that no integer can be both even and odd, which is indicative of an Action conception of mathematical logic. An example of this type of response follows.

When prompted to state the contradiction of the presented proof in Figure 4.14, group G14 stated "Oh they said that $2j$ is an even number but since it's plus 1, it's an odd number. So contradiction." In this response, G14 focused on the number $2j$ and stated why it is both even and odd. G14 did not describe why this would be a contradiction, however, and thus relied on a memorized mathematical theorem. Therefore, G14 exhibited an Action conception of mathematical logic when asked to justify the contradiction step.

4.2.2.2 Discussion of definition for proof by contradiction After the outline of the presented proof was discussed, groups were prompted to share their definitions of proof by contradiction from the previous phase. The purpose of this task was to agree on a formal definition for proof by contradiction based on the tasks and discussions of Activity 2. An ideal definition for proof by contradiction still included the two key steps of the method: assuming the statement is not true and using this assumption as well as other mathematical knowledge, arrive at a contradiction.

One group offered a formal definition, that was established in Classroom Discussion 1, and all groups agreed on this definition. Therefore, all groups were provided a formal definition of proof by contradiction.

4.2.2.3 Discussion of comparison between proof by contradiction procedures After formalizing the definition of proof by contradiction, teacher/researcher prompted groups to compare the outlines from teaching episodes 1 and 2 (see Figure 4.15) to develop a series of steps to write any proof by contradiction. The goal of this task was to encourage groups to construct a step-by-step procedure for any proof by contradiction and thus possibly develop a Process conception of proof by contradiction.

Classroom Discussion 1 **Statement:** $P \rightarrow Q$	**Classroom Discussion 2** **Statement:** $(\nexists x)(P(x))$
1. Assume $\sim (P \rightarrow Q)$ 2. $P \wedge \sim Q$ 3. $\sim Q_k$ 4. $(\sim Q_k \wedge P) \rightarrow Q_k$ 5. Q_k 6. $Q \wedge \sim Q$ 7. $\sim (\sim (P \rightarrow Q))$ 8. $P \rightarrow Q$	1. Assume $\sim (\nexists x)(P(x))$ 2. $(\exists x)(P(x))$ 3. $P(n)$ 4. Using $P(n)$, get to a contradiction. 5. $\sim (\sim (\nexists x)(P(x)))$ 6. $(\nexists x)(P(x))$

Figure 4.15 Side-by-side outlines from Classroom Discussion 1 (left) and Classroom Discussion 2 (right).

After groups compared and contrasted the outlines above in general, they were prompted to compare specific pairs of lines from the outlines to generalize the purpose of these lines in the outline. By looking across the two proof outlines, one may obverse the following commonalities among some lines or groups of lines. For example, lines 1 and 2 can be grouped together to describe assuming the negation of the statement is true while the final two lines can be grouped together to describe the assumption as false due to arriving at a contradiction.

A desired series of steps would contain three general purposes or roles: (1) Assuming the negation of the statement is true, (2) Using the initial assumption and mathematical knowledge to arrive at a contradiction, and (3) Since assuming the negation of the statement arrived at a contradiction, the statement must be true. These roles synthesis multiple lines of a particular proof to describe the purpose of a collection of lines. How groups explained these general purposes or roles would indicate their conception of proof by contradiction. A detailed description of how the series of steps would differ between a group with an Action conception and a group with a Process conception of proof by contradiction follows.

A group with an Action conception of proof by contradiction would describe the necessary steps of the method by cue words and without any further description of the logic behind the proof method. For example, step 2 may simply be stated as ‘contradiction’ or step 3 may just refer to the double negation of a statement and therefore the statement is true. The group would need to write the details for each step of a particular proof and could not skip steps in the procedure. In addition, they would not be able to give the logical relation between steps of the procedure.

A group with a Process conception of proof by contradiction would describe the necessary steps of the proof but would also describe, in their own words, intermediate steps between these necessary steps that provide reasoning behind the method. For example, a step after the initial assumption could state "Use the assumption and mathematical knowledge to derive a contradiction." This example step would provide logical reasoning for a group of steps in the context of the proof method. Using the reasoning between steps, the group could describe how to proceed in a particular proof without the need to write the details of that proof.

Groups' responses to the task of comparing the outlines in Figure 4.15, both in general and by specific pairs of lines, are provided in Table 4.1. The symbol "S" was provided by the teacher/researcher to represent any statement.

Table 4.1 General procedures for a proof by contradiction created by each group during Class Discussion 2.

G5-G12	**G13**	**G14**
1. Assume $\sim S$. 2. $\vdots$ 3. $\rightarrow\leftarrow$ 4. $\sim(\sim S)$ 5. S	1. Assume $\sim S$. 2. Rewrite $\sim S$. 3. Look at specific value of step 2. 4. Work (Algebra). 5. Get Contradiction. 6. $\sim$ Assumption. 7. S	1. Assume $\sim S$. 2. Negate variables one at a time. 3. Use theorems to manipulate it until you find a contradiction. 4. $\sim(\sim S)$. 5. S

When prompted to compare the outlines in Figure 4.15, groups G5, G6, G7, G8, G9, G10, G11, and G12 first generalized the initial assumption ($\sim S$), the contradiction ($\rightarrow\leftarrow$), and the completion of the proof ($\sim\sim S$, therefore S). However, these groups did not describe any of the intermediate steps. When prompted to compare specific pairs of lines in the outlines in Figure 4.15, these groups continued to state the steps that were already provided were sufficient. These groups did not describe the intermediate steps nor the reasoning for the proof method and thus exhibited an Action conception of proof by contradiction.

When prompted to compare the outlines in Figure 4.15, groups G13 and G14 generalized the initial assumption ($\sim S$), the contradiction ($\rightarrow\leftarrow$), and the completion of the proof ($\sim\sim S$, therefore S) in a similar as the other groups. However, when prompted to compare specific pairs of lines in the outlines in Figure 4.15, these groups described, in their own words, intermediate steps that satisfied the second general purpose of lines in a proof by contradiction (Using the initial assumption and mathematical knowledge, arrive at any contradiction). The ability to explain, in one's own words, the key steps and how these steps related is indicative of a Process conception of proof by contradiction. An example of this explanation follows.

Group G14 (consisting of Nora and Ren) began by comparing the two outlines line-by-line. Their first two comparisons are provided below.

Nora: So you are assuming that something is not true. Yeah that's basically the same thing, obviously.

Teacher: Yeah! So then what?

Nora: It looks like it breaks it down, that is...

Teacher: What do you mean by breaks it down?

Nora: Like, it negates it one set at a time.

Nora's first comparison, "so you are assuming something is not true", generalizes the first line in each outline. The phrase "negates it one set at a time" refers to a step-by-step negation of $P \rightarrow Q$ (i.e., P stays the same, '$\rightarrow$' becomes an $\wedge$, and Q becomes $\sim Q$) and a step-by-step negation of $\nexists x)(P(x))$ (i.e., $\nexists x$ becomes $\exists x$ and $P(x)$ stays the same). In other words, it referred to the rule that negating a complex statement requires negating each part of the statement separately and is indicative of an Action conception of mathematical logic.

Group G14 then compared lines 3-6 from Classroom Discussion 1 to lines 3-4 from Classroom Discussion 2 to describe, in general, how the method arrives at a contradiction. Their discussion is provided below.

Teacher: Okay, I'd agree with that. Then what? *[long pause]*

Nora: Use theorems to... *[pause]*

Teacher: Okay, use theorems to... to what? *[pause]*

Nora: Umm *[pause]* to figure out if it's true or not?

Teacher: Okay. What about *[points at Ren]*

Ren: You use theorems to manipulate it until you find a contradiction.

Nora synthesized these lines as "use theorems to figure out if it's true or not." This is not a desirable generalization as it did not provide a clear explanation of the purpose of these lines (to arrive at a contradiction). Ren's generalization "You use theorems to manipulate it until you find a contradiction" was desired as it clearly provided the purpose of these lines in the proof.

Finally, when asked to generalize the steps after the contradiction, Ren stated "You went back to your first statement and with the contradiction, you concluded that your first statement cannot be true." The phrase "your first statement" referred to the initial assumption. In this way, Ren's generalization describes the two key steps (initial assumption and contradiction) and described, in his own words, how these steps relate to prove the statement is true ("with your contradiction, you concluded that your first statement cannot be true"). Group G14 then wrote the final two lines of their general procedure to represent this generalization: $\sim(\sim S)$ and S. Therefore, G14 described, in their own words, the two key steps of a proof by contradiction and explained how these steps related to prove the statement, which is indicative of a Process conception of proof by contradiction.

4.2.3 Reinforcing formalization of proof by contradiction with quantification

This subsection is a report about groups' responses to tasks from Exercise 2 (see Appendix A for the complete set of tasks) - the last phase of teaching episode 2. The goal of this phase was to reinforce groups' conceptions of mathematical logic and proof by contradiction through a series of eight tasks aligned with the proof comprehension assessment model by Mejía-Ramos et al. (2012). The comprehension questions were written for the presented proof (Figure 4.9) and are provided in Figure 4.16.

1. Please give an example of an integer that is odd and explain why it is odd.
2. What kinds of numbers can be expressed in the form $4j - 1$?
3. Why exactly can we assume "there is an odd integer n such that $n = 4j - 1$ and $n = 4k + 1$ for integers j and k."?
4. What is the purpose of the statement "Note that $2j$ is an even number and, since $2j = 2k + 1$, $2j$ is an odd number."?
5. Summarize in your own words the main idea of this proof.
6. What do you think are the key steps of the proof?
7. In the statement, we have $4j - 1$ and $4k + 1$. Would the proof still work if we instead say no odd integer can be expressed in the form $4j - 3$ and in the form $4k + 3$? Why or why not?
8. Using the method of this proof, show that there is no odd integer that can be expressed in the form $8j - 1$ and in the form $8k + 1$ for integers j and k.

Figure 4.16 Comprehension questions for the presented proof in Exercise 2.

Groups were first given approximately 20 minutes to respond to the comprehension questions. Then, the teacher/researcher asked groups to share their answers and describe the reasoning behind their response to each question. After eliciting the groups' response and reasoning behind their response, the teacher/researcher provided a desired response to the question (if necessary).

The remainder of this subsection will be organized by comprehension questions one through eight. For each question, I will first describe what it was meant to assess in terms of the proof comprehension assessment model by Mejía-Ramos et al. (2012) (described on page 23). Then, I will describe the goal of the assessment in terms of the students' understanding of the presented proof, mathematical logic, or proof by contradiction. Finally, I will present an analysis and interpretation of groups' responses to the question.

Question one assessed whether groups could identify the meaning of terms and statements in the presented proof. In particular, the question asked groups to provide an example of an odd integer and explain why this integer is odd. The goal of this question was to determine whether groups understood one of the mathematical terms in the proof (odd integer) that was necessary to understand the logical argument of the proof, as the literature suggested students struggle with proof comprehension due to a lack of mathematical knowledge (e.g. Moore, 1994). In other words, the goal of this question was to assess prerequisite knowledge necessary to understand the particular proof.

All groups provided an example and a variation of the explanation "A number of the form 2n+1" without any further clarification. This definition was sufficient for the purposes of the proof, as the proof only required the knowledge that $2k+1$ is an odd integer. Therefore, all groups possessed the prerequisite knowledge of odd integers necessary to understand the particular proof.

Question two assessed whether groups could identify the meaning of terms and statements in the presented proof. The goal of this question was to assess whether students recognized that not all odd integers could be written in the form $4j - 1$. Responses to this question would indicate whether groups understood the statement being proved: There is no odd integer that can be expressed in the form $4j-1$ and in the form $4k+1$ for integers j and k. In other words, the goal of this question was to assess prerequisite knowledge necessary to understand the statement being proved.

Some groups (G5, G7, G10) noted the first few positive odd integers of the form $4j-1$: 3, 7, 11, 15, etc. By writing out specific examples, these groups indicated that not all odd integers could be written in the form $4j - 1$. This suggests these groups could identify the meaning of the statement proved. However, all other groups responded that all odd numbers could be written in the form $4j - 1$, which suggested the statement proved would be false. Therefore most groups could not identify the meaning of the statement being proved.

Question three assessed whether groups could identify the logical status of a statement in the proof. In particular, the question focused on the logical status of the statement that a particular odd integer existed that was both in the form $4j - 1$ and $4k + 1$, which is represented as $P(n)$ in the outline in Figure 4.10 (see page 107). The goal of this question was to assess whether groups recognized this statement was false (in general) and could therefore describe the logical purpose of the statement instead. Responses to this question would indicate groups' conception of mathematical logic as it required students to logically justify a false statement within the context of the proof. A desired response would indicate the statement follows from the initial assumption.

Most groups (G6, G8, G9, G11, G12, G14) justified the statement with the claim that all odd integers could be written in both forms. According to this claim, a single odd integer existed that could be written in both forms. These groups did not recognize the statement was false (in general) and, because of this, justified the false statement with a false mathematical claim. Justifying a false statement with a false mathematical claim is indicative of a Pre-Action conception of mathematical logic. For example, G6 stated "That any number plugged in for either j or k is odd." This response suggested G6 considered each equation separately (e.g., $n = 4j - 1$ and $n = 4k + 1$) and only checked that each value was odd. In other words, they did not coordinate these values to determine they cannot be equal and thus consider a logical justification for the statement.

Some groups (G5, G7, and G10) recognized not all odd integers could be written in the form $4j - 1$ but did not indicate the statement followed from the initial assumption. Instead, these responses suggested that some odd integers could be written in the form $4j - 1$ and that others can be written in the form $4k + 1$. That is, the groups did not recognize that the statement referred to a single odd integer that could simultaneously be written in both forms. For example, G5 stated "Because not all numbers can be formed using $4j-1$ but other odd numbers can be written as $4k+1$." The phrase "other odd numbers" suggests the groups recognized that *some* of the odd integers (but not all) could be written as $4k+1$ and that this may overlap with the odd integers that can be written as $4j-1$. In other words, the group considered that while the two forms did not completely overlap, they could have some overlap and thus such an odd integer could exist.

Only one group, G13, provided a desirable logical justification for the statement. G13 stated "b/c we're doing proof by contradiction and we assume the statement is not true." The phrase "we assume the statement is not true" referred to the initial assumption (i.e., that the statement is not true) and thus provided the desired logical justification for assuming the statement is true. The phrase "b/c we're doing proof by contradiction" suggests the logical justification relied directly on the group's procedure for proof by contradiction, which is indicative of an Action conception of the proof method.

Question four assessed whether groups could identify the logical status of a statement in the proof. This time, groups were asked for the purpose of the statement "Note that $2j$ is an even number and, since $2j = 2k+1$, $2j$ is an odd number" which described the contradiction of the proof. The goal of this question was to assess whether groups could describe the logical purpose of the contradiction statement. Responses to this question would indicate groups' conception of proof by contradiction, as the contradiction step is one of two key steps for the proof method.

All groups identified that this statement was the contradiction without further explanation. That is, no group described the purpose of this contradiction in the presented proof (i.e., that the initially assumption and contradiction together imply the statement is true). Therefore, all groups exhibited an Action conception of proof by contradiction.

Question five assessed whether groups could summarize the proof via high-level ideas. The purpose of this question was to assess whether groups could summarize the proof in their own words and, if they could, how they summarized the proof. For example, a summary for the presented proof could be similar to the following:

> It's a proof by contradiction, where you assume the statement isn't true. So then there is a number that can be written in both forms and you can set them equal to each other. But then you get that an even number is an odd number, which is a contradiction, so the statement is true.

This summary describes the specific procedure of the presented proof and would illustrate an Action conception of proof by contradiction. In contrast, another summary could be similar to the following:

> If the statement were false, then there would be a number that was both even and odd, which cannot be true. Therefore, the statement must be true.

This response describes the main idea of the proof (e.g., a number in both forms would necessarily mean there is another number that is both even and odd) and provides a generalization of the two key steps of a proof by contradiction, which is indicative of a Process conception of the proof method. Therefore, responses to this question would indicate groups' conception of proof by contradiction.

All written responses either rephrased the initial statement (e.g., G6 stated "That there is no odd integer that can be express *[sic]* by $4j-1$ and $4k+1$.") or noted the proof method used (e.g., G7 stated "The main idea of this proof is to show a proof by contradiction, i.e. to show that... *[does not finish though]*"). These responses indicated groups could not describe the proof, neither by reciting the procedure of the proof nor by citing the general ideas of the proof. In addition, the majority of groups left this question blank and did not respond. Therefore, all groups exhibited a Pre-Action conception of proof by contradiction.

Question six assessed whether groups could identify the modular structure of the proof. In other words, whether groups could group sentences in the proof together and provide a purpose for each of these groups. By phrasing the question as "key steps", the teacher/researcher attempted to illicit the purpose of particular lines and how these purposes provided a complete proof. Responses to this question would indicate groups' conception of proof by contradiction. In particular, groups that responded with lines from the desired outline (see Figure 4.10 on page 107) and did not describe the purpose of these lines in the overall argument of the proof would have exhibited an Action conception of proof by contradiction, while groups that responded with lines from the desired logical outline and described the purposes of these lines in the overall proof would have exhibited a Process conception of proof by contradiction.

All groups responded with at least the assumption line and contradiction line: the two key steps of a proof by contradiction. These responses were not accompanied by any explanation of the purpose of these lines in the overall proof and thus all groups exhibited an Action conception of proof by contradiction. For example, G13 stated "Assume $\sim$ statement, Get a contradiction." This response indicated the two key steps of a proof by contradiction but did not describe how these steps related to prove the statement is true.

Question seven assessed whether groups could transfer the general idea or method to another context. In particular, the question assessed whether groups appreciated the scope of the method (i.e., recognized the assumptions that needed to be in place to allow the method to be carried out). Changing $4j-1$ and $4k+1$ to $4j-3$ and $4k+3$ does not change the procedure of the proof (i.e., the two forms can be set equal and it is still shown that $2j$ is an even and odd number) nor does it change the meaning behind the proof (e.g., integers of the form $4j-1$ are integers of the form $4k+3$). Responses to this question would indicate groups' conception of mathematical logic. In particular, groups that responded the procedure would not change would have exhibited an Action conception of mathematical logic, while groups that responded the meaning of the statements would not change would have exhibited a Process conception of mathematical logic.

Groups responded to question seven in two ways: provided an inappropriate justification for why the change of forms was invalid (G5, G6, G7, G11, G12) and provided an appropriate response to justify that the procedure would still work (G8, G9, G10, G13, G14). Groups that provided an inappropriate justification for why the proof would not work focused on modifying $4j-1$ and $4k+1$ to arrive at the suggested forms $4j-3$ and $4k+3$ and therefore exhibited a Pre-Action conception of mathematical logic. For example, G6 responded "No, because both sides don't remain balanced. $4j-1$ when down 2 but $4k+3$ when up by 2." In other words, G6 compared $4j-1$ to $4j-3$ and $4k+1$ to $4k+3$ as sides of an equation. In contrast, groups that stated the procedure would still work provided the algebra of the modified proof and thus exhibited an Action conception of mathematical logic. For example, G13 set $4j-3=4k+3$ and showed that $2j$ is still an even and odd number. These groups did not describe whether $4j-3$ and $4k+3$ are still odd numbers or if these odd numbers are the same as $4k-1$ and $4j+1$.

Question eight also assessed whether groups could transfer the general idea or method to another context, this time in the form of writing a similar proof. This allowed groups to utilize either the specific outline of the similar proof in Figure 4.10 or the general outline for any proof by contradiction in Figure 4.1 to write a proof by contradiction. Responses to this question would indicate groups' conception of proof by contradiction. In particular, groups with an Action conception of proof by contradiction would follow one of the two step-by-step outlines to prove the similar statement, while groups with a Process conception of proof by contradiction would use their internalized steps to prove the statement and may skip explicit steps of the outline in either Figure 4.10 or Figure 4.1. A desired proof for question 8 is presented in Figure 4.17.

Statement: There is no odd integer that can be expressed in the form $8j-1$ and in the form $8k+1$ for integers j and k.

Proof: Assume the statement is false. That is, assume there is an odd integer that can be expressed in the form $8j-1$ and in the form $8k+1$ for integers j and k [**1**]. Let n be that integer; that is, $n \in \mathbb{Z}$ such that $n = 8j-1$ and $n = 8k+1$ for $j, k \in \mathbb{Z}$. Then $8j-1 = 8k+1$, so $8j = 8k+2$, and thus $4j = 4k+1$. Note that $4j$ is an even number and, since $4j = 4k+1$, $4j$ is an odd number. A number cannot be both even and odd and thus this is a contradiction [**2**]. As the assumption lead to a contradiction, the assumption is false [**3**]. In other words, there is no odd integer that can be expressed in the form $4j-1$ and in the form $4k+1$ for integers j and k.

Figure 4.17 Desired proof for question eight in Exercise 2.

The proof above includes both key steps of a proof by contradiction: assuming the statement is not true (denoted as [**1**]) and arriving at a contradiction (denoted as [**2**]). In addition, the proof explains how these key steps logically imply the statement is true (denoted as [**3**]). Finally, the proof is valid in that all lines are properly justified. Therefore, proofs that satisfy these three criteria, such as the proof in Figure 4.17, are considered to be desired proofs for the statement in question 8.

Groups responded[6] to question eight in three ways: with an undesired use of the outline from Activity 2 (G9, G12), desired use of the outline from Activity 2 (G8, G10), and only provide the algebra for the contradiction (G5, G6, G13). The first type of response to question eight was an undesired use of the outline from Activity 2. These responses copied the outline for the three general steps of a proof by contradiction (the initial assumption, that a contradiction was reached, and the statement is thus true) but did not modify the similar proof. That is, these groups did not complete the algebra manipulation to arrive at a number that was both even and odd. This inability to utilize an external set of steps to write a similar proof is indicative of a Pre-Action conception of proof by contradiction. An example of this type of response follows.

Group G9 provided the most clear use of the outline from Activity 2 and did not modify the outline for the particular proof (see Figure 4.18).

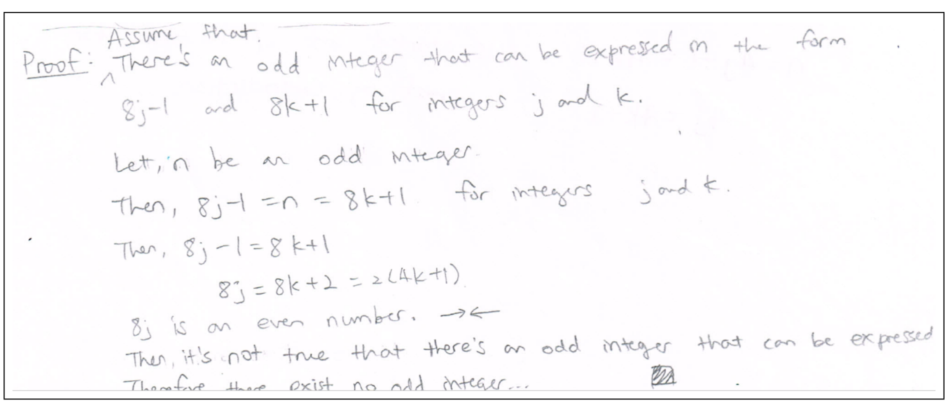

Proof: Assume that There's an odd integer that can be expressed in the form
$8j-1$ and $8k+1$ for integers j and k.
Let, n be an odd integer.
Then, $8j-1=n=8k+1$ for integers j and k.
Then, $8j-1=8k+1$
$8j=8k+2=2(4k+1)$
8j is an even number. →←
Then, it's not true that there's an odd integer that can be expressed
Therefore there exist no odd integer...

Figure 4.18 G9's proof for question 8 in Exercise 2.

[6]Groups G7, G11, and G14 did not respond to this question and thus exhibited a Pre-Action conception of proof by contradiction.

All three of the general steps to prove a statement by contradiction are present: the initial assumption that the statement is not true ("Assume that there's an odd integer that can be expressed in the form $8j - 1$ and $8k + 1$ for integers j and k"), that a contradiction was reached ("$8j$ is an even number. $\rightarrow\leftarrow$"), and therefore the initial assumption is false and so the statement is true ("Then it's not true that there's an odd integer that can be expressed. Therefore there exist no odd integer..."). However, that $8j$ is an even number is not a contradiction - the group was expected to then divide $8j$ by 2 and thus show that $4j$ is both even and odd. The additional step to divide $8j$ by 2 was not included in the original proof and thus was not noted in the outline of the proof. The absence of the extra step to arrive at a contradiction shows that G9 focused on the outline of the particular proof in Activity 2 without describing the meaning of the statements.

The second type of response to question eight was a desired use of the outline from Activity 2. In these responses, groups modified the specific algebra for the new proof to arrive at a contradiction. In addition, it was evident these responses utilized the outline as the group either told the teacher/researcher that they did or it was evident that they used the same symbolic representation of the argument and only modified the algebra steps. Responses that utilized a specific set of step-by-step instructions to write a proof illustrated an Action conception of proof by contradiction. An example of this type of response follows.

Group G8 provided the most clear use of the outline from Activity 2 with its appropriate modification for the particular proof (see Figure 4.19).

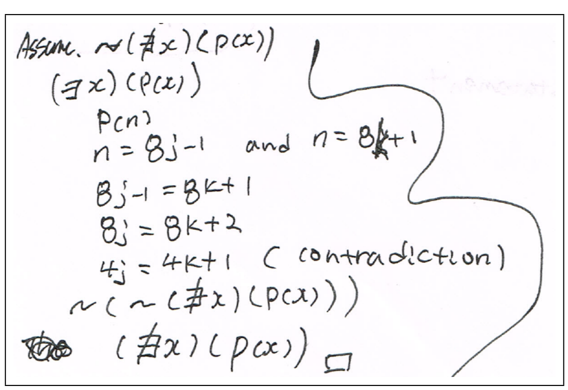


Figure 4.19 G8's proof for question 8 in Exercise 2.

The specific presentation of the symbolic representations of statements from the outline in Activity 2 were present. That is, the first three lines of G8's outline began in the same way as the outline from Activity 2: assume $\sim (\nexists x)(P(x))$ for line 1, $(\exists x)(P(x))$ for line 2, and $P(n)$ for line 3. In addition, the final two lines of G8's outline ended in the same way as the outline from Activity 2: $\sim (\sim (\nexists x)(P(x))$ for the penultimate line and $(\nexists x)(P(x))$ for the final line. Also, note that lines 3 and 4 represent the statement "Let n be that integer; that is, $n \in \mathbb{Z}$ such that $n = 8j - 1$ and $n = 8k + 1$ for $j, k \in \mathbb{Z}$" in different ways: line 3 represents the statement using predicate symbols and line 4 represents the statement using algebraic symbols. This redundancy suggests G8 copied the first three lines and final two lines from the outline in Activity 2, then appropriately modified the specific algebra between these lines to arrive at a contradiction.

The final type of response to question eight was to only provide the algebra to show a contradiction. These responses modified the specific algebra for the new proof to arrive at a contradiction as well as stated that the proof would otherwise follow the presented proof (see Figure 4.9 on page 106). Responses that provided a partial proof and could describe how the proof would be written without completing every line exhibited a Process conception of proof by contradiction. An example of this type of response follows.

Group G13 wrote the details for only one of the two key steps of a proof by contradiction: arriving at a contradiction (see Figure 4.20).

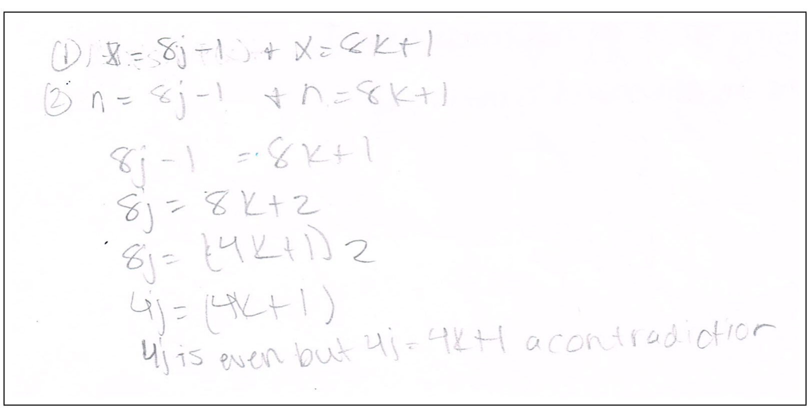


Figure 4.20 G13's proof for question 8 in Exercise 2.

The algebra in Figure 4.20 comprised steps 2-5 in the general procedure for proof by contradiction G13 constructed during Classroom Discussion 2 (see Figure 4.1 on page 121) which provided the details of how the proof arrived at a contradiction. In addition, G13 wrote only the lines that changed for the proof in question eight. This indicates the group skipped lines that were the same in each proof and only described the differences between the proofs. That is, the group did not need to write out every line of the proof and, in particular, did not need to explicitly write both key steps of a proof by contradiction to consider the proof completed.

4.2.4 Summary of results

Teaching episode 2 was designed to achieve three goals. First, groups were guided to construct a set of step-by-step instructions for proofs by contradiction for a nonexistence statement (i.e., $(\nexists x)(P(x))$). Groups could then use this set of step-by-step instructions to prove similar statements by contradiction and thus possibly develop an Action conception of proof by contradiction for this type of statement. Secondly, groups were encouraged to focus on the roles of groups of lines in a proof and on the key steps of the proof method. They could then relate the roles of lines or collections of lines to interiorize the previous step-by-step instructions into general steps for a nonexistence statement and thus possibly develop a Process conception of proof by contradiction for this type of statement. Thirdly, groups were encouraged to compare their general steps for an implication statements (from teaching episode 1) to their general steps for a nonexistence statement. Groups could then coordinate these general steps for specific structures of statements to construct general steps for any proof by contradiction and thus possibly develop a Process conception of proof by contradiction.

Subsection 4.2.1 reported on groups' work during Activity 2. The goals of this phase reflected the three goals of teaching episode 2: (1) guide groups to construct a set of step-by-step instructions for proofs by contradiction of a nonexistence statement, (2) prompt groups to relate the roles of lines, or collections of lines, to interiorize the previous step-by-step instructions into general steps for a nonexistence statement, and (3) encourage groups to coordinate general steps for an implication statement (from teaching episode 1) and a nonexistence statement to construct general steps for *any* proof by contradiction. Thus, the phase consisted of three tasks: (4.2.1.1) outlining a given proof by contradiction for a nonexistence statement, (4.2.1.2) defining proof by contradiction, and (4.2.1.3) comparing the presented proofs from Activity 1 and the previous task. A brief summary of the results from each task follows.

Three categories of responses emerged from an analysis of groups' outlines and, in particular, outline of how the proof arrived at a contradiction. The first was an undesired and/or unclear representation, which is indicative of a Pre-Action conception of mathematical logic. The second was a reliance on the outline from Classroom Discussion 1, which is indicative of an Action conception of proof by contradiction. The third was a desired representation that relied on algebraic manipulation, which is indicative of a Process conception of mathematical logic.

Four categories of responses emerged from an analysis of groups' definitions for proof by contradiction. The first definition explained the proof method as a generic indirect proof, which is indicative of a Pre-Action conception of proof by contradiction. The second definition explained the proof method as one that start with the negation of a statement, which is indicative of a Pre-Action conception of proof by contradiction. The third definition explained the proof method as one that contradict itself, which is indicative of an Action conception of proof by contradiction. Finally, the fourth definition explained the proof method in the same way as the formal definition presented during Classroom Discussion 1, which is indicative of an Action conception of proof by contradiction.

When asked to compare and contrast the two statements that were proved by contradiction, most groups either responded superficially or did not respond at all, which is indicative of a Pre-Action conception of proof by contradiction. However, three groups were able to compare the form of particular lines in the two presented proofs, which is indicative of an Action conception of proof by contradiction.

Subsection 4.2.2 reported on groups' responses during Classroom Discussion 2. The purpose of this phase was to formalize groups' conceptions of proof by contradiction based on their work and conjectures during Activity 2. Thus, the tasks for this phase were based on the three tasks of Activity 2. A brief summary of the results from each task follows.

When asked to provide and agree upon an outline of the presented proof, all groups initially agreed on an incomplete outline of the presented proof as it did not provide the logical justification necessary for how the proof arrived at a contradiction. Groups logically justified this line (the contradiction) in one of two ways. The first way was to state the contradiction was directly related to the initial assumption, which is indicative of an Action conception of proof by contradiction. The second way was to mathematically justify the contradiction, which is indicative of an Action conception of mathematical logic.

When asked to present their definitions of proof by contradiction, one group offered a formal definition that was established in Classroom Discussion 1. All groups subsequently agreed on this definition.

When asked to compare and contrast the outlines from Activity 1 and the previous task, the majority of groups generalized the initial assumption, the contradiction, and the completion of the proof. These groups did not describe any of the intermediate steps, which is indicative of an Action conception of proof by contradiction. Some groups, however, described the key steps of a proof by contradiction and how these steps related to prove the statement, which is indicative of a Process conception of proof by contradiction.

Subsection 4.2.3 reported on groups' work during Exercise 2. The goal of this phase was to reinforce groups' conceptions of mathematical logic and proof by contradiction through a series of eight comprehension questions (i.e., eight tasks) aligned with the proof comprehension assessment model by Mejía-Ramos et al. (2012). A short summary of results for each question follow.

Question one: When asked to provide an example of an odd integer and explain why it was odd, all groups provided an example and a variation of the explanation "A number of the form $2n + 1$ without any further clarification. Therefore, all groups possessed the prerequisite knowledge of odd integers necessary to understand the particular proof.

Question two: When asked what kind of numbers could be written in the form $4j-1$, some groups desirably indicated that not all odd integers could be written in this form and thus could identify the meaning of the statement being proved. However, the majority of groups responded that all odd numbers could be written in this form and thus could not identify the meaning of the statement proved.

Question three: When asked why the proof could assume an n existed such that $n = 4j-1$ and $4k+1$, most groups justified the statement with the false claim that all odd integers could be written in both forms, which is indicative of a Pre-Action conception of mathematical logic. Some groups recognized not all odd integers could be written in the form $4j-1$ and did not indicate the statement followed from the initial assumption, which is also indicative of a Pre-Action conception of mathematical logic. Finally, one group provided a desirable, logical justification for the statement based on the procedure of a proof by contradiction, which is indicative of an Action conception of the proof method

Question four: When asked for the purpose to the line in the presented proof that introduced the contradiction, all groups desirably identified the statement and did not provide further explanation of the purpose of the statement in the proof, which is indicative of an Action conception of proof by contradiction.

Question five: When asked to summarize the main idea of the proof, all groups either rephrased the initial statement, noted the proof method, or did not provide any response. All three of these types of responses are indicative of a Pre-Action conception of proof by contradiction.

Question six: When asked to provide the key steps of the proof, all groups responded with the two key steps of a proof by contradiction, which is indicative of an Action conception of proof by contradiction.

Question seven: When asked whether the proof would still be valid if $4j-1$ and $4k+1$ were changed to $4j-3$ and $4k+3$, groups responded in two ways. Some groups provided an inappropriate justification for why the proof would not work by algebraically manipulating

$4j - 1$ and $4k + 1$ in order to arrive at $4j - 3$ and $4k + 3$, which is indicative of a Pre-Action conception of mathematical logic. Other groups provided an appropriate justification for why the proof would work by setting the forms equal to each other and performing the same algebraic manipulations as featured in the proof, which is indicative of an Action conception of mathematical logic.

Question eight: When asked to write a similar proof using the method of the presented proof, groups responded in one of three ways. Some groups described the presented proof without appropriately modifying the algebraic manipulation of the new proof and thus did not reach a contradiction, which is indicative of a Pre-Action conception of proof by contradiction. Other groups described the previous proof and appropriately modified the algebraic manipulation of the new proof, which is indicative of an Action conception of proof by contradiction. Finally, groups provided only the appropriately modified algebraic manipulation of the new proof and skipped the parts of the proof that were nearly identical, which is indicative of a Process conception of proof by contradiction.

4.3 Teaching Episode 3: Refining proof by contradiction with quantification

Teaching episode 3 was designed to achieve three goals. First, the episode introduced a set of step-by-step instructions for students to use to construct proofs by contradiction for a uniqueness statement (i.e., $(\exists! x)(P(x))$). Students could then use this set of step-by-step instructions to prove similar statements by contradiction and thus possibly develop an Action conception of proof by contradiction for this type of statement. Secondly, students were encouraged to focus on the roles of groups of lines in a proof and on the key steps of the proof method. They could then relate the roles of lines or collections of lines to interiorize the previous step-by-step instructions into general steps for a uniqueness statement and thus possibly develop a Process conception of proof by contradiction for this type of statement. Thirdly, students reflected on both their prior knowledge of proof by contradiction and a specific example of proof by contradiction to assimilate[7] or accommodate[8] the new logical outline into their general procedure for a proof by contradiction.

This section will report on how students' understanding emerged from an analysis of their individual responses to the tasks and teacher/researcher prompts during teaching episode 3. In particular, data analysis for this section will focus on Wesley and Yara - representative students chosen for case study analysis (see Section 3.5 for details on why these two students were chosen).

The following subsections will be organized by the three phases of the teaching episode. Subsection 4.3.1 will focus on students' initial conjectures about proof by contradiction of a uniqueness statement. Subsection 4.3.2 will focus on how the teacher/researcher guided students to formalize the initial conjectures from the previous phase. Subsection 4.3.3 will focus on responses to proof comprehension questions designed to reinforce the formalized conjectures from the previous phase. For each subsection, I will begin by describing the lesson plan of the phase and then proceed by presenting the analysis and interpretation of data collected during the phase. The tasks used for this teaching episode can be found in Appendix A.1 and will be reproduced as needed to describe the lesson plan for each phase of teaching episode 3.

4.3.1 Initial conjectures on uniqueness proofs by contradiction

This subsection is a report about students' work during Activity 3 (see Appendix A.1 for the complete set of tasks). The goal of Activity 3 was to encourage students to construct a set of step-by-step instructions to prove uniqueness statements as well as encourage students to reflect on their knowledge of proof by contradiction and a specific example of proof by contradiction to assimilate or accommodate the new logical outline into their general definition of proof by contradiction. Thus, this phase consisted of two tasks: outline of a

[7] *Assimilation* occurs when an individual can apply a cognitive structure (proof by contradiction procedure) with minimal change in order to deal with a new situation.

[8] *Accommodation* occurs when an individual needs to reconstruct and modify a cognitive structure in order to deal with a new situation.

presented proof by contradiction of a uniqueness statement and provide a definition for proof by contradiction. For each task, I will first present the goal and reasoning behind the task. Then, I will present an analysis and interpretation of students' responses to the task.

4.3.1.1 Outline of a uniqueness proof by contradiction Students were asked to read a presented proof of the statement "The equation $5x-4=1$ has a unique solution." and subsequently outline this statement and proof (see Figure 4.21) utilizing propositional or predicate logic. As Wesley and Yara participated individually, the teacher/researcher acted as another student with incomplete knowledge during this phase. That is, the teacher/researcher provided memorized rules (e.g., $\sim(P \vee Q) \cong \sim P \wedge \sim Q$) when necessary and did not provide any desired lines for their outline. Any explanations for particular lines were reserved for the second phase of teaching episode 3.

Statement: The equation $5x-4=1$ has a unique solution.

Proof: Assume the equation $5x-4=1$ does not have a unique solution. Then either there is no solution to the equation $5x-4=1$ or there are at least two distinct solutions to the equation $5x-4=1$. Note $x=1$ is a solution of $5x-4=1$. Thus there are at least two distinct solutions to the equation $5x-4=1$, call them y and z. As both y and z are solutions of the equation $5x-4=1$, $5y-4=1$ and $5z-4=1$. Then $5y-4=5z-4$ and so $y=z$. Therefore it is not true that there are at least two distinct solutions to the equation $5x-4=1$. This is a contradiction, as we assumed that either there is no solution to the equation $5x-4=1$ or there are at least two distinct solutions to the equation $5x-4=1$. Therefore it is not true that the equation $5x-4=1$ does not have a unique solution. In other words, the equation $5x-4=1$ does have a unique solution.

Figure 4.21 Presented proof for Activity 3.

The purpose of outlining the logical structure of this presented proof was to encourage students to generalize this specific proof by contradiction into a series of steps that they could then use to write proofs by contradiction for uniqueness statements (e.g., $(\exists!x)(P(x))$. In addition, proving a uniqueness claim is commonly treated as a separate proof method in textbooks (Smith et al., 2015). By relating the procedure to other procedures by contradiction, students could either construct a new procedure for a general proof by contradiction or enhance their already existing general procedure for the proof method. That is, students could possibly assimilate or accommodate their procedure for proof by contradiction based on the new series of steps for the new type of statement and thus possibly further develop a Process conception of proof by contradiction.

A desired logical outline of the proof is illustrated in Figure 4.22.

Statement: $(\exists! x)(P(x))$

1. Assume $\sim (\exists! x)(P(x))$
2. $(\nexists x)(P(x)) \vee (\exists x, y)(P(x) \wedge P(y) \wedge x \neq y)$
3. Show $P(n)$ for some n.
4. $(\exists x, y)(P(x) \wedge P(y) \wedge x \neq y)$
5. $P(x) \wedge P(y) \rightarrow x = y$
6. $(\nexists x, y)(P(x) \wedge P(y) \wedge x \neq y)$
7. $\rightarrow\leftarrow$ (between lines 2, 3, and 6)
8. $\sim (\sim (\exists! x)(P(x)))$
9. $(\exists! x)(P(x))$

Figure 4.22 Desired logical outline of presented proof during Activity 3.

The outline above is a desired for four main reasons. First, the outline contains the two key steps to a proof by contradiction: assuming the negation of the statement is true (line 1) and arriving at a contradiction (line 6). Secondly, the phrase "Show $P(n)$ for some n" suggests line 3 does not logically follow from line 2. Thirdly, line 7 makes explicit the logical justification of the contradiction as written in the presented proof. Finally, the outline can be used for any proof by contradiction of a uniqueness statement.

A student's outline of the presented proof and explanation of their outline would indicate their conception of proof by contradiction. A student with an Action conception of proof by contradiction would describe the specific lines of the presented proof and represent every line of the proof. In addition, their outline would focus on the specifics of the presented proof, such as detailing the algebraic manipulation that is represented in line 5 in Figure 4.22. When asked to describe their outline, a student would describe each step in order and would not be able to describe the purposes of pairs or groups of steps. In contrast, a student with a Process conception of proof by contradiction would describe the purpose of groups of lines in the presented proof and represent these lines together. When asked to describe their outline, a student would discuss the purposes of parts of the outline (e.g., the assumption part, contradiction part, and conclusion) rather than describe every line of the outline. These two descriptions will be used to describe Wesley and Yara's evidenced conceptions of proof by contradiction.

