Universidade do Minho
Escola de Ciências

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Emergence and Self-organization of Cooperation

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Emergence and Self-organization of Cooperation

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do
Professor Doutor Francisco C. Santos
e da
Professora Doutora Marta M.D. Ramos

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DECLARAÇÃO DE INTEGRIDADE

Declaro ter atuado com integridade na elaboração da presente tese. Confirmo que em todo o trabalho conducente à sua elaboração não recorri à prática de plágio ou a qualquer forma de falsificação de resultados.

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ABSTRACT

Emergence and self-organization of cooperation

This dissertation reports the main work I developed during my Ph.D. program. It contains my contributions to the field of population dynamics and a study of a global problem of cooperation.

Evolutionary game theory (EGT) and stochastic population dynamics have proven to be powerful tools to describe frequency-dependent dynamics in evolutionary biology. More recently, EGT has become increasingly popular in the study of social settings and conflict resolution among humans, calling for an extension of the initial framework towards the boundless complexity of human reasoning. Here, I discuss the necessity of introducing different levels of rationality and prospective strategies, proving under which circumstances the equations that govern both rational and rationally-bounded players coincide. Moreover, since decision-making often relies in a continuum of possible options, I propose a novel equation to describe the evolution of populations with a continuum of strategies, analyzing why and when we can discretize the sets of possible strategies. Finally, when finite populations and stochastic effects are considered, the increasing population size or increasing number of individual configurations rapidly renders the analysis of stationary states prohibitive. Here, I also discuss a novel framework that allows us to define a hierarchy of approximations to the stationary distribution of any population dynamics described by a Markov process, overcoming the limitations of existing approaches. These results and methods are general in the sense that they are applicable to the study of different dilemmas and their respective game-theoretical representation.

In the last part of this dissertation, I focus on problems related with global coordination for the preservation of a common good, such as climate change governance. Indeed, preventing global warming requires overall cooperation. Contributions will depend on uncertainty of future losses, which plays a key role in decision-making. Here, I discuss an evolutionary game theoretical model – and its stochastic dynamics in finite populations – in which decisions within small groups under high risk and stringent requirements toward success are shown to significantly raise the chances of coordinating to save the planet's climate. This result calls for a decentralized or polycentric way of coordinating efforts to tame the planet's climate.

I further discuss whether a polycentric structure of multiple small-scale sanctioning institutions provides a viable solution to solve global dilemmas. Such structure is shown to help deterring non-cooperative behavior (when compared with a single global institution), even though it suffers, to a smaller extent, from most of the same problems as the top-down approach: sensitivity to risk perception and to overall uncertainty. Furthermore, I also discuss how world's wealth inequality may influence the outcome of this type of collective dilemmas, studying how the segregation between rich and poor players harms cooperative behavior, even if rich tend to, at first, compensate for contributions (or lack of them) from the poor. Finally, I discuss in which conditions the establishment of pre-play contracts may help to overcome part of these problems. The results indicate that contracts are more effective if voluntary and more prevalent if small, acting as a costly signaling mechanism for a naturally cooperative group of individuals sharing common goals. This, in turn, if combined with some partnership advantages, creates more incentives to join, allowing both cooperation and the total membership to grow.

Keywords: Emergence of cooperation; Evolution of Cooperation and Institutions; Collective action; Tragedy of the commons; Climate change; Complex systems; Evolutionary game theory; Evolutionary dynamics.

RESUMO

Emergência e auto-organização da cooperação

Esta dissertação é uma coletânea do principal trabalho desenvolvido durante o meu doutoramento. Contém as minhas contribuições para o ramo da dinâmica de populações e o estudo de um problema global de cooperação.

A Teoria de Jogos Evolutiva (EGT) e a dinâmica estocástica de populações são identificadas como ferramentas poderosas para descrever a dinâmica evolutiva em Biologia Evolutiva. Mais recentemente, a EGT tem-se tornado mais popular no estudo de sistemas sociais de resolução de conflitos entre humanos pedindo por uma extensão das ferramentas originais de forma a acomodar a grande complexidade humana. Nesta dissertação, eu discuto a necessidade de introduzir diferentes níveis de racionalidade e estratégias que recorrem a previsões, mostrando em que circunstâncias as equações que governam estratégias racionais e com racionalidade limitada coincidem. Além disso, uma vez que a tomada de decisão muitas vezes incide num contínuo de estratégias possíveis, proponho uma nova equação para descrever a evolução de populações com um contínuo de estratégias. Finalmente, quando as populações são finitas e são considerados os seus efeitos estocásticos, o aumento do tamanho da população ou do número de configurações individuais possíveis rapidamente torna impraticável a análise de estados estacionários. Aqui, eu também discuto uma nova ferramenta que permite definir uma hierarquia de aproximações para a distribuição estacionária de qualquer dinâmica de populações descrita por um processo de Markov, ultrapassando as atuais limitações. Estes resultados e métodos são gerais, no sentido de serem aplicáveis ao estudo de diferentes dilemas e da respetiva representação em termos de teoria de jogos.

Na última parte desta dissertação, foco-me em problemas relacionados com a coordenação global para a preservação de um bem comum, como a prevenção das alterações climáticas. De facto, a prevenção do aquecimento global requer cooperação a nível global. Contudo, as contribuições vão depender da incerteza sobre as perdas futuras, o que joga um papel crucial na tomada de decisão dos responsáveis. Aqui discuto um modelo de EGT – e os seus efeitos estocásticos em populações finitas – com o qual mostro que as hipóteses de coordenação para salvar o clima do planeta aumentam significativamente se as decisões forem tomadas no seio de pequenos grupos sobre problemas locais que, por um lado, reflitam menor incerteza e, por outro, onde os requisitos para a tomada de ação possam ser apertados. Este resultado pede uma forma de coordenar os esforços para domar o clima do planeta que seja descentralizada, ou policêntrica.

Ainda nesta parte, discuto se uma estrutura policêntrica de múltiplas instituições para sancionar comportamentos de pequena escala providencia uma solução viável para resolver problemas globais. Mostro que essa estrutura ajuda a prevenir comportamentos não cooperativos (quando comparada com uma única instituição global), mesmo que sofra, em menor escala, dos mesmos problemas da alternativa top-down: sensibilidade à perceção do risco de desastre e incerteza, em geral. Além disso, também discuto como é que a desigualdade de capacidade contributiva no mundo pode influenciar o resultado deste tipo de dilemas coletivos, estudando como é que a segregação entre jogadores ricos e pobres prejudica a cooperação, mesmo que os ricos, a princípio, tendam a compensar a falta de contribuições dos pobres. Finalmente, discuto em que condições a criação de contratos pode ajudar a ultrapassar parcialmente estes problemas. Os resultados indicam que os contratos são mais eficientes se voluntários e mais prevalentes se entre poucos membros, funcionando como um mecanismo de sinalização com custo para grupos de indivíduos naturalmente cooperativos. Isto, por sua vez, combinado com vantagens intra-contrato, cria mais incentivos para novas adesões o que torna possíveis o aumento tanto da cooperação como do número de membros.

Palavras Chave: Emergência da Cooperação; Evolução da Cooperação e Instituições; Ação Coletiva; Tragédia dos Comuns; Alterações Climáticas; Ciências da Sustentabilidade; Redes Complexas; Sistemas Complexos; Teoria de Jogos Evolutiva; Dinâmica Evolutiva.

Há um tempo em que é preciso abandonar as roupas usadas, que já têm a forma do nosso corpo, e esquecer os nossos caminhos, que nos levam sempre aos mesmos lugares. É o tempo da travessia: e se não ousarmos fazê-la, teremos ficado para sempre à margem de nós mesmos.

— Fernando Teixeira de Andrade, in *O Medo: o Maior Gigante da Alma*

PUBLICATIONS

The content of this dissertation is inspired and borrows directly from some of the work I have developed with my collaborators during my Ph.D. studies. Publications with a • are reprinted in the appendix as they contribute directly to the contents of this thesis, while clearly complementing the main text.

- • Francisco C. Santos, Vítor V. Vasconcelos, Marta D. Santos, P. N.B. Neves, and Jorge M. Pacheco. «Evolutionary Dynamics of Climate Change Under Collective-Risk Dilemmas.» In: Mathematical Models and Methods in Applied Sciences (M3AS) 22.Supplementary Issue 1 (2012), p. 17.
- • Vítor V. Vasconcelos, Francisco C Santos, and Jorge M Pacheco. «A bottom-up institutional approach to cooperative governance of risky commons.» In: Nature Climate Change 3.9 (2013), pp. 797–801.
- ◇ Vítor V. Vasconcelos, Flávio L. Pinheiro, Francisco C. Santos, and Jorge M. Pacheco. «Bootstrapping back the climate with self-organization.» In: Advances in Artificial Life, ECAL. Vol. 12. 2013, pp. 182–183.
- ◇ Jorge M Pacheco, Vítor V. Vasconcelos, and Francisco C Santos. «Climate change governance, cooperation and self-organization.» In: Physics of life reviews 11.4 (2014), pp. 573–586.
- ◇ Jorge M Pacheco, Vítor V. Vasconcelos, and Francisco C Santos. «Climate governance as a complex adaptive system Reply to comments on "Climate change governance, cooperation and self-organization".» In: Physics of life reviews 11.4 (2014), pp. 595–597.
- • Vítor V. Vasconcelos, Francisco C Santos, Jorge M Pacheco, and Simon A Levin. «Climate policies under wealth inequality.» In: Proceedings of the National Academy of Sciences 111.6 (2014), pp. 2212– 2216.
- ◇ Phillip M. Hannam, Vítor V. Vasconcelos, Simon A. Levin, and Jorge M. Pacheco. «Incomplete cooperation and co-benefits: deepening climate cooperation with a proliferation of small agreements.» In: Climatic Change (2015), pp. 1–15.
- ◇ Vítor V. Vasconcelos, Francisco C Santos, and Jorge M Pacheco. «Cooperation dynamics of polycentric climate governance.» In: Mathematical Models and Methods in Applied Sciences 25.13 (2015), pp. 2503– 2517.
- ◇ Vítor V. Vasconcelos, P. Fernando Santos, Francisco C. Santos, and Jorge M. Pacheco. «Stochastic dynamics through hierarchically embedded Markov chains (submitted).» In: Physical Review Letters (2016).

C'est le temps que tu as perdu pour ta rose qui fait ta rose si importante.

— Antoine de Saint-Exupéry, in *Le Petit Prince*

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Finally, ou melhor, por último, tenho de agradecer aos meus pais por nunca se terem poupado a esforços para que eu fosse feliz, para que estudasse e para que o fizesse tranquilamente. A eles, ao Ricardo, ao Gabriel e também à Sandrina pelos momentos familiares, de perspectiva do passado e do futuro, por serem o porto seguro onde posso sempre regressar.

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CONTENTS

Part I

INTRODUCTION

The foundations are a necessary piece of all that is to be developed.

1 INTRODUCTION

MY GOAL IS TO UNDERSTAND THE EMERGENCE AND EVOLUTION OF COOPERATIVE BEHAVIOR IN ORDER TO PROMOTE OVERALL COOPERATIVE RESPONSES FROM INDIVIDUALS THAT FACE SOCIAL DILEMMAS OPPOSING SELF AND COLLECTIVE INTEREST. Interactions are the center of many disciplines. In Physics, they are not only the core of research but also the basis of its scientific knowledge. Gravity, electromagnetism, weak and strong interactions, together with all the other effective forces derived from those, are part of, and many times define, the fields of Physics. On one hand, they can serve as means of studying the behavior of the elements in which they act upon but they can also be the object of study themselves. At a different level, both natural ecosystems and human economies are also structured by the interactions between their elements [136]. Typically, in these, the interactions in question are ones of competition: in the first case, the organisms that are better adapted to the environment grow or reproduce and replace those that do not and, in the second one, individuals or companies with the most successful formulas thrive and grow to take over those that are less successful. However, many other interactions act upon these complex systems and, again, they can be used to study and model the behavior of individuals and organisms or they can be the target of the study themselves.

Physics is not only about Physics anymore. Deeply, it is about organization, an exploration of the laws of pure form.[15]

Due to this common ground, modeling techniques are shared across these different scales, from atoms to world's economies. Many times, even the same models are applicable to diverse topics, when taken with different interpretation. This means that, no matter the scale, the insights obtained in the search of a solution of two apparently completely different problems can, if not provide immediate answers, at least pave way to the development of one another. In this dissertation, I will turn my attention to frameworks, methods, and models developed in the context of social and biological interactions. In particular, the elements of which system I want to study are individuals and their behavior. I will refrain from dwelling into the similarities and bridges across methods and models in different subjects, as they can either be completely clear, as in the case of, for instance, spin alignment and individuals imitation of behaviors in a network, or they can be hard to grasp, sometimes through some non-intuitive change of variables, or can be even unknown. This, of course, would be a hard and interesting exercise in itself, which, however, would deviate me from my goal.

THE STRUCTURE OF MY DISSERTATION is accomplished in three parts. In the first part, I lay down the frameworks used to study such complex systems in a general fashion. I start with the simplest equation that has been used to study the dynamics of behavior adoption, review where it arises from and use it as a motivation and reference frame. In the second part, I discuss the underlying assumptions, in this way creating new frameworks that either justify them or allow for their advancement. Lastly, in the third part, I will take these frameworks, equations and methods to explore the particular systems I am interested in, and over which I have raised my actual theses – and over which I have said little to nothing so far.

The prototypical scenario over which the solution of a global problem is directly against individual self-interest is that which relates to the mitigation of climate change effects. In fact, it has been 7 years since the United Nations Secretary-General said, before the Copenhagen summit in 2009, to nearly 100 world leaders: "There is little time left. The opportunity and responsibility to avoid catastrophic climate change is in your hands".

In a dance that repeats itself cyclically, countries and citizens raise significant expectations every time a new International Environmental Summit is settled. Unfortunately, few solutions have come out of these colossal and flashy meetings. This represents a challenge to our current understanding of models on decision-making: more effective levels of discussion, agreements and coordination must become accessible. From Montreal and Kyoto to Paris summits, it is by now clear how difficult it is to coordinate efforts [4, 5].

Climate is a public good, and, probably, the welfare of our planet accounts for the most important and paradigmatic example of a public good: a global good from which every single person profits, whether she contributes or not to maintain it. However, these summits failed to recognize the well-studied difficulties of cooperation in public-good games [47, 60, 88]. Often, individuals, regions or nations opt to be free riders, hoping to benefit from the efforts of others while choosing not to make any effort themselves. Most cooperation problems faced by humans share this setting, in which the immediate advantage of free riding drives the population into the tragedy of the commons [47], the ultimate limit of widespread defection [11, 28, 47, 60, 74, 77, 88, 122]. When dealing with such an essential public good as climate, many efforts need to be made to avoid this, being a major concern to countries. Indeed, efforts ought to be shared between all and balanced measures should then be taken. The strive to identify and improve the mechanisms that allow this will be the goal of this work.

One of the multiple fatal flaws often pointed to such agreements is a deficit in the overall perception of risk of widespread future losses, in particular the perception by those occupying key positions in the overall political network that underlies the decision process [54, 77, 115]. Another problem relates to the lack of sanctioning mechanisms to be imposed on those who do not contribute (or stop contributing) to the welfare of the planet [5, 91, 101]. Moreover, agreeing on the way punishment should be implemented is far from reaching a consensus, given the difficulty in converging on the *pros* and *cons* of some procedures against others. Many possibilities have been under consideration - from financial penalties, trade sanctions, to emissions penalties under future climate change agreements

– but their details have not been well established and negotiations are usually slow and difficult [10]. A deadlock over these measures is expected since their consequences do not have a solid theoretical or even experimental background.

To address this and other cooperation conundrums, ubiquitous at all scales and levels of complexity, the last decades have witnessed the discovery of several core mechanisms responsible to promote and maintain cooperation at different levels of organization [32, 33, 45, 47, 60, 76, 80–82, 90, 110, 119, 122, 149, 150].

Most of these key principles have been studied within the framework of two-person dilemmas, such as the Prisoner's Dilemma, which constitute a powerful metaphor to describe conflicting situations often encountered in the natural and social sciences. Many real-life situations, however, are associated with collective action based on joint decisions made by a group involving more than two individuals [47, 60, 90, 135]. These types of problems are best dealt-with in the framework of N-person dilemmas and Public Goods games, involving a larger complexity that only recently started to be unveiled [39, 49, 60, 96, 116, 118, 119, 128].

In this dissertation, reproducing the works my coauthors and I have developed, I will model the decision making process as a dynamical process, in which behaviours evolve in time [104, 119], taking into consideration decisions and achievements of others, which influence one's own decisions [16, 34, 103]. We implement such behavioural dynamics in the framework of Evolutionary Game Theory, in which the most successful (or fit) behaviours will tend to spread in the population. This way, one is able to describe strategic interactions between individuals, complemented by evolutionary (dynamical) principles. In particular, I will focus on the work done in finite populations, where such fitness driven dynamics occurs in the presence of errors (leading to stochastic effects), both in terms of errors of imitation [138] as well as in terms of behavioural mutations [141], the latter accounting for spontaneous exploration of the possible behaviors available. I expect that by the end of this dissertation it is clear that the emergence of overall cooperative behavior, specially in this prototypical example of climate change mitigation, is full of subtleties that make it hard to happen, but its chances can be ameliorated if one faces the problem as a complex system and tackles it from the bottom-up.

2 FRAMEWORK

Most of the phenomena in nature evolve in an intricate way. For sure, individual decision making processes are so deeply related to the ways of our brain – the most complex kilogram of matter in the universe, as a teacher of mine used to say – that models have no hope in computing all variations and variables in detail. Even if one would succeed, the results would be so cumbersome – like having all trajectories of all electrons of everyone's brain over time – that, by themselves, would be useless. Often, what is of use is how average properties evolve and are established and, often, these are described by much simpler laws. For this reason, I will make use of a variety of concepts related to the theory of probability and statistics.

In this chapter the framework used throughout the text is presented. Most of the notation is defined also here. From the deterministic replicator equation to stochastic processes and the M-equation, this chapter sets itself as an overview and collection of concepts and results that are relevant further on.

2.1 GROWTH AND COMPETITION

Growth and competition are many times synonyms. They happen in nature or economic contexts that are characterized by a limiting resource, may it be time, investment, space or nutrients. [64, 94, 136]. However, let us think of competition in a more general way. Let me start with the simplest model of growth I can think of. Imagine a population, A, of Z_A individuals that have a growth rate f_A. Many times, this growth rate is called fitness and is directly related (and is many times defined resorting to) the equation

$$\frac{dZ_A}{dt} = Z_A f_A. \tag{1}$$

Clearly, f_A can be extremely complex, dependent on Z_A and other external variables. Furthermore, usually, individuals of a given population, characterized by a given trait, are not alone, and they may compete or simply be considered as part of a larger population. Similarly, a population B, with some trait different of that of A, will also have its own fitness, f_B and its growth can be described by an equation identical to Eq.(1). In this case, if we look at the fraction of individuals with a given trait, say A, on the totality of individuals, $Z_A + Z_B$, we can look at the variation of $x_A = \frac{Z_A}{Z_A+Z_B}$ and get that

Competition does not necessarily mean interaction. It is not more than a balance of powers that are being measured in the same scale.

$$\frac{dx_A}{dt} = x_A(f_A - \langle f \rangle) = x_A(1 - x_A)(f_A - f_B), \tag{2}$$

where $\langle f \rangle$ is the average growth-rate of the population of the distribution of individuals. Eq.(2) is the so-called *replicator equation* with the first equality being generalizable to any number of different traits: A, B, C, etc.. Here, even though the traits are not directly competing, the nature of the equation is such that the faster rate of growth of one trait results in a decrease in the fraction of the other trait. Notice that this happens even if both traits are increasing in number (or decreasing). The competition is for the fraction of elements in the total population.

This equation will be of foremost importance in all the discussion that follows and we will have the chance to see it derived in other contexts, other than growth, with different assumptions and as limiting case of other equations.

One of the first aspects that may turn the replicator equation inviable is the fact that not all properties grow continuously. In fact, the number of individuals, which was the motivation in this derivation, is a discrete property and, thus, may not be appropriately described by it in some cases. To treat the discrete case, we need to move on to the study of stochastic processes, which we will in **Section** 2.3. However, before doing so, in the next section I will introduce the basic concepts of *game* and *Nash-equilibrium* and how these relate to population dynamics.

2.2 GAME THEORY AND ITS POPULATION WIDE COUNTERPART

Game theory is a wide term to classify the science that studies the behavior of decision makers, usually called *players*, from a rational viewpoint [1]. It is often used in Economics in the pursuit of optimal behavior from the purely strategic perspective. It consist in the study and development of models of conflict and cooperation that are competition problems, which, as we have seen, reach far beyond human competition.

The simplest of those models, called *game*, is a 2-person game, in which two players interact with each other and have to decide to act according to one of two strategies, C, typically associated with cooperation, and D, with defection. Their outcome, or *payoff*, is given the matrix bellow

$$\begin{array}{c} \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cc} C & D \end{array} \\ \begin{bmatrix} R & S \\ T & P \end{bmatrix} \end{array}$$

where the values are for the player playing the strategy represented on the line against an opponent playing the strategy on the column. If the game is symmetric, the same matrix describes both players.

Originally game theory was developed as a theory of human strategic behavior based on an idealized picture of rational decision making. It often concerns the search of equilibria of a mistake-free population.

Indeed, one of the most studied scenarios is the one called *Prisoners' dilemma*. This game is characterized by a conflict between self-interest and global optimum: i) the best move is always to defect but ii) players are worse of by defecting than if they would both cooperate. From i) we have $T > R$ and $P > S$ and ii) implies $R > P$. The fact that no player can unilaterally change strategy to be better of makes the configuration where both players play D a Nash-equilibrium.

Definition 1 Nash-equilibria

Consider a game with N players, where $S^{(i)}$ is the strategy space of player i, and $f(x) = (f_1(x), \dots, f_N(x))$ is its payoff function evaluated at $x \in S$, S being the set of strategies space. Let $x^{(i)}$ be the current strategy of player i and $x^{(\setminus i)}$ be the set of strategies of all players except for player i. A strategy profile $x^* \in S$ is a Nash equilibrium (NE) if no unilateral deviation in strategy by any single player is profitable for that player, that is:
$\forall i, x^{(i)} \in S^{(i)} : f_i(x^{*(i)}, x^{*(\setminus i)}) \geqslant f_i(x^{(i)}, x^{*(\setminus i)})$.

Depending on the relative values of the parameters in the 2-player matrix, different games (and dilemmas) can arise. To analyze them, let us move on from the idea that i) players are purely rational and that ii) they play these games once and isolated. For this, we will enter in the realm of population dynamics, more specifically, of Evolutionary Game Theory.

Evolutionary garne theory arises in Biology does not rely on rationality assumptions but on the idea that the Darwinian process of natural selection drives the population dynamics when fitness derives from the game indivudals play.

Evolutionary game theory applies the concept of games into population dynamics. Suppose we have a very large population. The individuals in that population will interact with each other and receive a payoff in each interaction, e. g. given by the matrix above. Let us say that x is the fraction of individuals playing C in that population and $1 - x$ the fraction of those that play D. Then, given that players interact in pairs and with all others with the same probability, the average payoff each strategy gets can be written as

$$f_C = xR + (1 - x)S \tag{3}$$
$$f_D = xT + (1 - x)P. \tag{4}$$

In well-mixed populations, 2-person games give rise linear fitness and linear fitness differences.

As time evolves, players will tend to adopt different strategies, in particular through imitation. If we assume that at any given time all players are equally likely to change strategy, the probability that a C changes strategy is proportional to the number of Cs, x, and the probability that he looks at a D to imitate is proportional to the number of Ds, $1 - x$. Finally, if the C player's fitness is larger than that of a D, the number of Cs increases, and it decreases otherwise. With this simple ingredients, we can write

$$\dot{x} = x(1 - x)(f_C - f_D) = x(1 - x)(x(R - T) + (1 - x)(S - P)) \tag{5}$$

which is exactly the replicator equation we saw last section. A more formal and detailed description will be given in next section. Notice that if both $R - T$ and $S - P$ are negative, as in the Prisoners' dilemma, the whole population will go into full defection. If that is not the case, three other topologically different scenarios can emerge, with the right hand side of Eq.(5) fully characterizing the dynamics and being called *gradient of selection*. **Figure** 1 illustrates those four scenarios: Prisoners'dilemma (PD), characterized by a dominance of D, Snowdrift (SD) characterized by a coexistence

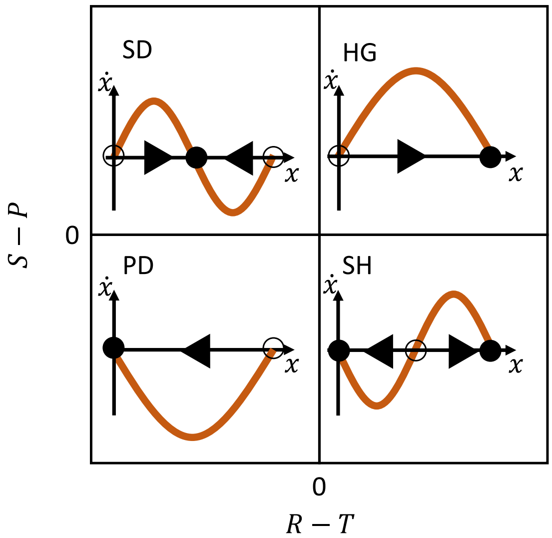


Figure 1: **Linear fitness differences game dynamics** The four different classes of games that result from symmetric 2-player games where $R > P$, depending on two parameters, $S - P$ and $R - T$ and their sign. The full dots represent stable equilibria of the system while the open dots represent unstable equilibria.

of both strategies, the Stag Hunt (SH) characterized by a coordination between both strategies and finally, but less interesting since it does not pose a dilemma, the Harmony Game (HG) where the whole population will always end up in the C state which is also a global maximum [22, 27, 118, 122].

We will be studying in more details both the SD and the SH in **Chapter** 6 and we will recover the PD many times.

2.3 STOCHASTIC PROCESSES

Individuals in large populations are often named, or even categorized, according to their preferences, behaviors, physical and psychological resemblance and many other characteristics that make people alike. This can happen for the most various reasons, from hierarchisation to discrimination, but is often a way to simplify the line of thought by means of disregarding individuality. In social problems, people, nations, leaders and other social entities can be classified according to their acts, to their strategy in some problem.

Throughout this dissertation, I will make use of a crucial assumption that will render all this setup possible: a *well-mixed*, or *mean-field*, approximation. This means that the outcome of an individual interaction is averaged over all possible interactions. In turn, because all players interact with all other equally, this allows one to fully classify and individual by his/her strategy – effects of personal history of interaction, position on a not complete graph, or other heterogeneous factors, are disregarded. Nonetheless, the effects of different sources of heterogeneity in players can sometimes be mapped into the well-mixed scenario with the game parameters changing into (different) effective parameters, making it a good frame of reference.[83, 93, 111, 113, 117]

The interference of diverse complex behaviors, such as those present in human interaction, may actually render the description and evolution of average salient properties compreensible. For this, a variety of concepts related to the statistical mechanics of non-equilibrium stochastic processes is required.

Therefore, it is convenient to consider a set of Z elements, each of these can be in one of S states. I will think of the elements of this system as the individuals of a population who can adopt different strategies: $\sigma_1, ..., \sigma_S$. Let $i_k(t)$ be, at a given time t, the number of individuals with the strategy σ_k, where $k = 1, ..., S$. The number of individuals of a given type will randomly evolve in time according to some rule. To study these quantities I need the proper framework; probability theory has the right objects.

Let i be a *random* or *stochastic variable* defined by a set Ω_i and a function $P_i(x)$. Ω_i represents the set of possible states of i; I will be calling it indiscriminately "phase space", "domain" or, ususally as a reference to its graphical representation, "simplex". $P_i(x)$ is a probability distribution defined over Ω_i. However, when studying a system, one may need more than just a single snapshot of $P_i(x)$. One is usually interested in describing its evolution in time. In order to do so, one needs more than one stochastic variable. Suppose I build a set according to Def.(2). It depends on the random variable i and on a parameter, t, that represents time.

Definition 2 Stochastic Process

Let Y be a a set of stochastic variables indexed by a parameter, t: i_{t_1}, i_{t_2}, ..., i_{t_m}. If t represents time, Y is a Stochastic Process. Whenever t does not represent time, Y is called a *random function*.

This set can be written in a compact way which easily takes into account all possible values of the indexing parameter: $Y_i(t)$. To cut out notation, often the set is named after the stochastic variable, $i_i(t)$ or, simply, $i(t)$.

All i_k, mentioned before as the number of individuals with a given strategy, perfectly fit into this definition and so can be treated as a stochastic process. Furthermore, note that

We will be looking at bounded phase-spaces.

$$i_1 + ... + i_S = Z. \tag{6}$$

Since Eq.(6) determines, for example, i_S in terms of the remaining i_k, I only need to retain $d = S - 1$ of them, making d the dimensionality of phase-space. Strictly speaking, one defines $\mathbf{i}(t) = \{i_1, ..., i_d\}$ as multivariable stochastic process over an d-dimensional sample space, $\Omega_{\mathbf{i}}$, given in Eq.(7).

$$\begin{aligned} &i_1 \in \{0, 1, ..., Z\} \\ &i_2 \in \{0, 1, ..., Z - i_1\} \\ &... \\ &i_d \in \{0, 1, ..., Z - i_1 - ... - i_{d-1}\} \end{aligned} \tag{7}$$

Within the framework of stochastic processes, there is a specific subclass of processes called Markov processes. Being important, Markov processes are, by far, the most well known and used stochastic processes. In part due to their manageability but, ultimately, because any isolated system is a Markov process once one considers all microscopic variables. Evidently, this is not always possible or even desirable. The task is to find a *small* collection of variables with the Markov property at a given time scale.

These are formally defined in Def.(3), which states that the conditional probability of a future event at t_n, $T_{1|n}$, is independent of the knowledge of the values at times earlier than t_{n-1}. Informally, one can say that the information required to compute future statistical properties depends only on the actual state; the system has no memory.

Definition 3 Markov Property

Let $\boldsymbol{i}(t)$ be a Markov process, indexed by a set of n successive times, i.e. $t_1 < t_2 < ... < t_n$. The conditional probability of getting $\boldsymbol{i}_n$ at t_n, given the set of observed values $\boldsymbol{i}_{n-1}$ at $t_{n-1}, \dots, \boldsymbol{i}_1$ at t_1 is given by:

$$T_{1|n-1}(\boldsymbol{i}_n, t_n | \boldsymbol{i}_1, t_1; ...; \boldsymbol{i}_{n-1}, t_{n-1}) = T_{1|1}(\boldsymbol{i}_n, t_n | \boldsymbol{i}_{n-1}, t_{n-1}).$$

The evolution of a population need not be a purely random process: there may be mechanisms that generate,e. g., a global tendency for individuals. From a practical standpoint, the Markov property is the simplest way of introducing statistical dependence into the models built and, hence, such tendency.

For example, individuals that are part of a population with a given configuration of strategies can opt to change their own whenever they feel their outcome is not the best: they can compare themselves with the present situation and choose a better strategy. Therefore, one supposes that, at a give state, the evolution of the system depends only on the present configuration so that $\mathbf{i}(t) = \{i_1, ..., i_d\}$ is a Markov process.

Fortunately, we will be able to numerically solve for the stationary distribution of the M-Equation in most cases. When the number of configurations grows too large, and that is no longer directly possible, the method proposed by my coauthors and I, which I show in part II, allows one to hierarchicaly obtain better estimations of the solution.

The study of a Markov process often consists in determining its probability density function (PDF) evolution, $P_{\mathbf{i}}(t)$. Since $\mathbf{i}(t)$ has the Markov property, its transition probability $T_{1|1}$ respects the discrete time M-Equation, Eq.(8), and consequently, with a delta-shaped initial condition, its PDF also respects it [57]. This is a gain-loss equation that allows one to compute $P_{\mathbf{i}}(t)$ given the transition probability from the configuration $\mathbf{i}$ to the configuration $\mathbf{i}'$, $T_{\mathbf{i}'\mathbf{i}}$.

$$P_{\mathbf{i}}(t+\tau) - P_{\mathbf{i}}(t) = \sum_{\mathbf{i}'} \{T_{\mathbf{i}\mathbf{i}'} P_{\mathbf{i}'}(t) - T_{\mathbf{i}'\mathbf{i}} P_{\mathbf{i}}(t)\} \tag{8}$$

As a result, the problem of modeling this system is reduced to the computation of $T_{\mathbf{i}\mathbf{i}'}$. Furthermore, making the left side zero, in the search for a stationary $P_{\mathbf{i}}(t)$, one falls into an eigenvector search problem [57].

2.4 FROM THE M-EQUATION TO THE LANGEVIN EQUATION

Before I proceed with modeling $T_{i'i}$, I would like to give further insight on its meaning and derive some related quantities. Therefore, in this section I will review some important equations together with their interpretation, which will be of use later on. By the end of the section, I will derive these equations, in the framework presented so far, in order to relate their well known quantities to $T_{i'i}$ and these to the replicator equation.

The M-equation, Eq.(8), can be rewritten in an equivalent formulation, the Kramers-Moyal (KM) expansion, Eq.(9) [57]. The functions $D^{(n)}_{i_1,\ldots,i_n}(\mathbf{x},t)$ characterize the process and each one is called the n-th Kramers-Moyal coefficient.

$$\frac{\partial P}{\partial t}(t) = \sum_{n=1}^{+\infty}(-1)^n \sum_{i_1,\ldots,i_n}^{d}\left[\prod_{l=1}^{n}\frac{\partial}{\partial x_{i_l}}\right] D^{(n)}_{i_1,\ldots,i_n}(\mathbf{x})P(\mathbf{x},t) \tag{9}$$

This is an equation for the time evolution of the PDF, $P(t)$, of an d-dimensional continuous Markov process, $\mathbf{X}(t)$. It often allows one to make use of perturbation theory and, consequently, the effect of its successive terms can be more easily studied. Hence, the equation containing only the first two terms is well discussed in the literature and is called the Fokker-Planck Equation (FPE), Eq.(10). Pawula's theorem reinforces the study of this equation stating that stochastic processes obey this equation not only as an approximation but exactly as long as one of the even KM coefficients is zero [105].

$$\frac{\partial P}{\partial t}(t) = -\sum_{i=1}^{d}\frac{\partial}{\partial x_i}\left[D^{(1)}_i(\boldsymbol{x},t)P\right] + \sum_{i=1}^{d}\sum_{j=1}^{d}\frac{\partial^2}{\partial x_i \partial x_j}\left[D^{(2)}_{ij}(\boldsymbol{x},t)P\right] \tag{10}$$

The first KM coefficient, $D^{(1)}_i(\boldsymbol{x},t)$, is called the Drift, which we will see matches the gradient of selection, $g(x,t)$, and the second, $D^{(2)}_{ij}(\boldsymbol{x},t)$, the Diffusion.

Associated with the FPE (10) is a system of S coupled Itô-Langevin equations, which can be written as [37, 105]

The Langevin equation is the simplest way to interpret the evolution of a stochastic system.

$$\frac{d\mathbf{X}}{dt} = \mathbf{h}(\mathbf{X}) + \mathbf{G}(\mathbf{X})\boldsymbol{\Gamma}(t) \tag{11}$$

Here, $\boldsymbol{\Gamma}(t)$ is a set of d normally distributed random variables fulfilling

$$\langle \Gamma_i(t)\rangle = 0, \quad \langle \Gamma_i(t)\Gamma_j(t')\rangle = 2\delta_{ij}\delta(t-t') \quad . \tag{12}$$

These equations drive the stochastic evolution of $\mathbf{X}(\mathbf{t})$. The vectors $\mathbf{h}$ and the matrices $\mathbf{G} = \{g_{ij}\}$ for all $i, j = 1, \ldots, s$ are connected to the local Drift and Diffusion function through

$$D_i^{(1)}(\mathbf{X}) = h_i(\mathbf{X}) \quad \text{and} \tag{13}$$

$$D_{ij}^{(2)}(\boldsymbol{X}) = \sum_{k=1}^{d} g_{ik}(\boldsymbol{X}) g_{jk}(\boldsymbol{X}) \quad . \tag{14}$$

While the FPE describes the evolution of the joint distribution of the d variables statistically, the system of Langevin equations in Eq.(11) models individual stochastic trajectories of a system. In Eq. (11) the term $\mathbf{h}(\mathbf{X})$, related to the Drift, contains the deterministic part of the macroscopic dynamics, while the functions $\mathbf{G}(\mathbf{X})$, related to Diffusion, account for the amplitudes of the stochastic forces mirroring the different sources of fluctuations due to all sorts of microscopic interactions within the system.

Notice, however, that the $d \times d$ matrix $\mathbf{G}$ cannot be uniquely determined from the symmetric diffusion matrix $\mathbf{D}^{(2)}$ for $s \geqslant 2$: the number of unknown elements in $\mathbf{G}$ exceeds the number of known elements in $\mathbf{D}^{(2)}$ leading to $d^2 - \frac{1}{2}d(d+1) = \frac{1}{2}d(d-1)$ free parameters. However, a simple method can be used to obtain $\mathbf{G}$ from $\mathbf{D}^{(2)}$ in such a way that a Langevin equation can be extracted from a FPE, but is not unique [58, 144]. Symbolically I will write this particular $\mathbf{G}$ as $\sqrt{\mathbf{D}^{(2)}}$. Furthermore, in general, the eigenvalues of these matrices indicate the amplitude of the stochastic force and the corresponding eigenvector indicates the direction toward which such force acts, in the Langevin point of view. Even more interesting features, however, can be extracted from the eigenvalues and eigenvectors.[144].

Now with some intuition on these equations, I will perform a Kramers-Moyal expansion in order to identify the KM coefficients of our system, namely the Drift and Diffusion – this is a classical procedure that can be found in many textbook of statistical mechanics[57, 105] but it is useful to set notation and to bring about results that will be of use. I will be able to prove that, in the dynamics of the fraction of traits in a population, the coefficients are sequentially smaller and therefore justify the analysis using the FPE and its Langevin interpretation.

The evolution of the fraction of individuals of a given trait is well described by a Langevin equation. The evolution of its distribution can be approximated by a Fokker-Planck equation. That will be of use and will help our interpretation of the results.

Consider that all transition probabilities and statistical properties defined so far as functions of the number of individuals of the different strategies, $\mathbf{i}$, are redefined as functions of the fraction of individuals in the total population, $\mathbf{X} = \mathbf{i}/Z$. The changes and notation are introduced in Eqs.(15).

Configurations are described by $\mathbf{x}$, with a PDF $\rho(\mathbf{x}, t)$, and $T^{\delta}(\mathbf{x})$ represents a transition from configuration $\mathbf{x}$ in the direction $\boldsymbol{\delta}$ such that it gets to configuration $\mathbf{x}' = \mathbf{x} + \boldsymbol{\delta}$.

$$x_k \equiv \frac{i_k}{Z} \tag{15a}$$

$$P_{\mathbf{i}}(t) \to p(\mathbf{x}, t) \tag{15b}$$

$$T_{\mathbf{i}'=\{i_1+\Delta_1,\ldots,i_d+\Delta_d\}\mathbf{i}=\{i_1,\ldots,i_d\}} \to T^{\delta}(\mathbf{x}) \tag{15c}$$

$$\rho(\mathbf{x}, t) \equiv Z^d p(\mathbf{x}, t) \tag{15d}$$

Finally, we can rewrite Eq.(8) in terms of the PDF of $\mathbf{X}$, $\rho(\mathbf{x}, t)$, using $\Delta\rho(\mathbf{x}, t) = \rho(\mathbf{x}, t+\tau) - \rho(\mathbf{x}, t)$ as follows.

$$\begin{aligned} \Delta\rho(\mathbf{x}, t) = \sum_{\delta \neq 0} \Big[&\rho(\mathbf{x}+\boldsymbol{\delta}, t) T^{-\delta}(\mathbf{x}+\boldsymbol{\delta}) - \\ &-\rho(\mathbf{x}, t) T^{\delta}(\mathbf{x}) \Big] \end{aligned} \tag{16}$$

With a population model in mind, it is reasonable to assume that, as the population increases, the frequency in which an individual of some type changes strategy also increases with Z. Thus the update time decreases with $1/Z$. Also, at this point, we do not have to neglect simultaneous transitions from different individuals as long as one guarantees that $\boldsymbol{\delta}$ scales with $1/Z$. Then, taking into account that one has two small parameters, τ and $\boldsymbol{\delta}$, which scale with the same $1/Z$ factor, and using a Taylor expansion, we can expand Eq.(16) to the desired order.

The left side of Eq.(16) is easily computed to be

$$\Delta\rho(\mathbf{x}, t) = \tau \frac{\partial \rho}{\partial t} + \frac{\tau^2}{2} \frac{\partial^2 \rho}{\partial t^2} + \mathcal{O}(\tau^3). \tag{17}$$

The right side requires further analysis. Writing the terms inside the sum in Eq.(16), using the Taylor expansion to isolate the terms without derivatives, performing the sum over the terms and reordering them, we identify the KM coefficients $D^{(n)}(\mathbf{x})$ as

$$D^{(n)}_{i_1,\ldots,i_n}(\mathbf{x}) = N \frac{(-1)^n}{n!} \sum_{\delta \neq 0} \left[\prod_{m=1}^{n} \delta_{i_m} \right] T^{-\delta}(\mathbf{x}). \tag{18}$$

This can be used to calculate the Drift, $D^{(1)}_k(\mathbf{x})$, and Diffusion, $D^{(2)}_{kl}(\mathbf{x})$. Notice that the n-th KM coefficient contains a factor which is roughly $Z\,|\boldsymbol{\delta}|^n$, which proves that the terms in this KM expansion are increasingly small. In addition, it proves that the dynamics of infinite populations is deterministic since all coefficients but the first tend to zero and, therefore, the Langevin equation, Eq.(11), becomes an ordinary differential equation.

2.5 ONE-STEP PROCESSES

Populations evolve when individuals change their strategy. When an individual with a given strategy decides to change, the number of individuals with this strategy is reduced by one and the strategy he adopts gains a new member. This way one has a *birth-death* or *one step process*, keeping the total number of elements. The underlying assumption when using this kind of processes is that the probability that two different individuals change strategy in a time interval τ is $\mathcal{O}\left(\tau^2\right)$ [57], making Def.4 appropriated.

Definition 4 One Step Process

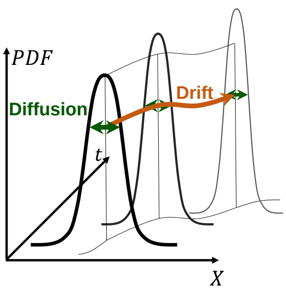
The process $\boldsymbol{i}$ is a *one step process* if its transition probability per unit time between states $\boldsymbol{i}$ and $\boldsymbol{i}' \neq \boldsymbol{i}$, $T_{\boldsymbol{i}\boldsymbol{i}'}$, is zero for all non adjacent configurations.

One-step processes are widely used, specially because even if we increase the time scale of observation, thus allowing for more intermediate step, we can redefine our states with meta-states over which definition 4 is valid.

When referring to the process, I will eventually introduce the extra strategy index just to keep track of what I am doing; symbolically, $\mathbf{i} = \{i_1, ..., i_d\} = (i_1, ..., i_{S-1}, i_S = N - i_1 - ... - i_{S-1})$. Then, if one considers all S strategies, the configuration of strategies at a given time is $\mathbf{i} = (i_1, ..., i_{S-1}, i_S)$ and it can only move to a configuration $\mathbf{i}' = (i'_1, ..., i'_{S-1}, i'_S) = (i_1 + \Delta_1, ..., i_S + \Delta_S)$, where, either all Δ_k are null, or only two of them are non-zero and, respectively, 1 and -1, which identifies the adjacent states. The probability that the system changes into states $\mathbf{i}'$ that do not obey these conditions is zero, $T_{\mathbf{i}'\mathbf{i}} = 0$. When all $\Delta_k = 0$, the system remains unchanged, $\mathbf{i}' = \mathbf{i}$, and the transition probability correspondent to this event can be calculated from the remaining as $T_{\mathbf{ii}} = 1 - \sum_{\mathbf{i}' \neq \mathbf{i}} T_{\mathbf{i}'\mathbf{i}}$. Hence, the determination of the transition probabilities between adjacent states, allows one to solve the problem. This will be left for next chapter.

In the next section, I will present and derive several general results related to these processes. Then, I will restrict the analysis to populations with only two strategies so I can introduce concepts as *fixation probability* and *fixation time* and motivate the study of stationary distributions.

2.5.1 *Drift and Diffusion*



The Drift and Diffusion will be of crucial importance to what comes next. Imagining a drifting peak of probability, the drift corresponds to the direction of where that probability tends to move to and the diffusion corresponds to the widening of that distribution.

Let me start by explicitly computing the Drift and Diffusion coefficients for this kind of processes. Considering birth-death processes, notice that the possible transition directions, $\boldsymbol{\delta}$, are under the assumptions of the derivation since all its entries are null except two of them which are, respectively, $1/Z$ and $-1/Z$, see Sec.2.4. Their explicit form is: $\boldsymbol{\delta} = \{\ldots, \delta_k, \ldots, \delta_l, \ldots\} = \{0, \ldots, 0, \pm 1/Z, 0, \ldots 0, \mp 1/Z, 0, \ldots, 0\}$. Using Eq.(18) one writes

$$D_k^{(1)}(\mathbf{x}) = \left(T^{\sigma_k +}(\mathbf{x}) - T^{\sigma_k -}(\mathbf{x})\right) \tag{19}$$

and

$$D_{kk}^{(2)}(\mathbf{x}) = \frac{1}{2N}\left(T^{\sigma_k -}(\mathbf{x}) + T^{\sigma_k +}(\mathbf{x})\right) \tag{20}$$

$$D_{kl}^{(2)}(\mathbf{x}) = -\frac{1}{2N}\left(T_{\sigma_k \to \sigma_l}(\mathbf{x}) + T_{\sigma_l \to \sigma_k}(\mathbf{x})\right), \tag{21}$$

where I have used the definitions in Eq.(22) and Eq.(23). $T^{\sigma_k \pm}(\mathbf{x})$ is the sum of all transition probabilities that increase or decrease the strategy k, respectively, and $T_{\sigma_k \to \sigma_l}(\mathbf{x})$ is the probability that an individual with strategy k changes into a strategy l.

$$T^{\sigma_k \pm}(\mathbf{x}) := \sum_{\delta:\delta_k = 1/N} T^{\pm\delta}(\mathbf{x}) \tag{22}$$

$$T_{\sigma_k \to \sigma_l}(\mathbf{x}) := T^{\{\ldots,\delta_k=-1,\ldots,\delta_l=1,\ldots\}}(\mathbf{x}) \tag{23}$$

We finally have Drift and Diffusion written in terms of quantities that can be computed in the finite system. We can also write a Langevin equation, which has now a very intuitive interpretation, Eq.(24). The deterministic trend and the most probable direction in phase space, given by the Drift, are a balance between the probability of increasing and decreasing a given strategy. The dispersion of the fraction of individuals with a given strategy across the configurations in the phase space decreases with the increase of population and the major terms in the Diffusion are a sum of the transitions in opposite directions.

$$\frac{d\mathbf{x}}{dt} = \mathbf{T}^{+}(\mathbf{x}) - \mathbf{T}^{-}(\mathbf{x}) + \frac{1}{\sqrt{Z}}\sqrt{Z\mathbf{D}^{(2)}}\boldsymbol{\Gamma}. \tag{24}$$

In the unidimentional case this is particularly easy to notice, Eq.(25).

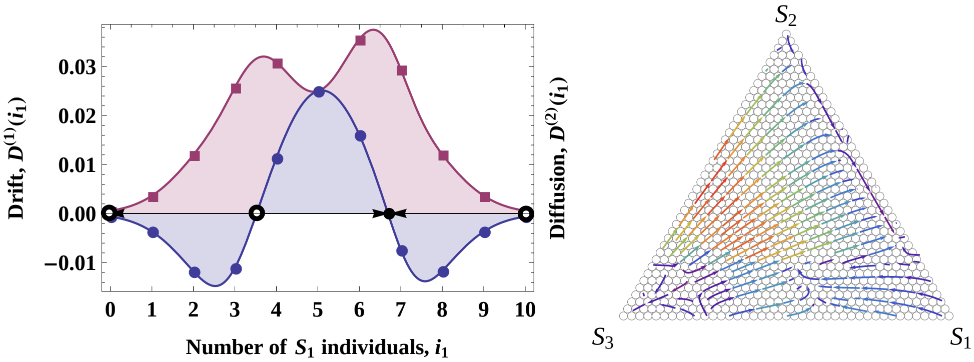
$$\frac{dx}{dt} = T^{+}(x) - T^{-}(x) + \frac{1}{\sqrt{Z}}\sqrt{\frac{T^{+}(x) + T^{-}(x)}{2}}\,\Gamma. \tag{25}$$

In any case, the study of the Drift and Diffusion will prove key to understand the population dynamics. In what follows I will discuss briefly how to interpret these functions and how to relate them to the population dynamics.

For large enough populations the fluctuations in the individuals of a given strategy will be small compared to the global trend. Therefore, neglecting the stochastic term in the Langevin equation, one finds a system of ordinary differential equations that, as we will see next, we can map to the replicator equation, Eq.(2): $\frac{d\mathbf{x}}{dt} = \mathbf{D}^{(1)}(\mathbf{x})$, where $\mathbf{D}^{(1)}(\mathbf{x}) = \mathbf{T}^{+}(\mathbf{x}) - \mathbf{T}^{-}(\mathbf{x})$. This is an intuitive way to motivate the Drift as the central direction in phase space. Formally, for prediction of the system's evolution, in general, short time propagators need to be taken into account, $T_{1|1}(\mathbf{x}, t+\tau|\mathbf{x}', t)$, which involve both the stochastic (Diffusion) and the deterministic (Drift) parts of the dynamics [105]; computing the gradient in $\mathbf{x}$ for the each point $\mathbf{x}'$ one finds $\boldsymbol{D}^{(1)}(\mathbf{x}')$ as the most probable direction.

Let me get back to finite populations. To make this transition clear, I will use the **i** variables whenever we are in the finite system and the **x** variables when we are talking about the infinite limit. The functional dependence distinction is done indexing the variable whenever we are talking about the discrete system's functions.

If the elements in the population can only adopt two different strategies, say σ_1 and σ_2, we have a one-dimensional problem. Let i_1 be the number of individuals with a given strategy σ_1 (then $Z - i_1$ is the number of individuals with strategy σ_2) and our configuration space be represented as a set of points confined in a line segment such that $0 \leqslant i_1 \leqslant Z$, see Eq.(7). Whenever $D^{(1)}_{i_1} > 0$ (< 0) the population will tend to see the number of elements with strategy σ_1 increase (decrease). In infinite populations, if $D^{(1)}(x_1) = 0$ one finds a fixed point which can be stable or unstable, as in traditional dynamical systems. However, since populations are finite, stochasticity is present and there is no fixation. These fixed point analogues act as attractors or repellers, respectively. Diffusion is also a function of i_1 and is always non-negative, so it can be plotted similarly to the Drift if one is interested in it. **Figure** 2a shows an example of a simple way of gathering all this information. Notice that **Figure** 1 has something similar to panel A but for an infinite population.



(a) (b)

Figure 2: **Drift and Diffusion representation** (a) Illustration of Drift and Diffusion representation for a population whose individuals can opt between two different strategies. The line with circles represents the Drift and the line with the squares represents the Diffusion. When the Drift is positive (negative) an arrow pointing right (left) is placed in the configuration axis, indicating the general tendency of the population. The dots in this axis represent fixed points analogues: a filled dot represents a fixed point which acts as an attractor and an empty dot acts as a repeller. Notice that those fixed points do not necessarily belong to the configuration state space, for it is discrete, and that Diffusion in those points is necessarily non-zero (see, eg. Eq.(25)). (b) Illustration of Drift representation for a population whose individuals can opt between three different strategies. The vector field indicates the direction of preferential motion and a color scale is used to indicate its strength. Fixed point analogues are also visible in this image though they are not represented to avoid overloading the image. The dynamics illustrated here are characteristic of the N-person games ($N > 2$) that we will see later on: they have multiple internal fixed points with different stabilities, contrary to the scenarios of the 2-player games in Fig. 1

However, if the elements in the population can adopt more than two different strategies, the representation can get a little bit tricky. Notice that, in general, the Drift is a vector field but the Diffusion is a tensor field that is represented by a matrix. Therefore, using its eigenvectors and making their magnitude equal to the corresponding eigenvalue, one is able plot the Diffusion matrix [144, 154].

2.5.2 *Fixation Problem and Stationary Distributions*

In this subsection I will consider one-dimensional processes, i.e. populations whose individuals can opt for two different strategies: σ_1 and σ_2. This way, $\mathbf{i} = i_1 = i$.

Suppose the population dynamics has some stable and unstable fixed points analogues, some $\{x \in \Re : D^{(1)}_{xZ} = 0\}$: attractors and repellers. The population will, therefore, spend most of its time around the attractors and little time near repellers but, since it is finite, stochasticity will allow all configurations to be explored. Eventually, the configuration in which the whole population adopts strategy σ_1 or σ_2 will occur and the imitation process, which drives the dynamics, will end: an individual can only imitate his own strategy. Essentially, if only the imitation process is considered, evolutionary dynamics in finite populations will (only) stop whenever the population reaches a state in which all individuals have the same strategy – a monomorphic state [84, 138].

Hence, in addition to the analysis of the shape of the drift, or gradient of selection, often one of the quantities of interest in studying the evolutionary dynamics in finite populations is the probability ϕ_i that the system fixates in a monomorphic σ_1 state, starting from, for instance, a given number i of σ_1's.

The *fixation probability* of i σ_1's, ϕ_i, depends on the ratio $\lambda_i = T^-_i / T^+_i$, being given by [58]

Even though computing fixation probabilities alone may disregard important information, they can be extremely useful. In chapter 6 I will make use of them to define approximations to invariant distributions of population Markov chains.

$$\phi_i = \sum_{l=0}^{i-1} \prod_{l'=1}^{l} \lambda_{l'} / \sum_{l=0}^{Z-1} \prod_{l'=1}^{l} \lambda_{l'} \tag{26}$$

Under neutral selection (that is, when the probability of changing strategy is 1/2), the fixation probability trivially reads $\phi^0_i = i/Z$, providing a convenient reference point [30, 58, 81, 138]. For a given i, whenever $\phi_i > \phi^0_i$, selection will favor σ_1 behavior, the opposite being true when $\phi_i < \phi^0_i$.

Yet, even if fixation in one of the two absorbing states is certain ($i = 0$ and $i = Z$), the time required to reach it can be arbitrarily long. This is particularly relevant in the presence of basins of attraction containing polymorphic stable configurations, which correspond to finite population analogues of co-existence equilibria in infinite populations. For example, the existence of a stable equilibrium may turn the analysis of the fixation probability misleading and, therefore, fixation probabilities may fail to characterize in a reasonable way the evolutionary dynamics under general conditions. Moreover, as I mentioned before, stochastic effects in finite populations can be of different nature, going beyond errors in the imitation process. In addition to social learning by imitation dynamics, one can also consider the so-called mutations: random exploration of strategies or any other reason that leads individuals to change their behavior [141]. Under these circumstances, the population will never fixate in any of the two possible monomorphic states (see **Chapter** 6, where I propose an extension to this approach).

The proper alternative, which overcomes the drawbacks identified in both fixation times and gradient alone, consists in the analysis of the distributions of the complete Markov process as mentioned in the very beginning, Sec. 2.3. In general, for the complete solution of the problem, one would solve the M-Equation, Eq.(8), to compute the PDF evolution. This can be very time and resource consuming and, furthermore, the analysis of the results would not be simple. However, I am not interested in transient distributions and, thereafter, we can look for stationary solutions of this equation. In general, this is obtained from the eigenvector associated to the eigenvalue 1 of the $T_{i'i}$ matrix. Notice that as long as the states are numerable this matrix is always possible to build for an arbitrary number of strategies.

To finish this subsection I will derive a solution that avoids solving this eigenproblem for one-dimensional one-step processes. For an alternate derivation see [57]. I also reinterpret this solution in a way that I conjecture it can be extended to any dimension and phase-space.

Let i be a limited one-step process: $i = 0, 1, ..., Z$. The transition probability from state i to i' is $T_{i'i}$.

