Hypercomplex polynomials, Vietoris' rational numbers and a related integer numbers sequence

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Dedicated to Frank Sommen on the occasion of his 60th birthday

Abstract. This paper aims to give new insights into homogeneous hypercomplex Appell polynomials through the study of some interesting arithmetical properties of their coefficients. Here Appell polynomials are introduced as constituting a hypercomplex generalized geometric series whose fundamental role sometimes seems to have been neglected. Surprisingly, in the simplest non-commutative case their rational coefficient sequence reduces to a coefficient sequence $\mathcal{S}$ used in a celebrated theorem on positive trigonometric sums by L. Vietoris in 1958. For $\mathcal{S}$ a generating function is obtained which allows to derive an interesting relation to a result deduced in 1974 by Askey and Steinig about some trigonometric series. The further study of $\mathcal{S}$ is concerned with a sequence of integers leading to its irreducible representation and its relation to central binomial coefficients.

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No matter how you twist and turn, you cannot avoid using some definite analytic representation, such as power series.
K.Weierstrass
Math. Seminar 28. Mai 1884[1]

[1]Quoted by R. Siegmund-Schultze in: Ausgewählte Kapitel aus der Funktionenlehre, Teubner, Leipzig, 1988, p. 253, including the re-publication of Weierstrass' "Zur Funktionentheorie," published in Mittag-Leffler's commemorative issue of Acta Mathematica, 45 (1925), pp. 1-10.

1. Introduction

Polynomials are an indispensable tool in all fields of multivariate analysis. They can have some peculiarities due to the large variety of different possible structures and the aim of their application. This is particularly true if we look to power series expansions in the context of Hypercomplex Function Theory (cf. [5, 10, 16]) where, in general, bases of homogeneous polynomials are used. For example, considering Clifford algebra valued polynomials as polynomials of one paravector variable (isomorphic to a vector of $(n+1)$ real variables) is different from studying them as polynomials of several hypercomplex variables like n Fueter variables [15, 16]. A more detailed discussion of both approaches and their role in the definition of holomorphic functions in the sense of Hypercomplex Function Theory (traditionally called *monogenic* functions) can be found in [19].

In general, the algebraic (symmetry, invariance etc.) or analytical properties (growth, zeros etc.) as well as their relevance to geometric applications (in quasi-conformal mapping problems, for example) are object of research on different types of polynomials in the context of Hypercomplex Function Theory. But also the arithmetical properties of the coefficients, which characterize and distinguish special classes of polynomials are of great interest. Last but not least, properties expressed in terms of differentiability or integrability relations, being central questions in the Umbral calculus of polynomials, are also challenging fields for research on hypercomplex polynomials. To stress herein the role of Appell polynomial sequences we refer, without being exhaustive, to [1, 21] for the classical one dimensional case and [3, 7, 8, 11, 13, 17] for the hypercomplex case.

A particular motivation for this paper was the observation that in the simplest non-commutative case, $n = 2$, a sequence of rational numbers $\mathcal{S}$ arising as coefficients of hypercomplex Appell polynomials of arbitrary (n+1) real variables plays also an essential role in other fields. In 1958, L. Vietoris used it in connection with positivity problems of trigonometric sums [24]. Positivity as an interdisciplinary subject was an active research field of that time. Later on, in 1974, Askey and Steinig [2] showed the embedding of Vietoris' results in general problems for Jacobi polynomials and their relation to other subjects in Harmonic Analysis. Thirty years after the publication of [2], Ruscheweyh and Salinas demonstrated in [20] the relevance of the celebrated result of Vietoris for a complex function theoretic result in the context of subordination of analytic functions. By showing their relationship to the hypercomplex generalized geometric series we try to stress a different field where these numbers arise, namely as number sequence which uniquely defines a hypercomplex Appell sequence of homogeneous polynomials.

Let us note that our special attention to some specific arithmetical properties of the numbers that constitute the terms of the sequence $\mathcal{S}$ has still another ground. From a pure number theoretic point of view they simply can be seen as some kind of weighted central binomial coefficients with a certain power of 2 as weight. Of course, from one side, this reveals some similarity with a very famous kind of *weighted central binomial coefficients*, namely the well known *Catalan numbers*. Curiously, both types of weights also have a strong relationship to the Pascal triangle. The weight in Catalan numbers is the number $(k+1)$ of binomial coefficients in the $k-$th row of the Pascal triangle, whereas the number 2^k is the sum of all binomial coefficients in the $k-$th row of the Pascal triangle. From the other side, looking for their irreducible representation by a sequence of odd integers divided by a certain power of 2, this problem reminds the Erdös Squarefree Conjecture (see [12, p. 71]). Of course, we have no pretension to compare our problem with the one of Erdös, but the reader will understand that any question on the divisibility of the central binomial coefficients shall come across Erdös' conjecture.