Wesley's outline (Figure 4.23) described the two key steps of a proof by contradiction. In addition, he represented nearly every line in the proof and did not generalize the purposes of particular lines or sets of lines. Rather than presenting how he constructed every line in the outline, I will focus on the second line of the outline as it is indicative of his conception of both mathematical logic and proof by contradiction. Then, I will describe how the outline, in total, indicated his conception of proof by contradiction.

Statement: $(\exists! x)(P(x))$

1. Assume $\sim (\exists! x)(P(x))$
2. $\sim (\exists x)(P(x)) \vee (\exists x, y)(P(x) \wedge P(y) \wedge x \neq y)$
3. $P(1)$
4. $P(y) \wedge P(z) \rightarrow y \neq z$
5. $5y - 4 = 5z - 4$
6. $y = z$
7. $\rightarrow\leftarrow$
8. $\sim (\sim (\exists! x)(P(x)))$
9. $(\exists! x)(P(x))$

Figure 4.23 Wesley's logical outline of the presented proof during Activity 3.

Wesley started the outline by referring to the first step of his procedure for proof by contradiction: assume the statement is not true. However, he exhibited difficulty implementing this step of his outline, which required rewriting the negation of the statement. A short excerpt of his attempt to negate the statement is provided below.

Wesley: Okay. *[long pause]* Okay, so I guess we are going to assume that for all x, P(x).

Teacher: So what are you doing there?

Wesley: Well because it's the *[pause]* contradiction of there exists a unique.

[Teacher reiterates the directions for the task]

Teacher: But you were saying the opposite of there is exactly one solution is that...

Wesley: All.

Two phrases stand out in terms of his conception of mathematical logic. First, he used 'contradiction' to mean 'negation'. While these words are related (a contradiction involves a statement and the negation of that statement), the use of contradiction in place of negation would suggest he considered these two logical terms to be nearly identical. Secondly, he stated multiple times that the negation of $(\exists! x)$ was $(\forall x)$. In addition, he only provided cue words for the negation such as when he stated "...it's the *[pause]* contradiction of there exists a unique." These two comments suggest he relied on improperly memorized rules and cues to define logical terms and negate statements, which is indicative of a Pre-Action conception of mathematical logic.

The excerpt also stands out in terms of his conception of proof by contradiction. When asked why he stated "so I guess we are going to assume that for all x, $P(x)$", he responded "because it's the *[pause]* contradiction of there exists a unique." That is, when asked why line 2 was a 'for all' statement, he responded it was the negation of the symbols $\exists!$ in line 1. This response is devoid of reasoning behind the line (e.g., unpacking/rewriting the initial assumption) and suggests he followed the procedure without considering the underlying reasoning between lines, which is indicative of an Action conception of proof by contradiction.

After Wesley was provided the appropriate logical negation of a uniqueness statement, $\sim (\exists x)(P(x)) \vee (\exists x, y)(P(x) \wedge P(y) \wedge x \neq y)$, he was again reminded to represent the statements in the proof rather than prove the statement himself. He then stated:

> So... *[long pause]* you wouldn't put a unique y and a unique z? As opposed to there exists a y and there exists a z? You couldn't do unique? I guess it doesn't matter.

His comment suggested the negation of exactly one solution was either 0 solutions or exactly 2 solutions. However, he then stated "I guess it doesn't matter", which signified that the logical term 'unique' did not change the meaning of the sentence. Indeed, this explains why he previously considered the negation of 'unique' as 'for all' - a partial memorization of the logical equivalence $\sim (\exists x)(P(x)) \cong (\forall x)(\sim P(x))$ suggests that the negation of 'there exists' (and in his case, also 'unique') was 'for all'. In other words, the phrase 'unique' did not hold any logical meaning and thus he exhibited a Pre-Action conception of mathematical logic.

Overall, Wesley's outline contained the two key steps of a proof by contradiction: assuming the negation of the statement is true (line 1) and arriving at a contradiction (line 7). However, line 7 stated 'contradiction' and does not specify the contradiction in the presented proof (the assumption implies there are either 0 or 2 solutions, which lines 3 and 4-6 contradict). His outline also described the specific lines of the presented proof (such as the algebraic manipulations in lines 5-6) and did not generalize these lines to a single purpose. Therefore, Wesley exhibited an Action conception of proof by contradiction as his outline described the two key steps of a proof by contradiction and did not describe the logical relation between these steps.

Yara's outline (Figure 4.24) described the two key steps of a proof by contradiction (assuming the statement is false and arriving at a contradiction). In addition, she represented nearly every line in the proof, after which she then generalized the purposes of particular lines or sets of lines. First, I will focus on the second line of the outline as it is indicative of her conception of mathematical logic. Then, I will describe the how the outline, in total, indicated her conception of proof by contradiction.

Statement: $\exists!x$ s.t. $P(x)$

1. Assume $\sim (\exists!x \text{ s.t. } P(x))$
2. $R \vee Q$
3. $\sim R$
4. Q
5. $5y - 4 = 1 \wedge 5z - 4 = 1$ (Algebra)
6. More algebra ($y = z$)
7. $\sim Q$
8. $\sim (R \vee Q)$
9. $\sim (\sim (\exists!x \text{ s.t. } P(x)))$
10. $\exists!x$ s.t. $P(x)$

Figure 4.24 Yara's logical outline of the presented proof during Activity 3.

Yara provided a desired representation of the statement as $(\exists!x)(P(x))$ and first line of the outline as $\sim (\exists!x)(P(x))$. Then, she switched to propositional logic and represented the statement "then either there is no solution to the equation $5x - 4 = 1$ or there are at least two distinct solutions to the equation $5x - 4 = 1$" as $P \vee Q$. An excerpt of her thought process behind this representation is provided below.

Yara: And then *[long pause]* and then either there is no solution to the equation or there is at least 2 disinked, I mean, 2 distinct solutions to the equation $5x - 4 = 1$. *[pause]* So it would be the or? Like P or Q?

Teacher: Alright. *[pause for writing]* P or Q. So do these have any relation to the original one?

Yara: No?

Teacher: So if this one doesn't have a relation, then maybe we should call it something else. Like R or Q.

Y: Oh! To separate that P from that $P(x)$.

Note that she saw the 'or' in the statement and immediately suggested the representation $P \vee Q$. She then clarified that this P should be changed to separate it from $P(x)$. This suggests the cue word 'or' prompted the representation $P \vee Q$ as a standard representation for an 'or' statement, after which she recognized the need to differentiate the P from the original statement. In addition, this suggests she interpreted each line in the proof separately and did not describe the logical relation between the lines in her representation. Therefore, Yara exhibited an Action conception of mathematical logic.

Overall, her outline contained the two key steps of a proof by contradiction: assuming the negation of the statement is true (line 1) and arriving at a contradiction (line 8). In addition, she verbally described the logical argument of the proof and how lines in the proof related. For example, when she reached the contradiction line in the presented proof, she stated:

> Then... this is a contradiction as we assumed that there is either no solution to the equation $5x-4=1$ or there are at least two distinct solutions to the equation $5x-4=1$. So it would be Q and not Q? Or would we not have to put that because we have it *[$R \vee Q$]*... It's already labeled out. *[...]* Okay, so then not... I was just trying to make sure I had it in my head right like, that *[$R \vee Q$]* would go into not R and not Q.

Her first sentence quoted the line from the presented proof. She then immediately considered the representation $Q \wedge \sim Q$ - the standard representation of a contradiction. Representing a statement by focusing on cue words (i.e., contradiction means $Q \wedge \sim Q$) is indicative of an Action conception of proof by contradiction. However, she then recognized that she would not represent this particular contradiction in this way as she already represented part of this contradiction as $R \vee Q$. Indeed, her final comment "I was just trying to make sure I had it in my head right like, that *[$\sim (R \vee Q)$]* would go into not R and not Q" suggests that she recognized the logical equivalence $\sim (R \vee Q) \cong \sim R \wedge \sim Q$ and thus recognized that a contradiction of the representation $R \vee Q$ was reached. That is, she recognized and verbally described the logical reasoning behind how a contradiction was reached in this particular proof, which is indicative of a Process conception of proof by contradiction. In addition, she generalized lines 5 and 6 in her outline as “algebra” and thus described the purpose of the algebraic manipulations in the overall argument. Therefore, Yara exhibited a Process conception of proof by contradiction as she described the logical relation between lines in her outline and generalized specific lines to describe their purpose in the overall argument.

4.3.1.2 Definition of proof by contradiction with quantification After reading the presented proof and writing an outline of the proof, Wesley and Yara were asked to write a definition for proof by contradiction. The goal of this task was to encourage students to reflect on both their knowledge of proof by contradiction and a specific example of proof by contradiction to assimilate or accommodate the new logical outline into their general steps for a proof by contradiction. In addition, students were asked to explain the logic behind the proof method (if their definition did not already include such an explanation). While it was not included in the written task, the teacher/researcher prompted students to look at their definitions from teaching episodes 1 and 2 and their outline of the presented proof to write a definition for the proof method.

An ideal definition for proof by contradiction for this task would include the two key steps of the proof method as well as describe the general logic behind the method. For example, an ideal definition of proof by contradiction may state “A proof by contradiction is when you start by assuming a statement is false and then you use that and other mathematical knowledge to show the assumption is false, thus the statement is true.” This example definition includes both key steps and describes how these key steps logically relate to prove the statement is true.

A student's definition would indicate their level of understanding of the proof method. A student with an Action conception of proof by contradiction would describe the specific lines of a proof by contradiction without providing any reasoning behind the method. For example, a student may state ”Start with an assumption and get a contradiction.” This definition does not provide a relation between these two key steps and does not provide a complete description of each key step. Students with an Action conception of proof by contradiction may also focus on two cues in their definition: assumption and contradiction. In contrast, a student with a Process conception of proof by contradiction would describe the relation between the two key steps of a proof by contradiction. For example, a student may state "Assume the statement is not true and, using this assumption, arrive at a contradiction. This would mean the assumption is not true and thus the statement is true." This definition would describe the role of the two key steps completely rather than focus on cues such as 'assumption' and 'contradiction'.

When asked to define proof by contradiction, Wesley stated "Have your statement and you assume the opposite and then prove it false." This definition described one of the two key steps of the proof method: assuming the statement is false. In addition, the definition read as a list of steps: (1) statement, (2) assume the opposite *[of the statement]*, and (3) prove step 2 is false. When asked how, exactly, to prove the opposite of the statement is false, he only stated "contradiction?" In this case, he recited the cue word but did not provide any additional description of the contradiction or how this proves the statement is true. Therefore, Wesley exhibited an Action conception of proof by contradiction.

When asked to define proof by contradiction, Yara stated "Because you assume the statement wasn't true and then it would be *[pause]* you reached a contradiction" to justify that the presented proof was a contradiction. She then stated "Because we assumed our statement was not *[pause]* not true, then we, then we brought, arrived at a contradiction, proving that the statement was true" as a general definition. This definition included the two key steps of the proof method: assuming the statement is false and arriving at a contradiction. When asked to explain why arriving at a contradiction would prove the statement is true, she stated "Because you get a contradiction, you negate the assumption, which gives you the statement." This explanation, listed as a series of steps, logically relates the two key steps of a proof by contradiction with the statement to be proved. Therefore, Yara exhibited a Process conception of proof by contradiction.

4.3.2 Formalization of proof by contradiction with quantification

This subsection is a report about Wesley and Yara's responses during Classroom Discussion 3 (see Appendix A.1 for general questions that guided this discussion). The purpose of this phase was to guide these students into formalizing their conceptions of proof by contradiction based on their work and conjectures during Activity 3. Therefore, the tasks for this phase are based on the tasks[9] of Activity 3 as well as two additional tasks, comparing two proof by contradiction procedures and writing an outline of a proof, that reinforced the formalization of their conceptions. For each task, I will first present the goal and reasoning behind the task. Then, I will present an analysis and interpretation of students' responses to the task.

4.3.2.1 Discussion of outline of a uniqueness proof by contradiction The teacher/researcher began the second phase of teaching episode 3 by asking Wesley and Yara to logically justify each line of their outline for the presented proof in Figure 4.21 (see page 140). The goal of this task was to examine each student's conception of mathematical logic within the context of proof by contradiction. An analysis and interpretation of each student's responses that most exhibited their conception of mathematical logic follows.

As previously mentioned when describing how Wesley wrote an outline for the presented proof, he recognized the initial steps to be taken (i.e., begin by assuming the negation of the statement) but did not provide a proper logical negation of the statement $(\nexists x)(P(x))$. In addition, Wesley could not describe the logical justification for line 4 in his outline (see Figure 4.23 on page 142). This representation, $P(y) \wedge P(z) \rightarrow y \neq z$, logically followed from lines 2 and 3. Since Wesley could not provide any logical justification for lines in his outline of the presented proof, he exhibited a Pre-Action conception of mathematical logic.

As previously mentioned when describing how Yara wrote an outline for the presented proof, she verbally recognized that lines 3 ($\sim R$) and 7 ($\sim Q$) implied line 8 ($\sim (R \vee Q)$).

[9]Since both Wesley and Yara produced a formal definition of proof by contradiction in Activity 3, the teacher/researcher did not revisit the definition during this phase.

She did so without first writing lines 3 and 7 as $\sim R \wedge \sim Q$ to apply the logical equivalence $\sim R \wedge \sim Q \cong \sim (R \vee Q)$. After Yara noted that lines 2 $(R \vee Q)$ and 8 together formed a contradiction, she paused and then stated "I was just trying to make sure I had it in my head right like, that *[*$\sim (R \vee Q)$*]* would go into not R and not Q." This comment reinforced the interpretation that Yara first made the connection between lines 3 and 7 to imply line 8, then checked this step through memorized logical equivalences. Since Yara could provide logical justification for lines in her outline and describe the logical relationship between multiple lines in the proof, she exhibited a Process conception of mathematical logic.

4.3.2.2 Discussion of comparison between proof by contradiction procedures After discussing the outline for the presented proof in Activity 3, students were prompted to review their outlines from Activities 1, 2, and 3 to write a list of steps to prove any statement by contradiction. The goal of this task was to encourage students to reflect on both their prior knowledge of proof by contradiction and a specific example of proof by contradiction to assimilate or accommodate the new logical outline into their general procedure for a proof by contradiction. Desired outlines for each presented proof (Figure 4.25) were provided to students.

Activity 1 **Statement:** $P \to Q$	**Activity 2** **Statement:** $(\nexists x)(P(x))$	**Activity 3** **Statement:** $(\exists! x)(P(x))$
1. Assume $\sim (P \to Q)$ 2. $P \wedge \sim Q$ 3. $\sim Q_k$ 4. $(\sim Q_k \wedge P) \to Q_k$ 5. Q_k 6. $Q_k \wedge \sim Q_k$ 7. $\sim (\sim (P \to Q))$ 8. $P \to Q$	1. Assume $\sim (\nexists x)(P(x))$ 2. $(\exists x)(P(x))$ 3. $P(n)$ 4. Using $P(n)$, get to a contradiction. 5. $\sim (\sim (\nexists x)(P(x)))$ 6. $(\nexists x)(P(x))$	1. Assume $\sim (\exists! x)(P(x))$ 2. $\sim (\exists x)(P(x)) \vee (\exists x, y)$ $(P(x) \wedge P(y) \wedge x \neq y)$ 3. Show $P(n)$ for some n. 4. $(\exists x, y)(P(x) \wedge P(y) \wedge x \neq y)$ 5. $P(x) \wedge P(y) \to x \neq y$ 6. $(\nexists x, y)(P(x) \wedge P(y) \wedge x \neq y)$ 7. $\to\leftarrow$ (lines 2, 3, and 6) 8. $\sim (\sim (\exists! x)(P(x)))$ 9. $(\exists! x)(P(x))$

Figure 4.25 Side-by-side desired outlines introduced in Activity 1 (left), Activity 2 (center) and Activity 3 (right).

By looking across the three proof outlines, one may obverse the following commonalities among some lines or groups of lines. For example, lines 1 and 2 can be grouped together to describe assuming the negation of the statement is true while the final two lines can be grouped together to describe the assumption as false due to arriving at a contradiction. As in teaching episode 2, a desired series of steps would contain three general purposes or roles: (1) Assuming the negation of the statement is true, (2) Using the initial assumption and mathematical knowledge to arrive at a contradiction, and (3) Since assuming the negation of the statement arrived at a contradiction, the statement must be true. These roles synthesize multiple lines of a particular proof to describe the purpose of a collection of lines. A detailed description of how the series of steps would differ between a student with an Action conception and a student with a Process conception of proof by contradiction follows.

A student with an Action conception of proof by contradiction would describe the necessary steps of the method by cue words and without any further description of the logic behind the proof method. For example, step 2 may simply be stated as 'contradiction' or step 3 may just refer to the double negation of a statement and therefore the statement is true. The student would need to write the details for each step of a particular proof and could not skip steps in the procedure. In addition, the student would not be able to give the logical relation between steps of the procedure.

A student with a Process conception of proof by contradiction would describe the necessary steps of the proof but would also describe, in their own words, intermediate steps between these necessary steps that provide reasoning behind the method. For example, a step after the initial assumption could state "Use the assumption and mathematical knowledge to derive a contradiction." This example step would provide logical reasoning for a group of steps in the context of the proof method. Using the reasoning between steps, the student could describe how to proceed in a particular proof without the need to write the details of that proof. These descriptions will be used to describe Wesley and Yara's evidenced conceptions of proof by contradiction.

When prompted to compare the outlines in Figure 4.25, Wesley stated that he recalled the following general procedure from Classroom Discussion 2 (see Figure 4.26).

Statement: S
1. Assume $\sim S$
2. $\vdots$
3. $\rightarrow\leftarrow$
4. S

Figure 4.26 Wesley's procedure for proof by contradiction during Classroom Discussion 3.

The procedure above is nearly the same as the procedure for groups G5 through G12 in Figure 4.1 (see page 121) and provided the minimal steps for a proof by contradiction. That is, the procedure did not describe any intermediate steps nor described the relation between lines. Therefore, Wesley exhibited an Action conception of proof by contradiction.

When prompted to compare the outlines in Figure 4.25, Yara grouped lines together and described a general purpose for each group of lines (see Figure 4.27).

Classroom Discussion 2 **Statement:** S	**Classroom Discussion 3** **Statement:** P
1. Assume $\sim S$ 2. Rewrite $\sim S$ 3. Look at specific value of step 2. 4. Work (Algebra) 5. Get contradiction 6. $\sim$ Assumption 7. S	1. Assume $\sim P$ 2. Negate P (Rewrite $\sim P$) 3. Use math skills to get to a contradiction 4. $\sim$ Assumption 5. P

Figure 4.27 Yara's Side-by-side procedures for proof by contradiction constructed during Classroom Discussion 2 (left) and Classroom Discussion 3 (right).

Yara's procedure for proof by contradiction contained both the key steps of a proof by contradiction and descriptions of how these key steps are logically related (e.g., that lines 3 and 7 logically implied line 8). In addition, it appears that Yara assimilated the procedure from Classroom Discussion 2 to help her deal with the presented proof during Activity 3. Consider the following exchange as Yara compared the desired outlines from Activities 2 and 3 to generalize the contradiction step.

Yara: So I guess it just, maybe it likes, depends on the proof, and what you are trying to prove? Whether you do algebra or... umm... *[pause]*

Teacher: So what do we do in that one *[outline during Activity 2]*?

Yara: In this one, it says to use $P(x)$, get a contradiction. So we did algebra, right? So this one you do... which math skills do you use? Because math skills could mean plenty of things. It could be, like, one of them induction whatever...

Yara decided that the phrase 'math skills' included algebra as well as possibly other proof methods, such as induction. This is noteworthy as the presented proof does not include another proof method as a sub-proof, yet Yara generalized the step to include utilizing other proof methods. In doing so, Yara produced a minimal change on her previous procedure for proof by contradiction and thus assimilated the new outline into her Schema for proof by contradiction. Therefore, Yara exhibited an enhanced Process conception of proof by contradiction via assimilation of her previous outline during Activity 3.

4.3.2.3 Writing a proof outline using a general procedure After creating a list of steps to prove any statement by contradiction, Wesley and Yara were asked to write a proof outline of "The multiplicative inverse of a non-zero real number x is unique" - a new, related statement. The goal of this task was to see if and how students can use their current understanding of proof by contradiction when given a slightly new problem statement. To this end, the new statement needed to be sufficiently different from the previous statement in Activity 3 to discourage students from utilizing the outline written during Activity 3 and, instead, to utilize their general procedure for proof by contradiction. The new statement was chosen as it contained a hidden 'for all' quantifier and could be rewritten as: For all non-zero real numbers x, there exists a unique multiplicative inverse y. Therefore, while the statement still described the uniqueness of an element, is was similar in structure to the statement in Activity 3 as it required the coordination of two variables.

A desired proof outline is illustrated in Figure 4.28.

Statement: $(\forall x \neq 0, x \in \mathbb{R})(\exists! y)(xy = 1)$

1. Let $x \neq 0$ be an arbitrary real number.
2. Assume the statement (without $\forall x$) is not true.
3. Then there is either 0 inverses or at least 2 distinct inverses.
4. For $y = \frac{1}{x}$, $xy = 1$, so there are not 0 inverses.
5. If there are 2 distinct inverses, we can show they are equal, so there are not at least 2 inverses.
6. Line 3 and lines 4, 5 are contradictory.
7. So the statement (without $\forall x$) is true.
8. Since x was arbitrary, the original statement is true.

Figure 4.28 Desired proof outline during Classroom Discussion 3.

This proof outline is desired for three main reasons. First, it included the two key steps of a proof by contradiction: assuming the statement is not true (line 2) and arriving at a contradiction (line 6). Secondly, it addressed the hidden quantifier 'for all' with lines 1 and 8. Finally, it provided the logical argument without providing the details of the proof. For example, line 5 described the general argument to show that there are not at least 2 multiplicative inverses of a non-zero real number x.

A student's proof outline and explanation of their outline would indicate their conception of proof by contradiction. A student with an Action conception of proof by contradiction would rely on a list of steps to write an outline for the statement. This list of steps would either be an outline of the necessary steps of any proof by contradiction or from an outline of a specific statement (e.g., outlines from Activities 1, 2, and 3). In contrast, a student with a Process conception of proof by contradiction would describe the general argument of the proof without writing the details of each line. This student would rely on a general procedure for proof by contradiction that could then be modified for the statement as well as describe the purposes of lines between the assumption step and the contradiction step.

A student's outline of a proof is not only influenced by a student's conception of proof by contradiction. That is, while a student's conception of proof by contradiction would influence how he or she outlines a proof, a student's conception of mathematical logic influences how he or she enacts the outline. The interaction between a student's conception of proof by contradiction and understanding of mathematical logic would thus produce different outlines of a proof. Therefore, I will describe how Wesley and Yara's responses illustrated their conception of both proof by contradiction and mathematical logic.

Wesley constructed an outline (Figure 4.29) for the statement by explicitly following the step-by-step outline for a uniqueness statement from Activity 3. Rather than describe how he constructed every line of the outline, I will focus on particular lines that especially illustrated his conception of mathematical logic or proof by contradiction. Then, I will describe how the outline, overall, illustrated his conception of proof by contradiction.

Statement: $(\forall x)(\exists! y)(P(x,y))$
$P(x,y)$: The mult inverse y of x is unique.

1. Assume $\sim (\forall x)(\exists! y)(P(x,y))$
2. $(\exists x)(\sim P(x,y)) \vee (\exists r,t)(P(x,r) \wedge P(x,t) \wedge r \neq t)$
3. $P(x, \frac{1}{x})$
4. $xr = xt$ so $r = t$, $\rightarrow\leftarrow$
5. $\sim\sim (\forall x)(\exists! y)(P(x,y))$
6. $(\forall x)(\exists! y)(P(x,y))$

Figure 4.29 Wesley's proof outline during Classroom Discussion 3.

When prompted to provide an outline of the proof, Wesley first converted the statement into predicate logic. Wesley's thought process is provided below:

> Okay. Umm so you are going to *[long pause]* for every x *[long pause]* $P(x)$? Wait, you have to account for y. So, for every x, there is a unique y *[pause]* x, our $P(x)$. Our $P(x)$ would be *[long pause]* $x = y$. But I can't... hold on. *[flips through to previous proof]* Alright, let's write the statement. The statement is for every x, there is a unique y, right? *[...]* There is a unique y. I guess, $P(x, y)$?

He arrived at the representation $(\forall x)(\exists! y)(P(x, y))$. He then wrote on his paper "$P(x, y)$: The mult inverse y of x is unique." This suggests he first used a memorized rule to replace 'unique' with the symbols $\exists!$. He then included uniqueness again in his representation $P(x, y)$. This redundancy - including uniqueness twice - indicates he did not describe the meaning of the symbols $\exists!$. Therefore, Wesley exhibited an Action conception of mathematical logic when writing the statement in predicate logic.

Wesley then stated "So you are going to assume not all of that" and thus utilized the first step in his general steps for a proof by contradiction. However, Wesley exhibited difficulty representing the negation of the statement. His attempt is provided below:

Wesley: So... then you would, that would be *[long pause]* there exists an x *[long pause]*

Teacher: So there exists an x with these special qualities?

Wesley: Right. *[long pause]* With *[pause]* no unique... I guess it's with... *[long pause]*

Teacher: Without... there does not exist a y such that x times y equals 1... right? *[pause]* Either there is an x where there is no multiplicative inverse, just like before, or *[pause]*

Wesley: Okay, so. If I use an *[pause]* there exists a unique, I am definitely using an 'or' statement.

His first two statements, with a number of incomplete phrases and long pauses, suggest he has difficulty representing the negation of the statement. In addition, his statement "If I use an *[pause]* there exists a unique, I am definitely using an 'or' statement" indicated that he did not have a logical rule to interpret the negation of a uniqueness statement. At this point, the teacher/researcher confirmed that the negation of exactly one is either none (less than 1) or at least 2 (more than 1). It is important to note that he was explicitly given this rule in Activity 3 and the rule is included in the outline from Activity 3, which the teacher/researcher implied with the phrase "just like before." In other words, he had access to the external rule and could not negate the statement with this rule, which is indicative of a Pre-Action conception of mathematical logic.

Wesley then utilized the outline from Activity 3 to continue, as indicated by his statement "So then the first one *[pause]* the first one is there exists, there is not a y. So you can just put $P(1,1)$? Right?". This comment referred to step 3 from Figure 4.23 (see page 142), which stated $P(1)$. In other words, he noted that step 3 inserted 1 into the representation and thus initially wrote line 3 of his new outline as $P(1,1)$. This suggests he followed the step-by-step outline from Activity 3 without describing the meaning of the statements. Therefore, Wesley exhibited an Action conception of proof by contradiction.

Overall, Wesley's proof outline included the two key steps for a proof by contradiction: assuming the negation of the statement is true (line 1) and arriving at a contradiction (line 4). In addition, the outline provided the specific details for parts of the proof, such as the algebraic manipulation in line 4, and did not describe the general logical argument of the proof, such as the logical relation between lines 2 and 3. Logical relations between lines were also not described when he constructed the proof outline. Instead, he focused on cues such as "so you are going to assume all of that" and "So step 5 would be not not *[pause]* all that." These incomplete phrases suggest he followed the outline in Activity 3 without describing the relations between lines. Therefore, Wesley exhibited an Action conception of proof by contradiction with his outline of a similar proof.

When prompted to write an outline of the proof, Yara instead began writing the entire proof without representations (see Figure 4.30). This proof followed her general procedure for proof by contradiction and did not rely on the outline for a uniqueness statement from Activity 3. Rather than describing how she constructed every line of the outline, I will focus on particular lines that especially illustrated her conception of mathematical logic or proof by contradiction. Then, I will describe how the outline, overall, illustrated her conception of proof by contradiction.

Statement: The multiplicative inverse of a nonzero real number r is unique.

Proof: Assume the multiplicative inverse of a nonzero real number r is not unique. Then either there is no multiplicative inverse of r or it has two or more multiplicative inverses. Let $t \neq 0$ be an arbitrary real number. Let $r = \frac{1}{t}$. Then $rt = \frac{t}{t} = 1$. So t has at least two distinct mult. inverses, call them r and m. Then $rt = 1$ and $mt = 1$. $rt = mt \to r = m$, a contradiction. So there is a unique multiplicative inverse.

Figure 4.30 Yara's proof during Classroom Discussion 3.

Consider the following excerpt as Yara described the initial assumption and rewriting this assumption.

Yara: Then it would be, well we basically negate that *[the statement]*.

Teacher: Uh huh. So we are going to negate this. So what does not unique actually mean?

Yara: That it can have zero or more than *[pause]* 2 or more. Since unique means just one.

She started the meaning of 'unique' as just one and was able to explain why the negation of 'unique' meant either zero or at least one. She later returned to the meaning of 'not unique' after showing that there existed at least one multiplicative inverse and stated "Because we showed that it didn't have none, so that scratches that off. So now we got to see if it has another one." In other words, she utilized the meaning of 'not unique' to logically justify a line in the proof, which is indicative of a Process conception of mathematical logic.

Overall, Yara's proof included the two key steps of a proof by contradiction and addressed the hidden quantifier 'for all' by including the line "Let $t \neq 0$ be an arbitrary real number." In addition, she could verbally describe the logical relation between lines of the proof when prompted, as illustrated in the above paragraph. Finally, she made no reference to the outline in Activity 3 and, instead, referred to initial assumption (either there are zero or more than one multiplicative inverse) to motivate continuing in the proof. Therefore, Yara exhibited a Process conception of proof by contradiction.

4.3.3 Reinforcing formalization of proof by contradiction with quantification

This subsection is a report about students' responses to tasks from Exercise 3 (see Appendix A.1 for the complete set of tasks) - the last phase of teaching episode 3. The goal of this phase was to reinforce students' conceptions of mathematical logic and proofs by contradiction with quantification through eight tasks aligned with the proof comprehension assessment model by Mejía-Ramos et al. (2012) as well as an additional question not aligned with the assessment model. All nine questions were written for the presented proof (Figure 4.31) and are provided in Figure 4.32.

Statement: The multiplicative inverse of a non-zero real number r is unique.

Proof: Assume the multiplicative inverse of an arbitrary non-zero real number r is not unique. Then either there is no multiplicative inverse of r or there are at least two distinct multiplicative inverses of r. Note $x = \frac{1}{r}$ is a multiplicative inverse of r. Thus there are at least two distinct multiplicative inverses of r, call them x and y. As both x and y are both multiplicative inverses of r, $rx = 1$ and $ry = 1$. Then $rx = ry$ and so $x = y$. Therefore it is not true that there are at least two distinct multiplicative inverses of r. This is a contradiction, as we assumed that either there is no multiplicative inverse of r or there are at least two distinct multiplicative inverses of r. Therefore it is not true that the multiplicative inverse of an arbitrary non-zero real number r is not unique. In other words, the multiplicative inverse of a non-zero real number r is unique.

Figure 4.31 Presented proof for Exercise 3.

1. Compare the outline of your proof in question 4 *[in Classroom Discussion 3]* to the proof above. Explain how your proof compares to the given proof in terms of: (1) general structure, (2) specific lines, and/or (3) overall approach to the proof.
2. Please give an example of a multiplicative inverse of a non-zero real number and explain why it is a multiplicative inverse.
3. Why does r have to have a multiplicative inverse?
4. Why exactly can one conclude that $x = y$?
5. What is the purpose of the statement "Then either there is no multiplicative inverse of r or there are at least two distinct multiplicative inverses of r."?
6. Summarize in your own words the main idea of this proof.
7. What do you think are the key steps of the proof?
8. Would the proof still work if we instead say the multiplicative inverse of a real number x is unique? Why or why not?
9. Using the method of this proof, show that: if a and b are real numbers and $a \neq 0$, then there is a unique real number r such that $ar + b = 0$.

Figure 4.32 Comprehension questions for the presented proof during Exercise 3.

Students were first given approximately 20 minutes to individually respond to the comprehension questions above. Then, the teacher/researcher asked students to share their answers and describe the reasoning behind their response to each question. After eliciting the student's response and reasoning behind their response, the teacher/researcher provided a desired response to the question (if necessary).

Question one, which was not aligned with the proof comprehension assessment model, prompted students to compare their outline from Classroom Discussion 3 to the presented proof in Figure 4.31. The goal of this question was to compare an outline and a proof in order to reinforce students' conception of proof by contradiction by identifying common key steps and general procedure of the outline and proof. A desired response would describe a correspondence between the two key steps of a proof by contradiction as well as describe how the proof arrived at a contradiction. These three ideas would then be used to summarize the procedure and main idea of the proof. However, responses to this question did not provide sufficient evidence of students' conception of proof by contradiction or mathematical logic and therefore will not be discussed further.

The remainder of this subsection will be organized by comprehension questions two through nine. For each question, I will first describe what it was meant to assess in terms of the proof comprehension assessment model by Mejía-Ramos et al. (2012) (described on page 23). Then, I will describe the goal of the assessment in terms of the students' understanding of the presented proof, mathematical logic, or proof by contradiction. Finally, I will present an analysis and interpretation of Wesley and Yara's responses to the question.

Question two assessed whether students could identify the meaning of terms and statements in the presented proof. In particular, the question asked students to provide an example of a multiplicative inverse of a real number and explain why this number is a multiplicative inverse. The goal of this question was to determine whether students understood one of the mathematical terms in the proof (multiplicative inverse) that was necessary to understand the logical argument of the proof, as the literature suggested students struggle with proof comprehension due to a lack of mathematical knowledge (e.g. Moore, 1994). In other words, the goal of this question was to assess pre-requisite knowledge necessary to understand the particular proof. A desired response would present two numbers and state that the product of these numbers is one.

Both Wesley and Yara responded with a pair of numbers and explained that the product of these numbers was 1. For example, Yara stated "I did 4 and $\frac{1}{4}$ because 4 times $\frac{1}{4}$ equals 1 and that's the... inverse for 4." Her response clearly indicated that that 4 and $\frac{1}{4}$ are multiplicative inverses as multiplying these numbers together equals 1. Wesley's response was similar to Yara's and therefore both students possessed the prerequisite knowledge of multiplicative inverses necessary to understand the particular proof.

Question three assessed whether students could justify a statement in the presented proof. The goal of this question was to assess whether students recognized that r has at least one multiplicative inverse due to a generated example. In other words, the question assessed whether students recognized the line "Note $x = \frac{1}{r}$ is a multiplicative inverse of r" did not follow from the previous statement. Responses to this question would indicate students' conception of mathematical logic as this question required a mathematical justification (as opposed to a logical justification). A desired response to this question would indicate that to validate the existence claim "there exists at least one multiplicative inverse", one needs to produce and verify a multiplicative inverse. That is, given an arbitrary x, that there exists a y such that $xy = 1$.

Wesley did not initially respond to this question and, when asked why, stated "I don't know what you are looking for in that. I don't know if we're looking for... if you are looking for something specific in math... Is it just because we said it does?" The phrase "just because we said it does" suggests an authoritarian justification (i.e., it is true because we say it is true) for the statement "r has at least one multiplicative inverse" rather than a mathematical justification (i.e., it is true as we produced such an example) or a memorized fact (i.e., it is true by definition). The appeal to authority to justify a statement as true (i.e., exhibiting an external conviction proof scheme[10]) has been noted in the literature as a proof validation difficulty (e.g., Harel & Sowder, 1998) that may make students prone to accepting false proof verifications on the basis of ritual and form. For this question, an appeal to authority is not a valid justification and therefore Wesley exhibited a Pre-Action conception of mathematical logic.

Yara's initial response to the question was "because it was assigned?" As it was unclear what this phrase meant, she was prompted to clarify. She then stated "It was just, that's what it said in the statement. It was a given." The phrase "It was a given" colloquially refers to an assumption that is part of the statement to be proved. That is, an assumption that is 'given' to be used in the proof. For example, in the statement "Let x and y be positive real numbers such that $x - 4y < y - 3x$. Prove that if $3x > 2y$, then $12x^2 + 10y^2 < 24xy$" the phrase "Let..." indicates the 'given' that should be used to prove the implication. This suggests her phrase "that's what it said in the statement" referred to the initial statement to be proved. In other words, she justified a line in the proof with the statement to be proved -

[10]A proof scheme is that which convinces the individual a statement is true (Harel & Sowder, 1998).

a form of circular logic. Therefore, Yara exhibited a Pre-Action conception of mathematical logic.

Question four assessed whether students could justify a statement in the proof. In particular, the question focused on the algebraic implication $rx = ry \rightarrow x = y$. The goal of this question was to assess whether students recognized this statement as true for all real numbers except zero. Responses to this question would indicate students' conception of mathematical logic as it required students to identify the necessary conditions to utilize the mathematical theorem: For real numbers $r \neq 0, x$, and y, $rx = ry$ implies $x = y$. A desired response would indicate that r must not be zero and that r, x, and y are real numbers.

When asked to justify that $x = y$ in the proof, Wesley stated "Because $\frac{1}{r}$ only has one solution regardless therefore x and y must be the same number." The phrase "Because $\frac{1}{r}$ only has one solution regardless" referred to the equation $x = \frac{1}{r}$ in the previous line of the presented proof. Thus the phrase "therefore x and y must be the same number" refers to the equation to construct a multiplicative inverse of r and how this equation has only one value for every r. This is circular logic as Wesley justified the algebra within the proof by citing the statement itself. Self-reference when proving a well-known mathematical theorem (such as the uniqueness of a real multiplicative inverse) has been identified in the literature as a proof construction difficulty (e.g., Antonini & Mariotti, 2008) as students could not separate their mathematical knowledge known from the mathematical knowledge available to utilize for the specific theorem. In this specific case, Wesley was not able to separate the entirety of his mathematical knowledge from the mathematical knowledge available to prove the statement. Therefore, Wesley exhibited a Pre-Action conception of mathematical logic.

When asked to justify that $x = y$ in the proof, Yara stated "Because $rx = 1$ and $ry = 1$, therefore $rx = ry$ so $x = y$." This justification is valid for real numbers when $r \neq 0$. Since she did not provide this detail, the teacher/researcher followed up with the specific question "So we can always divide both sides by r?" to elicit whether she was aware of this condition. Her response, "Mmhmm and get the same thing," suggests she was not consciously aware that $r \neq 0$ was a required condition for the algebraic manipulation to be valid. Therefore, Yara exhibited a Pre-Action conception of mathematical logic.

Question five assessed whether students could identify the logical status of a statement in the proof. This time, students were asked what the purpose of the statement "Then either there is no multiplicative inverse of r or there are at least two distinct multiplicative inverses of r" which described the negation of the assumption. The goal of this question was to assess whether students could describe the purpose of the statement: that it rephrases the initial assumption. Responses to this question would indicate students' conception of proof by contradiction as the assumption step is one of two key steps for the proof method. A desired response would describe how the statement is logically equivalent to the previous statement and is now in a form that can be used. That is, the proof implicitly used that there are either $0, 1$, or more than 1 multiplicative inverses and the statement in the question follows logically from assuming that there is not 1 multiplicative inverse.

When asked for the purpose of the statement, Wesley responded "It is the contradiction[11] for ($\exists!$) there is only one." Note his response cited a rule (the negation of 'there is only one') that aligned with line 2 in his outline for a proof by contradiction of a uniqueness statement (see Figure 4.23 on page 142). In addition, the phrase does not describe the role of the statement in the context of the proof. Therefore, Wesley exhibited an Action conception of proof by contradiction.

When asked for the purpose of the statement, Yara responded "Because you can either have 0, 1, or 2 (or more) solutions." Since this response indicated she recognized the underlying reasoning behind the proof, the teacher/researcher asked why the statement ignored the '1 solution case' to elicit whether she recognized the statement in the question followed logically from the initial assumption. She responded "I think we were assuming that it wasn't one?" In other words, Yara described the general argument of the proof (there are either 0, 1, or more than 1 multiplicative inverses) and stated it followed logically from the initial assumption. Therefore, Yara exhibited a Process conception of proof by contradiction.

[11]Wesley continued to use 'contradiction' in place of 'negation' throughout the teaching episode even after the teacher/researcher addressed the difference between the two terms.

Question six assessed whether students could summarize the proof via high-level ideas. The purpose of this question was to assess whether students could summarize the proof in their own words and, if they could, how they summarized the proof. For example, a summary for the presented proof could be similar to the following:

> It's a proof by contradiction, where you assume the statement isn't true. So then either there are zero distinct inverses or there are at least two distinct inverses of r. We know $\frac{1}{r}$ is an inverse of r, so there aren't 0. So there must be at least two distinct inverses. But when we set the equations equal to each other, we get the inverses are equal to each other, which is a contradiction. Therefore your statement is true.

This summary describes the specific procedure of the presented proof and would illustrate an Action conception of proof by contradiction. In contrast, another summary could be similar to the following:

> If the statement were false, then there would be either zero inverses of at least two inverses. But we know there are not zero and if there are at least two, then they are equal. Therefore, the statement must be true.

This response describes the main idea of the proof (e.g., there are not zero inverses and there are not at least two inverses) and provides a generalization of the two key steps of a proof by contradiction, which is indicative of a Process conception of the proof method. Therefore, responses to this question would indicate students' conception of proof by contradiction.

Both Wesley and Yara summarized the proof by rephrasing the statement: Wesley responded "To show that there is only one inverse" and Yara responded "To show that the multiplicative... that the multiplicative inverse is unique." These students did not summarize the proof by describing the procedure of the proof nor did they summarize the proof by describing the general ideas in the proof and thus both students exhibited a Pre-Action conception of proof by contradiction.

Question seven assessed whether students could identify the modular structure of the proof. In other words, whether students could group sentences in the proof together and provide a purpose for each of these groups. By phrasing the question as "key steps", the teacher/researcher attempted to illicit the purpose of particular lines and how these purposes related to form a complete proof. Responses to this question would indicate students' conception of proof by contradiction. In particular, students that responded with the two key steps of a proof by contradiction (assuming the statement is false and arriving at a contradiction) and did not describe the purpose of these steps in the overall argument of the proof would have exhibited an Action conception of proof by contradiction, while students that responded with lines from the desired logical outline and described the purposes of these lines in the overall proof would have exhibited a Process conception of proof by contradiction.

When asked to identify the key steps of the presented proof, Wesley responded "The contradiction. Proving $Q \neq \sim Q$." This response described one of the two key steps in a proof by contradiction: arriving at a contradiction. Therefore, Wesley exhibited a Pre-Action conception of proof by contradiction.

When asked to identify the key steps of the presented proof, Yara responded "Assuming $\sim$ statement, get a contradiction." These two statements are general (in that they do not refer to the specific proof) and describe the two key steps of a proof by contradiction. However, her response does not describe how these two statements are related and the teacher/researcher did not prompt her to describe the relation between these two key steps. Therefore, Yara exhibited an Action conception of proof by contradiction.