Using Eq.(8), we search for a stationary solution making the left-hand side equal to 0 for all i. Thus, one can write a recurrence relation:

$$i = 0 \Rightarrow P_1 = P_0 T_{10}/T_{01} \tag{27}$$

$$0 < i < Z \Rightarrow P_{i+1} = (P_i(T_{i-1i} + T_{i+1i}) - P_{i-1}T_{ii-1})/T_{ii+1} \tag{28}$$

$$i = Z \Rightarrow P_Z = P_{Z-1}T_{ZZ-1}/T_{Z-1Z} \tag{29}$$

which allows one to write all P_i as a function of $P_0 = \alpha$. However, from Eq.(28) and using Eq.(27), one has $T_{12}P_2 = P_1 T_{21}$, which is the same kind of relation as in Eq.(27). Thus, it is natural to assume the hypothesis:

$$P_n = P_{n-1}T_{nn-1}/T_{n-1n}, \tag{30}$$

Valid for $n = 1$ via Eq.(27). Using Eq.(28) one writes:

$$T_{nn+1}P_{n+1} = P_n(T_{n-1n} + T_{n+1n}) - P_{n-1}T_{nn-1} \tag{31}$$

$$\overset{hip}{=} P_n(T_{n-1n} + T_{n+1n}) - P_n T_{n-1n} \tag{32}$$

$$= P_n T_{n+1n} \tag{33}$$

which validates the hypothesis, by induction, for $i \geqslant 1$. Recurrently one has:

$$P_i = P_{i-1}T_{ii-1}/T_{i-1i} = P_{i-2}T_{i-1i-2}/T_{i-2i-1}T_{ii-1}/T_{i-1i} = \ldots = \\ = \alpha\frac{T_{10}}{T_{01}}\frac{T_{21}}{T_{12}}\ldots\frac{T_{i-1i-2}}{T_{i-2i-1}}\frac{T_{ii-1}}{T_{i-1i}} \quad (34)$$

Finally, each P_i is determined as a function of the ratio between $T_l^+ = T_{l+1l}$ and $T_{l+1}^- = T_{ll+1}$, R_l, and P_0 determined by normalization:

$$P_i = \alpha\prod_{l=0}^{i-1}\frac{T_{l+1l}}{T_{ll+1}} \equiv \alpha\prod_{l=0}^{i-1}R_l,\ \alpha = \left(\sum_{i=0}^{Z}\prod_{l=0}^{i-1}R_l\right)^{-1}. \quad (35)$$

Now notice that, apart from normalization, P_i has i products. However, if we multiply all P_i by $\prod_{i=1}^{Z} T_{i-1i}$, all terms loose the quotients and all end up with the same amount of products. Because beauty and symmetry in Math is seldom achieved by chance, let me give it a geometric interpretation.

If we represent $T_{\mathbf{ii}'}$ with an arrow from $\mathbf{i}'$ to $\mathbf{i}$, we get that the (stationary) probability of being in state $\mathbf{i}$ is given by **Figure** 3. The statement is that the stationary distribution at any given point can be computed as the sum of the different paths that fill the phase-space and end at that point. The paths are products of transition probabilities, one from each single state, consequently meaning that paths will always have the size of the phase-space minus 1. In the 1D case, there is only one path for each state. However, as Figure 4 shows, in a 2D phase-space, the number of paths that lead to the same configuration increases rapidly.

Now, clearly, some paths contribute more than others. In fact, in the light of this interpretation it is easy to understand why a stable fixed point in the deterministic dynamics (keeping only the drift, or gradient of selection, in the KM expansion) is an attractor of probability in the full system. If we look at the phase-portrait of the deterministic system, it will overlap with the transitions in **Figure** 4 giving different weights to the different paths. Notice, however, that in the full system, that stable point contains the path whose streams are given by the drift and thus have the largest transitions, making it the more probable point. On the other hand, if its basin of attraction is small, those streams are still multiplied by transitions that are contrary to the drift, and thus are small, making the probability of being in that attractor smaller.

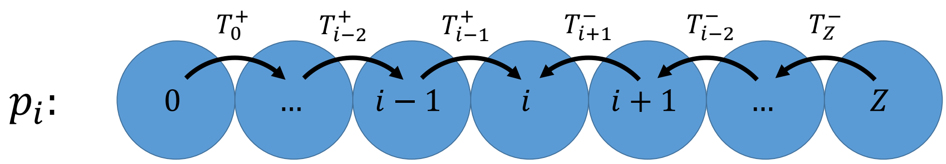


Figure 3: **Stationary distribution graphical solution** The solution of a stationary distribution in a 1D can be computed by the paths that fill the phase-space and lead to each point. Each arrow corresponds to a transition and a path is the product of those transitions. Each patch contains $|s|-1$ transitions, where $|s|$ is the size of the configuration space. In this case, there is only one path that is $T_0^+ \ldots T_{i-2}^+ T_{i-1}^+ T_{i+1}^- T_{i+2}^- \ldots T_X^-$.

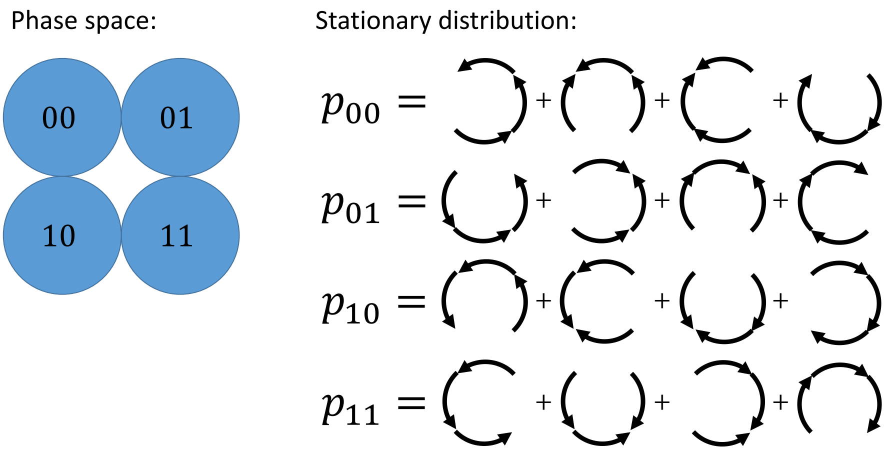


Figure 4: **Stationary distribution graphical solution for a 2D phase space** Apart from normalization, the right panel shows the solution of a stationary distribution in a 2D phase space. It has several paths that lead to the same configuration.

3 GAMES IN FINITE POPULATIONS

So far, I have reduced the problem of building up a model that describes the evolution of a finite population to the ability of writing some transition probabilities. To do this, we must include microscopic details into the model, i. e., what makes individuals change their own strategy. As already mentioned, an imitation principle will be employed but it is not enough, for it would systematically lead the population to homogeneous states.

Imitation reflect individuals' tendency to copy others whenever these appear to be more successful. Contrary to strategies defined by a contingency plan, which, as some argue [70], are unlikely to be maintained for a long time scale, this social learning (or evolutionary) approach allows traits or strategies to change as time goes by [104, 119, 137]. Likely, these strategies will be influenced by the behavior (and achievements) of others, as it happens in the context of donations to public goods [16, 34, 103].

Moreover, one should also consider random exploration of strategies or any other reason that leads individuals to change their behavior without imitating anyone, *mutation*. In the simplest scenario, this creates a modified set of transition probabilities, with an additional random factor encoding the probability of a mutation, μ, in each update step. Under these circumstances, the population will never fixate in any of the possible monomorphic states and will evolve preferentially according to such imitation process.

To this point, the missing transition probabilities correspond to the probability that an element with a given strategy, σ_l, changes into another specific strategy, σ_k, $k, l = 1, \ldots, S$. Evidently this may depend on the strategies but I will assume a common functional form for all pairs of strategies; the process of decision has the same rule. A rule one can use to compute $T_{\sigma_l \to \sigma_k} \equiv T_{\{\ldots,i_k+1,\ldots,i_l-1,\ldots\}\{\ldots,i_k,\ldots,i_l,\ldots\}}$ is the *Fermi update rule with mutation*, or *pairwise comparison rule* [138]:

Fermi update rule is but one of the many possible updates and defines a contact process. It is extremely useful not only due to analythical tractability but also because it allows one to push intensity of selection to a non-linear regime in a very familiar way, decreasing the temperature/errors of the system as $\beta \to \infty$.

- ◇ Considering a birth-death process in a well-mixed population, all individuals are equally likely to change of strategy and all the others are equally likely to be selected as role model, resulting in a contact probability between strategy σ_l and σ_k of $i_l/Z \times i_k/(Z-1)$.
- ◇ In the comparison process, the individual more likely changes strategy if his strategy is worse than the one he is comparing to. This is accomplished using a Fermi distribution, $(1+\exp(\beta\Delta_{\sigma_l\sigma_k}))^{-1}$, which introduces errors in the imitation process, where β represents the intensity of this selection and $\Delta_{\sigma_l\sigma_k}$ quantifies how better is strategy σ_l compared to σ_k. For $\beta << 1$, selection is weak and its effect is but a small perturbation to random drift in behavioral space.

- Additionally, one may introduce a parameter μ, the *mutation*, that allows transitions between strategies independent of how *good* they are. When μ has its maximum value, 1, individuals change (or not) to any strategy with equal probability.

These three ingredients build Eq.(36) [138, 139, 141].

$$T_{\sigma_l \to \sigma_k} = \frac{i_l}{Z}\left(\frac{i_k}{Z-1}\frac{1-\mu}{1+\exp(\beta\Delta_{\sigma_l\sigma_k})} + \frac{\mu}{s}\right) \tag{36}$$

This formulation allows one to explicitly compute the Drift and Diffusion for a Fermi update rule using Eq.(19) and Eqs.(21), see **Section** 3.1 below.

3.1 DRIFT AND DIFFUSION FOR FERMI UPDATE RULE

In this section I will calculate all Drift and Diffusion using the Fermi update rule. I use the finite system notation.

Consider the quantities $T_i^{\sigma_k+} \pm T_i^{\sigma_k-}$, whose definitions are given, in Eq.(22), in terms of $T_{\sigma_l \to \sigma_k}$, Eq.(36). Then, again with $d = S-1$,

$$T_i^{\sigma_k+} \pm T_i^{\sigma_k-} = \sum_{l\neq k}^{s} \left[\frac{i_k(1-\mu)}{N(N-1)} i_l \left(\frac{1}{1+\exp(\beta\Delta_{\sigma_l\sigma_k})} \pm \right.\right.$$
$$\left.\left. \pm \frac{1}{1+\exp(\beta\Delta_{\sigma_k\sigma_l})}\right) + \frac{\mu}{dZ}(i_l \pm i_k)\right]. \tag{37}$$

Using the conceptual symmetry $\Delta_{\sigma_l\sigma_k} = -\Delta_{\sigma_k\sigma_l}$ and the identities $(1+\exp(x))^{-1} - (1+\exp(-x))^{-1} \equiv \tanh(-x/2)$ and $(1+\exp(x))^{-1} + (1+\exp(-x))^{-1} \equiv 1$ I can write Eqs.(38).

$$T_i^{\sigma_k+} - T_i^{\sigma_k-} = \frac{i_k(1-\mu)}{Z(Z-1)} \sum_{l\neq k}^{s+1} \left[i_l \tanh\left(\frac{\beta}{2}\Delta_{\sigma_k\sigma_l}\right)\right] + \frac{\mu}{dZ}(Z - s i_k) \tag{38a}$$

$$T_i^{\sigma_k+} + T_i^{\sigma_k-} = \frac{i_k(1-\mu)}{Z(Z-1)}(Z - i_k) + \frac{\mu}{dZ}(Z + (d-1)i_k). \tag{38b}$$

In the same way, one computes

$$T_{\sigma_k \to \sigma_l} + T_{\sigma_l \to \sigma_k} = \frac{i_k i_l (1-\mu)}{Z(Z-1)} + \frac{\mu}{dZ}(i_k + i_l) \tag{39}$$

$T_i^{\sigma_k+} - T_i^{\sigma_k-}$ is the Drift vector and the remaining functions are part of the Diffusion matrix.

Notice that $T_i^{\sigma_k+} + T_i^{\sigma_k-}$ and $T_{\sigma_k \to \sigma_l} + T_{\sigma_l \to \sigma_k}$ are independent of the strategies. These elements build the diffusion matrix and, therefore, Diffusion is the same for all conceivable games and strategies once one chooses the Fermi update rule. If one is interested in the study of fluctuations of strategies in populations, the mutation should follow some more complex rule, otherwise no contributions from the games themselves will arise.

In the limit of very large populations, one can write a set of replicator-like equations including mutations with non-linear update and noise as

$$\frac{dx_i}{dt} = (1-\mu)\frac{Z}{Z-1}x_i\sum_{j\neq i} x_j \tanh\left(\frac{\beta}{2}\Delta_{\sigma_k\sigma_l}\right) + \frac{\mu}{d}(1-Sx_i) + \sum_j \frac{\Gamma_i}{Z}\sqrt{\mathbf{D}}_{ij}, \tag{40}$$

akin to what has been deduced for two strategies in [138].

3.2 interaction

Up to now, I have not defined how subjects compare their strategies. I have already mentioned "*better* strategies" but, so far, all I did was packing this information in $\Delta_{\sigma_l\sigma_k}$. Actually, $\Delta_{\sigma_l\sigma_k}$ is indexed to a given configuration too, since a strategy is only said to be good in a given situation. I will consider that each strategy, σ_k, for each configuration, has a well defined fitness, $f_{\sigma_k\mathbf{i}}$. The greater its fitness, the better succeeded is the strategy. In this sense, one writes $\Delta_{\sigma_l\sigma_k} = f_{\sigma_l} - f_{\sigma_k}$. The way the whole procedure is built assumes this fitness to be accessible to individuals since it is part of their decision process. Evidently, this can only be done via interaction with the other players.

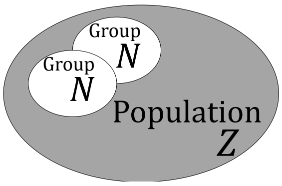
In a previous section, namely in **Section** 2.2, I mentioned some examples of 2-player games. Clearly, in general, interactions do not need to happen in pairs. Along this dissertation, I will consider examples of N-person interactions as they will repeatedly occur when dealing with public goods with $N > 2$. Indeed, according to the well-mixed ansatz, individuals in a population may interact with all other $Z-1$ players with equal probability but they do so in groups of size N. From that interaction results a payoff to each player. Indeed, since individuals with the same strategy are described in the same way, the payoff is a characteristic of the strategy and, ultimately, the payoff, P_{σ_k}, is what defines the strategy σ_k.

If the number of interactions is large enough, the central limit theorem assures that the average payoff obtained by each player will be close to the expected payoff. That, in turn, given an unbiased partition of the groups, can be analytically expressed for each configuration of the population $\mathbf{i}$, population size Z and groups size N.

Definition 5 Fitness

The *fitness* of a strategy σ_l, $f_{\sigma_l\mathbf{i}}$, is the average payoff a single individual with strategy σ_l obtains over all possible games that can be played in a given configuration of the whole population.

For finite, well-mixed populations of size Z, this average is accomplished using a hypergeometric sampling (without replacement) [107]. Let $\mathbf{j} = \{j_1, ..., j_d\} = (j_1, ..., j_S)$ be the configuration of players in the group of size N with identical definition to $\mathbf{i}$ but replacing Z for N, i. e., the configuration of the small group. Then, using Def.5, $f_{\sigma_k \mathbf{i}}$ is given by Eq.(41).



Randomly selected groups of size N are sampled from a population of size Z.

$$f_{\sigma_k \mathbf{i}} = \binom{Z-1}{N-1}^{-1} \sum_{\substack{j_1,\ldots,j_S \\ j_1+\ldots+j_S \leqslant N-1}} \Pi_{\sigma_k \mathbf{j}} \binom{i_k - 1}{j_k} \prod_{l \neq k}^{S-1} \binom{i_l}{j_l} \tag{41}$$

Where $\Pi_{\sigma_k \mathbf{j}}$ is the payoff of an individual with strategy σ_l in an N-person game with configuration $\mathbf{j}$, which must be such that it contains at least one individual with strategy σ_k.

It is common to assume that players interact enough times such that their average payoff is close to the expected payoff. For a not so large number of interaction, deviations in the expected fitness difference start to occur, in which an effective decrease in β, i. e. an increase in imitation errors, is expected.

If one assumes an infinite population, in a configuration $\mathbf{x} = \{x_1, ..., x_{S-1}\} = (x_1, ..., x_S = 1 - x_1 - ... - x_{S-1})$, where also every individual can potentially interact with everyone else, the fitness of each individual can be obtained from a random sampling of groups. The latter leads to groups whose composition follows a binomial distribution. Hence, for groups of size N and configuration $\mathbf{j}$, I may write the fitness of a given strategy $f_{\sigma_k}(\mathbf{x})$ using Eq.(42) [48, 96, 128], which, for $N = 2$ and the 2-player symmetric games discussed before, returns Eqs.(3,4).

$$f_{\sigma_k}(\mathbf{x}) = \sum_{\substack{j_1,\ldots,j_S \\ j_1+\ldots+j_S \leqslant N-1}} \frac{(N-1)!}{j_1! \ldots j_{s+1}!} \Pi_{\sigma_k \mathbf{j}} \prod_{l}^{S} x_l^{j_l} \tag{42}$$

From now on, my mission will be to define proper strategies by giving the behavior of an individual in a group of size N, attribute payoffs for the groups depending on their strategy composition, and compare the effects that different strategies have on the behavior of the whole population, using the tools laid so far.

Before proceeding, however, I will introduce a deeper reflection on the methods used. My goal is to justify for hypotheses used and motivate the extensions proposed.

Part II

DEVELOPING THE FRAMEWORK

This part represents my contribution to the development of the field of population dynamics in a general way. I propose frameworks to study

- the effects of discounting the future, with the use of forecasting and different orders of theory of mind,
- the evolution of the distribution of continuous traits in infinite populations with non-zero mutation and selection,
- evolution of the distribution of discrete sets of traits in finite populations with non-zero mutation and selection

I also use this part to question the usual assumptions of Evolutionary Game Theory and show how one could lift those assumptions. Namely, I will be concerned with the assumptions of

- Infinite discounting of future events,
- Bounded rationality,
- Discrete strategies,
- Rare exogenous events.

4 A DISCUSSION ON RATIONALITY

All the sophisticated abilities often associated with human reasoning should at least be mentioned in a dissertation on cooperation. In particular when we are dealing with benefits that, many times, only arise in the future. The description of the methods of the last part has, so far, also lacked a bridge between the purely rational and the simple imitation-based social learning process that is implicitly described by the replicator equation. However, the two need not be separated. Humans face different decisions everyday, some of which pose dilemmas that involve the provision of public goods[106]. These dilemmas typically concern the ability of individuals to suppress their selfish calculations – that only take into account their immediate benefits – in the interest of achieving an end result that provides a collective benefit. Many times, solutions involve the establishment of insurance arrangements whose implementation revolves on how individuals discount the future.

Discounting requires Forecasting which needs a Theory of Mind.

On one hand, discounting – the phenomenon of weighting present benefits more than future ones – is accounted as one of the main reasons for the lack of cooperation in what concerns resource consumption [65, 66]. On the other, evolutionary game theory is one of the main tools to deal with the study of cooperation and one of the few that can deal explicitly with time, modeling a complex system as a dynamic one. Here I look at – and give the equations on – how to extend Evolutionary Game Theory to analyze the effects of discounting. I will argue that in order to consider discounting, players need the ability of forecasting so as to make predictions about the future. The limiting cases of predicting no changes and considering only one's direct impact are considered and the equations for it are derived. Finally, I explore the relationship between the ability to make predictions and the need of a theory of mind. We will consider some ways to make predictions and superficially discuss the order of theory of mind it takes. Overall, this is not, and does not intend to be, a deep study of the effects of discounting, forecasting or theory of mind, but a way to clearly ascertain how the methods and equations used contain or not any of it. Indeed, this chapter deeply contrasts with the remainder of this dissertation in what level of simplicity of the assumptions, and consequent modeling, is concerned. Nonetheless, it sets aside some concerns about our ability to deal with human behavior.

4.1 DISCOUNTING

Let us consider a population of size Z whose individuals weight the future into their decisions. For simplicity, the individuals in this population can adopt one of two strategies, C or D. Individuals receive a given payoff depending on their strategy, on the number of Cs and Ds that they interact with and also on some resource that can have a dynamics of its own, B. As an example, B can stand for a water resource and Cs are the players that extract a smaller amount compared to Ds. In general, it represents additional state variables.

The average payoff of an individual, also called fitness, will then depend on the fraction of individuals of each strategy, say x for the fraction of individuals playing C, and on the resource level, B – i.e., $f_X \equiv f_X(x, B)$, with $X = C, D$.

A strategical individual will change strategy if she sees herself getting an advantage at a given time by doing so. The weight she gives to that advantage depends on how much importance she gives to that time relative to the present or future time. However, in order to perceive an advantage in any time that is not the present one, say τ units of time after the present, the individual has to estimate her fitness at that time. Thus, at a given time t, a focal individual with strategy X changes to strategy Y by eventually choosing a player with that strategy to imitate and computing the weighted fitness difference over future time, also called present value[66]

$$\widetilde{\Delta f_{XY}} = \sum_{\tau} w(\tau)\left(\widetilde{f_Y}^X(\tau) - \widetilde{f_X}^X(\tau)\right), \tag{43}$$

where $\widetilde{f_Y}^X$ is the fitness of strategy Y estimated by player (with strategy) X at time t for time $t+\tau$ and $w(\tau)$ is the weight given to that moment in the future [35, 40, 59, 63]. The bigger $\widetilde{\Delta f_{XY}}$, the more likely the player is to change strategy. We can use the logistic/Fermi function we used before, $F(x) = \left(1 + e^{-\beta x}\right)^{-1}$, to implement the Fermi update process under discounting players, setting the probability that the player with strategy X changes to strategy Y as $F\left(\widetilde{\Delta f_{XY}}\right)$. Notice that I use the tilde to reinforce the idea that such quantity is estimated. Also, I should emphasize that, despite the players being strategic, this is a socially driven update as, for now, players act by potential imitation, not being able to come up with new strategies themselves.

Before proceeding, let us navigate through some of the equations that govern this kind of dynamics so that we can get familiarized with it before getting into specificities. Neglecting stochastic effects, this leads to a dynamics akin to a replicator dynamics, which we have deduced before in Eq.(40).

$$\dot{x} = x(1-x)\left(F\left(\widetilde{\Delta f_{DC}}\right) - F\left(\widetilde{\Delta f_{CD}}\right)\right). \tag{44}$$

Because we are setting this from scratch, let us look at a linearized version making β small (i.e.in the limit of *weak selection*) – using the Taylor series [of] $F(x) = 1/2 + \beta x/4 + O\left(x^2\right)$ – and getting rid of it by rescaling the time

$$\dot{x} = x(1-x)\frac{1}{2}\left(\widetilde{\Delta f_{DC}} - \widetilde{\Delta f_{CD}}\right). \tag{45}$$

This equation is already quite similar to the replicator equation. Naturally, it can be obtained through other update rules, like the Moran process [85] or replicator dynamics [52] or any other F function that grows monotonically. In order to make sure everything is in place, we can try to recover the replicator equation from it. Naturally, there should be no weight on the future. Let us set $w(\tau) = \delta_{\tau 0}$, where δ_{ij} is the Kronecker delta that is 1 if $i = j$ and 0 otherwise. Furthermore, players assume that their fitness playing the other strategy will be the same as that of the other player and it will not change in the immediate next round, making $\widetilde{f_Y}^X(\tau) = \widetilde{f_Y} = f_Y$. This means that the players do not even have to know the game at stake or any kind of shape for the fitness functions. It also means that $\widetilde{\Delta f_{DC}} = -\widetilde{\Delta f_{CD}} = f_C - f_D$, leading to the replicator equation

$$\dot{x} = x(1-x)(f_C - f_D) . \tag{46}$$

Assuming no changes in fitness leads to the replicatior equation.

Notice that we would have gotten the exact same equation if we just hypothesized that players assume the fitness to be unchangeable, having only to get rid of the $\sum_\tau w(\tau)$ in the same fashion as we did with $\beta/2$, rescaling time, and making no hypotheses on $w(\tau)$. Let us stick to Eq.(43) and Eq.(45) and move on. We will keep checking in which cases we get the replicator equation back as we make more assumptions.

4.2 MOVING THE TILDES

We have briefly addressed discounting and we immediately see that in order to continue we have to move deeper into $\widetilde{f_Y}^X(\tau)$, the fitness of strategy Y estimated by player (with strategy) X for τ into the future. Without much loss of generality let us write it as $\widetilde{f_Y}^X(\tau) \equiv \widetilde{f_Y}^X\left(\tilde{x}^{X\to Y}(\tau), \tilde{B}^{X\to Y}(\tau)\right)$, meaning that, to estimate the fitness, players set some (potential) dependencies over which they understand their impact – in this case, they could understand their impact on the fraction of Cs, x, or in other external variable, that we called the resource, B. We say that the player with strategy X will estimate a fraction of Cs at time $t+\tau$ of $\tilde{x}^{X\to Y}(\tau)$ and a resource $\tilde{B}^{X\to Y}(\tau)$ when she changes to strategy Y (which could be the same as X or not).

For the sake of sanity – ours, definitely not that of the players – we assume that they know the rules of the game exactly and, thus, they know, or at least are very good at estimating, the fitness $f_Y(x, B)$ for all values of x, B and strategies. This means that we assume from now on that f_Y is a known function, which, in terms of what is known (no tildes) and what is predicted (with tildes), means $\widetilde{f_Y}^X(\tau) = f_Y\left(\tilde{x}^{X\to Y}(\tau), \tilde{B}^{X\to Y}(\tau)\right)$. Also, because we will develop on an arbitrary τ, I will drop it and recover it when necessary, keeping in mind that it is associated with the tilde. So, finally, we can go back to Eq.(43) and look at the quantity

$$\widetilde{f_Y}^X - \widetilde{f_X}^X = f_Y\left(\tilde{x}^{X\to Y}, \tilde{B}^{X\to Y}\right) - f_X\left(\tilde{x}^{X\to X}, \tilde{B}^{X\to X}\right) . \tag{47}$$

Well, so far we have not done much, apart from setting arguments for moving the estimations – and thus the tildes – from the fitness into something different. This way, the players may estimate the fraction of Cs and the resource τ in the future. As you were probably guessing from the first tilde, discounting needs forecasting and that is where we are getting at. Let us keep on going, now with Eq.(45), Eq.(43) and Eq.(48) in our mind and $\tilde{x}^{X\to Y}$, $\tilde{B}^{X\to Y}$ un...specified.

4.3 FORECASTING

Whenever we talk about forecasting, we have a reference value, typically the current value of some property and we worry about its variation. Assume the present value is accurate and common knowledge. In this case, the current value is that of the state variables x and B. Evidently the idea is to get this forecast as a function of those state variables as they should contain all the necessary information for the evolution of the system. Thus, we write

$$\tilde{x}^{X\to Y} = x + \widetilde{\Delta x}^{X\to Y}(x, B) \text{ and}$$

$$\tilde{B}^{X\to Y} = B + \widetilde{\Delta B}^{X\to Y}(x, B). \tag{48}$$

Clearly, if players assume nothing will change, making $\widetilde{\Delta x}^{X\to Y} = \widetilde{\Delta B}^{X\to Y} = 0$, we recover the replicator equation. In general, this will not be the case. In fact, one could argue that in order to make predictions we often look at the past but, first, we (really) do not want to dwell into delayed differential equations as it is fairly complicated[79] and, second, when we do so is to come up with the functional form to our estimations. For now, we will skip that discussion and, instead, discuss how one computes $\widetilde{\Delta x}^{X\to Y}$ and $\widetilde{\Delta B}^{X\to Y}$, which will come right after.

Let us look at the questions we need to answer. Starting with $\widetilde{\Delta B}^{X\to Y/X}(x, B)$, we want to answer the question “Given the actual configuration, how will the resource change if I do/do not change strategy?”. Similarly, in $\widetilde{\Delta x}^{X\to Y/X}$ we ask “Given the actual configuration, how will the fraction of Cs change if I do/do not change strategy?” .

With these questions we can grasp a very subtle difference in the two forecasts. The difference concerns the fact that both questions include the hypothetical change of strategy and only one of the forecasts is directly about the strategies in the population, $\widetilde{\Delta x}^{X\to Y}$. Let us make this clearer in our equations by setting a reference change over time and specifying the players' change of strategy in the hypothetical scenario.

$$\tilde{x}^{X\to Y} = x + \widetilde{\Delta x}(x, B) + \widetilde{\delta x}^{X\to Y}(x, B) + \frac{(-1)^{\delta_{YD}}}{Z} \text{ and}$$

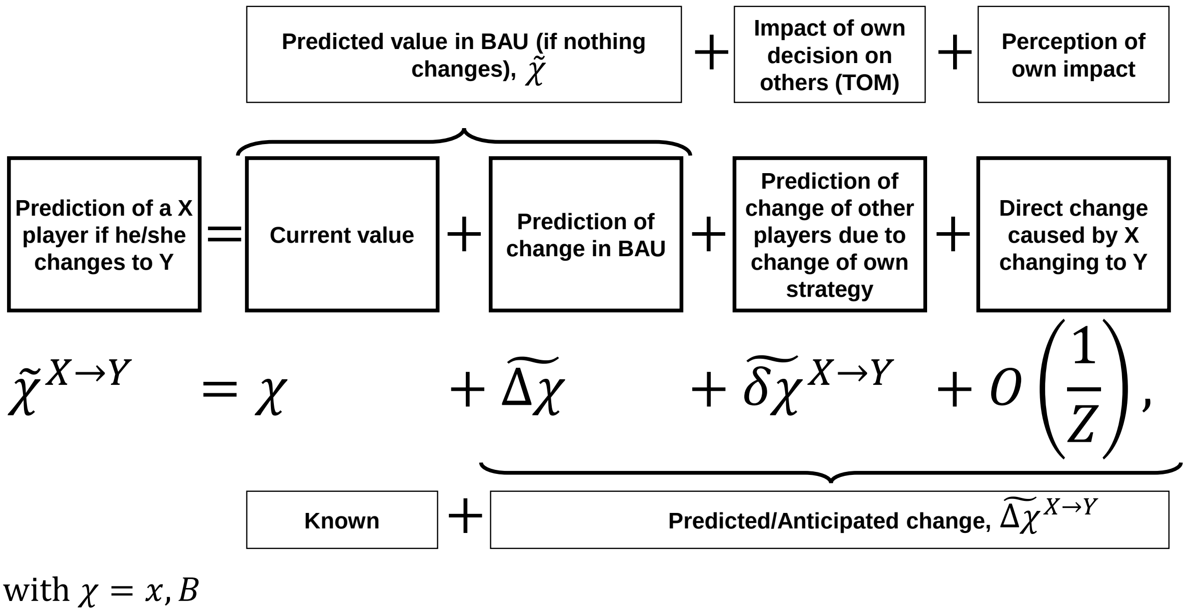


Figure 5: **Prediction decomposition** The decomposition of the prediction made by each player for a given moment in time, $t+\tau$, where the symbols with the tilde depend on τ.

$$\tilde{B}^{X\to Y} = B + \widetilde{\Delta B}(x,B) + \widetilde{\delta B}^{X\to Y}(x,B). \tag{49}$$

$\widetilde{\Delta x}(x,B)$ and $\widetilde{\Delta B}(x,B)$ are the scenarios in which the player at stake does not change strategy, i.e., the projections of change with the individual performing *Business As Usual* (BAU)[19]. Additionally, we look at $\widetilde{\Delta x}^{X\to Y}(x,B) - \widetilde{\Delta x}(x,B)$, which reads as the change of the fraction of Cs that is caused by changing the strategy. We isolate the direct effect of the (singular) player that changes strategy and necessarily changes the fraction of Cs by $\pm 1/Z$, i.e., writing $\widetilde{\Delta x}^{C\to D}(x,B) - \widetilde{\Delta x}(x,B) = \widetilde{\delta x}^{C\to D} - \frac{1}{Z}$ and $\widetilde{\Delta x}^{D\to C}(x,B) - \widetilde{\Delta x}(x,B) = \widetilde{\delta x}^{D\to C}(x,B) + \frac{1}{Z}$. The same for the resource, which has no direct effect, $\widetilde{\Delta B}^{X\to Y}(x,B) - \widetilde{\Delta B}(x,B) = \widetilde{\delta B}^{X\to Y}(x,B)$.

Before posing the next and final problem, let us be a bit redundant, and rewrite and reread this equation. So, the first two terms in the equations, $\tilde{x} = x + \widetilde{\Delta x}(x,B)$ and $\tilde{B} = B + \widetilde{\Delta B}(x,B)$, are the projections with BAU and everything else is caused by the decision of changing: an immediate effect of 1 more or 1 less C player – that is the player that is thinking about changing strategy –, eventually, $\widetilde{\delta x}^{X\to Y}$ of the perception of influencing the others and $\widetilde{\delta B}^{X\to Y}$ an effect of the perception of the consumption of more or less resource. All the quantities considered are the following

$$\tilde{x}^{C\to D} = \tilde{x} + \widetilde{\delta x}^{C\to D}(x,B) - \frac{1}{Z}, \quad \tilde{x}^{D\to C} = \tilde{x} + \widetilde{\delta x}^{D\to C}(x,B) + \frac{1}{Z},$$

$$\tilde{B}^{C\to D} = \tilde{B} + \widetilde{\delta B}^{CD}(x, B), \qquad \tilde{B}^{D\to C} = \tilde{B} + \widetilde{\delta B}^{DC}(x, B). \tag{50}$$

Figure 5 systematizes all the information and notation here exposed. It is a good time now to recover our equations. Putting all the ingredients together – Eq.(50) goes into Eq.(48) which goes into Eq.(43) which goes into Eq.(45), or alternatively Eq.(44) – we get

$$\begin{aligned}\dot{x} =& x(1-x)\sum_{\tau} w(\tau)\frac{1}{2}\Bigg(f_C\left(\tilde{x}(\tau)\ ,\tilde{B}(\tau)\right) - f_D\left(\tilde{x}(\tau),\tilde{B}(\tau)\right) + \\ & f_C\left(\tilde{x}(\tau) + \widetilde{\delta x}^{D\to C}(\tau, x, B) + \frac{1}{Z}, \tilde{B}(\tau) + \widetilde{\delta B}^{DC}(\tau, x, B)\right) - \\ & f_D\left(\tilde{x}(\tau) + \widetilde{\delta x}^{C\to D}(\tau, x, B) - \frac{1}{Z}, \tilde{B}(\tau) + \widetilde{\delta B}^{CD}(\tau, x, B)\right)\Bigg)\end{aligned} \tag{51}$$

This is not exactly pretty and one might ask what we can extract from this. Let me try to answer. On one hand, we have underlined the hypotheses assumed on building it and that is useful in itself. On the other hand, we can make additional assumptions to pursue more realistic behaviors or to understand in which cases we are led back to a replicator-like dynamics. We will do the latter before continuing with exploring how one can specify the forecast and how that will depend on the mental development of the players.

Whenever the players believe their actions will not affect that of the others, $\widetilde{\delta x}^{D\to C} = \widetilde{\delta x}^{C\to D} = 0$. Furthermore, if it is their belief that their action will not impact the resource dynamics $\widetilde{\delta B}^{CD} = \widetilde{\delta B}^{DC} = 0$. In that scenario, and using the difference operator $\Delta_x^{\delta} f(x, y) = f(x+\delta, y) - f(x, y) \approx \delta\partial_x f(x, y)$, we get

$$\begin{aligned}\dot{x} = x(1-x)\sum_{\tau} w(\tau)\Bigg(& f_C\left(\tilde{x}(\tau)\ ,\tilde{B}(\tau)\right) - f_D\left(\tilde{x}(\tau),\tilde{B}(\tau)\right) + \\ & \frac{1}{2}\Delta_x^{\frac{1}{Z}}\left(f_C\left(\tilde{x}(\tau)\ ,\tilde{B}(\tau)\right) + f_D\left(\tilde{x}(\tau),\tilde{B}(\tau)\right)\right)\Bigg).\end{aligned} \tag{52}$$

Additionally, if we weight only the immediate step, $w(\tau) = \delta_{\tau 0}$, this reduces to

$$\dot{x} = x(1-x)\left(f_C - f_D + \frac{1}{2}\Delta_x^{\frac{1}{Z}}(f_C + f_D)\right). \tag{53}$$

Considering one's own imediate impact, leads to a modified replicator equation.

Surprisingly, we do not get exactly the replicator equation. We have lost it along the way. In fact, we lost it in the moment we let the player take into account her own – eventually small – impact on the fraction of Cs when she changes strategy. Notice that if the fitness are smooth functions, this is an effect that disappears as the population goes to infinity, in which case one recovers exactly the replicator equation. This simple result, as we will see, is highly important. Not only because it can be directly applied to the studies that use the standard replicator equation to social systems but, mostly, because it introduces an additional term that is well known to exist when one turns to experiments and traditional game theory.

Take, for instance, the N-person prisoners' dilemma, in which N players choose to contribute, or not, to a common pool an amount c. Let Cs be those that contribute. The total contribution is then multiplied by a factor F and divided by everyone. If this group consists of the whole population – and that is a very important if – then the number of contributors is just xN. This makes average payoff of a C $f_C(x) = -c + Fcx$ and that of a D $f_D(x) = Fcx$. Then, using Eq.(53), the dynamics will be given by

$$\dot{x} = x(1-x)\left(-c + \frac{Fc}{N}\right). \tag{54}$$

Whenever the marginal per capita return F/N is smaller than 1, the population will evolve towards full defection, and we recover the Prisoners'Dilemma scenario, and whenever $F/N > 1$ the population will evolve towards full cooperation, as in the Harmony Game (see **Figure** 1). This is a triviality for many specialists but it is one that could not be obtained with the replicator dynamics whenever a single group was interacting, which, taking Eq.(46), would result in

$$\dot{x} = x(1-x)(-c). \tag{55}$$

And that is an important difference.

Later on, in the next chapter, I will clarify under which circumstances this result creates an important difference and when it only adds a minor shift to the replicator-like dynamics.

The difference between considering one's own impact or not can be big. But doing so also implies the knowledge of the game being played.

4.4 THEORY OF MIND

We are now under conditions of exploring different ways of forecasting and how those relate to different orders of theory of mind (TOM) [17, 102, 126].

Definition 6 Theory of Mind (loose)

A Theory of Mind is the ability to attribute mental states to oneself and others. It allows us to understand that the information we have, and our self, is different from that of the others and, more, allows us to assume what are those of the other. As we increase the order of Theory of Mind, we understand that the others can also do the same about us, and so on.

As we have done with the update rules, we will not dwell deeply into why-this-way-and-not-others or in all the bells and whistles we can add to the mix to make it more accurate. As the application of Eq.(52), resulting in Eq.(53), has proven, adding the right bell already provides additional important results.

To systematize before proceeding, let us recover the decompositions we have made so far for each of the estimated state variables, x and B.

$$\begin{aligned} \tilde{x}^{C\to D} &= x + \widetilde{\Delta x}\,(x,B) + \left(\widetilde{\delta x}^{C\to D}\,(x,B) - \frac{1}{Z}\right), \\ \tilde{x}^{D\to C} &= x + \widetilde{\Delta x}\,(x,B) + \left(\widetilde{\delta x}^{D\to C}\,(x,B) + \frac{1}{Z}\right), \end{aligned} \tag{56}$$

$$\begin{aligned} \tilde{B}^{C\to D} &= B + \widetilde{\Delta B}\,(x,B) + \left(\widetilde{\delta B}^{CD}\,(x,B)\right), \\ \tilde{B}^{D\to C} &= B + \widetilde{\Delta B}\,(x,B) + \left(\widetilde{\delta B}^{DC}\,(x,B)\right). \end{aligned} \tag{57}$$

The terms in parenthesis are the ones perceived as caused by the change of strategy and, again, $\widetilde{\Delta x}\,(x,B)$ and $\widetilde{\Delta B}\,(x,B)$ are the BAU predictions, made under the assumption that the player does not change strategy and x and B are known in each present time.

In fact, we have already used two different estimators – or different absences of estimations. We have set the BAU terms to zero and, in the first reasoning, we excluded the terms in parenthesis, which led to the replicator equation. In the second one, we left the $1/Z$ term and dropped the estimators. The difference between the two can be seen – taking it lightly – as one of different orders of theory of mind. In the first case, the focal individual simply looks at the other's fitness and compares with her own. This makes sense if the individuals do not realize that different individuals can attain different fitness or that their action affects the fitness of others and their own in the future. In the second case, keeping the $1/Z$ term, individuals compare their fitness with that that they would obtain if they would be using the other's strategy, similar to a fictitious play [53] but socially driven. This means that individuals understand their impact on the immediate result and, also, they may realize that the fitness they attain using the same strategy can be different from that of the others – this is extremely important, not only for what we saw before, but even more if individuals are in a network or if there is an imbalance of power between them or, in fact, for any case of known heterogeneity between the players.

This said, so far, using some strict definition of TOM, one could easily rebut what was described as a TOM. In particular, a TOM must be something that can be used to make predictions about the behavior of others. That, of course, is present in both $\widetilde{\Delta x}$ and $\widetilde{\delta x}^{X\to Y}$. However, even in the absence of a TOM, individuals with access to data, or a memory, can develop a form for $\widetilde{\Delta x}$ by looking at the history or by looking at the present variation rates of x. Nevertheless, only a developed TOM allows one to understand that one's own actions affect the actions of others and that is exactly what is considered in $\widetilde{\delta x}^{X\to Y}$. Those are the terms that we have not specified.

Specifying them requires one to decide on the order of the TOM the individuals have and that modifies the equations used. Conversely, choosing an equation implies assuming a given order of TOM.

4.5 A FINAL EQUATION

Before finishing, let us set up an example with all the terms, leaving its analysis and the discussion of different levels of TOM to other writings.

A way to get $\widetilde{\Delta x}\,(x, B)$ could be setting $\widetilde{\Delta x}\,(\tau, x, B) = x\,(t) - x(t-\tau)$, assuming a constant variation, which would be accurate for high discounting rates, in which case we could even set $\widetilde{\Delta x}\,(\tau, x, B) = \tau\,\dot{x}\,(x, B)$. The same for $\widetilde{\Delta B}\,(\tau,\ x, B) = \tau\,\dot{B}\,(x, B)$, whose improvement depends mostly on the knowledge about the resource dynamics.

To get $\widetilde{\delta x}^{X \to Y}$ we pick up its definition in Eq.(57) and we rewrite it as

$$\widetilde{\delta x}^{\,\substack{C \to D \\ D \to C}}\,(\tau,\ x, B) = \widetilde{\Delta x}^{\,\substack{C \to D \\ D \to C}}\left(\tau, x \mp \frac{1}{Z}, B\right) - \widetilde{\Delta x}\,(\tau, x, B)\,. \tag{58}$$

One needs the knowledge of the evolution of the system when the player changes strategy. Indeed, players need to have a model of how the others will behave to that change, in our notation it would be some $\tilde{x}$ when the player changes strategy. This can be anything, from a constant to a replicator like dynamics or even a dynamics in which players understand the others as acting the same way as they do. Different orders of TOM result in different functional forms for this. Using that last case, which would need a completely developed TOM, we set:

$$\widetilde{\delta x}^{\,\substack{C \to D \\ D \to C}}\,(\tau, x, B) \approx \tau\,\dot{x}\left(x \mp \frac{1}{Z}, B\right) - \tau\,\dot{x}\,(x, B) \approx \mp\frac{\tau}{Z}\partial_x \dot{x}\,(x, B)$$

$$\widetilde{\delta B}^{\,\substack{C \to D \\ D \to C}}\,(\tau, x, B) \approx \tau\,\dot{B}\left(x \mp \frac{1}{Z}, B\right) - \tau\,\dot{B}\,(x, B) \approx \mp\frac{\tau}{Z}\partial_x \dot{B}\,(x, B) \tag{59}$$

Finally, we have an autonomous form, though implicit, for the dynamics of the system:

The order of TOM is related to the number of times one considers the impact of one's own decision on the decision of the others and how that influences the prediction one's and others' behavior.

$$\begin{aligned}\dot{x} = x\,(1-x)\sum_{\tau} w\,(\tau)\Bigg(&\Delta f_{CD}\,(\tilde{x}\,, \tilde{B}) + \\ &\frac{1}{2}\left[\Delta_{\tilde{x}}^{\frac{\tau}{Z}(\partial_x \dot{x}) + \frac{1}{Z}} + \Delta_{\tilde{B}}^{\frac{\tau}{Z}(\partial_x \dot{B})}\right]\Sigma f_{CD}\,(\tilde{x}, \tilde{B}) + \\ &\frac{1}{2}\Delta_{\tilde{B}}^{\frac{\tau}{Z}(\partial_x \dot{B})}\Delta_{\tilde{x}}^{\frac{\tau}{Z}(\partial_x \dot{x}) + \frac{1}{Z}}\Delta f_{CD}\,(\tilde{x}, \tilde{B})\Bigg)\end{aligned} \tag{60}$$

with $\Delta f_{CD}\,(x\,, B) = f_C\,(x, B) - f_D\,(x, B)$, $\Sigma f_{CD}\,(x, B) = f_C\,(x, B) + f_D\,(x, B)$, $\tilde{x} = x + \tau\dot{x}\,(x, B)$ and $\tilde{B} = B + \dot{B}\,(x, B)$. Or, neglecting $1/Z^2$ terms,

$$\dot{x} = x\,(1-x)\sum_{\tau} w\,(\tau)\Bigg(\Delta f_{CD}\,(\tilde{x}\,,\tilde{B}) + \frac{1}{2Z}\left[\partial_x + \tau\left((\partial_x \dot{x})\,\partial_x + \left(\partial_x \dot{B}\right)\partial_B\right)\right]\Sigma f_{CD}\,(\tilde{x},\tilde{B})\Bigg). \tag{61}$$

Notice that, in this case, some truncation or an explicit equation for some derivative of order n in x, $\partial_x^{(n)}\dot{x}$, is required. This term originated from the estimation of the behavior of others and its truncation or direct estimation represents the limits up to which individuals understand and compute the behavior of others.

4.6 RATIONAL VERSUS BOUNDED-RATIONAL

The rationality of the subjects under study is often a subject of interest in itself.

The previous sections were a discussion on players' abilities and sophisticated reasoning with potential to solve an intricate problem. However, clearly, there are situations in which being rational has advantages and others in which the costs of it, in terms of time, energy, or, in general, resource consumption, overcome the marginal benefits when compared to a more instinctive, or simply based in less information, decision.

Indeed, for sixty years now, economists have identified what is often referred to as "bounded rationality" in decision making processes [120, 121]. This idea reflects the inability of individuals to access and process all the information needed to make an optimal decision, in the classical economical sense. However, this does not necessarily mean that individuals do not tend to choose an optimal decision (in whatever sense). As I will show next, different levels of "rationality" can lead to the same outcome, depending on the configuration of the decision making process.

Overall, I show that if an individual interacts in different groups with possibly different configurations, he/she can access the marginal advantages of each strategy, being "bounded" or not.

4.6.1 *Setup*

Let us assume we have a population of size Z. The individuals in this population engage in some interactions in groups of size N. These interactions consist on a general investment game in which players can decide to invest, paying a cost c, or not, paying no cost. After all investments are done, every player in the group gets a benefit b, that depends on the total group investment. Let us say that in the population there is a fraction x of players that invest, or cooperate, Cs, and the remaining players do not, D. In general, the return or benefit of the individuals depends on the number of C players in that group, yN, and on the group size N. Thus, Ds only get the benefit, which makes their payoff $\Pi_D = b(y, N)$, whereas Cs pay some cost, contributing to the benefit, meaning that their payoff is $\Pi_C = -c + b(y, N)$.

Assuming the well-mixed approximation, the average payoff, or fitness, of each strategy is given by

$$f_C(x) + c = \sum_{y=0,\frac{1}{N-1},\dots}^{1} P\left(x\frac{Z}{Z-1} - \frac{1}{Z-1}, Z-1; y, N-1\right) b\left(y\frac{N-1}{N} + \frac{1}{N}, N\right) \tag{62}$$

$$f_D(x) = \sum_{y=0,\frac{1}{N-1},\dots}^{1} P\left(x\frac{Z}{Z-1}, Z-1; y, N-1\right) b\left(y\frac{N-1}{N}, N\right), \tag{63}$$

where $P(x, Z, y, N) = \binom{Z}{N}^{-1}\binom{Zx}{Ny}\binom{Z(1-x)}{N(1-y)}$ is the hypergeometric distribution, giving the probability of success when sampling from a pool of Z elements, of which a fraction x can be successful selections, a set of N elements of which a fraction y are successful selections (see **Section** 3.2 for more details on fitness computation). This formulation is general and includes, under the well-mixed assumption, all 2-player games, and all of the common N-player games, such as N-person Prisoners Dilemma (NPD), Stag Hunt (NSH) and other threshold games, and many others.

4.6.2 *Simple imitation*

Now that we have laid down the game, let us introduce some dynamics. Here, we can differentiate individuals' rationality. I will choose the two simplest scenarios discussed in **Chapter** 4. Let us start by the update with the least information, with those individuals we identified in the previous chapter as being associated with the traditional replicator and evolutionary game theory, "bounded" or, strictly in that sense, "irrational". Each time an individual, A, is revising his strategy, he compares his fitness, f_A, to that of a randomly chosen individual, f_B. Then, he changes strategy according to a probability that is an increasing function of that difference, $p(f_B - f_A)$. Thus, whenever a D is changing to C, he only needs to know his own fitness and that of the other, without any knowledge of the game or even explicitly knowing the history of the games. The probability that the number of Cs increases is, thus, $T^+ = x(1-x)p(f_C(x) - f_D(x))$, and that it decreases is $T^- = x(1-x)p(f_D(x) - f_C(x))$. The direction of evolution of x will be determined by the sign of $f_C(x) - f_D(x)$ which is given by

$$\begin{aligned} f_C - f_D = &-c \\ &+ \sum_{y=0,1/(N-1),\dots}^{1} P\left(\left(x - \frac{1}{Z}\right)\frac{Z}{Z-1}, Z-1; y, N-1\right) \Delta_y^{\frac{1}{N}} b\left(y\frac{N-1}{N}, N\right) \\ &- \sum_{y=0,1/(N-1),\dots}^{1} \Delta_x^{\frac{1}{Z-1}} P\left((x-1/Z)\frac{Z}{Z-1}, Z-1; y, N-1\right) b\left(y\frac{N-1}{N}, N\right) \end{aligned} \tag{64}$$

where Δ_x^δ is a difference operator that acts in a function $f(x,\ldots)$ such that $\Delta_x^\delta f(x,\ldots) = f(x+\delta,\ldots) - f(x,\ldots) = \delta\frac{\partial f}{\partial x} + O(\delta^2)$ (see **Chapter** 2 for details on how to go from the transition probabilities into a drift or gradient of selection).

The first thing we notice is that as $Z \to \infty$

$$f_C - f_D = -c \qquad (65)$$
$$+ \sum_{y=0,\frac{1}{N-1},\ldots}^{1} P\left(x;y,N-1\right)\Delta_y^{\frac{1}{N}} b\left(y\frac{N-1}{N},N\right)$$

with $P\left(x;y,N\right) = \binom{N}{yN}x^{yN}(1-x)^{(1-y)N}$ being the limiting distribution of the hypergeometric distribution and Z increases, making replacement irrelevant and leading to the binomial distribution.

When the number of different groups increases, one's strategies is always present in all those groups, allowing for a permanent contribution (or lack of it) that is reflected in the average payoff.

We can see that even in this "bounded" players scenario, the average return of the benefit (its derivative) in relation to each unit of investment is what determines the tendency to invest, a result that is identical to the classical economics result. However, this is only the case when the population size is much larger than the interaction group. Indeed, if we make $N = Z$, in Eq.(64), since $P\left(\left(x-\frac{1}{Z}\right)\frac{Z}{Z-1}, Z-1; y, Z-1\right) = \delta_{y\ \left(x-\frac{1}{Z}\right)\frac{Z}{Z-1}}$ and $\Delta_x^{\frac{1}{Z}} P\left(\left(x-\frac{1}{Z}\right)\frac{Z}{Z-1}, Z-1; y, Z-1\right) = \delta_{y\ x\frac{Z}{Z-1}} - \delta_{y\ \left(x-\frac{1}{Z}\right)\frac{Z}{Z-1}}$, we get

$$f_C - f_D = -c. \qquad (66)$$

This means when the groups are of the size of the population (and thus all groups formed are the same), myopic players cannot access the marginal advantages by directly comparing fitness (and this is in line with the result in Eq.55 in the previous chapter). On the other hand, when they take part in different groups, even though they have limited information, and making no computation whatsoever, they are able to understand the return of the investment. How does this compare with more rational players, who are able to compute their fitness in each hypothetical configuration?

4.6.3 *Sophisticated update*

Let us now assume the players in the population revise their strategy according to a more complex rule. A player will change his strategy if the average payoff he will get when he changes is larger than that he has now. So a D will change to a C with a probability that is an increasing function of $f_C\left(x+\frac{1}{Z}\right) - f_D(x)$ and a C will change with a probability that is an increasing function of $f_D\left(x-\frac{1}{Z}\right) - f_C(x)$. To make both dynamics socially driven, let us assume that players act according to a *contextual best-response*, which does not allow them to come up with non-existing strategies, only computing the fitness differences when they encounter someone with the other (already existing) strategy, making the probability that a D changes to C $T^+ = x(1-x)p(f_C\left(x+\frac{1}{Z}\right) - f_D(x))$ and that of a C changing to D

$T^- = x(1-x)p(f_D\left(x-\frac{1}{Z}\right) - f_C(x))$. Thus, evolution of x will be determined, for a close to linear p, by the sign of

$$\frac{1}{2}\left(f_C\left(x+\frac{1}{Z}\right) - f_D\left(x-\frac{1}{Z}\right) + f_C(x) - f_D(x)\right) = \tag{67}$$
$$-c+$$
$$+\left(1+\frac{1}{2}\Delta_x^{\frac{1}{Z-1}}\right) \times$$
$$\sum_{y=0,1/(N-1),\ldots}^{1} P\left(\left(x-\frac{1}{Z}\right)\frac{Z}{Z-1}, Z-1; y, N-1\right)\Delta_y^{\frac{1}{N}} b\left(y\frac{N-1}{N}, N\right)$$

which turns into Eq.(65) when $Z \to \infty$. However, when we make $N = Z$ we get

$$\frac{1}{2}\left(f_C\left(x+\frac{1}{Z}\right) - f_D\left(x-\frac{1}{Z}\right) + f_C(x) - f_D(x)\right) = \tag{68}$$
$$= -c + \frac{1}{2}\left(b\left(x+\frac{1}{Z}, Z\right) - b\left(x-\frac{1}{Z}, Z\right)\right)$$
$$= -c + \frac{1}{Z}\frac{\partial b}{\partial x}(x, Z) + O\left(\frac{1}{Z^3}\right), \tag{69}$$

which has the same economical interpretation as Eq.(65).

4.6.4 *Match*

With this I have shown that both rational and irrational players have the same dynamics when $N << Z$, i.e., when players are part of different groups, making them able to access the marginal advantages of cooperating even without knowing the game. This is of relevance as one could have thought that the more different interactions there are the more complex the update rule should be in order to achieve a reasonable dynamics. However, that is not the case, on the contrary. Moreover, this many groups scenario is also closer to real world applications given that individuals are part of different circles and interact with different people in different circumstances, such as their social network, and professional network, and family and so on. Perhaps this sheds light into explaining why classical economical theory still provides so many results and research even though we have identified ourselves as beings with "bounded rationality". Furthermore, the scenario of a polycentric solution for climate, the one I am interested in, and that we will deal with later on, is one in which multiple interacting instances are key to solving the pieces of a global problem.

When the game's payoffs are hard to compute, for there being a lot of different interactions, and thus taking into account one own's impact whenever changing strategy is complicated, averaging leads to the same conclusion.

For this, and also because it is both easier to handle analytically and a more common practice, I will later on use the rationally bounded players that compare their fitness directly with one another without additional information. Furthermore, since my problem concerns understanding global behavior with the common goal of promoting cooperation, it is only fair that I choose the hardest scenario for cooperation to strive and, comparing Eq.(64) and Eq.(67), it is very clear which one is. Naturally, I will take special care setting $N < Z$ and recover the discussion of the update rule only in those settings that require $N \sim Z$.

4.7 SUMMING UP

Considering discounting effects on the importance of the provision of a public good in the future requires the use of forecasting abilities. As soon as this forecasting abilities are not for the immediate future, or the discounting is not complete, one needs to consider the Theory of Mind each individual has about the others in order to anticipate the influence of one's own choices in those of the others.