Besides giving an answer to our problem in a corollary to Theorem 4 at the end of the paper, the integer sequence of the exponents of 2 by itself attracted our attention, too. Becoming more and more involved in the world of special number sequences during our studies on hypercomplex polynomials, we noticed that in the last decades a huge amount of integer number sequences have been collected and classified in the famous On-Line Encyclopedia of Integer Sequences® [23] of N. Sloane. Surprisingly, we saw that the previously mentioned integer sequence of exponents of 2 had not been referred there until now. Of course, no tools of Hypercomplex Function Theory are necessary to derive such a sequence of integers. However, in our opinion, it is worth mentioning these facts as an example of how far problems originated by hypercomplex polynomials can sometimes lead us.

This paper is divided in the following way. A reader not familiar with the basics of Hypercomplex Function Theory will find the corresponding tools in the next section. Section 3 is concerned with the hypercomplex generalized geometric series written in terms of hypercomplex homogeneous Appell polynomials as generalizations of x^k or z^k, $k = 0, 1, \dots$ in the case of $\mathbb{R}$ resp.$\mathbb{C}$. Some remarks on the seemingly most natural generation of homogeneous polynomials in $\mathbb{R}^{n+1}$ by Cauchy-Kovalevskaya extension of corresponding polynomials in $\mathbb{R}^n$ are introducing to this rather seldom applied approach. After this, Section 4 starts by introducing the sequence $\mathcal{S}$ for the simplest non-commutative case as consequence of the construction of the hypercomplex homogeneous Appell polynomials in Section 3. Moreover, Vietoris' Theorem and its relation to $\mathcal{S}$ is discussed, followed by a theorem about its generating function. In Section 5 the aforementioned arithmetical properties of $\mathcal{S}$ and related sequences of rational and integer numbers are proved. The paper finishes with some final remarks in Section 6.

2. Preliminaries

Following the notations introduced in [5], let $\{e_1, e_2, \dots, e_n\}$ be an orthonormal basis of the Euclidean vector space $\mathbb{R}^n$ endowed with a non-commutative product according to the multiplication rules

$$e_i e_j + e_j e_i = -2\delta_{ij},\ i, j = 1, 2, \dots, n,$$

where δ_{ij} is the Kronecker symbol. This generates the associative 2^n-dimensional Clifford algebra $\mathcal{C}\ell_{0,n}$ over $\mathbb{R}$, whose elements are of the form

$$a = \sum_A a_A e_A, \ a_A \in \mathbb{R}, \text{ with } A \subseteq \{1, \cdots, n\}, \ e_A = e_{l_1} e_{l_2} \cdots e_{l_r},$$

where $1 \leq l_1 < \cdots < l_r \leq n$ and $e_\emptyset = e_0 = 1$. In general, the vector space $\mathbb{R}^{n+1}$ is embedded in $\mathcal{C}\ell_{0,n}$ by identifying the element $(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1}$ with the so-called paravector

$$x = x_0 + \sum_{k=1}^{n} e_k x_k = x_0 + \underline{x} \in \mathcal{A}_n := \operatorname{span}_{\mathbb{R}}\{1, e_1, \ldots, e_n\} \subset \mathcal{C}\ell_{0,n}.$$

Its conjugate is $\bar{x} = x_0 - \underline{x}$ and the norm of x is given by $|x| = (x\bar{x})^{1/2} = (\bar{x}x)^{1/2} = \left(\sum_{k=0}^{n} x_k^2\right)^{1/2}$. Consequently, any non-zero paravector x has an inverse defined by $x^{-1} = \frac{\bar{x}}{|x|^2}$.