Question eight assessed whether students could transfer the general idea or method to another context. In particular, the question assessed whether students recognized the scope of the method (i.e., recognized the assumptions that needed to be in place to allow the method to be carried out). Beyond the use of proof by contradiction, the condition that r is a non-zero real number is necessary for the algebraic manipulation of the proof and for the statement itself to be true. Responses to this question would indicate students' conception of mathematical logic. A desired response to this question would indicate that the method does not work since zero does not have a multiplicative inverse and thus the initial assumption "Then either there is no multiplicative inverse of r or there are at least two distinct multiplicative inverses of r" is true.

When asked if the proof would still work, Wesley stated "No, because zero is undefined." The phrase "zero is undefined" did not make clear whether this referred to dividing by 0 or the multiplicative inverse $\frac{1}{r}$ would be undefined. After the teacher/researcher stated "Zero is undefined? Zero is zero", he retorted "Okay, yeah division by zero." This response made clear that Wesley focused on the algebraic manipulation of the proof (i.e., that $rx = ry$ implies $x = y$) rather than that the multiplicative inverse of zero, $\frac{1}{0}$, would be undefined, which is indicative of an Action conception of mathematical logic.

When asked if the proof would still work, Yara stated "No, because it would be undefined." The phrase "it would be undefined" also did not make clear whether this referred to dividing by 0 or the multiplicative inverse $\frac{1}{r}$ would be undefined. She then clarified "Like, I guess zero, if you are being technical, like zero is like, zero over 1. And you can't flip it; you'll get 1 over zero. And that's undefined." The phrase "you can't flip it" refers to the general form of a real multiplicative inverse, $\frac{1}{r}$. That is, her response described the meaning of the statement rather than algebraic manipulation of the proof, which is indicative of a Process conception of mathematical logic.

Question nine also assessed whether students could transfer the general idea or method to another context, this time in the form of writing a similar proof for the statement "If a and b are real numbers and $a \neq 0$, then there is a unique real number r such that $ar + b = 0$." This question encouraged students to utilize either the specific outline of the similar proof in Figure 4.22 or their general outline for any proof by contradiction (see Figure 4.27 on page 152 for Yara and Figure 4.26 on page 152 for Wesley) to write a proof by contradiction. Responses to this question would indicate students' conception of proof by contradiction. In particular, students with an Action conception of proof by contradiction would follow one of the two step-by-step outlines to prove the similar statement, while students with a Process conception of proof by contradiction would use their internalized steps to prove the statement and may skip explicit steps of their procedure. A desired proof for question 8 is presented in Figure 4.33.

Statement: If a and b are real numbers and $a \neq 0$, then there is a unique real number r such that $ar + b = 0$.

Proof: Let $a \neq 0, b \in \mathbb{R}$ and assume that there is not a unique real number r such that $ar + b = 0$. Then there are either no such r or there are at least two such r [**1**]. Note that $r = -\frac{b}{a}$ is a solution to the equation $ar + b = 0$ and so there are not no such r. Then there are at least two such r, which we will denote as x and y. That is, $ax + b = 0$, $ay + b = 0$, and $x \neq y$. Then $ax + b = ay + b$, which implies $ax = ay$. Since $a \neq 0$, $ax = ay$ implies $x = y$ and thus there are not more than two such r. This is a contradiction as we assumed there are either no such r or at least two such r and have shown there are not no such r and there are not at least two such r [**2**]. Thus are initial assumption is false [**3**] and therefore there is a unique real number r such that $ar + b = 0$. Since $a \neq 0$ and b were arbitrary real numbers, this statement holds for all such real numbers.

Figure 4.33 Desired proof for question eight in Exercise 3.

The proof above includes both key steps of a proof by contradiction: assuming the statement is not true (denoted as [**1**]) and arriving at a contradiction (denoted as [**2**]). In addition, the proof explains how these key steps logically imply the statement is true (denoted as [**3**]). Finally, the proof is valid in that all lines are properly justified and complete as it focused on the two quantifiers for a and b. Therefore, proofs that satisfy these three criteria, such as the proof in Figure 4.17, are considered to be desired proofs for the statement in question 8.

Wesley initially responded with $(\forall a)(\forall b)(\exists! r)(P(r))$, a symbolic representation of the statement, and defined $P(r)$ as the solution to the equation $ar+b = 0$. When the teacher/researcher noted this was not a full proof, he responded:

> Now hold on, I didn't give you an answer. I just put my statement. *[...]* well I guess I didn't prove it, but I wrote the statement. If that's my statement, then I could follow that *[points at outline in Figure 4.22]*.

This suggested that once he represented the statement in a predicate form, he could find the outline that fit the statement structure and would follow that outline, which would indicate an Action conception of proof by contradiction. He then indicated that he had to leave the session and did not attempt to write the proof. It is unclear whether he could have, in fact, appropriately modified the outline to write the new proof and thus have exhibited an Action conception of proof by contradiction.

Yara initially responded by solving the equation for r. When the teacher/researcher prompted her to interpret the statement to be proved, she responded "It's an if, then statement." She then asked "Would it be... easier to do the... what's it called *[pause]* not that I know how to do it, but the contraposition?" Even though the question prompted her to use the same method of the presented proof, she seemed to not know how to begin the proof. This proof construction difficulty, the need for strategic knowledge[12], has been noted in the literature (e.g. Weber, 2001) and suggested that students exhibited difficulty writing a proof even when they understand the mathematical content and methods necessary to write the proof. That is, the student understands the mathematical knowledge and proof methods disjointly and does not know strategies necessary to use this knowledge in conjunction with each other. Therefore, the teacher/researcher provided prompts to guide her into beginning and continuing in the proof.

Yara continued by representing the statement using informal predicate logic (Figure 4.34). That is, she represented some parts of the statement using predicate logic (e.g., $\exists! r \in \mathbb{R}$) and did not change the representation of other parts of the statement (e.g., logical operators '$\rightarrow$' and '$\wedge$'). She then represented this using propositional logic $P \rightarrow Q$. When prompted to explain how one directly proves an implication statement, she responded "Prove P to get to Q?" which indicated she could not describe the steps to prove an implication statement and exhibited a Pre-Action conception of direct proof.

[12]Knowledge of how to choose which facts and theorems to apply.

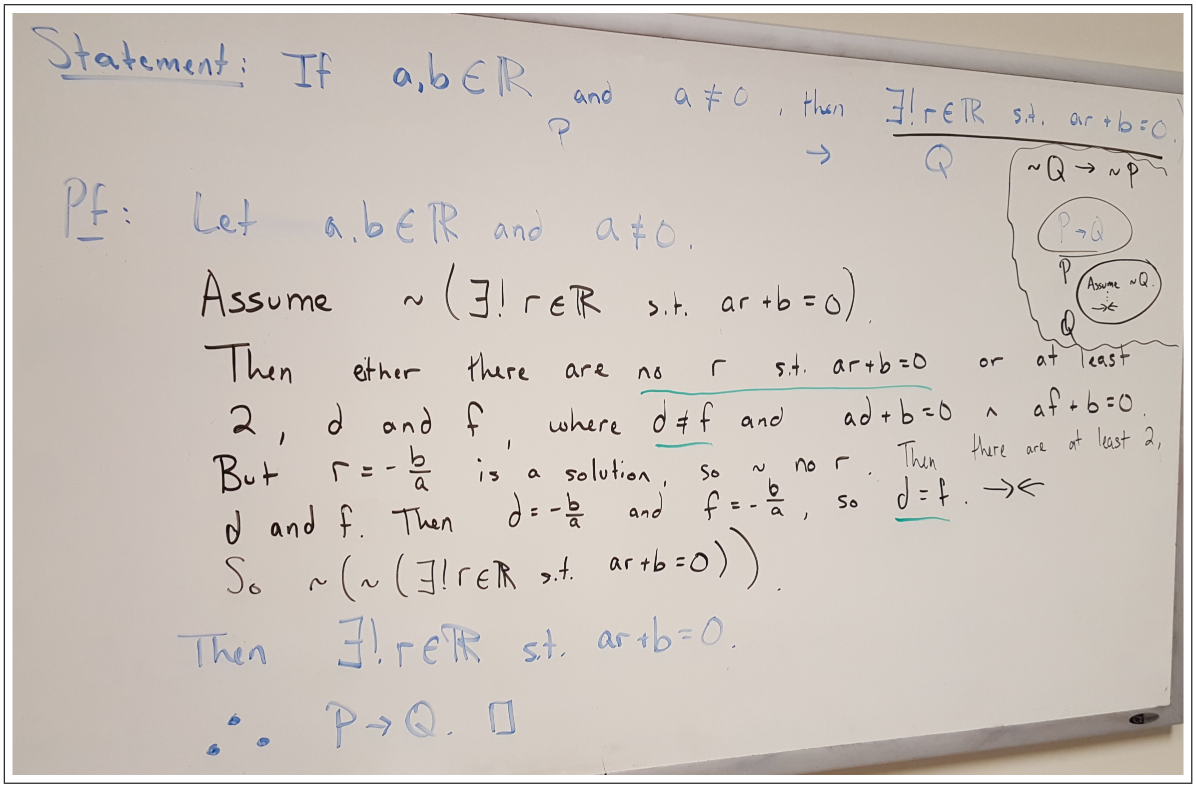


Figure 4.34 Yara's proof for question 8 during Exercise 3.

Since question nine was meant to examine her conception of proof by contradiction and not her conception of direct proof, the teacher/researcher explained that to prove $P \rightarrow Q$, one needs to start by assuming P and arrive at Q. These steps were written in blue (see Figure 4.34 below) to distinguish them from the steps for a proof by contradiction. This would allow Yara to consider the direct proof separately from the proof by contradiction as well as encourage her to consider the proof by contradiction as a subproof and thus possibly develop an Object conception of the proof method.

The researcher/teacher then prompted Yara to reread the part of question nine that said "Using the method of this proof *[...]*." She then began to read the first two sentences of the presented proof in Figure 4.31 and provided an appropriately modified line for the proof. For example, after reading the statement "Then either there is no multiplicative inverse of r or there are at least two distinct multiplicative inverses of r", she stated "that would be that it equals one solution, I mean zero solutions or at least two solutions." She then had difficulty continuing after representing the line "Then either there is no multiplicative inverse of r or there are at least two distinct multiplicative inverses of r." An excerpt of this difficulty is provided below.

Yara: It's not exactly the same proof because it's not talking about multiplicative inverses. So then, *[pause]* what do we have to show that *[pause]* d and f equal 0?

Teacher: First, let's deal with there are no r such that $ar + b = 0$.

Yara: Couldn't you just [pause] plug it in?

Teacher: And that is everything you wrote on your paper, right? *[...]* So this *[*$r = -\frac{b}{a}$*]* is a solution right?

Yara: So there is NOT, NO solution.

Teacher: Alright, so what did you say we were going to do with d and f?

Yara: Ah... just like *[pause]* solve for d? That would still give you negative b over a. And f is negative b over a. They are equal to each which is the same things as saying you've just got one solution. So contradiction because d and f can't equal each other but they equal each other.

Yara first responded that the proof needed to show that d and f were equal to zero. This response suggested she did not know how to proceed in the proof, so the teacher/researcher guided her with the vague prompt "First, let's deal with there are no r such that $ar+b = 0$." This prompt provided her with enough guidance to complete the proof. In particular, she was prompted to recall what she initially suggested (to show that d and f are equal to zero) and instead appropriately showed that $d = f$, which was a contradiction.

After completing the contradiction part of the proof, Yara questioned how the proof would be classified. A short excerpt from this discussion is provided below.

Yara: And then you get a contradiction inside a contraposition *[pause]* umm... *[pause]* so... So it would still be proof by contradiction even though we were starting with the not Q? Or is it like a contradiction inside a... [interrupted]

Teacher: So think about it like this: *[pause]* I started with P. I wanted to get to Q. So in between here I said assume not Q. Dot dot dot contradiction.

Yara: You still got a contradiction.

Teacher: Therefore Q.

Yara: So it's still a regular proof by contradiction.

Teacher: Because I did not *[pause]* end by not P. Thats why its not a contraposition. If I had assumed not Q and got to not P rather than a contradiction, then it would have been a contraposition.

The discussion above suggested Yara had difficulty coordinating the two proof methods within a single proof. To alleviate this difficulty, the teacher/researcher focused her attention on one proof method at a time by using different colored markers. In addition, to convince her that the proof did not use the contraposition proof method, the teacher/researcher generalized the proof as starting with P, assuming $\sim Q$, arriving at a contradiction and therefore Q, thus $P \to Q$. That is, the teacher/researcher clarified that the proof did not start with $\sim Q$ and arrive at $\sim P$ and thus couldn't be a proof by contraposition.

Overall, Yara required two significant prompts in order to begin the proof: how to prove an implication statement directly and to return to the previous proof by contradiction of a uniqueness statement. She then focused on a line-by-line revision of the previous proof in order to write the initial assumption and to rephrase this initial assumption as either zero solutions to the equation or more than one solution to the equation. The line-by-line revision of these two steps is indicative of an Action conception of proof by contradiction. After a vague prompt to focus on the case where there are zero solutions to the equation, she was able to complete the rest of the proof without explicitly referring to the previous proof or a list of steps she was following, which is indicative of a Process conception of proof by contradiction. Finally, she had difficulty describing the completed proof as it required her to consider proof by contradiction as a subproof within another proof, which suggests she could not coordinate the direct proof with a proof by contradiction. Considering this all together, Yara exhibited at least an Action conception of proof by contradiction with her initial line-by-line revision of the previous proof and did not provide enough evidence of a Process conception of proof by contradiction with how she described completing the proof.

4.3.4 Summary of results

Teaching episode 3 was designed to achieve three goals. First, the episode introduced a set of step-by-step instructions for students to use to construct proofs by contradiction for a uniqueness statement (i.e., $(\exists x!)(P(x))$). Students could then use this set of step-by-step instructions to prove similar statements by contradiction and thus possibly develop an Action conception of proof by contradiction for this type of statement. Secondly, students were encouraged to focus on the roles of groups of lines in a proof and on the key steps of the proof method. They could then relate the roles of lines or collections of lines to interiorize the previous step-by-step instructions into general steps for a uniqueness statement and thus possibly develop a Process conception of proof by contradiction for this type of statement. Thirdly, students reflected on both their prior knowledge of proof by contradiction and a specific example of proof by contradiction to assimilate[13] or accommodate[14] the new logical outline into their general procedure for a proof by contradiction.

Subsection 4.3.1 reported on students' work during Activity 3. The goals of this phase reflected the two of the three goals of teaching episode 3: (1) guide students to construct a set of step-by-step instructions for proofs by contradiction of a uniqueness statement and (2) prompt students to relate the roles of lines, or collections of lines, to interiorize the previous step-by-step instructions into general steps for a uniqueness statement. Thus, the phase consisted of two tasks: (4.3.1.1) outlining a given proof by contradiction for a uniqueness statement and (4.3.1.2) defining proof by contradiction. A brief summary of the results from each task follows.

[13] *Assimilation* occurs when an individual can apply a cognitive structure (proof by contradiction procedure) with minimal change in order to deal with a new situation.

[14] *Accommodation* occurs when an individual needs to reconstruct and modify a cognitive structure in order to deal with a new situation.

Wesley's outline included the two key steps of a proof by contradiction and did not describe the logical relation between these steps, which is indicative of an Action conception of proof by contradiction. Yara's outline included the two key steps of a proof by contradiction as well as described the logical relation between lines and generalized specific lines in the proof to describe their purpose in the overall argument, which is indicative of a Process conception of proof by contradiction.

Wesley's definition of proof by contradiction described the two key steps of a proof by contradiction as cue words and did not describe how these steps logically related to prove the statement, which is indicative of an Action conception of proof by contradiction. Yara's definition of proof by contradiction described the two key steps of a proof by contradiction as well as described how these steps logically related to prove the statement, which is indicative of a Process conception of proof by contradiction.

Subsection 4.3.2 reported on students' responses during Classroom Discussion 3. The purpose of this phase was to formalize students' conceptions of proof by contradiction based on their work and conjectures during Activity 3. Thus, the tasks for this phase were based on the two tasks of Activity 3 as well as an additional task, writing an outline of a proof, that reinforced the formalization of their conceptions. A brief summary of the results from each task follows.

When Wesley was asked to logically justify each line in his outline, he could not provide any logical justification for some lines in the proof and thus exhibited a Pre-Action conception of mathematical logic. When Yara was asked to logically justify each line in her outline, she provided logical justification for all lines in her outline as well as described the logical relationship between collections of lines and thus exhibited a Process conception of mathematical logic.

When prompted to compare three outlines of specific types of proof by contradiction, Wesley instead recalled a general procedure that described the key steps of a proof by contradiction and did not describe any intermediate steps nor described the logical relation between lines, which is indicative of an Action conception of proof by contradiction. When prompted to compare three outlines of specific types of proof by contradiction, Yara produced a minimal change on her previous procedure for proof by contradiction and thus assimilated the new outline into her Schema for proof by contradiction, which is indicative of an enhanced Process conception of proof by contradiction.

When prompted to write a proof outline for a similar proof, Wesley constructed an outline by explicitly following the step-by-step outline for a uniqueness statement from Activity 3 and thus exhibited an Action conception of proof by contradiction. When prompted to write a proof outline for a similar proof, Yara instead wrote a complete proof that followed her general procedure for proof by contradiction and did not rely on the outline for a uniqueness statement from Activity 3 and thus exhibited a Process conception of proof by contradiction.

Subsection 4.3.3 reported on students' work during Exercise 3. The goal of this phase was to reinforce students' conceptions of mathematical logic and proof by contradiction through a series of eight tasks aligned with the proof comprehension assessment model by Mejía-Ramos et al. (2012) as well as an additional question not aligned with the assessment model. A short summary of results for questions two through nine follow.

Question two: When asked to provide an example of a multiplicative inverse and explain why it was an inverse, both Wesley and Yara responded with a pair of numbers and explained that the product of these numbers was 1. Therefore, both students possessed the prerequisite knowledge of multiplicative inverses necessary to understand the particular proof.

Question three: When asked to mathematically justify a statement in the presented proof, Wesley made an appeal to authority and thus exhibited a Pre-Action conception of mathematical logic. When Yara was asked the same, she justified the line with the statement to be proved and thus also exhibited a Pre-Action conception of mathematical logic.

Question four: When asked to logically justify a statement in the proof, Wesley was not able to separate the entirety of his mathematical knowledge from the mathematical knowledge available to prove the statement and thus exhibited a Pre-Action conception of mathematical logic. When Yara was asked the same, she utilized a mathematical theorem without recognizing the conditions for the theorem to be valid were not met and thus exhibited a Pre-Action conception of mathematical logic.

Question five: When asked to identify the purpose of a statement in the presented proof, Wesley cited a rule that aligned with his procedure for a proof by contradiction and thus exhibited an Action conception of the proof method. When Yara was asked the same, she described the general argument of the proof and how the statement followed logically from the initial assumption and thus exhibited a Process conception of proof by contradiction.

Question six: When asked to summarize the main idea of the proof, both Wesley and Yara rephrased the statement to be proved and thus exhibited a Pre-Action conception of proof by contradiction.

Question seven: When asked to provide the key steps of the proof, Wesley responded with one of the two key steps of a proof by contradiction and thus exhibited a Pre-Action conception of the proof method. When Yara was asked the same, she described the two key steps of a proof by contradiction and did not describe how the proof arrived at a contradiction and thus exhibited an Action conception of the proof method.

Question eight: When asked if the proof would still work when the initial conditions were modified, Wesley described how the new conditions affected the algebraic manipulation of the proof and thus exhibited an Action conception of mathematical logic. When asked the same, Yara described how the new conditions affected the meaning of the statement and thus exhibited a Process conception of mathematical logic.

Question nine: When asked to write a similar proof using the method of the presented proof, Wesley responded that he could follow his procedure for proof by contradiction but did not do so and thus possibly exhibited an Action conception of proof by contradiction. When asked the same, Yara initially relied on the previous proof and then switched to rely on her general procedure for proof by contradiction. Thus, she exhibited at least an Action conception of proof by contradiction with her initial reliance on the previous proof and not enough evidence of a Process conception when she switched to her general procedure.

4.4 Teaching Episode 4: Generalizing procedure to prove infinity statements by contradiction

Teaching episode 4 was designed to achieve three goals. First, the episode introduced a set of step-by-step instructions for students to use when constructing proofs by contradiction for an infinity statement (e.g., There are infinitely many prime numbers). Students could then use these instructions to prove similar statements by contradiction and thus possibly develop an Action conception of proof by contradiction for this type of statement. Secondly, students were encouraged to relate the roles of lines or collections of lines in the proof to interiorize the previous step-by-step instructions into general steps for an infinity statement and thus possibly develop a Process conception of proof by contradiction for this type of statement. Thirdly, students reflected on both their knowledge of proof by contradiction and a specific example of proof by contradiction to assimilate or accommodate the new logical outline into their general procedure for a proof by contradiction. All tasks for the episode were designed around these three goals.

This section will report on how students' understanding emerged from an analysis of their individual responses to the tasks and teacher/researcher prompts during teaching episode 4. As in teaching episode 3, the data analyzed and presented for this section will focus on Wesley and Yara - representative students chosen for case study analysis (see Section 3.5 for details on why these two students were chosen).

The following subsections will be organized by the three phases of the teaching episode. Subsection 4.4.1 will focus on students' initial conjectures about proof by contradiction of a uniqueness statement. Subsection 4.4.2 will focus on how the teacher/researcher guided students to formalize the initial conjectures from the previous phase. Subsection 4.4.3 will focus on responses to proof comprehension questions designed to reinforce the formalized conjectures from the previous phase. For each subsection, I will begin by describing the lesson plan of the phase and then proceed by presenting the analysis and interpretation of data collected during the phase. The tasks used for this teaching episode can be found in Appendix A.1 and will be reproduced as needed.

4.4.1 Initial conjectures on infinity proofs by contradiction

This subsection is a report on students' work during Activity 4 (see Appendix A.1 for the complete set of tasks). The goal of Activity 4 was to encourage students to construct a set of step-by-step instructions to prove infinity statements as well as prompt students to reflect on their knowledge of proof by contradiction and a specific example of proof by contradiction to assimilate or accommodate the new logical outline into their general definition of proof by contradiction. Thus, this phase consisted of two tasks: outlining a presented proof by contradiction of an infinity statement and providing a definition of proof by contradiction. For each task, I will first present the goal and reasoning behind the task. Then, I will present an analysis and interpretation of students' responses to the task.

4.4.1.1 Outline of an infinity proof by contradiction Students were asked to read a presented proof of the statement "The set of natural numbers is not finite" and subsequently outline this statement and proof (see Figure 4.35) utilizing predicate logic.

Statement: The set of natural numbers is not finite.

Proof: Assume there are not infinitely many natural numbers. Then there are finitely many natural numbers. Let $N = \{n_1, n_2, n_3, \ldots, n_k\}$ be all the natural numbers, where $n_1 < n_2 < n_3 < \ldots < n_k$. Then $n = n_k + 1$ is a natural number and $n \notin N$, which is a contradiction. Therefore there are infinitely many natural numbers.

Figure 4.35 Presented proof for Activity 4.

As Wesley and Yara participated individually, the teacher/researcher acted as another student with incomplete knowledge during this phase. That is, the teacher/researcher provided memorized rules (e.g., $\sim (P \vee Q) \cong \sim P \wedge \sim Q$) when necessary and did not provide any desired lines for their outline, with an exception for the statement that was proved. A representation for the statement is necessary to utilize any external set of rules to outline the statement. Explanations for any other lines were reserved for the Classroom Discussion phase of teaching episode 4.

The purpose of outlining the logical structure of this presented proof was to encourage students to generalize the specific proof by contradiction into a series of steps that they could then use to write a proof by contradiction and thus possibly develop an Action conception of proof by contradiction for this type of statement. This particular proof was chosen as it is relatively short and provided the basic argument for this type of statement: constructing a new element that should be in the set but, by assumption, is not. In addition, the construction in this proof ($n = n_k + 1$) is relatively simple and easily identifiable as a natural number. This is important as infinity proofs by contradiction have been identified in the literature as especially difficult for students to comprehend due to the seemingly arbitrary constructions these proofs employ (e.g., Antonini & Mariotti, 2008; Leron, 1985). It was thus desirable to present a relatively simple construction that would not inhibit students from generalizing this specific type of proof into a series of general steps to describe the purpose of these seemingly arbitrary constructions and thus possibly develop a Process conception of proof by contradiction for this type of statement.

A desired logical outline of the proof is illustrated in Figure 4.36.

Statement: $P(\mathbb{N})$

1. Assume $\sim P(\mathbb{N})$.
2. Then $\mathbb{N} = \{n_1, n_2, \ldots, n_k\}$ where $n_1 < n_2 < \ldots < n_k$.
3. Let $n = n_k + 1$
4. $n \in \mathbb{N}$ by definition of natural numbers.
5. $n \notin \mathbb{N}$ since $n_k < n$.
6. $\rightarrow\leftarrow$ (lines 4 and 5)
7. $\sim (\sim P(\mathbb{N}))$
8. $P(\mathbb{N})$

Figure 4.36 Desired logical outline of presented proof during Activity 4.

This outline is desired for three main reasons. First, it contains the two key steps of a proof by contradiction (lines 1 and 6). Secondly, the outline represents the statement as $P(\mathbb{N})$. For this proof, students were not aware of a quantifier symbol that represented 'infinitely many'[15] (as opposed to knowing symbols for 'there exists' and 'for all'). This restricted students to representing the statement as either $P(X)$ (X is a set with infinitely many elements) or simply as P. While P would be a valid representation, $P(X)$ could be used to represent similar infinity statements and was thus more desirable. Finally, the outline contains all necessary mathematical and logical justifications to provide a complete argument for the proof. For example, line 5 provides a mathematical justification for why the step is true while line 6 justifies the contradiction by citing lines 4 and 5.

A student's outline of the presented proof and explanation of their outline would indicate their conception of proof by contradiction. A student with an Action conception of proof by contradiction would describe the specific lines of the presented proof and represent every line of the proof. In addition, their outline would not provide mathematical or logical justifications for particular lines. When asked to describe their outline, this student would describe each step, in order, and would not be able to describe the purposes of pairs or collections of lines. In contrast, a student with a Process conception of proof by contradiction would describe the purpose of groups of lines in the presented proof and represent these lines together. In addition, their outline would provide any necessary mathematical or logical justifications to complete the argument, such as the justification for why $n \notin \mathbb{N}$. When asked to describe their outline, this student would describe the purposes of parts of the outline (e.g., the assumption part, contradiction part, and conclusion) rather than describe every line.

Wesley and Yara exhibited different conceptions of proof by contradiction based on their outlines to the presented proof. A detailed description and interpretation of their outlines is provided below.

[15] Infinitely many elements sharing a property can be represented as $(\exists^\infty x)(P(x))$, though this representation is not common as it does not make clear whether the set contains countably or uncountably many elements.

Wesley was able to outline the key steps of the proof after he was provided a representation for the statement ($P(X)$, where the set X has infinitely many elements) in order to utilize his procedure for proof by contradiction. He did not provide justifications for these lines nor could he describe, in general, the purposes of these lines. Details of how he constructed this outline follow.

Statement: $P(\mathbb{N})$

1. Assume $\sim P(\mathbb{N})$
2. $\mathbb{N} = \{n_1, n_2, n_3, \ldots, n_k\}$
3. $n = n_k + 1$ is a natural number
4. n is not an element of $\mathbb{N}$
5. $\sim (\sim P(\mathbb{N}))$
6. $P(\mathbb{N})$

Figure 4.37 Wesley's outline of presented proof during Activity 4.

Wesley was first asked to represent the statement using either propositional or predicate symbols. When he could not do so, the teacher/researcher presented him with a desired representation of the statement. However, he did not understand the meaning of this representation, as illustrated in the excerpt below.

Teacher: So do we have a way to write the statement: the set of natural numbers is not finite? *[pause]*

Wesley: *[raises hand as a half arm shrug, long pause]* N is greater than zero. *[laughs]* *[Prompts Wesley to think about what* N *greater than zero would mean.]*

Teacher: So maybe if we just say something like $P(X)$ means *[pause]* X has infinitely many elements. *[long pause]* So whatever we put in there we say that set has infinitely many elements. So if we use that, what would our statement be, in terms of $P(X)$?

Wesley: Umm *[pause]* what would your statement be for $P(X)$? *[long pause]* I always make this harder than it has to be. Okay, so I mean, are you just looking for this $n = n_x + 1$? *[sic]*

First, Wesley improperly represented the statement as "N is greater than zero" as a set cannot be greater than a number (improper use of symbols). The teacher/researcher then explicitly gave and defined the representation $P(X)$. Even so, Wesley described the next line of the proof (the construction $n = n_k + 1$) as the representation for the previous line. Therefore, Wesley could not represent the statement "The set of natural numbers is not finite" and thus exhibited a Pre-Action conception of mathematical logic.

After representing the statement as $P(\mathbb{N})$, he stated "So... so the proof would be, if you are going to prove that by contradiction it would be... you are going to assume that it's not [pause] $P(N)$." The phrase "assume it's not $P(N)$" is a desired first step of the proof and suggests he followed a general procedure that began by negating the assumption, which is indicative of an Action conception of proof by contradiction He then stated:

> Then let N equal the set n_1, n_2, n_3, n_k are natural numbers. *[long pause]* So how do you... how... okay *[pause]* so I have not a set of... I have not a set of natural numbers and we're... I'm assuming that how I am going to contradict that to itself.

The phrase "I have not a set of natural numbers" suggests he interpreted $P(\mathbb{N})$ as $\mathbb{N}$, and thus suggests he would not have interpreted $\sim (P(\mathbb{N}))$ as $\mathbb{N} = \{n_1, n_2, n_3, \ldots, n_k\}$ without the presented proof. This undesired representation (indicative of a Pre-Action conception of mathematical logic) strengthens the interpretation that he followed an external procedure for the method as he does not understand what $\sim P(\mathbb{N})$ represents.

He then has difficulty justifying the statement that introduced the construction. Consider the following excerpt as he discussed how he justified this construction.

Wesley: And then it just tells you... and I'm assuming that n is equal to n_k plus 1 is a natural number... I'm assuming that's a proof from something else?

Teacher: I mean, how do we define our natural numbers?

Wesley No, and I understand, it's 1 and 2, 3, 4... I understand the concept. I guess what's hurting me is, like I said *[pause]* So, I get the $n_k + 1$ is a natural number. That... I get it. Is that something that we are using... something similar to the 2 divides, or the a divides b is equal to b times some k equals a? I mean, is this some rule that's supposed to be out there...

In particular, the phrases "So I get the $n_k + 1$ is a natural number. *[...]* Is that something that we are using *[...]* the a divides b is equal to b times some k equals a?" suggests a distinction between believing a claim is true and justifying that the claim is true. In this case, his comparison of $n_k + 1$ being a natural number to the definition of division ($a \mid b$ iff $bx = a$) suggests that while he could give examples of natural numbers and believed $n_k + 1$ was a natural number, he was not aware of the formal definition of the natural numbers. In other words, his non-formal conception of natural numbers impeded his justification of the statement and thus indicated a Pre-Action conception of mathematical logic.

Wesley then read the next line of the proof (n is not an element of $\mathbb{N}$) and exhibited difficulty reconciling the contradiction caused by the construction. In particular, he stated:

> And let's see... and n is not an element of N, which is a contradiction. *[long pause, mumbling and reading next lines to self]* But that's *[pause]* but how do you... how do you map then $n = n_k + 1$ is a natural number and n is not an element in N?

He recognized that the proof stated there was a contradiction ($n \in \mathbb{N} \wedge n \notin \mathbb{N}$) and wrote this down. However, the phrase "how do you map then $n = n_k + 1$ is a natural number and n is not an element in N?" suggests he could not justify why n is not an element in $\mathbb{N}$ and thus reconcile how n is simultaneously both in and not in the set. In other words, he represented the contradiction desirably without understanding why the proof arrived at a contradiction, which is indicative of an Action conception of mathematical logic.

Finally, after arriving at a contradiction, he returned to his procedure for proof by contradiction and stated "So then you would have not not *[pause]*." The phrase "not not", with no other description of the assumption, suggests he focused on key words to describe his procedure for proof by contradiction. That is, after arriving at a contradiction, the next step is "not not" the statement. Describing a key step in the proof procedure using cue words is indicative of an Action conception of proof by contradiction.

Overall, Wesley described his previously defined procedure for proof by contradiction (see Figure 4.26 on page 152) to outline the proof, which is indicative of an Action conception of proof by contradiction. However, he could not describe what these steps meant in the context of this proof (e.g. what $\sim P(\mathbb{N})$ represented for this proof) and he could not justify intermediate steps outside of his procedure (e.g., why $n \notin \mathbb{N}$), which is indicative of a Pre-Action conception of mathematical logic.

Yara used her general procedure for proof by contradiction to first give an outline of the procedure of the proof, after which she outlined the details of the proof (see Figure 4.38). In particular, she described the three general roles of lines in a proof by contradiction: (1) assume the statement is false, (2) using this assumption and other mathematical knowledge, arrive at a contradiction, and (3) since the assumption led to a contradiction, the assumption is false and thus the statement is true. Details for how she constructed this outline follow.

Statement: $P(\mathbb{N})$
Proof: Assume $\sim P(\mathbb{N})$
$\mathbb{N}$ is finite.
$\mathbb{N} = \{n_1, n_2, n_3, \ldots, n_k\}$; $n_1 < n_2 < n_3 < \ldots < n_k$
$n = n_k + 1$, $n \in \mathbb{N} \wedge n \notin \mathbb{N}$
$\sim (\sim P(\mathbb{N}))$
$P(\mathbb{N})$

Figure 4.38 Yara's outline of presented proof during Activity 4.

Yara began the outline by desirably representing the statement as $P(\mathbb{N})$, as illustrated in the following discussion.

Teacher: Let's go ahead and try to get our structure down for this proof. So statement. *[pause]*

Yara: It's like one of those $P(x)$ kind of things.

Teacher: So what does $P(x)$ mean?

Yara: $P(x)$... x is a natural number? No? Set of natural numbers is not finite?

While she expressed some doubt in her second interpretation ("Set of natural numbers is not finite?"), she was able to desirably represent the statement in predicate form. In addition, she had not previously represented a set as the object in a predicate form. This suggests she described the meaning of $P(x)$ rather than a rule that can only input a number, which is indicative of a Process conception of mathematical logic.

After she expressed the statement in the form $P(\mathbb{N})$, she proceeded to outline the first two general purposes of a proof by contradiction. The teacher/researcher noted she skipped lines in the outline and thus prompted her for the last general purpose of lines in the proof. This discussion is reproduced below.

Yara: So we assume not?

Teacher: Assume not P of the natural numbers, okay.

Yara: And... *[long pause]* we need to get a contradiction.

Teacher: Yeah, so we'll keep going... contradiction. Therefore not *[long pause]*

Yara: Not $P(N)$.

The teacher/researcher's phrase "therefore not" may have prompted Yara to state "Not $P(N)$". However, the rest of the discussion suggests Yara relied on her general procedure for proof by contradiction to outline the procedure of the particular proof, which is indicative of a Process conception of proof by contradiction.

Overall, she first desirably represented the statement without relying on a previous example, which is indicative of a Process conception of mathematical logic. She used her general conception of proof by contradiction to group parts of the outline into three general purposes: assumption, contradiction, and conclusion. She then provided the details of one of these purposes: how the proof arrived at a contradiction. Therefore, Yara exhibited a Process conception of proof by contradiction.

4.4.1.2 Definition of proof by contradiction with quantification After reading the presented proof and writing an outline of the proof, Wesley and Yara were asked to write a definition for proof by contradiction. The goal of this task was to encourage students to reflect on both their knowledge of proof by contradiction and a specific example of proof by contradiction to assimilate or accommodate the new logical outline into their general steps for a proof by contradiction. In addition, students were asked to explain the logic behind the proof method (if their definition did not already include such an explanation). While it was not included in the written task, the teacher/researcher prompted students to consider their previous definitions and outlines of presented proofs to write a possibly new definition for the proof method.

An ideal definition for proof by contradiction for this task would include the two key steps of the proof method as well as describe the general logic behind the method. For example, an ideal definition of proof by contradiction may state “A proof by contradiction is when you start by assuming a statement is false and then you use that and other mathematical knowledge to show the assumption is false, thus the statement is true.” This example definition includes both key steps and describes how these key steps logically relate to prove the statement is true.

A student’s definition of proof by contradiction would indicate their level of understanding of the proof method. A student with an Action conception of proof by contradiction would focus on the specific lines of a proof by contradiction without providing any reasoning behind the method. For example, a student may state ”Start with an assumption and get a contradiction.” This definition does not provide a relation between these two key steps and does not provide a complete description of each key step. Students with an Action conception of proof by contradiction may also focus on two cues in their definition: assumption and contradiction. In contrast, a student with a Process conception of proof by contradiction would describe the relation between the two key steps of a proof by contradiction. For example, a student may state "Assume the statement is not true and, using this assumption, arrive at a contradiction. This would mean the assumption is not true and thus the statement is true." This definition would describe the role of the two key steps completely rather than focus on cues such as 'assumption' and 'contradiction'. Wesley and Yara's definitions follow.

When asked for a definition of proof by contradiction, Wesley stated "To prove a statement by assuming it's contradiction is false." Again, he uses 'contradiction' in place of 'negation' and he does not refer to the second key step of the proof method (arriving at a contradiction). By only noting one of the two key steps in a proof by contradiction, Wesley exhibited a Pre-Action conception of proof by contradiction. When asked how the particular proof exemplified his definition, he stated "Well, you assumed that *[pause]* the natural numbers were not an infinite set and you proved that was a false assumption by proving $n = n_k + 1$." The phrase "proving $n = n_k + 1$" suggests he does not understand the logical role of this construction in the proof and thus exhibited a Pre-Action conception of mathematical logic.

When asked for a definition of proof by contradiction, Yara stated "Assuming *[pause]* the negation of the statement and getting a contradiction." This definition describes both key steps of the proof method but does not describe how these steps imply the statement is true, which is indicative of an Action conception of proof by contradiction. When asked to explain why the presented proof is a proof by contradiction, she stated "Because you are assuming the negation and you got a contradiction because you showed that n is *[pause]* a natural number but it's not an element in... in the set of natural numbers." Again, this response refers to the key steps of a proof by contradiction but does not describe how these steps imply the statement is true. Therefore, Yara exhibited an Action conception of proof by contradiction.

4.4.2 Formalization of proof by contradiction with quantification

This subsection is a report about Wesley and Yara's responses during Classroom Discussion 4 (see Appendix A.1 for general questions that guided this discussion). The purpose of this phase was to guide students to formalize their conceptions of proof by contradiction based on their work and conjectures during Activity 4. In particular, the goal of this phase was to develop a general procedure for the method that students could then utilize, along with mathematical logic, to prove statements by contradiction. Therefore, the tasks for this phase are based on developing and testing a general procedure for proof by contradiction. For each task, I will first present the goal and reasoning behind the task. Then, I will present an analysis and interpretation of students' responses to the task.

4.4.2.1 Discussion of comparison between proof by contradiction procedures Students were prompted to review their outlines from Activities 1, 2, 3, and 4 to write a list of steps to prove any statement by contradiction. The goal of this task was to encourage students to reflect on both their knowledge of proof by contradiction and a specific example of proof by contradiction to assimilate or accommodate the new logical outline into their general procedure for a proof by contradiction. Desired outlines for each presented proof (Figure 4.39) were provided to students.

By looking across the four proof outlines, one may observe commonalities among some lines or groups of lines. For example, lines 1 and 2 can be grouped together to describe assuming the negation of the statement is true while the final two lines can be grouped together to describe the assumption as false due to arriving at a contradiction. As in teaching episode 3, a desired series of steps would contain three general purposes or roles: (1) Assuming the negation of the statement is true, (2) Using the initial assumption and mathematical knowledge to arrive at a contradiction, and (3) Since assuming the negation of the statement arrived at a contradiction, the statement must be true. Each role is necessary to write a proof by contradiction. In addition, these roles synthesize multiple lines of a particular proof to describe the purpose of a collection of lines.

Activity 1 **Statement:** $P \to Q$	**Activity 2** **Statement:** $(\nexists x)(P(x))$	**Activity 3** **Statement:** $(\exists! x)(P(x))$	**Activity 4** **Statement:** $P(\mathbb{N})$
1. Assume $\sim (P \to Q)$	1. Assume $\sim (\nexists x)(P(x))$	1. Assume $\sim (\exists! x)(P(x))$	1. Assume $\sim P(\mathbb{N})$
2. $P \wedge \sim Q$	2. $(\exists x)(P(x))$	2. $\sim (\exists x)(P(x)) \vee (\exists x, y)(P(x) \wedge P(y) \wedge x \neq y)$	2. $\mathbb{N} = \{n_1, n_2, \ldots, n_k\}$ where $n_1 < n_2 < \ldots < n_k$
3. $\sim Q_k$	3. $P(n)$	3. Show $P(n)$ for some n	3. Let $n = n_k + 1$.
4. $(\sim Q_k \wedge P) \to Q_k$.	4. Using $P(n)$, get to a contradiction.	4. $(\exists x, y)(P(x) \wedge P(y) \wedge x \neq y)$	4. $n \in \mathbb{N}$ by definition of natural numbers.
5. Q_k	5. $\sim (\sim (\nexists x)(P(x)))$	5. $P(x) \wedge P(y) \to x = y$	5. $n \notin \mathbb{N}$ since $n_k < n$.
6. $Q_k \wedge \sim Q_k$	6. $(\nexists x)(P(x))$	6. $(\nexists x, y)(P(x) \wedge P(y) \wedge x \neq y)$	6. $\to\leftarrow$ (lines 4 and 5)
7. $\sim (\sim (P \to Q))$ 8. $P \to Q$		7. $\to\leftarrow$ (lines 2, 3, and 6)	7. $\sim\sim P(\mathbb{N})$
		8. $\sim (\sim (\exists! x)(P(x)))$	8. $P(\mathbb{N})$
		9. $(\exists! x)(P(x))$	

Figure 4.39 Side-by-side desired outlines introduced in Activities 1 through 4.

Students' explanations of these general steps would indicate their conception of proof by contradiction. A student with an Action conception of proof by contradiction would focus on the necessary steps of the method by cue words and without any further description of the logic behind the proof method. For example, step 2 may simply be stated as 'contradiction' or step 3 may just refer to the double negation of a statement and therefore the statement is true. The student would need to write the details for each step of a particular proof and could not skip steps in the procedure. In addition, the student would not be able to give the logical relation between steps of the procedure. In contrast, a student with a Process conception of proof by contradiction would describe the necessary steps of the proof but would also describe, in their own words, intermediate steps between these necessary steps that provide reasoning behind the method. For example, a step after the initial assumption could state "Use the assumption and mathematical knowledge to derive a contradiction." This example step would provide logical reasoning for a group of steps in the context of the proof method. Using the reasoning between steps, the student could describe how to proceed in a particular proof without the need to write the details of that proof. Wesley and Yara's general procedures and explanations of these procedures follow.