We can learn from this, even if we set discounting to the maximum. In an evolutionary game theory setting we are now able to recover traditional game theory results that were not possible before due to the bounded rationality with the advantage that we have a very specific temporal dynamics that can be mapped into reality. This, however, opens up a panoply of questions and considerations. In fact, we are now under the conditions of studying and revisiting different scenarios and the possible branches are almost uncountable.

What is the impact of forecasting and discounting in games that have a fixed public good provision? Is this impact dependent of the order of TOM considered? What happens when we specify a resource dynamics? Does it depend on whether the resource is renewable or non-renewable? What if players in the same population have different TOM?

Another branch one can develop is the connection between discounting and decision making in event that are uncertain. Can we explain the shape of the discounting function with a selection process given uncertainty? Can setting the weights of discounting as an inverse uncertainty (as we do in weighted least square, e.g.) be "optimal" in some evolutionary sense?

Many assumptions have been laid down when writing Eq.(61). Which ones make the most sense? For instance, we assumed that the players were projecting with a fixed strategy. Assuming strategies that allow in the forecast for fluctuation over time is a crucial improvement, especially for low discounting rates. If the player is so good at forecasting the behavior of others she can anticipate that at a certain point she will change strategy. When or why is it important to make such a big sophistication?

I will not dwell on these questions as they fall much deeper into how the human mind works and that is far away from my goal. Nonetheless, I believe this discussion to be of crucial importance as it sets boundaries on the scenarios we can explore and sets itself as a useful tool for further development. A way around this discussion is to show that rational players, with high level of TOM, and myopic individuals, with no TOM, can be governed by the same equation under certain circumstances; and that is what I will do next.

5 CONTINUOUS OR DISCRETE STRATEGIES?

So far, we have only dealt with finite sets of strategies. In fact, most of Evolutionary Game Theory deals with a finite set of strategies. However, biological and social traits can, many times, be continuous[44, 109]. An alternative framework, called adaptive dynamics [73], is typically used in these situations but it assumes the resident population already to be in a dynamical equilibrium, usually monomorphic[14, 26]. These monomorphic states are found by an invasion process: when a new mutant type appears, the fate of such mutant invasions can be inferred from the initial growth rate of the mutant population. If every mutant is prevented from growing, the population is in equilibrium, otherwise it is not. This allows one to find the different equilibria of a population but it does not say much about its weight in terms of probability of occupancy of such states. Furthermore, this process assumes a separation of time scales between selection and mutation that may not be appropriate to social systems. We will dwell more on this in next chapter.

In general, in a large population, a continuous distribution over that trait is expected. However, as I will show, whenever selection over those traits is high and mutation is low enough, specific values are selected. Those values coincide with (potentially Nash [52]) equilibria whose predominance in the population is induced as a critical transition into a condensate as errors, both in terms of mutation and selection, decrease. Clearly, when errors are small, on should recover the states identified in adaptive dynamics but with additional information as to their occupancy probability. As a consequence, this procedure serves as an argument to discretize the strategies considered when selection is high and reveals the importance of considering the full spectrum of traits for weak selection, whenever we are interested in a detailed characterization of the population.

Let me start by introducing a game that is a continuous version of the group contribution game we already encountered. This is of particular interest given that i) is simple and ii) later on, we will be dealing with public goods games, contribution games that involve groups of individuals.

5.1 A CONTINUOUS PRISONERS' DILEMMA

Let us assume we have an infinite population of individuals. Individuals engage in interactions in groups of size N. Each individual has a choice to make in terms of contributions to a pool, $0 \leqslant c \leqslant 1$. The total amount of contributions is multiplied by an enhancement factor F and split. At a given time, t, the distribution of contributions in the population is given by $\rho(c,t)$ – or simply $\rho(c)$. The average payoff of an individual with contribution c, its fitness, is given by

$$f(c) = \int dc_1 \cdots \int dc_{N-1} \left(\rho(c_1) \ldots \rho(c_{N-1}) (c_1 + \cdots + c_{N-1} + c) \frac{F}{N} - c \right)$$
$$= (N-1) \langle c \rangle \frac{F}{N} - c \left(1 - \frac{F}{N}\right) \tag{70}$$

which gives a fitness difference between a player that contributes c and one that contributes c' of

$$\Delta f(c, c') = f(c) - f(c') = -(c - c') \left(1 - \frac{F}{N}\right). \tag{71}$$

The process of changing strategy follows a Fermi update with mutation (see **Chapter** 3 for a deeper description). With probability $(1-\mu)$ the number of individuals contributing between c and $c + dc$ (c' and $c' + dc$) will increase (decrease) by, first, selecting a player A with contribution between c' and $c' + dc$ which, in turn, selects a player B contributing between c and $c + dc$ to imitate, resulting in a contact probability of $\rho(c', t)\, dc' \rho(c, t)\, dc$. Then, player A compares his fitness with that of player B and changes to his strategy with probability $p(c', c) = (1 + \exp(\beta \Delta f(c', c)\,)^{-1}$. Finally, with probability μ a randomly selected individual will change to any strategy. So, the probability that the traits between c and $c + dc$ increases is

$$T^{+}(c) = (1-\mu) \int_0^1 p(c', c)\, \rho(c', t)\, dc' \rho(c, t)\, dc + \mu \left(\frac{1 - \rho(c, t)\, dc}{\frac{1}{dc}} \right)$$

and the probability that the traits between c and $c + dc$ decreases is

$$T^{-}(c) = (1-\mu)\, \rho(c, t)\, dc \int_0^1 p(c, c')\, \rho(c', t)\, dc' + \mu \rho(c, t)\, dc.$$

Thus, in the light of what we deduced for an infinite population in **Section** 24, the rate of change of that probability can be written as

$$\dot{\rho}(c, t)\, dc = T^{+}(c) - T^{-}(c)$$

A continuous replicator equation allows one to study the distribution of continuous traits even in the presence of selection errors and mutations.

which, letting $dc \to 0$, turns into

$$\dot{\rho}(c, t) = (1-\mu)\, \rho(c, t) \int_0^1 \rho(c', t) \tanh\left(\frac{\beta}{2} \Delta f(c, c')\right) dc' + \mu (1 - \rho(c, t)). \tag{72}$$

As a consistency check, let us see what happens at $c = 0$. Setting $B \equiv \frac{\beta}{2}\left(1 - \frac{F}{N}\right)$,

$$\dot{\rho}(0, t) = (1-\mu)\, \rho(0, t) \int_0^1 \rho(c', t) \tanh(B\, c')\, dc' + \mu (1 - \rho(0, t)).$$

The stationary solution for this point entails, for $M \equiv \frac{\mu}{1-\mu}$,

$$\rho(0) = \frac{1}{1 - \frac{\langle \tanh(B\ c') \rangle}{M}}$$

Notice that as $B \to +\infty$, $\langle \tanh(B\ c) \rangle_c \to 1$, in which limit, for $M < 1$ (i.e., $\mu < 1/2$), $\rho(0)$ results in a negative value, which clearly violates the conditions for a probability density function. So, what is happening?

5.2 CORRECTING THE EQUATION

Notice that the state $c = 0$ corresponds to a (Nash-)equilibrium for any B as long as $B > 0$, i.e., as long as F/N, the marginal per capita return, is smaller than 1. This means, in classical economic theory, that an individual has no advantage in unilaterally changing from $c = 0$ to any other strategy and coincides with the result we already deduce in population dynamics in the previous chapters. So, let us consider the probability of being in state $c = 0$, $p_0(t)$, independently. Using the same reasoning as before we can write

Even though the probability that a mutant has an exact value of the trait is zero, selection can sistematically push probability into that state. Those states correspond to stable (robust to invasion) equilibria of the distribution in the absence of mutation.

$$\begin{aligned} \dot{p_0}(t) &= (1-\mu)\, p_0(t) \left(\int_0^1 \rho(c', t) \tanh(Bc')\ dc' \right) - \mu\, p_0(t) \\ \dot{\rho}(c,t) &= -(1-\mu)\, \rho(c,t) \left(\int_0^1 \rho(c', t) \tanh(B(c-c'))\ dc' + p_0(t) \tanh(Bc) \right) \\ &\quad + \mu(1 - \rho(c,t)), \quad 0 < c \leqslant 1 \end{aligned} \tag{73}$$

Here, we have that $p_0(t) + \int_0^1 \rho(c,t)\, dc = 1$ and we can solve for the stationary distribution, separating the positive and negative terms and, again, setting $M = \frac{\mu}{1-\mu}$

$$0 = p_0 \int_0^1 \rho(c') \tanh(Bc')\ dc' \ - M\, p_0 \Leftrightarrow p_0 = 0 \vee M = \int_0^1 \rho(c') \tanh(Bc')\ dc'$$

$$0 = \rho(c) \left(-\int_0^1 \rho(c') \tanh(B(c-c'))\ dc' - p_0 \tanh(Bc) \right) + M(1 - \rho(c)).$$

Let us rewrite using $\langle f(c') \rangle = \int_0^1 \rho(c') f(c')\, dc'$ – notice that this is not an average but it is a linear operator such that $\langle const \rangle = (1 - p_0)\, const$. We get

$$\begin{aligned} p_0 &= 0 \vee p_0 = 1 - M - \langle 1 - \tanh(Bc') \rangle \\ \rho(c) &= \frac{M}{\langle \tanh(B(c-c')) \rangle + M} \end{aligned} \tag{74}$$

which describes the complete stationary distribution and is a Fredholm integral equation of the second kind. This can be solved numerically and exactly in some cases. Let us do it.

5.3 SOLVING THE EQUATION

If we let $B \to +\infty$, and define $l(c) = \int_0^c \rho(c')\,dc'$ and $r(c) = \int_c^1 \rho(c')\,dc' = 1 - p_0 - l(c)$ we get

$$p_0 = 0 \vee p_0 = 1 - M \tag{75}$$
$$0 = \rho(c)\,(1 - 2l(c) - 2p_0) + M\,(1 - \rho(c)) \tag{76}$$

Thus, we have

$$p_0 = 1 - M \wedge \rho(c) = \frac{M}{1 - M + 2l(c)}, \quad M < 1 \tag{77}$$
$$p_0 = 0 \wedge \rho(c) = \frac{M}{M - 1 + 2l(c)}, \quad M \geqslant 1 \tag{78}$$

Finally, we can differentiate the equation for $\rho(c)$, noticing that $\frac{dl}{dc}(c) = \rho(c)$, for the Eq.(77) and Eq.(78), respectively,

$$\rho(c) = \frac{M}{1 - M + 2l(c)} \Leftrightarrow \frac{d\rho}{dc}(c) = -\frac{2}{M}\rho^3(c) \wedge \rho(0) = \frac{M}{1-M}, \quad M < 1 \tag{79}$$
$$\rho(c) = \frac{M}{M + 2l(c) - 1} \Leftrightarrow \frac{d\rho}{dc}(c) = -\frac{2}{M}\rho^3(c) \wedge \rho(0) = \frac{M}{M-1}, \quad M \geqslant 1 \tag{80}$$

which have the same solution

$$\rho(c) = M\Big((1-M)^2 + 4Mc\Big)^{-\frac{1}{2}}. \tag{81}$$

Notice that, as a proper probability density function, $\int_0^1 \rho(c)\,dc = M$, for $M < 1$ and $\int_0^1 \rho(c)\,dc = 1$ for $M \geqslant 1$, which makes it the complete stationary solution.

In terms of the mutation rate, we have

$$p_0 = \begin{cases} 4\frac{|\mu - \mu_c|}{1 + 2|\mu - \mu_c|} & , \mu < \mu_c = \frac{1}{2} \\ 0 & , \mu \geqslant \mu_c = \frac{1}{2} \end{cases} \tag{82}$$
$$\rho(c) = \mu(1 - 4\mu\,(1-\mu)\,(1-c))^{-\frac{1}{2}}. \tag{83}$$

Figure 6, panel 6a, shows the solution for different values of μ. Notice that for high mutation rates, high μ, the distribution if close to flat and its value close to 1. As noise in terms of mutations decreases, the configurations near $c = 0$ get more populated and the integral of $\rho(c)$ remains 1 (see the extreme case at the transition where $\mu = 1/2$ or $M = 1$). When we pass the transition point, as mutation rate keeps decreasing, $\rho(c)$ loses its normalization and starts flattening and vanishing, with the probability being absorbed by the single state $c = 0$. Panel 6b shows how the probability of that state increases as mutation decreases.

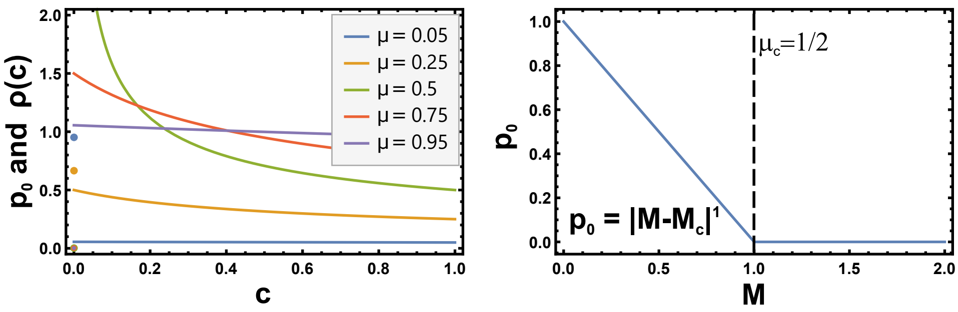


Figure 6: **Stationary distribution for** $B \to \infty$ (a) The lines represent $\rho(c)$ and the dot at $c = 0$ represents p_0. (b) Represents the condensation into $c = 0$ at $M = 1$, where the probability of being at $c = 0$ increases as the mutation rate decreases.

Clearly this is a condensation into the equilibrium ($c = 0$) for small random exploration rates ($\mu < \mu_c = 1/2$) with the ratio $M = \mu/(1-\mu)$ controlling the transition with a critical exponent of 1. This result for very high selection pressure ($B \to +\infty$) immediately suggests that, as long as $M < 1$, the selection pressure, B, can also induce a condensation in the equilibrium with that state containing up to $1-M$ of the probability.

To investigate that let us first take the opposite limit, $B \to 0$. Going back to Eqs(74), we get

$$p_0 = 0 \vee M = B\langle c \rangle \tag{84}$$
$$M = \rho(c)\left(Bc - B\langle c \rangle + M\right). \tag{85}$$

But if $M = B\langle c \rangle$ then $\rho(c)$ has no finite integral. Thus, $p_0 = 0$ is the only possible solution and

$$\rho(c) = \frac{M}{B\left(c - \langle c \rangle\right) + M}. \tag{86}$$

Clearly, $\langle c \rangle < \frac{M}{B}$ and, equating $\int_0^1 \rho(c)dc$ to 1, we get $\langle c \rangle = \frac{1}{2} + \frac{M}{B} - \frac{1}{2}\coth\left(\frac{B}{2M}\right) \approx \frac{1}{2} - \frac{B}{12M}$, which is a reflection of a flat distribution as $B \to 0$.

Finally, given that $M = \langle \tanh(Bc') \rangle \leqslant 1 - p_0$, if $M \to 0$, $p_0 \to 1$ for any B and, on the other hand, when $M = 1$, $p_0 = 0$. So at some point there should be a transition where the state $c = 0$ goes from 0 to $1 - M$.

Let us analyze it numerically. **Figure** 7 clearly shows a transition for some critical value of B, B_c.

5.4 ESTIMATING THE CRITICAL TRANSITION

Let me try to make a rough estimation of the critical value of the transition. For high B, let me set $\tanh(x) \approx 1 - \Theta(x-1)(1-x)$, where $\Theta(x)$ is the Heaviside function that is 1 is $x \geqslant 0$ and 0 otherwise. Then,

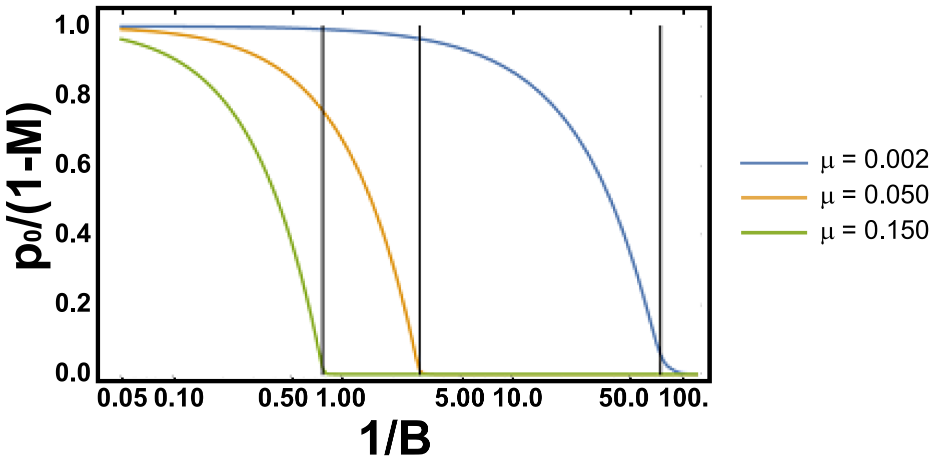


Figure 7: **Condensation into $c = 0$ for finite B** The value of $1/B$ controls the number of errors in the imitation process. As these decrease, the population goes from occupying all states between 0 and 1 with probability $\int_0^1 \rho(c)dc = 1$ to ocupying a given state, $c = 0$, with non-zero probability.

$$p_0 \approx 1 - M - \left\langle \Theta\left(Bc' - 1\right)\left(1 - Bc'\right)\right\rangle$$

And if I assume that $\rho(c)$ does not change much, we can use the $\rho(c)$ we computed for $B \to \infty$, in Eq.(81). Then, we get for $B > 1$

$$p_0 \approx 1 - M - M\int_0^{\frac{1}{B}} \left(1 - Bc'\right)\left((1-M)^2 + 4Mc'\right)^{-\frac{1}{2}} dc' \qquad (87)$$

which has an ugly but simple expression that can be solved for the critical value of B, B_c, that makes $p_0 = 0$, which results in

$$B_c = \alpha \frac{M}{(1-M)^2}, \text{ with } \alpha = \frac{7\sqrt{105} - 69}{12} \approx 0.227388. \qquad (88)$$

Just for the sake of completeness, for $B < 1$, where the approximation of $\rho(c)$ should not be valid,

$$p_0 \approx 1 - M - M\int_0^1 \left(1 - Bc'\right)\left((1-M)^2 + 4Mc'\right)^{-\frac{1}{2}} dc' = 1 - \frac{1}{6}\left(12 - B\left(3 - M\right)\right) M$$

which gives $B_c = \frac{6(2M-1)}{(3-M)M}$, and completes the approximation for high B_c.

In the opposite limit, of $B \to 0$, we know that the distribution will be close to flat, so we can use that $\rho(c)$ for the limit of small B. In that case we get

$$p_0 \approx 1 - M - (1 - p_0)\int_0^1 \left(1 - \tanh\left(Bc'\right)\right) dc' \Leftrightarrow p_0 \approx 1 - \frac{BM}{\log\cosh B} \qquad (89)$$

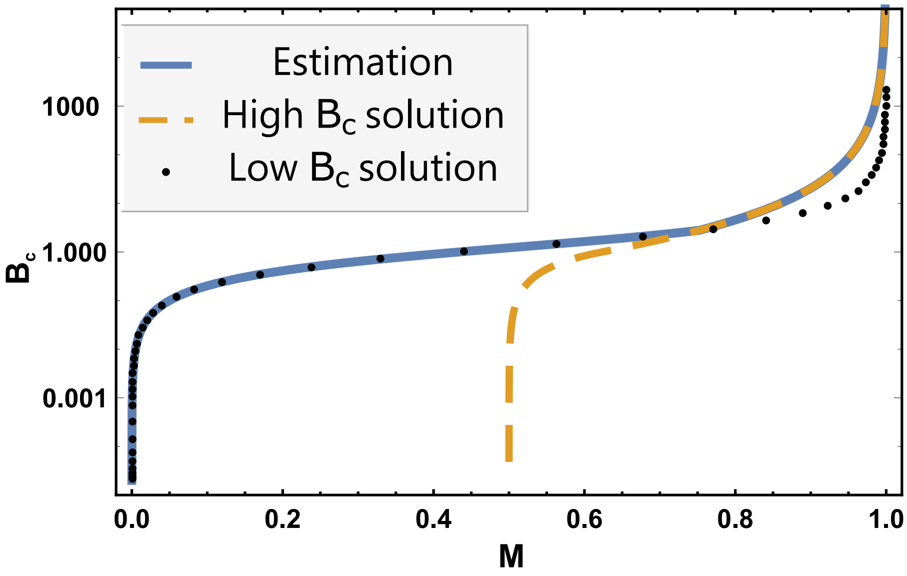


Figure 8: **Critical value of B_c above which there is condensation** The estimation is obtained by flipping from the low B_c solution to the high B_c solution when they coincide. This is useful as the numerical solutions are very sensitive to the discretization of the continuous parameter.

which we can solve for the critical value of M, $M_c(B) = \frac{\log\cosh B}{B}$, or $B_c = {M_c}^{-1}(M)$.

Apart from being an application for the equation that describes the evolution of continuous traits in the presence arbitrary rates of selection errors and mutations, this proof shows how this kind of system tends to choose particular values even if the traits being considered are continuous. This happens in the regime of *strong selection*. In the case discussed this was controlled by $B = \beta/2(1-F/N)$. In general, i. e., for different cases, this is still controlled by β. In fact, we can tell that as long as β times the scale of the fitness differences is at least of the order of 1 and mutation is not so large ($\mu < 1/2$), then selection is strong. In the opposite regime, of *weak selection*, the population is distributed across the range of traits if mutation is positive. This means that, if we want to describe a population with continuous traits and its dynamics, it is worth setting β high and consider specific values of that trait or, otherwise, for weak selection, the description can only be complete with the full spectrum of strategies.

For this, from now on, we will focus on sets of discrete traits and take into account that selection should not be weak. In the next chapter, the last before going into the application of all this, I take the idea that populations tend to adopt specific configurations further and develop a mechanism that allows one to study finite populations with finite but large sets of traits.

6 STOCHASTIC DYNAMICS THROUGH HIERARCHICALLY EMBEDDED MARKOV CHAINS

This chapter is based on the manuscript "Stochastic dynamics through hierarchically embedded Markov chains" by Vítor V. Vasconcelos, Fernando P. Santos, Francisco C. Santos and Jorge M. Pacheco, submitted to Physical Review Letters.

Even though in the previous chapter we were able to find a justification to discretize traits, even if they could be continuous, the study of dynamical processes in (finite) populations often requires the consideration of Markov processes of significant mathematical and/or computational complexity, which rapidly becomes prohibitive with increasing population size or increasing number of individual strategies. A ubiquitous example is found in the evolutionary game models of social systems that we have been studying and whose strategy space is sizeable. Tractability often leads one to compute the Markov stationary distribution in the so-called *small mutation approximation* (SMA), maximally reducing the configuration space (see below). Despite its popularity the SMA cannot account for several scenarios emerging at the macro-level, e.g., when individual decisions are subject to unanticipated random exploration, or when the real dynamics encompasses the existence of stable polymorphic configurations. Here, my coauthors and I develop a general and widely applicable procedure that allows one to calculate a hierarchy of approximations to the stationary distribution of general systems at progressive levels of approximation. The first-order in this procedure scales with the population size in the same way as the SMA (zeroth-order in this scheme), thus providing an efficient method for studying social and biological communities at non-vanishing mutation rates. In general, the method developed here is able to provide a hierarchy of estimations of the stationary distribution at the boundaries of the phase space of a discrete population Markov process with time invariant transition probabilities.

Many complex time-dependent processes, from the evolution of cooperation [98, 132, 133] to genetic drift and evolution of eco-systems [20], flocking behavior [21], voter dynamics [62], disease spread [3], diffusion of innovations [78], consensus formation [127] and peer-influence [100], have been modeled by means of finite-population stochastic Markov processes. Whenever the number of possible traits (or behaviors) of each individual increases, so does the complexity of the associated Markov chain. As we discussed in **Chapter 2**, even when these processes are approximated by their continuous infinite population limit, disregarding finite-size effects that may prove important[2, 29], the non-linear nature of the dynamics may preclude a full-analysis, in which case (quasi-)stationary distributions of the Markov chain still provide insightful information[36, 49, 55, 95, 119, 145]. However, determining these distributions may become impractical given that the size of the chains involved rapidly becomes prohibitive (see below). As a result, an approximation has been introduced – the so-called Small Mutation Approximation (SMA) – that relies on the development of a minimal (embedded) Markov chain whose solution estimates the limiting stationary distribution of the population [36]. We have already alluded to this, but let me describe it in more detail.

6.1 THE SMALL MUTATION APPROXIMATION

To reveal the important savings brought about by the SMA, let us consider a population of finite size Z in which each individual may adopt one of S different strategies/traits, $\sigma_1, \sigma_2, \ldots, \sigma_S$. In the (conventional) mean-field approximation, population configurations are characterized by the number of individuals adopting each strategy, $\{i_1, i_2, \ldots, i_S\}$ adding up to $|s| = \binom{Z+S-1}{S-1} \approx \frac{Z^{S-1}}{(S-1)!}$. The complexity obtained for a large number of strategies (large S) thus turns the complete analysis unfeasible, even for small populations (small Z). In the absence of mutations ($\mu = 0$), this stochastic process has S absorbing states, the so-called monomorphic [97, 115], pure [36], or homogeneous [36, 118] configurations, in which all individuals play the same strategy. Positive values of μ, in turn, allow escaping from the absorbing states. In this context, it is easy to introduce the SMA and its intuition: Assume that the population starts from a monomorphic configuration. The population will remain there until a mutation happens that flips the behavioral state of one individual. This new behavior either spreads in the population, ultimately invading and leading to another monomorphic configuration or, alternatively, it will go extinct, such that the population returns to the monomorphic configuration it started from. To the extent that mutations are rare, the time scales of selection (fast) and mutation (slow) become separated [49]. This allows us to define an embedded Markov chain consisting only of monomorphic configurations (S states). The transition matrix can then be found by computing the fixation probability of a single mutant in a population of resident individuals all in the same state [84]. Clearly, the SMA brings about a remarkable simplification, since now the (embedded) *configuration space has size* S and, importantly, all transition probabilities are computed by means of an invasion/extinction process *involving only two strategies at a time.*

The SMA is hardly applicable to highly noisy social systems and, even for low mutation systems, quickly fails for dynamics with coexistence equilibria.

Since its development, the SMA has been employed with success in different areas of research [49, 55, 97, 99, 108, 115, 119, 143]. However, the validity of the SMA requires the mutation probability μ to be small [151], depending on the population size and underlying dynamics of the system.

Here we develop a new method that provides a hierarchical selection of the configuration space which, starting from the SMA as its 0th-order (H0), introduces incremental levels of complexity. This hierarchical construction not only enlarges the range of mutation rates at which "low-cost" approximations remain valid, allowing also the explicit inclusion of mutations in the population dynamics, but also provides a means to assess the validity of the SMA at barely no additional cost. Given that many time-dependent processes of interest are not amenable to be described in the SMA[24, 25, 99, 108], in the sense that the mutation probability does not allow for a time scale separation, an approximate yet more reliable framework, such as the one developed here is highly desirable [25, 31].

6.2 GETTING THE FEEL OF IT

The goal is to separate the set of all possible configurations $s = \{s_1, s_2, \dots, s_{|s|}\}$ into two disjoint sets – $A = \{a_1, a_2, \dots, a_{|A|}\}$, the Configurations of Interest (CoI) and $B = \{b_1, b_2, \dots, b_{|B|}\}$, the neglected configurations, such that $|s| = |A| + |B|$ – in a way that it is possible to accurately infer the full dynamics by solely considering the behavior of the CoI.

The SMA (H0) described above provides the largest contraction of the configuration space, as the reduced CoI space is limited to the set of monomorphic configurations, making $|A| = S$. The next level of the hierarchical approach (1st-order, H1) implies enlarging the set A. For simplicity, let us start with the one-dimensional case described below ($S = 2$), using it to point out the limitations of the SMA and motivate the 1st-order extension (H1). Specifically, let us consider the evolutionary dynamics of a population of finite size Z in which individuals interact by means of a 2-person game that posits a social dilemma of cooperation, reflected in the associated payoff-matrix (compare with **Section** 2.2)

$$\begin{array}{c} \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cc} C & D \end{array} \\ \begin{bmatrix} 1 & f \\ 1+g & 0 \end{bmatrix} \end{array}$$

where f and g are assumed to vary in the interval $[-1, 1]$ [114]. Individual behavior is either **C** or **D**, and individual fitness, that drives the dynamics of the population, is associated with the average payoff acquired from interacting with all other members of the population. Any possible configuration of this population, s_i, is defined in terms of the number i of individuals that use strategy **C** (such that $Z - i$ individuals use strategy **D**). The temporal evolution is usually implemented by means of a discrete-time *birth-death* process [38], characterized by the probabilities that, in each time-step, the number of individuals adopting strategy **C** changes by ± 1 or remains the same. We model the one-step transitions as the outcome of a stochastic imitation process described in **Chapter** 3, inspired in the Fermi distribution of statistical physics [138], $T_i^{\pm} \equiv T_{i \to i\pm 1} = (1-\mu)\frac{i(Z-i)}{Z(Z-1)}\frac{1}{1+e^{\pm\beta(f_C(i)-f_D(i))}} + \mu\left(\frac{Z-i}{Z}\delta_{1,\pm 1} + \frac{i}{Z}\delta_{-1,\pm 1}\right)$, where the inverse temperature, $\beta \geqslant 0$, mimics here the intensity of selection [139], with added mutation probability μ that accounts for the possibility of (unanticipated) random exploration of strategies, an important process in social and cultural evolution [12, 13].

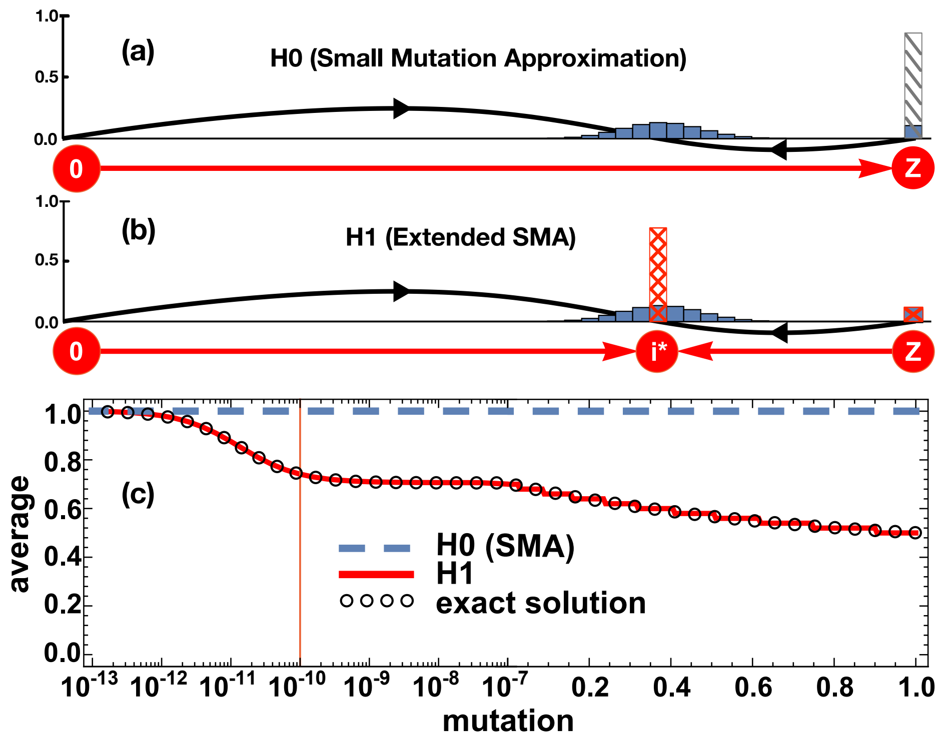


Figure 9: **Stationary distributions associated with a Coexistence game** ($f = 0.7$, $g = 0.3$**)** The gradient of selection is represented by the black curves, whereas the exact stationary distribution, computed for $\mu = 10^{-10}$, is depicted by solid (blue) bars. Two levels of approximation are also considered: i) the small mutation approximation (SMA or zeroth-order, H0, dashed gray bars) and ii) the extended SMA (first-order, H1 crossed red bars). In panel **(a)** we compare the exact solution with the SMA. In the SMA, only the configurations $i = 0$ and $i = Z$ are CoI, and the most probable transitions between CoI – indicated by orange arrows – suggest why the full strength is concentrated at $i = Z$. Clearly, $\mu = 10^{-10}$ in a population of size $Z = 50$ is not small enough to bring the exact result into the SMA domain of validity. In panel **(b)**, the crossed red bars represent the stationary distribution yielded by the H1, where i^* was added to the CoI. In panel **(c)** we compare the three methods by plotting the average of the different stationary distributions. The vertical line marks the value of μ employed in **(a)** and **(b)**. Other parameters: $Z = 50$, $\beta = 10$.

In this one-dimensional case, additional configurations to be included in A must belong to the line joining the two monomorphic states, as this line exhausts the phase-space (simplex) available to the system.

In **Figure** 9A we provide an example of a 2-strategy coexistence game, which we have already encountered in **Section** 2.2, known in Physics and Economics as the Snowdrift Game [131], in Evolutionary Biology as the Hawk-Dove game [124], and in other contexts as the Chicken game [153]. We consider a finite population of size $Z = 50$, whose full stationary distribution is depicted by the blue histogram bars ($\mu = 10^{-10}$). There are two monomorphic states, associated with configurations in which all individuals play either strategy **D** ($i = 0$) or strategy **C** ($i = Z$). The existence of a probability attractor at $i^* = 35$ justifies the coexistence nature of this evolutionary dynamics, where selection drives the population towards i^*. The SMA approximation leads to the distribution depicted in **Figure** 9A with a solid gray bar. The solid blue line represents the so-called gradient of selection, given by $g(i) = T_i^+ - T_i^-$. Clearly, the SMA leads to a distribution that differs substantially from the full distribution. In other words, the existence of an interior attractor means that $\mu = 10^{-10}$ is still too large a mutation probability for the SMA to lead to a reliable estimate of the distribution. This is easily confirmed by inspection of **Figure** 9C, where we change the mutation probability across 13 orders of magnitude. Indeed, one still needs to reduce the mutation probability by 3 orders of magnitude to obtain a good agreement between the SMA and the full distribution. Importantly, as μ increases, deviations start to occur, and the SMA quickly fails to account for the changes introduced in the stationary distribution by non-zero mutations.

Figure 9B, in turn, shows the result of adopting the first-order term in this hierarchical approach (H1). To this end, we now add i^* (the root of G and coexistence equilibrium of the continuous analogue problem) to the set A of CoI, which already contained the two monomorphic configurations included in the SMA. This additional configuration is trivial to find [138]. The transition probabilities $\rho_{a_i \to a_j}$, between CoI configurations a_i and a_j in A, merely require a re-partitioning of the terms already computed in the SMA, thus bringing no additional overhead to the computation. Similarly to setting $T_k^{\pm} = T_{k \to k\pm 1}$, an ordering can be defined over configurations in A such that the transition probabilities from configuration a_i to configuration $a_{(i\pm 1)}$ are written as $\rho_{a_i}^{\pm} \equiv \rho_{a_i \to a_j}$. Those, in turn, can be written as the probability of having a one-step transition in the direction of $a_{(i\pm 1)}$ and then being absorbed by it, resulting in $\rho_{a_i}^{\pm} = T_{a_i}^{\pm} \left(1 + \sum_{j=a_i \pm 1}^{a_{i\pm 1} \mp 1} \prod_{k=a_i \pm 1}^{j} \frac{T_k^{\mp}}{T_k^{\pm}}\right)^{-1}$ (see proof bellow, **Section** 6.9).

With the transition probabilities $\rho_{a_i}^{\pm}$, the unnormalized stationary distribution over configurations in A, $p(a_i)$, is then calculated through an eigenvector search [61]. In order to obtain the histogram shown with red bars in **Figure** 9B, however, a proper renormalization procedure is required. Indeed, as a_1 is a probability attractor, it should carry an associated strength that mimics the weight of the full stationary distribution (blue bars) in its

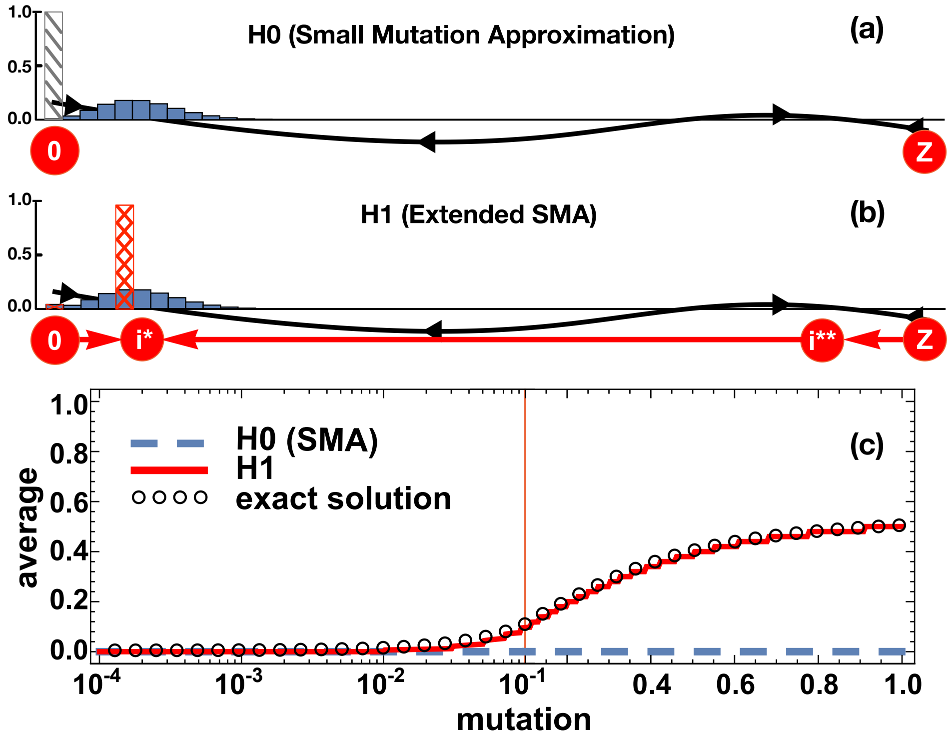


Figure 10: **Stationary distributions resulting from a Coordination game (**$f = -0.7$**,** $g = -0.3$**)** We use the same notation and conventions as in **Figure** 9.

vicinity. The natural choice is to derive this renormalization factor from the Kramers-Moyal expansion of the associated master equation [38, 95] (the same expansion we did in **Chapter** 2). We may thus write the distribution as

$$P(i) \approx \alpha^{-1}\left(p(a_0)\,\delta_{ia_0} + Z\sqrt{2\pi\sigma^2}p(a_1)\,\delta_{ia_1} + p(a_2)\,\delta_{ia_2}\right)$$

where $\alpha = p(a_0) + Z\sqrt{2\pi\sigma^2}p(a_1) + p(a_2)$, and σ^2 can be derived from the transition probabilities T_x^+ and T_x^- around a_1/Z as $\sigma^2 = F/|J|$ with $J = d\left(T_x^+ - T_x^-\right)/dx|_{x=a_1}$ and $F = \left(T_{a_1}^+ + T_{a_1}^-\right)/(2Z)$. Below we provide a proof of this results and discuss the validity and accuracy of this estimate.

Whenever the game at stake is not one of coexistence but, instead, one of coordination, the approach remains unchanged. In **Figure** 10A we provide an example of a 2-strategy coordination game or Stag Hunt (see **Section** 2.2), embodying both payoff and risk-dominant equilibria, of ubiquitous importance in areas as diverse as Economics, Econo-Physics, Biology and Philosophy [67, 69, 122, 123, 131]. Once again, there are two monomorphic states. Now, however, the evolutionary dynamics is characterized by the occurrence of a probability repeller at $i^* = 35$, such that selection drives the population away from i^* towards $i = 0$ and $i = Z$.

Unlike the scenario associated with **Figure** 9, coordination leads to a much better scenario for the SMA, as shown in **Figure** 10C by changing the

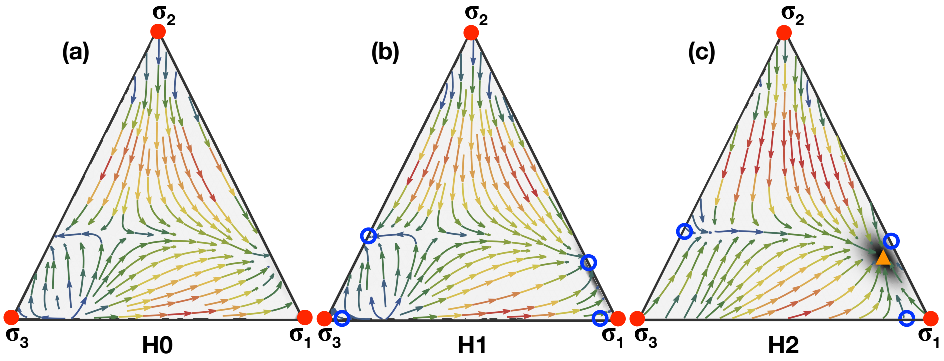


Figure 11: **CoI at different levels of the hierarchical approximation** The arrows represent the most likely direction of evolution of the system (the gradient of selection, G; warm colors represent larger magnitudes whereas the background gray shading represents the stationary distribution (darker areas correspond to states with higher probability). As mutation increases the population explores configurations deviating gradually from the vertices. For the dynamics and parameters specified below, this trend starts at $\mu = 10^{-4}$ (panel **(a)**), exploration of the phase space extending mostly along the edges up to $\mu = 10^{-2}$ (panel **(b)**), to finally explore the interior of the simplex for higher values of μ ($\mu = 10^{-1}$ in panel **(c)**). The probabilities of updates from strategy σ_i to σ_j are given by $T_{\sigma_i \to \sigma_j} = \frac{Z}{Z-1} x_i x_j \frac{1-\mu}{1+e^{\Delta f_{\sigma_i \sigma_j}}} + x_i \frac{\mu}{d}$, with $\Delta f_{\sigma_i \sigma_j} = \beta_{ijx}(x_1 - x_1^*) + \beta_{ijy}(x_2 - x_2^*)$, x_i being the frequency of strategy σ_j, and d the number of accessible strategies given the restriction of the phase space considered. Parameters: $Z = 50$, $\beta_{12x} = 2$, $\beta_{12y} = \beta_{13x} = 10$, $\beta_{23y} = -10$, $\beta_{13y} = \beta_{23x} = 0$, $x_1^* = 2/10$ and $x_2^* = 3/10$.

mutation probability across 5 orders of magnitude: Now the SMA leads to very good results up to $\mu \approx 1/Z$. However, as μ increases, the deviations start to occur (**Figure** 10A), and the SMA fails to account for the changes introduced in $g(i)$ by sizeable ($> 10^{-0.5}$) mutations. **Figure** 10B shows the resulting distribution after adding two configurations to the set A of the CoI. As **Figure** 10C shows, extending the set A leads to an excellent agreement between the hierarchical approximation and the exact solution. To reach this agreement, we include the two probability attractors that emerge naturally out of this coordination game when we add a sizeable mutation probability; indeed, these are associated with the roots of $g(i)$ close to the monomorphic states. Naturally, these two roots will depend on μ and their location should be determined for each value of μ. Once this choice is made, we merely repeat the procedure already adopted in connection with the coexistence game discussed before.

It is worth pointing out at this stage that augmenting the CoI allows us to include explicitly the role of mutations, whose occurrence is ubiquitous in (highly noisy) social systems [142]. Indeed, whereas in the SMA the notion of mutation is used as a tool to enforce that the embedded Markov chain

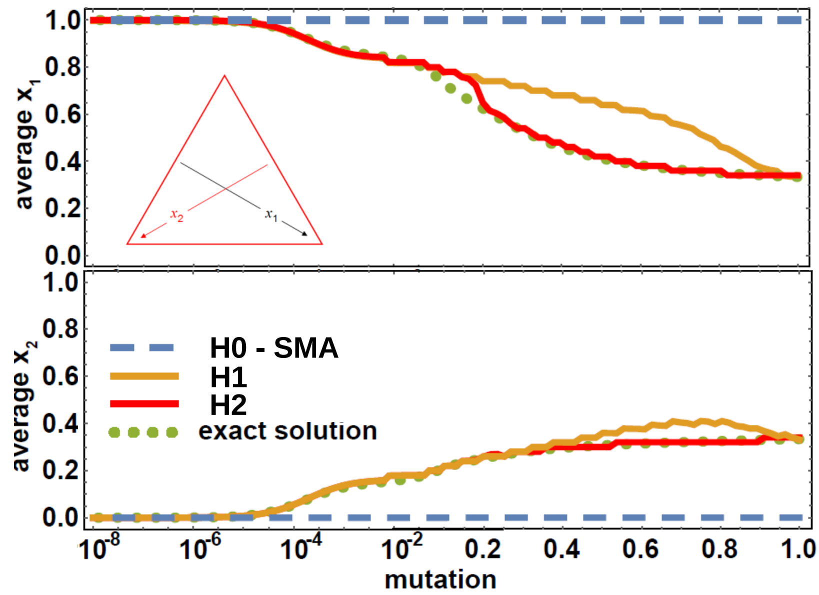


Figure 12: **The second order of approximation** We plot the average of the stationary distributions as a function of mutation probability, for the system depicted in **Figure** 11, and at different levels of the hierarchical approximation. At the H0 level (SMA), accurate average values are obtained whenever $\mu < 10^{-5}$. As mutation increases, the polymorphic configurations attract higher probabilities, requiring the inclusion of CoI along the edges of the simplex (H1) and in its interior (H2).

is irreducible and it does not occur in transient states, in this case nothing prevents mutations to occur *at par* with the selection process.

The first order extension (H1) is enough to provide an accurate description of the population in the one-dimensional case discussed up to now. This happens because a single mutation is enough to make the system explore the entire phase space starting from the vertices. Yet, higher dimensions may call for higher order approximations, as we discuss in the following for a two-dimensional problem (see next sections for the general case).

Let us now consider a population in which Z individuals may adopt (be in) one of three possible strategies (σ_k, $k = 1, 2, 3$). Each configuration $s_i = (i_1, i_2)$, is one in which i_1 (i_2) individuals have strategy σ_1 (σ_2) (and $Z - i_1 - i_2$ individuals with strategy σ_3). Both the gradient of selection and stationary distribution can be represented using the two-dimensional simplex portrayed in **Figure** 11.

In this case, the H0 (SMA) leads to a set of CoI comprising the three monomorphic configurations (associated with the vertices of the triangles, red solid circles in **Figure** 11A), confining the evolutionary trajectories to the edges of the simplex. At the next level of approximation (H1), trajectories retain the same constraints, but the CoI set now includes additional configurations in the edges, as illustrated with blue solid circles in **Figure** 11B. Importantly, the criteria adopted in choosing these CoIs follows straightforwardly from the one-dimensional cases already discussed before, given the constraints that apply to the trajectories at this level of approximation.

At level H2, we add to the CoI configurations in the interior of the simplex (in which three strategies are present), as illustrated with the solid yellow circle shown in **Figure** 11C. The results obtained are shown in **Figure** 12, where we compare the exact solution (green circles) with the results provided by successive orders of approximation, for a wide range of mutation values. We considered a generic two-dimensional evolutionary dynamics whose interior encloses a saddle point, while exhibiting both coexistence and coordination dynamics along the edges (see caption to **Figure** 11).

As expected, the SMA is unable to provide an accurate description of the stationary distribution across a wide range of mutation values. As μ increases, the stable fixed points move away from the vertices, first along the edges, a feature which is nicely captured at the level of H1, and subsequently to the interior of the simplex, requiring one to move to the H2 level.

It is noteworthy that both H1 and H2 levels are capable of providing accurate results regarding the average adoption of strategies. Naturally, however, the outcome of the two approximations is different. Intuitively, if we consider the extreme case of $\mu = 1$, clearly the stationary distribution will be concentrated in the center of the simplex – an interior stable fixed point $\mathbf{s}_{\mathbf{i}^*} = \left(\frac{Z}{3}, \frac{Z}{3}\right)$. This feature is nicely captured at the H2 level. At the H1 level, instead, since the CoI does not contain interior configurations, the result suggests that most of the time is spent under states $\mathbf{s}_{\mathbf{i}_1^*} = \left(\frac{Z}{2}, \frac{Z}{2}\right)$, $\mathbf{s}_{\mathbf{i}_2^*} = \left(0, \frac{Z}{2}\right)$ and $\mathbf{s}_{\mathbf{i}_3^*} = \left(\frac{Z}{2}, 0\right)$, which, on average, leads to the same result as obtained under H2 (simply by symmetry). This correspondence, nonetheless, would not occur considering, e.g., the variance of the distribution.

6.3 A discussion on the generality of the method

Restricting the phase-space only makes sense looking at conditional probabilities. Clearly, if a single parameter controls the probability of going from one region to others, the smaller it is, the more similar the conditional and full probability distributions will be.

The examples used in the text above belong to evolutionary game theory, the one I will be interested in the last part of the dissertation, which provides a very convenient framework to explore the approximation procedure that we propose given that, on one hand, different games lead to very different phase-space dynamics, while on the other hand a single parameter – mutation – controls the probability of jumping between the different topologically distinct regions in phase space. More precisely, mutation allows the system to move from the vertices to the edges of the phase space simplex, as well as from the edges to the faces, and so on. In this sense, it is natural that the simplest (and, in this notation, the 0th-order) estimate of the stationary distribution at the vertices originated in this context and was coined as the small mutation approximation (SMA).

Despite my arguments being built on evolutionary models, we believe the scope of the method reaches far beyond that and can be applied to a broader class of problems, of potential interest in several branches of science, as mentioned in the beginning of this chapter. In fact, the method we propose allows one to estimate the stationary distribution of a generic discrete population Markov process through a corresponding embedded Markov chain whose states are selected (starting) from the boundaries of the original phase-space (or simplex) and moving inward. The approximation is hierarchical as one may tune the recursion level of the state-space reduction, starting from the vertices (0th-order), to the edges (1st-order), faces (2nd-order) and so on, with a given level n of the approximation lending results in $O(\mu^n)$, where in general μ controls the probability of jumping from one hyperface to the other. This means that it can be adapted to the needs and features of different problems. Clearly, if there is no natural μ separating different regions of phase space, the results neglect the interaction between regions that should be taken into account. In that case, we can think of the results of a given order as a projection of the system dynamics to the considered region.

The success of 0th-order approximation – the SMA – is testified by the many applications employing it [36, 55, 97, 99, 108, 115, 143], meaning that, in many cases, the resulting embedded Markov chain leads to a distribution that provides useful information.

Next we provide the framework in which to develop this hierarchical approximation in the general case.

6.4 TRANSITION PROBABILITIES BETWEEN COI (ρ) IN AN ARBITRARY MARKOV CHAIN

Let us start by laying down the general framework for an arbitrary chain, defining the transition probabilities between the selected CoI using the transitions stemming from the full chain. To begin with, it is worth noting that in any finite chain, the configurations can be counted and ordered. Thus, we consider a finite set of configurations $\{s_1, s_2, \dots, s_{|A|+|B|}\}$ that, as clarified below, can be divided in those of interest (CoIs, here belonging to set A) and those to be neglected (set B). A time-homogeneous Markov process happens over this set such that the system moves from configuration s_k to configuration s_l, in each step, with a time invariant probability $T_{s_k \to s_l}$ (or $T_{s_k s_l}$ in matrix notation) independent of the configurations it occupied in all previous times.

The goal now is to reduce the number of considered configurations to those of interest (CoIs), $A = \{a_1, \dots, a_{|A|}\}$, and calculate the transition probabilities between each of these selected configurations, $\rho_{a_i \to a_j}$ (or $\rho_{a_i a_j}$). For this we also need the transition probabilities $T_{b_n b_m}$ between neglected configurations $B = \{b_1, \dots, b_{|B|}\}$ and also the transitions between neglected configurations and CoIs, which we call $P_{b_n \to a_j}$ (or $P_{b_n a_j}$). Thus, we can write that

$$\rho_{a_i a_j} = \sum_{\{b_n\}} T_{a_i b_n} P_{b_n a_j} + T_{a_i a_j} \text{ and} \tag{90}$$

$$P_{b_n a_j} = \sum_{\{b_m\}} T_{b_n b_m} P_{b_m a_j} + T_{b_n a_j}. \tag{91}$$

If we conveniently order the configurations, placing the neglected configurations first and then the CoI, we can make use of matrix notation and algebra. We may write

$$\mathbf{T} = [T_{kl}] = \begin{bmatrix} \mathbf{Q} & \mathbf{R} \\ \mathbf{U} & \mathbf{V} \end{bmatrix}, T_{kl} \equiv T_{s_k s_l} \tag{92}$$

with $Q_{mn} = T_{b_m b_n}$ representing one-step transitions between neglected configurations, $R_{mi} = T_{b_m a_i}$ representing one-step transitions from neglected configurations to CoI, $U_{im} = T_{a_i b_m}$ representing one-step transitions from CoI to neglected configurations and $V_{ij} = T_{a_i a_j}$ representing one-step transitions between CoI. Notice that if there are no adjacent CoI, $\mathbf{V} = 0$. This turns Eqs.(90,91) into

$$\boldsymbol{\rho} = \mathbf{UP} + \mathbf{V} \text{ and} \tag{93}$$

$$\mathbf{P} = \mathbf{QP} + \mathbf{R}. \tag{94}$$

Finally, we can write for the transitions between CoI

$$\boldsymbol{\rho} = \mathbf{U}(1 - \mathbf{Q})^{-1}\mathbf{R} + \mathbf{V}. \tag{95}$$

Notice, however, that the calculation of these transitions involves a matrix inversion, which, if calculated numerically, may be ill-conditioned.

Computing probabilities of reaching a CoI is equivalent to the computation of fixation probabilities, with an appropriate ajustment for different rates at which the different absorption processes happen.

Nonetheless, when considering a one dimensional chain – which turns out to be the cornerstone of the 1st-order approximation – this expression becomes fairly simple and exempts the numerical inversion of matrices or the eigenvector search. After presenting the general framework, we devote a whole a section (**Section** 6.9) to explore this scenario.

6.5 USING ρ TO COMPUTE STATIONARY DISTRIBUTIONS OVER COI AND THE NECESSITY OF ESTIMATING THE REMAINING DISTRIBUTION

Let us distinguish between the general process we defined, with transition probability between configurations s_k and s_l, $T_{s_k s_l}$, and the process that happens only over the set of CoI that one gets from using $\boldsymbol{\rho}$ as its transition matrix. Because we calculate transitions between CoI, $\boldsymbol{\rho}$, from the microscopic transitions between all states, $T_{s_k s_l}$, the two processes must be related. Let us figure out that relation.

Consider the full Markov chain with transition probability between configurations s_k and s_l, $T_{s_k s_l}$. We have seen that we can write $\mathbf{T}$ as in Eq.(92)

and, thus, according to the Master Equation [38], the stationary distribution of the system can be obtained as the vector $\mathbf{p} = \{\mathbf{p}_b, \mathbf{p}_a\}$ whose sum of the entries is 1 and is given by

$$\begin{bmatrix} \mathbf{p}_b \\ \mathbf{p}_a \end{bmatrix} = \begin{bmatrix} \mathbf{Q} & \mathbf{R} \\ \mathbf{U} & \mathbf{V} \end{bmatrix}^{T} \begin{bmatrix} \mathbf{p}_b \\ \mathbf{p}_a \end{bmatrix}. \tag{96}$$

Solving Eq.(96) by substitution we get that, apart from normalization, the probability of finding the system in the subset of configurations a_i, $\mathbf{p}_a$, is given by

$$\mathbf{p}_a = \boldsymbol{\rho}^{\mathbf{T}} \mathbf{p}_a. \tag{97}$$

This equation for $\mathbf{p}_a$ is exactly the equation of the stationary distribution of the process concerning only the CoIs and $\boldsymbol{\rho}$ as its transition matrix, and this defines the relation between the two processes.

This result shows that the transitions $\boldsymbol{\rho}$, computed in the previous section as Eq.(95), when used as transition matrices for the calculation of the stationary distribution, turn out to give the probability of finding the full system in each CoI apart from a normalization constant, which depends on the neglected configurations. Consequently, i) the ratios between the probabilities of being in each CoI can be easily computed as the ratios of the unnormalized $\mathbf{p}_a$ obtained in Eq.(97). Moreover, ii) correct normalization of $\mathbf{p}_a$ requires, at least, an estimate of the distribution over the neglected configurations.