In what follows, we consider $\mathcal{C}\ell_{0,n}$-valued functions defined in some open subset $\Omega \subset \mathbb{R}^{n+1} \simeq \mathcal{A}_n$, i.e. functions of the form $f(z) = \sum_A f_A(z) e_A$, where $f_A(z)$ are real valued. The generalized Cauchy-Riemann operator in $\mathbb{R}^{n+1}$ is given by

$$\overline{\partial} := \frac{1}{2}(\partial_0 + \partial_{\underline{x}}), \ \text{ with } \ \partial_0 := \frac{\partial}{\partial x_0}, \ \text{ and } \ \partial_{\underline{x}} := \sum_{k=1}^{n} e_k \frac{\partial}{\partial x_k}.$$

A function f is called *left (right) monogenic*, or simply *monogenic* in $\mathbb{R}^{n+1}$ if it is a solution of the differential equation $\overline{\partial} f = 0$ ($f\overline{\partial} = 0$). We underline that the conjugate generalized Cauchy-Riemann operator

$$\partial := \frac{1}{2}(\partial_0 - \partial_{\underline{x}})$$

acts as a hypercomplex derivative operator. Indeed the hypercomplex derivative of a monogenic function is given by

$$\partial f = \frac{1}{2}(\partial_0 - \partial_{\underline{x}}) f = \partial_0 f, \ \text{if } \overline{\partial} f = 0,$$

resp.

$$f\partial = \frac{1}{2} f(\partial_0 - \partial_{\underline{x}}) = \partial_0 f, \ \text{if } f\overline{\partial} = 0,$$

i.e. in the same way as the complex derivative of a holomorphic function and independently from being left resp. right monogenic.

3. Two ways of generating monogenic polynomials: Cauchy-Kovalevskaya extension and geometric series

Let us start with some remarks on the seemingly most natural generation of homogeneous monogenic polynomials in $\mathbb{R}^{n+1}$, $n > 1$, by Cauchy-Kovalevskaya's extension of corresponding polynomials in $\mathbb{R}^n$ (cf. [4, 5, 6, 10]). It allows a particular simple procedure to obtain a k-homogeneous polynomial in the $(n+1)$ variables $(x_0, x_1, \ldots, x_n)$ from the k-th power of the vector part $\underline{x}$ of the paravector $x \in \mathcal{A}_n$.

In the complex case, the Cauchy-Kovalevskaya (CK) extension of the $k-$th power of the imaginary part, i.e. $(x_1e_1)^k = \underline{x}^k$ (e_1 is naturally identified with the imaginary unit i), leads to the k-power of the complex variable $z = x_0 + x_1e_1 \cong x + iy$, i.e.

$$\mathrm{CK}[(x_1e_1)^k] = ((x_1 - x_0e_1)e_1)^k = (x_0 + x_1e_1)^k. \tag{3.1}$$

It coincides naturally also with the complex continuation of x_0^k to the complex power z^k which would be the procedure normally discussed when a power series in the real variable $x = x_0$ is continued to the corresponding power series in the complex variable $z = x_0 + ix_1$. The formal difference between both cases is that in (3.1) the real variable x_1 is substituted by $x_1 - x_0e_1$ (cf. [19]), whereas in the ordinary case x_0 is simply substituted by $z = x_0 + x_1e_1$.

From (3.1) we see also that the ordinary complex case reveals an essential difference to the Cauchy-Kovalevskaya procedure of continuation (extension) from the "imaginary" to the complex. To "come back" from the extended power $(x_0 + x_1e_1)^k$ to its "origin" $(x_1e_1)^k$ we have to consider the restriction to $x_0 = 0$, whereas in the ordinary complex approach the "origin" of $(x_0 + x_1e_1)^k$ is obtained by restriction to $x_1e_1 = 0$ to obtain $(x_0 + x_1e_1)^k\big|_{x_1e_1=0} = x_0^k$. A particularly interesting and concrete simple example is already obtained from $n = 2$ and $k = 2$. From (3.1) (see [19]) we obtain

$$\begin{aligned}\mathrm{CK}[(x_1e_1 + x_2e_2)^2] = \mathrm{CK}[-x_1^2 - x_2^2] &= -(x_1 - x_0e_1)^2 - (x_2 - x_0e_2)^2 \\ &= 2x_0^2 - x_1^2 - x_2^2 + 2x_0(x_1e_1 + x_2e_2)\end{aligned} \tag{3.2}$$

and "return" by restriction to $x_0 = 0$, i.e.

$$\mathrm{CK}[(x_1e_1 + x_2e_2)^2]\big|_{x_0=0} = -x_1^2 - x_2^2 = (x_1e_1 + x_2e_2)^2. \tag{3.3}$$

But following the complex case to obtain the "origin" of (3.2) in $\mathbb{R}$ by the restriction to $\underline{x} = 0$ it leads to

$$\mathrm{CK}[(x_1e_1 + x_2e_2)^2]\big|_{\underline{x}=0} = 2x_0^2, \tag{3.4}$$

instead of giving the expected x_0^2.