When asked to compare the four outlines to construct a new list of steps, Wesley instead described a general procedure (see Figure 4.40) similar to the one he developed during Classroom Discussion 3 (see Figure 4.26 on page 152).

Statement: S

1. Assume $\sim S$ is true
2. Attempt to prove $\sim S$ is true
3. $\rightarrow\leftarrow$ of $S = \sim S$
4. $\sim(\sim S)$
5. S

Figure 4.40 Wesley's general procedure for proof by contradiction during Activity 4.

This outline described the two key steps of a proof by contradiction: assuming the negation of the statement is true (line 1) and arriving at a contradiction (line 3). However, lines 2 and 3 are not desirable as they both indicate circular reasoning. In addition, he previously represented a contradiction as $Q = \sim Q$ during Exercise 3 in response to Question 8, which asked for the key steps of a particular proof. The teacher/researcher thus focused on line 3 in order to formalize the procedure.

First, the teacher/researcher pointed out that $S = \sim S$ was not present in any of the four outlines Wesley was presented to compare, to which he responded:

> So... and I don't see S equals not S, so I understand that. How many times, or I shouldn't say how many times... how *[pause]* often is that kind of, are you going to come to this kind of conclusion where you are not coming to a... you get your contradiction but it's not really just an S doesn't equal not S?

This suggests Wesley is in a state of cognitive dissonance. His first comment suggests that none of the proofs seen up until this point had a contradiction represented as $S = \sim S$. His second comment then suggests a contradiction represented as $S = \sim S$ is common with the phrase "but it's not really just an S doesn't equal not S?" The teacher/researcher then focused Wesley's attention specifically on the contradiction in the presented proof from Activity 2 (that a number was both even an odd) to clarify that the contradiction was not directly related to the statement and thus not representable as $S \wedge \sim S$. He then responded "But you got an odd and an even. *[...]* I'm used to seeing a comparison in that step." This response suggests that the representation $S = \sim S$ does not refer to the original statement and instead refers to a general comparison of two statements that cannot both be true. In other words, Wesley exhibited a Pre-Action conception of mathematical logic as he misrepresented his meaning of a general contradiction ($Q \wedge \sim Q$) for the proof method.

After formalizing step 3, Wesley continued with the next task of the phase and thus exhibited an Action conception of proof by contradiction as his outline focused on the key steps of the proof method without explaining how these steps logically imply the statement is true. In addition, he exhibited a Pre-Action conception of mathematical logic as he included circular reasoning and improper representations of statements.

When asked to compare the four outlines to construct a new list of steps, Yara responded:

> It would be *[long pause]* you are assuming the negation of the statement *[long pause]* and then *[pause]* you'd *[pause]* do some work, like that one right there *[Points to outline from Activity 4]*. And you get a contradiction. Which ultimately shows that your statement was true.

This procedure is similar to her procedure during Classroom Discussion 3 (see Figure 4.41).

Classroom Discussion 3 **Statement:** P	**Classroom Discussion 4** **Statement:** S
1. Assume $\sim P$ 2. Negate P (Rewrite $\sim P$) 3. Use math skills to get to a contradiction 4. $\sim$ Assumption 5. P	1. Assume $\sim S$ 2. Do work 3. $\rightarrow\leftarrow$ 4. S

Figure 4.41 Side-by-side procedures for proof by contradiction constructed during Classroom Discussion 3 (left) and Classroom Discussion 4 (right).

The new procedure describes the three general purposes in a proof by contradiction: assuming the statement is not true (line 1), using this assumption and other mathematical knowledge, arrive at a contradiction (lines 2 and 3), and that since the assumption arrived at a contradiction, the statement is true (line 4). Compared to Classroom Discussion 3, she no longer needed to specify rewriting the negation of the statement and no longer needed to write that the contradiction implies the assumption is false. Therefore, Yara assimilated the presented proof into her previous procedure and thus exhibited a Process conception of proof by contradiction.

4.4.2.2 Writing a proof outline using a general procedure After creating a list of steps to prove any statement by contradiction, Wesley and Yara were asked to write a proof for the statement "The set of primes is infinite" - a new, related infinity statement. The goal of this task was to see how students can use their current understanding of proof by contradiction when given a slightly new problem statement. This statement was chosen as it is one of the two classic proofs by contradiction covered in transition-to-proof courses[16].

A desired proof for the statement is illustrated in Figure 4.42.

[16]The other being a proof of the statement "$\sqrt{2}$ is an irrational number."

Statement: The set of primes is infinite.

Proof: Suppose the set of primes is finite. Let $p_1, p_2, p_3, \ldots, p_k$ be all those primes with $p_1 < p_2 < p_3 < \cdots < p_k$. Let n be one more than the product of all of them. That is, $n = (p_1 p_2 p_3 \ldots p_k) + 1$. Then n is a natural number greater than 1, so n has a prime divisor q. Since q is prime, $q > 1$. Since q is prime and $p_1, p_2, p_3, \ldots, p_k$ are all the primes, q is one of the p_i in this list. Thus, q divides the product $p_1 \cdot p_2 \cdot p_3 \cdot \ldots \cdot p_k$. Since q divides n, q divides the difference $n - (p_1 p_2 p_3 \ldots p_k)$. But this difference is 1, so $q = 1$. From the contradiction, $q > 1$ and $q = 1$, we conclude that the assumption that the set of primes is finite is false. Therefore, the set of primes is infinite.

Figure 4.42 Desired proof (Smith et al., 2015, p. 44) for the statement in Classroom Discussion 4.

This proof is desired for three main reasons. First, it included the two key steps of a proof by contradiction: assuming the statement is not true and arriving at a contradiction. Secondly, it provides a description of the relation between the two key steps and why these steps prove the statement is true. Finally, every line in the proof is either mathematically or logically justified.

A student's proof and explanation of this proof would indicate his or her conception of proof by contradiction. A student with an Action conception of proof by contradiction would rely on a list of steps to write the necessary steps for the proof. In particular, the student would not provide reasoning between steps or describe why arriving at a contradiction implies the statement is true. If asked to summarize the proof, this student would describe each line sequentially and not describe the proof with the general procedure of a proof by contradiction. In contrast, a student with a Process conception of proof by contradiction would describe the general argument of the proof without writing the details of each line (similarly to the desired proof outline). This student would rely on a general procedure for proof by contradiction that could then be modified for the statement. He or she could also provide a description of the purposes of lines between the assumption step and the contradiction step.

A student's outline of a proof is influenced by more than his or her conception of proof by contradiction. That is, while a student's conception of proof by contradiction would influence how he or she outlines a proof, a student's conception of mathematical logic influences how he or she enacts the outline. In addition, this particular proof requires enough knowledge of prime numbers to construct a prime number from a given set of numbers. Therefore, I will describe how Wesley and Yara's responses illustrated their conception of proof by contradiction and mathematical logic as well as their knowledge of prime numbers.

Both Wesley and Yara relied on the outline from Activity 4 to write a proof for the statement. For example, Yara stated "So then *[pause]* is it the same set up? Basically? It's the same thing, but instead of the natural numbers, it's the prime numbers." and explicitly referenced that the procedures would be the same. This suggests both students exhibited an Action conception of proof by contradiction. The difference between Wesley and Yara's proofs centered around the construction of a new prime number. How each student responded to the need of this construction, and how it illustrated their conception of proof by contradiction as well as knowledge about prime numbers, follows.

After writing the assumed set of all prime numbers as $P = \{p_1, p_2, \ldots, p_k\}$, Wesley stated that he knew of an equation that generated primes, but could not remember it at the moment. He then suggested $n + 1$, the construction from Activity 4. Details of this conversation are provided below.

Wesley: Umm *[pause]* so I'm sure there is some equation I am not aware of that you do the primes; what is it? Uhh *[pause]*

Teacher: A way to generate primes?

Wesley: I know I've done this before. There's an equation, it would be like *[pause]* I don't know, you are going to have to tell me. You are going to have to tell me because I honestly don't know. Can we do n plus 1 whatever you n is, you do n plus 1? And then you prove it's prime.

Teacher: So if I take any prime and add 1, is that also prime?

Wesley: No.

Teacher: Okay. Why? Give me an example, or a counterexample.

Wesley: 3. 3 is prime, you add 1 it's 4 and that's divisible by 2.

Since he did not recall how to generate primes, he suggested $n + 1$: the construction from Activity 4. After prompting, he recognized that this would not work and could give an example of why it would not work. This suggests Wesley focused on the previous argument, including the construction, and did not describe the meaning of statement in the current proof until he was prompted to do so. At that point, he recalled his knowledge about primes to discard the construction $n+1$ and could not provide an adequate construction to complete the proof, which indicates he did not have the knowledge of prime numbers necessary to complete the proof and thus exhibited an Action conception of proof by contradiction.

After writing the assumed set of all primes as $P = \{p_1, p_2, \ldots, p_k\}$, Yara stated "Now this is the part that's kind of tricky where it's not exactly the same. Because with a natural number, we can just add 1 and be fine but with prime numbers, just adding 1 doesn't mean that the next number is going to be prime." This response illustrates she compared the context of the previous construction ("we can just add 1 and be fine but with prime numbers") to the context of the new construction ("just adding 1 doesn't mean that the next number is going to be prime") and recognized the exact construction would not work. However, she did not give any other possible constructions. This suggests her knowledge of prime numbers was insufficient to construct a new prime number and that if she had such knowledge, she could have completed the proof. Therefore, Yara exhibited an Process conception of proof by contradiction as she described the new context of the proof and did not follow the previous procedure without modification.

4.4.3 Reinforcing formalization of proof by contradiction with quantification

This subsection is a report about students' responses to tasks from Exercise 4 (see Appendix A.1 for the complete set of tasks) - the last phase of teaching episode 4. The goal of this phase was to reinforce students' conceptions of mathematical logic and proof by contradiction through eight questions (i.e., eight tasks) aligned with the proof comprehension assessment model by Mejía-Ramos et al. (2012) as well as an additional question not aligned with the assessment model. All nine questions were written for a presented proof (Figure 4.42 on page 193) and are provided in Figure 4.43.

Students were first given approximately 20 minutes to individually respond to the comprehension questions above. Then, the teacher/researcher asked students to share their answers and describe the reasoning behind their response to each question. After eliciting the student's response and reasoning behind their response, the teacher/researcher provided a desired response to the question (if necessary).

1. Compare the outline of your proof in question 4 *[from Activity 4]* to the proof above. Explain how your proof compares to the given proof in terms of: (1) general structure, (2) specific lines, and/or (3) overall approach to the proof.
2. Please give an example of a set that is infinite and explain why it is infinite.
3. Why does n have to have a prime divisor?
4. Why exactly can one conclude that q divides the difference $n - (p_1 \cdot p_2 \cdot p_3 \cdot \ldots \cdot p_k)$?
5. What is the purpose of the statement "Let $p_1, p_2, p_3, \ldots, p_k$ be all those primes with $p_1 < p_2 < p_3 < \cdots < p_k$."?
6. Summarize in your own words the main idea of this proof.
7. What do you think are the key steps of the proof?
8. In the proof, we define $n = (p_1p_2p_3 \ldots p_k) + 1$. Would the proof still work if we instead defined $n = (p_1p_2p_3 \ldots p_k) + 31$? Why or why not?
9. Define the set $S_k = \{2, 3, 4, \ldots, k\}$ for any $k > 2$. Using the method of this proof, show that for any $k > 2$, there exists a natural number greater than 1 that is not divisible by any element in S_k.

Figure 4.43 Comprehension questions for the presented proof during Exercise 4.

Question one, which was not aligned with the proof comprehension assessment model, prompted students to compare their proof from Classroom Discussion 4 to the presented proof in Figure 4.42. The goal of this question was to compare two similar proofs in order to reinforce students' conception of proof by contradiction by identifying common key steps and general procedure of the two proofs. A desired response would describe a correspondence between the two key steps of a proof by contradiction as well as describe how the proof arrived at a contradiction. These three ideas would then be used to summarize the procedure and main idea of the proof. However, responses to this question did not provide sufficient evidence of students' conception of proof by contradiction or mathematical logic and therefore will not be discussed further.

The remainder of this subsection will be organized by comprehension questions two[17] through nine. For each question, I will first describe what it was meant to assess in terms of the proof comprehension assessment model by Mejía-Ramos et al. (2012) (described on page 23). Then, I will describe the goal of the assessment in terms of the students' understanding of the presented proof, mathematical logic, or proof by contradiction. Finally, I will present an analysis and interpretation of Wesley and Yara's responses to the question.

Question two assessed whether students could identify the meaning of terms and statements in the presented proof. In particular, the question asked students to provide an example of a set that is infinite and explain why this set is infinite. The goal of this question was to determine whether students understood one of the mathematical terms in the proof (infinite) that was necessary to understand the logical argument of the proof, as the literature suggested students struggle with proof comprehension due to a lack of mathematical knowledge (e.g. Moore, 1994). In other words, the goal of this question was to assess prerequisite knowledge necessary to understand the particular proof. A desired response would present a set, such as the rational numbers, and state that all elements in the set cannot be listed.

[17]Question one did not provide sufficient evidence of students' conception of proof by contradiction or mathematical logic.

Both Wesley and Yara exhibited an intuitive understanding of infinity but could not provide a formal definition for their examples. Welsey provided $\mathbb{Z}$ and stated that it was infinite "Because it's everything. It's all integers. So you can always do k and then $k+1$." The phrase "you can always do k and then $k+1$" suggests he focused on the description of why $\mathbb{N}$ was infinite. Yara provided $\mathbb{R}$ and stated that it was infinite "Because there is a lot of them. So it can go from negative infinity alllllll the way up to positive infinity. You don't know where to stop so..." The phrase "you don't know where to stop" suggests the same intuitive understanding as Wesley exhibited. That is, a set is infinite if, given a 'stopping point', one can find another number past this point. This conception of infinite is sufficient to understand the logic of the proof - that given a finite set of primes, a larger prime can be created that is outside the set.

Question three assessed whether students could justify a statement in the presented proof. The goal of this question was to assess whether students recognized that by the Fundamental Theorem of Arithmetic[18], every natural number greater than 1 has a prime divisor. In other words, the question assessed whether students recognized why the conditions 'natural number' and 'greater than 1' implied n had a prime divisor. Responses to this question would indicate students' conception of mathematical logic as this question required a mathematical justification (as opposed to a logical justification). A desired response to this question would indicate the statement follows from the Fundamental Theorem of Arithmetic, either by name or content.

Both Wesley and Yara were not able to provide a desired justification for the statement. Wesley responded "To show that n is not prime." All numbers have prime divisors and so n having a prime divisor cannot be justification for showing the number is not prime. Yara responded "Because it's greater than 1? I don't really know what that has to do with it." While she indicated one of the conditions to invoke the Fundamental Theorem of Arithmetic, it was clear she could not justify the statement beyond what was written in the proof. Therefore, both Wesley and Yara exhibited a Pre-Action conception of mathematical

[18] Every natural number greater than one can be factored into a unique set of prime numbers.

logic as they could not mathematically justification the claim.

Question four assessed whether students could justify a statement in the proof. In particular, the question focused on the following mathematical theorem: For integers a, b, and c, if a divides b and a divides c, then a divides $b - c$. The goal of this question was to assess whether students recognized this theorem and could mathematically justify why q divided the difference of n and the product of all prime numbers. Responses to this question would indicate students' conception of mathematical logic as it required students to identify that if an integer divides two other integers, it must divide the difference of these integers. A desired response would indicate and adequately explain the previously described theorem that makes the statement true.

Both Wesley and Yara were not able to provide a desirable justification for the statement. Wesley did not initially respond when working individually. When asked if he had a response after discussing questions 1 through 3, he stated "Now, I'm looking here - n is a natural number greater than 1, so n has a prime divisor q. And based upon what you just said there *[pause]* it could be itself, potentially. I was going to put n is prime, but... *[pause]*." By the phrase "based upon what you just said there", he referred to the discussion during question three about what the divisors of q could be (i.e., that $q = n$ when n is prime). That n is prime is not a necessary condition to invoke the division theorem and thus he did not provide an valid justification for the statement. Yara initially responded "I'm not even sure where q comes from but maybe b/c $n - (p_1 \cdot \ldots \cdot p_k) = 1$." This suggests she justified that q divides the difference as the difference is equal to one, which is not a valid justification for the statement. When she was asked to clarify her response, she stated "The only way to get 1 is to divide it by itself." This response suggests she justified the statement by the consequence of the statement - a form of circular reasoning. Therefore, both Wesley and Yara exhibited a Pre-Action conception of mathematical logic.

Question five assessed whether students could identify the logical status of a statement in the proof. This time, students were asked what the purpose of the statement "Let $p_1, p_2, p_3, \ldots, p_k$ be all those primes with $p_1 < p_2 < p_3 < \cdots < p_k$." The goal of this question was to assess whether students could describe the purpose of the statement as a whole (defining the primes to construct n) and the purpose of the strict inequalities (indicate that the primes are distinct). Responses to this question would indicate students' conception of mathematical logic as it required the student to identify the logical necessity of the two parts of the statement. A desired response would indicate the statement rephrases the initial assumption as well as indicate that the primes listed must be distinct otherwise n would not be prime.

Wesley initially responded the purpose of the statement was "To show an increasing set." This phrase does not relate the line statement to the initial assumption nor does it describe why the statement would need to show that the set is increasing. He did not provide any further explanation or description of the line when asked to describe the reasoning behind his response. Therefore, Wesley exhibited a Pre-Action conception of mathematical logic.

Yara did not initially respond to the question when working individually. When prompted to respond by the teacher/researcher, she stated "I don't know. I don't know why it necessarily matters they were in order less than and greater than... *[trails off]*." This response focuses on the purpose of the strict inequalities and suggests she did not recognize their purpose in the proof. The teacher/researcher then focused her attention and asked for the purpose of the first part of the statement (let $p_1, p_2, p_3, \ldots, p_k$ be all of those primes). She responded "Because you are listing out the primes... oh! That's why! Because if you list them in a random order, you probably don't pick one that's bigger than the other one, right?" This response suggests she did not recognize the purpose of listing the primes to construct the prime n. In addition, the phrase "you probably don't pick one that's bigger than the other one" suggests she focused on the reasoning provided for the strict inequality for the presented proof during Activity 4. In that proof, the construction was built from only the largest value. As all primes were used in the construction of this proof, her justification is not valid. Therefore, Yara exhibited a Pre-Action conception of mathematical logic.

Question six assessed whether students could summarize the proof via high-level ideas. The purpose of this question was to assess whether students could summarize the proof in their own words and, if they could, how they summarized the proof. For example, a summary for the presented proof could be similar to the following:

> It's a proof by contradiction, where you assume the statement isn't true. So then we can list all the prime numbers. But if we multiply them all together and add 1, we make another prime number that is not in our list, which is a contradiction. Therefore your statement is true.

This summary describes the specific procedure of the presented proof and would illustrate an Action conception of proof by contradiction. In contrast, another summary could be similar to the following:

> If the statement were false, then there would be finitely many prime numbers. But given a list of primes, we can always make a number that is relatively prime to all of the primes simultaneously. Therefore, the statement must be true.

This response describes the main idea of the proof (e.g., given a finite set of primes, one can create a new number relatively prime to all elements in the set) and provides a generalization of the two key steps of a proof by contradiction, which is indicative of a Process conception of the proof method. Therefore, responses to this question would indicate students' conception of proof by contradiction.

As in Exercise 3, both Wesley and Yara rephrased the statement as the summary of the proof. The teacher/researcher then explicitly stated this was a summary of the statement and not of the proof, prompting the students to attempt to provide a summary for the proof. Again, both students rephrased the statement. For example, consider the discussion with Yara below.

Teacher: Summarize, in your own words, the main idea of this proof.

Yara: To show that there are a lot a lot of primes. An infinite amount of primes. Yeah, I think that's what I put, yeah.

Teacher: So that's, kind of the main idea of the statement.

Yara: Statement, oh.

Teacher: What about the proof itself?

Yara: To show that it's not finite?

Even after prompting, her response "To show that it's not finite" suggests she could not summarize the proof via the procedure nor the general ideas of the proof. As both Wesley and Yara responded in this way, they both exhibited a Pre-Action conception of proof by contradiction.

Question seven assessed whether students could identify the modular structure of the proof. In other words, whether students could group sentences in the proof together and provide a purpose for each of these groups. By phrasing the question as "key steps", the teacher/researcher attempted to illicit the purpose of particular lines and how these purposes related to form a complete proof. Responses to this question would indicate students' conception of proof by contradiction. In particular, students that responded with the two key steps of a proof by contradiction (assuming the statement is false and arriving at a contradiction) and did not describe the purpose of these steps in the overall argument of the proof would have exhibited an Action conception of proof by contradiction, while students that responded with lines from the desired logical outline and described the purposes of these lines in the overall proof would have exhibited a Process conception of proof by contradiction.

When asked to describe the key steps of the proof, Wesley stated "Generating a relative prime. I'm... see, I could be wrong, I think I have the idea of the proof by contradiction. I'm having problems with this whole prime thing." This response suggests that his difficulty with comprehending the proof is rooted in the mathematical content of the proof (i.e., "Generating a relative prime") and not in following his general procedure for proof by contradiction. Therefore, Wesley exhibited an Action conception of proof by contradiction.

When asked to describe the key steps of the proof, Yara stated "That uhhh assume the negation of the statement and *[pause]* prove *[pause]* that *[pause]* the new prime number was not part of the set of prime numbers. And then that gets us a contradiction, so the statement is true." This response suggests she can summarize the proof by citing the procedure of the proof ("assume the negation of the statement ... that gets us a contradiction, so the statement is true") and the purpose of the construction ("the new prime number was not part of the set of prime numbers"). In particular, this response suggests Yara *can* summarize the proof (unlike question six) by utilizing her general conception of proof by contradiction and therefore Yara exhibited a Process conception of the proof method.

Question eight assessed whether students could transfer the general idea or method to another context. In particular, the question assessed whether students recognized the scope of the method (i.e., recognized the assumptions that needed to be in place to allow the method to be carried out). Beyond the use of proof by contradiction, the construction $n = (p_1p_2p_3 \dots p_k) + 1$ relies on the fact that 1 is not a prime number. Responses to this question would indicate students' conception of mathematical logic. A desired response to this question would indicate that the method does not work since p_i could be 31 for some $i \in \{1, 2, 3, \dots, k\}$.

When asked if the proof would still work, Wesley responded "Yes, it would, still generates elements outside of the set." The teacher/researcher then focused his attention specifically on the case when $p_i = 31$ for some i, to which he responded "It, and I thought about that. And what did I think? I thought *[pause]* you are still... but you still have other *[pause, snaps fingers]* other items within the set." The phrase "other items within the set" suggests the construction only relies on some, but not all, primes in the set. In addition, it does not address how the new construction changes the proof (that the difference of n and $p_1p_2p_3 \dots p_k$ is now 31) and no longer results in a contradiction. Therefore, Wesley exhibited a Pre-Action conception of mathematical logic.

When asked if the proof still worked, Yara responded with hesitation and said "No, because 31 is a prime number?" This is a valid justification, though the hesitation in her response suggested she could not explain this justification. When she was asked to explain her justification, she stated "Because prime numbers are only divisible by itself and 1. I mean, not divisible, but it's like... I don't know, it's prime maybe?" The phrase "it's prime maybe" along with the phrase "the new prime number" from her description of the procedure in question seven suggests she considered the number n to be a prime number and recognized that adding 31 instead of 1 may not result in a prime number. In other words, she thought that it *may* cause the proof to fail rather than that it would necessarily cause the proof to fail. Therefore, Yara exhibited a Pre-Action conception of mathematical logic.

Question nine also assessed whether students could transfer the general idea or method to another context, this time in the form of writing a similar proof. This question encouraged students to utilize a similar construction of n to write a new proof by contradiction[19]. This new statement can be represented as $(\forall k > 2, k \in \mathbb{Z})(\exists n \in \mathbb{N}\backslash\{1\})(\forall s \in S_k)(s \nmid n)$ and has three quantifiers. As mentioned previously, the literature has identified that students exhibit difficulties understanding and negating multi-level quantified statements (e.g., Dubinsky & Yiparaki, 2000). Therefore, responses to this question would indicate a student's mathematical logic (negating a multi-level quantified statement) and a students' conception of proof by contradiction (approach and summary of the proof). A desired proof for question 8 is presented in Figure 4.44.

Statement: Let $S_k = \{2, 3, 4, \ldots, k\}$ for any $k > 2$. Then there exists a natural number greater than 1 that is not divisible by any element in S_k.

Proof: Let $k > 2$ and $s_2, s_3, \ldots, s_k$ denote the elements of S_k. Consider the natural number $n = s_2 \cdot s_3 \cdot \ldots s_k + 1$. First, note $n > 1$ as $k > 2$ and thus 2 and 3 must be elements of S_k. Also, for any s_j with $2 \leq j \leq k$, $s_j \nmid n$ as $s_j \nmid 1$. Therefore, there exists a natural number greater than 1 that is not divisible by any element in S_k.

Figure 4.44 Desired proof for question eight in Exercise 4.

The proof above is desired for four main reasons. First, it used a similar construction to the presented proof in Exercise 4 - multiplying all elements of a set and adding 1. In this way, it uses the same construction method but not the same proof method. This is desirable as methods for particular proofs by contradiction are not necessarily unique to the proof

[19]The proof can be completed directly and does not require a proof by contradiction. However, students interpreted "method" as proof method and thus all students completed the proof by contradiction.

method and thus illustrated how different proof methods can be compared - by the methods used within the proofs. Thirdly, directly proving the statement provides the shortest proof, as utilizing a proof by contradiction will necessarily prove that no element of S_k divides the constructed natural number n. That is, this proof will be present in any proof of the statement. Finally, this proof addressed all three quantifiers and provides a complete logical argument.

Wesley initially did not respond to this question when working individually. When prompted by the teacher/researcher to give his reasoning behind not responding, he stated "I didn't understand what this means. 1 is not divisible by any of those numbers. 1 is only divisible by itself." This suggests he focused on only part of the statement (1 *[that]* is not divisible by any element in S_k) and that he did not understand the meaning of the statement to be proved.

Once he considered the full statement, he began to represent it using predicate logic. Using cue words, such as 'for any' and there exists, he represented the statement as: $(\forall k > 2)(\exists n \in \mathbb{N}, n > 1)(\forall S_k)(S_k \nmid n)$. This representation is inappropriate, however, as a set S_k cannot divide a number n. He acknowledged this may not be acceptable and stated "Is S_k divides n? I mean, would it be, is that how it's..." After the teacher/researcher confirmed this was not an acceptable representation, he desirably represented the statement as $(\forall k > 2)(\exists n \in \mathbb{N}, n > 1)(\forall s \in S_k)(s \nmid n)$. While he initially had difficulty converting the final part of the statement, he was able to focus on cue words in order to desirably represent the statement. Therefore, Wesley exhibited an Action conception of mathematical logic.

He then relied on his procedure for proof by contradiction to continue with the proof. In particular, after he was prompted to continue the proof, he stated "Let's see, it's going to be... *[pauses, begins to read the symbolic representation again]* so it would be there *[pause]* no, it's for all, right?" The phrase "it's for all" suggests he focused on cue words in order to begin negating the representation, though it is not clear which 'for all' in the statement he was referring to. He then continued:

> Well that's going to be for... there exists a k greater than 2 *[pause]* for all n

> greater than 1, yeah greater than 1 *[pause]* there exists an s in S_k where s is the...*[pause]* where s divides n. So why are we... why do we get to negate our set? Because k is undefined.

First, he appropriately negated the statement and wrote $(\exists k > 2)(\forall n \in \mathbb{N}, n > 1)(\exists s \in S_k)(s \mid n)$. He then suggested he negated the set S_k, though it is unclear what he meant by this comment. When asked to clarify what he meant, he stated "I don't... why are the set... why are we contradicting the set?" Again, he exchanged 'contradict' for 'negate' and otherwise repeated his previous unclear statement. These comments suggest that while he desirably negated the representation, he did not understand why he was able to negate parts of the statement. In other words, he could rely on cue words to complete the negation but did not understand the reasoning behind the negation. Therefore, Wesley exhibited an Action conception of mathematical logic.

After choosing an arbitrary value for k, he stated "For any, there exists an s in S_k *[long pause]*. So we have to find an s that doesn't divide n." This comment suggests he considered the representation $(\forall n \in \mathbb{N}, n > 1)(\exists s \in S_k)(s \mid n)$ and thought to prove this statement true. This step would align with his previously conceived procedure for proof by contradiction and, specifically, the step "Prove $\sim S$ is true." In other words, he does not recognize how to use a construction to prove this statement false and thus arrive at a contradiction. Therefore, Wesley exhibited a Pre-Action conception of proof by contradiction.

After prompting Wesley for an explanation of how to contradict this statement with a construction, the teacher/researcher explained how a specific construction of n such that $\forall s \in S_k$, $s \nmid n$ would prove the assumption to be false and thus the statement to be true. Wesley then suggested "So, I mean, you can pick 7." Suggesting a specific value of n, rather than a construction of n that uses the elements in S_k suggests that while the teacher/researcher described how to contradict the statement, Wesley still could not do so. Therefore, he exhibited a Pre-Action conception of proof by contradiction.

After properly constructing n to arrive at a contradiction and finish the proof, the teacher/researcher commented that the proof was difficult due to the numerous quantifiers in the statement, to which Wesley responded:

> No and I, it's not the number of quantifiers; I was just looking at your set and *[pause]* taking the contradiction of the set to begin with. Because I thought you would... because we didn't do that with the primes either, did we? We didn't do anything with the set.

Two phrases are of note in his comments above. First, the phrase "we didn't do that with the primes" suggests he relied on the previous proof as an outline to complete the current proof, which is indicative of an Action conception of proof by contradiction. Secondly, the phrase "We didn't do anything with the set" provided clarity to his previous comments about negating the set S_k. That is, his previous comment "why are we negating the set" referred to negating the quantifier $(\forall k > 2)$ and how this changed the set S_k. His question thus asked why the set S_k changes due to the negation when it did not in the previous proof. Thus the phrase "We didn't do anything with the set" is more evidence that he focused on the previous proof as an guide to write the current proof. Therefore, Wesley exhibited an Action conception of proof by contradiction as he focused on the specific example proof and could not modify his proof when the context of the proof changed.

Yara initially wrote "$\sim$ (There exists a natural # > 1 that is not divisible by any element in S_k)" as her response to question nine. This suggests she focused on her procedure for proof by contradiction and could not continue the proof due to an inability to negate the statement. Indeed, when asked to negate this statement, she said "There does not exist a natural number greater than 1 *[pause]* that is [pause] that is not divisible by any number in S_k?" This is an improper negation of the statement and confirms she could not negate the statement. She was then prompted to represent the initial statement using predicate logic, to which she wrote $(\exists x \in \mathbb{N}, x > 1)((\forall y \in S_k)(y \nmid x))$. She then asked "So that part, since we negated it, that, like the existential, does that second part of the proof not negate?" This question referred to the representation $(\forall y \in S_k)(y \nmid x)$ as the 'second part' and questioned the rule for negating multiple quantifiers. However, rather than providing how to negate multiple quantifiers, the teacher/researcher prompted her to try negating the full statement. She then stated "For all x is [pause] greater than 1 and there exists a y in S_k such that y does not divide it?" In other words, she negated the representation as $(\forall x > 1)(\exists y \in S_k)(y \nmid x)$. This suggests she was able to negate using cue words (e.g., 'there exists' and 'for all') and did not understand the meaning of negating multiple quantifiers. Therefore, Yara exhibited an Action conception of proof by contradiction as she focused on the procedure and an Action conception of mathematical logic to negate the statement and enact the procedure.

After negating the assumption, Yara was not able to proceed in the proof. After she was prompted that the proof would contradict the statement represented by $(\forall x > 1)(\exists y \in S_k)(y \nmid x)$, she stated " So wouldn't you say *[pause]* there... umm... *[pause]* you pick a y in S_k and show that it divides x?" This comment suggests that y should be constructed to show it divides all x, which would prove the statement assumed to be true. In other words, she could not again negate the statement and recognize the construction should focus on the natural number x, which is indicative of a Pre-Action conception of mathematical logic.

At this point, the teacher/researcher noted Yara needed to leave the session and gave an explanation of the rest of the proof. Thus, she was not given an opportunity to describe the proof after its completion and so did not have another opportunity to exhibit her conception of proof by contradiction or mathematical logic.

4.4.4 Summary of results

Teaching episode 4 was designed to achieve three goals. First, the episode introduced a set of step-by-step instructions for students to use when constructing proofs by contradiction for an infinity statement (e.g., There are infinitely many prime numbers). Students could then use these instructions to prove similar statements by contradiction and thus possibly develop an Action conception of proof by contradiction for this type of statement. Secondly, students were encouraged to relate the roles of lines or collections of lines to interiorize the previous step-by-step instructions into general steps for an infinity statement and thus possibly develop a Process conception of proof by contradiction for this type of statement.

Thirdly, students reflected on both their knowledge of proof by contradiction and a specific example of proof by contradiction to assimilate or accommodate the new logical outline into their general procedure for a proof by contradiction. All tasks for the episode were designed around these three goals.

Subsection 4.4.1 reported on students' work during Activity 4. The goals of this phase reflected two of the three goals of teaching episode 3: (1) guide students to construct a set of step-by-step instructions for proofs by contradiction of an infinity statement and (2) prompt students to relate the roles of lines, or collections of lines, to interiorize the previous step-by-step instructions into general steps for an infinity statement. Thus, the phase consisted of two tasks: (4.4.1.1) outlining a given proof by contradiction for an infinity statement and (4.4.1.2) defining proof by contradiction. A brief summary of the results from each task follows.

Wesley's outline included the two key steps of a proof by contradiction and did not describe the logical relation between these steps, which is indicative of an Action conception of proof by contradiction. In addition, he could not justify intermediate steps outside his procedure, which is indicative of a Pre-Action conception of mathematical logic. Yara's outline included the two key steps of a proof by contradiction as well as described the logical relation between lines, which is indicative of a Process conception of proof by contradiction. In addition, she desirably represented the statement without relying on a previous example, which is indicative of a Process conception of mathematical logic.

Wesley's definition of proof by contradiction described only one of the two key steps of a proof by contradiction using a cue word, which is indicative of a Pre-Action conception of proof by contradiction. When describing how the example proof illustrated his definition, his explanation suggested he did not understand the logical role of the construction in the proof, which is indicative of a Pre-Action conception of mathematical logic. Yara's definition of proof by contradiction described the two key steps of a proof by contradiction and, unlike Activity 3, did not describe the relation between these key steps, which is indicative of an Action conception of proof by contradiction.

Subsection 4.4.2 reported on students' responses during Classroom Discussion 4. The purpose of this phase was to formalize students' conceptions of proof by contradiction based on their work and conjectures during Activity 4. In particular, the goal of this phase was to develop a general procedure for the method that students could then utilize, along with mathematical logic, to prove statements by contradiction. Therefore, the tasks for this phase are based on developing and testing a general procedure for proof by contradiction. A brief summary of the results from each task follows.

When Wesley was prompted to compare the four outlines to construct a new list of steps for a general proof by contradiction, he instead described a general procedure similar to the one he developed during Classroom Discussion 3. This outline described the two key steps of a proof by contradiction without explaining how these steps logically imply the statement is true, which is indicative of an Action conception of proof by contradiction. In addition, this outline included circular reasoning and improper representations of statements, which is indicative of a Pre-Action conception of mathematical logic. When Yara was prompted to compare the four outlines, she assimilated the fourth proof outline into her previous procedure and thus exhibited a Process conception of proof by contradiction.

When prompted to write a proof outline for a similar proof, Wesley constructed an outline by explicitly following the step-by-step outline for an infinity statement from Activity 4 and, in particular, suggested the same construction without modification and is indicative of an Action conception of proof by contradiction. Yara also followed the step-by-step outline for an infinity statement from Activity 4. However, she also recognized, without prompting, the previous construction would not work for the new proof. This suggests that while she focused on the previous outline, she also described the new context of the proof and thus exhibited a Process conception of proof by contradiction.

Subsection 4.4.3 reported on students' work during Exercise 4. The goal of this phase was to reinforce students' conceptions of mathematical logic and proof by contradiction through a series of eight questions (i.e., eight tasks) aligned with the proof comprehension assessment model by Mejía-Ramos et al. (2012) as well as an additional question not aligned with the assessment model. A short summary of results for questions two through nine follow.

Question two: When asked to provide an example of a set that is infinite and explain why it is infinite, both Wesley and Yara responded with an intuitive understanding of infinity but could not provide a formal definition for their examples. Therefore, both students possessed the prerequisite knowledge of infinity necessary to understand the particular proof.

Question three: When asked to mathematically justify a statement in the presented proof, both Wesley and Yara did not provide a desired justification and thus both students exhibited a Pre-Action conception of mathematical logic.

Question four: When asked to mathematically justify another statement in the proof, both Wesley and Yara did not provide a desired justification and thus both students again exhibited a Pre-Action conception of mathematical logic.

Question five: When asked to identify the purpose of a statement in the presented proof, Wesley instead described the statement and thus exhibited a Pre-Action conception of mathematical logic. When Yara was asked the same, she provided the purpose for a similar statement in Activity 4 (which was not the same in this proof) and thus exhibited a Pre-Action conception of mathematical logic.

Question six: When asked to summarize the main idea of the proof, both Wesley and Yara rephrased the statement to be proved (even after prompting that this was not a desired response) and thus exhibited a Pre-Action conception of proof by contradiction.

Question seven: When asked to provide the key steps of the proof, Wesley responded with the main idea of the proof as he understood his procedure for proof by contradiction and thus exhibited an Action conception of the proof method. When Yara was asked the same, she described the two key steps of a proof by contradiction as well as described how the proof arrived at a contradiction, which is indicative of a Process conception of the proof method.

Question eight: When asked if the proof would still work if the construction were inappropriately modified, Wesley did not describe how the construction would affect the proof and no longer result in a contradiction, which is indicative of a Pre-Action conception of mathematical logic. When asked the same, Yara provided a valid response that, after prompting, was revealed to be a guess and not a valid justification, which is indicative of a Pre-Action conception of mathematical logic.

Question nine: When asked to write a similar proof using the method of the presented proof, Wesley focused on the outline from Activity 4 and could not describe how to use the contradiction to prove the statement, which is indicative of a Pre-action conception of proof by contradiction. When asked the same, Yara initially relied on her general procedure for proof by contradiction but could not proceed beyond her first step (assuming the negation of the statement is true), which is indicative of a Pre-Action conception of mathematical logic. She needed to leave shortly after properly negating the statement and thus did not have another opportunity to exhibit her conception of proof by contradiction.

4.5 Teaching Episode 5: Validating the contradiction proof method

Teaching episode 5 was designed to achieve two goals. First, the episode introduced a general proof by contradiction that could be considered proving the proof method (i.e., showing the proof method is valid) and thus encourage students to encapsulate their Process for proof by contradiction into an Object. Secondly, the episode introduces a new statement type that can be proved by contradiction - property claim statements (e.g., $\sqrt{2}$ is an irrational number). Students were encouraged to use their general procedure for proof by contradiction to prove a particular statement and thus assimilate or accommodate the new type of proof into their Schema for the method. All tasks for this episode were designed around these two goals.

This section will report on how students' understanding emerged from an analysis of their individual responses to the tasks and teacher/researcher prompts during teaching episode 5. As in teaching episodes 3 and 4, the data analyzed and presented for this section will focus on Wesley and Yara - representative students chosen for case study analysis (see Section 3.5 for details on why these two students were chosen).

The following subsections will be organized by the three phases of the teaching episode. Subsection 4.5.1 will focus on students' initial conjectures about validating the proof method and incorporating a new statement type into their procedure for proof by contradiction. Subsection 4.5.2 will focus on how the teacher/researcher guided students to formalize the initial conjectures from the previous phase. Subsection 4.5.3 will focus on responses to proof comprehension questions designed to reinforce the formalized conjectures from the previous phase. For each subsection, I will begin by describing the lesson plan of the phase and then proceed by presenting the analysis and interpretation of data collected during the phase. The tasks used for this teaching episode can be found in Appendix A.1 and will be reproduced as needed to describe the lesson plan for each phase of teaching episode 5.

4.5.1 Initial conjectures on validating the contradiction proof method

This subsection is a report on students' work during Activity 5 (see Appendix A.1 for the complete set of tasks). The goal of Activity 5 were to encourage students to validate the contradiction proof method. Thus, this phase consisted of two main tasks: (1) outlining a proof of the method and (2) defining proof by contradiction. For each task, I will first present the goal and reasoning behind the task. Then, I will present an analysis and interpretation of students' responses to the task.

4.5.1.1 Outline of a general proof by contradiction Students were asked to read a presented proof of the statement "Statement P is true" and subsequently outline this statement and proof (see Figure 4.45) utilizing propositional or predicate logic.

Statement: Statement P is true.

Proof: A statement P is either true or false. Assume P is false; that is, the negation of P, $\sim P$, is true. If $\sim P$ leads to a contradiction, then $\sim P$ must be false. This implies our initial assumption was not true; that is, it is not true that P is false. Since P is either true or false and P is not false, P is true.

Figure 4.45 Presented proof for Activity 5.

As Wesley and Yara participated individually, the teacher/researcher acted as another student with incomplete knowledge during this phase. That is, the teacher/researcher provided memorized rules (e.g., $\sim (P \vee Q) \cong \sim P \wedge \sim Q$) when necessary and did not provide the representation for any line. Explanations for any line were reserved for the Classroom Discussion phase of teaching episode 5.

The purpose of outlining the logical structure of this presented proof was to encourage students to validate the contradiction proof method. That is, rather than consider the method as a procedure that should be followed, students could consider the underlying logical reasoning that makes the proof method valid. In doing so, they can consider using the method as a subproof within other proofs and thus possibly develop an Object conception of proof by contradiction.

A desired logical outline of the proof is illustrated in Figure 4.46.

Statement: P

1. $P \vee \sim P$
2. Assume $\sim P$
3. If $\sim P$ implies $\rightarrow\leftarrow$, then $\sim P$ is false.
4. $P \vee \sim P$ and $\sim P$ is false implies P is true.

Figure 4.46 Desired logical outline of presented proof during Activity 5.

This outline is desired for two main reasons. First, it contains the two key steps of a proof by contradiction (lines 2 and 3). Secondly, it describes how the *law of excluded middle* (line 1) and the two key steps of a proof by contradiction (lines 2 and 3) prove the statement is true (line 4).