These points formally clarify my arguments of the first section. i) clarifies why using the SMA – or, in general, an approximation level lower in hierarchy compared to the dimensionality of the system – is important, as in this way we already collect crucial information on the process; ii), in turn, calls for our comprehension of the shape of the distribution around the CoI. That is precisely what we discuss next.

6.6 RENORMALIZATION

Renormalization is appropriate when the properties condidered are not sensitive to the fluctuations near the fixed points.

Let us assume that we have $|A|$ CoI, a_i, of which S correspond to vertex configurations (also called monomorphic, see **Section** 6.8), and $|A| - S$ to internal stable fixed points. We can write a continuous stationary probability density for the state variable, $\mathbf{x} = (x_1, \ldots, x_{S-1})$, which we call $P(\mathbf{x})$. As shown already, we are able to compute $p(a_i)$, the non-normalized stationary distribution over configurations in A computed from the eigenvector of ρ, in Eq.(95). Since the probability of being in configuration a_i is $\alpha^{-1} p(a_i)$, where α^{-1} is a normalization constant, and given that this configuration has a range of $1/Z$ in each of the $S-1$ independent directions, the probability density at that point must be $\alpha^{-1} Z^{S-1} p(a_i)$. In other words, $P(a_i/Z) = \alpha^{-1} Z^{S-1} p(a_i)$, for $i = 1, \ldots, |A|$. A simple way of collecting this information is by writing

$$P(\mathbf{x})\,\alpha/Z^{S-1} = \sum_{i=1}^{|A|} p(a_i)\,\frac{f_i(\mathbf{x})}{f_i(a_i/Z)}. \quad (98)$$

The $f_i(x)$ functions are normalized probability densities that are non-zero in the neighborhood of $\mathbf{x} = a_i/Z$, $D[a_i]$.

In order to further simplify the procedure, let us compute the weight of each peak, $w_{a_i} = \int_{D[a_i]} P(\mathbf{x})\,d\mathbf{x}$, and then use it as the probability of being in configuration a_i.

Taking the weights of each peak as the value of the distribution in that CoI and setting the remaining states as unoccupied, we may write a discrete counterpart as

$$P(s) = \alpha^{-1} \sum_{i}^{|A|} p(a_i)\,\frac{Z^{S-1}}{f_i(a_i/Z)}\delta_{s a_i}. \quad (99)$$

We are left with computing a normalized function f_i in a single point that reflects the shape of the distribution around the CoI. Below we deduce the expressions for different functions in two different dimensionalities.

6.7 PDF SOLUTION NEAR AN INTERNAL STABLE FIXED POINT

Now that we have laid down the general framework, we can start making some assumption in order to produce some closed form formulas.

Let us estimate the shape of the probability distribution function (PDF) near an attractor that corresponds to a stable fixed point in the infinite dynamics and show that it can be approximated by a Gaussian function. To this end, let us keep considering a general phase space of arbitrary dimension. The system moves from configuration s_k to configuration s_l, in each step, with a probability $T_{s_k s_l}$. In general this corresponds to an $S-1$ dimensional process with the same properties.

The evolution of the PDF of the process $\mathbf{x} = \mathbf{i}/Z$, $p(\mathbf{x}, t)$, is given by the Master Equation [38] below (see **Chapter** 2 for details).

$$p(\mathbf{x}, t+\tau) - p(\mathbf{x}, t) = \sum_{\boldsymbol{\delta}} \left(p(\mathbf{x}+\boldsymbol{\delta}, t)\,T^{-\boldsymbol{\delta}}(\mathbf{x}+\boldsymbol{\delta}) - p(\mathbf{x}, t)\,T^{\boldsymbol{\delta}}(\mathbf{x}) \right), \quad (100)$$

where $T^{\boldsymbol{\delta}}(\mathbf{x}) \equiv T_{\mathbf{Zx},\mathbf{Z}(\mathbf{x}+\boldsymbol{\delta})}$. This, in turn, can be expanded in power series in $\delta \sim 1/Z$ to give

$$\begin{aligned} \frac{\partial p}{\partial t}(\mathbf{x}, t) &= -\sum_k \frac{\partial}{\partial x_k}\left(D_k^{(1)}(\mathbf{x})\,p(\mathbf{x}, t)\right) \\ &\quad + \sum_k \sum_{k'} \frac{\partial^2}{\partial x_k \partial x_{k'}}\left(D_{kk'}^{(2)}(\mathbf{x})\,p(\mathbf{x}, t)\right) \\ &\quad + \ldots \end{aligned} \quad (101)$$

where

$$\begin{aligned} D_k^{(1)}(\mathbf{x}) &= T^{k+}(\mathbf{x}) - T^{k-}(\mathbf{x}), \text{with} \\ T^{k\pm}(\mathbf{x}) &\equiv \sum_{\delta:\delta_k=\pm 1/Z} T^{\delta}(\mathbf{x}) \end{aligned} \tag{102}$$

and

$$D_{kk}^{(2)}(\mathbf{x}) = \frac{1}{2Z}\left(T^{k+}(\mathbf{x}) + T^{k-}(\mathbf{x})\right) \tag{103}$$

$$D_{kk}^{(2)}(\mathbf{x}) = -\frac{1}{2Z}\left(T^{k\to k'}(\mathbf{x}) + T^{k'\to k}(\mathbf{x})\right), k \neq k' \tag{104}$$

Here, $T^{t\to t'}$corresponds to a transition with $\boldsymbol{\delta} = \{0, \ldots, 0, \delta_t = -1, 0, \ldots, 0, \delta_{t'} = +1, 0, \ldots, 0\}$.

If we neglect the remaining terms of the expansion, we get a Fokker-Planck equation[95] as we did in **Section** 2.4. Thus, it is natural that the system spends a lot a time around the attractive roots of $D^{(1)}$ as long as diffusion is not too high. As a rough approximation, we linearize the gradient of selection, $\mathbf{D}^{(1)}(\mathbf{x}) = \mathbf{J}(\mathbf{x} - \mathbf{x}^*)$, with $\mathbf{J}$ the Jacobian matrix of $\mathbf{D}^{(1)}(\mathbf{x})$ at $\mathbf{x}^*$ and set $\mathbf{D}^{(2)}(\mathbf{x}) = \mathbf{D}^{(2)}(\mathbf{x}^*) = \mathbf{D}$ as a constant. Using the gradient, $\nabla = \left(\partial_{x_1}, \ldots, \partial_{x_{S-1}}\right)$, and Hessian, $\mathbf{H}$, operators, we can write

$$\begin{aligned} \frac{\partial p}{\partial t}(\mathbf{x}, t) &= -\nabla.\left(\mathbf{J}(\mathbf{x} - \mathbf{x}^*)\, p(\mathbf{x}, t)\right) \\ &\quad + \mathrm{Tr}\left(\mathbf{H}(p(\mathbf{x}, t))\mathbf{D}\right) \end{aligned} \tag{105}$$

which corresponds to a multidimensional Ornstein-Uhlenbeck process, whose stationary solution is a multivariate normal distribution [38]. This gaussian should, then, be plugged into Eq.(98) or Eq.(99) as the appropriate f function.

Below we explicitly write the equations and solutions of the normal distribution for 1D and 2D cases as a function of the transition probabilities, as well as the particular expression for the population dynamics models we used before, using results for the Drift and Diffusion of **Chapter** 3.

6.7.1 *One dimensional case (two strategies)*

The equation for a linear drift or gradient of selection, $D^{(1)}(x) = J(x - x^*)$, and constant diffusion D is

$$0 = -\frac{d}{dx}\left(J(x - x^*)\, p(x)\right) + D\frac{d^2 p}{dx^2}(x), \tag{106}$$

with solution $p(x) = \text{Normal}\left(x; \langle x\rangle = x^*, \sigma_x^2 = -D/J\right)$, where x^* is such that $T^+(x^*) - T^-(x^*) = 0$, $J = d(T^+(x) - T^-(x))/dx|_{x=x^*}$ and $D = (T^+(x^*) + T^-(x^*))/(2Z)$ deduce in **Section 3.1** which we will recover for the sake of completeness of the chapter

$$T^+(x) - T^-(x) = (1-\mu)\frac{Z}{Z-1}x(1-x)\tanh\left(\frac{\beta}{2}\Delta f_{12}\right) + \mu(1-2x), \quad (107)$$

$$T^+(x) + T^-(x) = (1-\mu)\frac{Z}{Z-1}x(1-x) + \mu, \quad (108)$$

with Δf_{12} representing the fitness difference driving the update of players of trait 1 in relation to trait 2. This completely defines the parameters of the Gaussion function for any given Δf_{12}. For this particular 1D scenario, the expression of the variance is fairly simple:

$$\sigma_x^2 = \frac{1}{2}\left(1 - \frac{1-\mu}{\mu}\frac{Z}{Z-1}\frac{\left(\frac{1}{2}-x^*\right)\sinh\beta\Delta f_{12}(x^*) + x^*(1-x^*)\frac{\beta\Delta f'_{12}(x^*)}{2}}{1+\cosh\beta\Delta f_{12}(x^*)}\right)^{-1} \quad (109)$$

6.7.2 *Bi-dimensional case (three strategies)*

In a population with three strategies, $\mathbf{x} = (x_1, x_2)$ is the configuration of a population with a fraction of individuals x_1 with strategy 1 and x_2 individuals with strategy 2 ($1 - x_1 - x_2$ is the fraction of individuals with strategy 3). If we let $\mathbf{y} = \mathbf{x} - \mathbf{x}^*$, the equation for a linear drift or gradient of selection, $\mathbf{D}^{(1)}(\mathbf{y}) = \mathbf{J}(\mathbf{y})$, and constant diffusion $\mathbf{D}$ is

$$0 = -\frac{\partial}{\partial y_1}(J_{11}y_1 + J_{12}y_2)p(\mathbf{y},t) - \frac{\partial}{\partial y_2}(J_{21}y_1 + J_{22}y_2)p(\mathbf{y},t) \quad (110)$$

$$+ D_{11}\frac{\partial^2 p(\mathbf{y},t)}{\partial y_1^2} + 2D_{12}\frac{\partial^2 p(\mathbf{y},t)}{\partial y_1 \partial y_2} + D_2\frac{\partial^2 p(\mathbf{y},t)}{\partial y_2^2},$$

$$\text{with solution } p(\mathbf{x},t) = \frac{1}{\sqrt{\pi^2 \det\boldsymbol{\Sigma}}}e^{-(\mathbf{x}-\mathbf{x}^*)^T\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\mathbf{x}^*)}, \quad (111)$$

where

$$\Sigma_{11} = \left(2D_{12}J_{12}J_{22} - D_{22}J_{12}^2 - D_{11}\left(J_{22}^2 + \det\mathbf{J}\right)\right)/\left((\mathrm{tr}\mathbf{J}\det\mathbf{J})/2\right) \quad (112)$$

$$\Sigma_{12} = \Sigma_{21} = (2D_{12}J_{11}J_{22} - D_{11}J_{21}J_{22} - D_{22}J_{11}J_{12})/\left((\mathrm{tr}\mathbf{J}\det\mathbf{J})/2\right) \quad (113)$$

$$\Sigma_{22} = \left(2D_{12}J_{21}J_{11} - D_{11}J_{21}^2 - D_{22}\left(J_{11}^2 + \det\mathbf{J}\right)\right)/\left((\mathrm{tr}\mathbf{J}\det\mathbf{J})/2\right) \quad (114)$$

where

$$\mathbf{J} = \left.\begin{bmatrix} \partial_{x_1}\left(T^{1+}(\mathbf{x}) - T^{1-}(\mathbf{x})\right) & \partial_{x_2}\left(T^{1+}(\mathbf{x}) - T^{1-}(\mathbf{x})\right) \\ \partial_{x_1}\left(T^{2+}(\mathbf{x}) - T^{2-}(\mathbf{x})\right) & \partial_{x_2}\left(T^{2+}(\mathbf{x}) - T^{2-}(\mathbf{x})\right) \end{bmatrix}\right|_{\mathbf{x}=\mathbf{x}^*}, \quad (115)$$

$$\mathbf{D} = \frac{1}{2Z}\begin{bmatrix} T^{1+}(\mathbf{x}^*) + T^{1-}(\mathbf{x}^*) & T^{1\to 2}(\mathbf{x}^*) + T^{2\to 1}(\mathbf{x}^*) \\ T^{1\to 2}(\mathbf{x}^*) + T^{2\to 1}(\mathbf{x}^*) & T^{2+}(\mathbf{x}^*) + T^{2-}(\mathbf{x}^*) \end{bmatrix} \tag{116}$$

and $\mathbf{x}^*$ is such that $T^{1+}(\mathbf{x}^*) - T^{1-}(\mathbf{x}^*) = 0 \wedge T^{2+}(\mathbf{x}^*) - T^{2-}(\mathbf{x}^*) = 0$. Again, for a pairwise Fermi update rule, we can write

$$T^{k+}(\mathbf{x}) - T^{k-}(\mathbf{x}) = (1-\mu)\frac{Z}{Z-1}x_k \tanh\left(\frac{\beta}{2}\Delta f_{kl}\right)\sum_{l\neq 1}^{3} x_l + \frac{\mu}{2}(1-3x_k) \tag{117}$$

$$T^{k+}(\mathbf{x}^*) + T^{k-}(\mathbf{x}^*) = (1-\mu)\frac{Z}{Z-1}x_k^*(1-x_k^*) + \frac{\mu}{2}(1+x_k^*) \tag{118}$$

$$T^{1\to 2}(\mathbf{x}^*) + T^{2\to 1}(\mathbf{x}^*) = (1-\mu)\frac{Z}{Z-1}x_1^* x_2^* + \frac{\mu}{2}(x_1^* + x_2^*) \tag{119}$$

with Δf_{kl} representing the fitness difference driving the update of players of trait k in relation to trait l. This set of equations completely defines the parameters of the Gaussian for any given set of Δf_{kl}.

6.8 PDF SOLUTION NEAR MONOMORPHIC CONFIGURATIONS

In the previous section we gave functional forms for the functions that can be used to weight the internal CoIs. Now we look at what happens in the boundaries, which can also act as attractors and that is why some of its configurations are usually included in the CoIs.

The boundaries are characterized by having at least one absent strategy, while the remaining coexist. Notice that each direction is defined by a pair of interacting strategies. This means that the shape of the function in the directions of the coexisting strategies falls into the previous section, with a reduced number of total strategies, since in those direction the CoIs are internal. Thus, we are left with computing the shape of the function in the directions where absent strategies invade others.

Because new strategies can only appear through mutation, and the boundaries are only occupied when mutation is small, the simplest shape we can assume for the distribution around the CoI in the boundary is that it decays immediately to zero, requiring no renormalization, as the range of the function in that direction is $1/Z$ and thus its value is Z, canceling out. This is the shape we implicitly used above.

However, one might want to improve this estimation. Whenever those CoI act as attractors and mutation is non-zero a tail arises in the distribution, conferring strength to configurations near the CoI. The solution is to study the behavior of the system close to the boundaries, which will depend on the system at hands.

Because the configurations corresponding to the vertices only have transitions into the edges, we can start analyzing the 1D case. There, the evolution of the PDF of the process i, $p(i,t)$, is given by the appropriate Master Equation (100),

$$p(i, t+\tau) - p(i, t) = \sum_{\delta} \left(p(i+\delta, t) T^{-\delta}(x+\delta) - p(i, t) T^{\delta}(i) \right), \quad (120)$$

which has an analytical solution $p(i)$, for its stationary distribution when considering a one step process (that we have deduced and interpreted already in the end of **Section** 2.5),

$$p(i) = p(0) \prod_{l=0}^{i-1} \frac{T_l^+}{T_{l+1}^-},\ p(0) = \left(1 + \sum_{i=1}^{Z} p(i) \right)^{-1}. \quad (121)$$

Typically, when a configuration from the CoI – in this case a monomorphic configuration – is an attractor, the stationary probability of the adjacent configuration is smaller than that of the CoI. In this case, and assuming that the attractor is state 0, we have that $p(0) > p(1) = p(0) T_0^+ / T_1^-$ or $T_0^+ < T_1^-$ or, equivalently, $\mu < T_1^-$, reflecting the balance between mutation and selection of a mutant in a population.

Let us assume a resident (and dominant) strategy R and a mutant strategy M. If i is the number of individuals with strategy R, the assumption that configuration 0 is an attractor means that the fitness of a mutant in the population is smaller than that of the individuals in the populations by an amount Δf. For simplicity, let us make it frequency independent (at least in the region where the function decays). With these assumptions we can calculate $p(i)$ and use it to renormalize the probability of being in the considered monomorphic state ($i = 0$).

Assuming a Fermi-update process, we have that $T_i^+ = \frac{Z-i}{Z} \left(\frac{i}{Z-1} \frac{1-\mu}{1+e^{\beta \Delta f}} + \mu \right)$ and $T_i^- = \frac{i}{Z} \left(\frac{Z-i}{Z-1} \frac{1-\mu}{1+e^{-\beta \Delta f}} + \mu \right)$. Using Eq.(101) we obtain a decaying function. Recalling Eq.(99), we need both the value of the function, $p(i)$, at 0 and the cumulative function to guaranty a distribution (with a normalization of one) in the selected range $D[0]$. In this case, the value of the distribution at zero is a hypergeometric function,

$$p(0) = {}_2F_1 \left[-Z, \left(1 + e^{-\beta \Delta f}\right)(Z-1) \frac{\mu}{1-\mu}; -\frac{(Z-1)\left(1 + e^{\beta \Delta f} \mu\right)}{1-\mu}; e^{\beta \Delta f} \right]^{-1} \quad (122)$$

which can be obtained using Eq.(121) and reads, approximately, $\left(1 - e^{\beta \Delta f}\right)^{-\left(1+e^{-\beta \Delta f}\right) Z \mu}$. The cumulative function must be used whenever the decay is not fast enough and the function penetrates the domain of other attractors. With the knowledge of the transition to be computed between CoI and the right renormalizations to the distributions, we have shown all the ingredients present in the discussion above.

In general, when there are several strategies in place, this means that, in order to renormalize the probability of the monomorphic states, one needs the fitness of each of the mutants. This fitness should be smaller than that of the resident population and satisfy $T_{mutant \to resident} > \frac{\mu}{S-1}$,

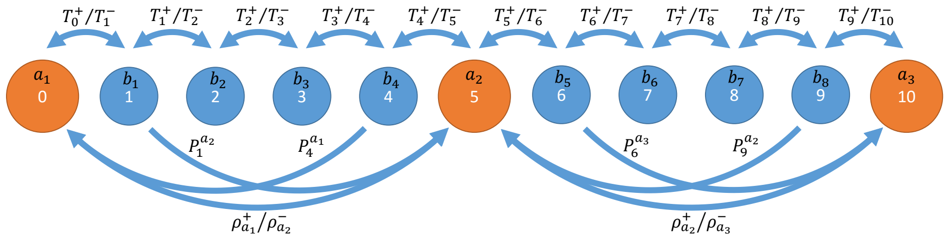


Figure 13: **Illustration of the transition and notation considered in this section** T denotes the one-step transition probabilities between any pair of adjacent configurations, whereas ρ and P denote the long run transition probabilities to configurations in A departing from, respectively, configurations in set A in set B.

for all the $S-1$ mutant strategies. If this is not the case, the state will not be an attractor and no renormalization is needed. The same procedure can be used to calculate more accurately the shape of the attractor in any boundary in the direction caused solely by a single mutation.

6.9 THE ONE DIMENSIONAL SYSTEM: TRANSITION PROBABILITIES BETWEEN COI

As promised, we deduce next the expression for Eq.(95) in a one-dimensional Markov chain.

For a one dimensional one-step Markov chain the phase space, of size $Z+1$, is simply $\{0, 1, \ldots, Z\}$. This phase-space is a single line, corresponding to an edge of a multi-dimensional problem. Because in 1st order only edge points are included, the following treatment corresponds to the 1st order correction to the SMA.

The one dimensional process is characterized by allowing transitions only between adjacent states, contrary to what happens in the general scenario of the previous section and the different notation in the labeling of states is done to reflect that.

Consider $|A|$ non-adjacent ordered CoIs $\{a_1, \ldots, a_{|A|}\}$, and, the remaining, $|B| = Z+1-|A|$ neglected configurations $\{b_1, \ldots, b_{|B|}\}$. Once again, the goal is to reduce the Markov chain containing the whole phase space to one containing only the CoIs $\{a_i\}$ and calculate the transition probabilities $\rho_{a_i a_j}$ between those configurations as a function of the one-step transition probabilities of the whole 1D chain T_{kl}. Notice that the new simpler chain of CoI is also a 1D one-step Markov chain, in the sense that each configuration has two transitions outward, apart from the two boundary configurations that have only one.

Let $T_k^{\pm} \equiv T_{kk\pm1}$ and $T_k^0 \equiv T_{kk}$, $\rho_{a_i}^{\pm} \equiv \rho_{a_i a_{(i\pm1)}}$ and $P_b^a \equiv P_{ba}$ (see **Figure** 13). In what follows we will use configurations and the value of the process in that configuration interchangeably.

Given two non-adjacent configurations $a_i \equiv L < a_j \equiv R$ (Left and Right), we have

$$\rho_L^+ = T_L^+ P_{L+1}^R, \tag{123}$$

meaning that the probability of moving from L to R, requires a (one step) transition to the right from configuration L (to the neglected configuration $L+1$), T_L^+, and then eventually going to R from $L+1$, P_{L+1}^R. To calculate this term, we write the equations for the probability of being absorbed by R from any intermediate configuration $b \in \{L+1, \ldots, R-1\}$

$$\begin{aligned} P_b^R = \quad & T_b^- P_{b-1}^R (1 - \delta_{b\,L+1}) + T_b^0 P_b^R \\ & + T_b^+ P_{b+1}^R (1 - \delta_{b\,R-1}) + \delta_{b\,R-1} T_b^+, \end{aligned} \tag{124}$$

which are known and whose solution is [58]

$$P_{L+1}^R = \left(1 + \sum_{j=L+1}^{R-1} \prod_{k=L+1}^{j} \frac{T_k^-}{T_k^+}\right)^{-1}. \tag{125}$$

With this we can define ρ_L^+ as a function of the one step transitions. Using the same procedure for ρ^- we can finally write

$$\rho_{a_i}^{\pm} = T_{a_i}^{\pm} P_{a_i \pm 1}^{a_{(i\pm1)}} = T_{a_i}^{\pm} \left(1 + \sum_{j=a_i\pm1}^{a_{(i\pm1)}\mp1} \prod_{k=a_i\pm1}^{j} \frac{T_k^{\mp}}{T_k^{\pm}}\right)^{-1}. \tag{126}$$

Now the constraint to non-adjacent CoIs can be lifted, in which case we may write $\rho_{a_i}^{\pm} = T_{a_i}^{\pm}$ for adjacent configurations.

6.10 THE ONE DIMENSIONAL SYSTEM: AN INTERNAL FIXED POINT EXAMPLE

Let us continue with the one dimensional case with phase space $\{0, 1, \ldots, Z\}$, of size $Z+1$. Let us assume that we have three CoI, the two monomorphic configurations, $a_1 = 0$ and $a_3 = Z$ and a coexistence (stable fixed point) at a_2. Following the general procedure, we can write a continuous stationary probability density for the fraction of individuals playing strategy σ_1, $x = i/Z$, which we call $P(x)$. As shown already, we are able to compute $p(a_i)$, the non-normalized stationary distribution over configurations in A computed from the $\rho_{a_i}^{\pm}$, in Eq.(126). Since the probability of being in configuration a_i is $\alpha^{-1} p(a_i)$, and given that this configuration has a range of $1/Z$, the probability density at that point must be $\alpha^{-1} Z p(a_i)$, where α^{-1} is a normalization constant. In other words, $P(a_i/Z) = \alpha^{-1} Z p(a_i)$, for $i = 1, 2, 3$. A simple way of writing this is

$$P(x)\,\alpha/Z = p(a_1)\frac{f_1(x)}{f_1(a_1/Z)} + p(a_2)\frac{f_2(x)}{f_2(a_2/Z)} + p(a_3)\frac{f_3(x)}{f_3(a_3/Z)}. \tag{127}$$

The $f_i(x)$ functions are normalized probability densities that are non-zero in the neighborhood of a_i/Z, $D[a_i]$. The simplest choice for $f_1(x)$ and $f_3(x)$ is to let them reflect the peaking in the monomorphic configurations as a constant with value Z in a range $1/Z$ – see **Section** 6.8 for further refinements – whereas for $f_2(x)$ we consider a Normal distribution, $N(x)$, centered at point a_2/Z with variance σ^2. This defines an estimate of the distribution. In order to further simplify the procedure, let us compute the weight of each peak, $w_{a_i} = \int_{D[a_i]} P(x)\,dx$, and then use it as the probability of being in configuration a_i.

$$w_1 = \int_0^{1/Z} \alpha^{-1} Z p(a_1) \frac{1/Z}{1/Z} dx = \alpha^{-1} p(a_1), \tag{128}$$

$$w_2 = \int_{1-1/Z}^{1/Z} \alpha^{-1} Z p(a_2) \frac{N(x)}{N(a_2/Z)} dx \tag{129}$$
$$= \alpha^{-1} Z p(a_2) \sqrt{2\pi\sigma^2} \frac{1}{2} \left(\mathrm{Erf}\left[\frac{a_2/Z - 1/Z}{\sqrt{2\sigma^2}}\right] + \mathrm{Erf}\left[\frac{1 - a_2/Z - 1/Z}{\sqrt{2\sigma^2}}\right] \right),$$

$$w_3 = \int_{1-1/Z}^{1} \alpha^{-1} Z p(a_3) \frac{1/Z}{1/Z} dx = \alpha^{-1} p(a_3). \tag{130}$$

Notice that if $0 < a_2/Z \pm 2\sigma < 1$, then $w_2 \approx \alpha^{-1} Z p(a_2) \sqrt{2\pi\sigma^2}$. Taking the weights of each peak, we may write a discrete counterpart as

$$P(i) = \alpha^{-1} \left(p(a_0)\, \delta_{i a_0} + Z\sqrt{2\pi\sigma^2} p(a_1)\, \delta_{i a_1} + p(a_2)\, \delta_{i a_2} \right) \tag{131}$$

where σ^2 can be derived from the transition probabilities $T_x^+$and T_x^- around a_2/Z as $\sigma^2 = F/|J|$ with $J = d(T_x^+ - T_x^-)/dx|_{x=a_2/Z}$ and $F = \left(T_{a_2/Z}^+ + T_{a_2/Z}^-\right)/(2Z)$ (see **Section** 6.7.1).

Notice also that collapsing the weight of the distribution around the fixed point is particularly useful if one is interested in computing the first moment.

Furthermore, if $0 < a_2/Z \pm 2\sigma < 1$, then $w_2 = \alpha^{-1} Z p(a_2) \sqrt{2\pi\sigma^2}$. Taking the weights of each peak, we may write a discrete counterpart as

$$P(i) = \alpha^{-1} \left(p(a_1)\, \delta_{i a_1} + Z\sqrt{2\pi\sigma^2} p(a_2)\, \delta_{i a_2} + p(a_3)\, \delta_{i a_3} \right) \tag{132}$$

where $\alpha = \left(p(a_1) + Z\sqrt{2\pi\sigma^2} p(a_2) + p(a_3) \right)$.

Indeed, for the average we get

$$\langle i/Z \rangle = \alpha^{-1} \left(\sqrt{2\pi\sigma^2} p(a_2)\, a_2 + p(a_3) \right). \tag{133}$$

Also, one could use Eq.(132) to compute the second moment, but since we already know the variance of the Gaussian, we can use Eq.(127) to get an estimation of the variance of any 1D process with a single fixed point. For $0 < a_2/Z \pm 2\sigma < 1$ the expression is can also be simplified to

$$\mathrm{var}_{i/Z} = \alpha^{-1}\left(\langle i/Z\rangle^2\, p(a_1) + \sigma^3 Z\sqrt{2\pi}p\,(a_2) + (1-\langle i/Z\rangle)^2 p(a_3)\right). \quad (134)$$

6.11 ESTIMATING NEGLECTED CONFIGURATIONS WITH A GRID

So far, in order to compute the stationary distribution at different orders of μ (the mutation parameter), we estimate the behavior of that distribution employing an approximation of the Master Equation as a Fokker-Planck equation near the stable fixed points. As we show above, this consists in using a normal distribution near the stable fixed points and delta – or rapidly decaying – functions on the borders. Naturally, these estimates can be improved by studying, for instance, the discrete version of the equations, or by taking into account the influence of the borders. Nonetheless, this approach may not be so easy to do in general, especially for higher orders in μ, which increase the dimensionality of the phase space taken in consideration and, thus, make it more difficult to find all fixed points and their stability. Under such circumstances, the method we now discuss may prove rewarding.

Selection of the appropriate interpolation method can reduce the number of necessary CoI.

Let us go back to the first case presented in **Section** 6.2 – a one-dimensional process with a single stable fixed point – and employ a straightforward method to estimate the distribution. Instead of finding the polymorphic stable fixed points and including these as CoIs, we define the CoIs by setting up a homogeneous grid that includes 1, 3 or 5 intermediate configurations. We compute the probability of being in each of these configurations, using the standard method already described. The distribution over the configurations not considered in the CoIs may be approximated assuming a completely flat distribution between each of those configurations and the middle points of its neighbors, creating a piecewise discontinuous function. **Figure** 14 depicts the average of the distribution estimated this way and shows how with 5 intermediate CoIs the distribution already matches the exact solution. In general, the number of points needed will vary depending on the properties of the distribution (e.g. the asymmetry of the distribution curve) and on the method for estimating the distribution. By using this approach, we frame the problem as an interpolation exercise (that in general is well studied), refraining from studying partial differential equations or their discrete equivalent, which usually require a case by case analysis and study. On the other hand, it is worth pointing out the advantage of the study of the differential equations as a way to minimize the number of extra CoIs – in this example, we obtain the same quality result with only 1 extra CoI, to the extent that this point is the fixed point. Needless to say, mixed approaches can also be used.

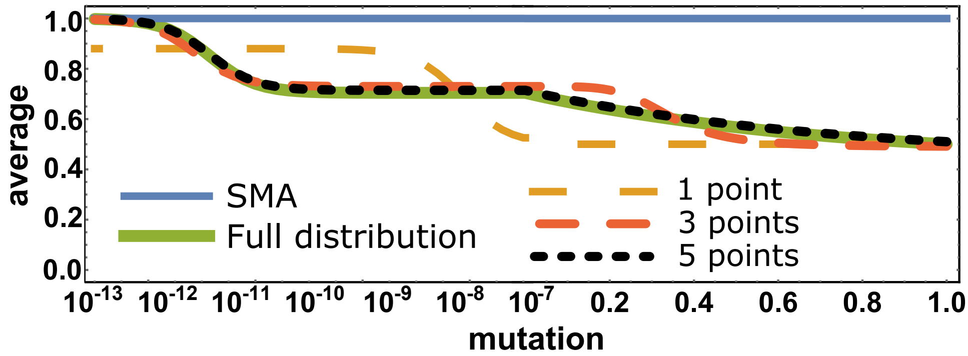


Figure 14: **Estimating with a grid** Same system as **Figure** 9 in **Section** 6.2 with 1,3 and 5 intermediate points assuming a flat distribution around each CoI. With five equidistant points one is able to reproduce the full distribution in this scenario.

6.12 SUMMARY AND GENERAL PROCEDURE

This approach attempts to describe the dynamics and steady state solution of the system by following a series of systematic steps:

1. Study the system's behavior based on the gradient of selection dynamics and understand the dimensionality of the hyperface containing the stable fixed points. Based on that, choose the desired order of approximation (order n or Hn-approximation). The traditional SMA is the H0-approximation.

2. Investigate the existence of stable fixed points in which at most $n+1$ strategies are present in the population. These configurations, together with the monomorphic configurations, are named configurations of interest (CoIs) and compose the set called A.

3. Calculate the transition probabilities $\rho_{a_i a_j}$ between all configurations in A.

4. Calculate the (un-normalized) stationary distribution given the reduced Markov chain whose configurations and transition probabilities are, respectively, the configurations in A and the transitions calculated in point 3.

5. Build a normalized stationary distribution by renormalizing the stationary distributions calculated in point 4 with a normalization factor deduced from properly linearizing the system close to those points (presented in the text and detailed in **Sections** 6.6, 6.7 and 6.8, above).

Each of these points encompasses advantages and limitations. Point 1 reflects the limitations of stopping the expansion. Whenever there are stable fixed points more internal than the phase space considered in a given term of the expansion, its validity only holds for lower mutation rates, which was the original problem of the SMA. Nonetheless, we are able to incorporate, without significant increase in computational cost, the ability to take into account fixed points along the edges of the phase space. Together with point 2, this evidences the fact that higher order approximations come with a computational cost. It is worth mentioning that, in points 4 and 5, we assume that the distribution near the monomorphic configurations either adopts the shape of a delta function, or a decaying function resulting from frequency independent mutant disadvantage. Moreover, we regard the distribution near the stable fixed points as displaying the symmetry properties of a Gaussian curve, which may not be the case. Ways to improve both matters can be introduced in order to attain results with higher precision, whenever necessary.

To conclude: As is well known in many areas of Science, a judicious choice of the CoI proves instrumental in minimizing the workload necessary to reach a good description of a system. In this chapter we show how this concept applies to the dynamics of populations in which individuals may be equipped with complex physical, biological or social repertoires. We developed a hierarchy of approximations that exhibit an intuitive topological appeal. Topological criteria should be also employed in choosing the CoI in those more complex cases where the calculation of interior fixed points becomes too cumbersome. Starting from the SMA at zeroth-oder, this hierarchical approach provides a description that allows one to explicitly include the role of mutations while dramatically reducing the complexity of the multi-dimensional Markov processes we aim to describe, thus retaining analytical and/or numerical tractability. We believe that the present formalism is general in scope, being of potential applicability in a wide variety of problems that transcend pure Physics applications.

We are now in conditions to move on to the last part of this dissertation and deal with what I believe to be one of the greatest challenges of mankind.

Part III

PUBLIC GOODS

Global coordination for the preservation of a common good, such as climate, is one of the most prominent challenges of modern societies. In this part, I use the framework of evolutionary game theory and the tools developed so far to investigate whether a polycentric structure of multiple small-scale agreements provides a viable solution to solve global dilemmas as climate change governance. The text of this part is based on the work I published with my coauthors, indicated at the beginning of each chapter, and all of it originated from the Evolutionary Game Theoretical model of climate change under risk published in 2011 by Santos and Pacheco [112].

Polycentric governance is based on the concurrent learning and action of multiple agents that pursue their interests and that act at a smaller scale than that of the problem at stake.[90–92] In cases when a central, top-down, large scale agreement or treaty fails to be sought, the polycentric approach has been invoked[91] as a possible means to mitigate the issue or to pave way to a global solution, bringing progress to the stalled process of resolution. From here on, I will show the work my coauthors and I have develloped with, in turn, uses the framework developed in the previous parts to investigate whether polycentricity provides a viable solution, that is, whether polycentricity can be used to solve global dilemmas. In this context, we will be focusing on N-person Public Goods Games (PGG), and on the mechanisms that act to uphold cooperation based on joint decisions made by groups. For what we have seen so far, EGT reveals itself as a very appropriate tool to do this, given that it can make use of the learning process of a multiplicity of interacting agents, facing problems of cooperation. The individuals in these populations are able to revise their strategies depending on their outcome but they have limited information, observing only the acts of others and their final or average outcome, without ever knowing the whole process: this is a challenging setup for cooperation, making it hard to strive (see part II, **Chapter** 4). In what follows, we show that polycentricity allows for cooperation to emerge even in this adverse scenario.

Mechanisms that act to promote and maintain cooperation based on joint decisions made by groups involving more than two individuals have been thoroughly investigated, leading to the formulation of N-person Public Goods games (PGG), in which collective action often depends on the coordination, into cooperation, of a threshold number of group members.

Despite scientific consensus of its negative impacts in many natural ecosystems, with immediate consequences in human life,[72, 129, 130] climate change is one of the examples in which global treaties have failed so far and polycentricity comes up as an alternative. The dilemma over climate issues comes from the fact that regions or nations are tempted to make no effort themselves, while reaping the benefit from the possible efforts of others. Besides this dilemma, akin to many others that humans face, cooperation is sought by world leaders that every year try to reach some consensus. The failure of these global summits has been attributed to many factors, of which we distinguish: *i*) overall perception of risk is too small, with decision makers not taking fully into account the effects of missing the targets and discounting[65] its effects; *ii*) (scientific/political) uncertainties regarding the exact values of the targets to be met[4, 6, 7, 86, 89, 134]; *iii*) conflicting policies between rich and poor parties, with a possible segregation of behaviors involving developed countries, on one side, and developing countries, on the other and, *iv*) absence of institution(s) to monitor and sanction those not abiding to agreements. Here we will see how a polycentric approach may ameliorate the impact of this plethora of effects.

With this in mind, the base model[112] considers a threshold N-person game, in which individuals have to contribute a minimum to effectively contribute to reduce Green-House-Gas (GHG) emissions. However, players may perceive the negotiation devoid from risk: even if the threshold is not met there may still be a chance that nothing catastrophic happens and everyone keeps whatever they have; an effective value attributed to this risk allows EGT to operate for a population under a given risk perception: high levels of risk perception translate into a calculus where existing benefits will likely be lost when contributions are below the threshold, the opposite happening otherwise. Additionally, a second kind of uncertainty can be present[6]: uncertainty in the threshold to be met that defines the collective goal and concerns the amount of contributions required for individuals to be sure to retain whatever they have.

In **Chapter** 7, we will analyze the combined impact of each of these uncertainties when agreements are set at various scales, showing how a polycentric approach to such a global problem helps in reducing the detrimental effect created by uncertainty. Subsequently, in **Chapter** 8, we will augment the base model by splitting the population into two wealth classes — those with high endowments, metaphorically representing the "rich", and those with low endowments, representing the "poor" — describing the feedback dynamics between the poor and the rich, and how it acts to build up or diminish the chances of reaching cooperation in each class. We allow these classes to (partially or totally) segregate their behavior, and hence we can study the impact of homophily between these classes. Finally, in **Chapter** 9 and 10, we will investigate the impact and adoption of different kinds of sanctioning institutions to regulate the contribution to GHG reduction, and explain how those institutions, created locally, with a smaller range of effect, have a sizeable impact on the overall eagerness to cooperate.

7 EFFECT OF UNCERTAINTY & SCALE

This chapter is based in the manuscript "Evolutionary dynamics of Climate Change Under Collective-Risk Dilemmas" by F.C. Santos et.al.[107], which contains a detailed analysis of the the finite population dynamics of the original model by Santos and Pacheco [112]. Appendix a

On the issues of environmental sustainability and mitigation of the impacts of climate change, one must not overlook uncertainty. Especially when looking at Environmental Agreements, which are typically non-binding, [56, 86] uncertainty becomes ubiquitous, as collective investment and its forecast becomes truly unknown: whether it is due to political hesitation or to reservations regarding reversibility, the timings, goal temperatures and even consequences of GHG induced climate change. Whereas the political hesitations are somehow manageable, through negotiation and research, the others, since they refer to future events, lead to a lot more discussion resulting in (or from) a mindset in which the possibility that nothing damaging happens is non-negligible. Thus, the study of collective action cannot be detached from the overall perception of risk conveyed by climate change,[50, 51] a remark that has been reinforced by recent experiments [77, 134] and which the base model[112] captures as we will see next.

7.1 MODEL

Consider a population of Z individuals. Groups of size N are randomly sampled from that population. Each group is set to play a game in which a target of M contributions is to be met. Each individual starts with an initial endowment b that can be used to contribute. We start with only two kinds of players: those whose strategy is to contribute (only) to the public good, paying a cost c, the Coopeartors (Cs), and those who don't, Defectors (Ds). Consider the possibility that a given group does not reach the predetermined threshold: we call risk, r, the probability of losing one's endowment in that situation ($0 \leqslant r \leqslant 1$), such that $r = 0$ means endowments will never be lost, whereas $r = 1$ means loss of endowment is certain. Hence, the payoffs of players in a group with k Cs (and $N - k$ Ds) can be written as

Failure to reach a given minimum contribution may imply – also depending on the risk (r) of disaster – that cooperators invest in vain and all endowments are lost.

$$\begin{aligned} \Pi_D(k) &= bP + (1-r)b(1-P) \\ \Pi_C(k) &= \Pi_D(k) - c \end{aligned} \tag{135}$$

with P standing for the probability that the group achieves the threshold and $1 - P$ the probability that it does not. We start with the case when there is no uncertainty δ in the threshold, $\delta = 0$. In that case, $P = \Theta(k - M)$

, where $\Theta(x)$ is the Heaviside function that is 1 for $x \geqslant 0$ and 0 otherwise. Later in the chapter, we will relax this assumption.
As we have done so far, group interactions give individuals a certain payoff, depending on their strategy, whose average value is designated by fitness. We compute the fitness in a well-mixed scenario (the mean-field approximation), where each individual as a fixed (same) probability of interacting with all others and, in a population with i Cs (and $Z-i$ Ds), is given by (see **Chapter** 3 for details)

$$f_D(i) = \sum_{j=0}^{N-1} \binom{Z-1}{N-1}^{-1} \binom{i}{j} \binom{Z-i-1}{N-1-j} \Pi_D(k)$$
$$f_C(i) = \sum_{j=0}^{N-1} \binom{Z-1}{N-1}^{-1} \binom{i-1}{j} \binom{Z-i}{N-1-j} \Pi_C(k+1). \tag{136}$$

Assuming that time evolves in discrete steps, at every step one individual A compares her/his average payoff with that of another randomly chosen individual B and, the larger the payoff of individual B, selected as role model, the more likely it is that A imitates her/his behavior, with a probability given by $p(A,B) = (1+e^{\beta(f_A - f_B)})^{-1}$, where β controls for learning errors, which we have been calling intensity of selection. Additionally, individuals can explore the other strategies due to other exogenous factor(s) —- technically equivalent to a mutation —- controlled by μ. If we let i_A (i_B) be the number of individuals with strategy A (B), then the probability that an individual with strategy A changes to the (different) strategy of B is

$$T_{A\to B} = \frac{i_A}{Z}\left(\frac{i_B}{Z-1}(1-\mu)p(A,B) + \mu\right) . \tag{137}$$

In light of the framework we have discussed, the configuration of the population will evolve according to a birth-death process in discrete time, a Markov process with time invariant transitions, allowing us to describe the dynamics by means of a Markov chain characterized by the transition probabilities from a state with i Cs (and $Z-i$ Ds) to a state with i' Cs, $T_{i',i}$. The non-zero transitions are written in Eqs. 138.

A dynamical approach, in which individuals revise their strategies through peer-influence, copying others whenever these appear to be more successful, is a stamp of social learning (in the sense of cultural evolution), allowing policies to change in time as individuals are influenced by the behavior (and achievements) of others, something one actually witnesses in the context of donations to public goods [16, 104].

$$T_{i+1,i} = T_{D\to C}$$
$$T_{i-1,i} = T_{C\to D}$$
$$T_{i,i} = 1 - T_{D\to C} - T_{C\to D} \tag{138}$$

7.2 IMPACT OF RISK

We analyze the stationary distribution, given by the eigenvector of matrix $T_{i',i}$, corresponding to the eigenvalue one.[57] Additionally, we compute the most likely direction of evolution of the system (also called the gradient of selection[96, 107, 115]) as the first Kramers-Moyal coefficient of the expansion of the M-Equation of the process (the drift term). The remaining coefficients tend to zero as the population increases which means that it governs the dynamics in very large populations.[95, 138] This coefficient is computed as the difference of the probabilities that the number of individuals of a given strategy goes up and that it goes down, in each independent direction. In this case, it is just $g(i) = T_{i+1,i} - T_{i-1,i}$ (see part I, **Section** 2.5 for deduction of this result).

Stochastic effects do play an important role, in particular, for the case of the world summits where group and population sizes are comparable and of the order of the hundreds. They facilitate the tunneling through the coordination barrier.

Figure 15 depicts the dynamics and average behavior of a population of individuals for different values of risk for a dilemma played in groups of different sizes. Naturally, in the absence of risk of disaster, there is no point in contributing and thus, apart from random contributions due to some exogenous reason ($\mu > 0$), the dynamics will favor the demise of Cs, with the gradient of selection being always negative, a scenario akin to our first description of the prisoners' dilemma. As the risk increases, it leads to the emergence of two internal roots of the gradient, corresponding to unstable and stable mixed internal equilibria of the deterministic dynamics: a coordination between individuals to cooperate is necessary before a stable fraction is able to be robust to changes in strategy. Above a critical value, the unstable fixed point, the average fraction of Cs will increase steadily up to a certain value, the stable root. Furthermore, under high-risk, this collective coordination becomes easier to achieve, lowering the critical value for the transition to occur, and the final level of cooperation is also higher, increasing the final stable coexistence. These results, together with available behavioral experiments,[77, 134] demonstrates the key role played by risk perception in favoring the dynamics of Cs.

7.3 IMPACT OF SCALE

Given the intrinsic global nature of the problem of Climate Change, it is natural to extend this analysis to different group sizes. Larger group sizes implicitly consider less partitioning and, thus, decisions that involve simultaneously a larger and larger fraction of the population. In practice, one can think about group size as the scale at which the decision is being made: smaller group sizes consider several local decisions as opposed to a single large group with the world's fate on its hands. In this model no relation has been established between the level at which the decision is happening and the risk perception, considering both independent parameters. With this in mind, we can compare, for a given level of risk, if large groups do better or worse than small ones.

Cooperation is better dealt with within small groups, contrary to modern world attempts to solve the climate change problem.

Figure 15 covers two different groups sizes with a threshold fixed at 50% of Cs in the group and it is very clear that smaller groups do better. Not only the transition into a level of high cooperation happens for a lower value of risk for the smaller group but also the overall level that Cs attain is higher. Our results confirms that when the group size becomes comparable to the population size (N=Z), cooperation is effectively harder to achieve, suggesting that present world summits may set harder conditions for cooperation than, for instance, a combination of multiple, small-scale, agreements.[115] This effect becomes particularly relevant when collective

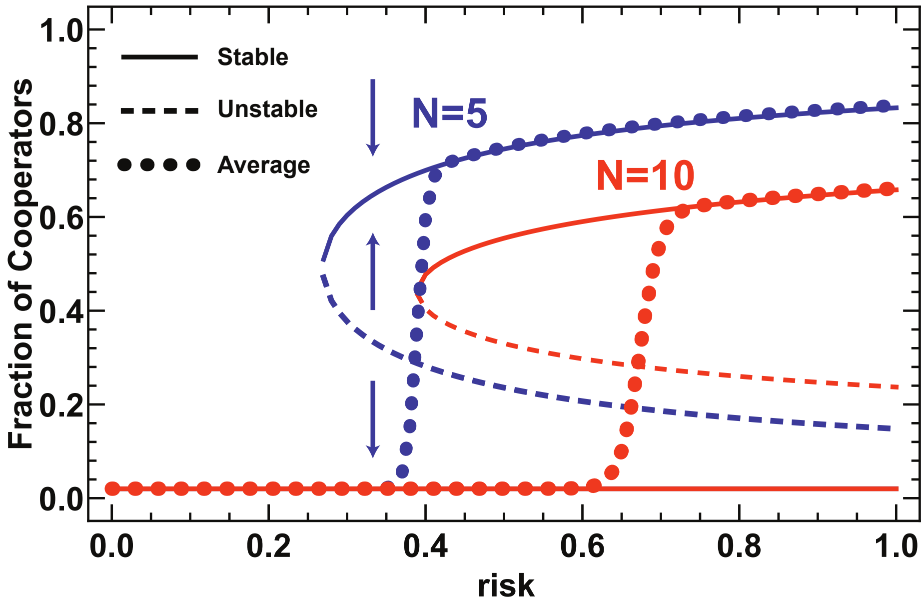


Figure 15: **Effect of risk and scale of the agreement** Evolution occurs in populations for different values of risk and group size (N). The dotted lines represent the average fraction of Cs. The solid and dashed lines, together with the arrows, are determined by the gradient of selection which rules the deterministic dynamics. Following the average fraction of Cs, one finds a rapid transition for a critical value of risk, bellow which there is no cooperation. This transition is triggered by the appearance of a stable fixed point in the deterministic dynamics. The blue and red data denote groups of different sizes, with larger groups being less cooperative both in the value of the transition and in the level of cooperation for high risk. Model parameters: $Z = 200$, $M = N/2$, $b = 1$, $c = 0.1$, $\beta = 5$, $\mu = 1/Z$.

perception of risk is low, and when,e. g., economic and technologic constraints still require sizeable costs from the parties involved, as it is most likely the case in climate negotiations.

7.4 THERSHOLD UNCERTAINTY

Before closing the chapter, a different type of uncertainty must be addressed. As discussed in the preamble to part III, the role played by uncertainties associated with incomplete information regarding targets is an unavoidable issue. Experiments observed that uncertainty in the threshold required for the achievement goal (and its respective benefit) is detrimental to cooperation.[6, 7] This is distinguishable from risk, since it acts not on the consequence of not achieving the threshold but in the threshold itself. Let us see how the model can easily capture this feature. With all else kept the same, let us now introduce variability on the threshold which is now being sampled from a uniform distribution, $u(x)$, with range $[M - \delta/2, M + \delta/2]$ (other distributions would produce identical results as it shall be clear next), leading to a change in P, in Eqs. 135, such

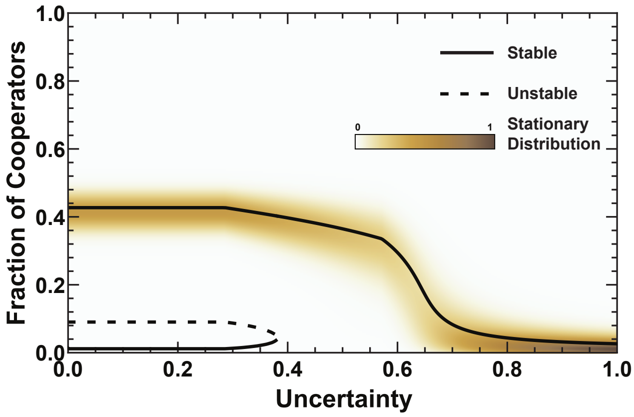


Figure 16: **Effect of threshold uncertainty** The solid (dashed) lines correspond to stable (unstable) fixed points of the deterministic dynamics. The color scheme represents the time the population spends in each configuration, given by the stationary distribution. Model Parameters: $Z = 200$, $N = 5$, $M = N/2$, $b = 1$, $c = 0.1$, $r = 0.4$, $\beta = 5$, $\mu = 1/Z$. Uncertainty is represents as δ/N.

that $P = \int u(m)\Theta(k-m)dm = \int_{M-\delta/2}^{M+\delta/2} \frac{1}{\delta}\Theta(k-m)dm$, which is 0 for $k < M-\delta/2$, 1 for $k \geqslant M+\delta/2$, and $\frac{1}{\delta}(k-M+\delta/2)$ otherwise. This changes the average payoffs of the players, and necessarily their behavior, introducing a region where individuals cannot know what will happen. This region's size in the number space of Cs is δ. As this range increases, we change the game from a coordination plus coexistence into a prisoners' dilemma, a result that is the same independently of the distribution chosen for the uncertainty in threshold. This is an immediate consequence of Eq.65 in **Chapter** 4.

Looking at **Figure** 16, we see that when uncertainty increases the probability that the population remains in the state with high levels of cooperation drops. This corroborates the impetus of the recent report of the Intergovernmental Panel for Climate Change[129] emphasizing research in order to narrow down the amount of threshold uncertainty, besides indicating that humans are the main cause of climate change and, consequently, our actions directly affect the levels needed to reach the targets.

7.5 DISCUSSION

When dealing with environmental sustainability one cannot overlook the uncertainty associated with the outcome of a collective investment. The simple form used to describe this problem and study its impact in behavioral evolution, is able to obtain an unambiguous agreement with recent experiments[77], together with several concrete predictions. This is achieved in the framework of non-cooperative N-person evolutionary game theory, an unusual mathematical tool within the framework of modeling of political decision-making. The new N-person game where both the risk of collective failure and threshold uncertainty is explicitly introduced by means of a simple collective dilemma. Moreover, instead of resorting to complex and rational planning or rules, individuals revise their behavior by peer-influence, creating a complex dynamics akin to many evolutionary systems. This framework allowed us to address the impact of risk in several configurations, from large to small groups. Overall, we saw how the emerging behavioral dynamics depends heavily on the perception of risk and threshold uncertainty. The impact of risk is enhanced in the presence of small behavioral mutations and errors and whenever global coordination is attempted in a majority of small groups under stringent requirements to meet coactive goals. This result calls for a reassessment of policies towards the promotion of public endeavors: Instead of world summits, decentralized agreements between smaller groups (small N), possibly focused on region-specific issues, where risk is high and goal achievement involves tough requirements (large relative M), are prone to significantly raise the probability of success in coordinating to tame the planet's climate. The model provides a "bottom-up" approach to the problem, in which collective cooperation is easier to achieve in a distributed way, eventually involving, e. g., regions, cities, NGOs and, ultimately, all citizens. Moreover, by promoting regional or sectoral agreements, we are opening the door to the diversity of economic and political structure of all parties, which, as showed before[114, 116] can be beneficial to cooperation.

Naturally, a model as simple as this has room for improvement. Next chapter, we will look at the effects of introducing one possible kind of heterogeneity, considering a division between rich and poor players.

8 EFFECT OF HOMOPHILY AND WEALTH INEQUALITY

This chapter is based on the manuscript "Climate policies under wealth inequality" by Vítor V. Vasconcelos, Francisco C. Santos, Jorge M. Pacheco and Simon A. Levin, published in the Proceeding of the National Academy of Science (PNAS). Appendix c

Besides risk and uncertainty, lack of consensus in climate summits has also been attributed to conflicting policies between developed and developing countries. My coauthors and I have introduced wealth inequality in the contributions to the Public Good, mimicking the world's patent wealth inequality and diversity of roles played by different countries. In the light of what was previously found in experiments,[77, 134] one might investigate how these roles influence both the distribution of contributions and the effect of homophily in the behavioral dynamics [18, 147]. The economic experiments of, e. g., Tavoni et. al [134] involved groups of 6 students from western, educated, industrialized, rich, and democratic (WEIRD) countries. They were performed by distributing endowments unequally among those groups of people who could reach a fixed target sum through successive money contributions, knowing that if they were to fail, they would lose all their remaining money with 50% probability. In this way, using a similar PGG to that we used in the previous chapter, they introduced different kinds of players: Rich and Poor, whose initial endowment was higher and lower, respectively. All groups were composed by half of poor individuals and the other half rich individuals, and showed that in some cases rich would compensate for the smaller tendency to cooperate by the poor. In the model that follows, we will consider a population with a 1:4 distribution of rich to poor players, roughly reflecting the present-day status in what concerns the wealth asymmetry of nations. Additionally, instead of a perfectly homogeneous imitation, let us allow for the two classes of wealth to limit their individuals' sphere of influence,[34, 71, 99] what is often called homophily [34, 71, 99]: high homophily means rich (poor) players only influence and are influenced by rich (poor) players, whereas low homophily means every player is equally likely to influence every other player, independently of their wealth status. This creates a weighted network of influence that results in the homogeneous influence scenario as homophily goes to zero.

One of the greatest challenges in addressing global environmental problems such as climate change, which involves public goods and common-pool resources, is achieving cooperation among peoples. There are great disparities in wealth among nations, and this heterogeneity can make agreements much more difficult to achieve.