Formula (3.3) and (3.4) together show that the expected real power of second degree x_0^2 as the result of the restriction to $\underline{x} = 0$ in $\mathrm{CK}[(x_1e_1+x_2e_2)^2]$ is not straightforward. For that reason one has to look for another procedure to come to the extension of $f(x_0) = x_0^2$ (or any other natural power of x_0) in the form of a monogenic polynomial of second degree in $\mathbb{R}^{2+1}$ (or any $\mathbb{R}^{n+1}$).

After having noticed the peculiarities of a straightforward use of Cauchy-Kovalevskaya's extension for generating simple monogenic hypercomplex power functions, we present now an approach based on a hypercomplex generalized geometric series which allows to overcome the aforementioned difficulties and includes as special cases both the expected real and complex powers.

It is well-known that the complex function

$$f(z) = \frac{1}{1-z} \;\text{ where }\; z = x_0 + e_1x_1,\; e_1 \cong i,$$

plays the essential role in the development of holomorphic functions into power series through the Cauchy integral formula. Of course, this function is not only the result of the appropriately specified Cauchy kernel $\mathcal{C}(\zeta, z) = (\zeta - z)^{-1}$, but above all it is the analytic continuation of the sum of the real geometric power series or, in other words, the sum of monomials of all degrees, i.e.

$$\frac{1}{1-x_0} = \sum_{k=0}^{+\infty} x_0^k, \text{ for } |x_0| < 1, \tag{3.5}$$

Since x^k in $\mathbb{R}$ or, resp., z^k in $\mathbb{C}$, $(k = 0, 1, \ldots)$ are the simplest examples of Appell polynomials, due to the special form of their real resp. complex derivatives (cf. [1])

$$\frac{d}{dx} x^k = k x^{k-1} \text{ resp. } \frac{d}{dz} z^k = k z^{k-1},$$

we intend to generalize (3.5) to the hypercomplex setting. In order to achieve our goal, we simply use now the Cauchy kernel

$$\mathcal{C}(y, x) = \frac{\overline{y - x}}{|y - x|^{n+1}}, \quad x, y \in \mathcal{A}_n,$$

in the hypercomplex Cauchy integral formula (cf. [5], [16]) and consider analogously for $|x| < 1$, the right and left monogenic function

$$\begin{aligned} g(x) &= \frac{1 - \bar{x}}{|1 - x|^{n+1}}, \\ &= (1 - x)^{-1} |1 - x|^{1-n}, \ x \in \mathcal{A}_n, \end{aligned} \tag{3.6}$$

which for $n = 1$ coincides with the function $f(z) = f(x_0 + x_1 e_1)$ that we mentioned at the beginning of this paragraph.

In view of the previous considerations, it is now evident to expect that the series development of g leads to a hypercomplex generalization of the geometric series.

Observing that

$$g(x) = (1 - \bar{x}) |1 - x|^{-n-1} = (1 - \bar{x}) \left[(1 - x)(1 - \bar{x})\right]^{-\frac{1}{2}(n+1)}$$

and taking into account that both $1 - x$ and $1 - \bar{x}$ commute and have the same scalar part $(1 - x_0)$, we obtain

$$g(x) = (1 - x)^{-\frac{n+1}{2}} (1 - \bar{x})^{-\frac{n-1}{2}} \tag{3.7}$$

Details on non-integer powers of paravectors that justify this operation the reader can find in [16]. We expand now each factor in the right-hand side of (3.7) into a binomial series and use the Cauchy product to obtain

$$g(x) = \sum_{k=0}^{+\infty} \left(\sum_{s=0}^{k} \frac{\left(\frac{n+1}{2}\right)_{k-s}}{(k-s)!} \frac{\left(\frac{n-1}{2}\right)_s}{s!} \right) x^{k-s} \bar{x}^s, \tag{3.8}$$

where $(.)_k$ stands for the Pochhammer symbol defined by $(a)_s = \frac{\Gamma(a+s)}{\Gamma(a)}$ or $(a)_s = a(a+1)(a+2)\cdots(a+s-1)$, $(a)_0 = 1$, $s \geq 0$.