A student's outline of the presented proof and explanation of their outline would indicate their conception of proof by contradiction. A student with an Action conception of proof by contradiction would describe the key steps of the method and would not describe how these key steps relate to prove the statement is true. In contrast, a student with a Process conception of proof by contradiction would describe the key steps of the method as well as how these key steps relate to prove the statement is true. That is, he or she would describe that by assuming the statement is false and arriving at a contradiction, the assumption must be false and so the statement is true. This explanation relies on the logical equivalence $\sim\sim P \cong P$, which is a consequence of the *law of excluded middle*. In other words, the description relies on a memorized rule and could not prove the method is logically valid. Finally, a student with an Object conception of proof by contradiction could describe the procedure for the method as well as prove the method is valid. He or she could then consider using the procedure in conjunction with other proof methods (i.e., as a sub-argument within a direct proof).

When asked to outline the proof, Wesley focused on his previous procedure for proof by contradiction (see Figure 4.47).

Statement: P

1. Assume $\sim P$
2. $\sim P = P$ $(\rightarrow\leftarrow)$
3. $\sim(\sim P)$
4. P

Figure 4.47 Wesley's outline of presented proof during Activity 5.

This outline does not represent the proof as written. For example, it does not represent the statement “A statement P is either true or false.” Instead, the outline focused on his previously constructed procedure for proof by contradiction, which is most evident by the inclusion of the improper representation $\sim P = P$. This equation was meant to represent the statement “If $\sim P$ leads to a contradiction, then $\sim P$ must be false” as illustrated by his comment:

> If not P leads to a contradiction, then P [sic] must be false. So I guess it would be *[long pause]* not P and just put arrows going to each other. *[gestures that the arrows point toward each other as $\rightarrow\leftarrow$]* So I guess it would be *[pause]* if not P leads *[pause]* so it would be not P equals P. And then you’d have the arrows going.

Wesley first recited the line he planned to represent. He then represented the phrase “not P leads to a contradiction” as $\sim P \rightarrow\leftarrow$, which suggests he changed the representation for the new context of the presented proof. However, his phrase “so it would be not P equals P” changes his initial representation to the improper registration $\sim P = P$ from his procedure in Activity 4 (Figure 4.37 on page 181). The teacher/researcher then followed up on this representation, as illustrated below.

Teacher: So, in this proof here, we didn’t *[pause]* get a direct opposite of the statement we were trying to prove. It was kind of like, a little piece, a little side thing of that. So if we are trying to make this even more general, we get rid of not P equals P and we just say ‘some contradiction’.

Wesley: So *[long pause]* this is the only one we’ve done like this where you are using an approach to the original statement that’s kind of off to the side.

As in Activity 4, the teacher/researcher prompted Wesley to recognize that the contradiction in this specific proof was not a direct negation of the statement (i.e., the contradiction was not $P \wedge \sim P$) by using non-mathematical jargon such as “direct opposite” and “little side thing” as the previous intervention was not successful. Wesley then retorted “this is the only one we've done like this" and implied all previously presented proofs had a contradiction that was directly related to the statement proved. This is false as he was made aware of multiple such proofs during Activity 4 (see page 190 for details). This suggests he did not recognize a need to change his improper representation of a contradiction, $\sim P = P$, even when presented contradictory evidence. Moreover, this response suggests the representation $\sim P = P$ was not solely an improper representation of a contradiction (i.e., improperly representing the general contradiction $\sim P \wedge P$). In other words, the step $\sim P = P$ was an external rule that he focused on to the exclusion of other relevant evidence, referred to as *centration* (Piaget & Garcia, 1989). The focus on this particular step prevented him from properly conceptualizing what a contradiction can be in a proof by contradiction, which is indicative of a Pre-Action conception of the proof method.

When asked to outline the proof, Yara also focused an abbreviated version of her previous procedure for proof by contradiction (see Figure 4.48).

Statement: P

1. Assume $\sim P$
2. $\rightarrow\leftarrow$
3. P

Figure 4.48 Yara's outline of presented proof during Activity 5.

This outline described the two key steps of a proof by contradiction in lines 1 and 2, after which the procedure was completed and thus the statement is true (line 3). This outline focused on the key steps of her previous procedure and did not represent statements such as"A statement P is either true or false," which is indicative of an Action conception of proof by contradiction. However, after outlining the proof, Yara stated "It's a proof for how to get a proof by contradiction?" This response suggests she considered acting on the proof method (i.e., proving the proof method), which is indicative of an Object conception of proof by contradiction. However, when asked to clarify what she meant, she instead stated it was slightly helpful "because it, like, kind of breaks down how you can get a proof by contradiction." The phrase "how you can get" may be a vague reference to validating the procedure, though it is not explicit enough to state she can consider acting on a proof procedure as a whole. Therefore, these two comments suggest Yara was developing toward an Object conception of proof by contradiction and could not yet describe how to validate the proof method.

4.5.1.2 Definition of proof by contradiction After reading the presented proof and writing an outline of the proof, Wesley and Yara were asked to write a definition for proof by contradiction. The goal of this task was to encourage students to reflect on both their knowledge of proof by contradiction and a presented validation of the method to possibly encapsulate their procedure so that it can be used as a sub-proof. In addition, students were asked to explain the logic behind the proof method (if their definition did not already include such an explanation).

An ideal definition for proof by contradiction for this task would include the two key steps of the proof method as well as describe how the *law of excluded middle* validates the proof method. For example, an ideal definition of proof by contradiction may state "A proof by contradiction shows that by assuming the statement is not true and arriving at a contradiction, the statement must be true as it is either true or false." This example definition includes both key steps and describes how these key steps logically relate to the *law of excluded middle* to prove the statement is true.

A student's definition of proof by contradiction would indicate their level of understanding of the proof method. A student with an Action conception of proof by contradiction would focus on the specific lines of a proof by contradiction without providing any reasoning behind the method. For example, a student may state "Start with an assumption and get a contradiction." This definition does not provide a relation between these two key steps and does not provide a complete description of each key step. Students with an Action conception of proof by contradiction may also focus on two cues in their definition: assumption and contradiction. In contrast, a student with a Process conception of proof by contradiction would describe the relation between the two key steps of a proof by contradiction. For example, a student may state "Assume the statement is not true and, using this assumption, arrive at a contradiction. This would mean the assumption is not true and thus the statement is true." This definition would describe the role of the two key steps completely rather than focus on cues such as 'assumption' and 'contradiction'. Finally, a student with an Object conception of proof by contradiction would also describe how the *law of excluded middles* logically proves that if the statement is not false, it must be true. Wesley and Yara's definitions follow.

When asked to provide a definition for proof by contradiction, Wesley stated "Well that's like, perfect example right there, isn't it?" referring to the outline of the presented proof. He was then asked to explain what he meant, to which he responded "I mean that's... *[trails off]*" and moved on to the next question. This was interpreted to mean the list of steps of a proof by contradiction was his entire definition and he had nothing more to add, which is indicative of an Action conception of proof by contradiction.

When asked to provide a definition for proof by contradiction, Yara stated "It's when you assume the statement isn't... the original... assuming the negation of the statement and getting a contradiction." The phrase "assuming the negation of the statement and getting a contradiction" described the two key steps of a proof by contradiction in complete phrases (i.e., did not focus on cue words such as "assume the negation and then contradiction"). She then began reading the next question and thus was not prompted to describe how these key steps logically related to prove the statement was true. Since she described the key steps in complete sentences and did not describe how they related to prove the statement was true, she exhibited at least an Action conception of proof by contradiction.

4.5.2 Formalization of the contradiction proof method

This subsection is a report about Wesley and Yara's responses during Classroom Discussion 5 (see Appendix A.1 for general questions that guided this discussion). The purpose of this phase was to guide students to formalize the initial conjectures from Activity 5. In particular, the goal of this phase was to continue developing students' procedure for proof by contradiction. Therefore, the tasks for this phase are based on enhancing and testing students' general procedures from the previous teaching episodes. For each task, I will first present the goal and reasoning behind the task. Then, I will present an analysis and interpretation of students' responses to the task.

4.5.2.1 Comparison of general proof with their procedure Wesley and Yara were asked to compare their general procedure for proof by contradiction to the presented proof in Activity 5. The goal of this task was to possibly modify the logical reasoning behind their general procedure for proof by contradiction. That is, the explicit inclusion of the *law of excluded middle* could possibly modify the logical reasoning behind their general procedure. For example, they could think of the procedure as "A statement is either true of false. If we show it cannot be false, then it must be true." However, neither student modified their logical reasoning behind the general procedure due to the outline. Each student's response follows.

When Wesley was asked to compare his general procedure to the outline of the presented proof, he stated "Assume not S is true, attempt to prove S is true. Why yes! It looks just like that *[sarcasm]*." The phrase "Assume not S is true, attempt to prove S is true" was in reference to the first two steps from his procedure of proof by contradiction. It was evident by his tone of voice and the exclamation "Why yes!" that he considered the outline and his own procedure to be exactly the same and thus did not need to be compared further. In this way, he continued to conceptualize a proof by contradiction as a series of steps to be performed, which is indicative of an Action conception of the proof method.

When Yara was asked to compare her general procedure to the outline of the presented proof, she stated "Yeah, it's uh... *[long pause]* For the most part, it's the same. Minus one little section in between the assume not P and the contradiction we had "Do work" to get the contradiction." The phrase "minus one little section" suggests she thought of the procedure as 'sections' and the line described the role of the 'section', which is indicative of a Process conception of proof by contradiction. However, she only compared the 'sections' of her procedure and did not re-conceptualize the logical argument of a proof by contradiction.

4.5.2.2 Writing a proof using a general procedure After considering their general procedure for proof by contradiction, Wesley and Yara were asked to write a proof for the statement "$\sqrt{2}$ is an irrational number" - a property claim statement. The goal of this task was to examine how students utilized their general procedure to prove a new type of statement by contradiction. This statement was chosen as it is one of the two classic proofs by contradiction covered in transition-to-proof courses[20]. In addition, the statement requires one to show that a specific mathematical object holds some property. This type of statement contains no quantification and thus outlining the proof (either with propositional or predicate symbols) does not illuminate how the proof should proceed. In this way, it is the most difficult type of statement to prove by contradiction.

A desired proof for the statement, a modified version of a proof by Tall (1979), is illustrated in Figure 4.49.

Statement: $\sqrt{2}$ is an irrational number.

Proof: Suppose $\sqrt{2}$ is a rational number. Then there exists $p, q \in \mathbb{Z}$ such that $\sqrt{2} = \frac{p}{q}, q \neq 0$, and $\gcd(p,q) = 1$. This implies $p^2 = 2q^2$ and so p^2 is even. Now, if p were odd, then p^2 would be odd. Thus p must be an even number and so $p = 2r$ where $r \in \mathbb{Z}$. Since $p = 2r$ and $p^2 = 2q^2$, $4r^2 = 2q^2$ and so $2r^2 = q^2$. By the same argument as above, q must also be even number. Since both p and q are even, $\gcd(p,q) \neq 1$. From the contradiction that $\gcd(p,q) = 1$ and $\gcd(p,q) \neq 1$, we conclude that the assumption that $\sqrt{2}$ is a rational number is false. Therefore, $\sqrt{2}$ is an irrational number.

Figure 4.49 Desired proof for the statement in Classroom Discussion 5.

This proof is desired for four main reasons. First, it included the two key steps of a proof by

[20]The other being a proof of the statement "There are infinitely many prime numbers."

contradiction: assuming the statement is not true and arriving at a contradiction. Secondly, it provides a description of the relation between the two key steps and why these steps prove the statement is true. Thirdly, the proof is self-contained. That is, the statement "If p^2 is even, then p is even" can be proved using Euclid's Lemma: If a prime p divides ab, then $p \mid a$ or $p \mid b$. However, the proof stated "if p were odd, then p^2 would be odd" which is an implicit proof by contradiction. Finally, every line in the proof is either mathematically or logically justified.

A student's proof and explanation of their proof would indicate their conception of proof by contradiction. A student with an Action conception of proof by contradiction would rely on a list of steps to write the necessary steps for the proof. In particular, the student would not provide reasoning between steps or describe why arriving at a contradiction implies the statement is true. If asked to summarize the proof, this student would describe each line sequentially and not describe the proof with the general procedure of a proof by contradiction. In contrast, a student with a Process conception of proof by contradiction would include both the necessary steps for the proof as well as descriptions for how these steps prove the statement (e.g., that arriving at a contradiction implies the initial assumption is false). If asked to summarize the proof, this student would describe the proof as having three parts: assuming the statement false, arriving at a contradiction, and that arriving at a contradiction implies the assumption is false and thus the statement is true. Finally, a student with an Object conception of proof by contradiction would describe the proof as having three parts (as a student with a Process conception) as well as identify the statement "Now, if p were odd, then p^2 would be odd" as a proof by contradiction. That is, the student could identify a proof by contradiction as a sub-proof in the argument.

A student's outline of a proof is not only influenced by a student's conception of proof by contradiction. That is, while a student's conception of proof by contradiction would influence how he or she outlines a proof, a student's conception of mathematical logic influences how he or she enacts the outline. In addition, this particular proof requires knowledge of the formal definition of rational numbers to proceed beyond the key steps of a proof by contradiction.

Therefore, I will describe how Wesley and Yara's responses illustrated their conception of proof by contradiction and mathematical logic as well as their knowledge of rational numbers.

First, Wesley was asked if he had seen a proof of the statement "$\sqrt{2}$ is an irrational number" before, to which he responded "Nope." This is important to note as he could not rely on a similar example to prove the statement. Indeed, he used his procedure for proof by contradiction to begin the proof and then could not continue, as illustrated below.

> So *[long pause]* so your statement is *[pause]* root 2 is irrational. *[long pause]* So proof. That would be assume it is rational. Okay! *[pause]* Well! Where to go from here? *[laughs]*

After reading the statement, he first assumed "it is rational" and thus described the first step of a proof by contradiction: assuming the negation of the assumption. He then stated "Well! Where to go from here?" which suggests that he did not know how to continue to the next step. The teacher/researcher then prompted Wesley to utilize the rest of his procedure for proof by contradiction to outline the proof. This prompt and its response is provided below.

Teacher: So we know, just based on our structure, we will eventually get to some kind of contradiction... *[Wesley interrupts]*

Wesley: Where square root of 2 is not rational. Where the square root of 2 is false.

The teacher/researcher prompted Wesley to return to the formal procedure step of arriving at "some kind of contradiction" (formalized from his initial step $\sim P = P$). He then interrupted to state what the contradiction would be - directly related to the initial statement. In particular, the phrase "Where the square root of 2 is false" referred to the non-formal step of showing the statement is false. That is, he again exhibited a Pre-Action conception of proof by contradiction by suggesting the contradiction is always directly related to the statement.

Before revising the contradiction, teacher/researcher focused Wesley's attention on writing the step after assuming the number is rational. He then stated:

> Welp *[long pause]* I guess one thing we can do is I'm sure there is a definition for a rational number. Some sort of... maybe... *[pause]* a rational number is like... I don't know!

The phrase "I'm sure there is a definition for a rational number" refers to unpacking the definition of a rational number - a valid proof writing strategy. However, he did not employ this strategy as he did not recall the formal definition of a rational number and suggests a lack of mathematical knowledge inhibited his proof writing.

Overall, Wesley focused on his non-formal procedure (Figure 4.47) for proof by contradiction to outline the proof, which is indicative of an Action conception of proof by contradiction. After assuming the negation of the statement was true, he suggested unpacking the definition of a rational number (a valid proof-writing strategy) that he did not employ due to not knowing this definition. In other words, a lack of mathematical knowledge inhibited his proof writing (as opposed to proof-writing strategies or his conception of mathematical logic).

First, Yara was asked if she had seen a proof of the statement "$\sqrt{2}$ is an irrational number" before, to which she responded "It was on the quiz, it was on the test, in the notes *[laughs]*." This is important to note as she could rely on a previously memorized proof of the statement when writing her own proof. This was immediately evident with how she proceeded in the proof, which is provided below.

> Assume the square root of 2 is a rational number? And if it's rational, then square root of 2 is equal to a over b. That's what you do. And I think you square both sides, if I'm remembering correctly. And then I think you solve for a? Like 2 b squared equals a squared. And then *[pause]* after that, I don't really know how to get to it. I still couldn't tell you. [laughs]

The phrases "That's what you do" and "if I'm remembering correctly" indicate a recall of some previously memorized procedure. She did not continue the proof past this point and thus relied on her memorized steps to complete the proof, which is indicative of an Action conception of proof by contradiction.

4.5.3 Reinforcing formalization of the contradiction proof method

This subsection is a report about students' responses to tasks from Exercise 5 (see Appendix A.1 for the complete set of tasks) - the last phase of teaching episode 5. The goal of this phase was to reinforce students' conceptions of mathematical logic and proof by contradiction through eight tasks aligned with the proof comprehension assessment model by Mejía-Ramos et al. (2012) as well as an additional question not aligned with the assessment model. All nine questions were written for a presented proof (Figure 4.49 on page 221) and are provided in Figure 4.50.

Students were first given approximately 20 minutes to individually respond to the comprehension questions below. Then, the teacher/researcher asked students to share their answers and describe the reasoning behind their response to each question. After eliciting the student's response and reasoning behind their response, the teacher/researcher provided a desired response to the question (if necessary).

1. Compare your proof in question 4 *[in Activity 5]* to the proof above. Explain how your proof compares to the given proof in terms of: (1) general structure, (2) specific lines, and/or (3) overall approach ot the proof.
2. Please given an example of a number that is irrational and explain why it is irrational.
3. What does $\gcd(p, q) = 1$ mean and why can we conclude $\gcd(p, q) = 1$?
4. Why exactly can one conclude that p^2 is an even number?
5. What is the purpose of the statement "Now, if p were odd, then p^2 would be odd."?
6. How exactly can one conclude that q is an even number?
7. Summarize in your own words the main idea of this proof.
8. What do you think are the key steps of the proof?
9. Using the method of this proof, show that for any prime number n, $\sqrt{n}$ is an irrational number.

Figure 4.50 Comprehension questions for the presented proof during Exercise 5.

Question one, which was not aligned with the proof comprehension assessment model, prompted students to compare their proof from Classroom Discussion 5 to the presented proof in Figure 4.49. The goal of this question was to compare two similar proofs in order to reinforce students' conception of proof by contradiction by identifying common key steps and general procedure of the two proofs. A desired response would describe a correspondence between the two key steps of a proof by contradiction as well as describe how the proof arrived at a contradiction. These three ideas would then be used to summarize the procedure and main idea of the proof. However, similar to Exercise 4, responses to this question did not provide sufficient evidence of students' conception of proof by contradiction or mathematical logic and therefore will not be discussed further.

The remainder of this subsection will be organized by comprehension questions two through nine. For each question, I will first describe what it was meant to assess in terms of the proof comprehension assessment model by Mejía-Ramos et al. (2012) (described on page 23). Then, I will describe the goal of the assessment in terms of the students' understanding of the presented proof, mathematical logic, or proof by contradiction. Finally, I will present an analysis and interpretation of Wesley and Yara's responses to the question.

Question two assessed whether students could identify the meaning of terms and statements in the presented proof. In particular, the question asked students to provide an example of an irrational number and explain why this number is irrational. The goal of this question was to determine whether students understood one of the mathematical terms in the proof (irrational) that was necessary to understand the logical argument of the proof, as the literature suggested students struggle with proof comprehension due to a lack of mathematical knowledge (e.g. Moore, 1994). In other words, the goal of this question was to assess prerequisite knowledge necessary to understand the particular proof. A desired response would present an irrational number such as $\sqrt{5}$ and explain that $\sqrt{5} \neq \frac{a}{b}$ for integers a and b.

Wesley's initial written response included the irrational numbers $\sqrt{2}, \sqrt{3}, \sqrt{5}$, and π without any explanation. When prompted to explain why these numbers were irrational, he stated:

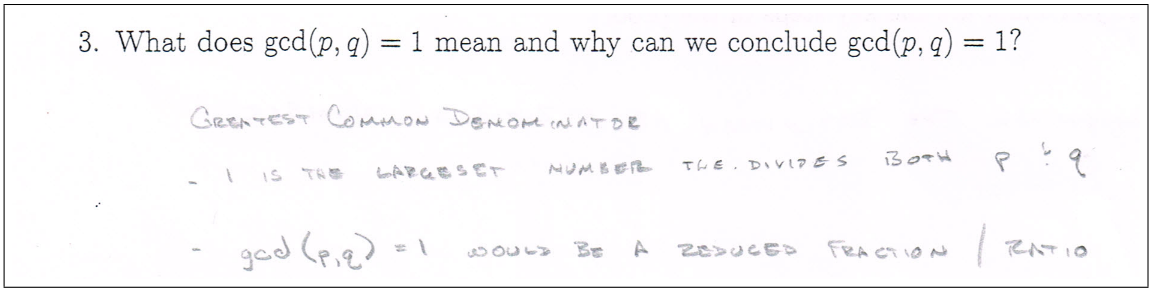
> I don't remember my definitions but is pi not irrational as well? *[Teacher again asks why]* It is because of that right there *[points to presented proof]*. Because you can't get a greatest common denominator equal to 1.

The phrase "I don't remember my definitions" suggests he still did not recall the formal definition of a rational number, even after reading the use of this definition in the presented proof. The phrase "greatest common denominator" referred to greatest common divisor (gcd) in the definition used in the proof while the phrase "you can't get a greatest common denominator equal to 1" improperly referenced this definition to explain why his examples were irrational. In other words, he did not possess the formal definition for irrational numbers that was necessary to understand the logical argument of the proof.

Yara's initial written response stated "$\sqrt{5}$ b/c it equals some fraction that can't be reduced." The phrase "equals some fraction that can't be reduced" referred to the formal definition of a rational number as a fraction where the numerator and denominator are in lowest terms. In this way, her reasoning that $\sqrt{5}$ is irrational was that $\sqrt{5}$ is rational. Therefore, she did not exhibit the knowledge of a formal definition for irrational numbers that was necessary to understand the logical argument of the proof.

Question three assessed whether students could identify the meaning of terms and statements in the presented proof. In particular, the question asked students to explain the phrase "$\gcd(p, q) = 1$" and why the proof states it is true. The goal of this question was to determine whether students understood one of the mathematical concepts in the proof (greatest common divisor) that was necessary to understand the logical argument of the proof, as the literature suggested students struggle with proof comprehension due to a lack of mathematical knowledge (e.g. Moore, 1994). In other words, the goal of this question was to assess prerequisite knowledge necessary to understand the particular proof. A desired response would describe $\gcd(p, q) = 1$ to mean that p and q have no common factors and that this is true by definition of a rational number.

Both Wesley and Yara could define gcd and state what $\gcd(p, q) = 1$ meant about the fraction $\frac{p}{q}$. In addition, both students implicitly related this to the formal definition of a rational number. For example, Wesley's written response is provided in Figure 4.51.


3. What does $\gcd(p, q) = 1$ mean and why can we conclude $\gcd(p, q) = 1$?

GREATEST COMMON DENOMINATOR

- 1 IS THE LARGESET NUMBER THE. DIVIDES BOTH p & q

- gcd (p,q) = 1 WOULD BE A REDUCED FRACTION / RATIO

Figure 4.51 Wesley's response to question three during Exercise 5.

The first item, "1 is the largest number the [sic] divides both p & q," was a proper definition of the greatest common divisor of p and q. The second item, "gcd $(p, q) = 1$ would be a reduced fraction/ratio" desirably related $\gcd(p, q) = 1$ to the fraction $\frac{p}{q}$ in lowest terms. This description was enough to understand the specific contradiction in the presented proof: $\gcd(p, q) = 1$ and $\gcd(p, q) \neq 1$. Therefore, both students exhibited the knowledge of greatest common divisors necessary to understand the logical argument of the proof.

Question four assessed whether students could justify a statement in the proof. In particular, the question focused on the equation $p^2 = 2q^2$ and why this equation meant that p^2 was even. The goal of this question was to assess whether students could provide justification for an omitted detail in the proof. Responses to this question would indicate whether students possessed a formal definition of 'even', which is necessary to understand the logic of the presented proof. A desired response would describe that q^2 is an integer and thus $p^2 = 2q^2$ is written in the form $2k$ for some integer k.

Both Wesley and Yara desirably noted that $p^2 = 2q^2$ meant that p^2 was even. For example, Yara responded "b/c $p^2 = 2q^2$ and an even # is any integer multiplied by two." The phrase "an even # is any integer multiplied by two" referred to the definition of an even number to justify why the equation $p^2 = 2q^2$ implied that p^2 was even. Therefore, both students exhibited the necessary mathematical knowledge of even numbers required to understand the presented proof.

Question five assessed whether students could identify the logical status of a statement in the proof. In particular, students were asked for the purpose of the statement "Now, if p were odd, then p^2 would be odd." The goal of this question was to assess whether students could recognize that this statement was a proof by contradiction (with the explicit contradiction omitted). Responses to this question would indicate students' conception of proof by contradiction as it contains the key steps of the method and does not follow the normal form of the method (i.e., cue words such as 'assume' and 'contradiction' are not present). A desired response would indicate the statement is a proof by contradiction as p^2 was shown to be even in the previous line of the proof and the next line of the proof concludes that p is even.

When asked for the purpose of the statement, Wesley initially wrote "b/c the the *[sic]* product of two (2) odd numbers is odd." This was the mathematical justification of the statement and not its purpose. After reading his written response, he began to describe his response to the next question. Thus, he did not recognize the statement as an implicit proof by contradiction and did not provide enough evidence to suggest why he did not recognize the statement as a proof by contradiction (e.g., was it that the proof did not follow every step of his procedure for proof by contradiction).

When asked for the purpose of the statement, Yara initially responded "I have no clue." The teacher/researcher then prompted Yara to consider that odd meant 2 did not divide the number. She then considered the purpose of the statement "If 2 does not divide p then 2 does not divide p^2." This discussion is provided below.

Teacher: So we are saying if 2 doesn't divide p, then 2 doesn't divide p squared. *[pause]* But it does, right?

Yara: Mmhmm. *[long pause]*

Teacher: Therefore 2 divides p. *[pause]* So what does this look like: 2 does not divide p implies 2 does not divide p squared. But 2 divides p squared. Therefore, 2 divides p.

Yara: Contradiction?

The teacher/researcher provided numerous pauses to give Yara a chance to recognize this rewritten statement as a proof by contradiction. After he listed the sentences to focus on, Yara only stated "Contradiction?" This suggests she recognized the two phrases "2 does not divide p squared" and "But 2 divides p squared" as a contradiction and did not recognize the procedure as a whole. In other words, Yara did not exhibit an Object conception of proof by contradiction as she did not recognize the procedure as a sub-proof within a proof.

Question six assessed whether students could justify a statement in the proof. In particular, the question focused on an omitted proof by contradiction as it would have been identical to the statement in question five. The goal of this question was to assess whether students could provide justification for an omitted detail in the proof. Responses to this question would indicate students' conception of mathematical logic as it required students to identify the argument the phrase "by the same argument as above" the proof referred to.

When Wesley was asked to justify the statement, he responded "And then how exactly can one conclude that q is an even number? Because q squared is equal to $2r$ squared which is again in the form of a equals $2k$ for a big number." This response explicitly referred to the need to justify that q as an even number and justified that q^2 is an even number. He then moved on to discuss his response to the next question and did not provide a justification for why if q^2 is even then q must be even, which is indicative of a Pre-Action conception of mathematical logic.

When Yara was asked to justify the statement, she also responded "b/c $q^2 = 2r^2$." However, the teacher/researcher followed up and, after a single prompt, she was able to justify that q^2 was even implied q was even. This discussion is provided below.

Yara: Is that because q squared equals $2r$ squared? So *[pause]* that means *[pause]* that 2 can divide q?

Teacher: Okay, why?

Yara: Beeeeecause it's even?

Teacher: So 2 times r squared equals q squared means 2 divides q squared. So why does it divide q?

Yara: Because if it divides q squared then it divides q. Euclids Lemma.

First, Yara responded that q^2 was even implied q was even and provided the vague justification "Beeeeecause it's even?" The teacher/researcher then read the line from the equation "$2r^2 = q^2$" and stated "means 2 divides q^2." In other words, he focused her attention on her original written response "b/c $q^2 = 2r^2$." She then responded with a valid justification by way of Euclid's Lemma: If p is prime and $p \mid ab$, then $p \mid a$ or $p \mid b$. As mentioned previously, she had seen a proof of the statement "$\sqrt{2}$ is irrational" previously - these previous proofs utilized Euclid's lemma rather than a sub-proof. This would suggest Yara was able to recall this justification from a previously memorized proof and thus exhibited an Action conception of mathematical logic.

Question seven assessed whether students could summarize the proof via high-level ideas. The purpose of this question was to assess whether students could summarize the proof in their own words and, if they could, how they summarized the proof. For example, a summary for the presented proof could be similar to the following:

> It's a proof by contradiction, where you assume the statement isn't true. So then $\sqrt{2} = \frac{p}{q}$ and $\gcd(p,q) = 1$. Then you solve for p^2 and get $p^2 = 2q^2$, so that means p^2 is even. But p^2 is even means p is even, so then $p = 2r$. We then plug that back in to our original equation and get $4r^2 = 2q^2$, which means $2r^2 = q^2$. So then q^2 is even which means q is even. That means $\gcd(p,q) \neq 1$, which is a contradiction with $\gcd(p,q) = 1$. That means the statement is true.

This summary describes the specific procedure of the presented proof and does not justify statements in the proof. Thus, this type of summary would illustrate an Action conception of proof by contradiction as a student has to go through a step-by-step procedure and include all details of the proof. In contrast, another summary could be similar to the following:

> If the statement were false, then we could solve for p^2 and q^2 to show these are both even. But we know that if a number squared is even, then the number is even. So p and q are both even, which is a contradiction to the assumption that the fraction is in lowest terms. Thus, our assumption is false and therefore the statement must be true.

This response describes the main idea of the proof (e.g., showing p^2 and q^2 are even, which implies that p and q are even) and provides a generalization of the two key steps of a proof by contradiction, which is indicative of a Process conception of the proof method as the student has interiorized details of the proof and is able to explain the proof in more general terms. Finally, a summary could be similar to the following:

> If the statement were false, then p^2 and q^2 are even. We can do then do a separate proof by contradiction within the proof that shows if a^2 is even, then a is even. That sub-proof means p and q are even, which contradicts the assumption that the fraction is in lowest terms. Therefore, the statement must be true.

This response is similar to the previous example in all ways except for how the main idea of the proof was summarized. In this example, the summary includes a separate proof by contradiction as a sub-proof to validate two claims made in the statement (in contrast to the mathematical justification in the previous example). Using a proof by contradiction as a sub-proof is indicative of an Object conception of the proof method. Therefore, responses to this question would indicate students' conception of proof by contradiction.

When asked to summarize the proof, Wesley responded "Proof by contradiction is not always a direct contradiction of the negated statement." This comment referred to the previous discussion of what the contradiction in a general proof by contradiction was (i.e., that it was not represented as $\sim P = P$) that took place during Classroom Discussion 5. His response here described what he believed was the teacher/researcher's goal of reading this particular proof and did not attempt to summarize the proof. Therefore, he did not provide evidence of his conception of proof by contradiction with this response.

When asked to summarize the proof, Yara responded "To show $\sqrt{2}$ is irrational" - which was the statement proved. As in Exercise 4, the teacher/researcher then explicitly stated this was the statement and not a summary of the proof. Yara then provided part of the main idea of the proof: showing that $\gcd(p, q) \neq 1$. This discussion is provided below.

Yara: To show that square root of 2 is irrational?

Teacher: Okay, but remember when we were talking about this, we were able to do a lot of this proof. So when it talks about the main idea of the proof, not the statement but the proof, well, it seems like this is where that main idea is coming from.

Yara: So it's to prove that there is more than one greatest common... something?

The phrase "more than one greatest common... something" referred to the how the proof arrived at a contradiction: showing $\gcd(p, q) \neq 1$. Her response was not stated properly (e.g., to prove that the greatest common divisor is not equal to 1) and thus she could not desirably describe how the proof arrived at a contradiction, which is indicative of a Pre-Action conception of proof by contradiction.

Question eight assessed whether students could identify the modular structure of the proof. In other words, whether students could group sentences in the proof together and provide a purpose for each of these groups. By phrasing the question as "key steps", the teacher/researcher attempted to illicit the purpose of particular lines and how these purposes related to form a complete proof. Responses to this question would indicate students' conception of proof by contradiction. In particular, students that responded with the two key steps of a proof by contradiction (assuming the statement is false and arriving at a contradiction) and did not describe the purpose of these steps in the overall argument of the proof would have exhibited an Action conception of proof by contradiction, while students that responded with lines from the desired logical outline and described the purposes of these lines in the overall proof would have exhibited a Process conception of proof by contradiction. Finally, students who stated that sub-proofs were key steps in the overall proof would have exhibited an Object conception of proof by contradiction.

When asked to describe the key steps of the proof, Wesley stated "Knowing the definition of a rational number. I have never seen that before. But because of that, you are not... it's not a direct... proving the negation." This is similar to his response to the key steps in the presented proof for Exercise 4 in that he described the mathematical knowledge necessary to utilize his proof by contradiction procedure. In addition, the phrase "you are not... it's not a direct... proving the negation" referred to the form of the contradiction and, specifically, that the contradiction did not include a negation of the statement (i.e., $S \wedge \sim S$). In other words, this comment also referred to his general procedure for proof by contradiction. Since his response describes the key steps of the proof in terms of his procedure (and form of steps in his procedure), Wesley exhibited an Action conception of proof by contradiction.

When asked to describe the key steps of the proof, Yara wrote "Assuming the negation statement and providing a contradiction by showing that the $\gcd(p, q) = 1$ and $\gcd(p, q) \neq 1$?" In particular, the phrase "providing a contradiction by showing that the $\gcd(p, q) = 1$ and $\gcd(p, q) \neq 1$" was what she tried to articulate when asked for a summary of the proof. Since this response contained both key steps of a proof by contradiction and a description of how the proof would arrive at a contradiction, Yara exhibited a Process conception of the proof method.

Question nine also assessed whether students could transfer the general idea or method to another context in the form of writing a similar proof for the statement "For any prime number n, $\sqrt{n}$ an irrational number. This proof encouraged students to write a similar proof by contradiction, this time a quantified 'property claim' statement. In addition to the small change to the structure of the statement, a proof would require the use of either a lemma (theorem to be called on as mathematical fact) or a sub-proof within the proof in order to show that if p^2 is divisible by a prime number n, then p is divisible by n. Responses to this question and, in particular, how students would incorporate these two changes into their proof, would indicate students' conception of proof by contradiction. A student with an Action conception of proof by contradiction would focus on the key steps of a proof by contradiction in a way similar to the previously presented proof. In addition, they would describe a procedure for proving 'for all' statements (i.e., let n be arbitrary [...] since n was arbitrary, this is true for all n) in order to satisfy that the property is true for all prime numbers n. A student with a Process conception of proof by contradiction would utilize a generalized proof by contradiction procedure (i.e., assume the statement is false; that is, there exists a prime n such that $\sqrt{n}$ is a rational number [...] since the assumption led to a contradiction, the statement is true). To complete the proof, this student would rely on a lemma or external theorem as mathematical justification verify that if p^2 is divisible by n, then p is divisible by n. Finally, a student with an Object conception of proof by contradiction would also rely on a generalized contradiction procedure to prove the statement as well as directly prove (as a sub-proof) the necessary theorem to complete the proof.

A desired proof for question nine is presented in Figure 4.52.

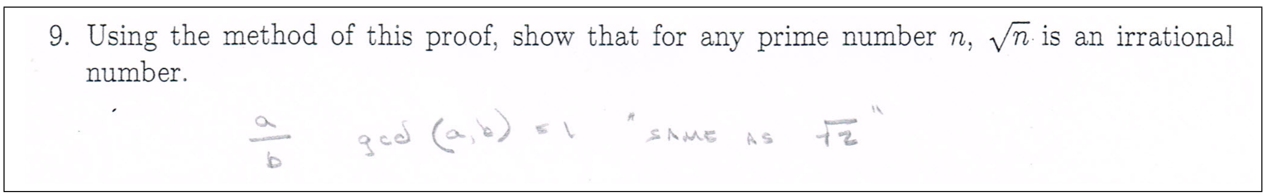
Statement: For any prime number n, $\sqrt{n}$ is an irrational number.

Proof: Assume the statement is not true [**1**]; that is, that there exists a prime number n such that $\sqrt{n}$ is a rational number. Then $\sqrt{n} = \frac{p}{q}$ with integers $p, q \neq 0$ such that $\gcd(p, q) = 1$. Then $nq^2 = p^2$ and thus $n \mid p^2$. Now, if $n \nmid p$, then when p is prime factorized, n is not a factor and therefore $n \nmid p^2$. This is a contradiction as we have already shown $n \mid p^2$ and thus our assumption, $n \nmid p$, is false. [**4**] This implies $n \mid p$ and thus $nk = p$ for some integer k. Plugging this into the previous equation, $nq^2 = n^2k^2$ and thus $q^2 = nk^2$. By a similar argument as before, $n \mid q^2$ implies $n \mid q$. Since $n \mid p$ and $n \mid q$, $\gcd(p, q) \geq n$. Since n is a prime number, $n > 1$. This implies that $\gcd(p, q) > 1$, which is a contradiction [**2**]. Thus, the assumption is false and therefore the statement is true [**3**].

Figure 4.52 Desired proof for question eight in Exercise 4.

This proof is desired for four main reasons. First, it includes both key steps of a proof by contradiction: assuming the statement is not true (denoted as [**1**]) and arriving at a contradiction (denoted as [**2**]). Secondly, it explains how these key steps logically imply the statement is true (denoted as [**3**]). Thirdly, the proof is valid in that all lines are properly justified. In particular, all steps of the sub-proof (denoted by [**4**]) by contradiction are fully justified. It would also be desirable to remove this sub-proof and prove it separately, though this would not change the argument. Therefore, proofs that satisfy these three criteria, such as the proof in Figure 4.52, are considered to be desired proofs for question nine.

When asked to write a similar proof, Wesley responded "same as $\sqrt{2}$" (see Figure 4.53).


9. Using the method of this proof, show that for any prime number n, $\sqrt{n}$ is an irrational number.

$\frac{a}{b}$ gcd $(a,b) = 1$ "SAME AS $\sqrt{2}$"

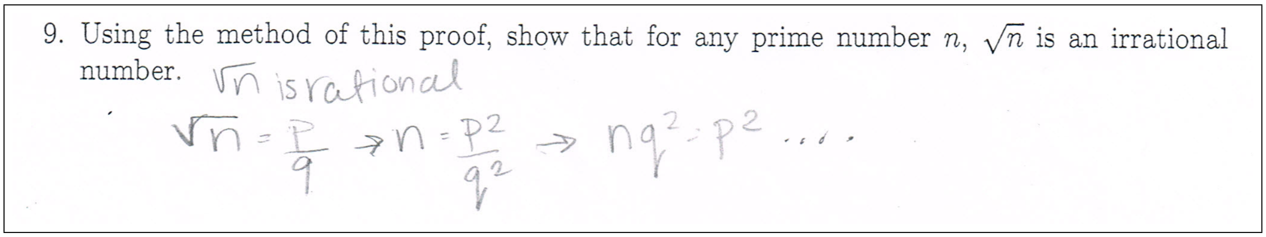
Figure 4.53 Wesley's response to question nine during Exercise 5.

After reading his written response, he added "Your greatest common denominator is not going to equal 1." He was then prompted to explain how a particular argument in the previous proof, that p^2 is even implies p is even, would change in this new proof to determine whether he would be able to complete the similar proof. He responded:

> Why... *[pause]* for *[long pause]* p squared to be prime, then *[pause]* you can only have a factor of itself and 1. So *[pause]* you are not going to get a *[pause]* you are not going to get a number that has a factor of... a square that has 1 and itself. Because it will always be itself twice.

The phrase "for *[long pause]* p squared to be prime *[...]*" suggests Wesley modified the statement "p^2 is even implies p is even" to "If p^2 is prime, then p is prime" as the number 2 was replaced by the prime number n. This is not a valid modification of the statement and suggests he would not be able to rewrite the proof. However, the rest of his response explained that p^2 is not prime as "you are not going to get a number that has a factor of... a square that has 1 and itself. Because it will always be itself twice." This suggests that Wesley did not recognize how the proof would change and could not provide a valid argument for these changes, which is indicative of a Pre-Action conception of proof by contradiction

When asked to write a similar proof, Yara wrote the same chain of implications that she had memorized for this particular type of proof (see Figure 4.54 for details).


9. Using the method of this proof, show that for any prime number n, $\sqrt{n}$ is an irrational number. $\sqrt{n}$ is rational

$\sqrt{n} = \frac{p}{q} \rightarrow n = \frac{p^2}{q^2} \rightarrow nq^2 = p^2 \ldots$

Figure 4.54 Yara's response to question nine during Exercise 5.

When asked to elaborate on her response, she stated:

> It's not an exact proof type order, but I put that the square root of n is rational, and then it means like square root of n equals p over q, where that goes to n equals p squared over q, where that goes to n times q squared equals p squared. And then I couldn't, I was really, I was trying to remember that lemma thing, I just didn't know that's what I was trying to remember.

The phrase "I was trying to remember that lemma thing" referred to Euclid's Lemma, which she used to provide the justification for the statement "If q^2 is even, then q is even" in question 6. She then continued:

> And then, so... the lemma would say that if n divides p squared then n divides p. So that would mean there is, like, more than one thing that can divide p? More than one factor than 1?

The phrase "more than one thing that can divide p" referred to the divisors of p in order to show that $\gcd(p, q) \neq 1$, which she stated in question eight was a key step for the presented proof. This suggests that had she recalled Euclid's Lemma (which her previously memorized proofs used) she would have been able to complete the similar proof. Since Yara was able to describe how the proof changed without sequentially writing the proof, she exhibited a Process conception of proof by contradiction.

4.5.4 Summary of results

Teaching episode 5 was designed to achieve two goals. First, the episode introduced a general proof by contradiction that could be considered proving the proof method (i.e., showing the proof method is valid) and thus encourage students to encapsulate their Process for proof by contradiction into an Object. Secondly, the episode introduces a new statement type that can be proved by contradiction - property claim statements (e.g., $\sqrt{2}$ is an irrational number). Students were encouraged to use their general procedure for proof by contradiction to prove a particular statement and thus assimilate or accommodate the new type of proof into their Schema for the method. All tasks for this episode were designed around these two goals.

Subsection 4.5.1 reported on students' work during Activity 5. The goals of this phase was to encourage students to validate the contradiction proof method. Thus, the phase consisted of two tasks: (4.5.1.1) outlining a proof of the method and (4.4.1.2) defining proof by contradiction. A brief summary of the results from each task follows.