8.1 MODEL

More specifically, when we introduce wealth inequality, we split the Z individuals into Z_R rich and $Z_P = Z - Z_R$ poor; b and c, in Eqs. 135, now depend on the class, with b_R (b_P) and c_R (c_P) standing for the initial endowment and cost paid by the rich (poor), respectively. The payoffs of the classes $X = R, P$ are thus written as

$$\Pi_D^X(k_R, k_P) = b_X P + (1-r) b_X (1-P)$$
$$\Pi_C^X(k_R, k_P) = \Pi_D^X(k_R, k_P) - c_X. \tag{139}$$

The different contributions must now add up to a certain value instead of to a number of contributors. This amounts to a change in P and, since we will ignore the effects of threshold uncertainty. In a group with k_R Rich Cs and k_P Poor Cs, we may write

$$P = \Theta(k_R c_R + k_P c_P - M\bar{c}), \tag{140}$$

with $Z\bar{c} = Z_R c_R + Z_P c_P$. The two classes also introduce a splitting in the sampling for the calculus of the fitness. Notice we do not restrict the fraction of Rich and Poor in the groups, despite the results being robust to that change.

$$\begin{aligned}
f_D^R(i_R, i_P) &= \sum_{j=0}^{N-1} \sum_{l=0}^{N-1-j} \binom{Z-1}{N-1}^{-1} \binom{i_R}{j} \binom{i_P}{l} \binom{Z - i_R - i_P - 1}{N-1-j-l} \Pi_D^R(j, l) \\
f_C^R(i_R, i_P) &= \sum_{j=0}^{N-1} \sum_{l=0}^{N-1-j} \binom{Z-1}{N-1}^{-1} \binom{i_R - 1}{j} \binom{i_P}{l} \binom{Z - i_R - i_P}{N-1-j-l} \Pi_C^R(j+1, l) \\
f_D^P(i_R, i_P) &= \sum_{j=0}^{N-1} \sum_{l=0}^{N-1-j} \binom{Z-1}{N-1}^{-1} \binom{i_R}{j} \binom{i_P}{l} \binom{Z - i_R - i_P - 1}{N-1-j-l} \Pi_D^P(j, l) \\
f_C^P(i_R, i_P) &= \sum_{j=0}^{N-1} \sum_{l=0}^{N-1-j} \binom{Z-1}{N-1}^{-1} \binom{i_R}{j} \binom{i_P - 1}{l} \binom{Z - i_R - i_P}{N-1-j-l} \Pi_C^P(j, l+1)
\end{aligned} \tag{141}$$

The imitation dynamics occurs in two sub-populations, eventually restricted by the homophily parameter, h, that incorporates the idea that individuals of a given class $X = P, R$ may be more likely to choose to imitate individuals of the same class than individuals of the opposite class Y. Thus, we can build the transition matrix such that going from a state with a given number of rich and poor Cs (i_R, i_P) to (i'_R, i'_P) , $T_{(i'_R, i'_P)(i_R, i_P)}$, can be written using $T_{(i_R \pm 1, i_P)(i_R, i_P)} = T_R^{\pm}$ and $T_{(i_R, i_P \pm 1)(i_R, i_P)} = T_P^{\pm}$, with

$$\begin{aligned}
T_X^+ &= \frac{Z_X - i_X}{Z} \left(\left(\frac{i_X}{Z-1-hZ_Y} p(D_X, C_X) + \frac{(1-h) i_Y}{Z-1-hZ_Y} p(D_X, C_Y) \right) (1-\mu_X) + \mu_X \right) \\
T_X^- &= \frac{i_X}{Z} \left(\left(\frac{Z_X - i_X}{Z-1-hZ_Y} p(C_X, D_X) + \frac{(1-h)(Z_Y - i_Y)}{Z-1-hZ_Y} p(C_X, D_Y) \right) (1-\mu_X) + \mu_X \right).
\end{aligned} \tag{142}$$

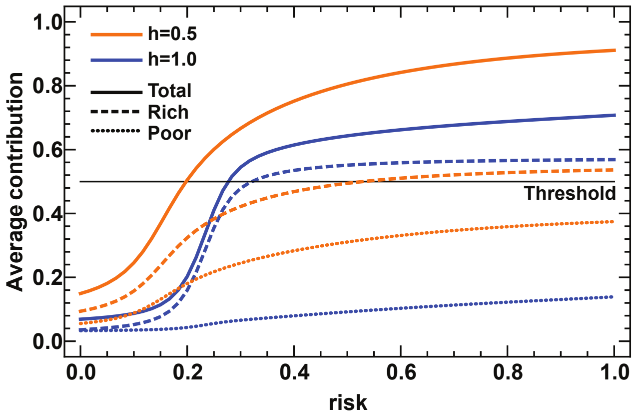


Figure 17: **Effect of homophily in heterogeneous populations** The solid lines represent the total average contribution per group normalized by the average maximum possible contribution of the groups. The dashed and dotted lines correspond to the decoupling of the total contribution into what is contributed by the rich and poor, respectively. $Z = 200, Z_R = 40, N = 6, M = 3c\bar{b}, c = 0.1, b_R = 1.7, b_P = 0.3, (\bar{b} = 1), \beta = 5, \mu_X = 1/Z_X$.

8.2 SPLITTING OF CONTRIBUTORS

The stochastic evolutionary dynamics of the population occurs in the presence of errors, both in terms of errors of imitation and in terms of behavioural mutations, the latter accounting for a free exploration of the possible strategies. We calculate the pervasiveness in time of each possible behavioural composition of the population, using the stationary distribution, which allows the computation of the average fraction of groups that successfully produce (or maintain) the public good – a quantity we designate as group achievement, η_G.

Figure 17 shows that, under the premises of our model, and in agreement with existing experiments,[75, 134] the rich generally contribute more than the poor. This effect is even stronger in the presence of high homophily, given that the contribution of the poor is very sensitive to homophily and tends to go down in that case. This, in turn, means that the rich will often compensate for the lower contribution of the other class, a feature which will happen to a limited extent, being dependent on risk. Overall, this also indicates that homophily, if widespread, may lead to a collapse of cooperation, especially in the transition of low to high risk.

In experiments, games comprised groups of fixed size where participation were equally split between rich and poor individuals, whose different wealth resulted from two different start-up amounts of money made available to group participants.

Figure 18 shows that, even in the absence of significant homophily bias ($h \leqslant 0.5$) a higher fraction of rich contribute (with average values of 57% for $r = 0.2$ and 78%, for $r = 0.3$), compared with the poor (with average values of 46% for $r = 0.2$ and 69% for $r = 0.3$), thus also protecting their greater wealth. This result does not depend on risk; however, for low risk, the overall contribution is limited, increasing significantly after a slight increase in overall risk perception. Indeed, in the absence of homophily, cooperation may prevail in a wealth-unequal world (e.g., Figure 18D).

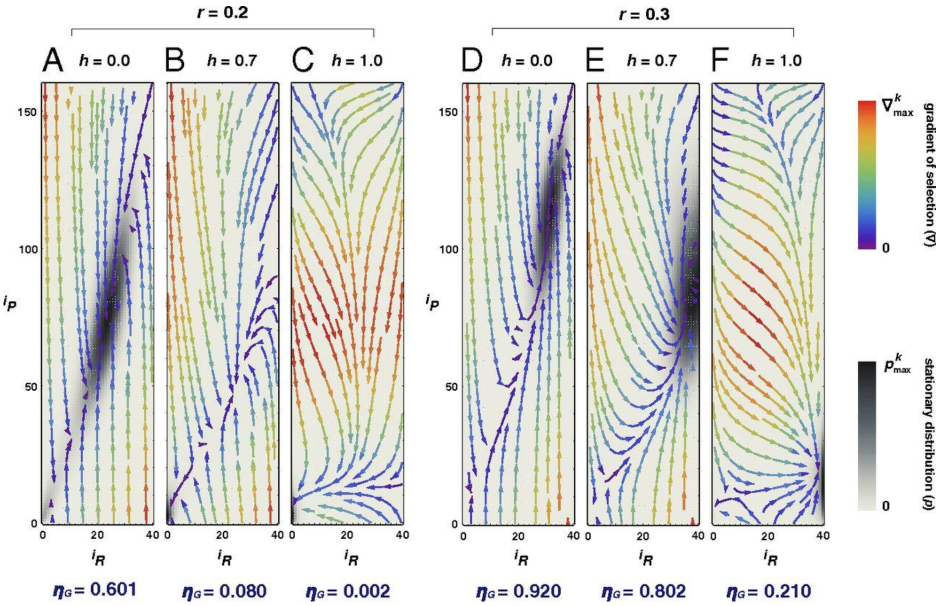


Figure 18: **Stationary distribution and gradient of selection for different values of risk r and of the homophily parameter h** (A–F) Each panel contains all possible configurations of the population (in total $Z_R \times Z_P$), each specified by the number of rich (i_R) and poor (i_P) it contains and represented by a gray-colored dot. Darker dots represent those configurations in which the population spends more time, thus providing a contour representation of the stationary distribution. The curved arrows show the so-called gradient of selection, which provides the most likely direction of evolution from a given configuration. We use a color code in which red lines are associated with higher speed of transitions. The behavioral dynamics of the population depend on the homophily parameter h in a nonlinear way. For $h \leqslant 0.5$, the results remain qualitatively similar to those depicted for $h = 0$, in which case everybody influences and is influenced by everybody else. In this case, the contribution of the rich is sizeable, which also leads the poor to contribute. For $h > 0.5$ the behavior changes abruptly, and one witnesses the rapid collapse of cooperation among the poor and, for low risk ($r = 0.2$, A–C), an ensuing disappearance of contributions to the overall PGG, with the population spending most of the time in full defection, leading to a dramatic impact on the overall group achievement η_G, indicated below each contour plot. However, a slight increase in overall risk perception (here $r = 0.3$, D–F) actually impels the rich to contribute, despite the fact that the poor still do not cooperate. Other parameters: and . $Z = 200, Z_R = 40, N = 6, M = 3c\bar{b}, c = 0.1, b_R = 2.5, b_P = 0.625, (\bar{b} = 1), \beta = 5, \mu_X = 1/Z_X$.

Qualitatively, one can now understand the results in **Figure** 17 if one takes into account that, in most cases, the dynamics both among the rich and among the poor can become dominated by basins of attraction that lead to a coexistence between Cs and Ds. Whenever the risk is moderate to high, there is an increase of the size of such basins, with a corresponding increase of the stationary fraction of cooperators, such that the feedback dynamics between the poor and the rich act to build up the cooperation levels among both subpopulations.

As also shown in **Figure** 18, this positive feedback between the two subpopulations is interrupted whenever homophily becomes dominant ($h \approx 1$). When rich and poor cease to be able to sway one another, we observe two distinct scenarios: At low risk ($r = 0.2$ in Figure 17) overall cooperation collapses. With a slight increase in risk perception, however ($r = 0.3$), the rich contribute, despite the fact that the poor do not. Together with risk, a lack of homophily plays an important role: As soon as the homophily constraint is relaxed – by adopting $h < 1$ – poor individuals start to be nudged by the successes of the rich, effectively inducing the poor players to contribute to the common good.

However, even in the absence of homophily ($h = 0$), this positive feedback between the two subpopulations does not always lead to an increase of cooperation – thus we obtain the coexistence dynamics shown in **Figure** 18. Indeed, whenever most poor opt for cooperation, the dynamics drive rich countries toward less cooperation, given that they may now profit from the larger overall contributions stemming from the poor. Similar dynamics may also occur among the poor. This reduction, however, not only does not prevent the majority of rich from engaging in cooperation, but also does not compromise the overall group achievement values. As a result of these coupled dynamics, the population will stay most of the time nearby a coexistence equilibrium (interior attractor, Figure 18 A, D, and E).

The result of less contribution by the poor has been identified in economics as the exploitation of the big by the small[87]. However, the effect of homophily and its impact on this is novel and shows how segregation between rich and poor policies worsens contributions to the public good.

This said, we are all aware that some individuals may be more receptive than others to change their mind, based on the influence of their peers. In fact, some individuals—for various reasons, as witnessed in the world summits on climate change that have taken place to date—may maintain the same behavior irrespective of their sphere of influence. Given the small size of the overall population, such an obstinate behavior may lead to sizable effects in the global dynamics. In the following we investigate how such obstinate behaviors (in both wealth classes) affect the overall dynamics. For simplicity, we assume that, in all cases, obstinate behavior amounts to 10% of individuals in one subpopulation—which corresponds to the same fixed contribution to the PGG, considering either rich or poor obstinate players.

Figure 19 shows that obstinate poor cooperators provide impressive improvements in the aggregate propensity of the population to achieve

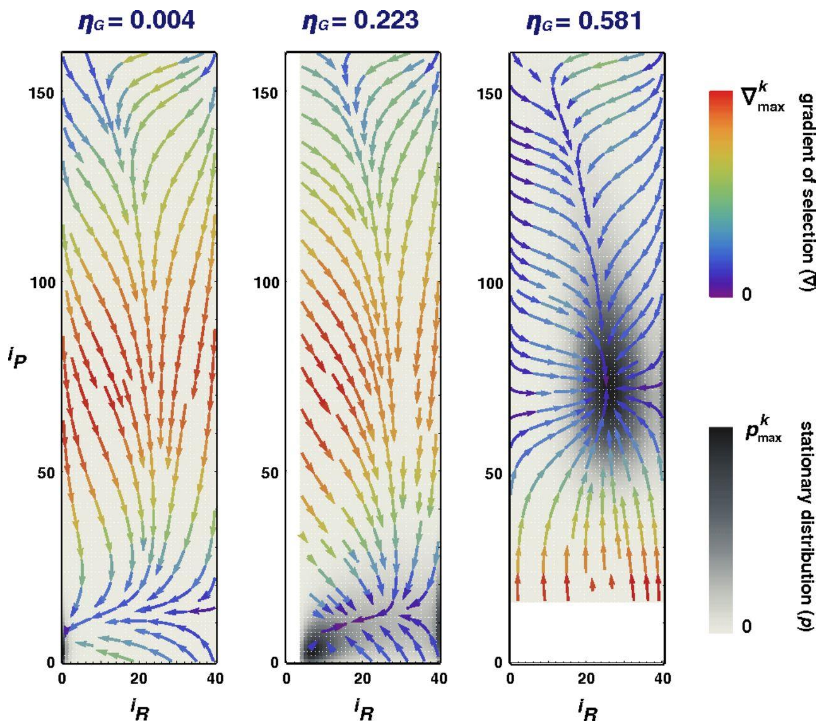


Figure 19: **Stationary distribution and gradient of selection for populations comprising 10% of individuals exhibiting an obstinate cooperative behavior** Same notation as in **Figure** 18 is used. Whenever 10% of individuals exhibit obstinate cooperative behavior (Center and Right contours), the number of configurations of the population in which the evolutionary dynamics proceed is correspondingly reduced (white areas in contours). The Left contour contains no obstinate individuals and is displayed for reference. In the Center contour, 10% of the rich individuals behave as obstinate cooperators; that is, they never change their behavior. In the Right contour, 10% of poor individuals exhibit such behavior. A small fraction of obstinate rich and obstinate poor cooperators lead to very different outcomes, also for the average group achievement η_G. Indeed, the chances of success are significantly enhanced whenever obstinate cooperator behavior occurs among the poor. The effect is most pronounced whenever individuals are homophilic, as is the case here (h = 1). Other parameters: $Z = 200$; $Z_R = 40$; $Z_P = 160$; $c = 0.1$; $N = 6$; $M = 3c\bar{b}$; $b_P = 0.625$; $b_R = 2.5$; $r = 0.2$; $\beta = 5.0$; $p^k_{max} = \{p^A_{max}, p^B_{max}, p^C_{max}\} = \{76, 4, 2\} \times 10^{-3}$ and $\nabla^k_{max} = \{\nabla^A_{max}, \nabla^B_{max}, \nabla^C_{max}\} = \{3, 3, 4\} \times 10^{-2}$. ($\bar{b} = 1$), $\beta = 5$, $\mu_X = 1/Z_X$.

coordination ($\eta_G = 0.581$ in the presence of obstinate individuals compared with $\eta_G = 0.004$ in the absence of obstinate individuals), larger than obstinate rich cooperators, who lead to less pronounced enhancements ($\eta_G = 0.223$). This effect, which extends qualitatively to all values of h, is more pronounced when $h = 1$, as is the case in Figure 19.

8.3 DISCUSSION

Homophily generally impels the rich to compensate for the poor. Given that contributions from the poor are crucial to solving the climate change problem we face, it is then imperative that homophilic behavior is avoided. Moreover, a small fraction of obstinate poor cooperators leads to sizeable increases in the overall prospects for success, mostly when homophily rules.

Conventional wisdom would lead one to believe that wealth inequality and homophily would constitute important obstacles regarding overall cooperation in climate change negotiations. Our results predict that, as long as (i) risk perception is high; (ii) climate negotiations are partitioned in smaller groups agreeing on local, short-term targets; and (iii) individuals are influenced by their more successful peers, whom they imitate—irrespective of their wealth class—and making errors while doing so, the prospects are not that grim. On the contrary we find that, under such conditions, cooperation may outcompete defection. Moreover, the qualitative nature of the results obtained here remains robust if we assume that, instead of proportional contributions, poor and rich contribute the same amount, when cooperating[147].

In this chapter, however, we ignored an important factor: that the thresholds may be intrinsically uncertain. As we have seen, this uncertainty, if sizeable, can destroy cooperation. Likely, to the extent that agreements aim at short-term targets involving smaller groups, it will also be easier to narrow down threshold uncertainties.

Finally, the recent report of the Intergovernmental Panel for Climate Change[129], besides emphasizing that climate change is real and humans are the main cause of it, urging countries to stop the warming of the planet, has also attempted to narrow down the threshold uncertainty. However, given that risk perception is low and that a bottom–up approach – as defended by the late Elinor Ostrom[91] and also by the results collected in the present dissertation – has yet to spread globally, it is perhaps not surprising that today's prospects remain gloomy. Clearly it is urgent that individuals become aware of the true risk that we face. Indeed, an increase in risk perception will surely promote the development of local initiatives that may foster overall cooperation by extending the bottom–up approach to all players of the global game.

Next chapters introduce the approach of sanctioning to solve this global dilemma.

9 SANCTIONING INSTITUTIONS

This chapter is based on the manuscript "A bottom-up institutional approach to cooperative governance of risky commons" by Vítor V. Vasconcelos, Francisco C. Santos and Jorge M. Pacheco. Appendix b

To conclude the list of effects I proposed to address, let us explore the effects of global punishment institutions versus locally arranged ones and investigate at which scale sanctioning should happen and discuss these two major configurations addressed by my coauthors and I.[145] Naturally, given the pros and cons of some procedures against others, agreeing on the way punishment should be implemented is far from reaching a consensus.[140] Institutions need not be global (such as the United Nations), supported by all members willing to punish/sanction, or punishers, that overview all group interactions in the population; they may also be local, group-wide institutions, created to enforce cooperation within a particular group of individuals [90]. While the establishment of global institutions will depend on the total number of those willing to sanction in the population, setting up local institutions relies solely on those that exist within a group. Moreover, one does not expect that all the parties (e. g.countries, regions or cities) will be willing to incur in a cost in order to sanction others, despite being willing to undertake the necessary measures to mitigate the climate change effects (or, in the language used so far, to cooperate). In other words, one may expect to witness, in general, the three behaviors simultaneously in the population. **Figure** 21 represents these three behaviors, cooperate, defect and punish as C, D and P providing an overall portrait of the evolutionary dynamics of the population, in the presence of these three possible behaviors.

9.1 MODEL

With this in mind, in this section we go back to a population comprising players of the same average wealth and explore the effects of this additional strategy, the punishers (Ps). As before, Cs, but now also, Ps contribute a certain fraction of their endowment, in order to reach a common goal, whereas Ds do not contribute. Ps will also contribute to an institution incurring in an additional cost c_s (cost of sanctioning) adding to that associated with cooperation. This cost is paid to the institution to make it able to punish defectors by an amount p (punishment fine), whenever the institution reaches a total of contributions $M_I c_s$. This creates a second game, by introducing an additional efficiency threshold that must be achieved, now in terms of punishers that contribute both to the public good and to the sanctioning institution. Thus, the punishment institution acts as a second order public good that indirectly increases the investment in the original public good, which, as before, is seen as the health and stability of climate. This leads to a modification of the payoffs in Eqs. 135 such that for a group with k_C Cs, k_P Ps and $N - k_C - k_P$ Ds the payoffs are

$$\begin{aligned} \Pi_D(k_C, k_P) &= bP + (1-r)b(1-P) - \Pi_{\text{scale}} \\ \Pi_C(k_C, k_P) &= bP + (1-r)b(1-P) - c \\ \Pi_P(k_C, k_P) &= \Pi_C(k_C, k_P) - c_s \end{aligned} \tag{143}$$

with $P = \Theta(k_C + k_P - M)$ and the scale being either local or global. In the first case, $\Pi_{\text{local}} = p\Theta(k_P - M_I)$ and in the later $\Pi_{\text{global}} = p\Theta(i_P - M_I)$, with i_P being the number of Punishers in the whole population. The averages are also computed with the hypergeometric sampling for a given number of Cs, i_C, and Ps, i_P (and Ds, $i_D = Z - i_C - i_P$) in the population:

$$\begin{aligned} f_D(i_C, i_P) &= \sum_{j_1=0}^{N-1} \sum_{j_2=0}^{N-1-j_1} \binom{Z-1}{N-1}^{-1} \binom{i_C}{j_1} \binom{i_P}{j_2} \binom{Z-i_C-i_P-1}{N-1-j_1-j_2} \Pi_D(j_1, j_2) \\ f_C(i_C, i_P) &= \sum_{j_1=0}^{N-1} \sum_{j_2=0}^{N-1-j_1} \binom{Z-1}{N-1}^{-1} \binom{i_C-1}{j_1} \binom{i_P}{j_2} \binom{Z-i_C-i_P}{N-1-j_1-j_2} \Pi_C(j_1+1, j_2) \\ f_P(i_C, i_P) &= \sum_{j_1=0}^{N-1} \sum_{j_2=0}^{N-1-j_1} \binom{Z-1}{N-1}^{-1} \binom{i_C}{j_1} \binom{i_P-1}{j_2} \binom{Z-i_C-i_P}{N-1-j_1-j_2} \Pi_P(j_1, j_2+1) \end{aligned} \tag{144}$$

Finally, the transitions that build the transition matrix are given by the transition between any pair of strategies. The probability that an individual with strategy $A = C, D, P$ changes to another strategy, B, is given by Eq. 137, with the mutation term $i_A/Z\mu$ now divided by two. This, again, allows us to build the transition matrix from which the stationary distribution is extracted.

9.2 ACTING LOCALLY

Using the same measure of group success used in the previous chapter, the average group achievement (η_G), we can compare the effectiveness of the institutions. The empirical results obtained for the risk dependence[77] (in the absence of any sanctioning) show that the group achievement increases with the value of risk, correlating nicely with the dependence shown in **Figure** 20 with black lines and symbols that establishes the baseline comparison. Indeed, in **Figure** 20 the behavior of η_G as a function of risk is shown in the absence of any institutions (in black), under one global institution (in red) and under local institutions (in blue).

Comparison of the black and red curves shows that global institutions provide, at best, a marginal improvement compared with no institutions at all. This result is surprising, given that most climate agreements attempt to

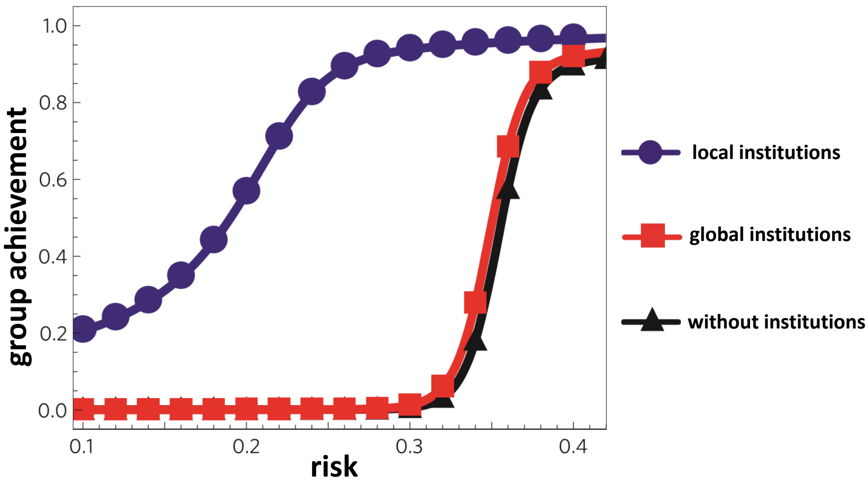


Figure 20: **Group achievement** The average fraction of groups that attain the public good (η_G) as a function of perception of risk (r). Sanctions are enacted by a global institution (red lines and squares) or by local institutions (blue lines and circles). Black lines and triangles: results obtained in the absence of any institution. Unlike global institutions, often associated with marginal improvements of cooperation, local institutions promote group coordination to avoid a collective disaster, mostly for low perception of risk. The coordination threshold M is set to 75% of the group size, whereas local (global)institutions are created whenever 25% of the group (population) contributes to its establishment. Other parameters: $Z = 100, N = 4, c/b = 0.1$, $\mu = 1/Z$, $p = 0.3$, $c_s = 0.03$.

involve all countries at once[4, 5], in which case a single, global institution constitutes the most natural candidate.

On the contrary, under local, group-wide, sanctioning institutions, associated with a distributed scenario in which global sanctions will result from the joint role of a variety of institutions, group achievement is substantially enhanced, in particular when it is most needed: for low values of the perception of risk and whenever individuals face stringent requirements to avoid a collective disaster, as has been pointed out to be the case in the context of climate treaties[4]. This aspect is particularly important, as the group size (N) now defines both the scale at which agreements should be attempted and the overall scope of each institution as the local institutions are more efficient than the global one.

Local institutions prevail for longer periods than a (single) global one, promoting systematically more widespread cooperation than global ones.

The success of local institutions is closely connected with their resilience. **Figure** 21 contains the key elements of the dynamics of this complex system. In the depicted case, the local institutions are able cancel the effect of the attractor close to the configuration where everyone defects, pushing the population to a cooperative state with the average number of punishers just over the threshold. On the other hand, the global institution keeps that attractor, since it exists when the institution is not working. For high risk though, the institution seems to work quite on the verge, allowing for sequences of small invasions of defectors before becoming effective, creating three areas of behavior: full defection, on one side and, on the other, full cooperation with enough punishers to maintain the institution which sequentially goes to a mixed state of defectors and Cs that goes back to an effective institution. Thus, local institutions act as a second order public good, thus being more effective locally.

Before I conclude, I will dedicate a last chapter extending this analysis and explore the adoption of agreements that include punishing institutions.

9.3 DISCUSSION

These results support the conclusion that a decentralized, polycentric, bottom-up approach, involving multiple institutions instead of a single global one, provides better conditions both for cooperation to thrive and for ensuring the maintenance of such institutions. This is particularly relevant whenever perception of risk of collective disaster, alone, is not enough to ensure global cooperation. In this case, local sanctioning institutions may provide an escape hatch to the tragedy of the commons humanity is facing. This is a consequence of the institution providing a second order public good, thus have similar properties of the climate public good.

In this context, it is worth stressing that the mechanisms discussed here operate optimally whenever groups are small. Present-day local initiatives, such as the *Western Climate Initiative*, have started with a small group of US states. As time went by, the Western Climate Initiative group size has grown to include additional Canadian states and Mexican provinces. Although the reasons and motivations for such an evolution are comprehen-

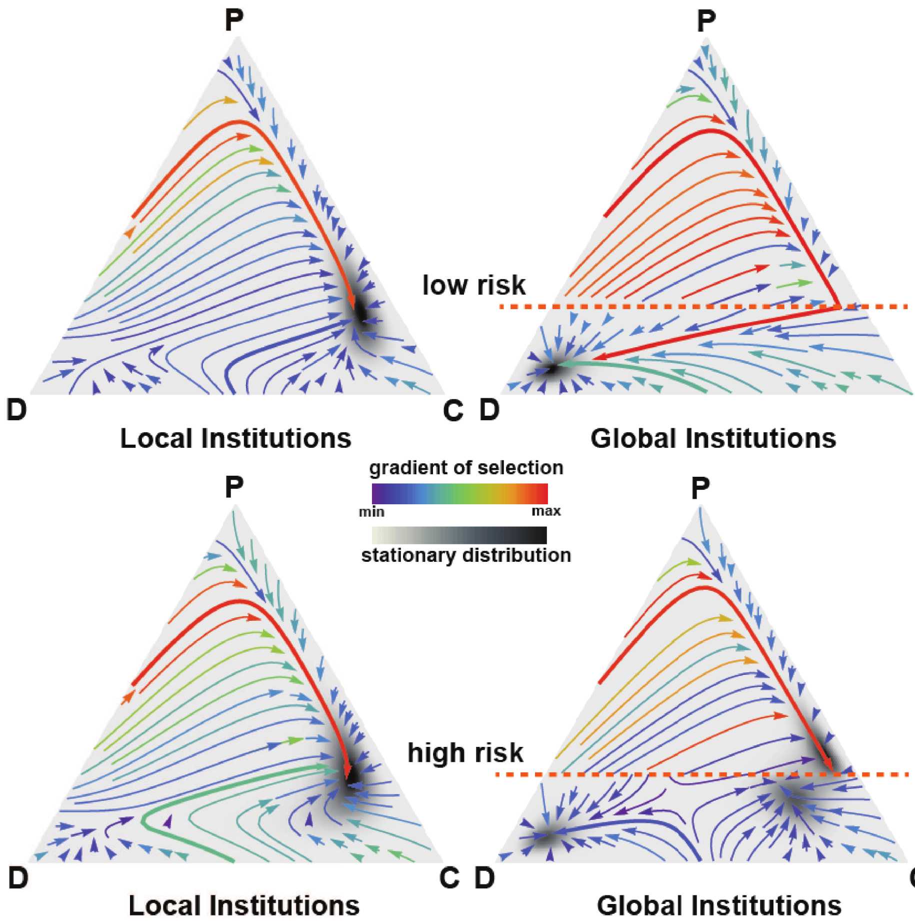


Figure 21: **Effect of local versus global sanctioning institutions** Left panels represent the dynamics with local institution and right panels the dynamics with global institutions, with the orange line representing the threshold in the creation of the institution. The top panels are evaluated for low risk and the bottom panels for high risk. The gradient of selection is represented as a stream indicating in each point the most likely direction and which corresponds, in the deterministic dynamics, to paths of the behavior of the population. $Z = 200$, $\mu = 0.05$, $N = 8$, $M = 6$, $c = 0.1$, $b = 1$, $\beta = 5$, $M_I = 25\%$, $c_s = 0.03$, $p = 0.3$, $r = 0.2$ (low risk), $r = 0.5$ (high risk).

sible, one should not over look that larger groups are more difficult to coordinate into widespread cooperation. Similar dynamics, in which cooperation nucleating in a small group expands into a larger and larger group, can be found in policies beyond climate governance with mixed results, from the major transitions in evolution[125] to the recent evolution of the European Union, stressing the common ground shared by governance and a variety of ecosystems[68]. In this context, it might be easier to seek a multi-scale (and multi-step) process, in which coordination is achieved in multiple small groups or climate blocks[101], before aiming, if needed, at agreements encompassing larger groups (or, alternatively, inter-group agreements). Hence, although most causes of climate change result from the combined action of all inhabitants of our planet, the solutions for such complex and global dilemma may be easier to achieve at a much smaller scale[91].

10 CONTRACTS AND SPILLOVERS EFFECTS IN CLIMATE CHANGE DILEMMAS

This chapter is based on the work that is being developed with Francisco C. Santos, Alessandro Tavoni and Jorge M. Pacheco.

The failure of consecutive attempts to reach global cooperation and avoid effects of climate change has been associated with a lack of sanctioning institutions and lack of other mechanisms to deal with those who do not contribute to the welfare of the planet or fail to abide by agreements[4, 101, 146, 148, 152]. These agreements, when empirically successful, tend to occur in the form of clubs or other polycentric structures and are seen as effective means of producing public goods, despite spillovers [46]. This contrasts with the previous chapter since there we lack an explicit agreement with an associated cost and benefit and the mechanism to sanction defectors was assumed universal, in the sense that it was applied to all defectors within the chosen setting (global or local), something that will unlikely happen in reality. Contracts adjacent to the club structure are often voluntary and impose mechanisms to prevent free riding, acting as a signaling mechanism for a naturally cooperative club [8, 9, 41–43].

The legally-binding Oslo (1972) and Paris (1974) Conventions, known collectively since 1992 as the OSPAR Convention, sought to reduce ship, aircraft and land-based pollution among European countries in the north-east Atlantic. Achieving significant cooperative agreements on the North Sea, seen as a particularly pressing area of environmental concern, was hobbled by these conventions' unanimous consent requirements. Non-North Sea states including Spain and Portugal, which had little interest in North Sea pollution, and the U.K., which was the largest North Sea polluter, objected to stronger binding protections.[46]

Using the Collective Risk Dilemma we have been discussing so far, let us incorporate the effects of pre-play contracts and spillovers or synergistic club interactions to systematize club dynamics for public good provisioning and to understand the importance of the signaling systems introduced by such constructions. My coauthors and I have investigated the emergence and impact of voluntary pre-play contracts to deter non-cooperative behavior in climate agreements. Here, I will show that i) voluntarily signing a binding agreement and contributing can be a robust strategy ii) Cooperation even if stable is never global, with defectors outside the agreement reaping the benefits of the spillovers produced within the agreement, iii) incomplete cooperation within the agreement is a transient state with defectors leaving and iv) the contract may resolve the second order free riding, acting as an auto-organized signaling mechanism, boosting cooperation through voluntary participation. Finally, we analyze the adoption behavior over time of such agreements, finding a rapid growth of the number of signatories after a broad transient state.

10.1 THE MODEL

We again model a population in which players are asked to contribute to a common good but now to either do it within a binding agreement setup with enforced supervision or outside it. Thus, players act upon two decisions: signing the agreement and cooperating. This can be systematized defining a strategy in two steps, $X = (s, c)$, where s stands for, the first, signing the contract ($s = 1$) or not ($s = 0$) and, the second, c, for cooperating ($c = 1$) or defecting ($c = 0$). Overall we can have four strategies: Signatories Cooperators, $C = (1, 1)$, Signatories Defectors $D = (1, 0)$, Non-signatories Cooperators, $\tilde{C} = (0, 1)$, and Non-signatories Defectors, $\tilde{D} = (0, 0)$. The common good is shared among all players with the possibility of enhancement for the members who choose to be within the contract. In turn, the contract consists of a punishment mechanism to defectors within the club, given that the club is big enough. Notice that this is a scenario more demanding (and also more realistic [46]) than, for instance, forcing insiders' contributions to go directly to the good and, thus, providing more public good.

Individuals will interact in groups of size N and receive a payoff depending on their own and on the others' strategies. Each group contains players signing the contract ($s = 1$) and those that do not ($s = 0$). Everyone will get the benefit produced, $b\ P$, which may be none, when $P = 0$, depending on how many contributed. Those who sign the contract will always pay a cost c_s which may provide an excludable benefit increment, α, and an institution to enforce the contract if and only if the number of signatories n_s is greater than M_s. Cooperators ($c = 1$) as usual contribute to the good an amount c_c and if the total contribution, $n_c\ c_c$, is greater than $M_c\ c_c$, then $P = 1$, providing the benefit. As in the previous chapters, and as given by the original model of Santos and Pacheco [112], players perceive the risk of disaster with probability r leading to a modified probability of getting the benefit, $P = \Theta(n_c - M_c) + (1 - r)(1 - \Theta(n_c - M_c))$, with $\Theta(x) = \{1 \text{ for } x \geqslant 0,\ 0 \text{ o.w.}$ the Heaviside function. The payoffs can be written as

$$\Pi_{C=(1,1)}(n_s, n_c) = -c_s - c_c + b\ (1 + \alpha\,\Theta(n_s - M_s))\,P(n_c), \quad (145)$$

$$\Pi_{D=(1,0)}(n_s, n_c) = -c_s + b\ (1 + \alpha\,\Theta(n_s - M_s))\,P(n_c) - p\Theta(n_s - M_s), \quad (146)$$

$$\Pi_{\tilde{C}=(0,1)}(n_s, n_c) = -c_c + b\ P(n_c), \quad (147)$$

$$\Pi_{\tilde{D}=(0,0)}(n_s, n_c) = b\ P(n_c), \quad (148)$$

where p stands for the punishment that may be induced on Ds when the contract is enforced.

In each time step, an individual with strategy $X = \{C, D, \tilde{C}, \tilde{D}\}$ may revise their strategy comparing their average payoff (or fitness), f_X, with that of another individual with strategy Y, f_Y, and change strategy with a probability $p(X, Y) = (1 + \exp\beta(f_X - f_Y))^{-1}$. Furthermore, exogenous factors can induce a change of strategy with probability μ and, thus, the probability that some individual changes from strategy X to Y is $(1 - \mu)\,\frac{i_X}{Z}\left(\frac{i_Y}{Z-1}p(X, Y) + \mu/3\right)$, where i_X is the number of individuals with strategy X. Again, for further description see **Chapter** 3.

10.2 DYNAMICS

The complete dynamics happens in a simplex represented in a tetrahedron of which each of the four vertex represents a configuration of the population in which all individuals have the same strategy (all C, all D, all C̃, all D̃). In order to gain intuition of the complete dynamics, let us analyze what happens between each pair of strategies, studying the edges of the tetrahedron, where only two of the strategies coexist at any given time (**Figure** 22). Due to the continuity properties of the gradient of selection, the properties on the edges condition the behavior of the population on the faces, where three strategies are present at each time, and consequently the behavior on the interior, where all strategies coexist. Thus, some critical parameters of the game can be obtained through this pairwise analysis. For simplicity, we set $Z \to \infty$ and use Eq.(65) to get the pairwise dynamics.

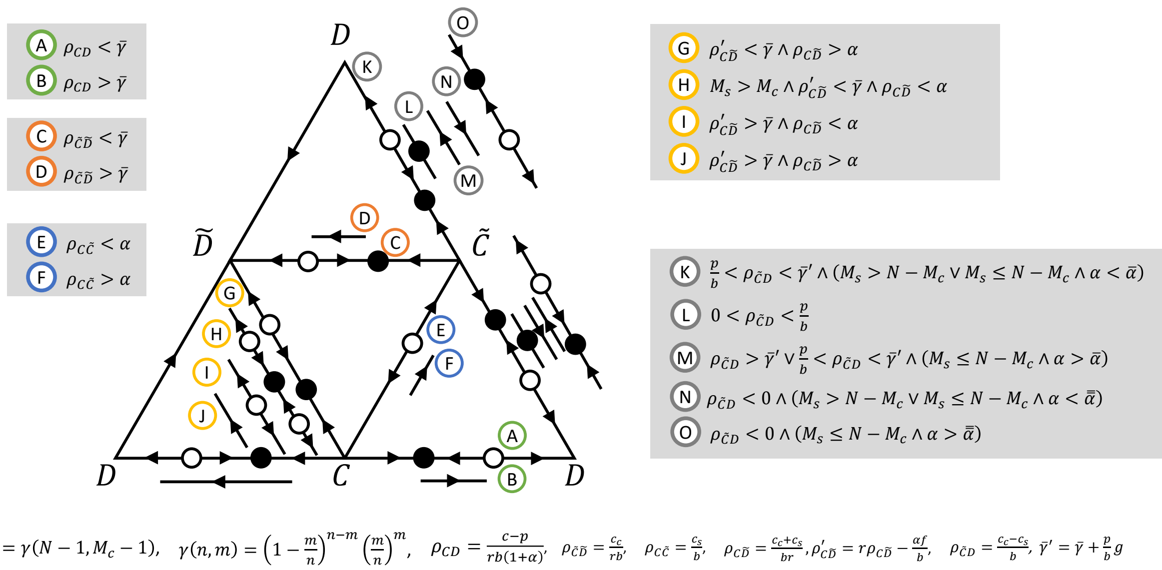


Figure 22: **Conditions that govern the interactions of the pairs of strategies** $\bar{\alpha}$ and $\bar{\bar{\alpha}}$ are critical values of α and f and g are scaling factors between 0 and 1.

We can understand what governs the dynamics of cooperation by looking at the fitness difference between cooperative and non-cooperative strategies. In fact, apart from risk perception and uncertainty, cooperation is driven by the usual settings of the agreements, namely, group size N and threshold M_c, with $\bar{\gamma} = \binom{N-1}{M_c - 1} \left(\frac{M_c - 1}{N-1}\right)^{M_c - 1} \left(\frac{N - M_c}{N-1}\right)^{N - M_c}$ controlling what are high or low cost-to-benefit ratios (see condition **A/B** and **C/D** in **Figure** 22). In the same way, the dynamics of signing or not the contract can be grasped by analyzing the pairs of strategies that change signing behavior (see conditions **E/F**). This analysis reveals that the enhancement the contract provides to the generated good (α) – or equivalently but opposite, the amount of spillover the players outside the contract identify coming from the good generated by the contract – governs signing. Hence, if there is no enhancement or there is full spillover ($\alpha = 0$), players tend not to sign the contract. This reveals that individuals need to effectively obtain additional outcome for being part of the contract, with reinforcement through the institution being insufficient to provide such marginal benefit right away. Effects that capture $\alpha > 0$ can be achieved, for instance, through expectations, as discussed in **Chapter** 4, with individuals inside the contract estimating an increased benefit; or can be thought as a political reinforcing, strengthening of relationships between signatories. Naturally, interplay between these two decisions, cooperating and signing, is a non-linear one (see conditions **G/H/I/J** and **K/L/M** and **Figure** 23).

Overall, one can notice that the pairwise conditions show how, when the game is not favorable outside the contract, the contract can serve its purpose and pave way for a favorable setting. In fact, the contract not only establishes a coordination of signing, typical of contracts, but it imposes both an effective reduction of costs of cooperation together with an increased benefit that can counterbalance the effect of a reduced risk (with an effective risk of $r(1 + \alpha)$).

Excludable benefits can sustain incomplete agreements and are central to both hegemonic stability theory and the dominant actor models of cooperation.

With that in mind, we can pose the question of whether signing and contributing can or cannot be a robust strategy. Furthermore, we would like to understand what kind of dynamics allows for this. **Figure** 23 shows the dynamics of the population on the faces of the simplex, where each trio of strategies is represented under conditions of stable cooperation (conditions **A**, **E** and **G/H** in Figure 22) with high risk ($r = 1$) and without exogenous factors ($\mu = 0$). Under these conditions, we can identify two distinct scenarios: one in which cooperation both inside and outside of the contract is somewhat stable, coexisting with defection, and the other where cooperation is stable only inside the contract.

When the game is less stringent, whether for higher risk or smaller cost to contribution, there are two coordination problems, those of cooperation and of signing, leading to three different stable configurations of the population (right panel). These configurations consist of full defection ($\tilde{D}$), a mix of cooperation and defection in the absence of contracts ($\tilde{D}$ and $\tilde{C}$) or a mix of cooperation and defection within the contract (C and D). As cooperation becomes less favorable (left panel), cooperation will tend to only happen within the contract (C): full cooperation outside the contract ($\tilde{C}$) starts by leading to defection, but with some players joining the club as defectors (D) before either becoming cooperators within the club (C) or before they leave ($\tilde{D}$). On the other hand, signing the contract for those who do not contribute ($\tilde{D}$) becomes a coordination problem: a certain amount of contributors and signatories is needed for them to join the contract, creating a bistability on the population. Depending on the marginal advantages of the contract, there can be a tendency for the absence of non-signatories but, generally, a mixed scenario of non-signatories defectors ($\tilde{D}$) and signatories (C and D) arises, in one hand, due to the slow dynamics on the face comprising those strategies and, on the other, due to exogenous factors ($\mu > 0$) as random exploration pushes the dynamics to the inside of the simplex, as we saw in **Chapter** 6.

Altogether, this means that cooperation within the contract can then emerge and be stable though never global, with defectors outside the contract reaping the benefits of the spillovers produced. We are now ready to show how the previous analyzes hold in the general setting of the four strategies interplaying.

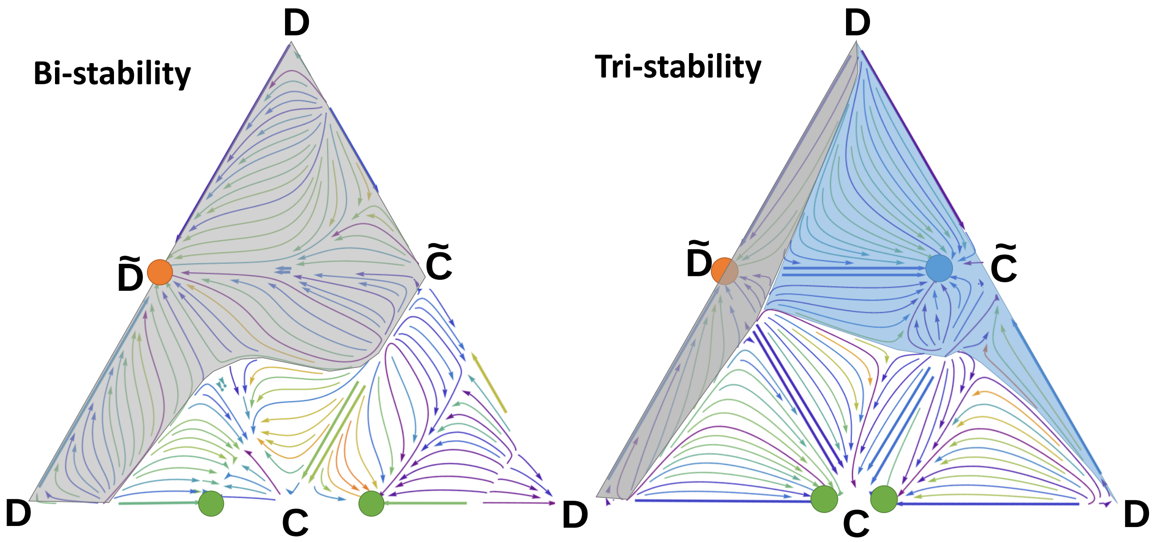
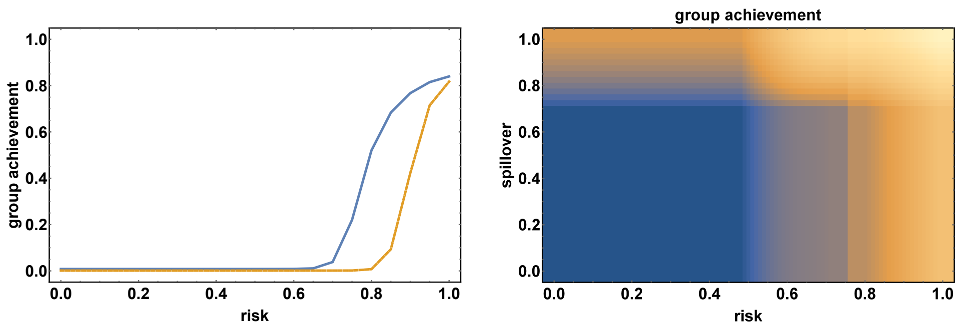


Figure 23: **Tristability and Bistability of the population represented on the faces of the tetrahedron** The arrows represent the most likely direction of evolution of the population composed of a finite number of individuals, the gradient of selection. $Z = 50, n = 5, \mu = 0, M_c = 3, c_c = 1, \beta = 1$, $M_s = 3, p = 0.5$, $r = 1$, $\epsilon = \delta = 0$, $c_s = b/10$. Left: $b = 10$, $\alpha = 0.25$. Right: $b = 5, \alpha = 0.3$. Conditions **A**, **E** and **G** in **Figure** 22 are fulfilled.

10.3 THE PERKS OF AMBIGUOUS STRATEGIES

In the previous section we saw that, apart from all $\tilde{D}$, the two remaining stable fixed points occur in the edges containing only strategies that sign or do not sign the contract. Even though we can weigh the probability that the players are inside and outside the contract, this, in one hand, does not allow us to immediately grasp the general difference from a scenario without contracts or, more precisely, where there is no option of entering or leaving a contract (like in **Chapter** 7). On the other hand, the ability to choose to ratify a contract has been shown to promote cooperation (see [8] and for a review see [23]). One of the reasons pointed out for this effect is related with the ability of the players to perceive the choice as a signal for the players' preferences and their intended behavior.

In order to understand the importance of the contract, we need to take into account the complete dynamics of the system, allowing for all the strategies to be present in the population. We then compare the achievement of the two scenarios: *i*) ability to contribute outside the contract or defect inside it and *ii*) contributing implies signing. For this, we use the average group achievement of the full dynamics (the complete tetrahedron) and that of the dynamics that happens only between Cs and D̃s. **Figure** 24 depicts that unclear strategies, where players can cooperate outside the contract or can defect even though paying to guarantee cooperation, allow for the promotion of cooperation when compared to a scenario in which players either cooperate inside the contract of defect outside.



(a) (b)

Figure 24: **Average group achievement in the complete dynamics** (a) Group achievement as a function of risk (blue is full dynamics, orange is dynamics with only Cs and D̃s). (b) Group achievement dependence of spillovers (the color gradient grows from blue (0.0) to light orange(1.0)). $Z = 45$, $N = 5$, $\mu = 2/15$, $M_c = 3$, $c_c = 2$, $\beta = 10$, $b = 10$, $c_s = 0.5$, $M_s = 4$, $p = 0.5$, $\alpha = 0.1$, $\epsilon = \delta = 0$.

10.4 GROWTH OF A STRUCTURE

The initiation of cooperation commonly involves a grouping of cooperators, in a process akin to the formation of coalitions. In this model this is reflected by the coordination barrier illustrated in **Figure** 23 and that is also present in the dynamics between D̃s and Cs – it is only when the coalition grows to a certain point that others will systematically join, up to a certain point. Thus, in a scenario where cooperation can strive, we expect to see cooperation burst after a period of where a core of members is being created.

Two possible scenarios are depicted in **Figure** 25, where we can distinguish between trajectories that overcome the coordination barrier by the end of the simulation time, the blue shade, and those that did not, the orange shade. The time it takes the groups to form a critical mass of signatories can be very large and its distribution is broad. In order to characterize the time series growth rate, we analyze the time of passage at half way

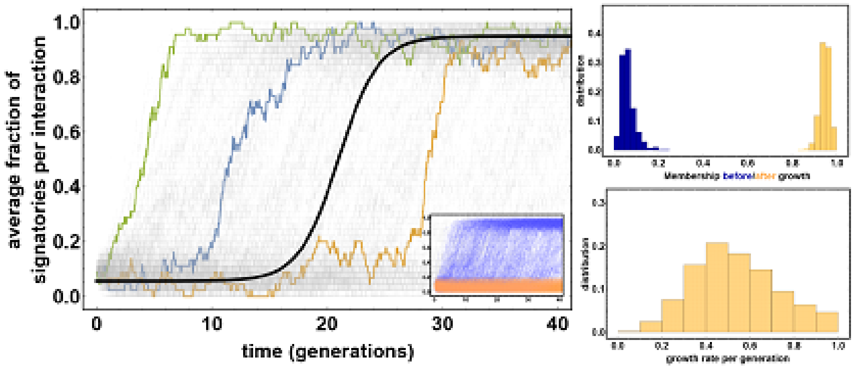


Figure 25: **Temporal dynamics of membership** The membership dynamics contains a coordination point, resulting in two types of trajectories: those in which no players adhere to the agreement and those in which most of the groups are constituted by signatories (inset). In the left panel three different trajectories are highlighted to show the triggering effect characteristic of a coordination threshold. The solid black line represents a sigmoidal function, $\frac{max-min}{1+e^{-\rho(t-m)}} + min$, with the average parameters. As one can see from the inset, the time it takes to reach the coordination threshold, m, has a very broad distribution, going far beyond the generations considered. On the right panels we see the distribution of the different parameters: (top) the membership before growth (min) has a very narrow distribution close to zero and the membership after growth (max) settles at high levels of adoption possibly reaching 100%; (bottom) the rate of adoption (ρ) is very high as soon as the threshold is reached with an average growth of 50% per generation. $Z = 45,\ N = 5,\ \mu = 2/Z, M_c = 3, c_c = 1, \beta = 1, b = 10, c_s = 1, M_s = 3, p = 0.5, \alpha = 0.25,\ r = 1, i_C(0) = i_D(0) = i_{\tilde{C}}(0) = 2, \epsilon = \delta = 0$

and allow for an exponential growth to saturation, obtaining a distribution of growth rates with an average close to 50% of adoption per generation, which consists in a very fast adoption of the agreement once the coordination threshold is met.

10.5 DISCUSSION

The demand for cooperative behavior in the international system is high, though designing effective institutions to enable and sustain cooperation is complex. The problem structure of climate change makes it particularly challenging, given demands for sharp reductions in emissions that will be costly, and incentives by individual countries to free-ride on the mitigation actions of others. The diversity inherent in climate change politics complicates efforts to achieve a deep centralized agreement. Despite these challenges, history offers a range of examples where clubs initially composed of a few countries did succeed in cooperating to achieve shared goals despite free-riding risks, but initially motivated by partially excludable club goods[46]. This model supports the hypothesis of the contract as a coordination problem in which a certain amount of signatories is required to stabilize it. Indeed, some excludability in the good produced, i. e., a portion of the good produced that is exclusive to the signatories ($\alpha > 0$) partially solves the second order free riding problem. This can be thought as political reinforcing, strengthening of relationships between signators, or increased payoff expectation.

While simplifying to homogenous actors with simple preferences and representing an ideal type of agreement formation (bottom-up club growth vis-à-vis top-down inclusive construction, as we saw in last chapter), the model lends more optimism to the potential for small agreements with managed growth to achieve substantial provision of both excludable (domestic) benefits as well as non-excludable public goods. In fact, it shows that cooperation inside the contract is easier with the enhancement it provides reducing the effective risk and punishment reducing the effective cost. Nonetheless, even though signing and contributing can be a robust strategy, incomplete cooperation within the contract is a stable state, making cooperation, even if stable, never global, with defectors in or outside the contract reaping the benefits of the spillovers produced by the contract.

The analysis shown, in which agreements start small before growing, is conceptually consistent with the stated goals of the building blocks approach, the dynamics being a mix between coordination and coexistence. For this, the group size tends to increase as soon as a significant amount of signatories is reached. This recalls the importance of contracts which can strengthen relations between the actors and allow for the formation of coalitions that can then overcome the coordination point. Firstly, though domestic interests that motivate coalitions may be limited in the long term, they nonetheless may catalyze sufficient participation to surpass coordination points that attract outsiders to reap the rewards of fuller cooperation. Secondly, while keeping agreements small may lead to redundancy, it is also likely that overlapping institutions in the sequential construction of an international agreement allows actors to build trust and networks, and exploit synergies that can overcome the eventual challenge of aggregation into a single universal agreement.

Overall, this supports the idea of club formation and of polycentricity: voluntary clubs are a means of creating a signaling mechanism for future cooperation of its members. As long as the club has the right structure, being small and creating enough excludable benefits to its members, they will remain cooperative and they will sprawl. The important work for policymakers will be to find issue areas where opportunities for new building blocks exist.

In light of all our results, the widely repeated motto "Think globally, act locally" would hardly seem more appropriate.

Part IV

CONCLUSIONS

11 CONCLUSIONS

In this dissertation I started by laying down the framework of Evolutionary Game Theory on finite well-mixed populations. I use it to developed intuition and to serve as base to build new tools.

First, I elaborate on a framework to study discounting and forecasting which, so far, only served as a means to assess the different assumptions that are usually implicit in the equations. Indeed, I have shown that the traditional framwork is exactly recovered only when either discounting is very high or when forecasting is invariant to the present. I have also shown that the simplest form of rationality maps exactly into the typical evolutionary player when groups are smaller than the population.

Then, I proposed an equation to describe the evolution of continuous traits in the presence of imitation errors and mutations and used it to show that if selection is high, quantitatively specifying high, the evolutionary populations with continuous traits tend to collapse in specific values of those traits.

Finally, I expose a framework particularly useful to access the invariant distribution when the number of strategies is high. It consists of a hierarchy of approximations that exhibit an intuitive topological appeal. Topological criteria is employed in choosing "Configurations of Interest", providing a description that allows one to explicitly include the role of mutations while dramatically reducing the complexity of the multi-dimensional Markov processes.

This part consisted on my contribution to the field in a more pure form, laying down the tools, thoughts and reasoning developed to study discounting, rationality, continuous strategies and exploration of large phase-spaces, in the familiar framework of EGT with the objective of creating a reference frame and to open up new research questions or to revisit old ones.

In the final part of this dissertation, I use the mathematical framework of Evolutionary Game theory to study what has been coined as “polycentric governance”[90, 91]. The results show that high levels of cooperation can indeed be reached globally via such a polycentric approach, when a multiplicity of small groups face local dilemmas.[107, 115] However, this mechanism is not bullet proof and suffers from fragilities which are of the same stance as those stemming from the usual top-down approach. I discuss this in the context of Climate Change negotiations and introduce four of the main issues that are pointed as causes to their failure. I show how uncertainty is a key factor to this threshold game, whether in terms of the perception of disaster or in terms of the targets to be met. Both low levels of risk perception or high threshold uncertainty induce a critical transition into a state where everyone involved ceases to cooperate. Additionally, I consider the possibility that individuals have different abilities to contribute, taking into account wealth inequality. I show how homophily in terms of the network of influence of the Rich and the Poor leads to an added effort from the Rich and, is some cases, to a collapse of cooperation. In any of these cases, all of which are detrimental to cooperation, the co-operative state is more robust the smaller the groups, an observation that strongly supports the polycentric approach. Finally, I study the possibility of supervision of these dilemmas via institutions and contracts and, again, pose the question of at which scale should it be implemented. First, comparing, in the setup of EGT, the efficiency of a global institution overviewing all interactions versus that of locally arranged ones, then analyzing the dynamics of voluntary agreement. The results show that polycentric sanctioning is more efficient in monitoring the multiple interactions. Then, I studied the creation of climate contracts and show that the fact that entering the agreement is voluntary tends to increase overall cooperation, especially in smaller groups, where those remain cooperative and can, then, proliferate.