Relation (3.8) shows that a *hypercomplex generalized geometric power series*[2] can be obtained as the series expansion of (3.6) in terms of certain homogeneous hypercomplex polynomials in x and $\bar{x}$ with coefficients formed by expressions from the inner sum in (3.8). To guarantee that the hypercomplex generalized geometric series (3.8) for $\underline{x} = 0$ leads to a corresponding generalized geometric series in the real variable x_0 (see the discussion in the beginning of this section) we must calculate the result of $g(x)\big|_{\underline{x}=0}$ in (3.8). Straightforward we get

$$g(x) = (1-\bar{x})\,|1-x|^{-n-1}\,\Big|_{\underline{x}=0} = (1-x_0)^{-n} = \sum_{k=0}^{+\infty} \frac{(n)_k}{k!}\,x_0^k \tag{3.9}$$

and we are now able to formulate

Theorem 1. *Let $n \geq 1$ be fixed and for each $k = 0, 1, \dots,$ consider the sets of real numbers $\{T_s^k(n)\}_{(0\leq s\leq k)}$ defined by*

$$T_s^k(n) := \binom{k}{s} \frac{(\frac{n+1}{2})_{k-s}(\frac{n-1}{2})_s}{(n)_k} \tag{3.10}$$

and the homogeneous polynomials of degree k in x and $\bar{x}$

$$\mathcal{P}_k^n(x) := \sum_{s=0}^{k} T_s^k(n) x^{k-s} \bar{x}^s. \tag{3.11}$$

Then the series

$$\mathcal{P}_0^n(x) + \frac{n}{1!}\,\mathcal{P}_1^n(x) + \frac{n(n+1)}{2!}\,\mathcal{P}_2^n(x) + \frac{n(n+1)(n+2)}{3!}\,\mathcal{P}_3^n(x) + \cdots \tag{3.12}$$

is the hypercomplex generalized geometric series whose restriction to $\underline{x} = 0$ coincides with the right hand side of (3.9) *and whose sum in $|x| < 1$ is given by*

$$g(x) = (1-x)^{-1}\,|1-x|^{1-n} = \sum_{k=0}^{+\infty} \frac{(n)_k}{k!}\,\mathcal{P}_k^n(x). \tag{3.13}$$

Proof. It is clear that the definition of $T_s^k(n)$ in the form (3.10) together with (3.11) leads to the right hand side of (3.13). Choosing now in the homogeneous polynomials $\mathcal{P}_k^n(x)$ of (3.11) $\underline{x} = 0$ results in

$$\mathcal{P}_k^n(x_0) = \sum_{s=0}^{k} T_s^k(n) x_0^k = x_0^k \sum_{s=0}^{k} T_s^k(n). \tag{3.14}$$

Applying Chu-Vandermonde's convolution identity for Pochhammer symbols (cf. [9])

$$(a+b)_{(k)} = \sum_{s=o}^{k} \binom{k}{s} a_{(k-s)} b_{(s)},$$

[2]This is different from the case of several complex variables where in the expansion of the Cauchy kernel a multiple geometric series as generalization of the ordinary geometric series arises.

with $a = \frac{n+1}{2}$ and $b = \frac{n-1}{2}$ we get automatically after division of the left hand side that

$$1 = \sum_{s=0}^{k} T_s^k(n) = \sum_{s=0}^{k} \binom{k}{s} \frac{(\frac{n+1}{2})_{k-s}(\frac{n-1}{2})_s}{(n)_k}, n \geq 1; k \geq 0; s = 0, \ldots, k.$$

This implies in (3.14) that $\mathcal{P}_k^n(x_0) = x_0^k$ and (3.9) together with (3.13) proves the rest of the assertion. □

Notice that if we look to the case $n = 1$ with $x := z = x_0 + e_1 x_1$, and consider the left hand side of (3.9) as not restricted to $\underline{x} = 0$ but to $x = z$, we end up with the ordinary complex geometric series since

$$\begin{aligned} g(x) =& (1 - \bar{x}) \left|1 - x\right|^{-n-1} \Big|_{x=z} \\ =& (1 - \bar{z}) \left|1 - z\right|^{-2} \\ =& (1 - z)^{-1} = \sum_{k=0}^{+\infty} \frac{(1)_k}{k!} z^k = \sum_{k=0}^{+\infty} z^k. \end{aligned} \tag{3.15}$$

Moreover, we notice an essential difference in the way the real case ($\underline{x} = 0$ *not* $n = 0$) and the complex case ($n = 1$) fit into the hypercomplex generalization of the geometric series of singularity degree n (resulting from the Cauchy kernel). Whereas the former is nothing else than the multiple ordinary geometric series (3.9), the latter reduces to the ordinary complex geometric series (3.15).