Wesley's outline focused on his previous procedure for proof by contradiction rather than represent the specific statements in the proof. In addition, he included the step $\sim P = P$: a persistent, improper representation that a contradiction is always made with the initial statement. Attending to a memorized, step-by-step procedure with improper steps is indicative of a Pre-Action conception of proof by contradiction. Yara's outline focused on an abbreviated version of her previous procedure for proof by contradiction rather than represent the specific statements in the proof. However, after outlining the proof, she provided evidence of considering the presented proof as a validation of proof by contradiction, though it was not explicit enough to state she considered acting on a proof procedure as a whole. Therefore, Yara was developing toward an Object conception of proof by contradiction and could not yet describe how to validate the proof method.

Wesley did not provide a definition when asked and instead referred to the outline as a "perfect example." He also did not provide a definition when asked to clarify. Therefore, his reference to the outline was considered to be his entire definition, which is indicative of an Action conception of proof by contradiction. Yara's definition of proof by contradiction described the two key steps of a proof by contradiction. She then moved on before she could be prompted to explain how these steps logically imply the statement is true. Therefore, she exhibited at least an Action conception of proof by contradiction.

Subsection 4.5.2 reported on students' responses during Classroom Discussion 5. The purpose of this phase was to formalize students' conceptions of proof by contradiction based on their work and conjectures during Activity 5. In particular, the goal of this phase was to continue developing students' procedure for proof by contradiction. Therefore, the tasks for this phase are based on developing and testing students' general procedures from the previous teaching episodes. A brief summary of the results from each task follows.

Wesley and Yara were asked to compare their general procedure for proof by contradiction to the presented proof in Activity 5 to possibly modify the logical reasoning behind their general procedure for proof by contradiction. However, neither student modified their logical reasoning due to the outline.

When asked to write a proof for the statement "$\sqrt{2}$ is an irrational number," Wesley followed his general procedure for proof by contradiction to negate the statement. He expressed that he would have unpacked the definition of a rational number if he knew it, which illustrated that a lack of mathematical knowledge inhibited his proof writing. He was then prompted to utilize the rest of his procedure for proof by contradiction to outline the proof, to which he again suggested the contradiction was related to the statement, which is indicative of a Pre-Action conception of proof by contradiction. When asked to write a proof for the same statement, Yara recited a memorized portion of the proof and could not continue without these memorized steps, which is indicative of an Action conception of proof by contradiction.

Subsection 4.5.3 reported on students' work during Exercise 5. The goal of this phase was to reinforce students' conceptions of mathematical logic and proof by contradiction through a series of eight tasks aligned with the proof comprehension assessment model by Mejía-Ramos et al. (2012) as well as an additional question not aligned with the assessment model. A short summary of results for questions two through nine follow.

Question two: When asked to provide an example of a number that is irrational and explain why it is irrational, Wesley provided appropriate examples without any explanation. When prompted to provide an explanation, he improperly referenced the definition in the presented proof and therefore did not possess the formal definition for irrational numbers that was necessary to understand the logical argument of the proof. When Yara was asked the same question, she provided an appropriate example along with an explanation and therefore did possess the formal definition for irrational numbers that was necessary to understand the logical argument of the proof.

Question three: When asked to explain the meaning of the phrase "$\gcd(p,q) = 1$" and why it was true in the particular proof, both Wesley and Yara appropriately described the definition of the greatest common divisor of two numbers and related this claim to the definition of a rational number. Therefore, both students exhibited the prerequisite knowledge of greatest common divisors necessary to understand the logical argument of the proof.

Question four: When asked to justify why a number in the proof was even, both Wesley and Yara desirably noted $p^2 = 2q^2$ meant that p^2 was even. Therefore, both students exhibited the necessary mathematical knowledge of even numbers required to understand the proof.

Question five: When asked for the purpose of the statement "Now, if p were odd, then p^2 would be odd," Wesley instead explained why the statement was true and is indicative of a

Pre-Action conception of mathematical logic. In addition, he did not recognize the statement as an implicit proof by contradiction. When Yara was asked for the purpose of the same statement, she also did not recognize the statement as an implicit proof by contradiction.

Question six: When asked to provide the justification for an omitted detail in the proof, Wesley provided an improper and undesirable justification and thus exhibited a Pre-Action conception of mathematical logic. When Yara was asked to do the same, she was able to provide a valid justification from a previously memorized proof and thus exhibited an Action conception of mathematical logic.

Question seven: When asked to summarize the proof, Wesley instead responded with what he believed was the teacher/researcher's goal of reading the particular proof and thus did not provide evidence of his conception of proof by contradiction. When asked to summarize the proof, Yara initially responded with the statement proved. When prompted that this was not a desirable response, she referred to but did not completely describe how the proof arrived at a contradiction and thus exhibited a Pre-Action conception of the proof method.

Question eight: When asked to provide the key steps of the proof, Wesley stated the definition of a rational number, which is the mathematical knowledge necessary to utilize his proof by contradiction procedure. In addition, he appropriately noted the contradiction did not directly relate to the statement of the proof and therefore exhibited an Action conception of proof by contradiction. When Yara was asked for the key steps, she responded with both key steps of a proof by contradiction and a complete description of the contradiction, which is indicative of a Process conception of the proof method.

Question nine: When asked to write a similar proof, Wesley stated the proof would be the same as before. However, when asked to articulate how a particular argument would change in the new proof, he could not do so. Therefore, he did not recognize how the proof would change and could not provide a valid argument for these changes, which is indicative of a Pre-Action conception of proof by contradiction. When Yara was asked to write a similar proof, she wrote the same chain of implications she had memorized for the previous proof.

However, she also described how she had forgotten Euclid's Lemma and was then able to describe, in general, how to complete the proof. Since Yara was able to describe how the proof changed without sequentially writing the proof, she exhibited a Process conception of proof by contradiction.

4.6 Chapter summary

This chapter presented the analysis and interpretation of data collected throughout the course of the teaching experiment. This data was organized into sections by teaching episode and further into subsections by phases of the ACE teaching cycle.

Section 4.1 presented the results of teaching episode 1, which was designed to achieve two goals. First, groups were guided to construct a set of step-by-step instructions for proofs by contradiction of an implication statement (i.e., $P \to Q$). Groups could then use this set of step-by-step instructions to prove similar statements by contradiction and thus possibly develop an Action conception of proof by contradiction for this type of statement. Secondly, groups were encouraged to focus on the roles of collections of lines in a proof and on the key steps of the proof method. Groups could then relate these roles to interiorize the previous step-by-step instructions into general steps for an implication statement and thus possibly develop a Process conception of proof by contradiction for this type of statement.

This section focused on how students' understanding emerged from an analysis of groups' responses to the tasks and teacher/researcher prompts during teaching episode 1. A summary of the results from tasks during each phase follows.

Subsection 4.1.1 reported on groups' responses during Activity 1. This phase consisted of two tasks: outlining a given proof by contradiction for an implication statement and defining proof by contradiction. Both categories of groups' responses to the outline task were indicative of a Pre-Action conception of proof by contradiction and, in particular, no group provided a complete logical outline of the presented proof. Of the three categories of responses to the definition task, two were indicative of a Pre-Action conception of proof by contradiction and one was indicative of an Action conception of the proof method.

Subsection 4.1.2 reported on groups' responses during Classroom Discussion 1. The tasks for this phase were based on the two tasks from Activity 1. All groups initially agreed on an incomplete outline of the presented proof that did not include the justification for two lines: 4 and 6. All three types of responses to the logical justification for line 4 were undesirable and indicative of a Pre-Action conception of mathematical logic. One type of response to the logical justification for line 6 identified the appropriate lines and could not describe the reasoning behind the justification, which is indicative of an Action conception of mathematical logic. The other type of response to the logical justification for line 6 both appropriately identified the necessary lines and provided a reasoning behind the justification, which is indicative of a Process conception of mathematical logic. After groups presented their initial conjectures of a definition for proof by contradiction, the teacher/researcher presented the following formal definition: prove that the statement is true by assuming the negation and proving this assumption is false with a contradiction.

Subsection 4.1.3 reported on groups' work during Exercise 1. The tasks for this phase were eight comprehension questions aligned with the proof comprehension assessment model by Mejía-Ramos et al. (2012). Responses to these tasks revealed the following about:

Mathematical Knowledge

- Nearly all groups possessed the prerequisite knowledge of primes numbers with their responses to question one.

Conception of Mathematical Logic

- All groups exhibited a Pre-Action conception of mathematical logic with their improper responses to questions two and four that asked to justify (mathematically or logically) statements in the proof. In addition, some groups exhibited a Pre-Action conception with their incomplete response to question seven that asked groups to consider the effects of a modification to the statement.
- Some groups exhibited an Action conception of mathematical logic with their complete response to question three that asked students to justify a statement in the proof.

- Some groups exhibited a Process conception of mathematical logic with their complete response and explanation of the response to question seven that asked students to consider the effects of a modification to the statement.

Conception of Proof by Contradiction

- All groups exhibited a Pre-Action conception of proof by contradiction with their responses to questions five (summarize the proof) and eight (write a similar proof). In addition, some groups exhibited a Pre-Action conception of proof by contradiction with their incomplete response to question six that asked for the key steps of the proof.
- Some groups exhibited an Action conception of proof by contradiction with their complete response to question six that asked for the key steps of the proof.

Section 4.2 presented the results of teaching episode 2, which was designed to achieve three goals. First, groups were guided to construct a set of step-by-step instructions for proofs by contradiction for a nonexistence statement (i.e., $(\nexists x)(P(x))$). Groups could then use this set of step-by-step instructions to prove similar statements by contradiction and thus possibly develop an Action conception of proof by contradiction for this type of statement. Secondly, groups were encouraged to focus on the roles of groups of lines in a proof and on the key steps of the proof method. They could then relate the roles of lines or collections of lines to interiorize the previous step-by-step instructions into general steps for a nonexistence statement and thus possibly develop a Process conception of proof by contradiction for this type of statement. Thirdly, groups were encouraged to compare their general steps for an implication statements (from teaching episode 1) to their general steps for a nonexistence statement. Groups could then coordinate these general steps for specific structures of statements to construct general steps for any proof by contradiction and thus possibly develop a Process conception of proof by contradiction.

This section focused on how students' understanding emerged from an analysis of groups' responses to the tasks and teacher/researcher prompts during teaching episode 2. A summary of the results from tasks during each phase follows.

Subsection 4.2.1 reported on groups' work during Activity 2. This phase consisted of three tasks: outlining a given proof by contradiction for a nonexistence statement, defining proof by contradiction, and comparing the presented proofs from Activity 1 and the previous task. Each of the three types of responses to the outline task were indicative of a different conception of mathematical logic and proof by contradiction. In particular, an undesired and/or unclear representation was indicative of a Pre-Action conception of mathematical logic, a response that relied on the outline from Classroom Discussion 1 was indicative of an Action conception of proof by contradiction, and a desired response that relied on algebraic manipulation was indicative of a Process conception of mathematical logic. Of the four categories of responses to the definition task (three of which were present in Activity 1), two were indicative of a Pre-Action conception of proof by contradiction while the other two two were indicative of an Action conception of proof by contradiction. Finally, most responses to the comparison task were superficially or blank, which was indicative of a Pre-Action conception of proof by contradiction. However, three groups compared the forms of particular lines, which was indicative of an Action conception of proof by contradiction.

Subsection 4.2.2 reported on groups' responses during Classroom Discussion 2. The tasks for this phase were based on the three tasks of Activity 2. All groups initially agreed on an incomplete outline of the presented proof that did not include the justification for how the proof arrived at a contradiction. Some groups' responses to the justification task relied on the general outline of a proof by contradiction, which is indicative of an Action conception of the proof method. Other groups' responses to the justification task relied on mathematical justification without reasoning, which is indicative of an Action conception of mathematical logic. When asked to present their definitions of proof by contradiction, one group offered a formal definition that was established in Classroom Discussion 1 and all groups subsequently agreed on this definition. Most groups' responses to the comparison task described the initial assumption, contradiction, and completion of the proof without intermediate steps, which is indicative of an Action conception of proof by contradiction. Some groups' responses to the comparison task also described how these key steps related to prove the statement, which is indicative of a Process conception of proof by contradiction.

Subsection 4.2.3 reported on groups' work during Exercise 2. The tasks for this phase were eight comprehension questions aligned with the proof comprehension assessment model by Mejía-Ramos et al. (2012). Responses to these tasks illustrated the following about:

Mathematical Knowledge

- All groups possessed the prerequisite knowledge of odd integers with their responses to question one. In addition, some groups possessed the necessary prerequisite knowledge of odd integers in the form $4j - 1$ in order to understand the presented statement with their responses to question two.
- Most groups did not exhibit the necessary prerequisite knowledge of odd integers in the form $4j - 1$ with their responses to question two.

Conception of Mathematical Logic

- Most groups exhibited a Pre-Action conception of mathematical logic with their improper responses to questions three and seven that asked groups to justify (mathematically or logically) statements in the proof. In addition, some groups exhibited a Pre-Action conception of mathematical logic with their improper responses to question seven that asked groups to consider the effects of a modification to the statement.
- Some groups exhibited an Action conception of mathematical logic with their responses to question seven that relied on examining the algebraic manipulation in the proof when asked to consider the effects of a modification to the statement.

Conception of Proof by Contradiction

- All groups exhibited a Pre-Action conception of proof by contradiction with their responses to question five that asked students to summarize the proof.
- Some groups exhibited an Action conception of proof by contradiction with their responses to question three that relied on their procedure to justify a line in the proof

as well as question eight that relied on their procedure to write a similar proof. In addition, all groups exhibited an Action conception of proof by contradiction with their responses to question four that relied on their procedure to identify the logical status of a line in the proof as well as their responses to question six that relied on their procedure to identify the key steps of the proof.

- Some groups exhibited a Process conception of proof by contradiction with their responses to question eight as they provided a partial proof and could describe how the proof would be written without writing every line of the proof.

Section 4.3 presented the results of teaching episode 3, which was designed to achieve three goals. First, the episode introduced a set of step-by-step instructions for students to use to construct proofs by contradiction for a uniqueness statement (i.e., $(\exists x!)(P(x))$). Students could then use this set of step-by-step instructions to prove similar statements by contradiction and thus possibly develop an Action conception of proof by contradiction for this type of statement. Secondly, students were encouraged to focus on the roles of groups of lines in a proof and on the key steps of the proof method. They could then relate the roles of lines or collections of lines to interiorize the previous step-by-step instructions into general steps for a uniqueness statement and thus possibly develop a Process conception of proof by contradiction for this type of statement. Thirdly, students reflected on both their prior knowledge of proof by contradiction and a specific example of proof by contradiction to assimilateor accommodatethe new logical outline into their general procedure for a proof by contradiction.

This section focused on how Wesley and Yara's understanding emerged from an analysis of their responses to the tasks and teacher/researcher prompts during teaching episode 3. A summary of the results from tasks during each phase follows.

Subsection 4.3.1 reported on students' work during Activity 3. This phase consisted of two tasks: outlining a given proof by contradiction for a uniqueness statement and defining proof by contradiction. A summary of Wesley and Yara's results for each task follows.

Wesley's outline included the two key steps of a proof by contradiction and did not describe the logical relation between these steps, which is indicative of an Action conception of proof by contradiction. Yara's outline included the two key steps of a proof by contradiction as well as described the logical relation between lines and generalized specific lines in the proof to describe their purpose in the overall argument, which is indicative of a Process conception of proof by contradiction.

Wesley's definition of proof by contradiction described the two key steps of a proof by contradiction as cue words and did not describe how these steps logically related to prove the statement, which is indicative of an Action conception of proof by contradiction. Yara's definition of proof by contradiction described the two key steps of a proof by contradiction as well as described how these steps logically related to prove the statement, which is indicative of a Process conception of proof by contradiction.

Subsection 4.3.2 reported on students' responses during Classroom Discussion 3. The tasks for this phase were based on the two tasks of Activity 3 as well as an additional task, writing an outline of a proof, that reinforced the formalization of their conceptions. A summary of Wesley and Yara's results for each task follows.

When Wesley was asked to logically justify each line in his outline, he could not provide any logical justification for some lines in the proof and thus exhibited a Pre-Action conception of mathematical logic. When Yara was asked to logically justify each line in her outline, she provided logical justification for all lines in her outline as well as described the logical relationship between collections of lines and thus exhibited a Process conception of mathematical logic.

When prompted to compare three outlines of specific types of proof by contradiction, Wesley instead recalled a general procedure that described the key steps of a proof by contradiction and did not describe any intermediate steps nor described the logical relation between lines, which is indicative of an Action conception of proof by contradiction. When prompted to compare three outlines of specific types of proof by contradiction, Yara produced a minimal change on her previous procedure for proof by contradiction and thus assimilated the new outline into her Schema for proof by contradiction, which is indicative of an enhanced

Process conception of proof by contradiction.

When prompted to write a proof outline for a similar proof, Wesley constructed an outline by explicitly following the step-by-step outline for a uniqueness statement from Activity 3 and thus exhibited an Action conception of proof by contradiction. When prompted to write a proof outline for a similar proof, Yara instead wrote a complete proof that followed her general procedure for proof by contradiction and did not rely on the outline for a uniqueness statement from Activity 3 and thus exhibited a Process conception of proof by contradiction.

Subsection 4.3.3 reported on students' work during Exercise 3. The tasks for this phase were eight comprehension questions aligned with the proof comprehension assessment model by Mejía-Ramos et al. (2012) as well as one additional task that was not aligned with assessment model. Responses to these tasks revealed the following about:

Mathematical Knowledge

- Both students possessed the prerequisite knowledge of multiplicative inverses with their responses to question one.

Conception of Mathematical Logic

- Both Wesley and Yara exhibited a Pre-Action conception of mathematical logic with their improper justification of the statements in questions three and four.
- Wesley exhibited an Action conception of mathematical logic with his response to question seven that relied on examining the algebraic manipulation in the proof when asked to consider the effects of a modification to the statement.
- Yara exhibited a Process conception of mathematical logic with her response to question seven that relied on how the new conditions affected the meaning of the statement.

Conception of Proof by Contradiction

- Both students exhibited a Pre-Action conception of proof by contradiction with their improper responses to question five that asked them to summarize the proof. In addition, Wesley exhibited a Pre-Action conception of proof by contradiction with his improper response to question six that asked for the key steps of the proof.

- Wesley exhibited an Action conception of proof by contradiction with his response to question five that relied on his procedure to identify the purpose of a statement in the proof. In addition, Yara exhibited an Action conception of proof by contradiction with her response to question six that relied on her procedure to identify the key steps of the proof. Finally, both students exhibited an Action conception of proof by contradiction with their responses to question nine that relied on a step-by-step procedure to write a similar proof.
- Yara exhibited a Process conception of proof by contradiction with her response to question five that relied on the general argument of the proof to identify the purpose of a statement.

Section 4.4 presented the results of teaching episode 4, which was designed to achieve three goals. First, the episode introduced a set of step-by-step instructions for students to use when constructing proofs by contradiction for an infinity statement (e.g., There are infinitely many prime numbers). Students could then use these instructions to prove similar statements by contradiction and thus possibly develop an Action conception of proof by contradiction for this type of statement. Secondly, students were encouraged to relate the roles of lines or collections of lines to interiorize the previous step-by-step instructions into general steps for an infinity statement and thus possibly develop a Process conception of proof by contradiction for this type of statement. Thirdly, students reflected on both their knowledge of proof by contradiction and a specific example of proof by contradiction to assimilate or accommodate the new logical outline into their general procedure for a proof by contradiction.

This section focused on how Wesley and Yara's understanding emerged from an analysis of their responses to the tasks and teacher/researcher prompts during teaching episode 4. A summary of the results from tasks during each phase follows.

Subsection 4.4.1 reported on students' work during Activity 4. This phase consisted of two tasks: outlining a given proof by contradiction for an infinity statement and defining proof by contradiction. A summary of Wesley and Yara's results for each task follows.

Wesley's outline included the two key steps of a proof by contradiction and did not describe the logical relation between these steps, which is indicative of an Action conception of proof by contradiction. In addition, he could not justify intermediate steps outside his procedure, which is indicative of a Pre-Action conception of mathematical logic. Yara's outline included the two key steps of a proof by contradiction as well as described the logical relation between lines, which is indicative of a Process conception of proof by contradiction. In addition, she desirably represented the statement without relying on a previous example, which is indicative of a Process conception of mathematical logic.

Wesley's definition of proof by contradiction described only one of the two key steps of a proof by contradiction using a cue word, which is indicative of a Pre-Action conception of proof by contradiction. When describing how the example proof illustrated his definition, his explanation suggested he did not understand the logical role of the construction in the proof, which is indicative of a Pre-Action conception of mathematical logic. Yara's definition of proof by contradiction described the two key steps of a proof by contradiction and, unlike Activity 3, did not describe the relation between these key steps, which is indicative of an Action conception of proof by contradiction.

Subsection 4.4.2 reported on students' responses during Classroom Discussion 4. The tasks for this phase are based on developing and testing a general procedure for proof by contradiction. A summary of Wesley and Yara's results for each task follows.

When Wesley was prompted to compare the four outlines to construct a new list of steps, he instead described a general procedure similar to the one he developed during Classroom Discussion 3. This outline described the two key steps of a proof by contradiction without explaining how these steps logically imply the statement is true, which is indicative of a Pre-Action conception of proof by contradiction. In addition, this outline included circular reasoning and improper representations of statements, which is indicative of a Pre-Action conception of mathematical logic. When Yara was prompted to compare the four outlines, she assimilated the fourth proof outline into her previous procedure and thus exhibited a Process conception of proof by contradiction.

When prompted to write a proof outline for a similar proof, Wesley constructed an outline by explicitly following the step-by-step outline for an infinity statement from Activity 4 and, in particular, suggested the same construction without modification and is indicative of an Action conception of proof by contradiction. Yara also followed the step-by-step outline for an infinity statement from Activity 4. However, she also recognized, without prompting, the previous construction would not work for the new proof. This suggests that while she focused on the previous outline, she also described the new context of the proof and thus exhibited a Process conception of proof by contradiction.

Subsection 4.4.3 reported on students' work during Exercise 4. The tasks for this phase were eight comprehension questions aligned with the proof comprehension assessment model by Mejía-Ramos et al. (2012) as well as one additional task that was not aligned with assessment model. Responses to these tasks revealed the following about:

Mathematical Knowledge

- Both students possessed the prerequisite knowledge of infinity with their responses to question one.

Conception of Mathematical Logic

- Both students exhibited a Pre-Action conception of mathematical logic with their improper justification of the statements in questions three and four. In addition, both students exhibited a Pre-Action conception of mathematical logic with their improper response to the purpose of the statement in question five. Finally, both students exhibited a Pre-Action conception of mathematical logic with their improper response and explanation of question eight that required students to consider the effects of a modification to the statement.

Conception of Proof by Contradiction

- Both students exhibited a Pre-Action conception of proof by contradiction with their improper responses to question six that asked them to summarize the proof. In addition, both students exhibited a Pre-Action conception of proof by contradiction with their incomplete responses to question nine that relied on a step-by-step procedure to write a similar proof.
- Wesley exhibited an Action conception of proof by contradiction with his response to question seven that relied on her procedure to identify the key steps of the proof.
- Yara exhibited a Process conception of proof by contradiction with her response to question seven that described how the key steps of the procedure imply the statement is true.

Section 4.5 presented the results of teaching episode 5, which was designed to achieve two goals. First, the episode introduced a general proof by contradiction that could be considered proving the proof method (i.e., showing the proof method is valid) and thus encourage students to encapsulate their Process for proof by contradiction into an Object. Secondly, the episode introduces a new statement type that can be proved by contradiction - property claim statements (e.g., $\sqrt{2}$ is an irrational number). Students were encouraged to use their general procedure for proof by contradiction to prove a particular statement and thus assimilate or accommodate the new type of proof into their Schema for the method.

This section focused on how Wesley and Yara's understanding emerged from an analysis of their responses to the tasks and teacher/researcher prompts during teaching episode 5. A summary of the results from tasks during each phase follows.

Subsection 4.5.1 reported on students' work during Activity 5. This phase consisted of two tasks: outlining a proof of the method and defining proof by contradiction. A summary of Wesley and Yara's results for each task follows.

Wesley's outline focused on his previous procedure for proof by contradiction rather than represent the specific statements in the given proof. In addition, he included the step $\sim P = P$: a persistent, improper representation that a contradiction is always made with the initial statement. Attending to a memorized, step-by-step procedure with improper steps is indicative of a Pre-Action conception of proof by contradiction. Yara's outline focused on an abbreviated version of her previous procedure for proof by contradiction rather than represent the specific statements in the proof. However, after outlining the proof, she provided evidence of considering the presented proof as a validation of proof by contradiction, though it was not explicit enough to state she conceived the proof procedure as a whole. Therefore, Yara was developing toward an Object conception of proof by contradiction and could not yet describe how to validate the proof method.

Wesley did not provide a definition when asked and instead referred to the outline as a "perfect example." He also did not provide a definition when asked to clarify. Therefore, his reference to the outline was considered to be his entire definition, which is indicative of an Action conception of proof by contradiction. Yara's definition of proof by contradiction described the two key steps of a proof by contradiction. She then moved on before she could be prompted to explain how these steps logically imply the statement is true. Therefore, she exhibited at least an Action conception of proof by contradiction.

Subsection 4.5.2 reported on students' responses during Classroom Discussion 5. The tasks for this phase are based on developing and testing a general procedure for proof by contradiction. A summary of Wesley and Yara's results for each task follows.

Wesley and Yara were asked to compare their general procedure for proof by contradiction to the presented proof in Activity 5 to possibly modify the logical reasoning behind their general procedure for proof by contradiction. However, neither student modified their logical reasoning due to the outline.

When asked to write a proof for the statement "$\sqrt{2}$ is an irrational number, Wesley followed his general procedure for proof by contradiction to negate the statement. He expressed that he would have unpacked the definition of a rational number if he knew it, which illustrated that a lack of mathematical knowledge inhibited his proof writing. He was then prompted to utilize the rest of his procedure for proof by contradiction to outline the proof, to which he again suggested the contradiction was related to the statement, which is indicative of a Pre-Action conception of proof by contradiction. When asked to write a proof for the same statement, Yara recited a memorized portion of the proof (assuming not and the initial algebraic manipulations). She then could not arrive at a contradiction, which is indicative of an Action conception of proof by contradiction.

Subsection 4.5.3 reported on students' work during Exercise 5. The tasks for this phase were eight comprehension questions aligned with the proof comprehension assessment model by Mejía-Ramos et al. (2012) as well as one additional task that was not aligned with assessment model. Responses to these tasks revealed the following about:

Mathematical Knowledge

- Wesley did not possess the prerequisite knowledge of irrational numbers with his improper explanation of his response to question one while Yara exhibited the prerequisite knowledge of irrational numbers with her response.
- Both students possessed the prerequisite knowledge of greatest common divisors with their appropriate responses and explanations to question three.
- Both students possessed the prerequisite knowledge of even numbers with their appropriate responses and explanations to question four.

Conception of Mathematical Logic

- Wesley exhibited a Pre-Action conception of mathematical logic with his mathematical justification of question five that asked students to identify the logical status of a statement. In addition, Wesley exhibited a Pre-Action conception of mathematical logic with his improper and undesirable justification in question 6 that required students to justify an omitted detail in the proof.
- Yara exhibited an Action conception of mathematical logic with her memorized justification in question 6 that required students to justify an omitted detail in the proof.

Conception of Proof by Contradiction

- Yara exhibited a Pre-Action conception of proof by contradiction with her incomplete response to question six that asked students to summarize the proof. In addition, Wesley exhibited a Pre-Action conception of proof by contradiction with his incomplete response to question nine that asked students to write a similar proof.
- Wesley exhibited an Action conception of proof by contradiction with his response to question seven that relied on her procedure to identify the key steps of the proof.
- Yara exhibited a Process conception of proof by contradiction with her response to question seven that described how the proof arrived at a contradiction. In addition, Yara exhibited a Process conception of proof by contradiction with her complete description of a similar proof in question nine.

CHAPTER 5

DISCUSSION AND CONCLUSIONS

This study employed APOS Theory to investigate transition-to-proof students' understanding of proof by contradiction. The goal of this study was to explore students' understanding of the proof method based on a five-part teaching intervention, which was guided by an initial hypothesis of the necessary mental constructions a student should make in order to develop an understanding of proof by contradiction (referred to as a preliminary genetic decomposition). To do so, this study focused on (1) students' initial conceptualization of the proof method and (2) how students' developed an understanding of this particular proof method over time. This chapter will synthesize the results presented in the previous chapter to answer these two main research questions as well as discuss their implications.

First, Section 5.1 will present a synthesis of the results reported in Chapter 4 by responding to each of the research questions posed in Section 1.3. In addition, the results of this study are discussed in light of other findings that have been reported in the literature to situate this study in a broader body of research that reflects current advances in research of undergraduate mathematics education. Next, Section 5.2 will re-examine the researcher's conjectures, based on the results of the study, that provided the foundation for developing the teaching experiment. Then, Section 5.3 will consider the implications of this study for the curriculum and instruction of transition-to-proof courses. Subsequently, Section 5.4 will briefly discuss the limitations of this study. Finally, Section 5.5 will conclude the study with suggestions for future research on the topic of teaching and learning of proof comprehension and, specially, proof by contradiction.

5.1 Discussion of results

The results of this study provided answers to each of the research questions and gave insight into the mental constructions called for by the preliminary genetic decomposition. In this section, the results from Chapter 4 are discussed in light of each research question posed in Section 1.3 and with respect to the larger body of related literature pertaining to the concepts of proof construction, comprehension, validation, and evaluation.

5.1.1 Research question 1: Initial conceptions of proof by contradiction

Research question 1 asked how transition-to-proof students initially conceptualize proof by contradiction. By initial, I mean during the first two teaching interventions as this is a typical amount of instruction for introducing proof by contradiction (see page 63 for a description of how proof by contradiction is typically taught in this course). Results from teaching episodes 1 and 2 can be used to answer the following four sub-questions that address this larger research question:

1.1 What are students' initial conjectures about the proof method?

1.2 How does outlining the logical structure of a presented proof affect students' initial understanding of the proof method?

1.3 How do comprehension questions affect students' initial understanding of the proof method?

1.4 How does comparing logical outlines of proofs affect students' initial understanding of the proof method?

For each sub-question, I will provide relevant results from teaching episodes 1 and 2 that answer the question as well as relate any literature on the results. Then, I will use the answers to these sub-questions to answer the primary research question.

Sub-question 1.1: What are students' initial conjectures about the proof method?

Students' definitions during teaching episodes 1 and 2 could be categorized in one of four ways: (1) Generic indirect proofs; (2) Proofs that start with the negation of a statement; (3) Proofs that contradict the statement to be proved (i.e., to prove statement P, show $P \wedge \sim P$); and (4) a memorized, formal definition. A description of what each type of response indicates, in terms of students' understanding of proof by contradiction, follows.

The first category of response defined the method as a generic indirect proof (i.e., the proof shows a logically equivalent statement is true and thus the statement itself is true). While a proof by contradiction is an indirect proof, this type of response did not distinguish between a proof by contradiction and a proof by contraposition. Since this type of response was present in both teaching episodes (i.e., after considering two examples and having been provided the formal definition), it may be a prevalent, undesired initial definition of proof by contradiction.

The second category of response defined the method as a proof that starts with the negation of a statement. This type of response is consistent with authors such as Antonini and Mariotti (2008) who claim that students' conceptions of proof by contradiction and contraposition can be analyzed together as both methods begin with the negation of a statement. However, this definition does not attend to the second key step (and namesake) of the proof method: the contradiction. In addition, a proof that begins with the negation of a statement may or may not be valid. For example, consider the implication $P \to Q$. Proofs of the related implications $\sim Q \to \sim P$ and $\sim P \to \sim Q$ would both begin with the negation of a statement, yet the first is a valid proof method (proof by contraposition) while the second is not (known as *Denying the antecedent*). Therefore, while this definition may be prevalent, it is not desirable as it can include invalid proof methods. In addition, this definition focuses on the 'negative' aspect of the method, which some authors (e.g., Leron, 1985) have suggested cause cognitive difficulties for students.

The third category of response defined the method as a proof that contradicts itself.

That is, if S is the statement to be shown, then the proof will begin by assuming $\sim S$ is true, arrive at the statement S, and thus reach the contradiction $S \wedge \sim S$. This definition is valid in that it describes both of the key steps of a proof by contradiction: assuming the negation of the statement to be proved and arriving at a contradiction. However, this type of definition is not convincing in that to prove S is true (i.e., prove the statement), one must prove S is true (in order to reach a contradiction) - a form of circular reasoning. This type of definition may provide an alternative explanation for why some students do not find proof by contradiction convincing. That is, considering the procedure as a type of circular reasoning rather than difficulty working in a "false, impossible world" (Antonini & Mariotti, 2006, p. 2-65). Moreover, this alternative explanation bolsters Brown's (2016a) suggestion that proof comprehension may play a role in students' avoidance of the method.

The fourth category of response defined the method in a memorized, formal fashion. That is, during Classroom Discussion 1, the teacher/researcher provided a formal definition of proof by contradiction. Then, during Activity 2 when asked to provide a definition for proof by contradiction, some students responded with a near exact replica of the definition from Classroom Discussion 1. In addition, it relied on the general procedure of a proof by contradiction (assume the statement is false and arrive at a contradiction) without explaining why the method is valid. This is a desirable initial definition for the method as a memorized procedure (i.e., an Action conception of the method) has the potential to be interiorized into a general, flexible procedure for any type of statement (i.e., a Process conception of the method).

Overall, one of the four initial categories of definitions was desirable. This suggests that after the first two teaching episode, the majority of students did not develop an initial understanding of proof by contradiction beyond a Pre-Action conception and that those who did developed an Action conception of the proof method. In addition, this is evidence that proof by contradiction is a difficult proof method for transition-to-proof students to understand, as suggested by the literature (e.g., Antonini & Mariotti, 2008; Brown, 2017; Harel & Sowder, 1998).

Sub-question 1.2: How does outlining the logical structure of a presented proof affect students' initial understanding of the proof method?

Outlining tasks (i.e., tasks that asked students to logically outline a presented proof by contradiction) were included to prompt students to identify the logical argument within a presented proof by contradiction. According to the proof comprehension assessment model by Mejía-Ramos et al. (2012), both the logical status of statements and the modular structure of a proof play a role in students' proof comprehension. That is, examining the logical relation between one or more lines in a presented proof aids students in understanding the particular proof. Authors have substantiated this claim, such as Hodds et al. (2014) who developed a booklet containing self-explanation training focused on the logical relationships within a mathematical proof that improved students' proof comprehension. For the purposes of this study, it was conjectured that improving students' proof comprehension of particular proofs by contradiction would aid students' comprehension of the proof method in general. Results from teaching episodes 1 and 2 related to this conjecture follow.

During teaching episode 1, students provided one of two types of improper logical outlines for the proof: (1) attention to the algebraic manipulation over the logical argument of the proof and (2) an incomplete logical outline. The first type of outline, a focus on the algebraic manipulation over the logical argument of the proof, has been identified in the literature as a proof comprehension difficulty (e.g. Inglis & Alcock, 2012; A. Selden & Selden, 2003). The second type of outline, an incomplete logical outline, included a "logical leap" between the representations Q and $\sim Q$. This type of response indicated students did not consider the logical relation between these two steps. In particular for proof by contradiction, these responses indicated students did not recognize the underlying proof by contradiction procedure and how this procedure proved the statement. Later in the teaching episode (i.e., during the Classroom Discussion phase), the teacher/researcher prompted students to focus on the logical relation between each line and the previous line (or lines).

During teaching episode 2, nearly all students provided a desirable logical outline for the first part of the presented proof (assuming the statement is not true) and the last part of the presented proof (the assumption is false and therefore the statement is true). In this way, students implicitly partitioned the proof into three modules, each with a single purpose: (1) assuming the statement is false, (2) arriving at a contradiction, and (3) the assumption is false and thus the statement is true. This suggests that students could consider the underlying procedure, proof by contradiction, as the logical impetus of the proof.

Together, these results suggest that by prompting students to construct a logical outline of a presented proof, they can consider the proof procedures that motivate the proof. In addition, students implicitly partitioned the presented proof into three modules that each play a role in the procedure for proof by contradiction.

Sub-question 1.3: How do comprehension questions affect students' initial understanding of the proof method?

Comprehension tasks (i.e., questions that assessed students' understanding of a presented proof) focused on a presented proof or the student producing their own proof of a statement. These tasks included: (1) asking for the key steps of a presented proof, (2) stating the purpose of a statement related to their procedure, and (3) writing a complete proof by contradiction. In this way, comprehension tasks were conjectured to reinforce students' conceptions of proof by contradiction after they were formalized. Results from teaching episodes 1 and 2 related to this conjecture follow.

During teaching episode 1, all groups exhibited a Pre-Action conception of proof by contradiction with their responses to questions five (summarizing the proof) and eight (writing a similar proof). That is, given the formal definition of a proof by contradiction and a logical outline of a particular proof, students did not summarize the particular proof nor write a similar proof. Some groups did utilize the formal definition of proof by contradiction to consider the key steps of the presented proof.

During teaching episode 2, all groups again exhibited a Pre-Action conception of proof by contradiction with their responses to question five (summarize the proof). Unlike teaching episode 1, groups relied on a formal procedure for proof by contradiction to answer question three (justifying a line in the proof) question eight (writing a similar proof). In addition, all groups relied on a formal procedure for proof by contradiction to answer question four (identifying the logical status of a statement) and question six (identifying the key steps of the proof).

Together, these results suggest that after being presented the formal definition of proof by contradiction, students could subsequently use this definition to answer comprehension questions. This suggests that comprehension tasks can be used to reinforce a formal conception of proof by contradiction and therefore moving towards an Action conception of the proof method.

Sub-question 1.4: How does comparing logical outlines of proofs affect students' initial understanding of the proof method?

Comparison tasks (i.e., tasks that required students to compare two or more logical outlines of presented proofs) were used as a reflection tool for students to consider the necessary lines of a proof by contradiction and how these lines logically relate. The goal of these tasks was to encourage students to compare the purpose of lines in multiple outlines and develop a series of general steps that could be used as a proof technique whenever appropriate. Results from teaching episode 2 related to this conjecture follow.

The majority of groups compared the two logical outlines and generalized these into the following general steps:

1. Assume $\sim S$
2. $\vdots$
3. $\rightarrow\leftarrow$
4. $\sim(\sim S)$
5. S

This general procedure contained the key steps of a proof by contradiction: assuming the statement is not true (line 1) and arriving at a contradiction (line 3). It does not, however, describe the intermediate steps nor does it describe the reasoning for the proof method. Therefore, these students constructed a general procedure for proof by contradiction that relied on the key steps of a proof by contradiction, which is indicative of an Action conception of proof by contradiction.

Some groups compared the two logical outlines and, in addition to the steps above, described intermediate steps (see Figure 4.16 for a description of these steps). These groups were also able to verbally relate the two key steps of a proof by contradiction to describe how these steps imply the statement is true. For example, group G14 stated "You went back to your first statement and with the contradiction, you concluded that your first statement cannot be true." This explanation related the initial assumption ("first statement") with the contradiction to describe how these two steps related to prove the statement ("conclude that your first statement cannot be true"), which is indicative of a Process conception of proof by contradiction.

These results suggest that comparison tasks can aid students' initial understanding of proof by contradiction in one of two ways: by constructing a step-by-step procedure for *any* proof by contradiction (Action conception) and by constructing a general procedure for *any* proof by contradiction (Process conception). In addition, this task suggests that students can develop an understanding of the proof method without a need to rewrite the proof directly, as opposed to rewriting the statement to prove the secondary statement S^* as suggested by Antonini and Mariotti (2008).

Question 1: How do transition-to-proof students initially conceptualize proof by contradiction?

Results from this study suggest students initially conceptualize proof by contradiction in one of four ways: (1) as a generic indirect proof; (2) as a proof that starts with the negation of a statement; (3) as a proof that directly contradicts the statement to be proved; and (4) as a memorized list of steps to be performed. This suggests that two teaching interventions focusing on proof by contradiction are insufficient in developing a desirable conception for the majority of these transition-to-proof students.

Two types of tasks - Outlining and Comparison - were shown to aid students in developing a formal conception of proof by contradiction. In particular, the Comparison task in teaching episode 2 suggested that students can develop an understanding of proof by contradiction without a need to rewrite the proof directly, as suggested by Antonini and Mariotti (2008).

One type of task - Comprehension - was shown to reinforce students' conceptions of proof by contradiction. That is, students utilized their conception of the proof method and, if it was insufficient in completing the task, students did not provide any answer to the question. This suggests that Comprehension tasks, such as writing a proof by contradiction, may not aid students in *developing* a conception of the proof method and merely reinforce their current conception. Therefore, other tasks, such as Outlining and Comparison, should be used when introducing and teaching proof by contradiction and, once a formal conception of the proof method has been developed, should be reinforced with Comprehension tasks.

5.1.2 Research question 2: Developing an understanding of proof by contradiction over time

Research question 2 asked how transition-to-proof students develop an understanding of proof by contradiction over time. Results from the case study analysis of Wesley and Yara can be used to answer the following four sub-questions that address this larger research question:

2.1 How do students explain the underlying concept of a proof by contradiction as they develop an understanding of the method?

2.2 What types of tasks support transition-to-proof students' development of the proof method over time?

2.3 How do cognitive obstacles inhibit transition-to-proof students' understanding of the proof method over time?

For each sub-question, I will provide relevant results from Wesley and Yara's responses during teaching episodes 1 through 5 that answer the question as well as relate any literature on the results. Then, I will use the answers to these sub-questions to answer the primary research question.

Sub-question 2.1: What types of tasks support transition-to-proof students' development of the proof method over time?

As described previously, *Outlining tasks* asked students to logically outline a presented proof by contradiction. These tasks prompted students to construct new, more specialized procedures based on the statement structure, such as implication, nonexistence, and uniqueness. Each of these procedures contained the two key steps of a proof by contradiction as well as any steps specific to the type of statement (e.g., what the negation of a uniqueness statement looks like using predicate logic). These procedures could be used to possibly develop an Action and then Process conception of proof by contradiction for the specific type of statement. That is, as students constructed a new specific procedure, they enhanced their conception of proof by contradiction with a step-by-step procedure (Action) that could then be interiorized into a general procedure (Process) based on the structure of the statement and their previous knowledge. To illustrate this point, I will present two occasions Outlining tasks supported Wesley and Yara's development of proof by contradiction.