In any case, the chance of failing to solve the climate change problem is still very high, specially taking into account that the general dynamics is a mix between coordination and coexistence. This also reinforces the importance of contracts which can strengthen relations between the actors and allow for the formation of coalitions that allow overcoming the coordination point.

Furthermore, cooperation can be slightly amended whenever demanding thresholds are adopted.[107, 115] Moreover, if intermediate tasks are designated,[77] or if individuals have the opportunity to pledge their contribution before actual action,[134] cooperation is also more prominent.

Nonetheless, some of these results need to be tested in the field. This should happen not only in terms of their validity but also in order to pursue the identification of the critical parameters that define the regime we are in. How close are we to the transition in terms of risk perception in a specific issue? How much can we reduce threshold uncertainty? Up to which scale can we implement local institutions and keep the cost:size proportionality? Answering these practical questions is what, ultimately, will render feasible the empirical implementation of these results.

Part V

APPENDIX

Collective writing produces much more mature texts. Because I believe this thesis should also reflect the stage of my writing skills (or lack of them), I chose – while admittedly reaping some of what my coauthors and I have written – to put the detailed and matured documents in this final part.

a

EVOLUTIONARY DYNAMICS OF CLIMATE CHANGE UNDER COLLECTIVE-RISK DILEMMAS

Below follows the print of the document "Evolutionary dynamics of climate change under collective-risk dilemmas" by Francisco C. Santos, Vítor V. Vasconcelos, Marta D. Santos, P. N. B. Neves and Jorge M. Pacheco, published in *Mathematical Models and Methods in Applied Sciences* in 2012, in volume 22 1140004.

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EVOLUTIONARY DYNAMICS OF CLIMATE CHANGE UNDER COLLECTIVE-RISK DILEMMAS

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Preventing global warming requires overall cooperation. Contributions will depend on the risk of future losses, which plays a key role in decision-making. Here, we discuss an evolutionary game theoretical model in which decisions within small groups under high risk and stringent requirements toward success significantly raise the chances of coordinating to save the planet's climate, thus escaping the tragedy of the commons. We discuss both deterministic dynamics in infinite populations, and stochastic dynamics in finite populations.

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1. Introduction

In a dance that repeats itself cyclically, countries and citizens raise significant expectations every time a new International Environmental Summit is settled. Unfortunately, few solutions have come out of these colossal and flashy meetings, challenging our current understanding and models on decision-making, so that more effective levels of discussion, agreements and coordination become accessible. From Montreal and Kyoto to Copenhagen summits, it is by now clear how difficult it is to coordinate efforts.[1,2] Often, individuals, regions or nations opt to be *free riders*, hoping to benefit from the efforts of others while choosing not to make any effort themselves. Cooperation problems faced by humans often share this setting, in which the immediate advantage of free riding drives the population into the tragedy of the commons,[3] the ultimate limit of widespread defection.[3–12]

To address this and other cooperation conundrums, ubiquitous at all scales and levels of complexity, the last decade has witnessed the discovery of several core mechanisms responsible to promote and maintain cooperation at different levels of organization.[3,5,10,13–26] Most of these key principles have been studied within the framework of two-person dilemmas such as the Prisoner's dilemma, which constitutes a powerful metaphor to describe conflicting situations often encountered in the natural and social sciences. Many real-life situations, however, are associated with collective action based on joint decisions made by a group often involving more than two individuals.[3,5,13,27] These types of problems are best dealt with in the framework of N-person dilemmas and Public Goods games, involving a much larger complexity that only recently started to be unveiled.[5,14,22,28–33] The welfare of our planet accounts for possibly the most important and paradigmatic example of a public good: a global good from which everyone profits, whether or not they contribute to maintain it.

One of the most distinctive features of this complex problem, only recently tested and confirmed by means of actual experiments,[9] is the role played by the perception of risk that accrues to all actors involved when making a decision. Indeed, experiments confirm the intuition that the risk of collective failure plays central role in dealing with climate change. Up to now, the role of risk has remained elusive.[1,2,11] In addition, it is also unclear what is the ideal scale or size of the population engaging in climate summits — whether game participants are world citizens, regions or country leaders, such that the chances of cooperation are maximized. Here we address these two issues in the context of game theory and population dynamics.

The conventional Public Goods game — the so-called N-person Prisoner's dilemma — involve a group of N individuals, who can be either Cooperators (C) or Defectors (D). Cs contribute a cost "c" to the public good, whereas Ds refuse to do so. The accumulated contribution is multiplied by an enhancement factor that returns equally shared among all individuals of the group. This implies a collective return which increases linearly with the number of contributors, a situation that contrasts with many real situations in which performing a given task requires the cooperation of a minimum number of individuals of that group.[28–30,33–38] This is the case in international environmental agreements which demand a minimum number of ratifications to come into practice,[1,2,9,39–42] but examples abound where a minimum number of individuals, which does not necessarily equal the entire group, must simultaneously cooperate before any outcome (or public good) is produced.[28,29] Furthermore, it is by now clear that the N-person Prisoner's dilemma fails short to encompass the role of risk, as much as the nonlinearity of most collective action problems.

Here we address these problems resorting to a simple mathematical model, adopting unusual concepts within political and sustainability science research, such as peer-influence and evolutionary game theory.[14,43,44] As a result we encompass several of the key elements stated before regarding the climate change conundrum in a single dynamical model.

In the following we show how small groups under high risk and stringent requirements toward collective success significantly raise the chances of coordinating to save the planet's climate, thus escaping the tragedy of the commons. In other words, global cooperation depends on how aware individuals are concerning the risks of collective failure and on the pre-defined premises needed to accomplish a climate agreement. Moreover, we will show that to achieve stable levels of cooperation, an initial critical mass of cooperators is needed, which will then be seen as role models and foster cooperation.

We will start by presenting the model in Sec. 2. In Sec. 3, we discuss the situation in which evolution is deterministic and proceeds in very large populations. In Sec. 4 we analyze the evolutionary dynamics of the same dilemma in finite populations under errors and behavioral mutations. Finally, in Sec. 5 we provide a summary and concluding remarks.

2. Model

Let us consider a large population of size Z, in which individuals engage in an N-person dilemma, where each individual is able to contribute or not to a common good, i.e. to cooperate or to defect, respectively. Game participants have each an initial endowment b. Cooperators (Cs) contribute a fraction c of their endowment, while defectors (Ds) do not contribute. As previously stated, irrespectively of the scale at which agreements are tried, most demand a minimum number of contributors to come into practice. Hence, whenever parties fail to achieve a previously defined minimum of contributions, they may fail to achieve the goals of such agreement (which can also be understood as the benefit "b"), being this outcome, in the worst possible case, associated with an appalling doomsday scenario. To encompass this feature in the model we require a minimum collective investment to ensure success: If the group of size N does not contain at least M Cs (or, equivalently, a collective effort of Mcb), all members will lose their remaining endowments with a probability r (the *risk*); otherwise everyone will keep whatever they have. Hence,

$M < N$ represents a coordination threshold,[9,28] necessary to achieve a collective benefit. As a result, the average payoff of a D in a group of size N and kCs can be written as

$$\Pi_D(k) = b\{\theta(k-M) + (1-r)[1-\theta(k-M)]\}, \tag{1}$$

where $\theta(x)$ is the Heaviside step function ($\theta(x < 0) = 0$ and $\theta(x \geq 0) = 1$). Similarly, the average payoff of a $\boldsymbol{C}$ is given by

$$\Pi_C(k) = \Pi_D(k) - cb. \tag{2}$$

The risk r is here introduced as a probability, such that with probability $(1-r)$ the benefit will be collected independent of the number of contributors in a group.

This collective-risk dilemma represents a simplified version of the game used in the experiments performed by Milinski *et al.*[9] on the issue of the mitigation of the effects of climate change, a framework which is by no means the standard approach to deal with International Environmental Agreements and other problems of the same kind.[1,2,39,40] The present formalism has the virtue of depicting black on white the importance of risk and its assessment in dealing with climate change, something that Heal *et al.*[41,45] have been conjecturing for quite awhile. At the same time, contrary to the experiments in Ref. 9, our analysis is general and not restricted to a given group size.

Additionally, and unlike most treatments,[1] our analysis will not rely on individual or collective rationality. Instead, our model relies on evolutionary game theory combined with one-shot Public Goods games, in which errors are allowed. In fact, our model includes what we believe are key factors in any real setting, such as bounded rational individual behavior, peer-influence and the importance of risk assessment in meeting the goals defined from the outset.

We assume that individuals tend to copy others whenever these appear to be more successful. Contrary to strategies defined by a contingency plan which, as argued before,[46] are unlikely to be maintained for a long time scale, this social learning (or evolutionary) approach allows policies to change as time goes by,[22,47,48] and likely these policies will be influenced by the behavior (and achievements) of others, as previously shown in the context of donations to public goods.[44,49,50] This also takes into account the fact that agreements may be vulnerable to renegotiation, as individuals may agree on intermediate goals or assess actual and future consequences of their choices to revise their position.[1,2,7,39,40,45]

3. Evolution of Collective Action in Large Populations

In the framework of evolutionary game theory, the evolution or social learning dynamics of the fraction x of Cs (and $1-x$ of Ds) in a large population ($Z \to \infty$) is governed by the gradient of selection associated with the replicator dynamics equation[14,28,51]

$$g(x) \equiv \dot{x} = x(1-x)(f_C(x) - f_D(x)), \tag{3}$$

which characterizes the behavioral dynamics of the population, where $f_C(f_D)$ is the fitness of Cs (Ds), here associated with the game payoffs. According to the replicator equation, Cs (Ds) will increase in the population whenever $g(x) > 0$ ($g(x) < 0$). If one assumes an unstructured population, where every individual can potentially interact with everyone else, the fitness (or social success) of each individual can be obtained from a random sampling of groups. The latter leads to groups whose composition follows a binomial distribution. Hence, we may write the fitness of Cs, f_c, and Ds, f_D, as[28–30]

$$f_C(x) = \sum_{k=0}^{N-1} \binom{N-1}{k} x^k (1-x)^{N-1-k} \Pi_C(k+1) \tag{4a}$$

and

$$f_D(x) = \sum_{k=0}^{N-1} \binom{N-1}{k} x^k (1-x)^{N-1-k} \Pi_D(k), \tag{4b}$$

where $\Pi_C(k)(\Pi_D(k))$ stands for the payoff of a $C(D)$ in a group of size N and kCs, as defined above in Eqs. (1) and (2).

Figure 1 shows that, in the absence of risk, $g(x)$ is always negative. Risk, in turn, leads to the emergence of two mixed internal *equilibria*, rendering cooperation viable: for finite risk r, both Cs (for $x < x_L$) and Ds (for $x > x_R$) become disadvantageous when rare. Co-existence between Cs and Ds becomes stable at a fraction x_R which increases with r. Collective coordination becomes easier to achieve under high-risk and, once the coordination barrier (x_L) is overcome, high levels of cooperation will be reached.

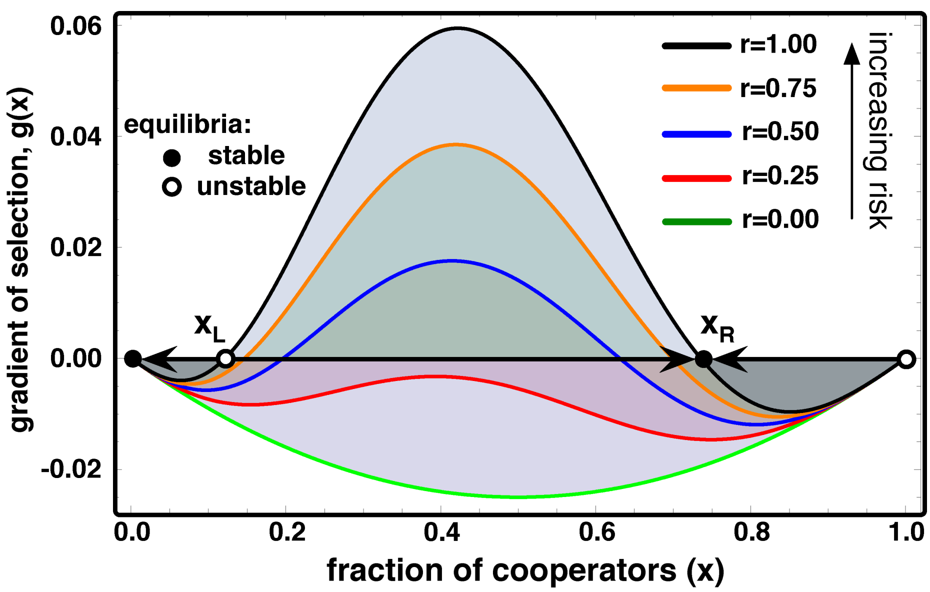


Fig. 1. For each fraction of Cs, if the gradient $g(x)$ is positive (negative) the fraction of Cs will increase (decrease). Increasing risk (r) modifies the population dynamics rendering cooperation viable depending on the initial fraction of Cs ($N = 6$, $M = 3$ and $c = 0.1$). The five curves correspond, from top to bottom to, $r = 1$, 0.75, 0.5, 0.25, 0.

The appearance of two internal *equilibria* under risk can be studied analytically, as the roots of the fitness difference $Q(x) \equiv f_C(x) - f_D(x)$ determines the occurrence of nontrivial *equilibria* of the replicator dynamics. From the equations above we may write, after some algebra, that

$$Q(x) = b\left[\binom{N-1}{M-1} x^{M-1}(1-x)^{N-M} r - c\right]. \tag{5}$$

Defining the cost-to-risk ratio $\gamma = c/r$, i.e. the ratio between the fraction of the initial budget invested by every C and the risk of losing it, the sign of $Q(x)$ is conveniently analyzed by using the polynomial

$$p(x) = \binom{N-1}{M-1} x^{M-1}(1-x)^{N-M} - \gamma, \tag{6}$$

which, in turn, can be used to determine the critical value $\bar{\gamma}$ below which an interior fixed point $x^* \in (0,1)$ emerges. Indeed, we can prove the following theorem.

Theorem 1. *Let* $\Gamma(x) = \binom{N-1}{M-1} x^{M-1}(1-x)^{N-M}$. *For* $1 < M < N$, *there exists a critical cost-to-risk ratio* $\bar{\gamma} = \Gamma(\bar{x}) > 0$ *and fraction of Cs* $0 < \bar{x} < 1$ *such that*:

(a) *If* $\gamma > \bar{\gamma}$, *the evolutionary dynamics has no interior equilibria.*
(b) *If* $\gamma = \bar{\gamma}$, *then* $\bar{x}$ *is a unique interior equilibrium, as this equilibrium is unstable.*
(c) *If* $\gamma < \bar{\gamma}$, *there are two interior equilibria* $\{x_L, x_R\}$, *such that* $x_L < \bar{x} < x_R, x_L$, *is unstable and* x_R *stable.*

Proof. Let us start by noticing that

$$\frac{d\Gamma(x)}{dx} = -\binom{N-1}{M-1} x^{M-2}(1-x)^{N-M-1} s(x),$$

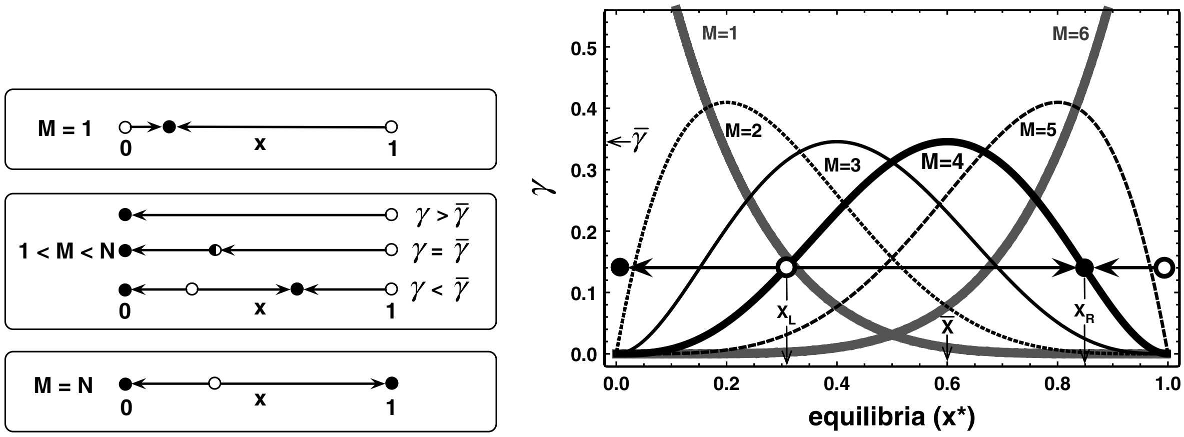
where $s(x) = 1 + (N-1)x - M$. Since $N > 2$ and $1 < M < N$, then $d\Gamma(x)/dx$ has a single internal root for $\bar{x} = (M-1)/(N-1)$. In addition, $s(x)$ is negative (positive) for $x < \bar{x}$ ($x > \bar{x}$), which means that Γ has a global maximum for $x = \bar{x}$.

(a) and (b) can now easily follow. Since Γ has a maximum at $\bar{x}$, it follows that $\Gamma(x) = 0$ has no solutions for $\gamma > \bar{\gamma}$ and a single one, at $\bar{x}$, for $\gamma = \bar{\gamma}$. Moreover, both when $x \to 0$ and $x \to 1$, $p(x) < 0$, making $x = 0$ a stable fixed point and $x = 1$ an unstable one. Therefore, if $\bar{x}$ is a root, it must be unstable.

To prove (c), we start by noticing that $\Gamma(0) = \Gamma(1) = 0$. From the sign of $s(x)$ (see above), $\Gamma(x)$ is clearly monotonic increasing (decreasing) to the left (right) of $\bar{x}$. Hence, there is a single root x_L (x_R) in the interval $0 < x < \bar{x}$ ($\bar{x} < x < 1$). Since $x = 0$ is stable and $x = 1$ unstable, x_R must be stable and x_L unstable. □

Theorem 2. *For* $M = 1$, *if* $\gamma < \bar{\gamma}$, *there is one stable interior equilibrium point in the interval* $0 < x < 1$.

Proof. If $M = 1$, $\Gamma(x) = (1-x)^{N-1}$, which is a monotonic decreasing function for $0 < x < 1$. This means that the function $p(x)$ has only one zero in that interval,



(a) (b)

Fig. 2. (a) Classification of all possible dynamical scenarios when evolving an infinitely large population of Cs and Ds as a function of γ, M and N. A fraction x of an infinitely large population adopts the strategy C; the remaining fraction $1 - x$ adopts D. The replicator equation describes the evolution of x over time. Solid (open) circles represent stable (unstable) *equilibria* of the evolutionary dynamics; arrows indicate the direction of selection. (b) Internal roots x^* of $g(x)$ for different values of the cost-to-risk ratio $\gamma = c/r$, at fixed group size ($N = 6$) and different coordination thresholds (M). For each value of γ one draws a horizontal line; the intersection of this line with each curve gives the value(s) of x^*, defining the internal *equilibria* of the replicator dynamics. The empty circle represents an unstable fixed point (x_L) and the full circle a stable fixed point (x_R) ($M = 4$ and $\gamma = 0.15$ in example).

i.e. there is only one $\bar{x}$ $(0 < \bar{x} < 1)$ such that $p(\bar{x}) = 0$. Given that $p(x)$ is positive (negative) for $x < \bar{x}\,(x > \bar{x})$ then $\bar{x}$ is a stable equilibrium point. □

Theorem 3. *For* $M = N$, *if* $\gamma < \bar{\gamma}$, *there is one unstable interior equilibrium point in the interval* $0 < x < 1$.

Proof. If $M = N$, $\Gamma(x) = x^{N-1}$, which is a monotonic increasing function for $0 < x < 1$. This means that the function $p(x)$ has only one zero in that interval, i.e. there is only one $\bar{x}$ $(0 < \bar{x} < 1)$ such that $p(\bar{x}) = 0$. Given that $p(x)$ is negative (positive) for $x < \bar{x}\,(x > \bar{x})$ then $\bar{x}$ is an unstable equilibrium point. □

In Fig. 2(a), we provide a concise scheme of all possible dynamical scenarios that emerge from collective-risk dilemmas, showing how the coordination threshold and the level of risk play a central role in dictating the viability of cooperation. Figure 2(b) also shows the role played by the threshold M: for fixed (and low) γ, increasing M will maximize cooperation (increase of x_R) at the expense of making it more difficult to emerge (increase of x_L).

4. Evolution of Collective Action in Small Populations

Real populations are finite and often rather small, contrary to the hypothesis underlying the dynamics portrayed in Sec. 3. In particular, this is the case of the famous world summits where group and population sizes are comparable and of the order hundreds, as individuals are here associated with nations or their respective leaders. For such population sizes, stochastic effects play an important role and the deterministic description of the previous section may be too simplistic.[52]

For finite, well-mixed populations of size Z, the binomial sampling in Eqs. (4) is replaced by a hypergeometric sampling (sampling without replacement). As a result, the average fitness of Ds and Cs in a population with kCs, is now written as

$$f_D(k) = \binom{Z-1}{N-1}^{-1} \sum_{j=0}^{N-1} \binom{K}{j} \binom{Z-k-1}{N-j-1} \Pi_D(j) \tag{7}$$

and

$$f_C(k) = \binom{Z-1}{N-1}^{-1} \sum_{j=0}^{N-1} \binom{k-1}{j} \binom{Z-k}{N-j-1} \Pi_C(j+1), \tag{8}$$

respectively. We adopt a stochastic birth–death process[53] combined with the pairwise comparison rule[54] in order to describe the social dynamics of Cs (and Ds) in a finite population. Under pairwise comparison, each individual i adopts the strategy of a randomly selected member of the population j with probability given by the Fermi function (from statistical physics)

$$p_{ij} = \frac{1}{1 + e^{-\beta(f_j - f_i)}}. \tag{9}$$

Here β controls the intensity of selection. For $\beta \ll 1$, selection is weak and individual fitness is but a small perturbation to random drift in behavioral space. Under this regime one recovers the replicator equation in the limit $Z \to \infty$.[54] For arbitrary β, the quantity $g(x)$ of Eq. (3), specifying the gradient of selection, is replaced in finite populations by[54]

$$G(k) \equiv T^+(k) - T^-(k) = \frac{k}{Z}\frac{Z-k}{Z} \tan h \left\{ \frac{\beta}{2}[f_C(k) - f_D(k)] \right\}, \tag{10}$$

where k stands for the total number of Cs in the population and

$$T^{+}(k) = \frac{k}{Z}\frac{Z-k}{Z}[1 + e^{\mp\beta[f_C(k) - f_D(k)]}]^{-1} \tag{11}$$

for the probabilities to increase and decrease the number of Cs in the population.

4.1. *Fixation probabilities*

The fact that, in finite populations, the continuous gradient of selection $g(x)$ is replaced by a discrete $G(k/Z)$ has implications in the overall evolutionary dynamics of the population. Importantly, in the absence of mutations evolutionary dynamics in finite populations will only stop whenever the population reaches a monomorphic state.[52,54] Hence, in addition to the analysis of the shape of $G(k/Z)$, often one of the quantities of interest in studying the evolutionary dynamics in finite populations is the probability ϕ_k that the system fixates in a monomorphic cooperative state, starting from, for instance, a given number k of Cs. The fixation probability of kCs (ϕ_k) depends on the ratio $\lambda_j = T^-(j)/T^+(j)$, being given by[53]

$$\phi_k = \sum_{i=0}^{k=1} \prod_{j=1}^{i} \lambda_j \bigg/ \sum_{i=0}^{Z-1} \prod_{j=1}^{i} \lambda_j. \tag{12}$$

Under neutral selection (that is, in the limit $\beta \to 0$) the fixation probability trivially reads $\phi_k^N = k/Z$, providing a convenient reference point.[17,53–55] For a given k, whenever $\phi_k > \phi_k^N$, natural selection will favor cooperative behavior, the opposite being true when $\phi_k < \phi_k^N$.

In Fig. 3 we plot the fixation probability as a function of the initial fraction of Cs for different values of risk, and a population of 50 individuals. Even if cooperators remain disadvantageous for a wide range of the discrete frequency of Cs (see Fig. 1), the fixation probability of kCs outperforms ϕ_k (picture as a dashed grey line) for most values of k/Z. This is due to the stochastic nature of the imitation

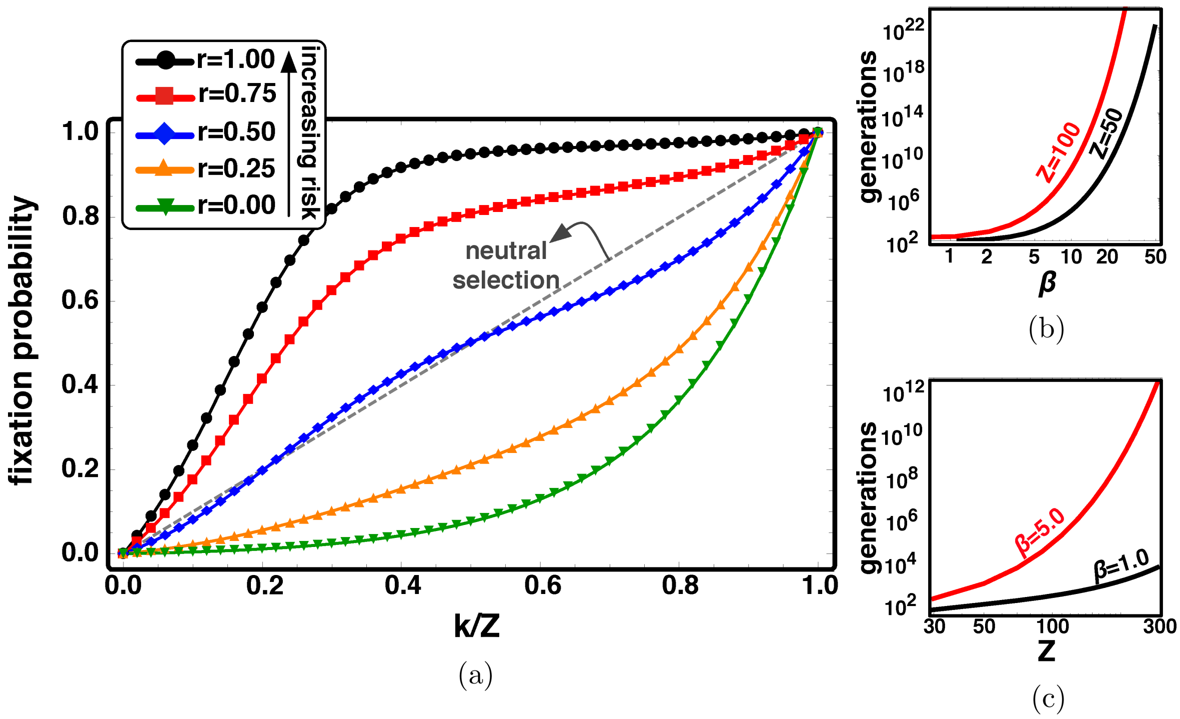


Fig. 3. Evolutionary dynamics for different values of risk in finite populations. In panel (a), we show the fixation probabilities for different values of risk (r) as a function of the number of Cs ($Z = 50$, $c = 0.1$, $N = 6 = 2\,M$, $\beta = 1.0$). In panels (b) and (c), we show the average number of generations (t_j/Z)[57,58] needed to fixate an initial fraction of 0.5 of cooperators, as a function of the intensity of selection β (panel (b)) and population size Z (panel (c)). We consider the case of maximum risk ($r = 1$) for both (b) and (c) panels and $c = 0.1$, $N = 6 = 2\,M$. Even if high risk can turn the fixation of cooperators almost certain (as shown in panel (a)), the time the population takes to reach such state can be arbitrarily long.

process, which allows the fixation of rare cooperators, even when they are initially disadvantageous. Hence, even without random exploration of strategies,[56] simple errors in the imitation process (finite β) are enough to overcome the unstable fixed point shown in Fig. 2 and reach a more cooperative basin of attraction on the right-hand side of the gradient (see Fig. 3). As a result, for high values of risk and large, but finite, populations, cooperation is by far the strategy most favored by evolution irrespectively of the initial fraction of cooperators.

As discussed above, in finite populations the evolutionary dynamics becomes stochastic. Yet, even if fixation in one of the two absorbing states is certain ($k = 0$ and $k = Z$), the time required to reach it can be arbitrarily long. This is particularly relevant in the presence of basins of attraction with polymorphic stable configurations, which correspond to finite population analogues of co-existence equilibria in infinite populations. For high intensities of selection and/or large populations, the time required for fixation (t_j) can increase significantly. Following Antal and Scheuring,[57] the average number of updates t_j the population takes to reach full cooperation, starting from j cooperators, can be written as[57,59]

$$t_j = -t_1 \frac{\phi_1}{\phi_j} \sum_{k=j}^{N-1} \prod_{m=1}^{k} \lambda_m + \sum_{k=j}^{N-1} \sum_{l=1}^{k} \frac{\phi_l}{T^+(j)} \prod_{m=l+1}^{k} \lambda_m, \tag{13a}$$

where

$$t_1 = \sum_{k=j}^{N-1} \prod_{l=1}^{k} \frac{\phi_l}{T^+(l)} \prod_{m=l+1}^{k} \lambda_m. \tag{13b}$$

This is illustrated in Figs. 3(b) and 3(c), where we compute average number of generations (t_j/Z) needed to attain monomorphic cooperative state as a function of the intensity of selection and population size, starting from 50% of Cs and Ds for a dilemma with highest risk ($r = 1$). These panels clearly indicate that even if high risk can turn the fixation of cooperators almost certain (as shown in the left panel), the time the population takes to reach such state can be arbitrarily long. In other words, while the computation of the fixation probabilities can be mathematically attractive, its relevance may be limited for large intensities of selection and/or large Z. In other words, the stochastic information built in ϕ_k shows how unstable roots of G may be irrelevant; however, the lack of time information in ϕ_k ignores the key role played by the stable roots of G.

Moreover, stochastic effects in finite populations can be of different nature, going beyond errors in the imitation process. One can also consider mutations, random exploration of strategies or any other reason that leads individuals to change their behavior, in addition to social learning by imitation dynamics.[56] In the simplest scenario, this creates a modified set of transition probabilities, with an additional random factor encoding the probability of a mutation (μ) in each update step. Under these circumstances, the population will never fixate in none of the two possible *monomorphic* states.

4.2. *Stationary distributions*

As discussed in the previous section, the existence of a stable equilibrium may turn the analysis of the fixation probability misleading. Not only fixation probabilities fail to characterize in a reasonable way the evolutionary dynamics under general conditions, if one considers other forms of stochastic effects as random exploration of strategies, the system will never fixate.

A proper alternative which overcomes the drawbacks identified in both ϕ_k and G consists in the analysis of the stationary distributions of the complete Markov chain $P(k/Z)$ (of size $Z+1$). The probabilities entering the tridiagonal transition matrix $S = [p_{ij}]^T$ are defined as $p_{k,k\pm 1} = T_{\mu}^{\pm}(k)$ and $p_{k,k} = 1 - p_{k,k-1} - p_{k,k+1}$, where $T_{\mu}^{\pm}$ stands for the transition probabilities for an arbitrary mutation rate μ, which are given by $T_{\mu}^{+}(k) = (1-\mu)T^{+}(k) + \mu(Z-k)/Z$ for the probability to increase from k to $k+1$ Cs and $T_{\mu}^{-}(k) = (1-\mu)T^{-}(k) + \mu k/Z$ for the probability to decrease to $k-1$.[56] The stationary distribution is then obtained from the eigenvector corresponding to the eigenvalue 1 of S.[53,60]

In Fig. 4 we show the stationary distributions for different values of risk, for a population of size $Z = 50$ where $N = 2\,\mathrm{M} = 6$. While the finite population gradient of selection $G(k/Z)$ shown in the inset exhibits a behavior qualitatively similar to x in Fig. 1, the stationary distributions show that the population spends most of the time in configurations where Cs prevail, irrespective of the initial condition. This is a direct consequence of stochastic effects, which allow the "tunneling" through the

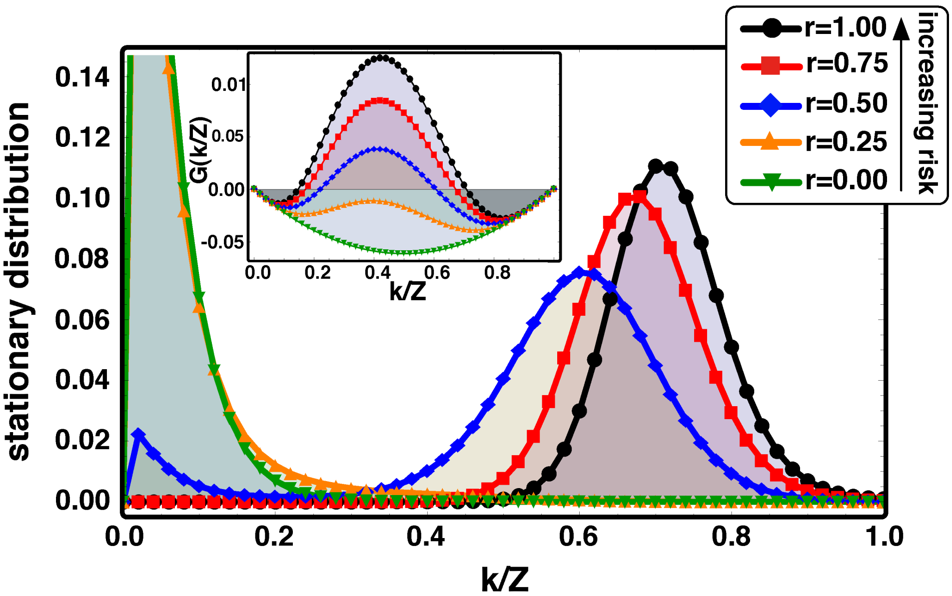
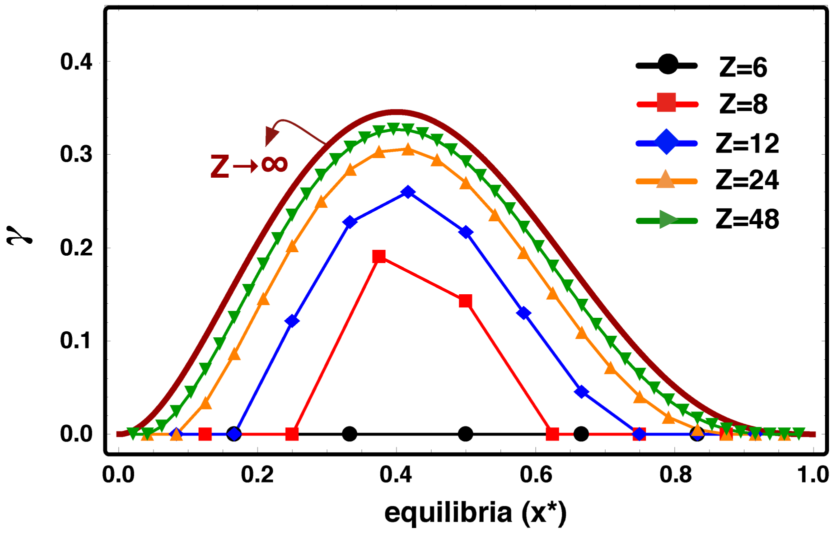
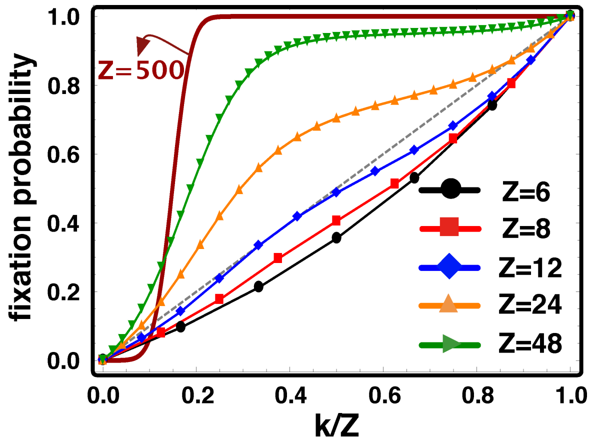


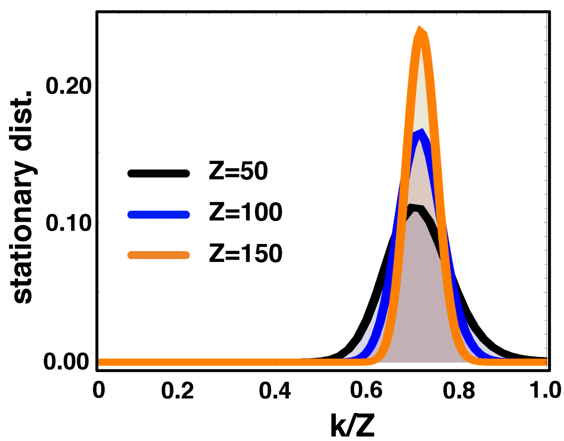
Fig. 4. Prevalence of cooperation in finite populations. The main panel pictures the stationary distribution corresponding to the prevalence of each fraction of Cs that emerges from the discrete gradient of selection G shown in inset. Whenever risk is high, stochastic effects turn collective cooperation into a pervasive behavior, rendering cooperation viable and favoring the overcome of coordination barriers, irrespective of the initial configuration ($Z = 50$, $N = 6$, $M = 3$, $c = 0.1$, $\mu = 0.005$).



(a)



(b)



(c)

Fig. 5. Population size dependence for $N = 6 = 2\,\text{M}$. (a) Roots of the gradient of selection for different values of the cost-to-risk ratio and population sizes. (b) Fixation probabilities for different values of the population size for a fixed cost-to-risk ratio ($\gamma = 0.1$) as a function of the number of Cs ($\beta = 5.0$). (c) We introduce a small mutation ($\mu = 0.005$) to show the stationary distribution for the same game parameters in (b) and different population sizes. As the population size increases, the system spends increasingly less time close to the monomorphic configurations. The three curves correspond, from top to bottom to, $Z = 150$, 100, 50.

coordination barrier associated with x_L, rendering such coordination barrier (x_L) irrelevant and turning cooperation into the prevalent strategy. On the other hand, the existence of a stable fixed root of G is triggered in P with a maximum at this position, unlike what one observes with ϕ_k.

Yet, until now the effect of the population size on the game itself remains uncharted. In Fig. 5(a), we plot the roots of $G(k)$ as a function of the *cost-to-risk ratio* for different values of population size Z. For large Z the general picture described for infinite populations remains qualitatively valid. As before, two interior roots of $G(k)$ characterize the evolutionary dynamics of the population. However,

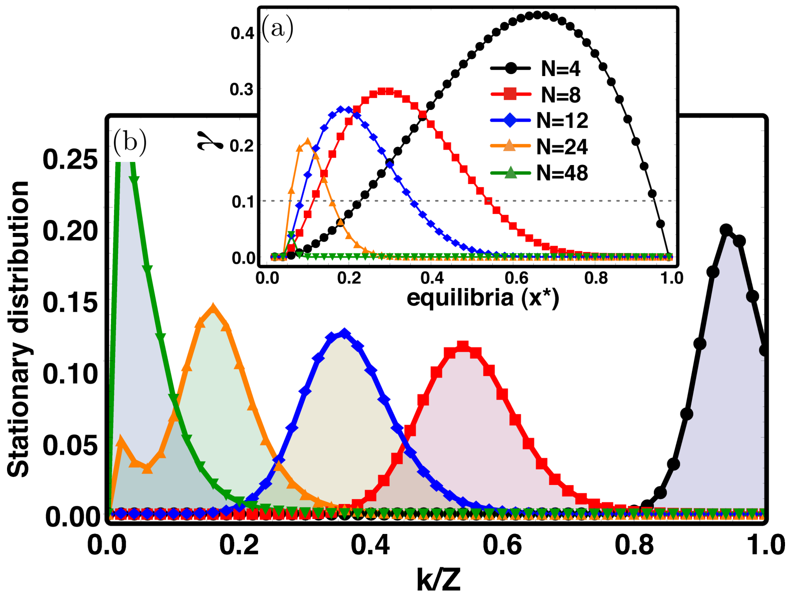


Fig. 6. Group size dependence for $M = 3$. (a) Roots of the gradient of selection for different values of the cost-to-risk ratio and group sizes. (b) Stationary distribution for different group sizes and $c/r = 0.15$. Cooperation will be maximized in small groups, where the risk is high and goal achievement involves stringent requirements.

the position of the interior fixed points can be profoundly altered by the population size. The range of k/Z in which Cs are advantageous is also strongly reduced for small populations. Moreover, while $\bar{x}$ (see Sec. 2) remains almost unchanged as we move from infinite to finite populations, the critical $\bar{\gamma}$ is drastically reduced for small populations that, in turn, reduces the interval of *cost-to-risk ratios* for which a defection dominance dilemma is replaced by a combination of coordination and co-existence dilemmas. In other words, the smaller the population size the higher the perception of risk needed to achieve cooperation. The population size also plays an important role on the shape of the stationary distribution: In Fig. 5(c) we plot the stationary distribution for $r = 1$ and $c = 0.1$, for different population sizes. Whenever the population size increases, a higher number of errors is needed to escape the equilibrium between Cs and Ds, leading the system to spend a higher fraction of time on the internal stable root of $G(k)$.

Naturally, the assessment of the effects of the population size should be carried out in combination with the number of parties involved in collective-risk dilemmas, i.e. the group size. Whether game participants are world citizens, world regions or country leaders, it remains unclear at which scale global warming should be tackled.[40,61] Indeed, besides perception of risk, group size may play a pivotal role when maximizing the likelihood of reaching overall cooperation. As shown by the stationary distributions in Fig. 6, cooperation is better dealt with within small groups, with the proviso that for higher M/N values, coordination is harder to attain, as shown by the position of the roots of G (see inset of Fig. 6).

5. Conclusions

Dealing with environmental sustainability cannot overlook the uncertainty associated with a collective investment. Here we propose a simple form to describe this problem and study its impact in behavioral evolution, obtaining an unambiguous agreement with recent experiments,[9] together with several concrete predictions. We do so in the framework of non-cooperative N-person evolutionary game theory, an unusual mathematical tool within the framework of modeling of political decision-making. We propose a new N-person game where the risk of collective failure is explicitly introduced by means of a simple collective dilemma. Moreover, instead of resorting to complex and rational planning or rules, individuals revise their behavior by peer-influence, creating a complex dynamics akin to many evolutionary systems. This framework allowed us to address the impact of risk in several configurations, from large to small groups, from deterministic towards stochastic behavioral dynamics.

Overall, we have shown how the emerging behavioral dynamics depends heavily on the perception of risk. The impact of risk is enhanced in the presence of small behavioral mutations and errors and whenever global coordination is attempted in a majority of small groups under stringent requirements to meet co-active goals. This result calls for a reassessment of policies towards the promotion of public endeavors: Instead of world summits, decentralized agreements between smaller groups (small N), possibly focused on region-specific issues, where risk is high and goal achievement involves tough requirements (large relative M),[62] are prone to significantly raise the probability of success in coordinating to tame the planet's climate. Our model provides a "bottom-up" approach to the problem, in which collective cooperation is easier to achieve in a distributed way, eventually involving regions, cities, *NGO*s and, ultimately, all citizens. Moreover, by promoting regional or sectorial agreements, we are opening the door to the diversity of economic and political structure of all parties, which, as showed before[32,63] can be beneficial to cooperation.

Naturally, we are aware of the many limitations of a bare model such as ours, in which the complexity of human interactions has been overlooked. From higher levels of information, to non-binary investments, additional layers of realism can be introduced in the model. Moreover, from a mathematical perspective, several extensions and complex aspects common to human socio-economical systems could be further explored.[64–67] On the other hand, the simplicity of the dilemma introduced here, makes it generally applicable to other problems of collective cooperative action, which will emerge when the risks for the community are high, something that repeatedly happened throughout human history,[68,69] from ancient group hunting to voluntary adoption of public health measures.[59,70,71] Similarly, other cooperation mechanisms,[10,13,15,18,22–26] known to encourage collective action, may further enlarge the window of opportunity for cooperation to thrive. The existence of collective risks is pervasive in nature, in particular in many dilemmas faced by humans. Hence, we believe the impact of these results go well beyond decision-making towards global warming.

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A BOTTOM-UP INSTITUTIONAL APPROACH TO COOPERATIVE GOVERNANCE OF RISKY COMMONS

Below follows the print of the document "A bottom-up institutional approach to cooperative governance of risky commons" by Vítor V. Vasconcelos, Francisco C. Santos and Jorge M. Pacheco, published in *Nature Climate Change* in July 14, 2013, online.

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A bottom-up institutional approach to cooperative governance of risky commons

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Avoiding the effects of climate change may be framed as a public goods dilemma[1], in which the risk of future losses is non-negligible[2–7], while realizing that the public good may be far in the future[3,7–9]. The limited success of existing attempts to reach global cooperation has been also associated with a lack of sanctioning institutions and mechanisms to deal with those who do not contribute to the welfare of the planet or fail to abide by agreements[1,3,10–13]. Here we investigate the emergence and impact of different types of sanctioning to deter non-cooperative behaviour in climate agreements. We show that a bottom-up approach, in which parties create local institutions that punish free-riders, promotes the emergence of widespread cooperation, mostly when risk perception is low, as it is at present[3,7]. On the contrary, global institutions provide, at best, marginal improvements regarding overall cooperation. Our results clearly suggest that a polycentric approach involving multiple institutions is more effective than that associated with a single, global one, indicating that such a bottom-up, self-organization approach, set up at a local scale, provides a better ground on which to attempt a solution for such a complex and global dilemma.

To investigate the role of sanctioning institutions, let us consider a finite (and small[1,3]) population of size Z where individuals interact through what has been coined the collective-risk dilemma (CRD), a threshold public goods game—akin to an N-person stag-hunt or coordination game[14]—that mimics the problem at stake[2–4,6]. Individuals organize groups of size N, in which each participant may act as a cooperator (C), defector (D) or punisher (P). Each individual starts with an initial endowment or benefit b. Cs and Ps contribute a fraction c of this benefit, the cost, to reach a common goal, whereas Ds do not contribute. If the overall contribution in the group is insufficient—that is, if the joint number of Cs and Ps in the group is below n_{pg}—everyone in that group will lose their remaining endowments with a probability r (here understood as the perception of risk of collective disaster[2]); otherwise, everyone will keep whatever they have.

The scenario of present-day summits, in which all countries meet in a single group with the aim of establishing long-term goals and commitments by which all must abide[3], is known to be detrimental to cooperation[6]. Hence, it is better to establish smaller groups focused on overcoming shorter-term goals, meant to be revised and reassessed frequently. To this end, we model individual decision-making as an interacting dynamical process, where individuals are embedded in a behavioural ecosystem[15–17], such that decisions and achievements of others influence one's own decisions through time[18–21] (Methods and Supplementary Information for further details). Behavioural experiments[4,5,22], as well as other theoretical models[23,24], have implemented thresholds through repeated interactions, and other authors have highlighted the role played by pledges and communication during negotiations[1,5,25], bringing about additional layers of complexity to this problem (details and comparison with other models in the Supplementary Information).

Besides contributing to this public good, Ps also contribute with a punishment tax (π_t) to an institution that, whenever endowed with enough funding ($n_p\pi_t$) will effectively punish Ds by an amount Δ. Hence, establishing an institution stands as a second-order public good[17,20], which is only achieved above a certain threshold number of contributors n_p (ref. 14). The fact that, in both cases, contributors may pay a cost in vain increases the realism (and the inherent complexity) of the decision process modelled here.

The institution need not be a global one (such as the United Nations)—supported by all Ps in the population—that overviews all group interactions in the population. Institutions may also be local, group-wide, created by Ps within each group to enforce cooperation in that group of individuals. Here we shall consider both cases.

In the absence of Ps, this model reduces to the evolutionary game theoretical model[6] developed to investigate the general role of risk in climate change agreements, and inspired in economic experiments[4] that provided evidence on the unavoidable role of risk perception in the context of climate change. Indeed, the theoretical model not only corroborates the results of the economic experiments[4], but also allows one to extend the analysis to arbitrary group size, risk perception and even group-networked agreements[6]. The new, fundamental changes stemming from the introduction of Ps in this behavioural ecosystem will allow us to assess the role of sanctioning institutions in the presence of risk, a feature that has not been studied before, neither theoretically nor experimentally.

The stochastic evolutionary dynamics of the population occurs in the presence of errors, both in terms of errors of imitation[21] and in terms of behavioural mutations[26], the latter accounting for a free exploration of the possible strategies. We calculate the pervasiveness in time of each possible behavioural composition of the population, the so-called stationary distribution (Methods), which allows the computation of the average fraction of groups that successfully produce (or maintain) the public good—a quantity we designate as group achievement, η_G—and the prevalence in time of a given type of institution—that is, the fraction of time the population witnesses the presence of sanctioning institutions (local or global)—a quantity we designate as institutions prevalence, η_I. It is important to note that both quantities can be directly

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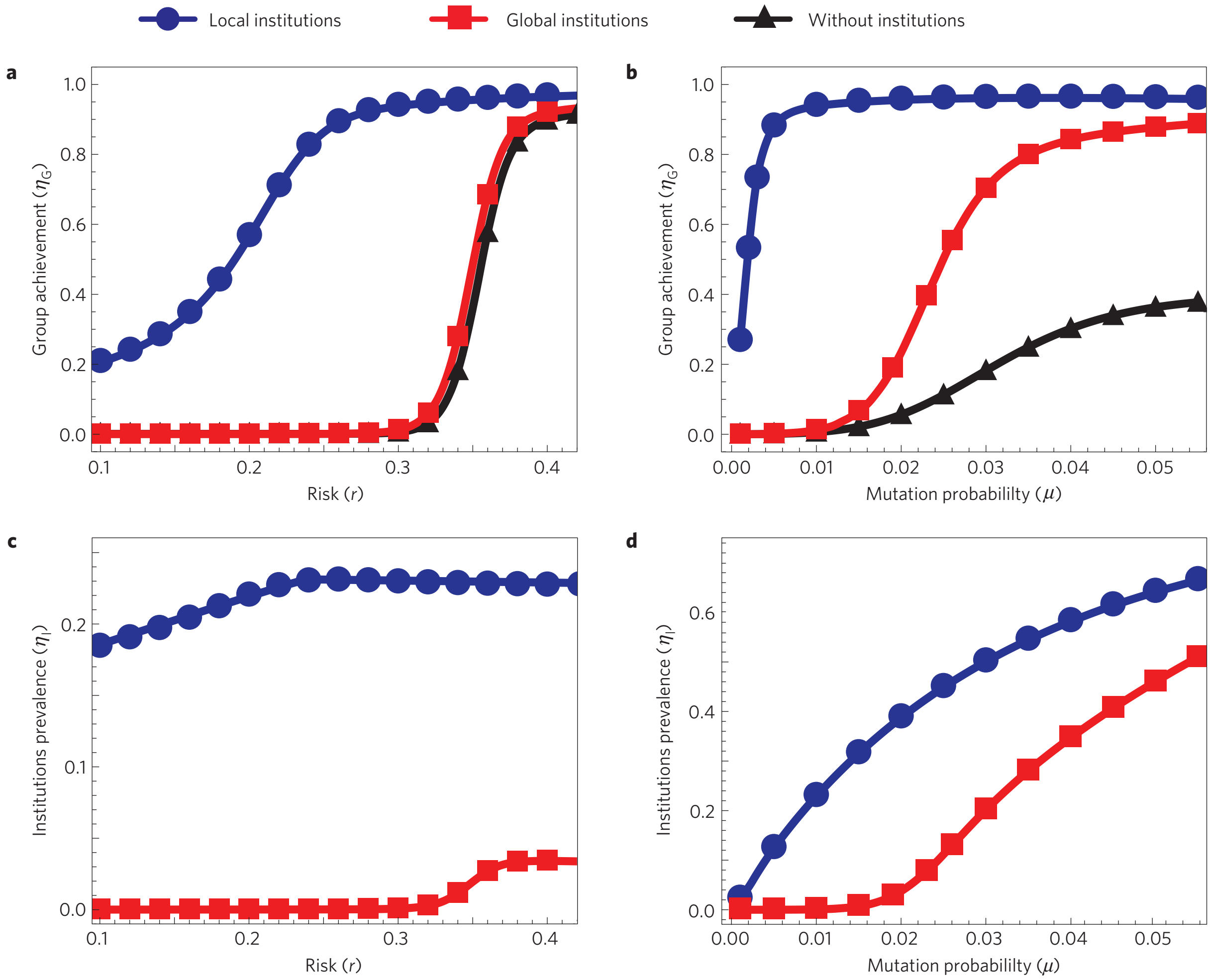

Figure 1 | Group achievement η_G and institutions prevalence η_I. a,**b**, The average fraction of groups that attain the public good (η_G) as a function of perception of risk (r; **a**) and behavioural exploration probability (μ; **b**). Sanctions are enacted by a global institution (red lines and squares) or by local institutions (blue lines and circles). Black lines and triangles: results obtained in the absence of any institution. **c**,**d**, Results for η_I as a function of risk r (**c**) and exploration μ (**d**). Unlike global institutions, often associated with marginal improvements of cooperation, local institutions promote group coordination to avoid a collective disaster, mostly for low perception of risk. The coordination threshold n_{pg} is set to 75% of the group size, whereas local (global) institutions are created whenever 25% of the group (population) contributes to its establishment. Other parameters: $Z=100$, $N=4$, $c/b=0.1$, $\mu=1/Z$, $\pi_f=0.3$, $\pi_t=0.03$ and $r=0.3$.

compared with data extracted from experiments[2,4]. In particular, the empirical results obtained for the risk dependence[4] (in the absence of any sanctioning) show that the group achievement (η_G in our model) increases with the value of risk, correlating nicely with the dependence shown in Fig. 1a with black lines and symbols.

In Fig. 1a the behaviour of η_G as a function of risk is shown in the absence of any institutions (in black), under one global institution (in red) and under local institutions (in blue). Comparison of the black and red curves shows that global institutions provide, at best, a marginal improvement compared with no institutions at all. This result is surprising, given that most climate agreements attempt to involve all countries at once[1,3,27], in which case a single, global institution constitutes the most natural candidate (further details in the Supplementary Information).

On the contrary, under local, group-wide, sanctioning institutions, associated with a distributed scenario in which global sanctions will result from the joint role of a variety of institutions, group achievement is substantially enhanced, in particular when it is most needed: for low values of the perception of risk and whenever individuals face stringent requirements to avoid a collective disaster (Fig. 1a), as has been pointed out to be the case in the context of climate treaties[1]. One can also show (Supplementary Information) that this result is even more pronounced in a scenario encompassing (many) small groups (and institutions). This aspect is particularly important, as the group size (N) defines both the scale at which agreements should be attempted and the overall scope of each institution.

The success of local institutions is closely connected with their resilience. As shown in Fig. 1c, local institutions prevail for longer periods than a (single) global one, always promoting more widespread cooperation than global ones. The efficiency and prevalence of both kinds of institution, however, can be significantly enhanced for high behavioural mutations (Fig. 1b,d), associated with situations in which participants change their decisions more frequently. This scenario may be relevant, given the multitude of (often conflicting) factors that contribute to the process of decision-making[12,13,19].