Remark 1. *As we have seen in Theorem* 1*, it was crucial to understand that because of the relationship of the hypercomplex generalized geometric series to the Cauchy kernel and its degree of singularity equal to n one has to consider* (3.12) *as the adequate generalized geometric series. In the paper* [22, p. 375]*, the author tried to connect the expansion of the Cauchy kernel to a hypercomplex generalized geometric series of the form*

$$\mathcal{Q}_0^n(x) + \mathcal{Q}_1^n(x) + \mathcal{Q}_2^n(x) + \mathcal{Q}_3^n(x) + \cdots$$

where the $\mathcal{Q}_n$ are also homogeneous monogenic polynomials. Obviously this was suggested by the ordinary geometric series

$$1 + x + x^2 + x^3 \cdots$$

But due to Theorem 1, this approach cannot directly be connected to the real or complex case through corresponding restrictions of the argument or to the concept of Appell sequences. Therefore the author needed to use other tools like the theory of monomial functions, Fourier multipliers and integral transforms.

Finally we recall Appell's concept [1] of power-like polynomial sequences, which in hypercomplex form reads as following (cf. [13]):

> A sequence of homogeneous monogenic polynomials $(\mathcal{F}_k)_{k\geq 0}$ of degree k is called a *generalized Appell sequence with respect to the hypercomplex differential operator* ∂ if $\mathcal{F}_0(x) \equiv 1$, $\mathcal{F}_k$ is of exact degree k and $\partial \mathcal{F}_k = k\, \mathcal{F}_{k-1},\ k = 1, 2, \ldots$.

Instead of considering a hypercomplex generalized geometric series this definition has been used in [13] to obtain homogeneous hypercomplex Appell sequences for the first time. Since the classical Appell approach relies on the behavior of the derivatives of the polynomials and $\partial = \frac{\partial}{\partial x_0}$ in the hypercomplex monogenic case (see Section 2), in [13] the same polynomials (3.11) have been obtained in a different form. It is easy to see that the relations $x_0 = (x+\bar{x})/2$ and $\underline{x} = (x-\bar{x})/2$, allow a transformation of (3.11) in polynomials with respect to x_0 and $\underline{x}$. Indeed, this leads (cf. also [7], [8]) to

$$\mathcal{P}_k^n(x) = \sum_{s=0}^{k} \binom{k}{s} c_s(n)\, x_0^{k-s}\, \underline{x}^s, \tag{3.16}$$

where the coefficients $c_s(n)$ $(s = 0, \ldots, k)$ are obtained as

$$c_k(n) := \sum_{s=0}^{k} (-1)^s T_s^k(n),\ \ k = 0, 1, \ldots. \tag{3.17}$$

In the next section the sequence of these coefficients $c_k(n)$ is in the center of our attention and their relation to the coefficients (3.10) will be studied in detail.

It is worth to notice that, for each k, the polynomial (3.11) is part of the orthogonal basis of the space M_k of k-homogeneous monogenic polynomials constructed in [17]. Notice also that, for each k, the coefficients $a(k,n) := \frac{(n)_k}{k!}$ in the expression (3.13), for fixed $n \in \mathbb{N}$, coincide with the dimension of M_k.

4. Vietoris' number sequence

In what follows we consider the special case $n = 2$ and focus our attention on the sequence of rational numbers $c_k(2)$ related to (3.10) by formula (3.17). Indeed, considering $x_0 = 0$ in (3.16) and taking into account that $\bar{\underline{x}} = -\underline{x}$, one obtains easily

$$\mathcal{P}_k^2(\underline{x}) = c_k \underline{x}^k,$$

where $c_k := c_k(2)$ is given by (3.17).