During teaching episode 3, Wesley utilized his Action conception of proof by contradiction to begin an outline of the presented proof of the statement "The equation $5x-4=1$ has a unique solution." In particular, he used his first step of a proof by contradiction (assume the statement is not true) and justified his second step of the outline by stating "because it's the *[pause]* contradiction[1] of there exists a unique" (see 142 for more details). That is, he justified the second step of the outline according to his general procedure for proof by contradiction, which is indicative of an Action conception of the proof method. After representing the contradiction of the presented proof, he again utilized his general step-bu-

[1]Wesley commonly switched 'negation' with contradiction.

step procedure to represent that the assumption statement was not true and therefore the statement was true. The intermediate steps (i.e., the steps that were not part of his general procedure for proof by contradiction) were considered to be a new kind of procedure for proof by contradiction that could be used to write similar proofs. This suggests he enhanced his understanding of the proof method by examining how to prove a specific type of statement by contradiction and incorporating this new step-by-step procedure into his Action conception of proof by contradiction.

During teaching episode 4, Yara utilized her Process conception of proof by contradiction to begin a proof outline for the statement "The set of natural numbers is not finite." In particular, she assumed the statement was not true, left space for the contradiction part of the proof, and then completed the outline by signifying the assumption is false and therefore the statement is true (see Figure 4.38 on page 184 for details). She then outlined how the proof arrived at a contradiction: rewriting the assumption, defining the finite set, and constructing a number that is both in and out of the set. These intermediate steps illustrated how "infinity" statements are proved by contradiction and thus Yara constructed a new step-by-step procedure for this specific type of statement based on her Process conception of proof by contradiction. This suggests she enhanced her understanding of the proof method by examining how to prove a specific type of statement by contradiction and incorporating this new procedure into her conception of proof by contradiction.

As described previously, *Comparison tasks* required students to compare two or more logical outlines of presented proofs. These comparison tasks prompted students to either assimilate or accommodate any new proof outlines for a specific type of statement in their general procedure for proof by contradiction. To illustrate how these tasks supported the development of students understanding over time, I will present how Wesley and Yara's procedures developed throughout the teaching experiment.

Wesley's procedures after each Comparison task are provided in Table 5.1 below.

Table 5.1 Wesley's procedure for proof by contradiction after each Comparison task (CD# refers to the Classroom Discussion the task was administered during).

Task 1 (CD2) **Statement:** S	**Task 2 (CD3)** **Statement:** S	**Task 3 (CD4)** **Statement:** S	**Task 4 (CD5)** **Statement:** S
1. Assume $\sim S$ 2. $\vdots$ 3. $\rightarrow\leftarrow$ 4. $\sim(\sim S)$ 5. S	1. Assume $\sim S$ 2. $\vdots$ 3. $\rightarrow\leftarrow$ 4. S	1. Assume $\sim S$ is true 2. Attempt to prove $\sim S$ is true 3. $\rightarrow\leftarrow$ of $S = \sim S$ 4. $\sim(\sim S)$ 5. S	1. Assume $\sim S$ is true 2. Attempt to prove $\sim S$ is true 3. $\rightarrow\leftarrow$ of $S = \sim S$ 4. $\sim(\sim S)$ 5. S

Procedures from tasks 1 and 2 were agreed upon by multiple groups, including Wesley's group: G5. These procedures attended to the key steps of a proof by contradiction and did not include any details on possible intermediate steps. However, task 3 allowed for Wesley to describe his personal procedure for proof by contradiction and include details on intermediate steps. For example, step 2 - "Attempt to prove $\sim S$ is true" - represented rewriting the assumption and using this assumption to arrive at a contradiction, though he described this part of the procedure as a form of circular logic (assuming a statement is true and then proving that statement is true). While this form of logic was not evident in his previous procedures, it was expressed in conversations with his groupmates, such as the following conversation from teaching episode 1.

Wesley: After you do this, you are saying for this to be true *[$\sim Q$]*, this has to be true *[$P \wedge \sim Q$]* and that has to be true for the statement *[$\sim (P \rightarrow Q)$]* to be true. I get it. So how am I going to prove $\sim Q$?

[Teacher explains that $\sim Q$ is a logical consequent of assuming $\sim (P \rightarrow Q) \cong P \wedge \sim Q$ is true.]

Wesley: Right, and I understand that, but I guess what I'm confused is... I'm saying that that... *[pause]* that I'm a little lost. At this point in my proof, don't I have to prove this is true, I have to prove $\sim Q$ is true?

Wesley's first statement explains that for $\sim Q$ to be true, $\sim (P \rightarrow Q)$ must be true. This suggests he understood the logical relations between $\sim (P \rightarrow Q)$ and $\sim Q$. However, he then stated "So how am I going to prove $\sim Q$?" That is, while he started the proof by assuming $\sim (P \rightarrow Q)$ was true, he still expressed the need to prove a logical consequent of this assumption ($\sim Q$) and, by extension of his first statement, therefore prove his assumption is true. This suggests the intermediate step "Attempt to prove $\sim S$ is true" was part of Wesley's personal procedure for proof by contradiction throughout all five teaching episodes and he did not feel the need to change this step. In addition, he did not feel the need to change step three in tasks 3 and 4 ($\rightarrow\leftarrow$ of $S = \sim S$) even when provided proofs in which he acknowledged the contradiction did not directly relate to the original statement (see pages 190 and 216 for a detailed discussion of this step). Overall, his procedure for proof by contradiction stayed largely the same after each Comparison task and therefore these tasks seemingly did not affect[2] his conception of proof by contradiction.

Yara's procedures for proof by contradiction after each Comparison task are provided in Table 5.2 below.

Table 5.2 Yara's procedure for proof by contradiction after each Comparison task (CD# refers to the Classroom Discussion the task was administered during).

Task 1 (CD2) **Statement:** S	**Task 2 (CD3)** **Statement:** P	**Task 3 (CD4)** **Statement:** S	**Task 4 (CD5)** **Statement:** P
1. Assume $\sim S$ 2. Rewrite $\sim S$ 3. Look at specific value of step 2. 4. Work (Algebra). 5. Get Contradiction. 6. $\sim$ Assumption. 7. S	1. Assume $\sim P$ 2. Negate P (Rewrite $\sim P$) 3. Use math skills to get to a contradiction. 4. $\sim$ Assumption 5. P	1. Assume $\sim S$ 2. Do work 3. $\rightarrow\leftarrow$ 4. S	1. Assume $\sim P$ 2. $\rightarrow\leftarrow$ 3. P

[2]Conjectures on why these tasks did not affect Wesley's conception will be discussed with sub-question 2.3 on cognitive obstacles.

Unlike Wesley's static procedures, Yara's became more refined over time. For example, consider her procedures after tasks 1 and 2. Steps 3, 4, and 5 after task 1 - Look at specific value in step 2; Work (Algebra); and Get Contradiction - were synthesized and written as a single step after task 2: Use math skills to get to a contradiction. Indeed, she attributed the need for a more general phrase ("math skills") to the comparison of a new specific procedure for proof by contradiction as illustrated below.

Yara: So I guess it just, maybe it likes, depends on the proof, and what you are trying to prove? Whether you do algebra or... umm... *[pause]*

Teacher: So what do we do in that one *[outline during Activity 2]*?

Yara: In this one, it says to use $P(x)$, get a contradiction. So we did algebra, right? So this one you do... which math skills do you use? Because math skills could mean plenty of things. It could be, like, one of them induction whatever...

The phrase "depends on the proof, and what you are trying to prove? Whether you do algebra or..." suggests that Yara considered the specific steps used to arrive at a contradiction as different and thus needed to generalize these steps for her own procedure. She then decided that the phrase 'math skills' included algebra as well as possibly other proof methods, such as induction. This is noteworthy as the presented proof did not include another proof method as a sub-proof, yet Yara generalized the step to include utilizing other proof methods. That is, she made a generalized step that included possible other steps that were not explicitly present in the proof she was considering. Therefore, the Comparison tasks were effective in prompting Yara to reflect on specific procedures and generalize these steps to include possibly new steps.

As described previously, *Comprehension tasks* called for students to use their general procedure for proof by contradiction to answer some comprehension question. These tasks included: (1) asking for the key steps of a presented proof, (2) stating the purpose of a statement related to their procedure, and (3) writing a complete proof by contradiction. The goal of Comprehension tasks was to reinforce[3] students' conception of proof by contradiction rather than to modify it (which was the goal of Outlining and Comparison tasks). To illustrate this, I will present two tasks (one for each student) that focused on reinforcing the procedures developed in the Outlining tasks previously described.

As mentioned previously when discussing how Wesley responded to the Outlining task during teaching episode 3, he developed a specific procedure for proving uniqueness statements by contradiction (see Figure 4.23 on page 142 for more details). During Classroom Discussion 3, he was asked to write a proof outline[4] for the statement "The multiplicative inverse of a non-zero real number x is unique" and thus reinforce the specific procedure he constructed during Activity 3. I will focus on one line in this specific procedure - P(1) - to illustrate how a formal, general representation of this line was reinforced through writing a proof outline.

The line $P(1)$ from the specific procedure during Activity 3 (see Figure 4.23 on page 142) referred to representation $P(x)$: x is a solution to the equation $5x-4=1$. The purpose of this step was to represent that there were not *no* solutions to the equation (i.e., negating half of the assumption that there was not exactly one solution). To write a similar proof for the statement "The multiplicative inverse of a non-zero real number x is unique", the same line would be represented as $P(x, \frac{1}{x})$ to signify that at least one multiplicative of x, namely $\frac{1}{x}$ exists. However, Wesley instead responded "So then the first one *[pause]* the first one is there exists, there is not a y. So you can just put $P(1,1)$? Right?". This comment suggests he noted that step 3 previously inserted 1 into the representation and therefore initially wrote line 3 of his new outline as $P(1,1)$. This new proof provided Wesley an opportunity to reflect on what a general step 3 would look like - showing that the statement is true for at least one value. In this way, the Reinforcement task prompted Wesley to generalize a specific step of his specialized procedure for uniqueness statements that he initially constructed during the

[3]Students who did not develop any conception (i.e. at the Pre-Action level) would not be able to complete the reinforcement tasks, as illustrated with the numerous blank responses to Comprehension tasks during Exercise 1.

[4]The key difference here is that students were writing a proof - whether in complete sentences or in a series of steps. This is different than Outlining tasks as the student were writing their own proof.

Outlining task.

As mentioned previously when discussing how Yara responded to the Outlining task during teaching episode 4, she developed a specific procedure for proving uniqueness statements by contradiction (see Figure 4.38 on page 184 for more details). During Classroom Discussion 4, she was asked to write a proof for the statement "There are infinitely many primes" and thus reinforce the specific procedure she constructed during Activity 4.

Yara started the proof by assuming the negation of the statement was true and thus that there are finitely many primes. She then utilized her specific procedure to write the set of all primes as $P = \{p_1, p_2, \ldots, p_k\}$ where $p_1 < p_2 < \ldots < p_k$. She then stated "Now this is the part that's kind of tricky where it's not exactly the same. Because with a natural number, we can just add 1 and be fine but with prime numbers, just adding 1 doesn't mean that the next number is going to be prime." This response illustrates she compared the context of the previous construction ("we can just add 1 and be fine but with prime numbers") to the context of the new construction ("just adding 1 doesn't mean that the next number is going to be prime") and recognized the exact same construction would not work. This reinforced the idea that while the construction of the procedure may not be the same, the next step in the proof is to provide some construction that would be both in and out of the set in question. Therefore, the proof provided her the opportunity to reflect on the general purpose of the construction step and thus reinforce her specialized procedure for proving uniqueness statements by contradiction.

In summary, the Outlining tasks were effective (for both students) in developing specific step-by-step procedures for different types of statements that can be proved by contradiction. In contrast, the Comparison tasks were not effective for modifying Wesley's conception of proof by contradiction and yet were effective for Yara. Therefore, both Outlining and Comparison tasks may affect how students develop an understanding of proof by contradiction over time. Finally, the Comprehension tasks were effective in reinforcing previously constructed procedures for proof by contradiction.

Sub-question 2.2: How do cognitive obstacles inhibit transition-to-proof students' understanding of proof by contradiction over time?

Two prominent cognitive obstacles emerged from the case study analysis of Wesley and Yara's responses to tasks during the teaching experiment: the contradiction and considering valid (but not sound) arguments. For each of these cognitive obstacles, I will first describe how it relates to students' conception of proof by contradiction. Then, I will illustrate how it inhibited a student's conception of the proof method.

The first cognitive obstacle, the contradiction, has been identified as one of the main cognitive difficulties of the proof method in that students exhibit difficulty identifying the contradiction of a proof, especially when it does not directly relate to the primary statement (Antonini & Mariotti, 2009; Barnard & Tall, 1997). However, another difficulty emerged from an analysis of Wesley's conception of proof by contradiction: the contradiction as a direct negation of the statement to be proved. That is, if S is the statement to be proved, then the contradiction would be $S \wedge \sim S$. To be clear, it is sometimes the case that a proof by contradiction will arrive at a contradiction that is directly related to the statement to be proved, such as when proving uniqueness claims. The belief that the contradiction will *always* be related to the statement, however, can appear to students as a form of circular reasoning and inhibit their conception of proof by contradiction as a valid proof method.

As noted previously when describing Wesley's responses to the Comparison tasks, he did not modify his general procedure for proof by contradiction throughout the entire teaching experiment. In addition, he remained steadfast in describing the contradiction in the procedure as directly related to the statement to be proved. For example, in teaching episode 4 he asked:

> So... and I don't see S equals not S, so I understand that. How many times, or I shouldn't say how many times... how *[pause]* often is that kind of, are you going to come to this kind of conclusion where you are not coming to a... you get your contradiction but it's not really just an S doesn't equal not S? (page 190)

The phrase "S doesn't equal not S" was an improper representation of a contradiction directly involving the statement to be proved. After this comment, the teacher/researcher provided Wesley with multiple example proofs where the contradiction was not directly related to the statement to be proved. However, during teaching episode 5 he directly contradicted this statement, as illustrated below.

Teacher: So, in this proof here, we didn't *[pause]* get a direct opposite of the statement we were trying to prove. It was kind of like, a little piece, a little side thing of that. So if we are trying to make this even more general, we get rid of not P equals P and we just say 'some contradiction'.

Wesley: So *[long pause]* this is the only one we've done like this where you are using an approach to the original statement that's kind of off to the side. (page 216)

The phrase "approach to the original statement that's kind of off to the side" referred to the contradiction $\gcd(p, q) = 1$ and $\gcd(p, q) \neq 1$ as not relating directly to the statement "$\sqrt{2}$ is irrational." Thus, this comment directly contradicts his previous admission (during teaching episode 4) that not all contradictions are directly related to the statement being proved. Together, these comments suggest that, in order to alleviate his cognitive dissonance, he ignored any examples of proofs that did not align with his conception that the contradict relate directly to the statement proved. Therefore, this cognitive difficulty inhibited the development of his conception of proof by contradiction.

The second cognitive obstacle, valid (and not sound) arguments, relates to a student's mathematical logic Schema. An argument is *valid* "If, and only if, whenever statements are substituted that make all the premises true, the conclusion is also true." (Epp, 2004, p. 29). In other words, the truth of the conclusion necessarily follows from the truth of the premises, but the premises need not be true. An argument is *sound* if it is valid and the premises are true. Proof by contradiction is one of two[5] prominent proof methods that rely on valid (and not sound) proof arguments. That is, a student must use a statement that is either false

[5]The other being proof by mathematical induction.

or of indeterminate truth-value in conjunction with their valid logical congruences and valid proof methods to prove some statement. Antonini and Mariotti (2006) suggested that when students use either a false or indeterminate truth-valued statement, they no longer know what theorems, congruences, and proofs methods are available to be used. Yet, this would not happen if students conceptualized their theorems, congruences, and proof methods as *valid* and not sound, as determining the truth value of the premises is not necessary in a valid argument. Instead, this suggests students consider their theorems, congruences, and proof methods as sound arguments that must then be questioned when dealing with non-true premises. To illustrate this cognitive difficulty, I will again present responses from Wesley that suggest he attended to proof by contradiction as a sound argument.

As noted previously when describing Wesley's responses to the Comparison tasks, he included the following two steps: (1) Assume $\sim S$ is true and (2) Attempt to prove $\sim S$ is true. This was explained as a form of circular reasoning as it "attempts to prove" a statement that was already assumed true. This need to prove an assumption may also be indicative of viewing proof by contradiction as a sound argument. Indeed, this would explain the motivation to prove a statement true that was already assumed to be true, as a sound argument requires the premises to be true. This study did not find explicit evidence of this cognitive difficulty though as no tasks were developed with it in mind. Therefore, this cognitive difficulty remains a theoretical explanation of difficulties exhibited by Wesley.

Sub-question 2.3: How do students explain the underlying concept of a proof by contradiction as they develop an understanding of the method?

To examine how students explained the underlying concept of proof by contradiction, students completed a Definition task (i.e., provided a definition and explanation of proof by contradiction) during the Activity and Classroom Discussion phase of each teaching episode. For both Wesley and Yara, I will present their responses to each task and then summarize what these responses meant in terms of how their conception of proof by contradiction developed throughout the teaching experiment.

Wesley's responses to the definition tasks are provided in Table 5.3.

Table 5.3 Wesley's definitions during teaching episodes 1 through 5.

Teaching Episode 1:	Prove statement is true by assuming the negation of the statement is true and proving the assumption is false w/ a contradiction *[Written during the Classroom Discussion]*
Teaching Episode 2:	You assume the contradiction is true then disprove the contradiction $\rightarrow$ the original statement must be true.
Teaching Episode 3:	Have your statement and you assume the opposite and then prove it false.
Teaching Episode 4:	To prove a statement by assuming it's contradiction is false.
Teaching Episode 5:	Well that's *[Outline of presented proof during Activity 5 (see page 215)]* like, perfect example right there, isn't it?

His first definition was copied after the teacher/researcher provided a formalized definition for all groups. In other words, he did not provide his own definition to the task during Activity 1 and thus did not exhibit a conception of the proof method before a formal step-by-step procedure was introduced for the class.

His second definition referred to 'negation' as 'contradiction' and thus made it difficult to definitely determine what he meant in the definition and, in particular, what the phrase "disprove the contradiction" meant. However, combined with the step from his general procedure "Attempt to prove $\sim S$ is true," the phrase "disprove the contradiction" likely referred to disproving the assumption by arriving at a contradiction. Using this interpretation, he explained proof by contradiction as (1) assuming the negation of the statement is true, (2) attempting to prove the negation of the statement is true, (3) arriving at a contradiction, (4) since one disproved the negation of the statement is true, the original statement is true. In particular, step 4 provides the explanation of how the previous steps relate to prove the statement - disproving the negation of a statement implies proving the statement itself.

His third and fourth definitions are a subset of his second definition. That is, they follow the same reasoning and do not describe all steps in the procedure. In addition, his fifth definition only referenced his list of steps in his general procedure and did not verbalize his reasoning behind the procedure.

Therefore, Wesley's description of proof by contradiction remained relatively stable throughout the teaching experiment as a proof that followed the following steps: (1) assuming the negation of the statement is true, (2) attempting to prove the negation of the statement is true, (3) arriving at a contradiction, and (4) since one disproved the negation of the statement is true, the original statement is true.

Yara's responses to the definition tasks are provided in Table 5.3.

Table 5.4 Yara's definitions during teaching episodes 1 through 5.

Teaching Episode 1:	Assuming something is not true, proving that it is.
Teaching Episode 2:	Because like we assumed *[long pause]* in our assumption, we assumed that our statement was true. So that negated that and then like, as you are doing the proof, we got to the contradiction, proving that our assumption was wrong and so our statement was true. *[Needed to refer to the example to describe a general proof by contradiction]*
Teaching Episode 3:	Because you assume the statement wasn't true and then it would be *[pause]* you reached a contradiction.
Teaching Episode 4:	Assuming the negation of the statement and getting a contradiction.
Teaching Episode 5:	It's when you assume the statement isn't... the original... assuming the negation of the statement and getting a contradiction.

Her first definition referred to one of the two key steps of a proof by contradiction and, in particular, aligns with a definition that begins by negating "something" and does not describe arriving at a contradiction.

Her second definition relied on scanning the example outline in order to describe the steps necessary to prove a statement by contradiction. That is, she describe the proof method as a series of steps (assuming the negation of the statement, getting a contradiction, and thus the assumption is false and the statement is true). In particular, she described how starting with an assumption and arriving at a contradiction proved the assumption was false.

Her third, fourth, and fifth definitions described both key steps of a proof by contradiction and did not describe how these steps related to prove the statement was true. However, descriptions of general procedures for proof by contradiction confirmed that she could, when prompted, provide an explanation (similar to her second definition) of how these key steps proved a statement was true.

Overall, both students could describe their understanding of proof by contradiction by teaching episode 2. For Wesley, this stayed his conception and, other than adding specific procedures based on the structures of statements, did not change. For Yara, her explanation did not exhibit the changes in her conception of proof by contradiction as much as her procedures for the proof method did. In other words, examining students' general procedures for the proof method exhibited their conception of proof by contradiction more than examining their definitions for the proof method.

Question 2: How do students develop an understanding of proof by contradiction over time?

Results from this study suggest two ways in which students may develop an understanding of proof by contradiction over time. First, Wesley's results suggest a student may develop an initial[6] Action conception for proof by contradiction as the following step-by-step procedure: (1) assuming the negation of the statement is true, (2) attempting to prove the negation of the statement is true, (3) arriving at a contradiction, and (4) since one disproved the negation of the statement is true, the original statement is true. The student would not see a need to change this procedure, even when presented proofs that do not follow this step-by-step procedure. In contrast, Yara's results suggest a student may instead develop (from an Action) an initial Process conception for proof by contradiction as the following general procedure: (1) assume the statement is not true and rewrite this statement, (2) look at specific values and do algebra to get a contradiction, and (3) thus the assumption is not true and therefore the statement is true. The student would then refine these general steps as they incorporate new specific procedures into their proof by contradiction Schema.

Yara's results suggest two tasks - Outlining and Comparison - may aid students in developing an understanding of proof by contradiction over time. Outlining tasks may aid

[6] 'Initial' is taken to mean within the first two teaching interventions.

students in developing new step-by-step procedures for specific types of statement (e.g., implication, nonexistence, and uniqueness). These procedures can then be called on and used when necessary to write proofs by contradiction as well as compared to develop a general procedure for any proof by contradiction. The comparison of specific procedures, Comparison tasks, also allowed students to assimilate or accommodate new step-by-step procedures for specific types of statement into their conception of proof by contradiction. In particular for Yara, each Comparison task resulted in a refinement of her general procedure for proof by contradiction as she synthesized multiple lines from her previous procedure to assimilate the new step-by-step procedure into her conception of proof by contradiction. Therefore, the Outlining and Comparison tasks may aid students in developing an understanding of proof by contradiction over time.

Wesley's results suggest two cognitive difficulties - Contradiction and Valid arguments - may inhibit students in developing an understanding of proof by contradiction over time. While identifying the contradiction of a proof has previously been recognized in the literature as a cognitive difficulty (Antonini & Mariotti, 2009; Barnard & Tall, 1997), considering the contradiction as directly related to the statement proved has not be recognized. The belief that the contradiction will *always* be related to the statement can appear to students as a form of circular reasoning and thus would prevent students from interiorizing the procedure. Therefore, considering the contradiction as directly related to the statement is a possible cognitive obstacle to comprehending proof by contradiction. For the second cognitive obstacle, valid (and not sound arguments), Antonini and Mariotti (2006) suggested that when students use either a false or indeterminate truth-valued statement, they no longer know what theorems, congruences, and proof method are available to be used. Yet, this would not happen if students conceptualized their theorems, congruences, and roof methods as *valid* and not sound, as determining the truth value of the premises is not necessary in a valid argument. Instead, this suggests students consider their theorems, congruences, and proof methods as sound arguments that must then be questioned when dealing with non-true premises, as is necessarily the case for any proof by contradiction. Furthermore, Weber and

Alcock (2005) reported that, when dealing with implications, students tend to focus on the truth value of the premise and conclusion over whether the implication was warranted. This report corroborates the suggestion that students' focus on the truth value of statements in a proof to the determent of the logical argument of a proof. Therefore, the consideration of theorems, congruences, and proof methods as necessarily sound arguments is a possible cognitive obstacle to comprehending proof by contradiction.

5.2 Re-examination of initial conjectures

This section focuses on re-examining the initial conjectures this study made about students' understanding of proof by contradiction based on the results outlined in Section 5.1. In particular, this section will focus on the preliminary triad of Schema development and the preliminary genetic decomposition. Then, I will describe how changes to these underlying conjectures would change the tasks and general structure of the teaching episode.

5.2.1 Triad of Schema development

As described in Subsection 3.1.2, the preliminary triad of Schema development for proof by contradiction focuses on the relationship between the mental procedures (either external or internal) constructed by students. However, the only tasks that asked students to relate their mental procedures were the Comparison tasks. In these tasks, students were prompted to relate two or more procedures to construct a list of general steps to prove any statement by contradiction. Instead, students seemed to relate the first two procedures to construct a list of general steps for a proof by contradiction (as desired) after which students compared each new procedure to this general procedure rather than with the others. In particular, the specific procedures for uniqueness, infinity, and property claim statements were only compared with the general procedure and were not compared to one another. This meant the tasks did not provide evidence to determine how students related the specific mental procedures between one another.

Moreover, the structure of the teaching experiment did not prompt students to relate the procedures. Each teaching episode was designed to focus on exactly one procedure. In this way, students were not required to consider which procedure could be used to write a proof for the given statement or if more than one procedure could be used. It would thus be useful to have tasks outside of these teaching episodes that asked students to choose which specific procedure they would use to write a proof and explain their choice.

These deficiencies in the tasks and design of the teaching episodes resulted in an incomplete examination of students' proof by contradiction Schema development in relation to the triad. To be clear - this does not mean students did not relate these mental constructions and thus were at an intra-contradiction stages of Schema development. Instead, the tasks (as written) did not provide sufficient evidence to claim students exhibited an intra-, inter-, or trans-contradiction stage of Schema development.

5.2.2 Genetic decomposition

As described in Subsection 3.1.3, the preliminary genetic decomposition for proof by contradiction focuses on the mental constructions students should make in order to develop an understanding of the concept. While the results of this study support the constructions called for by the preliminary genetic decomposition, they also suggest that some steps need to be refined and that new steps need to be added. This section presents a revision of the genetic decomposition for proof by contradiction in a transition-to-proof course as well as the reasoning behind these revisions.

The results of this study suggest additional prerequisite knowledge students should possess in order to begin developing an understanding of proof by contradiction: valid (and not sound) arguments. This knowledge is necessary to understand the validity of the proof method and, without it, may result in a cognitive rejection of the method. Moreover, proof by contradiction is one of the few proof methods that require valid and not sound arguments.

The steps that are new or revised in the following genetic decomposition are indicated in bold:

1. Action conception of propositional or predicate logic statements as specific step-by-step

instructions to construct proofs by contradiction for the following types of statements: (I) implication, **(II) non-existence, and (III) uniqueness;**

2. Interiorization of each Action in Step 1 individually as general steps to writing a proof by contradiction for statements of the form (I), (II), and (III).

3. Coordination of the Processes from Step 2 as general steps to writing a proof by contradiction.

4. **Assimilate or accommodate new proof by contradiction procedures (e.g., for infinity and property claim statements) into one's Process conception from Step 3 to enhance the general steps to writing a proof by contradiction.**

5. Encapsulate the Process in **Step 4** as an Object by utilizing the law of excluded middle to show proof by contradiction is a valid proof method. Alternatively, encapsulate the Process in Step 4 as an Object by comparing the contradiction proof method to other proof methods.

6. De-encapsulate the Object in **Step 5** into a Process similar to **Step 4** that then coordinates with a Process conception of **other proof methods to prove statements that require two or more proof methods**.

In Step 1, the types of statements to develop step-by-step instructions for were reduced to only implication, non-existence, and uniqueness. This was changed as students in this study were able to construct a general procedure for any type of statement after constructing and comparing these three specific procedures.

A new step was added between constructing a general procedure for any type of proof by contradiction and encapsulating these general steps into a static Object that can be acted on. This step described how students assimilated or accommodated the specific procedures for infinity and property claim statements into their existing general procedures for proof by contradiction (for an example, see how Yara assimilated the specific procedure for infinity statements into her existing Schema of proof by contradiction starting on page 192).

Finally, Step 5 was changed to describe how students could prove statements that required two or more proof methods. This change more accurately described how students attempted to write a proof for the statement "If a and b are real numbers and $a \neq 0$, then there is a unique real number r such that $ar + b = 0$." (for an example, see Yara's attempt to coordinate two proof procedures on page 171).

5.2.3 Teaching episodes

As described in Section 3.3, all teaching episodes were based on the preliminary genetic decomposition and developed to mimic the ACE teaching cycle - an instructional approach that consists of three phases: Activities, Classroom discussion, and Exercises. In the *Activities* phase, students worked in groups to complete tasks (e.g., Outlining presented proofs and defining proof by contradiction) designed to promote reflective abstraction. These tasks assisted students in making the specific procedures based on statement types suggested by the genetic decomposition. In the *Classroom discussion* phase, the teacher/researcher led a discussion about the mathematical concepts and tasks the Activities focused on. During this phase, students were provided formal responses to the tasks during the Activity as well as encouraged to compare specific procedures in order to develop a general procedure for proof by contradiction. In the *Exercises* phase, students work on standard problems (i.e., comprehension questions) designed to reinforce the Classroom discussion and support the continued development of the mental constructions suggested by the genetic decomposition.

Based on the refinements of the preliminary genetic decomposition and general results of the study, some tasks in the teaching episodes should be modified. For each teaching episode, I will describe modifications to the tasks and why they are necessary. A revised version of all tasks in the teaching experiment can be found in Appendix A.2.
For *teaching episode 1*, the following modifications should be made:

- Question 2 in Exercise 1 should be rephrased. As is, students were not clear on exactly what the question was assessing. Instead, it can be rewritten as: The statement "Every even natural number greater than 2 is the sum of two primes" is known as the Goldbach

Conjecture and has not been shown to be true. Yet, it is used in the proof as if it is true. How can this be?

- Question 5 in Exercise 1 had few responses and none were valid. Therefore, this question should be posed as a multiple choice question with two valid options: one that attends to the specific procedure of the proof and one that attends to the main idea of the proof. Students can then choose one of these two responses and explain why the response is valid (The proposed multiple choice options are included in Appendix A.2).
- Question 8 in Exercise 1 had no responses. Therefore, this question should be changed to as for a proof outline utilizing their procedure from the Classroom Discussion.

For *teaching episode 2*, the following modifications should be made:

- In Activity 2, the comparison task should be more explicit. Specifically, the tasks should ask students to compare the purpose of two particular statements, one in each proof. For example, a task should state: Consider the statement "Then there exists an odd natural number greater than 5 that is not the sum of three primes, call it k." from Proof 1 and the statement "Then there is an odd integer that can be expressed in the form $4j - 1$ and in the form $4k + 1$ for integers j and k." from Proof 2. What is the purpose of this line in each proof?
- During Classroom Discussion 2, students should be encouraged to make a side-by-side comparison of lines or pairs of lines from the first two procedures for implication and non-existence statements. In particular, students should be encouraged to describe the purpose of these lines in the entire argument of the proof.
- Question 2 in Exercise 2 should include the statement "What kinds of numbers can be expressed in the form $4k + 1$?" to prompt students to compare numbers in these two forms.
- Similar to Question 5 in Exercise 1, Question 5 in Exercise 2 should be posed as a multiple choice question with two valid responses.

For *teaching episode 3*, the following modifications should be made:

- Question 1 in Classroom Discussion 3 should be rephrased. Each of the formal procedures should be presented for students to compare without having to review their responses to previous teaching episodes.
- Question 1 in Exercise 3 should be removed as responses to it were not meaningful.

For *teaching episode 4*, the following modifications should be made:

- Question 1 in Classroom Discussion 4 should be rephrased. Each of the formal procedures should be presented for students to compare without having to review their responses to previous teaching episodes.
- Question 1 in Exercise 4 should be removed as responses to it were not meaningful.

For *teaching episode 5*, the following modifications should be made:

- Question 1 in Classroom Discussion 4 should be rephrased. It should ask students to first recall their list of steps and then to compare these steps with the outline in Activity 5.
- Question 1 in Exercise 5 should be removed as responses to it were not meaningful.

5.3 Implications for curriculum and instruction

This study has four main implications for the curriculum and instruction of transition-to-proof courses. First, this study purposed and refined a new model for how transition-to-proof students may develop an understanding of proof by contradiction as well as a series of lessons based on this understanding. This model is a promising alternative for transition-to-proof students' conceptions of proof by contradiction to the model by Lin et al. (2003) as it addresses numerous types of proofs by contradiction and describes the conceptual knowledge of the proof method as the *law of excluded middle*. In addition, this model is a promising alternative for transition-to-proof students' conceptions of proof by contradiction to the model by Antonini and Mariotti (2008) as it develops students' understanding of indirect proofs without translating these into direct arguments as well as provides a series of lessons based on how students develop an understanding of the proof method.

Secondly, this study bolsters the results by Hodds et al. (2014) that a focus on developing students' mathematical logic improves their proof comprehension. In particular for proof by contradiction, mathematical logic can improve their understanding of the concept through validation of the method as well as alleviating the two cognitive difficulties that emerged from an analysis of how Wesley's understanding of the method developed (that the contradiction must be made with the statement and the difference between valid and sound arguments).

Thirdly, this study tested and validated non-proof writing tasks that aided students in developing an understanding of proof by contradiction. These tasks, such as outlining the logical argument of a particular type of proof by contradiction (Outlining tasks) and comparing these outlines in order to develop general steps for the proof method (Comparison tasks), differed from the traditional "definition-theorem-proof" format of transition-to-proof courses where the only tasks are proving theorems (Weber, 2004). These tasks join the tasks based on proof reading strategies by Samkoff and Weber (2015) and the self-explanation training tasks by Hodds et al. (2014) as some of the first tasks specifically designed to improve students' proof comprehension. In addition, these are the first tasks designed to improve students comprehension of a particular proof method: proof by contradiction.

Finally, while this study utilized a specific instructional approach (the ACE teaching cycle) aligned with a particular theoretical framework (APOS Theory), these tasks are not dependent on the instructional approach and theoretical framework. That is, an instructor may utilize the Outlining and Comparison tasks without using the same instructional approach. For example, an instructor with a longer class period (e.g., 100 minutes) may ask students to complete multiple Outlining tasks that they could then compare together, with comprehension questions on a new proof assigned as homework. The point of this example is to illustrate that the use of these tasks is flexible and need not rely on a particular instructional approach or theoretical framework[7].

5.4 Limitations of the study

Four limitations, along with how the researcher attempted to address these limitations, follow. First, mental processes such as understanding are by their very nature impossible to observe directly. This study relied on students' observable actions in order to determine their understanding of particular proofs and proof by contradiction.

Secondly, the number of participants in this study was limited to twenty-seven, all of which came from two sections of a transition-to-proof course during Fall 2016 at a large, public university. The researcher acknowledges that results may be different under a different research setting and makes no claim that the results necessarily generalize to all transition-to-proof students. That is, the results suggest how some transition-to-proof students initially conceptualize proof by contradiction.

Thirdly, a limited number of students completed all five teaching episodes. Similar to the second limitation, the small retention rate suggests that not all types of conception were examined and thus this study only posits how some students may develop an understanding of proof by contradiction over time.

Finally, the teacher/researcher had no control of the curriculum and instruction of the transition-to-proof course beyond the teaching episodes completed in class. As the teaching experiment was conducted over multiple weeks, it is possible students' understanding was developed based on instructional tasks and not the tasks in the teaching episodes. Any instances of instructional tasks affecting responses to tasks for the study (such as Yara's memorized response to a proof that $\sqrt{2}$ is an irrational number) were noted.

[7] These assignments do rely on a Constructivist framework, though the particular framework need not be APOS Theory.

5.5 Future research

While this study laid the foundation for examining how students' understanding of a particular proof method may develop over time, it still left many questions unanswered. Three future directions for expanding on this study follow.

While this study suggested that developing students' Schema of mathematical logic may aid students in their understanding of proof by contradiction, the extent of which was not addressed. In particular, more research needs to be done on the effects of students' understanding valid (and not sound) arguments has on proof by contradiction.

Due to the types of tasks and structures of the teaching episodes, students' proof by contradiction Schema development, via the triad of Schema development, was not addressed completely. To examine their Schema development, tasks should be developed that ask students to consider a statement and provide an outline of the procedure they would use to prove the statement. This would allow the researcher to examine the relation of these procedures to one another. However, this type of comparison requires students to have already constructed multiple procedures for proof by contradiction and thus is more appropriate to occur in the later portion of a transition-to-proof course, as opposed to the beginning of the course when this study was conducted.

This study included a few proofs that required students to consider the coordination of proof method (e.g., the presented proof in Exercise 5). However, the coordination of proof methods, especially with proof by contradiction, can be further explored. The coordination of proof by contradiction with other methods (e.g., proof by cases and proof by exhaustion) would enhance a students' conception of the proof method, yet would require both proof methods to be sufficiently developed. Therefore, these types of tasks would be more appropriate to occur in the later portion of a transition-to-proof course.

Finally, this study examined one of the two most difficult proof methods for students to construct and comprehend, with the other being proof by mathematical induction. Therefore, the tasks and instructional design of this study could be modified to consider how students develop an understanding of proof by mathematical induction.

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Appendix A

INSTRUMENTS

A.1 Instruments used during fall 2016

The following instruments were used in Fall 2016. Response space was removed to conserve space.

Activity 1

Read the following mathematical statement and the proof of that statement. Then, answer the questions.

Statement 1: If every even natural number greater than 2 is the sum of two primes, then every odd natural number greater than 5 is the sum of three primes [$\boldsymbol{P \to Q}$].

Proof 1: Assume it is not true that if every even natural number greater than 2 is the sum of two primes, then every odd natural number greater than 5 is the sum of three primes [$\boldsymbol{\sim (P \to Q)}$]. Then every even natural number greater than 2 is the sum of two primes and it is not the case that every odd natural number greater than 5 is the sum of three primes [$\boldsymbol{P \wedge \sim Q}$]. Then there exists an odd natural number greater than 5 that is not the sum of three primes, call it k [$\boldsymbol{\sim Q}$]. Then $k = 2n + 1$. Since $k > 5$, $k - 3 > 2$. Thus $k - 3 = 2n - 2 > 2$ and $k - 3$ is even. By our assumption, $k - 3$ is then the sum of two primes: p and q. Thus $k - 3 = p + q$. Solving for k, we get $k = p + q + 3$ [$\boldsymbol{Q}$]. This is a contradiction, as we assumed k was not the sum of three primes [$\boldsymbol{Q \wedge \sim Q}$]. Therefore it is not the case that it is not true that if every even natural number greater than 2 is the sum of two primes, then every odd natural number greater than 5 is the sum of three primes [$\boldsymbol{\sim (\sim (P \to Q))}$]. In other words, if every even natural number greater than 2 is the sum of two primes, then every odd natural number greater than 5 is the sum of three primes [$\boldsymbol{P \to Q}$].

1. Using the symbols in the statement and proof above, outline the proof.
2. In your own words, state a definition of "proof by contradiction". Then, explain why Proof 1 is a proof by contradiction.

Classroom Discussion 1

Proof by Contradiction for Implications ($P \to Q$):

1. Assume $\sim (P \to Q)$
2. $P \wedge \sim Q$
3. $\sim Q_k$
4. $(\sim Q_k \wedge P) \to Q_k$
5. Q_k
6. $Q_k \wedge \sim Q_k$
7. $\sim (\sim (P \to Q))$
8. $P \to Q$

Things to discuss:

- Student's previous outline - Have students agree on an outline of the proof.
- Logical flow of structure - Have students describe the logical relation between steps. In particular, logically justify steps 4, 5, and 6.
- Definition of proof by contradiction - Definition should include the two key steps of a proof by contradiction: assume the statement is false and arrive at a contradiction.

Exercise 1

Read the following mathematical statement and the proof of that statement. Then, answer the questions.

Statement 1: If every even natural number greater than 2 is the sum of two primes, then every odd natural number greater than 5 is the sum of three primes.

Proof 1: Assume it is not true that if every even natural number greater than 2 is the sum of two primes, then every odd natural number greater than 5 is the sum of three primes. Then every even natural number greater than 2 is the sum of two primes and it is not the case that every odd natural number greater than 5 is the sum of three primes. Then there exists an odd natural number greater than 5 that is not the sum of three primes, call it k. Then $k = 2n+1$. Since $k > 5$, $k-3 > 2$. Thus $k-3 = 2n-2 > 2$ and $k-3$ is even. By our assumption, $k-3$ is then the sum of two primes: p and q. Thus $k-3 = p+q$. Solving for k, we get $k = p+q+3$. This is a contradiction, as we assumed k was not the sum of three primes. Therefore it is not the case that it is not true that if every even natural number greater than 2 is the sum of two primes, then every odd natural number greater than 5 is the sum of three primes. In other words, if every even natural number greater than 2 is the sum of two primes, then every odd natural number greater than 5 is the sum of three primes.

1. Please give an example of a prime number and explain why it is prime.
2. Why is every even natural number greater than 2 the sum of two primes?
3. Why exactly can one conclude that $k-3$ is the sum of two primes?
4. What is the purpose of the statement "Since $k > 5, 2n+1 > 5$ and so $n > 2$"?
5. Summarize in your own words the main idea of this proof.
6. What do you think are the key steps of this proof?
7. In this proof, we subtracted 3 and worked with $k-3$. Would the proof still work if we instead subtracted 5 and worked with $k-5$? Why or why not?
8. Using the method of this proof, show that: if every odd natural number greater than 5 is the sum of three primes and one of those primes is 3, then every even natural number greater than 2 is the sum of two primes.

Activity 2

Read the following mathematical statement and the proof of that statement. Then, answer the questions.

Statement 1: There is no odd integer that can be expressed in the form $4j-1$ and in the form $4k+1$ for integers j and k. $(\nexists \boldsymbol{x})(\boldsymbol{P}(\boldsymbol{x}))$

Proof 1: Assume it is not true that there is no odd integer that can be expressed in the form $4j-1$ and in the form $4k+1$ for integers j and k. Then there is an odd integer that can be expressed in the form $4j-1$ and in the form $4k+1$ for integers j and k. Let n be that integer; that is, $n \in \mathbb{Z}$ such that $n = 4j-1$ and $n = 4k+1$ for $j, k \in \mathbb{Z}$. Then $4j-1 = 4k+1$ and so $2j = 2k+1$. Note that $2j$ is an even number and, since $2j = 2k+1$, $2j$ is an odd number. A number cannot be both even and odd and thus this is a contradiction. Therefore, it is not true that it is not true that there is no odd integer that can be expressed in the form $4j-1$ and in the form $4k+1$ for integers j and k. In other words, there is no odd integer that can be expressed in the form $4j-1$ and in the form $4k+1$ for integers j and k.