The dynamics associated with each type of institution is best characterized by the full stationary distributions, plotted in Figs 2 and 3 and covering the entire configuration space mapped onto the triangular simplexes, in which each (discrete) configuration is represented by a circular dot. Darker dots indicate those configurations visited more often, according to the colour gradient scale indicated in each panel. In each dot the relative frequencies

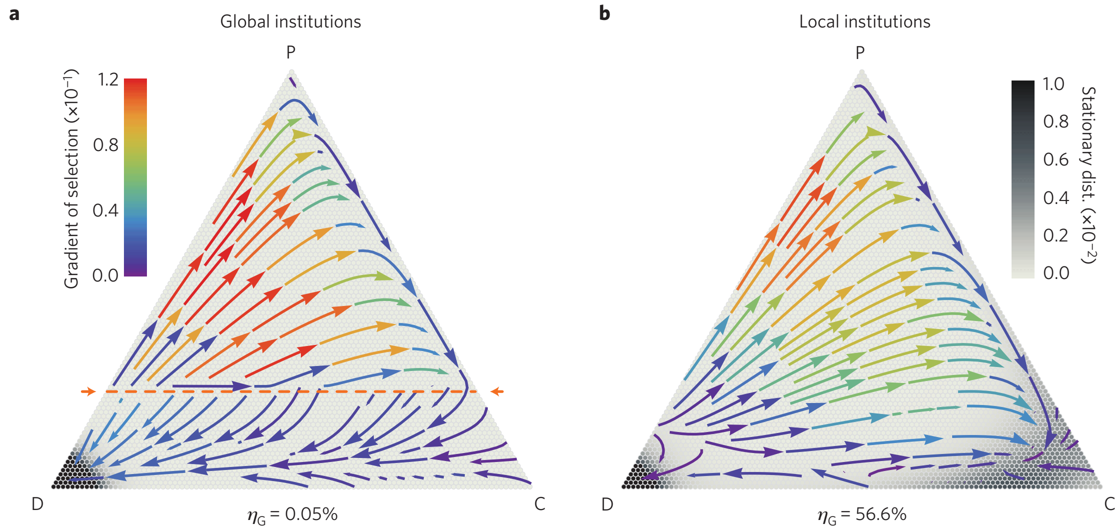

Figure 2 | Behaviour of the CRD in the full configuration space with three strategies—Cs, Ps and Ds—for the same parameters as in Fig. 1 and low risk ($r = 0.2$). a, Global institutions. **b**, Local institutions. The value of the stationary distribution at each configuration is shown using a grey scale; the magnitude of the gradient of selection is shown using the blue–yellow–red scale indicated. For global institutions, the population-wide threshold is indicated using a dashed orange line.

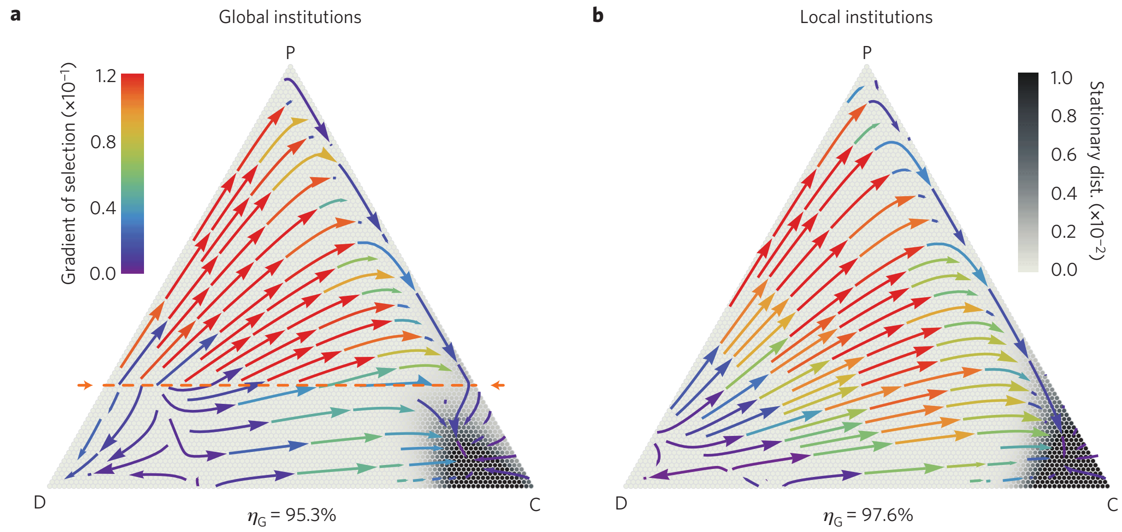

Figure 3 | Behaviour of the CRD in the full configuration space with three strategies: Cs, Ps and Ds. a, Global institutions. **b**, Local institutions. We use the same parameters as in Fig. 2, yet for high values of the perception of risk ($r = 0.5$, in this case), that is, a value of r above which most of the groups manage to coordinate their action, even in the absence of institutions (Fig. 1a).

of Cs, Ds and Ps sum up to one, whereas each vertex of the triangle is associated with monomorphic configurations. Arrows in each simplex represent the most probable direction of evolution, obtained from the computation of the two-dimensional gradient of selection (Methods). We used a continuous colour code associated with the likelihood of such transitions.

The two panels of Fig. 2 show representative examples of the behavioural dynamics of Cs, Ds and Ps under global institutions (Fig. 2a) and local institutions (Fig. 2b), for low values of the perception of risk ($r = 0.2$). For global institutions (Fig. 2a) and whenever the population starts below n_P (the punishment or institution threshold value indicated by a horizontal, orange dashed line), behavioural mutations allow the appearance of Ps in the population (Supplementary Information for further details), such that whenever the composition of the population lies above the threshold line, Ps rapidly outcompete Ds (see arrows), leading the population towards full cooperation, associated with the CP-edge of the simplex. Once in this situation, however, Ps will be outcompeted by Cs as now they contribute to support an institution that has become useless. Hence, the global institution becomes unstable, leading the population (slowly, as shown by the blue arrows along the whole path) to a configuration that falls below the threshold line again. Thus, for low perception of risk, a global institution cannot be maintained for long periods (Fig. 1c) and, as shown by the stationary distributions, the population will remain most of the time under widespread defection. This, in turn, leads to the small value of η_G reported in Fig. 2a.

For local institutions, however, the situation is quite different, as shown in Fig. 2b. Comparison between Fig. 2a and Fig. 2b shows that the role of the threshold line is not so pronounced in this case. Considering that we need the same fraction of Ps (compared to Fig. 2a) to make the institution efficient (25% in this example), but now at the level of the group (and no longer at the level of the population), it is possible that some (although not all) groups have enough Ps for sanctions to become effective. This leads to a marked increase of η_G, as in this case the population evolves towards regimes of widespread cooperation. This happens because the population will stabilize in configurations comprising a sizeable amount of Cs together with enough Ps to prevent Ds from invading. The fact that this happens for low values of risk r is important, given that, at present, the perception of risk regarding climate issues is low[3,7].

For high values of the perception of risk, shown in Fig. 3 ($r = 0.5$), both local and global institutions marginally enhance the positive prospects for cooperation already attained in the absence of any institution, as for high risk the dynamics occurs in the vicinity of the CD-edge of the simplex (Supplementary Information). Notwithstanding, and because local institutions are easier to emerge, they work as catalysers of collective action, while helping to prevent the invasion of Ds, as shown in Fig. 2. Neither local nor global institutions are robust to free-riding, a result that has been recently confirmed experimentally[28]. Finally, behavioural mutations enhance the prevalence of configurations in the inner part of the simplex, which in turn increases the chances of having enough Ps to establish institutions and cooperation, as previously shown in Fig. 1.

Our results support the conclusion that a decentralized, polycentric, bottom-up approach[10], involving multiple institutions instead of a single global one, provides better conditions both for cooperation to thrive and for ensuring the maintenance of such institutions. This result is particularly relevant whenever perception of risk of collective disaster, alone, is not enough to ensure global cooperation. In this case, local sanctioning institutions may provide an escape hatch to the tragedy of the commons humanity is facing. In this context, it is worth stressing that the mechanisms discussed here operate optimally whenever groups are small. Present-day local initiatives, such as the Western Climate Initiative[29], have started with a small group of US states. As time went by, the Western Climate Initiative group size has grown to include additional Canadian states and Mexican provinces. Although the reasons and motivations for such an evolution are comprehensible, one should not overlook that larger groups are more difficult to coordinate into widespread cooperation (Supplementary Information). Similar dynamics, in which cooperation nucleating in a small group expands into a larger and larger group, can be found in policies beyond climate governance with mixed results, from the major transitions in evolution[30] to the recent evolution of the European Union, stressing the common ground shared by governance and a variety of ecosystems[15]. In this context, it might be easier to seek a multi-scale (and multi-step) process, in which coordination is achieved in multiple small groups or climate blocks[12], before aiming, if needed, at agreements encompassing larger groups (or, alternatively, inter-group agreements). Hence, although most causes of climate change result from the combined action of all inhabitants of our planet, the solutions for such complex and global dilemma may be easier to achieve at a much smaller scale[10]. In light of our results, the widely repeated motto 'Think globally, act locally' would hardly seem more appropriate.

Methods

We consider a population of Z individuals, who set up groups of size N, in which they engage in the CRD public goods game[4,6], being capable of adopting one of the three strategies: C, P and D. Following the discussion in the main text, the payoff of an individual playing in a group in which there are j_C Cs, j_P Ps and $N - j_P - j_C$ Ds, can be written as $\Pi_C = -c + b\Theta(j_C + j_P - n_{pg}) + (1-r)b[1-\Theta(j_C + j_P - n_{pg})]$, $\Pi_P = \Pi_C - \pi_t$ and $\Pi_D = \Pi_C + c - \Delta$ for Cs, Ps and Ds, respectively. In the equations above, $\Theta(k)$ is the Heaviside function (that is, $\Theta(k) = 1$ whenever $k \geq 0$, being zero otherwise), $0 < n_{pg} \leq N$ is a positive integer not greater than N, and r (the perception of risk) is a real parameter varying between 0 and 1; the parameters c, π_t and b are all real positive; Δ corresponds to the punishment function, which depends on whether the institution is global or local. For local institutions, punishment acts at the group level, and Δ yields $\Delta_{local} = \pi_f\Theta(j_P - n_P)$, which means that a punishment fine π_f is applied to each D in the group whenever $N \geq j_P \geq n_P \geq 0$. For global institutions, punishment acts at the population level; in a population with i_C Cs, i_P Ps and $Z - i_P - i_C$ Ds, the punishment function for global institutions can be written as $\Delta_{global} = \pi_f\Theta(i_P - n_P)$, applying a punishment fine π_f now to every D in the population, whenever $Z \geq i_P \geq n_P \geq 0$. Finally, the fitness f_X of an individual adopting a given strategy, X, will be associated with the average payoff of that strategy in the population. This can be computed for a given strategy in a configuration $i = \{i_C, i_P, i_D\}$ using a multivariate hypergeometric sampling (without replacement; Supplementary Information for details). The number of individuals adopting a given strategy will evolve in time according to a stochastic birth–death process combined with the pairwise comparison rule[21], which describes the social dynamics of Cs, Ps and Ds in a finite population. Under pairwise comparison, each individual of strategy X adopts the strategy Y of a randomly selected member of the population, with probability given by the Fermi function $(1 + e^{\beta(f_X - f_Y)})^{-1}$, where β controls the intensity of selection ($\beta = 5.0$ in all figures). In addition, we consider that, with a mutation probability μ, individuals adopt a randomly chosen strategy. As the evolution of the system depends only on its actual configuration, evolutionary dynamics can be described as a Markov process over a two-dimensional space. Its probability distribution function, $p_i(t)$, which provides information on the prevalence of each configuration at time t, obeys a master equation, a gain–loss equation involving the transition rates between all accessible configurations. The stationary distribution $\bar{p}_i$ is then obtained by reducing the master equation to an eigenvector search problem (Supplementary Information for details). Another central quantity, which portrays the overall evolutionary dynamics in the space of all possible configurations, is the gradient of selection Δ_i. For each configuration i, we compute the most likely path the population will follow, resorting to the probability to increase (decrease) by one the number of individuals adopting a strategy S_k, $T_i^{S_k+}$ ($T_i^{S_k-}$) in each time step. In addition, for each possible configuration i, we make use of multivariate hypergeometric sampling to compute both the (average) fraction of groups that reach n_{pg} contributors, that is, that successfully achieve the public good—which we designate by $a_G(i)$—and the (average) fraction of groups that reach n_p punishers (for local institutions) or whether for that configuration i a global institution will be formed—in both cases, we designate this quantity by $a_I(i)$. Average group achievement—η_G—and institution prevalence—η_I—are then computed averaging over all possible configurations i, each weighted with the corresponding stationary distribution: $\eta_G = \sum_i \bar{p}_i a_G(i)$ and $\eta_I = \sum_i \bar{p}_i a_I(i)$.

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Author contributions

V.V.V., F.C.S. and J.M.P. have contributed equally to this work: they all designed and performed the research, analysed the data and wrote the paper.

Additional information

Supplementary information is available in the online version of the paper. Reprints and permissions information is available online at www.nature.com/reprints. Correspondence and requests for materials should be addressed to J.M.P.

Competing financial interests

The authors declare no competing financial interests.

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SUPPLEMENTARY INFORMATION

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A bottom-up institutional approach to cooperative governance of risky commons

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1. Collective risk dilemma and pool punishment

Following the discussion in the main text the payoff an individual within a group of j_C ***C***s, j_P ***P***s and $N - j_P - j_C$ ***D***s, can be written as

$$\Pi_C = -c + b\,\Theta(j_C + j_P - n_{pg}) + (1-r)b\left[1 - \Theta(j_C + j_P - n_{pg})\right] \quad (1)$$

$$\Pi_P = \Pi_C - \pi_t \quad (2)$$

$$\Pi_D = \Pi_C + c - \Delta \quad (3)$$

In the Equations above, $\Theta(k) = \{^{0\,(k<0)}_{1\,(k\geq 0)}$ is the Heaviside function, $0 < n_{pg} \leq N$ a positive integer, and r is a real number between 0 and 1; the parameters c, π_t, and b are all positive real numbers. In Eq. 3, Δ corresponds to the "punishment function", which depends on whether the *institution* is *global* or *local*. For *local institutions* punishment acts at the group level, and Δ yields

$$\Delta = \pi_f\,\Theta(j_P - n_p) \quad (4)$$

which means that a *punishment fine* π_f is applied to each ***D*** in the group whenever $j_P \geq n_p$.

For *global institutions* punishment acts at the population level; in a population with i_C ***C***s, i_P ***P***s and $Z - i_P - i_C$ ***D***s, the punishment function for *global institutions* can be written as

$$\Delta = \pi_f\,\Theta(i_P - n_p) \quad (5)$$

applying a *punishment fine* π_f now to every ***D*** in the population, whenever $i_P \geq n_p$.

If one models individual decision process as purely rational (as is usually done in conventional game theoretical analysis), one will ignore existing evidence that peer-influence plays a determinant role in strategy revision[1-3]. Hence, we assume here a simpler, short-term behavioural revision process conveniently modelled in the framework of evolutionary game theory. As a result, our framework allows agreements to become vulnerable to (or to benefit from) such short-term behavioural updates, as individuals assess the consequences of their choices. Thus, our approach contrasts with that implemented in behavioural experiments[4-7] and alternative theoretical models[8,9], where a repeated-game scenario is implemented, involving a wide repertoire of strategies and contingency plans[4-14]. As a result, theoretical models are no longer amenable to be dealt with analytically[8].

In fact, short-term commitments and strategy revision are presumably more realistic. The fact that different countries (and different political actors in each country) do not agree on long term policies[15], suggests that defining short term time scales may prove beneficial, giving decision makers more room to change their mind and (hopefully) reach a consensus. This is also the best framework in which to adopt an evolutionary game theoretical approach, as we do here. Such an approach has been employed before, although sanctioning institutions have not been analysed[16,17]. Needless to say, other mechanisms are certainly relevant, and may even prove crucial, given the time frame at stake[18], as discussed at length in Refs.[10,12,19-30]. In this context, our work provides the barest framework establishing conditions that naturally favour widespread cooperation in attempting to mitigate the adverse effects of global warming.

The variable n_{pg} imposes a minimum number of contributions needed to achieve a common goal[5,17] or an intermediate climate target[7]. In line with a previous model[17], individuals engage in several (preferably small scale) **CRD** games with the aim of coordinating to establish *short term* goals in each of them. The extent to which individuals cooperate in these games will ultimately dictate the solution (or not) of the (*long term*) problem of Climate Change.

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To let the entire population form a single group engaging in the **CRD** is detrimental to cooperation[17], and hence it is much better to establish smaller (eventually local) groups focussing on coordinating to overcome more modest, common interest and shorter term targets. Short-term commitments are meant to be revised and re-assessed frequently in subsequent instances of the **CRD** game, whereby individual decisions may naturally change in time. Because individual decisions are known to all in the population, it is natural to assume that previous decisions will influence future decisions, which also means that communication between participants actually takes place in such a setting. Although such type of communication is different from, e.g., that studied explicitly in Ref.[11], there is some correspondence between these two forms of "pre-play" signalling[31,32]. Needless to say, a detailed theoretical model of the process of pre-play communication would require signalling to be explicitly incorporated (honest signalling would perhaps suffice, in face of the results of Ref. [11]), which would render the model analytically intractable[31]. In this sense, the present model, making information of individuals' successes and failures available to all between different games, can be understood as a first (and much simpler) step in that direction. For this reason, we model individual decision making as an interacting dynamical process, such that decisions and achievements of others may influence one's own decisions through time[3,33-38]. Such (stochastic) dynamical system is discussed in detail in the following section.

2. Evolutionary dynamics in finite populations

We consider a population of Z individuals. Each individual can adopt one of $s+1$ strategies: $S_1,...,S_{s+1}$, such that, at each time-step t, we have a given configuration (or state) $\mathbf{i}(t)=\{i_1,\cdots,i_k,\cdots,i_s\}$ of a population, specified by the number of individuals adopting each particular strategy (we need only the first s strategies – and hence, an s-dimensional simplex – as the frequency of players using strategy s+1 can be obtained through normalization). The fitness of a strategy will be associated, as usual[39], with the average payoff of any individual using this strategy. Let $\mathbf{j}=\{j_1,\cdots,j_k,\cdots,j_s\}$ be the configuration of players in a group of size N and ($\mathbf{j}$; j_k=q) designating any group configuration in which there are specifically q players with strategy j_k; with these definitions, we may write down the average fitness of a strategy S_k in a population characterized by configuration $\mathbf{i}$, $f_{S_k}(\mathbf{i})$, as[9,17,40-43]

$$f_{S_k}(\mathbf{i})=\binom{Z-1}{N-1}^{-1}\sum_{(\mathbf{j}\,;\,j_k=0)}^{(\mathbf{j}\,;\,j_k=N-1)}\Pi_{S_k}(\mathbf{j})\binom{i_k-1}{j_k}\prod_{\substack{l=0\\(l\neq k)}}^{s+1}\binom{i_l}{j_l} \tag{6}$$

where $\Pi_{S_k}(\mathbf{j})$ stands for the payoff of a strategy S_k in a group with composition $\mathbf{j}$.

For each configuration $\mathbf{i}$, we may also compute other population-wide variables of interest making use of variants of Eq. 6. In particular, the average fraction of groups $a_G(\mathbf{i})$ that reach n_{pg} contributors (see Methods and previous section) is obtained replacing $\Pi_{S_k}(\mathbf{j})$ by $\Theta(j_C+j_P-n_{pg})$ in Eq. 6 for the case of 3 strategies (s=2, ***C***s, ***P***s and ***D***s) and by $\Theta(j_C-n_{pg})$ for the case 2 strategies (s=1, ***C***s and ***D***s). Similarly, the average fraction of groups $a_I(\mathbf{i})$ that reach n_p punishers (for local institutions) is also provided by Eq. 6 with $\Pi_{S_k}(\mathbf{j})$ replaced by $\Theta(j_P-n_p)$. For global institutions, $a_I(\mathbf{i})$ is simply given by $\Theta(i_P-n_p)$, as described in the previous section and main text.

Strategies evolve according to a mutation-selection process. At each time step, the strategy of one randomly selected individual X is updated. With probability μ, X undergoes a mutation, adopting a strategy drawn randomly from the space of available strategies. With probability $1-\mu$, another randomly selected individual Y acts as a

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potential role model of *X*. The probability that *X* adopts the strategy of *Y* equals $\varphi = \left[1 + e^{\beta(f_X - f_Y)}\right]^{-1}$, whereas *X* maintains the strategy with probability $1 - \varphi$. We use f_X and f_Y to denote the fitness of individual *X* and *Y*, respectively. This update rule is known as the pairwise comparison rule[35,44]. The parameter $\beta \geq 0$, measures the contribution of fitness to the update process, i.e., the selection pressure. In the limit of strong selection ($\beta \to \infty$), the probability φ is either zero or one, depending on how f_X compares with f_Y. In the limit of weak selection ($\beta \to 0$), φ is always equal to $1/2$, irrespective of the fitness of *X* and *Y*.

For the sake of mathematical convenience, analysis of evolutionary dynamics in finite populations and arbitrary number of strategies have been mostly dealt with either in the limit of rare mutations[9,38,40,45-47] — in which the population will never contain more than two different strategies simultaneously — and/or in the limit of weak selection ($\beta \to 0$)[34,35,44,48-54]. Here we do not restrict our analysis to any of these approximations. Instead, as the pairwise comparison relies solely on the present configuration of the population[35], the dynamics of $\mathrm{i}(t) = \{i_1, \ldots, i_s\}$ corresponds to a Markov process over a *s*–dimensional space[35,46,50,55-57], and hence its probability density function, $p_{\mathrm{i}}(t)$, i.e., the prevalence of each configuration at time *t*, evolves in time according to the Master-Equation[55],

$$p_{\mathrm{i}}(t+\tau) - p_{\mathrm{i}}(t) = \sum_{\mathrm{i}'} \{T_{\mathrm{ii}'} p_{\mathrm{i}'}(t) - T_{\mathrm{i}'\mathrm{i}} p_{\mathrm{i}}(t)\} \qquad (7)$$

a gain-loss equation that allows one to compute the evolution of $p_{\mathrm{i}}(t)$ given the transition probabilities per unit time between configurations $\mathbf{i}$ and $\mathbf{i}'$, $T_{\mathrm{i}'\mathrm{i}}$ and $T_{\mathrm{ii}'}$. The stationary distribution $\bar{p}_{\mathrm{i}}$ analysed in the main text, is obtained by making the left side zero, which transforms Eq. 7 into an eigenvector search problem[55], namely, the eigenvector associated with the eigenvalue 1 of the transition matrix $\Lambda = \left[T_{ij}\right]^T$.

Besides providing the prevalence in time of each configuration **i**, the stationary distribution $\bar{p}_{\mathrm{i}}$ also allows the direct computation of an average measure of group achievement (η_G) and institution prevalence (η_I) given by $\eta_G = \sum_{\mathrm{i}} \bar{p}_{\mathrm{i}} a_G(\mathrm{i})$ and $\eta_I = \sum_{\mathrm{i}} \bar{p}_{\mathrm{i}} a_I(\mathrm{i})$, respectively, where $a_I(\mathbf{i})$ and $a_G(\mathbf{i})$ were defined before.

We are then left with the task of computing the transition probabilities among all possible configurations that define Λ. The nature of the birth-death process we defined imposes that, if the configuration of strategies at a given time is

$$\mathbf{i} = \{i_1, ..., i_s, i_{s+1} = N - i_1 - ... - i_s\},$$

it can only move to a configuration

$$\mathbf{i}' = \{i_{1'}, ..., i_{s'}, i_{s+1}' = N - i_{1'} - ... - i_{s'}\} = \{i_1 + \delta_1, ..., i_{s+1} + \delta_{s+1}\},$$

where either all δ_k are zero, or only two of them are non-zero being, respectively, 1 and -1. When all $\delta_k = 0$, the configuration remains unchanged, $\mathbf{i}' = \mathbf{i}$, and the transition probability corresponding to this event can be calculated from the remaining transitions as $T_{\mathbf{ii}} = 1 - \sum_{\mathbf{i}' \neq \mathbf{i}} T_{\mathbf{i'i}}$. The missing transition probabilities are associated with events in which an element with a given strategy, S_l, changes into another specific strategy, S_k, which, for the pairwise comparison rule with an arbitrary mutation rate μ, is given by

$$T_{S_l \to S_k} = (1-\mu)\left[\frac{i_l}{Z}\frac{i_k}{Z-1}\left(1+e^{\beta(f_{S_l}-f_{S_k})}\right)^{-1}\right] + \mu\frac{i_l}{sZ}. \quad (8)$$

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Hence, for a given configuration $\mathbf{i}=\{i_1,...,i_s\}$, we can compute the probability to increase (decrease) by one the number of individuals adopting a strategy S_k, which we denote by $T_{\mathbf{i}}^{S_k+}$ and $T_{\mathbf{i}}^{S_k-}$, as

$$T_{\mathbf{i}}^{S_k\pm}=\sum_{i_{1'},...,i_{k-1}',i_{k+1}',...i_{s'}} T_{\mathbf{i}\{i_{1'},...,i_k\pm1,...,i_{s'}\}}\ . \tag{9}$$

These transitions constitute a central quantity to compute the gradient of selection ($\nabla_{\mathbf{i}}$) — i.e., the most likely path the population will follow when leaving configuration **i** — as pictured in the main text.

In a 2-strategy case, each configuration **i** — e.g., *i* ***C***s and *Z-i* ***D***s — would have two neighbours, and therefore two possible transitions, one with one ***C*** more ($i\rightarrow i+1$) and another with one ***C*** less ($i\rightarrow i-1$). Hence, we will have $\nabla_{\mathbf{i}}=T_{D\rightarrow C}(i)-T_{C\rightarrow D}(i)$, where $T_{D\rightarrow C}(i)=(1-\mu)\left[\frac{i}{Z}\frac{Z-i}{Z-1}\left(1+e^{\beta(f_D-f_C)}\right)^{-1}\right]+\mu\frac{Z-i}{Z}$ for the probability to increase from *i* to *i*+1 ***C***s and $T_{C\rightarrow D}(i)=(1-\mu)\left[\frac{i}{Z}\frac{Z-i}{Z-1}\left(1+e^{\beta(f_C-f_D)}\right)^{-1}\right]+\mu\frac{i}{Z}$ for the probability to decrease to *i*-1[58].

For the 3-strategy case – e.g., when configurations are given by $\mathbf{i}=\{i_C,i_P\}$, standing for the number of ***C***s, ***P***s (and Z–i_C–i_P ***D***s) — the evolutionary dynamics occurs in a 2-dimensional simplex (see Figures 2 and 3 in the main text). To every adjacent configuration **i'** in the simplex (the ones accessible in a single update), we associate a vector with magnitude $T_{\mathbf{i'i}}$ and with the direction of $\mathbf{i'}-\mathbf{i}$. Performing the sum of these vectors leads to a new local vector, $\nabla_{\mathbf{i}}$, which contains information about all possible transitions, and which can be written as

$$\nabla_{\mathbf{i}}=(T_{\mathbf{i}}^{C+}-T_{\mathbf{i}}^{C-})\mathbf{u_C}+(T_{\mathbf{i}}^{P+}-T_{\mathbf{i}}^{P-})\mathbf{u_P} \tag{10}$$

where $\mathbf{u_C}$ ($\mathbf{u_p}$) are unit vectors defining a basis of the 2 dimensional simplex in which evolution proceeds (see Fig. S1). The entries of $\nabla_\mathbf{i}$ correspond to a balance of transitions along each direction and, therefore, we call $\nabla_\mathbf{i}$ the gradient of selection (or *drift*)[*].

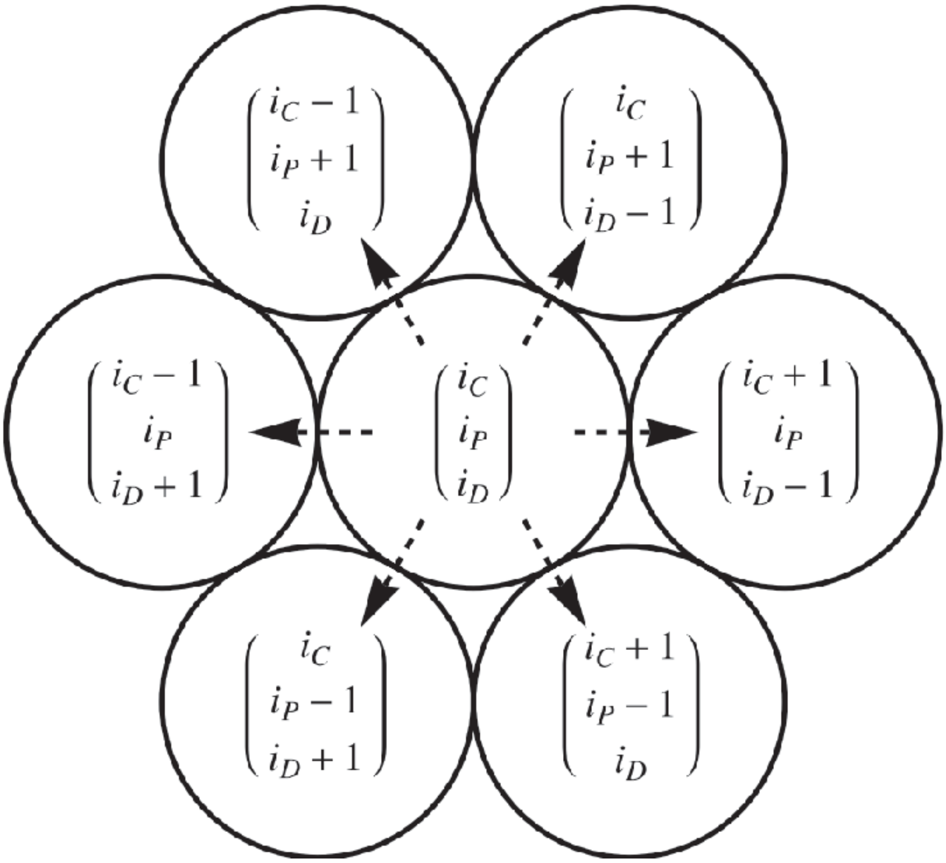

Figure S1 | Local representation of the phase space and possible transitions for a bi-dimensional one-step process. A vector can be associated with every transition between the element ***i*** and each adjacent element. The sum of these vectors corresponds to the gradient of selection, $\nabla_\mathbf{i}$, in the configuration ***i***, i.e., the *drift* (see above).

[*] It can be shown that $\nabla_\mathbf{i}$ corresponds to the first coefficient of the Kramers-Moyal expansion of the Master Equation for this birth-death process which, in the limit $N \to +\infty$, gives the drift term of the Fokker-Planck equation and the corresponding meaning of the most probable direction.

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3. Evolutionary dynamics in populations with two strategies

A natural first step to describe the role of sanctioning defectors is to understand in detail the evolutionary dynamics of populations in which only one of the cooperating strategies is present in the population — namely ***P***s and ***D***s, and ***C***s and ***D***s. In the absence of ***P***s, we recover the *N*-person **CRD** game recently proposed[17], where the risk of collective failure is explicitly introduced and where it is was shown how the perception of risk plays a central role in the emergence of cooperation. The different panels in Fig. S2 show the stationary probability distribution function together with the gradient of selection $\nabla_{\mathbf{i}}$ for populations in which only 2 strategies are present — ***C***s and ***D***s (left) and ***P***s and ***D***s (right). Whenever $\nabla_{\mathbf{i}} = 0$ is zero, at i_C^*, the transition probabilities to a configuration with more ***D***s and to a configuration with fewer ***D***s are the same. Furthermore, if the configuration to the right (decreasing the number of ***D***s) has a negative (positive) $\nabla_{\mathbf{i}}$ and the one to the left has a positive (negative) $\nabla_{\mathbf{i}}$, this system is preferentially pushed into (pulled out of) this configuration. Intuitively, the configurations with less ***C***s, to the left of i_C^*, tend to have their number of ***C***s increase (decrease), whereas the configurations to the right, with more ***C***s, tend to have less (more). In this sense, those configurations associated with i_C^* are analogous to the stable (unstable) fixed points obtained from the replicator equation[39], the stable analogues being probability attractors (repellers). Hence, the maximum of the stationary distribution is nearly coincident with the configuration i_C^*, whenever the gradient crosses zero with negative slope. Consequently, the population will spend most of the time around i_C^*.

Fig. S2 shows how risk (decreasing from top to bottom) plays a crucial role in the overall population dynamics, given the sensitivity of cooperation to risk. The left panels reproduce the scenarios obtained in the absence of ***P***s[17], which will be used here as references.

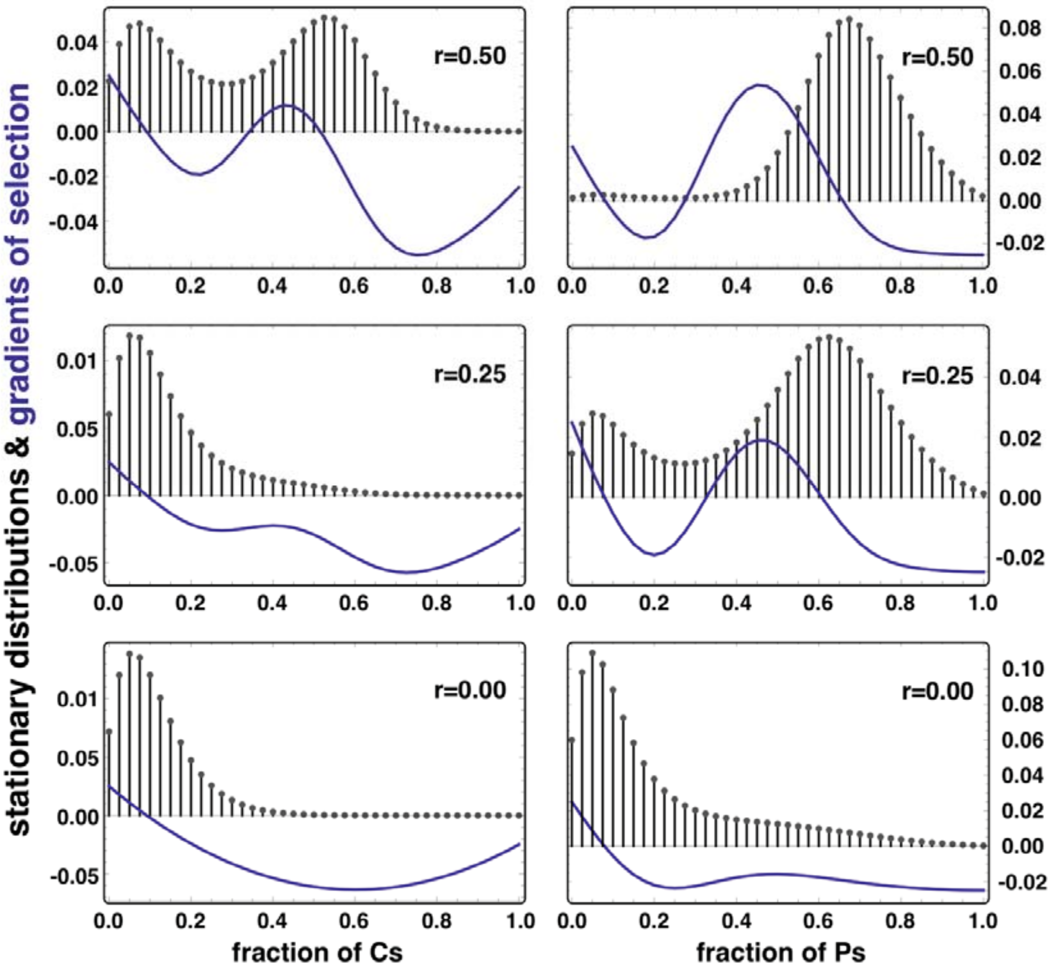

Figure S2 | The role of risk in populations made of ***C***s and ***D***s (left), and ***P***s and ***D***s (right) under local institutions. From top to bottom, each panel shows the stationary distributions (black vertical lines) and respective gradients of selection (blue solid curves) for r = 0.5, r = 0.25 and r = 0. (Z=40, N=10, n_{pg}=5, b=1, c=0.1, π_t=0.02, π_f=0.1, n_p=3, μ=1/Z and β = 5).

The right panels show the impact of punishment on the levels of cooperation — implemented here in the *local institution* version, see Eq. 4 — as ***P***s now co-evolve with ***D***s in the population. In the absence of risk (bottom), the gradient of selection is nearly half as negative compared to the reference scenario. This means that the strength with

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which the population is driven into defection is smaller and, as a result, the stationary distribution grows a heavy tail towards punishment, meaning that the population actually spends a significant amount of time in configurations with less than 50% ***D***s. This is a rather impressive result, revealing the power of punishment[20] in hindering (in this case) defection. As we increase risk (center and top panels) populations adopt more and more the punishment behaviour.

An overall view of the results is provided in Fig. S3, where we plot the internal roots of the gradient of selection for different values of *r*. We compare again the two strategies against ***D***s: ***C***s and ***P***s with *local institutions*.

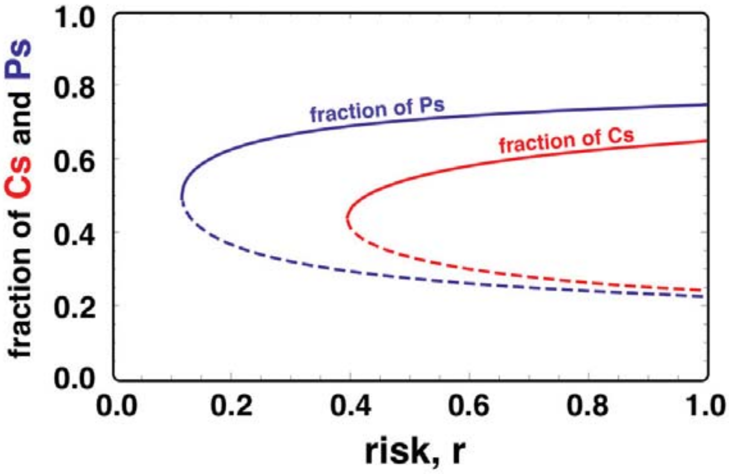

Figure S3 | Interior roots i_C^* of the gradient of selection for populations made of ***C***s and ***D***s (red lines), and ***P***s and ***D***s (blue lines) under local institutions. For each value of *r,* the solid (dashed) lines represent the finite analogues of stable (unstable) fixed points, that is, probability attractors (repellers). (Z=40, N=10, n_{pg} = 5, b=1, c=0.1, π_t = 0.02, π_f = 0.1, n_p = 3, μ = 1/Z and β = 5).

The **CRD** played between ***C***s and ***D***s, shows two kinds of behaviours. In the first scenario, for low values of the perception of risk, the system is driven into a configuration in which defection dominates. As one increases the perception of risk, one reaches a critical value above which the analogues of stable and unstable fixed points emerge, allowing the system to spend longer periods of time in more cooperative configurations. Note that the stable point drives the population into configurations in which ***D***s are a minority.

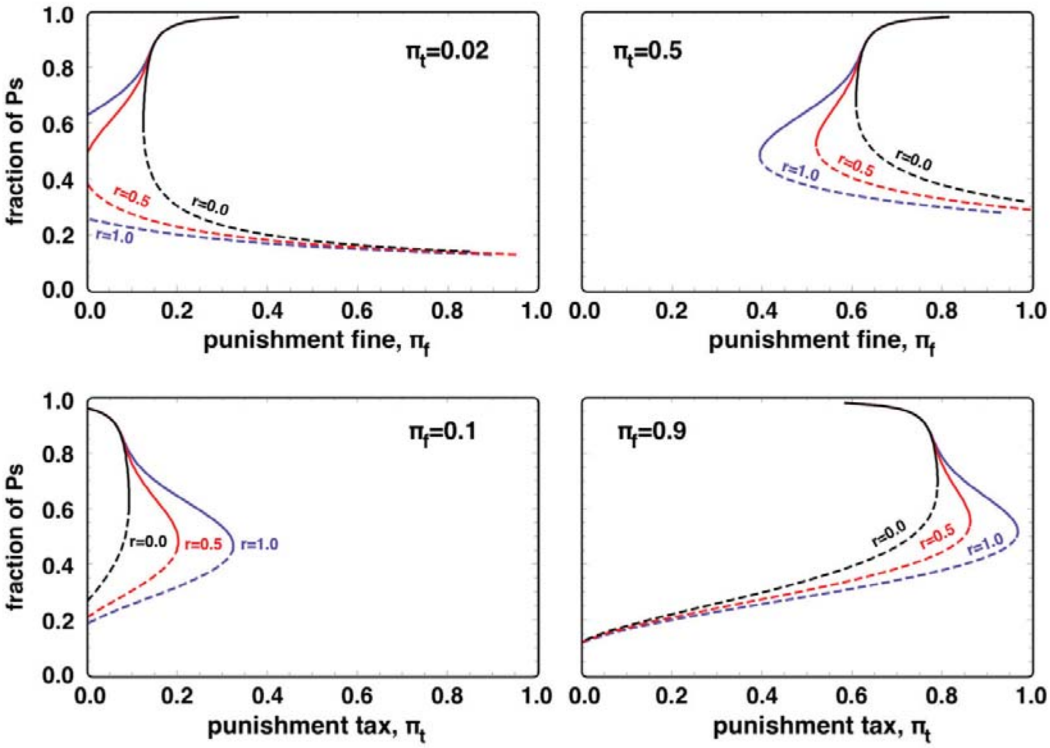

Figure S4 | Effect of punishment under local institutions and sensitivity to risk. The top panels show the internal roots of ∇_i as functions of the punishment fine π_f; the bottom panels show the same quantity now as a function of the punishment tax, π_t; left panels show results for low π_t (top) and low π_f,(bottom), respectively, whereas right panels show results for high values of these parameters. Different line colours represent increasing values of risk: r=0 with black lines, r=0.5 with red lines and r=1 with blue lines, respectively. (Z=40, N=10, c=0.1, b=1, n_{pg}=5, n_p=3, β=5.0 and μ=1/Z).

Whenever ***P***s co-evolve with ***D***s, we also obtain a change in the relative size of the basins of attraction, in particular for low values of risk, as the critical perception of risk r needed to create a cooperative basin of attraction decreases. Furthermore, with ***P***s, the stable equilibrium where few ***D***s co-exist with ***P***s occurs for lower fraction of ***D***s.

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Overall, this means that the population will spend a greater amount of the time in a more cooperative configuration. Additionally, and compared to ***C***s, ***P***s also push the unstable fix point to lower fractions of ***D***s, rendering collective coordination an easier task.

In Fig. S4 we adopt the same notation scheme of Fig. S3 to show the dependence of the position of the internal roots of $\nabla_{\mathbf{i}}$ on the punishment fine π_f and punishment tax π_t. This analysis is repeated for different values of risk (low risk, r=0.0, black lines; intermediate risk, r=0.5, red lines; high risk, r=1.0, blue lines).

In the top panels we see how π_f affects the positions of the fixed points, for both low (left panel) and high (right panel) *punishment fine* π_t. As expected, if the taxes for the maintenance of institutions are low, a considerable amount of punishers pervades; however, as we increase the tax, punishment eventually fades. When the *punishment fine* applied to the ***D***s is smaller, punishment vanishes for smaller values of the tax (left panel).

In the bottom panels we show how the *punishment tax* π_t affects the positions of the internal roots of $\nabla_{\mathbf{i}}$, for both low (left panel) and high (right panel) *punishment fine* π_f. If the tax for the punishment institution is low enough, a small *punishment fine* leads to the appearance of a coexistence root further away from the full defection configuration. However, if the *punishment fine* is high, once again we regain the two different scenarios: for very low (or none) *punishment tax*, the population falls into the tragedy of the commons, whereas above a critical value the coexistence point will arise. Both left and right panels show that a small increase on the *punishment tax* can drastically wipe-out defection. As a final point, all panels contain the location of the internal roots for the three values of risk indicated before, showing the importance of risk in the emergence collective action.

4. Group size dependence in the 3-strategy case

As discussed in the main text, one does not expect that all the parties (e.g. countries, regions or cities[59]) will be willing to pay in order to punish others, despite being perhaps willing to undertake the necessary measures to mitigate climate change effects (***C***s). In other words, an analysis of a 2-strategy case fails to provide a complete overview of the overall dynamics, as players who are willing to pay towards mitigating the climate change effects may free-ride in a 2nd order cooperation dilemma, by refusing to pay a tax in order to create an institution (local or global), able to punish defectors. Consequently, in the main text we discussed the evolutionary dynamics of the population considering the entire set of strategies (***C***s, ***P***s and ***D***s), from which we showed that the adoption of multiple institutions instead of a single, global one provides better conditions for cooperation to thrive.

In this context, the group size N constitutes an important variable, as it defines not only the scale at which agreements should be tried but also the overall dimension and scope of each institution. In particular, a local sanctioning system converges to a single global institution whenever $N \rightarrow Z$. Hence, one should expect that the evolutionary advantage provided by a polycentric approach vanishes for increasing N. Fig. S5 shows, however, that the results in the main text are robust, as local institutions provide a significant advantage in the promotion of cooperation for a wide range of values of N, when compared with a global institution. Furthermore, for a given group size, local institutions are always able to promote higher group achievements for lower values of risk.

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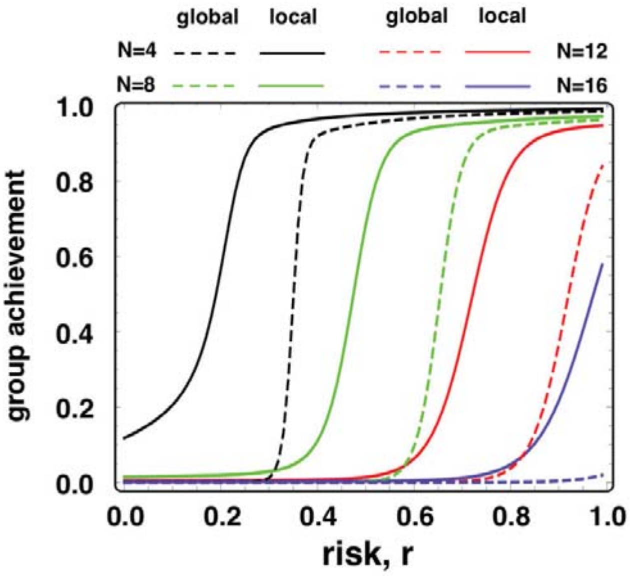

Figure S5 | Group achievement for different group sizes and institution types. Group achievement — η_G —standing for the average fraction of groups that are able to attain the public good, is shown as a function of the perception of *risk* (*r*) for global institutions (dashed lines) and local institutions (full lines). Each colour corresponds to a different group size, as indicated. The coordination threshold (n_{pg}) is set to 75% of the group size, whereas local (global) institutions are created whenever 25% of the group (population) contributes to its establishment. Punishment tax is π_t=0.03, whereas the punishment fine for defecting is π_f=0.3. Other parameters: *Z*=100, *N*=4, *c/b*=0.1, μ=1/*Z*=0.01.

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CLIMATE POLICIES UNDER WEALTH INEQUALITY

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Climate policies under wealth inequality

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Taming the planet's climate requires cooperation. Previous failures to reach consensus in climate summits have been attributed, among other factors, to conflicting policies between rich and poor countries, which disagree on the implementation of mitigation measures. Here we implement wealth inequality in a threshold public goods dilemma of cooperation in which players also face the risk of potential future losses. We consider a population exhibiting an asymmetric distribution of rich and poor players that reflects the present-day status of nations and study the behavioral interplay between rich and poor in time, regarding their willingness to cooperate. Individuals are also allowed to exhibit a variable degree of homophily, which acts to limit those that constitute one's sphere of influence. Under the premises of our model, and in the absence of homophily, comparison between scenarios with wealth inequality and without wealth inequality shows that the former leads to more global cooperation than the latter. Furthermore, we find that the rich generally contribute more than the poor and will often compensate for the lower contribution of the latter. Contributions from the poor, which are crucial to overcome the climate change dilemma, are shown to be very sensitive to homophily, which, if prevalent, can lead to a collapse of their overall contribution. In such cases, however, we also find that obstinate cooperative behavior by a few poor may largely compensate for homophilic behavior.

collective action | global warming | governance of the commons | environmental agreements | evolutionary game theory

Despite existing scientific consensus that anthropogenic greenhouse gas emissions (GHGE) perturb global climate patterns with negative consequences for many natural ecosystems (1–3), reaching a global agreement regarding reduction of GHGE remains one of the most challenging problems humans face (4). International climate negotiations have largely failed to reach consensus (5, 6), evidencing a conflict between rich and poor countries, which often do not agree on the urgency of emission reduction measures, given the scientific uncertainty regarding the impacts of climate change (7–10). Indeed, in the aftermath of the 15th Conference of Parties in Copenhagen/2009 one has observed a tendency of several governments to regard climate change as a problem of a distant future—2050—hence discounting (4) the actual risk of collective disaster—despite predictions that severe climate change consequences, such as increased occurrence of heat waves and droughts, for instance, may happen sooner (1).

The issue of reducing GHGE has been addressed recently, both experimentally and theoretically, by means of a threshold Public Goods Game (PGG) in which success requires overall cooperative collective action, and decisions must be made knowing that failure to cooperate implies a risk of overall collapse (10–18). Like many social dilemmas of collective action, any participant that curbs emissions pays a cost whereas the benefits are shared among everyone. Thus, the rational choice is to free ride on the benefits produced by others at their own expense (through abatement), leading to the well-known tragedy of the commons, where selfish behavior results in overexploitation of the public good (19, 20).

Both theory and experiment agree that risk perception plays a central role in escaping the tragedy of the commons (12, 13). Besides risk, the role of wealth inequality among participants has been recently investigated by means of economic experiments involving students from western, educated, industrialized, rich, and democratic (WEIRD) countries (a feature that may induce biases regarding behavior of subjects taking the role of poor countries) (10, 11). Games comprised groups of fixed size ($N = 6$) where participation was equally split between rich and poor individuals, whose different wealth resulted from two different start-up amounts of money made available to group participants. The insights provided by these experiments (10, 11) (using different methodologies and assumptions while using the same PGG) converge on the idea that resolution of the climate change policy problem stems from the rich compensating for the smaller contribution by the poor and, even when risk is very high (something that does not seem to apply to the present situation), there is still a very significant chance of failing to solve the climate change dilemma, a situation that is ameliorated whenever intermediate tasks are designated (11) or whenever individuals have the opportunity to pledge their contribution before actual action (10).

Here we address the issue of wealth inequality from a theoretical perspective. The model we extend here to deal with wealth inequality (13, 17) has been shown to lead to predictions that correlate nicely with previous economic experiments carried out in the absence of any wealth inequalities (10), with the added value of allowing a full exploration of how success in addressing the climate change dilemma depends on other important parameters, such as risk, group size, introduction of sanctioning institutions of global or local nature, etc. It is important to stress that,

Significance

One of the greatest challenges in addressing global environmental problems such as climate change, which involves public goods and common-pool resources, is achieving cooperation among peoples. There are great disparities in wealth among nations, and this heterogeneity can make agreements much more difficult to achieve (e.g., regarding implementation of climate change mitigation). This paper incorporates wealth inequality into a public goods dilemma, including an asymmetric distribution of wealth representative of existing inequalities among nations. Without homophily (imitation of like agents), inequality actually makes cooperation easier to achieve; homophily, however, can undercut this, leading to collapse because poor agents may contribute less. Understanding such effects may enhance the ability to achieve agreements on climate change and other issues.

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despite the limited number of scenarios realizable in the laboratory, data stemming from behavioral experiments have provided crucial insights, not only because they unravel human behavior when confronted with climate change issues, but also because they provide important guidelines that help in calibrating theoretical models, such as the one we use here. Indeed, here we model climate change negotiations in the framework of Evolutionary Game Theory, where individuals exhibit a well-defined behavior (C or D), as a result of which they accumulate a certain payoff resulting from the game group interactions. Regularly, every individual A compares her/his payoff with that of a randomly chosen individual B, imitating (or not) the behavior of B with a probability that is a growing function of the payoff difference between B and A. Naturally, the larger the payoff of individual B (randomly) selected as role model, the more likely it is that A imitates her/his behavior (see *Methods* for specific details of the update rule).

Let us consider a population of finite size Z of which Z_R individuals are rich and $Z_P = Z - Z_R$ individuals are poor. Individuals are randomly sampled from the population and organize into groups of size N. Each rich individual starts with an initial endowment b_R whereas each "poor" individual starts with b_P, with $b_R > b_P$. These endowments may be used (or not) by an individual to contribute to reducing GHGE in her own group. We distinguish two types of behavior: (*i*) cooperators (Cs), who contribute a certain fraction c of their endowment to help solve the group task, and (*ii*) defectors (Ds) that do not contribute anything to solve the group task. Hence, the endowment is directly related to what each participant will lose if the next intermediate target is not met: Cs will lose $b_{R/P}(1-c)$ whereas Ds will lose the entire endowment $b_{R/P}$. If the overall amount of contributions in the group is above a certain threshold $Mc\bar{b}$ (where $\bar{b}$ is the average endowment of the population), the target will be met. Otherwise, with a probability r—the perception of risk of collective disaster (10, 12, 13)—individuals in the group will lose whatever they have.

This framework creates an interdependent behavioral ecosystem, where each player in a group knows what all other members of the group will do and where decisions and achievements of others influence one's own decisions (21–24). In particular, decisions taken by the poor can be potentially influenced by the actions and achievements of the rich (and vice versa), adding an additional coupling between these two subpopulations (details in *Methods*). In standard conditions, anyone in this population may influence and be influenced by anyone else. This, however, may not always be the case, in the sense that individuals may be more receptive to the behavior and decisions of those in the same wealth class, thus selecting preferentially those of their wealth class as peers. To this end we define a homophily parameter ($0 \le h \le 1$), such that when $h = 1$, individuals are restricted to influence (and be influenced) by those of the same wealth status, whereas when $h = 0$, no wealth discrimination takes place. Naturally, such influence dynamics occur in the presence of action errors (24) as well as other stochastic effects, such as random exploration of the strategy space, akin to behavioral mutations (25).

In *SI Text*, we show that in populations with a mixed composition of rich and poor (and for different combinations of Cs and Ds in each wealth group), the nature of the overall public goods dilemma faced by the rich subpopulation differs qualitatively from that faced by the poor subpopulation. In these limiting cases where a decoupling of the timescales associated with the dynamics of the rich and of the poor takes place, the rich generally face an N-person coordination dilemma between Cs and Ds, whereas (for most combinations of parameters) poor Cs and poor Ds engage in a coexistence dilemma. Such a diversity in the nature of the games played—due to, e.g., heterogeneous interaction patterns or resource distributions—will have strong implications on the emerging social dynamics, often promoting the chances of achieving cooperation in structured populations (13, 26, 27).

In practice, however, no reason other than mathematical simplicity may justify such extreme scenarios. When analyzing the fully coupled dynamics on the entire configuration space represented by a two-dimensional simplex (see, e.g., Fig. 2), in which the y axis (x axis) portrays the fraction of Cs among the poor (rich), it is natural to ask to which extent the existence of rich and poor alters the dynamics of the risky PGG at stake, compared with the standard model where no wealth inequality is explicitly considered. To answer this question, we start by recognizing that 20% of the world's wealthier countries produce approximately the same gross domestic product as the remaining 80%. Thus, we break the population into two wealth classes, such that the poor comprise 80% of the population, whereas the rich constitute the remaining 20%. Concomitantly, we assume that poor countries contribute an amount proportional to their wealth (as reflected in their initial endowment) and similarly with the rich. As a result, different groups will exhibit, on average, different ratios of rich and poor, reflecting the intrinsic wealth asymmetry that one observes in the real world.

In Fig. 1 we compare the average group achievement (η_G), that is, the fraction of time a group succeeds in achieving $Mc\bar{b}$ as a function of risk (*Methods*), in the cases when there is no wealth inequality (gray line) and in the presence of wealth inequality (blue and red lines). The results show unequivocally that wealth inequality may promote group success. This result, however, depends strongly on the level of homophily (h): Whenever the rich and poor are evenly influenced by anyone else (no homophily, $h = 0$, blue line), group achievement is enhanced for all values of risk (r). However, when the rich (poor) influence and are influenced by rich (poor) only (homophily $h = 1$, red line), the chances of success are generally below those attained in the absence of wealth inequality.