1. First, assign symbols to the statement and proof above. Then, outline the proof using these symbols.
2. Write a definition of proof by contradiction. Then, explain why Proof 1 is a proof by contradiction.
3. Read the following statement:

   Statement 2: If every even natural number greater than 2 is the sum of two primes, then every odd natural number greater than 5 is the sum of 3 primes.

Compare Statement 1 and Statement 2. How are they similar? How are they different?

4. Read the following proof:

> **Proof 2:** Assume it is not true that if every even natural number greater than 2 is the sum of two primes, then every odd natural number greater than 5 is the sum of three primes. Then every even natural number greater than 2 is the sum of two primes and it is not the case that every odd natural number greater than 5 is the sum of three primes. Then there exists an odd natural number greater than 5 that is not the sum of three primes, call it k. Then $k = 2n+1$. Since $k > 5$, $k-3 > 2$. Thus $k-3 = 2n-2 > 2$ and $k-3$ is even. By our assumption, $k-3$ is then the sum of two primes: p and q. Thus $k-3 = p+q$. Solving for k, we get $k = p+q+3$. This is a contradiction, as we assumed k was not the sum of three primes. Therefore it is not the case that it is not true that if every even natural number greater than 2 is the sum of two primes, then every odd natural number greater than 5 is the sum of three primes. In other words, if every even natural number greater than 2 is the sum of two primes, then every odd natural number greater than 5 is the sum of three primes.

Compare Proof 1 and Proof 2. How are they similar? How are they different?

Classroom Discussion 2

Proof by contradiction for nonexistence $(\nexists x)[P(x)]$:

1. Assume $\sim ((\nexists x)(P(x)))$
2. $(\exists x)(P(x))$
3. $P(n)$
4. Using $P(n)$, get to a contradiction.
5. $\sim (\sim (\nexists x)(P(x)))$
6. $(\nexists x)(P(x))$

Things to discuss:

- Structure above - Students should agree on something similar to the structure above. Students should be encouraged to use quantification symbols in order to contrast the structures for nonexistence and uniqueness (the next type of statement).
- Logical flow of structure - In particular, focus on how step 1, 2, and 3 logically follow but how 4 is separate from these 3. Then focus on how steps 5 and 6 are logically equivalent.

- Definition of proof by contradiction - Definition should attend to the two key steps of a proof by contradiction: assume the statement is false and arrive at a contradiction. This contradiction should be general and not necessarily related to the statement/assumption.

Exercise 2

Read the following mathematical statement and the proof of that statement. Then, answer the questions.

Statement 1: There is no odd integer that can be expressed in the form $4j-1$ and in the form $4k+1$ for integers j and k.

Proof 1: Assume it is not true that there is no odd integer that can be expressed in the form $4j-1$ and in the form $4k+1$ for integers j and k. Then there is an odd integer that can be expressed in the form $4j-1$ and in the form $4k+1$ for integers j and k. Let n be that integer; that is, $n \in \mathbb{N}$ such that $n = 4j-1$ and $n = 4k+1$ for $j, k \in \mathbb{Z}$. Then $4j-1 = 4k+1$ and so $2j = 2k+1$. Note that $2j$ is an even number and, since $2j = 2k+1$, $2j$ is an odd number. A number cannot be both even and odd and thus this is a contradiction. Therefore, it is not true that it is not true that there is no odd integer that can be expressed in the form $4j-1$ and in the form $4k+1$ for integers j and k. In other words, there is no odd integer that can be expressed in the form $4j-1$ and in the form $4k+1$ for integers j and k.

1. Please give an example of an integer that is odd and explain why it is odd.
2. What kinds of numbers can be expressed in the form $4j-1$?
3. Why exactly can we assume "there is an odd integer n such that $n = 4j-1$ and $n = 4k+1$ for integers j and k."?
4. What is the purpose of the statement "Note that $2j$ is an even number and, since $2j = 2k+1$, $2j$ is an odd number."?
5. Summarize in your own words the main idea of this proof.
6. What do you think are the key steps of the proof?
7. In the statement, we have $4j-1$ and $4k+1$. Would the proof still work if we instead say no odd integer can be expressed in the form $4j-3$ and in the form $4k+3$? Why or why not?
8. Using the method of this proof, show that there is no odd integer that can be expressed in the form $8j-1$ and in the form $8k+1$ for integers j and k.

Activity 3

Read the following mathematical statement and the proof of that statement. Then, answer the questions.

Statement 1: The equation $5x - 4 = 1$ has a unique solution.

Proof 1: Assume the equation $5x - 4 = 1$ does not have a unique solution. Then either there is no solution to the equation $5x - 4 = 1$ or there are at least two distinct solutions to the equation $5x - 4 = 1$. Note $x = 1$ is a solution of $5x - 4 = 1$. Thus there are at least two distinct solutions to the equation $5x - 4 = 1$, call them y and z. As both y and z are solutions of the equation $5x - 4 = 1$, $5y - 4 = 1$ and $5z - 4 = 1$. Then $5y - 4 = 5z - 4$ and so $y = z$. Therefore it is not true that there are at least two distinct solutions to the equation $5x - 4 = 1$. This is a contradiction, as we assumed that either there is no solution to the equation $5x - 4 = 1$ or there are at least two distinct solutions to the equation $5x - 4 = 1$. Therefore it is not true that the equation $5x - 4 = 1$ does not have a unique solution. In other words, the equation $5x - 4 = 1$ does have a unique solution.

1. First, assign symbols to the statement and proof above. Then, outline the proof using these symbols.
2. Write a definition of proof by contradiction. Then, explain why Proof 1 is a proof by contradiction.

Classroom Discussion 3

Proof by contradiction for uniqueness $(\exists! x)(P(x))$:

1. Assume $\sim (\exists! x)(P(x))$
2. $(\forall x)(\sim P(x)) \ \vee (\exists x, y)(P(x) \ \wedge P(y) \ \wedge x \neq y)$
3. Show $P(n)$ for some n
4. $(\exists x, y)(P(x) \ \wedge P(y) \ \wedge x \neq y)$
5. $P(x) \wedge P(y) \to x = y$
6. $(\nexists x, y)(P(x) \ \wedge P(y) \ \wedge x \neq y)$
7. $\to\leftarrow$ (lines 2, 3, and 6)
8. $\sim (\sim (\exists! x)(P(x)))$
9. $(\exists! x)(P(x))$

Things to discuss:

- Structure above - Students may get bogged down in the quantification for uniqueness (especially line 2). If this happens, have students write the statement without quantifiers. While this will make the comparison with nonexistence weaker, it may help the student recognize the general structure for proof by contradiction.
- Logical flow of structure - In particular, focus on how step 1, 2, and 3 logically follow but how 4 is separate from these 3. Then focus on how steps 5 and 6 are logically equivalent.
- Definition of proof by contradiction - In addition to key steps, definition should describe how these key steps logically relate to prove the statement.
- Outline of the proof - Outline should reinforce procedure for proof by contradiction.

1. Look at your responses for question 1 in Activities 1, 2, and 3. Can you write a list of steps to prove *any* type of statement by contradiction?
2. Using the steps above, try to write an outline proof for the following statement:

    Statement 2: The multiplicative inverse of a non-zero real number x is unique.

Exercise 3

Read the following mathematical statement and the proof of that statement. Then, answer the questions.

Statement 2: The multiplicative inverse of a non-zero real number r is unique.

Proof 2: Assume the multiplicative inverse of an arbitrary non-zero real number r is not unique. Then either there is no multiplicative inverse of r or there are at least two distinct multiplicative inverses of r. Note $x = \frac{1}{r}$ is a multiplicative inverse of r. Thus there are at least two distinct multiplicative inverses of r, call them x and y. As both x and y are both multiplicative inverses of r, $rx = 1$ and $ry = 1$. Then $rx = ry$ and so $x = y$. Therefore it is not true that there are at least two distinct multiplicative inverses of r. This is a contradiction, as we assumed that either there is no multiplicative inverse of r or there are at least two distinct multiplicative inverses of r. Therefore it is not true that the multiplicative inverse of an arbitrary non-zero real number r is not unique. In other words, the multiplicative inverse of a non-zero real number r is unique.

1. Compare the outline of your proof in question 4 to the proof above. Explain how your proof compares to the given proof in terms of: (1) general structure, (2) specific lines, and/or (3) overall approach to the proof.

2. Please give an example of a multiplicative inverse of a non-zero real number and explain why it is a multiplicative inverse.
3. Why does r have to have a multiplicative inverse?
4. Why exactly can one conclude that $x = y$?
5. What is the purpose of the statement "Then either there is no multiplicative inverse of r or there are at least two distinct multiplicative inverses of r."?
6. Summarize in your own words the main idea of this proof.
7. What do you think are the key steps of the proof?
8. Would the proof still work if we instead say the multiplicative inverse of a real number x is unique? Why or why not?
9. Using the method of this proof, show that: if a and b are real numbers and $a \neq 0$, then there is a unique real number r such that $ar + b = 0$.

Activity 4

Read the following mathematical statement and the proof of that statement. Then, answer the questions.

Statement 1: The set of natural numbers is not finite.

Proof 1: Assume there are not infinitely many natural numbers. Then there are finitely many natural numbers. Let $N = \{n_1, n_2, n_3, \ldots, n_k\}$ be all the natural numbers, where $n_1 < n_2 < n_3 < \ldots < n_k$. Then $n = n_k + 1$ is a natural number and $n \notin N$, which is a contradiction. Therefore there are infinitely many natural numbers.

1. First, assign symbols to the statement and proof above. Then, outline the proof using these symbols.
2. Write a definition of proof by contradiction. Then, explain why Proof 1 is a proof by contradiction.

Classroom Discussion 4

Proof by contradiction for infinity $P(\mathbb{N})$:

1. Assume $\sim P(\mathbb{N})$.
2. Then $\mathbb{N} = \{n_1, n_2, \ldots, n_k\}$ where $n_1 < n_2 < \ldots < n_k$.

3. Let $n = n_k + 1$
4. $n \in \mathbb{N}$ by definition of natural numbers.
5. $n \notin \mathbb{N}$ since $n_k < n$.
6. $\rightarrow\leftarrow$ (lines 4 and 5)
7. $\sim (\sim P(\mathbb{N}))$
8. $P(\mathbb{N})$

Things to discuss:

- Structure above - Students may not know how to write infinitely many as a quantified symbol and should be encouraged to represent the statement as $P(\mathbb{N})$.
- Logical flow of structure - In particular, the logical role of the construction.
- Definition of proof by contradiction - Definition should attend to the key steps of the method as well as describe how these key steps relate to prove the statement is true.
- Outline of the proof - This reinforces their procedure for proof by contradiction.

1. Look at your responses for question 1 in Activities 1-4. Can you write a list of steps to prove *any* type of statement by contradiction?
2. Using the steps above, try to write an outline proof for the following statement:

   Statement: The set of primes is infinite.

Exercise 4

Read the following mathematical statement and the proof of that statement. Then, answer the questions.

Statement 1: The set of primes is infinite.

Proof 1: Suppose the set of primes is finite. Let $p_1, p_2, p_3, \ldots, p_k$ be all those primes with $p_1 < p_2 < p_3 < \cdots < p_k$. Let n be one more than the product of all of them. That is, $n = (p_1 p_2 p_3 \ldots p_k) + 1$. Then n is a natural number greater than 1, so n has a prime divisor q. Since q is prime, $q > 1$. Since q is prime and $p_1, p_2, p_3, \ldots, p_k$ are all the primes, q is one of the p_i in this list. Thus, q divides the product $p_1 \cdot p_2 \cdot p_3 \cdot \ldots \cdot p_k$. Since q divides n, q divides the difference $n - (p_1 p_2 p_3 \ldots p_k)$. But this difference is 1, so $q = 1$. From the contradiction, $q > 1$ and $q = 1$, we conclude that the assumption that the set of primes is finite is false. Therefore, the set of primes is infinite.

1. Compare the outline of your proof in question 4 to the proof above. Explain how your proof compares to the given proof in terms of: (1) general structure, (2) specific lines, and/or (3) overall approach to the proof.

2. Please give an example of a set that is infinite and explain why it is infinite.
3. Why does n have to have a prime divisor?
4. Why exactly can one conclude that q divides the difference $n - (p_1 \cdot p_2 \cdot p_3 \cdot \ldots \cdot p_k)$?
5. What is the purpose of the statement "Let $p_1, p_2, p_3, \ldots, p_k$ be all those primes with $p_1 < p_2 < p_3 < \cdots < p_k$."?
6. Summarize in your own words the main idea of this proof.
7. What do you think are the key steps of the proof?
8. In the proof, we define $n = (p_1 p_2 p_3 \ldots p_k) + 1$. Would the proof still work if we instead defined $n = (p_1 p_2 p_3 \ldots p_k) + 31$? Why or why not?
9. Define the set $S_k = \{2, 3, 4, \ldots, k\}$ for any $k > 2$. Using the method of this proof, show that for any $k > 2$, there exists a natural number greater than 1 that is not divisible by any element in S_k.

Activity 5

Read the following mathematical statement and the proof of that statement. Then, answer the questions.

Statement 1: Statement P is true.

Proof 1: A statement P is either true or false. Assume P is false; that is, the negation of P, $\sim P$, is true. If $\sim P$ leads to a contradiction, then $\sim P$ must be false. This implies our initial assumption was not true; that is, it is not true that P is false. Since P is either true or false and P is not false, P is true.

1. First, assign symbols to the statement and proof above. Then, outline the proof using these symbols.
2. Write a definition of proof by contradiction. Then, explain why Proof 1 is a proof by contradiction.

Classroom Discussion 5

Proof by Contradiction (General Form)
Statement: P

1. $P \vee \sim P$
2. Assume $\sim P$

3. If $\sim P$ implies $\rightarrow\leftarrow$, then $\sim P$ is false.

4. $P \vee \sim P$ and $\sim P$ is false implies P is true.

Things to discuss:

- Outline of the proof - This validates the procedure and encourages students to encapsulate the procedure into an Object that can be a sub-proof.
- Logical flow of structure - Try to get students to describe the logic of the proof method *as a whole* rather than line-by-line.
- Definition of proof by contradiction - Definition should describe the key steps and how these key steps logically imply the statement is true.
- Writing a proof - Encourage students to first use the structure of a proof by contradiction and then fill in the details.

1. Look at your response for question 1 in Classroom Discussion 4. Does the proof in Activity 5 follow the steps you wrote for a proof by contradiction? If so, explain why. If not, revise your steps and explain why the previous steps did not work.

2. Using the steps above, try to write a proof of the following statement:

   Claim: $\sqrt{2}$ is an irrational number.

Exercise 5

Read the following mathematical statement and the proof of that statement. Then, answer the questions.

Statement: $\sqrt{2}$ is an irrational number.

Proof: Suppose $\sqrt{2}$ is a rational number. Then there exists $p, q \in \mathbb{Z}$ such that $\sqrt{2} = \frac{p}{q}, q \neq 0$, and $\gcd(p, q) = 1$. This implies $p^2 = 2q^2$ and so p^2 is even. Now, if p were odd, then p^2 would be odd. Thus p must be an even number and so $p = 2r$ where $r \in \mathbb{Z}$. Since $p = 2r$ and $p^2 = 2q^2$, $4r^2 = 2q^2$ and so $2r^2 = q^2$. By the same argument as above, q must also be even number. Since both p and q are even, $\gcd(p, q) \neq 1$. From the contradiction that $\gcd(p, q) = 1$ and $\gcd(p, q) \neq 1$, we conclude that the assumption that $\sqrt{2}$ is a rational number is false. Therefore, $\sqrt{2}$ is an irrational number.

1. Compare your proof in question 4 to the proof above. Explain how your proof compares to the given proof in terms of: (1) general structure, (2) specific lines, and/or (3) overall approach ot the proof.

2. Please given an example of a number that is irrational and explain why it is irrational.

3. What does $\gcd(p, q) = 1$ mean and why can we conclude $\gcd(p, q) = 1$?
4. Why exactly can one conclude that p^2 is an even number?
5. What is the purpose of the statement "Now, if p were odd, then p^2 would be odd."?
6. How exactly can one conclude that q is an even number?
7. Summarize in your own words the main idea of this proof.
8. What do you think are the key steps of the proof?
9. Using the method of this proof, show that for any prime number n, $\sqrt{n}$ is an irrational number.

A.2 Revised instruments

The following instruments were revised as described in Subsection 5.2.3. Response space was removed to conserve space.

Activity 1

Read the following mathematical statement and the proof of that statement. Then, answer the questions.

Statement 1: If every even natural number greater than 2 is the sum of two primes, then every odd natural number greater than 5 is the sum of three primes [$\boldsymbol{P \rightarrow Q}$].

Proof 1: Assume it is not true that if every even natural number greater than 2 is the sum of two primes, then every odd natural number greater than 5 is the sum of three primes [$\boldsymbol{\sim (P \rightarrow Q)}$]. Then every even natural number greater than 2 is the sum of two primes and it is not the case that every odd natural number greater than 5 is the sum of three primes [$\boldsymbol{P \wedge \sim Q}$]. Then there exists an odd natural number greater than 5 that is not the sum of three primes, call it k [$\boldsymbol{\sim Q}$]. Then $k = 2n + 1$. Since $k > 5$, $k - 3 > 2$. Thus $k - 3 = 2n - 2 > 2$ and $k - 3$ is even. By our assumption, $k - 3$ is then the sum of two primes: p and q. Thus $k - 3 = p + q$. Solving for k, we get $k = p + q + 3$ [$\boldsymbol{Q}$]. This is a contradiction, as we assumed k was not the sum of three primes [$\boldsymbol{Q \wedge \sim Q}$]. Therefore it is not the case that it is not true that if every even natural number greater than 2 is the sum of two primes, then every odd natural number greater than 5 is the sum of three primes [$\boldsymbol{\sim (\sim (P \rightarrow Q))}$]. In other words, if every even natural number greater than 2 is the sum of two primes, then every odd natural number greater than 5 is the sum of three primes [$\boldsymbol{P \rightarrow Q}$].

1. Using the symbols in the statement and proof above, outline the proof.
2. In your own words, state a definition of "proof by contradiction". Then, explain why Proof 1 is a proof by contradiction.

Classroom Discussion 1

Proof by Contradiction for Implications ($P \rightarrow Q$):

1. Assume $\sim (P \rightarrow Q)$
2. $P \wedge \sim Q$

3. $\sim Q_k$
4. $(\sim Q_k \wedge P) \to Q_k$
5. Q_k
6. $Q_k \wedge \sim Q_k$
7. $\sim (\sim (P \to Q))$
8. $P \to Q$

Things to discuss:

- Student's previous outline - Have students agree on an outline of the proof.
- Logical flow of structure - Have students describe the logical relation between steps. In particular, logically justify steps 4, 5, and 6.
- Definition of proof by contradiction - Definition should include the two key steps of a proof by contradiction: assume the statement is false and arrive at a contradiction.

Exercise 1

Read the following mathematical statement and the proof of that statement. Then, answer the questions.

Statement 1: If every even natural number greater than 2 is the sum of two primes, then every odd natural number greater than 5 is the sum of three primes.

Proof 1: Assume it is not true that if every even natural number greater than 2 is the sum of two primes, then every odd natural number greater than 5 is the sum of three primes. Then every even natural number greater than 2 is the sum of two primes and it is not the case that every odd natural number greater than 5 is the sum of three primes. Then there exists an odd natural number greater than 5 that is not the sum of three primes, call it k. Then $k = 2n+1$. Since $k > 5$, $k-3 > 2$. Thus $k-3 = 2n-2 > 2$ and $k-3$ is even. By our assumption, $k-3$ is then the sum of two primes: p and q. Thus $k-3 = p+q$. Solving for k, we get $k = p+q+3$. This is a contradiction, as we assumed k was not the sum of three primes. Therefore it is not the case that it is not true that if every even natural number greater than 2 is the sum of two primes, then every odd natural number greater than 5 is the sum of three primes. In other words, if every even natural number greater than 2 is the sum of two primes, then every odd natural number greater than 5 is the sum of three primes.

1. Please give an example of a prime number and explain why it is prime.
2. The statement "Every even natural number greater than 2 is the sum of two primes" is known as the Goldbach Conjecture and has not been shown to be true. Yet, it is used in the proof as if it is true. How can this be?

3. Why exactly can one conclude that $k-3$ is the sum of two primes?
4. What is the purpose of the statement "Since $k>5, 2n+1>5$ and so $n>2$"?
5. Which of the following **best** summarizes the main idea of this proof? Explain your choice.

   (a) The main idea of the proof is to show that if there exists an odd natural number greater than 5 that is not the sum of three primes, one could find three primes that sum to be that number, contradicting the assumption.

   (b) The main idea of the proof is to assume that every even natural number greater than 2 is the sum of two primes and there exists an odd natural number greater than 5 that is not the sum of three primes, and then to show that the odd natural number can be written as three primes, which is impossible.

6. What do you think are the key steps of this proof?
7. In this proof, we subtracted 3 and worked with $k-3$. Would the proof still work if we instead subtracted 5 and worked with $k-5$? Why or why not?
8. Using the method of this proof or the outline developed during classroom discussion, write a proof outline to show that: if every odd natural number greater than 5 is the sum of three primes and one of those primes is 3, then every even natural number greater than 2 is the sum of two primes.

Activity 2

Read the following mathematical statement and the proof of that statement. Then, answer the questions.

Statement 2: There is no odd integer that can be expressed in the form $4j-1$ and in the form $4k+1$ for integers j and k. $(\nexists \boldsymbol{x})(\boldsymbol{P}(\boldsymbol{x}))$

Proof 2: Assume it is not true that there is no odd integer that can be expressed in the form $4j-1$ and in the form $4k+1$ for integers j and k. Then there is an odd integer that can be expressed in the form $4j-1$ and in the form $4k+1$ for integers j and k. Let n be that integer; that is, $n \in \mathbb{Z}$ such that $n = 4j-1$ and $n = 4k+1$ for $j, k \in \mathbb{Z}$. Then $4j-1 = 4k+1$ and so $2j = 2k+1$. Note that $2j$ is an even number and, since $2j = 2k+1$, $2j$ is an odd number. A number cannot be both even and odd and thus this is a contradiction. Therefore, it is not true that it is not true that there is no odd integer that can be expressed in the form $4j-1$ and in the form $4k+1$ for integers j and k. In other words, there is no odd integer that can be expressed in the form $4j-1$ and in the form $4k+1$ for integers j and k.

1. First, assign symbols to the statement and proof above. Then, outline the proof using these symbols.

2. Write a definition of proof by contradiction. Then, explain why Proof 1 is a proof by contradiction.
3. Read the following lines from proof 1 and proof 2:

   > Then there exists an odd natural number greater than 5 that is not the sum of three primes, call it k.
   >
   > Then there is an odd integer that can be expressed in the form $4j - 1$ and in the form $4k + 1$ for integers j and k.

   What is the purpose of each statement in their respective proof? Compare these purposes.
4. Read the following lines from proof 1 and proof 2:

   > Then $k = 2n + 1$. Since $k > 5$, $k - 3 > 2$. Thus $k - 3 = 2n - 2 > 2$ and $k - 3$ is even. By our assumption, $k - 3$ is then the sum of two primes: p and q. Thus $k - 3 = p + q$.
   >
   > Then $4j - 1 = 4k + 1$ and so $2j = 2k + 1$. Note that $2j$ is an even number and, since $2j = 2k + 1$, $2j$ is an odd number.

   What is the purpose of these collections of lines in their respective proof? Compare these purposes.

Classroom Discussion 2

Proof by contradiction for nonexistence $(\nexists x)[P(x)]$:

1. Assume $\sim ((\nexists x)(P(x)))$
2. $(\exists x)(P(x))$
3. $P(n)$
4. Using $P(n)$, get to a contradiction.
5. $\sim (\sim (\nexists x)(P(x)))$
6. $(\nexists x)(P(x))$

Things to discuss:

- Structure above - Students should agree on something similar to the structure above. Students should be encouraged to use quantification symbols in order to contrast the structures for nonexistence and uniqueness (the next type of statement).

- Logical flow of structure - In particular, focus on how step 1, 2, and 3 logically follow but how 4 is separate from these 3. Then focus on how steps 5 and 6 are logically equivalent.
- Definition of proof by contradiction - Definition should attend to the two key steps of a proof by contradiction: assume the statement is false and arrive at a contradiction. This contradiction should be general and not necessarily related to the statement/assumption.

1. Consider the following outlines for implication and nonexistence statements:

Activity 1 **Statement:** $P \to Q$	**Activity 2** **Statement:** $(\nexists x)(P(x))$
1. Assume $\sim (P \to Q)$	1. Assume $\sim (\nexists x)(P(x))$
2. $P \wedge \sim Q$	2. $(\exists x)(P(x))$
3. $\sim Q$	3. $P(n)$
4. $(\sim Q \wedge P) \to Q$	4. Using $P(n)$, get to a contradiction.
5. Q	5. $\sim (\sim (\nexists x)(P(x)))$
6. $Q \wedge \sim Q$	6. $(\nexists x)(P(x))$
7. $\sim (\sim (P \to Q))$	
8. $P \to Q$	

   Compare and contrast the purposes of steps in each outline.

2. Write a list of steps to prove either statement by contradiction. Be sure your new list of steps works for each type of statement above.

Exercise 2

Read the following mathematical statement and the proof of that statement. Then, answer the questions.

Statement 1: There is no odd integer that can be expressed in the form $4j - 1$ and in the form $4k + 1$ for integers j and k.

Proof 1: Assume it is not true that there is no odd integer that can be expressed in the form $4j - 1$ and in the form $4k + 1$ for integers j and k. Then there is an odd integer that can be expressed in the form $4j - 1$ and in the form $4k + 1$ for integers j and k. Let n be that integer; that is, $n \in \mathbb{N}$ such that $n = 4j - 1$ and $n = 4k + 1$ for $j, k \in \mathbb{Z}$. Then $4j - 1 = 4k + 1$ and so $2j = 2k + 1$. Note that $2j$ is an even number and, since $2j = 2k + 1$, $2j$ is an odd number. A number cannot be both even and odd and thus this is a contradiction. Therefore, it is not true that it is not true that there is no odd integer that can be expressed in the form $4j - 1$ and in the form $4k + 1$ for integers j and k. In other words, there is no odd integer that can be expressed in the form $4j - 1$ and in the form $4k + 1$ for integers j and k.

1. Please give an example of an integer that is odd and explain why it is odd.
2. What kinds of numbers can be expressed in the form $4j - 1$? What kinds of numbers can be expressed in the form $4k + 1$?
3. Why exactly can we assume "there is an odd integer n such that $n = 4j - 1$ and $n = 4k + 1$ for integers j and k."?
4. What is the purpose of the statement "Note that $2j$ is an even number and, since $2j = 2k + 1$, $2j$ is an odd number."?
5. Which of the following **best** summarizes the main idea of this proof? Explain your choice.
   (a) The main idea of the proof is to show that if an odd integer could be expressed in the form 4j-1 and 4k+1, one could find a number that is both even and odd, contradicting the assumption.
   (b) The main idea of the proof is to assume that an odd integer could be written in the form 4j-1 and 4k+1, and then to show 2j is both even and odd, which is impossible.
6. What do you think are the key steps of the proof?
7. In the statement, we have $4j - 1$ and $4k + 1$. Would the proof still work if we instead say no odd integer can be expressed in the form $4j - 3$ and in the form $4k + 3$? Why or why not?
8. Using the method of this proof, show that there is no odd integer that can be expressed in the form $8j - 1$ and in the form $8k + 1$ for integers j and k.

Activity 3

Read the following mathematical statement and the proof of that statement. Then, answer the questions.

Statement 1: The equation $5x - 4 = 1$ has a unique solution.

Proof 1: Assume the equation $5x - 4 = 1$ does not have a unique solution. Then either there is no solution to the equation $5x - 4 = 1$ or there are at least two distinct solutions to the equation $5x - 4 = 1$. Note $x = 1$ is a solution of $5x - 4 = 1$. Thus there are at least two distinct solutions to the equation $5x - 4 = 1$, call them y and z. As both y and z are solutions of the equation $5x - 4 = 1$, $5y - 4 = 1$ and $5z - 4 = 1$. Then $5y - 4 = 5z - 4$ and so $y = z$. Therefore it is not true that there are at least two distinct solutions to the equation $5x - 4 = 1$. This is a contradiction, as we assumed that either there is no solution to the equation $5x - 4 = 1$ or there are at least two distinct solutions to the equation $5x - 4 = 1$. Therefore it is not true that the equation $5x - 4 = 1$ does not have a unique solution. In other words, the equation $5x - 4 = 1$ does have a unique solution.

1. First, assign symbols to the statement and proof above. Then, outline the proof using these symbols.
2. Write a definition of proof by contradiction. Then, explain why Proof 1 is a proof by contradiction.

Classroom Discussion 3

Proof by contradiction for uniqueness $(\exists ! x)(P(x))$:

1. Assume $\sim (\exists ! x)(P(x))$
2. $(\forall x)(\sim P(x)) \ \vee (\exists x, y)(P(x) \ \wedge P(y) \ \wedge x \neq y)$
3. Show $P(n)$ for some n
4. $(\exists x, y)(P(x) \ \wedge P(y) \ \wedge x \neq y)$
5. $P(x) \wedge P(y) \rightarrow x = y$
6. $(\nexists x, y)(P(x) \ \wedge P(y) \ \wedge x \neq y)$
7. $\rightarrow\leftarrow$ (lines 2, 3, and 6)
8. $\sim (\sim (\exists ! x)(P(x)))$
9. $(\exists ! x)(P(x))$

Things to discuss:

- Structure above - Students may get bogged down in the quantification for uniqueness (especially line 2). If this happens, have students write the statement without quantifiers. While this will make the comparison with nonexistence weaker, it may help the student recognize the general structure for proof by contradiction.
- Logical flow of structure - In particular, focus on how step 1, 2, and 3 logically follow but how 4 is separate from these 3. Then focus on how steps 5 and 6 are logically equivalent.
- Definition of proof by contradiction - In addition to key steps, definition should describe how these key steps logically relate to prove the statement.
- Outline of the proof - Outline should reinforce procedure for proof by contradiction.

1. Consider the following outlines for implication, nonexistence, and uniqueness statements:

Activity 1 **Statement:** $P \to Q$	**Activity 2** **Statement:** $(\nexists x)(P(x))$	**Activity 3** **Statement:** $(\exists! x)(P(x))$
1. Assume $\sim (P \to Q)$	1. Assume $\sim (\nexists x)(P(x))$	1. Assume $\sim (\exists! x)(P(x))$
2. $P \wedge \sim Q$	2. $(\exists x)(P(x))$	2. $\sim (\exists x)(P(x)) \vee (\exists x, y)$ $(P(x) \wedge P(y) \wedge x \neq y)$
3. $\sim Q$	3. $P(n)$	3. Show $P(n)$ for some n.
4. $(\sim Q \wedge P) \to Q$	4. Using $P(n)$, get to a contradiction.	4. $(\exists x, y)(P(x) \wedge P(y) \wedge x \neq y)$
5. Q	5. $\sim (\sim (\nexists x)(P(x)))$	5. $P(x) \wedge P(y) \to x \neq y$
6. $Q \wedge \sim Q$	6. $(\nexists x)(P(x))$	6. $(\nexists x, y)(P(x) \wedge P(y) \wedge x \neq y)$
7. $\sim (\sim (P \to Q))$		7. $\to\leftarrow$ (lines 2, 3, and 6)
8. $P \to Q$		8. $\sim (\sim (\exists! x)(P(x)))$
		9. $(\exists! x)(P(x))$

Compare and contrast the purposes of steps in each outline.

2. Write a list of steps to prove either statement by contradiction. Be sure your new list of steps works for each type of statement above.

3. Using the steps above, try to write an outline proof for the following statement:

   Statement: The multiplicative inverse of a non-zero real number x is unique.

Exercise 3

Read the following mathematical statement and the proof of that statement. Then, answer the questions.

Statement 2: The multiplicative inverse of a non-zero real number r is unique.

Proof 2: Assume the multiplicative inverse of an arbitrary non-zero real number r is not unique. Then either there is no multiplicative inverse of r or there are at least two distinct multiplicative inverses of r. Note $x = \frac{1}{r}$ is a multiplicative inverse of r. Thus there are at least two distinct multiplicative inverses of r, call them x and y. As both x and y are both multiplicative inverses of r, $rx = 1$ and $ry = 1$. Then $rx = ry$ and so $x = y$. Therefore it is not true that there are at least two distinct multiplicative inverses of r. This is a contradiction, as we assumed that either there is no multiplicative inverse of r or there are at least two distinct multiplicative inverses of r. Therefore it is not true that the multiplicative inverse of an arbitrary non-zero real number r is not unique. In other words, the multiplicative inverse of a non-zero real number r is unique.

1. Please give an example of a multiplicative inverse of a non-zero real number and explain why it is a multiplicative inverse.
2. Why does r have to have a multiplicative inverse?

3. Why exactly can one conclude that $x = y$?
4. What is the purpose of the statement "Then either there is no multiplicative inverse of r or there are at least two distinct multiplicative inverses of r."?
5. Summarize in your own words the main idea of this proof.
6. What do you think are the key steps of the proof?
7. Would the proof still work if we instead say the multiplicative inverse of a real number x is unique? Why or why not?
8. Using the method of this proof, show that: if a and b are real numbers and $a \neq 0$, then there is a unique real number r such that $ar + b = 0$.

Activity 4

Read the following mathematical statement and the proof of that statement. Then, answer the questions.

Statement 1: The set of natural numbers is not finite.

Proof 1: Assume there are not infinitely many natural numbers. Then there are finitely many natural numbers. Let $N = \{n_1, n_2, n_3, \ldots, n_k\}$ be all the natural numbers, where $n_1 < n_2 < n_3 < \ldots < n_k$. Then $n = n_k + 1$ is a natural number and $n \notin N$, which is a contradiction. Therefore there are infinitely many natural numbers.

1. First, assign symbols to the statement and proof above. Then, outline the proof using these symbols.
2. Write a definition of proof by contradiction. Then, explain why Proof 1 is a proof by contradiction.

Classroom Discussion 4

Proof by contradiction for infinity $P(\mathbb{N})$:

1. Assume $\sim P(\mathbb{N})$.
2. Then $\mathbb{N} = \{n_1, n_2, \ldots, n_k\}$ where $n_1 < n_2 < \ldots < n_k$.
3. Let $n = n_k + 1$
4. $n \in \mathbb{N}$ by definition of natural numbers.
5. $n \notin \mathbb{N}$ since $n_k < n$.

6. $\rightarrow\leftarrow$ (lines 4 and 5)
7. $\sim(\sim P(\mathbb{N}))$
8. $P(\mathbb{N})$

Things to discuss:

- Structure above - Students may not know how to write infinitely many as a quantified symbol and should be encouraged to represent the statement as $P(\mathbb{N})$.
- Logical flow of structure - In particular, the logical role of the construction.
- Definition of proof by contradiction - Definition should attend to the key steps of the method as well as describe how these key steps relate to prove the statement is true.
- Outline of the proof - This reinforces their procedure for proof by contradiction.

1. Consider the following outlines for implication, nonexistence, uniqueness, and infinity statements:

Activity 1 **Statement:** $P \rightarrow Q$	**Activity 2** **Statement:** $(\nexists x)(P(x))$	**Activity 3** **Statement:** $(\exists! x)(P(x))$	**Activity 4** **Statement:** $P(\mathbb{N})$
1. Assume $\sim (P \rightarrow Q)$	1. Assume $\sim (\nexists x)(P(x))$	1. Assume $\sim (\exists! x)(P(x))$	1. Assume $\sim P(\mathbb{N})$
2. $P \wedge \sim Q$	2. $(\exists x)(P(x))$	2. $\sim (\exists x)(P(x)) \vee (\exists x, y)(P(x) \wedge P(y) \wedge x \neq y)$	2. $\mathbb{N} = \{n_1, n_2, \ldots, n_k\}$ where $n_1 < n_2 < \ldots < n_k$
3. $\sim Q$	3. $P(n)$	3. Show $P(n)$ for some n	3. Let $n = n_k + 1$.
4. $(\sim Q \wedge P) \rightarrow Q$.	4. Using $P(n)$, get to a contradiction.	4. $(\exists x, y)(P(x) \wedge P(y) \wedge x \neq y)$	4. $n \in \mathbb{N}$ by definition of natural numbers.
5. Q	5. $\sim (\sim (\nexists x)(P(x)))$	5. $P(x) \wedge P(y) \rightarrow x = y$	5. $n \notin \mathbb{N}$ since $n_k < n$.
6. $Q \wedge \sim Q$	6. $(\nexists x)(P(x))$	6. $(\nexists x, y)(P(x) \wedge P(y) \wedge x \neq y)$	6. $\rightarrow\leftarrow$ (lines 4 and 5)
7. $\sim (\sim (P \rightarrow Q))$ 8. $P \rightarrow Q$		7. $\rightarrow\leftarrow$ (lines 2, 3, and 6)	7. $\sim\sim P(\mathbb{N})$
		8. $\sim (\sim (\exists! x)(P(x)))$	8. $P(\mathbb{N})$
		9. $(\exists! x)(P(x))$	

Compare and contrast the purposes of steps in each outline.

2. Write a list of steps to prove either statement by contradiction. Be sure your new list of steps works for each type of statement above.

3. Using the steps above, try to write an outline proof for the following statement:

   Statement: The set of primes is infinite.

Exercise 4

Read the following mathematical statement and the proof of that statement. Then, answer the questions.

Statement 1: The set of primes is infinite.

Proof 1: Suppose the set of primes is finite. Let $p_1, p_2, p_3, \ldots, p_k$ be all those primes with $p_1 < p_2 < p_3 < \cdots < p_k$. Let n be one more than the product of all of them. That is, $n = (p_1 p_2 p_3 \ldots p_k) + 1$. Then n is a natural number greater than 1, so n has a prime divisor q. Since q is prime, $q > 1$. Since q is prime and $p_1, p_2, p_3, \ldots, p_k$ are all the primes, q is one of the p_i in this list. Thus, q divides the product $p_1 \cdot p_2 \cdot p_3 \cdot \ldots \cdot p_k$. Since q divides n, q divides the difference $n - (p_1 p_2 p_3 \ldots p_k)$. But this difference is 1, so $q = 1$. From the contradiction, $q > 1$ and $q = 1$, we conclude that the assumption that the set of primes is finite is false. Therefore, the set of primes is infinite.

1. Please give an example of a set that is infinite and explain why it is infinite.
2. Why does n have to have a prime divisor?
3. Why exactly can one conclude that q divides the difference $n - (p_1 \cdot p_2 \cdot p_3 \cdot \ldots \cdot p_k)$?
4. What is the purpose of the statement "Let $p_1, p_2, p_3, \ldots, p_k$ be all those primes with $p_1 < p_2 < p_3 < \cdots < p_k$."?
5. Summarize in your own words the main idea of this proof.
6. What do you think are the key steps of the proof?
7. In the proof, we define $n = (p_1 p_2 p_3 \ldots p_k) + 1$. Would the proof still work if we instead defined $n = (p_1 p_2 p_3 \ldots p_k) + 31$? Why or why not?
8. Define the set $S_k = \{2, 3, 4, \ldots, k\}$ for any $k > 2$. Using the method of this proof, show that for any $k > 2$, there exists a natural number greater than 1 that is not divisible by any element in S_k.

Activity 5

Read the following mathematical statement and the proof of that statement. Then, answer the questions.

Statement 1: Statement P is true.

Proof 1: A statement P is either true or false. Assume P is false; that is, the negation of P, $\sim P$, is true. If $\sim P$ leads to a contradiction, then $\sim P$ must be false. This implies our initial assumption was not true; that is, it is not true that P is false. Since P is either true or false and P is not false, P is true.

1. First, assign symbols to the statement and proof above. Then, outline the proof using these symbols.
2. Write a definition of proof by contradiction. Then, explain why Proof 1 is a proof by contradiction.

Classroom Discussion 5

Proof by Contradiction (General Form)
Statement: P

1. $P \vee \sim P$
2. Assume $\sim P$
3. If $\sim P$ implies $\rightarrow\leftarrow$, then $\sim P$ is false.
4. $P \vee \sim P$ and $\sim P$ is false implies P is true.

Things to discuss:

- Outline of the proof - This validates the procedure and encourages students to encapsulate the procedure into an Object that can be a sub-proof.
- Logical flow of structure - Try to get students to describe the logic of the proof method *as a whole* rather than line-by-line.
- Definition of proof by contradiction - Definition should describe the key steps and how these key steps logically imply the statement is true.
- Writing a proof - Encourage students to first use the structure of a proof by contradiction and then fill in the details.

1. Write down your general steps for any proof by contradiction. Does the proof in Activity 5 follow these steps? If so, explain why. If not, revise your steps and explain why the previous steps did not work.

2. Using the steps above, try to write a proof of the following statement:

   Claim: $\sqrt{2}$ is an irrational number.

Exercise 5

Read the following mathematical statement and the proof of that statement. Then, answer the questions.

Statement: $\sqrt{2}$ is an irrational number.

Proof: Suppose $\sqrt{2}$ is a rational number. Then there exists $p, q \in \mathbb{Z}$ such that $\sqrt{2} = \frac{p}{q}, q \neq 0$, and $\gcd(p, q) = 1$. This implies $p^2 = 2q^2$ and so p^2 is even. Now, if p were odd, then p^2 would be odd. Thus p must be an even number and so $p = 2r$ where $r \in \mathbb{Z}$. Since $p = 2r$ and $p^2 = 2q^2$, $4r^2 = 2q^2$ and so $2r^2 = q^2$. By the same argument as above, q must also be even number. Since both p and q are even, $\gcd(p, q) \neq 1$. From the contradiction that $\gcd(p, q) = 1$ and $\gcd(p, q) \neq 1$, we conclude that the assumption that $\sqrt{2}$ is a rational number is false. Therefore, $\sqrt{2}$ is an irrational number.

1. Please given an example of a number that is irrational and explain why it is irrational.
2. What does $\gcd(p, q) = 1$ mean and why can we conclude $\gcd(p, q) = 1$?
3. Why exactly can one conclude that p^2 is an even number?
4. What is the purpose of the statement "Now, if p were odd, then p^2 would be odd."?
5. How exactly can one conclude that q is an even number?
6. Summarize in your own words the main idea of this proof.
7. What do you think are the key steps of the proof?
8. Using the method of this proof, show that for any prime number n, $\sqrt{n}$ is an irrational number.