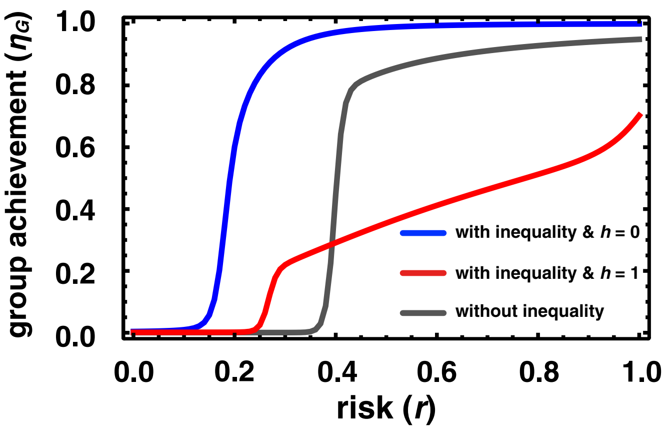

Fig. 1. Average group achievement η_G as a function of risk. The gray line shows the average group achievement in the case of no wealth inequality; that is, all individuals have an initial endowment $b = \bar{b} = 1$ and the cost of cooperation is $0.1b$. The blue line shows results for wealth inequality with the homophily parameter $h = 0$, whereas the red line shows results for $h = 1$. We split the population of $Z = 200$ individuals into $Z_R = 40$ rich (20%) and $Z_P = 160$ poor (80%); initial endowments are $b_R = 2.5$ and $b_P = 0.625$, ensuring that the average endowment $\bar{b}$ remains $\bar{b} = 1$ (used to generate the gray line); the cost of cooperation also remains, on average, $0.1\bar{b}$, which means $c_R = 0.1b_R$ and $c_P = 0.1b_P$. The results show that wealth inequality significantly enhances the average chance of group success in the absence of homophily ($h = 0$), whereas under homophily ($h = 1$) the fact that only like influences like brings the overall chances of success to levels generally below those under wealth equality. Other parameters (*Methods*) are $N = 6$ and $M = 3$.

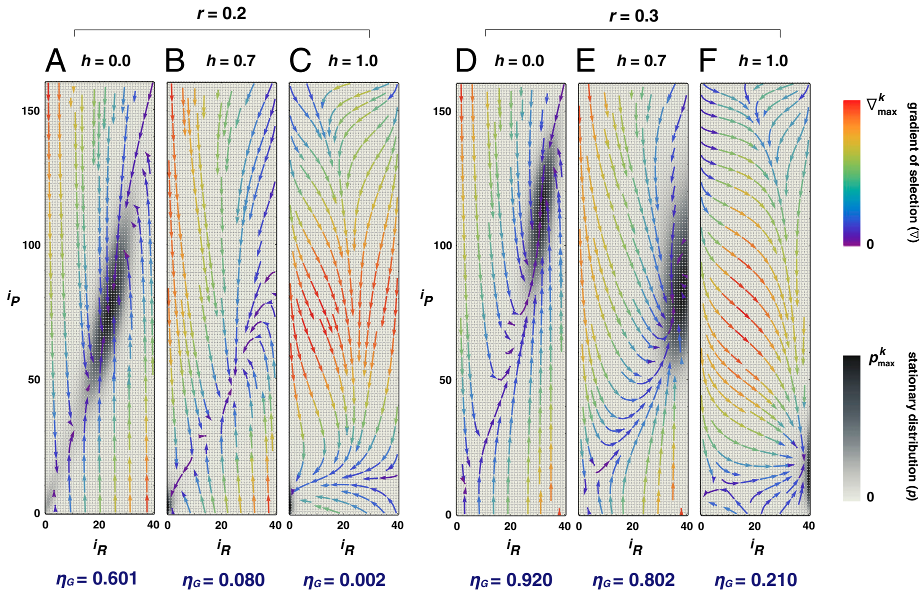

Fig. 2. Stationary distribution and gradient of selection for different values of risk r and of the homophily parameter h. (*A–F*) Each panel contains all possible configurations of the population (in total $Z_R \times Z_P$), each specified by the number of rich (i_R) and poor (i_P) it contains and represented by a gray-colored dot. Darker dots represent those configurations in which the population spends more time, thus providing a contour representation of the stationary distribution. The curved arrows show the so-called gradient of selection (∇), which provides the most likely direction of evolution from a given configuration. We use a color code in which red lines are associated with higher speed of transitions. The behavioral dynamics of the population depend on the homophily parameter h in a nonlinear way. For $h \leq 0.5$, the results remain qualitatively similar to those depicted for $h = 0$, in which case everybody influences and is influenced by everybody else. In this case, the contribution of the rich is sizeable, which also leads the poor to contribute. For $h > 0.5$ the behavior changes abruptly, and one witnesses the rapid collapse of cooperation among the poor and, for low risk ($r = 0.2$, *A–C*), an ensuing disappearance of contributions to the overall PGG, with the population spending most of the time in full defection, leading to a dramatic impact on the overall group achievement η_G, indicated below each contour plot. However, a slight increase in overall risk perception (here $r = 0.3$, *D–F*) actually impels the rich to contribute, despite the fact that the poor still do not cooperate. Other parameters: $Z = 200$; $Z_R = 40$; $Z_P = 160$; $c = 0.1$; $N = 6$; $M = 3c\bar{b}$ ($\bar{b} = 1$); $b_P = 0.625$; $b_R = 2.5$; $p^k_{max} = \{p^A_{max}, \cdots, p^F_{max}\} = \{2, 40, 75, 3, 2, 20\} \times 10^{-3}$; and $\nabla^k_{max} = \{\nabla^A_{max}, \cdots, \nabla^F_{max}\} = \{16, 6, 2, 16, 6, 3\} \times 10^{-2}$.

Fig. 1 also shows that η_G depends strongly on homophily, a feature that plays a central role in the overall dynamics, as is discussed in more detail below. However, it is also important to understand how the contributions are split between the rich and the poor, a feature that is not possible to grasp directly from η_G. To this end we now study in detail the stationary distributions associated with the dynamics of the two-subpopulation model. The results are shown in Fig. 2. Each (discrete) configuration is represented by a small circle, colored in gray tones. Darker circles indicate those configurations visited more often, providing a representation of the full stationary distribution ($\bar{p}$, *Methods*), i.e., the prevalence in time of each possible configuration of the population. Arrows in each simplex represent, in turn, the most probable direction of evolution starting from a given configuration (the gradient of selection ∇, *Methods*). For each arrow, we adopt a continuous color code associated with the likelihood of such a transition (brighter colors indicate more likely transitions).

Fig. 2 shows that, even in the absence of significant homophily bias ($h \leq 0.5$) a higher fraction of rich contribute (with average values of 57% for $r = 0.2$ and 78%, for $r = 0.3$), compared with the poor (with average values of 46% for $r = 0.2$ and 69% for $r = 0.3$), thus also protecting their greater wealth. This result does not depend on risk; however, for low risk, the overall contribution is limited, increasing significantly after a slight increase in overall risk perception. Indeed, in the absence of homophily, cooperation may prevail in a wealth-unequal world (e.g., Fig. 2*D*).

Qualitatively, one can now understand the results in Fig. 1 if one takes into account that, in most cases, the dynamics both among the rich and among the poor can become dominated by basins of attraction that lead to a coexistence between *C*s and *D*s (*SI Text*). Whenever the risk is moderate to high, there is an increase of the size of such basins, with a corresponding increase of the stationary fraction of cooperators, such that the feedback dynamics between the poor and the rich act to build up the cooperation levels among both subpopulations. In other words, the poor pave the way for the rich to cooperate, which, in turn, feeds back into the poor, also increasing their levels of cooperation. This feedback occurs because, perhaps counterintuitively, not only the poor imitate the rich, but also the rich imitate the poor. In fact, it is easy to prove that, for the model considered, the rich imitate the poor more often than the poor imitate the rich.

As also shown in Fig. 2, this positive feedback between the two subpopulations is interrupted whenever homophily becomes dominant ($h \sim 1$). When rich and poor cease to be able to sway one another, we observe two distinct scenarios: At low risk ($r = 0.2$ in Fig. 2) overall cooperation collapses. With a slight increase in risk perception, however ($r = 0.3$), the rich contribute, despite the fact that the poor do not. Together with risk, a lack of homophily plays an important role: As soon as the homophily constraint is relaxed—by adopting $h < 1$—poor individuals start to be nudged by the successes of the rich, effectively inducing the poor players to contribute to the common good.

However, even in the absence of homophily ($h = 0$), this positive feedback between the two subpopulations does not always lead to an increase of cooperation—thus we obtain the coexistence dynamics shown in Fig. 2. Indeed, whenever most poor opt for cooperation, the dynamics drive rich countries toward less cooperation, given that they may now profit from the larger overall contributions stemming from the poor. Similar dynamics may also occur among the poor. This reduction, however, not only does not prevent the majority of rich from engaging in cooperation, but also does not compromise the overall group achievement values. As a result of these coupled dynamics, the population will stay most of the time nearby a coexistence equilibrium (interior attractor, Fig. 2 *A*, *D*, and *E*).

This said, we are all aware that some individuals may be more receptive than others to change their mind, based on the influence of their peers. In fact, some individuals—for various reasons, as witnessed in the world summits on climate change that have taken place to date—may maintain the same behavior irrespective of their sphere of influence. Given the small size of the overall population, such an obstinate behavior may lead to sizeable effects in the global dynamics. In the following we investigate how such obstinate behaviors (in both wealth classes) affect the overall dynamics. For simplicity, we assume that, in all cases, obstinate behavior amounts to 10% of individuals in one subpopulation—which corresponds to the same fixed contribution to the PGG, considering either rich or poor obstinate players—see *SI Text* for a more detailed analysis.

Fig. 3 shows that obstinate poor cooperators provide impressive improvements in the aggregate propensity of the population to achieve coordination ($\eta_G = 0.581$ compared with $\eta_G = 0.004$ in the absence of obstinate individuals), larger than obstinate rich cooperators, who lead to less pronounced enhancements ($\eta_G = 0.223$). This effect, which extends qualitatively to all values of h, is more pronounced when $h = 1$, as is the case in Fig. 3.

The trend shown in Fig. 3 is qualitatively inverted in the case of obstinate defectors who, in general, are detrimental to overall cooperation and group achievement (details in *SI Text*). These results are largely independent of the parameters chosen and

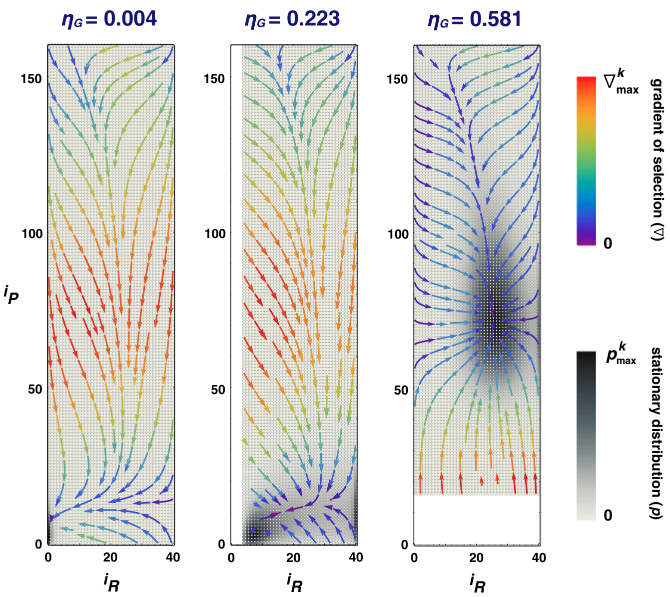

Fig. 3. Stationary distribution and gradient of selection for populations comprising 10% of individuals exhibiting an obstinate cooperative behavior. Same notation as in Fig. 2 is used. Whenever 10% of individuals exhibit obstinate cooperative behavior (*Center* and *Right* contours), the number of configurations of the population in which the evolutionary dynamics proceed is correspondingly reduced (white areas in contours). The *Left* contour contains no obstinate individuals and is displayed for reference. In the *Center* contour, 10% of the rich individuals behave as obstinate cooperators; that is, they never change their behavior. In the *Right* contour, 10% of poor individuals exhibit such behavior. A small fraction of obstinate rich and obstinate poor cooperators lead to very different outcomes, also for the average group achievement η_G. Indeed, the chances of success are significantly enhanced whenever obstinate cooperator behavior occurs among the poor. The effect is most pronounced whenever individuals are homophilic, as is the case here ($h = 1$). Other parameters: $Z = 200$; $Z_R = 40$; $Z_P = 160$; $c = 0.1$; $N = 6$; $M = 3c\bar{b}$ ($\bar{b} = 1$); $b_P = 0.625$; $b_R = 2.5$; $r = 0.2$; $\beta = 5.0$; $p^k_{\max} = \{p^A_{\max}, p^B_{\max}, p^C_{\max}\} = \{76, 4, 2\} \times 10^{-3}$; and $\nabla^k_{\max} = \{\nabla^A_{\max}, \nabla^B_{\max}, \nabla^C_{\max}\} = \{3, 3, 4\} \times 10^{-2}$.

highlight the important role that obstinate defectors among the rich and obstinate cooperators among the poor may play in the outcome of climate negotiations.

In summary, homophily generally impels the rich to compensate for the poor. Given that contributions from the poor are crucial to solving the climate change problem we face, it is then imperative that homophilic behavior is avoided (it is noteworthy that it is enough that overall behavior is not purely homophilic for homophily to be effectively avoided). Moreover, a small fraction of obstinate poor cooperators leads to sizable increases in the overall prospects for success, mostly when homophily rules. Perhaps unsurprisingly, when the contribution of the poor is widespread, the rich refrain from contributing. Certainly, David Hume would not be impressed by this feature that emerges from the game dynamics.

Conventional wisdom would lead one to believe that wealth inequality and homophily would constitute important obstacles regarding overall cooperation in climate change negotiations. Our results predict that, as long as (*i*) risk perception is high; (*ii*) climate negotiations are partitioned in smaller groups agreeing on local, short-term targets; and (*iii*) individuals are influenced by their more successful peers, whom they imitate—irrespective of their wealth class—and making errors while doing so, the prospects are not that grim. On the contrary we find that, under such conditions, cooperation may outcompete defection, benefiting from wealth inequality. Thus, hope remains that the problem may be overcome. Moreover, the qualitative nature of the results obtained here remains robust if we assume that, instead of proportional contributions, poor and rich contribute the same amount, when cooperating.

Our model, however, ignores an important factor: that the thresholds may be intrinsically uncertain. This uncertainty, if sizeable, can destroy cooperation, as sharply demonstrated recently, both theoretically and experimentally (9). Likely, to the extent that agreements aim at short-term targets involving smaller groups, it will also be easier to narrow down threshold uncertainties. Nonetheless, and in the absence of wealth inequality, introducing threshold uncertainty into our model leads to the same scenarios predicted (and confirmed) in ref. 9 (*SI Text*).

Finally, the recent report of the Intergovernmental Panel for Climate Change (28), besides emphasizing that climate change is real and humans are the main cause of it, urging countries to stop the warming of the planet, has also attempted to narrow down the threshold uncertainty. However, given that risk perception is low and that a bottom–up approach [as defended by the late Elinor Ostrom (29) and also, indirectly, by the results of the present model] has yet to spread globally, it is perhaps not surprising that today's prospects remain gloomy. Clearly it is urgent that individuals become aware of the true risk that we face. Indeed, an increase in risk perception will surely promote the development of local initiatives that may foster overall cooperation by extending the bottom–up approach to all players of the global game.

Methods

We consider a population of Z individuals, Z_R of which are considered rich (initial endowment b_R) and Z_P considered poor (initial endowment b_P) who, together, set up groups of size N, in which they engage in the climate change threshold PGG (12, 13). Each individual is capable of adopting one of the two strategies: C and D. Following the discussion in the main text, and given that rich Cs contribute $c_R = cb_R$ whereas poor Cs contribute $c_P = cb_P$, the payoff of an individual playing in a group in which there are j_R rich Cs, j_P poor Cs, and $N - j_R - j_C$ Ds, can be written as $\Pi^D_{R/P} = b_{R/P}\{\Theta(\Delta) + (1-r)[1-\Theta(\Delta)]\}$ and $\Pi^C_{R/P} = \Pi^C_{R/P} - c_{R/P}$ $(\Delta = c_R j_R + c_P j_P - Mc\bar{b})$, for rich/poor Ds and Cs, respectively. In the equations above, $\Theta(k)$ is the Heaviside function [that is, $\Theta(k) = 1$ whenever $k \geq 0$, being zero otherwise], $0 < M \leq N$ is a positive integer, $\bar{b}$ is the average endowment ($Z\,\bar{b} = Z_R b_R + Z_P b_P$), and r (the perception of risk) is a real parameter varying between 0 and 1; the parameters $c < 1$, $\bar{b}$, b_R, and b_P are all positive real numbers. Finally, the fitness f^X of an individual adopting a given strategy, X, will be associated with the average payoff of that strategy in the population. The average payoff can be computed for a given strategy in a configuration $\mathbf{i} = \{i_R, i_P\}$, using a multivariate hypergeometric sampling (without replacement) (details in *SI Text*). The number of individuals adopting a given strategy will evolve in time according to a stochastic birth–death process combined with the pairwise comparison rule (24, 30), which describes the social dynamics of rich *C*s, poor *C*s, rich *D*s, and poor *D*s in a finite population. Under pairwise comparison, each individual of strategy X adopts the strategy Y of another member of the population, with probability given by the Fermi function $(1+e^{\beta(f^X-f^Y)})^{-1}$, where β controls the intensity of selection ($\beta = 3$ in Figs. 1 and 2, $\beta = 5$ in Fig. 3). In the absence of homophily, the strategy Y is chosen at random with uniform probability. For a finite value of the homophily parameter h, individuals of the same wealth class are chosen with probability 1 whereas individuals of the other wealth class are chosen with probability $1 - h$; thus, when homophily is maximum, the choice occurs only among the individuals of the same wealth class (rich or poor) (details in *SI Text*). Additionally, we consider that, with a mutation probability μ ($\mu = 1/Z$ in Figs. 1–3), individuals adopt a randomly chosen strategy. As the evolution of the system depends only on its actual configuration, evolutionary dynamics can be described as a Markov process over a two-dimensional space. Its probability distribution function, $p_{\mathbf{i}}(t)$, which provides information on the prevalence of each configuration at time t, obeys a master equation (details in *SI Text*), a gain–loss equation involving the transition rates between all accessible configurations (24, 31, 32). The stationary distribution $\bar{p}_{\mathbf{i}}$ is then obtained by reducing the master equation to an eigenvector search problem (31) (details in *SI Text*). Another central quantity that portrays the overall evolutionary dynamics in the space of all possible configurations is the gradient of selection $\nabla_{\mathbf{i}}$. For each configuration $\mathbf{i}$, we compute the most likely path the population will follow, resorting to the probability to increase (decrease) the number of individuals adopting a strategy S_k, $T_{\mathbf{i}}^{S_k+}$ ($T_{\mathbf{i}}^{S_k-}$) in each time step. Additionally, for each possible configuration $\mathbf{i}$, we make use of multivariate hypergeometric sampling to compute the (average) fraction of groups that reach a total of $Mc\bar{b}$ in contributions, that is, that successfully achieve the public good—which we designate by $a_G(\mathbf{i})$. Average group achievement—η_G—is then computed, averaging over all possible configurations $\mathbf{i}$, each weighted with the corresponding stationary distribution $\eta_G = \sum_{\mathbf{i}} \bar{p}_{\mathbf{i}} a_G(\mathbf{i})$.

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Supporting Information

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SI Text

Theoretical and Experimental Studies of Behavior Regarding Climate Change

Several experiments have been performed to understand human behavior in dealing with global warming (1–4). Social dilemmas involving collective action were set up in a repeated game framework, where a given threshold had to be surpassed—otherwise, there was a variable risk (externally defined) of everyone losing all their endowments. The first results have shown that, most of the time, disaster was not avoided (1), risk being an important factor in promoting disaster avoidance. Later on the possibility to make pledges was introduced in the same repeated threshold public goods game, showing that pledges led to an increase of cooperation (despite the possibility of acting differently from what was pledged) (2). However, when the same treatments were run using players with different wealth, this improvement was demoted (2). Using the same game settings, two time horizons were introduced into the dilemma (3), showing that fixing intermediate goals in climate agreements is beneficial, although the final target is reached less often than the intermediate target. More recently, a nonrepeated threshold public goods game experiment was performed in which the effects of both impact uncertainty (where individuals could lose a random amount of money if the threshold was not met) and threshold uncertainty (where the threshold was a random value) were investigated (4). The authors set up the experiment based on results from a theoretical analysis of such a game (assuming fully rational individuals engaging in a one-shot threshold public goods game) that predicted the existence of a critical value for the threshold uncertainty above which cooperation would collapse. Experiments fully confirmed this catastrophic prediction.

Despite the limited number of scenarios realizable in the laboratory, data stemming from behavioral experiments have provided crucial insights, not only because they unravel human behavior when confronted with climate change issues, but also because they provide important guidelines toward developing theoretical models (5–9). Indeed, a dynamical approach to the problem of cooperating to tame the planet's climate was developed, inspired by the intriguing results that experiments were revealing (5–7), allowing one not only to generalize the experimental settings to scenarios that are more difficult to realize in the laboratory, but also to predict what the impact of different approaches to the solution of the climate change problem may bring. Namely, the effect of risk perception and the disruptive power of uncertainty have been captured in the models (*SI Text*, *Threshold Uncertainty*). Theoretical models also extended the experimental insights by predicting the importance of small groups and stringent requirements in improving cooperation, as well as the role and scale of sanctioning institutions in supervising agreements. In keeping with this discussion, the present model brings additional information to this important subject.

In particular, although conventional wisdom would lead one to believe that wealth inequality and homophily would constitute important obstacles regarding overall cooperation in climate change negotiations, the present model predicts that, as long as (*i*) risk perception is high; (*ii*) climate negotiations are partitioned into smaller groups agreeing on local, short-term targets; and (*iii*) individuals are influenced by their more successful peers, whom they imitate—irrespectively of their wealth class—and making errors while doing so, the prospects are not that grim. On the contrary we find that, under such conditions, cooperation may outcompete defection, benefiting from wealth inequality. On the other hand, and to the extent that agreements aim at short-term targets involving smaller groups, it may also become easier to narrow down threshold uncertainties that, if large, do haunt overall cooperation (4) (*SI Text*, *Threshold Uncertainty*).

Evolutionary Dynamics in Finite Populations Under Wealth Inequality, Uncertainty, and Homophily

Let us consider a population of Z individuals. As stated in the main text, each individual adopts one of the two possible strategies $X \in \{C, D\}$ and belongs to one of two possible wealth classes $k \in \{R, P\}$. Let us assume there are Z_R rich (with initial endowment b_R) and Z_P poor individuals (with an initial endowment b_P). These numbers will remain fixed. Individuals are given an initial endowment (with $b_P < b_R$) and play the climate threshold Public Goods Game (PGG) (1, 5), engaging in groups of size N. Following the discussion in the main text, and given that rich Cs contribute $c_R = cb_R$ whereas poor Cs contribute $c_P = cb_P$, the payoff of an individual playing in a group in which there are j_R rich Cs, j_P poor Cs, and $N - j_R - j_P$ Ds can be written as

$$\Pi^D_{R/P}(j_R, j_P) = b_{R/P}\{\Theta(c_R j_R + c_P j_P - Mc\bar{b}) + (1-r)[1 - \Theta(c_R j_R + c_P j_P - Mc\bar{b})]\}$$

and $\Pi^C_{R/P}(j_R, j_P) = \Pi^D_{R/P}(j_R, j_P) - c_{R/P}$, for rich/poor Cs and Ds, respectively. In the equations above, $\Theta(k)$ is the Heaviside function [that is, $\Theta(k) = 1$ whenever $k \geq 0$, being zero otherwise], $0 < M \leq N$ is a positive integer, $\bar{b}$ is the average endowment ($Z\bar{b} = Z_R b_R + Z_P b_P$), and r (the perception of risk) is a real parameter varying between 0 and 1; the parameters $0 < c < 1$, $\bar{b}$, b_R, and b_P are all real positive. Finally, the fitness f^X_k of an individual adopting a given strategy X in a population of wealth class k will be associated with the average payoff of that strategy in the entire population. This can be computed for a given configuration of strategies and wealth classes specified by $\mathbf{i} = \{i_R, i_P\}$, using a multivariate hypergeometric sampling (without replacement):

$$f^C_R(\mathbf{i}) = \binom{Z-1}{N-1}^{-1} \sum_{j_R=0}^{N-1} \sum_{j_P=0}^{N-1-j_R} \binom{i_R - 1}{j_R}\binom{i_P}{j_P} \times \binom{Z - i_R - i_P}{N-1-j_R-j_P} \Pi^C_R(j_R+1, j_P) \quad \textbf{[S1a]}$$

$$f^D_R(\mathbf{i}) = \binom{Z-1}{N-1}^{-1} \sum_{j_R=0}^{N-1} \sum_{j_P=0}^{N-1-j_R} \binom{i_R}{j_R}\binom{i_P}{j_P} \times \binom{Z-1-i_R-i_P}{N-1-j_R-j_P} \Pi^D_R(j_R, j_P) \quad \textbf{[S1b]}$$

$$f^C_P(\mathbf{i}) = \binom{Z-1}{N-1}^{-1} \sum_{j_R=0}^{N-1} \sum_{j_P=0}^{N-1-j_R} \binom{i_R}{j_R}\binom{i_P - 1}{j_P} \times \binom{Z - i_R - i_P}{N-1-j_R-j_P} \Pi^C_P(j_R, j_P+1) \quad \textbf{[S1c]}$$

$$f^D_P(\mathbf{i}) = \binom{Z-1}{N-1}^{-1} \sum_{j_R=0}^{N-1} \sum_{j_P=0}^{N-1-j_R} \binom{i_R}{j_R}\binom{i_P}{j_P} \times \binom{Z-1-i_R-i_P}{N-1-j_R-j_P} \Pi^D_P(j_R, j_P). \quad \textbf{[S1d]}$$

The number of individuals adopting a given strategy will evolve in time according to a stochastic birth–death process combined with the pairwise comparison rule (10, 11), which describes the social dynamics of rich *C*s, poor *C*s, rich *D*s, and poor *D*s in a finite population. Under pairwise comparison, each individual of strategy X adopts the strategy Y of a randomly selected member of the population, with probability given by the Fermi function $(1+e^{\beta(f_k^X-f_l^Y)})^{-1}$, for any wealth class $\{k,l\}\in\{R,P\}$, where β controls the intensity of selection. Additionally we consider that, with a mutation probability μ, individuals adopt a randomly chosen different strategy, in such a way that when $\mu=1$, the individual does change strategy. As the evolution of the system depends only on its actual configuration, evolutionary dynamics can be described as a Markov process over a two-dimensional space. Its probability distribution function, $p_{\mathbf{i}}(t)$, which provides information on the prevalence of each configuration at time t, obeys a master equation of the form

$$p_{\mathbf{i}}(t+\tau)-p_{\mathbf{i}}(t)=\sum_{\mathbf{i}'}\{T_{\mathbf{i}\mathbf{i}'}p_{\mathbf{i}'}(t)-T_{\mathbf{i}'\mathbf{i}}p_{\mathbf{i}}(t)\}, \qquad \textbf{[S2]}$$

a gain–loss equation that allows one to compute the evolution of $p_{\mathbf{i}}(t)$ given the transition probabilities per unit time τ from the configuration $\mathbf{i}$ to $\mathbf{i}'$, $T_{\mathbf{i}'\mathbf{i}}$ (11–13). The stationary distribution $\bar{p}_{\mathbf{i}}$ analyzed in the main text is obtained by making the left-hand side equal to zero, which transforms Eq. **S2** into an eigenvector search problem (12), namely, the eigenvector associated with the eigenvalue 1 of the transition matrix W (12) whose matrix elements W_{qp} are built from the transition probabilities per unit time $T_{\mathbf{i}'\mathbf{i}}$ in the following way: Let us enumerate each of all possible configurations $\mathbf{i}=\{i_R,i_P\}$ of the population by an integer number—we do so by defining a bijective function V such that $p=V(\mathbf{i})$ and $q=V(\mathbf{i}')$ and, therefore, $\mathbf{i}=V^{-1}(p)$ and $\mathbf{i}'=V^{-1}(q)$. Then, we may write $W_{qp}=T_{\mathbf{i}'\mathbf{i}}$. The transition probabilities $T_{\mathbf{i}'\mathbf{i}}$ can all be written in terms of the following expression, which gives the probability that an individual with strategy $X\in\{C,D\}$ in the subpopulation $k\in\{R,P\}$ changes to a different strategy $Y\in\{C,D\}$, both from the same subpopulation k and from the other population l (that is, $l=P$ if $k=R$, and $l=R$ if $k=P$):

$$T_k^{X\to Y}=\frac{i_k^X}{Z}\left((1-\mu)\left[\frac{i_k^Y}{Z_k-1+(1-h)Z_l}\left(1+e^{\beta(f_k^X-f_k^Y)}\right)^{-1}+\frac{(1-h)i_l^Y}{Z_k-1+(1-h)Z_l}\left(1+e^{\beta(f_k^X-f_l^Y)}\right)^{-1}\right]+\mu\right).$$

Thus, if the homophily is maximum ($h=1$), the imitation occurs only between individuals of the same wealth class (rich or poor), whereas $h=0$ means that everyone influences and may be influenced by anyone else.

Another central quantity—which portrays the overall evolutionary dynamics in the space of all possible configurations—is the gradient of selection $\nabla_{\mathbf{i}}$ (GoS). For each configuration $\mathbf{i}=\{i_R,i_P\}$, we compute the most likely path each subpopulation $k\in\{R,P\}$ will follow, resorting to the probability to increase (decrease) by one, in each time step, the number of cooperators for that configuration $\mathbf{i}$ of the population, which we denote by $T_{\mathbf{i},k}^{+}$ ($T_{\mathbf{i},k}^{-}$), such that

$$\nabla_{\mathbf{i}}=\left\{T_{\mathbf{i},R}^{+}-T_{\mathbf{i},R}^{-},\ \ T_{\mathbf{i},P}^{+}-T_{\mathbf{i},P}^{-}\right\}.$$

Finally, for each possible configuration $\mathbf{i}$, we make use of multivariate hypergeometric sampling (Eq. **S1**) to compute the (average) fraction of groups that reach a total of $Mc\bar{b}$ in contributions, that is, that successfully achieve the public good—which we designate by $a_G(\mathbf{i})$. Average group achievement—η_G—is then computed by averaging over all possible configurations $\mathbf{i}$, each weighted with the corresponding stationary distribution $\eta_G=\sum_{\mathbf{i}}\bar{p}_{\mathbf{i}}a_G(\mathbf{i})$.

Timescale Separation: Games Among the Rich and Among the Poor

To assess what games the rich play in the presence of the poor (*C*s and *D*s) and the poor play in the presence of the rich (*C*s and *D*s), we let each subpopulation evolve assuming that the rate of evolution of the other subpopulation is zero. The results are shown in Fig. S1, where we compute the gradient of selection (∇) that governs the evolutionary dynamics of the rich in the presence of frozen, mixed configurations of the poor (Fig. S1 *A* and *B*) and vice versa (Fig. S1 *C* and *D*). We consider the cases in which the population is subdivided into subpopulations of equal size ($Z_P=Z_R$, Fig. S1 *A* and *C*) or not ($Z_P=4Z_R$, as in the main text, Fig. S1 *B* and *D*).

The results in Fig. S1 show that the rich tend to be more cooperative as the difference between the endowments of both classes increases, whereas cooperation among the poor largely remains unaffected. Moreover, whereas the poor engage in a coexistence game in which overall cooperation decreases as cooperation among rich decreases, the dynamics of the rich are influenced by the relative size of the poor subpopulation. In general, the rich engage in an N-player stag-hunt game (14) with different degrees of coordination and coexistence, depending on the (fixed) fraction of poor cooperators. As a result, different combinations of parameters may transform the original N-player stag-hunt dilemma (14)—characterized by two internal roots—into a pure coordination or coexistence dilemma or even a defection dominance dilemma (Fig. S1 *A* and *C*). However, as discussed before (5, 6) and illustrated in Fig. 2 of the main text, the unstable fixed point can be overcome by stochastic effects—such as errors in imitation and random exploration of the strategy space—such that the population spends most of its time in the vicinity of the coexistence points. Thus, the prevalent levels of cooperation among the rich will be ultimately defined by the size of the cooperative basin attraction and the position of the respective coexistence root, both influenced by the dynamics occurring among the poor. It is also noteworthy that the gradients of the rich are 10 times smaller than those of the poor. This means the rate of response of the rich to changes is (on average) slower than that of the poor. In practice, the poor will adjust their behavior more rapidly to changes of the configuration of the rich, thus quickly shifting between the corresponding levels of coexistence between poor *C*s and *D*s.

Evolutionary Dynamics for the Same Amounts of Rich and Poor

In all experimental settings carried out to date, the fraction of rich and poor in each group was kept equal. Here we compute the analog situation in our model; that is, we compute the stationary distribution in the case when $Z_P=Z_R$. The results are shown in Fig. S2.

Comparison with Fig. 2 in the main text shows unequivocally that, for low risk ($r=0.2$), rich and poor populations of the same size lead to more pessimistic prospects concerning overall cooperation (compare the values of η_G below each panel). For a corresponding increase of risk perception (Fig. 2), we observe that, overall, cooperation remains below that observed for asymmetric subpopulation sizes. In particular, the rich cooperate less and are no longer able to compensate for the collapse of cooperation among the poor, a feature that becomes more pronounced with increasing homophily.

Threshold Uncertainty

In the absence of wealth inequality, all individuals are equivalent. This has been, to date, the most studied situation in the laboratory. In particular, in ref. 4 it has been demonstrated how, in a situation that all individuals in the group are equivalent, threshold uncertainty has a disruptive effect on the overall chances of cooperation. Below we show that this is also the case in our model.

Therefore, we modify the individual payoffs so that the games played have a random threshold, with a value drawn from a uniform probability distribution in the interval $[Mc\overline{b}-\delta,\ Mc\overline{b}+\delta]$. The larger the value of δ is, the larger the uncertainty associated with the threshold. Fig. S3*A* shows how this uncertainty induces a regime shift in the overall behavior of the population, from an N-player coordination game (5, 14, 15) where cooperators do have a chance toward a defection dominance dilemma. This shift leads, in turn, to a radical change in the profile of the stationary distribution, also shown in Fig. S3*B*. The impact of this threshold uncertainty on group achievement, $\eta_G(r)$, is shown in Fig. S4, corroborating the results obtained in ref. 4. The study of the effects of threshold uncertainty in the presence of wealth inequality, which is more complex given the problem that the threshold does not affect in the same way the rich and the poor, will be deferred to a future study.

Evolutionary Dynamics in the Presence of Obstinate Cooperators and Defectors

Here we investigate in more detail the role played by obstinate individual behavior in the population. We assume that a fixed fraction of individuals in the population exhibits obstinate behavior; that is, these individuals are not susceptible to changing their behavior in time. We compute, for all possible combinations, the overall group achievement η_G (*Methods* and *SI Text*, *Evolutionary Dynamics in Finite Populations Under Wealth Inequality, Uncertainty, and Homophily*) shown in Fig. S5 for three different values of risk (0.2, 0.3, and 0.4, each associated with a different line color) and for the fractions of obstinate individuals indicated in Fig. S5 *A–G*, *Insets*.

We carry out this analysis as a function of the homophily parameter h. The results corroborate the idea that obstinate Cs generally lead to positive effects in what concerns overall group achievement, whereas obstinate Ds lead to negative effects. Among these, obstinate poor Cs play a crucial role in sustaining cooperation, mostly when homophily is high ($h \sim 1$), whereas obstinate Ds are generally detrimental to overall cooperation.

Robustness of Results as a Function of *N*

In the following we investigate the dependence of our model results when we change group size and group threshold. To this end we compute, as a function of risk, the same curves that we plot in Fig. 1 of main text, for two group sizes ($N = 6$ and $N = 12$) and for several combinations of M and N, leading to six different scenarios. We use the same parameters as those used in Fig. 1; namely, we split the population of $Z = 200$ individuals into $Z_R = 40$ rich (20%) and $Z_P = 160$ poor (80%); initial endowments are $b_R = 2.5$ and $b_P = 0.625$, ensuring that the average endowment $\overline{b}$ remains $\overline{b} = 1$ (used to generate the gray line in Fig. 1 and Fig. S6); the cost of cooperation also remains, on average, $0.1\overline{b}$, which means $c_R = 0.1b_R$ and $c_P = 0.1b_P$. The results are shown in Fig. S6. Clearly, group size constitutes a very important parameter, because smaller groups lead to higher chances of success (5). Nonetheless, what we observe, in all cases, is that the message contained in Fig. 1 remains valid for all combinations of parameters shown: Wealth inequality without homophily (blue line) systematically fosters overall cooperation for lower values of risk than what is observed under wealth equality (gray line). Finally, homophilic and wealth-unequal subpopulations lead to the grimmest prospects for overall cooperation.

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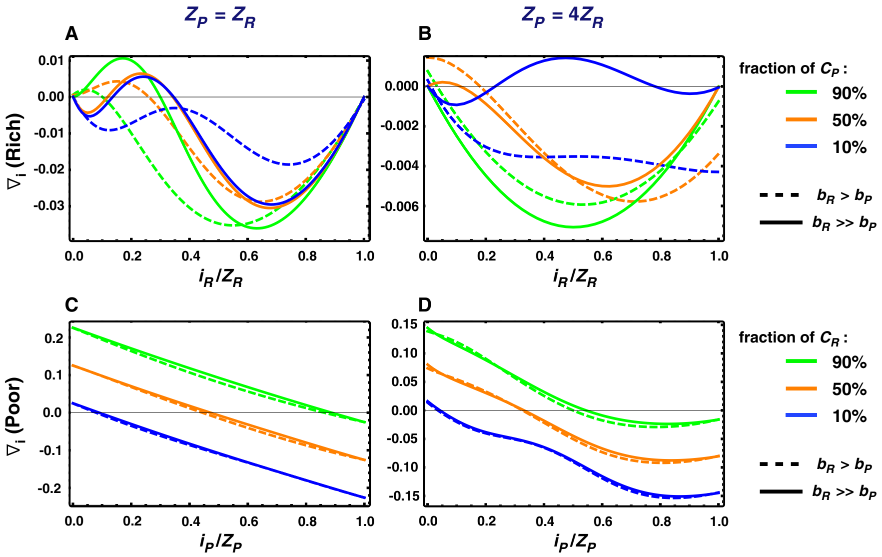

Fig. S1. Rich and poor play different games. (*A*–*D*) The gradient of selection (∇, *Methods*) in the case when one subpopulation (rich or poor) evolves while the rate of evolution of the other subpopulation is zero, in this way assessing which effective game each subpopulation engages in. (*A* and *B*) Number of poor (Z_P) and the fraction of *C*s (i_P/Z_P) in the poor population are kept constant (at three different values, legend at *Right*) while the fraction of *C*s (i_R/Z_R) and *D*s in the rich population is allowed to evolve. (*C* and *D*) Here we reverse the timescales and evolve the poor subpopulation, freezing the composition of the rich subpopulation. For each case we consider two regimes associated with different ratios for the initial endowments of rich and poor: $b_R > b_P$ ($b_R = 1.35$) and $b_R >> b_P$ ($b_R = 1.75$); imposing that $(b_R Z_R + b_P Z_P)/Z = \overline{b} = 1$ fixes the values of b_P for each case. (*A* and *C*) $Z_P = Z_R$. (*B* and *D*) $Z_P = 4Z_R$. Other model parameters: $Z = 200$, $N = 10$, $M = 3$, $\beta = 10$, $h = 0$, $r = 0.3$, $c_R = 0.1b_R$, and $c_P = 0.1b_P$.

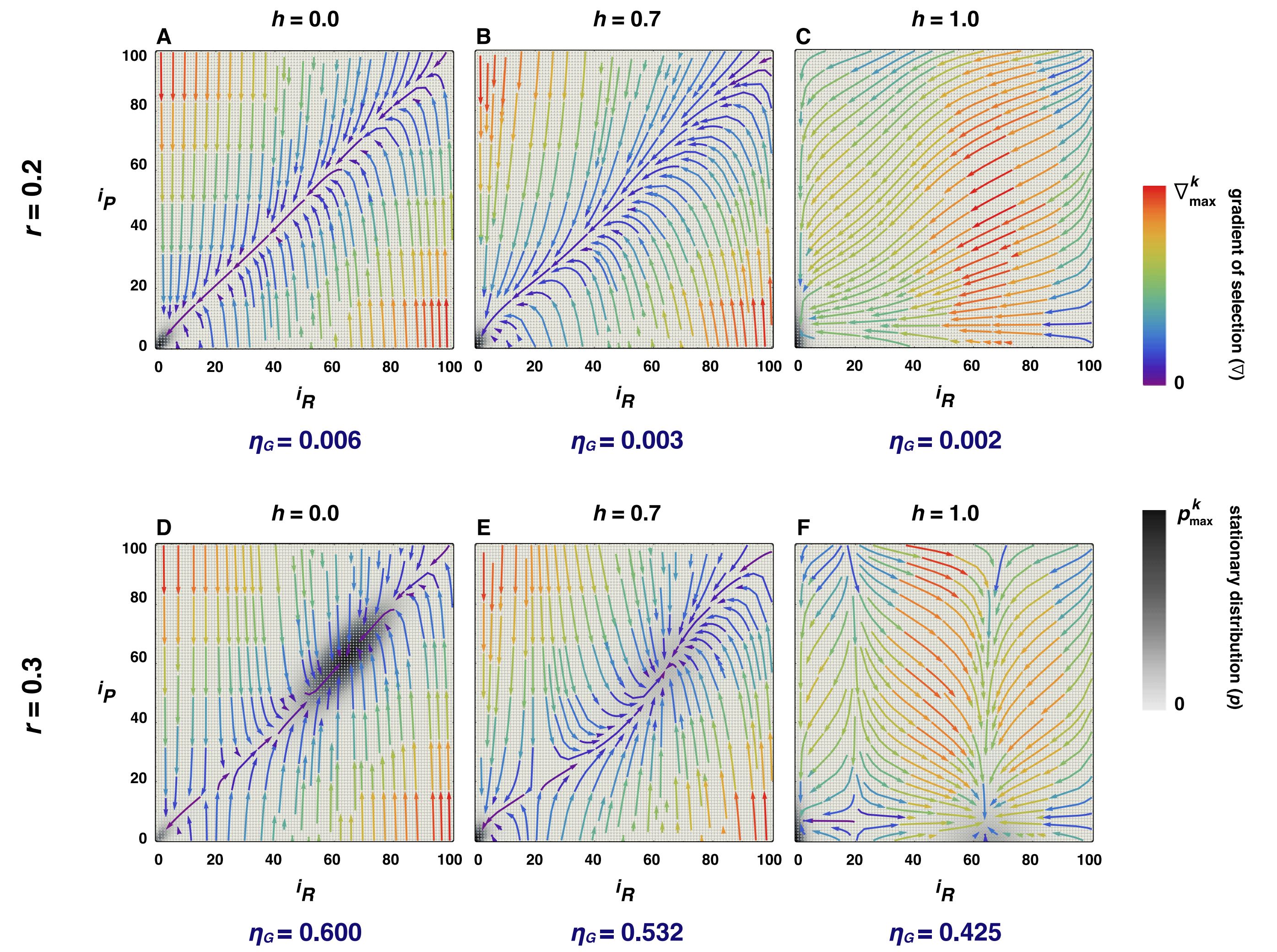

Fig. S2. Evolutionary dynamics for equal fractions of rich and poor. (*A–F*) The gradient of selection (∇, *Methods*) in the case when each subpopulation (rich or poor) evolves with the same number of individuals (compare with Fig. 2). Other model parameters: $Z = 200$, $Z_P = Z_R$, $N = 6$, $M = 3c\overline{b}$, $\beta = 5$, $\mu = 1/Z$, $c_R = 0.1b_R$, $c_P = 0.1b_P$, $b_R = 1.7$, and $b_P = 0.3$, ensuring that $(b_R Z_R + b_P Z_P)/Z = \overline{b} = 1$; $p_{\max}^{k=A\ldots F} = \{4.2, 5.3, 3.4, 0.2, 0.7, 1.6\} \times 10^{-2}$; and $\nabla_{\max}^{k=A\ldots F} = \{0.25, 0.12, 0.02, 0.25, 0.12, 0.02\}$.

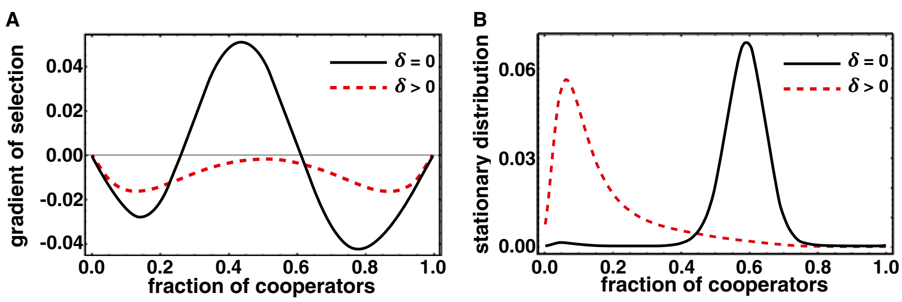

Fig. S3. Threshold uncertainty effect. (*A*) The gradient of selection; (*B*) the stationary distribution, that is, the fraction of time the population spends in each population composition specified by x. The black lines provide results for no threshold uncertainty ($\delta = 0$) whereas the red lines show results for $\delta = 2.75$. Other parameters are $Z = 200$, $N = 8$, $M = 4$, $c = 0.1$, $\overline{b} = 1.0$, $\beta = 6.0$, $\mu = 1/Z$, and $r = 0.6$.

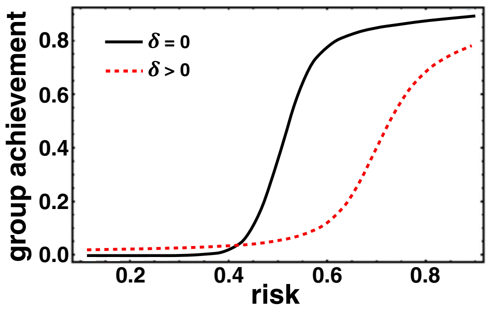

Fig. S4. Group achievement under threshold uncertainty. Results for the group achievement η_G as a function of the risk r are shown. Same parameters as in Fig. S3 are used. Clearly, there is no chance of cooperation before the risk promotes cooperation in an otherwise defection-dominant dilemma imposed by threshold uncertainty.

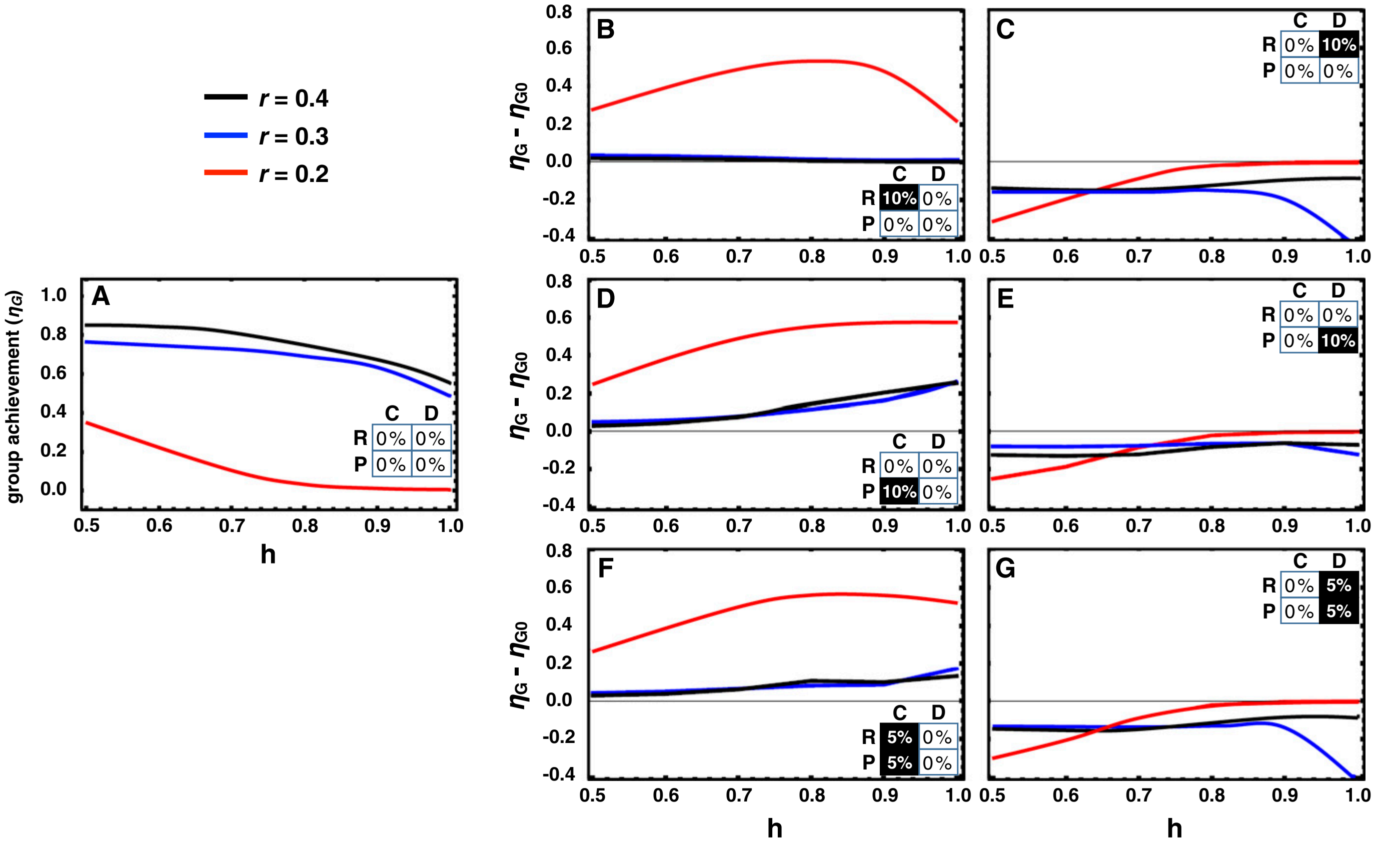

Fig. S5. (*A–G*) Average group achievement in the presence of obstinate players. *B–G* show the difference between the average group achievement (η_G, *Methods* and *SI Text, Evolutionary Dynamics in Finite Populations Under Wealth Inequality, Uncertainty, and Homophily*) in the presence of players whose (obstinate) behavior remains unchanged throughout the evolution of the population and the average group achievement (η_{G0}) computed in the absence of these types of individuals (*A*). Results are shown for the different values of risk indicated. Sizable positive effects (in particular for low risk) are obtained whenever obstinate behavior occurs among cooperators. Other parameters: $Z = 200$, $Z_P = 4Z_R$, $N = 6$, $M = 3$, $\beta = 10$, $\mu = 1/Z$, $c_R = 0.1b_R$, $c_P = 0.1b_P$, $b_R = 2.5$, and $b_P = 0.625$.

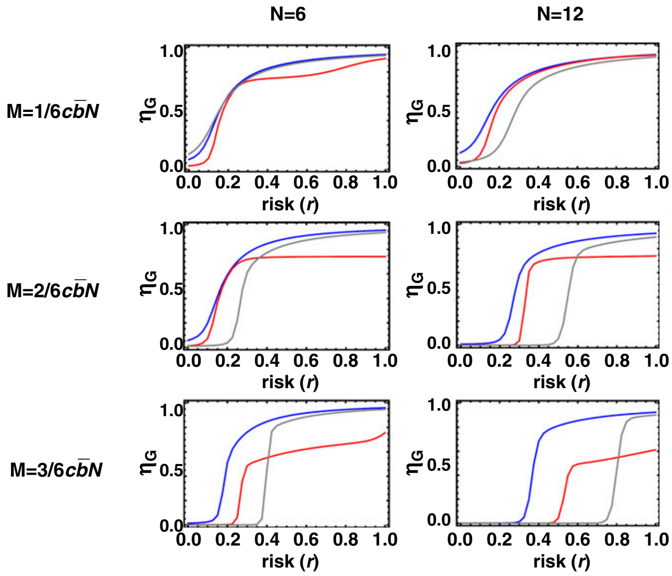

Fig. S6. Robustness of results with respect to N and M. Different panels show the average group achievement (η_G, *Methods* and *SI Text*, *Evolutionary Dynamics in Finite Populations Under Wealth Inequality, Uncertainty, and Homophily*) as a function of risk r, for different group sizes ($N = 6$, *Left*; and $N = 12$, *Right*) and, in each case, for three different values of the sharp threshold M: $M = 1$ (*Top*), $M = 2$ (*Middle*), and $M = 3$ (*Bottom*). As group size N increases, cooperation blooms for larger values of r, a behavior that is more pronounced for larger values of M. Nonetheless, and irrespective of the dependences in N and M exhibited, the overall behavior remains qualitatively identical to that found in Fig. 1 of the main text: Wealth inequality without homophily (blue line) systematically fosters overall cooperation for lower values of risk than what is observed under wealth equality (gray line). Finally, wealth-unequal but homophilic groups lead to the grimmest prospects for overall cooperation. Other parameters: $Z = 200$, $Z_P = 4Z_R$, $\beta = 10$, $\mu = 1/Z$, $c_R = 0.1b_R$, $c_P = 0.1b_P$, $b_R = 2.5$, $b_P = 0.625$.

D FLASHBACK AND OTHER WORKS

Being part of an interdisciplinary group as the ATP-group provided me opportunities that I could not even dream of before joining in. Some of the opportunities I was able to grasp, many others I was not, but I was always immersed in a creative, diverse and fruitful environment.

In fact, in the day I officially started my Ph.D., I was with my advisors, Jorge M. Pacheco and Francisco C. Santos, and also Flávio L. Pinheiro, at the time another Ph.D. student and ATP-group member as I was, in a Program at the Kavli Institute for Theoretical Physics, at the University of California Santa Barbara. There, I had the chance to meet people with diverse interests and ability to speak the language of different subjects. In no time, I was involved in both computing parities of numbers in sequences, with Greg Huber, and, at the same time, studying and trying to comprehend the biology of choanoflagellates, and the origins of multicellularity, in a problem introduced Mimi Koehl. In that same trip, I had the pleasure to share an office with my advisor, Jorge Pacheco, and directly work with Brian Skyrms at the Department of Logic and Philosophy of Science, University of California Irvine. Out of those discussions – that would many times happen with the taste of a cappuccino – a paper was born:

Jorge M. Pacheco, Vítor V. Vasconcelos, Francisco C. Santos, and Brian Skyrms. «Co-evolutionary Dynamics of Collective Action with Signaling for a Quorum.» In: PLoS Comput Biol 11.2 (2015), e1004101.

The knowledge I acquired during this project was immense, and the technical part was my hands-on Small Mutation Approximation. Only later, I would find out that it was crucial in the development on the hierarchical approach we have now developed to extend it and that is part of this dissertation.

Clearly, those are not the only examples of the opportunities I had. I can recall that about a year later, this time at Princeton University, while I was working with Phillip M. Hannam on a model he and Jorge M. Pacheco developed with Simon A. Levin[46], I had the pleasure to co-advise, with Simon A. Levin, a senior undergraduate student Kashyap Rajagopal in a project called "Homophily in Climate Change Negotiations: A Probabilistic SIR Model". Even though I was familiar with the Climate Negotiations by then, it was fun to learn some connections between the EGT models and epidemic models. From those days, I remember one evening at the hall of Simon's Lab trying to sketch a proof for the dependence of diffusion on an N-step process. It was when Lisa C. McManus showed up and we talked about her project. An unexpected discussion, about coral and how it grows in a network of reefs, ended up in an on-going collaboration,

Lisa C. McManus, James R. Watson, Vítor V. Vasconcelos, Simon A. Levin, «Larval Dispersal as a mechanism for coral persistence on reef communities» at 13th International Coral Reef Symposium.

During most of the period of my Ph.D., the office was never one place, whether in location, or configuration of ideas. It had Physicists, a Computer Scientist and a Geographer – and their collaborators. Lots of conversations always flowed, on different topics, whether at office hours, lunch time, or coffee time. I ended up getting interested in a lot of topics, but also in other branches or approaches of the study of cooperation:

Flavio L. Pinheiro, Vitor V. Vasconcelos, Francisco C. Santos, Jorge M. Pacheco. «Self-organized game dynamics in complex networks.» In: Advances in Artificial Life, ECAL 12 pp61-62;

Flávio L. Pinheiro, Vítor V. Vasconcelos, Francisco C. Santos, Jorge M. Pacheco. «Evolution of All-or-None Strategies in Repeated Public Goods Dilemmas.» In: PLoS Comput Biol 10.11 (2014), e1003945.

I cannot say how much I enjoyed all the small things I did, like learning how to run parallel simulations in the ATP-group's cluster; implementing fractal analysis tools for cities with Sara Encarnação; how to program cooperatively – and, after all, my thesis is about cooperation – using tools like github to develop packages with Fernando P. Santos; or simply all the discussions the whole ATP-group I know had.

Overall, and even though I remember the times of studying, programming and bug finding, writing and rewriting, I still feel like all of those successful projects and collaborations happened naturally and I know this was only possible for being where I was and with the people with whom I was.

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