As shown in [14, Theorem 3.9], the elements of the number sequence $\mathcal{S} = (c_k)_{k\geq 0}$ can be written using the *generalized central binomial coefficient* $\binom{k}{\lfloor \frac{k}{2} \rfloor}$ in the form

$$c_k = \frac{1}{2^k} \binom{k}{\lfloor \frac{k}{2} \rfloor},\ \ k \geq 0. \tag{4.1}$$

This leads to the sequence

$$1,\ \tfrac{1}{2},\ \tfrac{1}{2},\ \tfrac{3}{8},\ \tfrac{3}{8},\ \tfrac{5}{16},\ \tfrac{5}{16},\ \tfrac{35}{128},\ \tfrac{35}{128},\ \tfrac{63}{256},\ \tfrac{63}{256},\ \tfrac{231}{1024},\ \tfrac{231}{1024},\ \ldots. \tag{4.2}$$

In the context of positive trigonometric sums and without relying on the generalized central binomial coefficients, Vietoris used in his celebrated paper [24] the sequence

$$a_{2k} = \frac{1}{2^{2k}}\binom{2k}{k}, \ k \geq 0, \tag{4.3}$$

demanding explicitly that $a_{2k+1} = a_{2k}$. i.e.

$$1,\ 1,\ \tfrac{1}{2},\ \tfrac{1}{2},\ \tfrac{3}{8},\ \tfrac{3}{8},\ \tfrac{5}{16},\ \tfrac{5}{16},\ \tfrac{35}{128},\ \tfrac{35}{128},\ \tfrac{63}{256},\ \tfrac{63}{256},\ \tfrac{231}{1024},\ \tfrac{231}{1024},\ldots.$$

More concretely, we recall Vietoris' theorem in Askey and Steinig's version [2].

Theorem 2 (L. Vietoris). *If a_k are given by (4.3) and $a_{2k+1} = a_{2k}$, then for $0 < t < \pi$,*

$$\sum_{k=1}^{n} a_k \sin kt > 0 \qquad \textit{and} \qquad \sum_{k=0}^{n} a_k \cos kt > 0.$$

In [2] we can also find the following interesting result on trigonometric series.

Theorem 3 (Askey and Steinig). *If a_k is defined as in Theorem 2, then for $0 < t < \pi$,*

$$\sum_{k=1}^{\infty} a_k \sin kt = \sum_{k=0}^{\infty} a_k \cos kt = \left(\tfrac{1}{2}\cot\tfrac{t}{2}\right)^{\frac{1}{2}}. \tag{4.4}$$

We call attention to the fact that the coefficients in the sine series of (4.4) are exactly the elements of the sequence $(c_k)_{k\geq 0}$ and the sequence (4.3) differs from (4.2) by the inclusion of $a_0 = 1$ and the shift of the indices, i.e.

$$a_0 = 1 \quad \text{and} \quad a_{k+1} = c_k, \ k \geq 0. \tag{4.5}$$

For that reason we call $(c_k)_{k\geq 0}$ *Vietoris' number sequence*. An elementary procedure can be used to derive a generating function of this sequence.

Theorem 4. *The Vietoris' number sequence $(c_k)_{k\geq 0}$ defined by (4.1) is generated by the function*

$$F(t) = \frac{\sqrt{1+t} - \sqrt{1-t}}{t\sqrt{1-t}}, \ \ 0 < |t| < 1. \tag{4.6}$$

Proof. The proof is based on the expansion of the binomial function $(1-y)^l$, for rational positive or negative l. In fact, considering $y = t^2$ we obtain

$$\frac{1}{(1-t^2)^m} = \sum_{k=0}^{\infty} \frac{(m)_k}{k!}\, t^{2k}, \ \ m \geq 0, \ \ |t| < 1$$

and taking $m = \frac{1}{2}$, we get

$$\begin{aligned} \frac{1}{(1-t^2)^{1/2}} &= 1 + \frac{(\frac{1}{2})_1}{1!}t^2 + \frac{(\frac{1}{2})_2}{2!}t^4 + \frac{(\frac{1}{2})_3}{3!}t^6 + \ldots \\ &= 1 + c_2\, t^2 + c_4\, t^4 + c_6\, t^6 + \ldots. \end{aligned} \tag{4.7}$$

Therefore, for $t \neq 0$,

$$\frac{1}{t}\,\frac{1}{(1-t^2)^{1/2}} = \frac{1}{t} + c_2\,t + c_4\,t^3 + c_6\,t^5 + \dots$$
$$= \frac{1}{t} + c_1\,t + c_3\,t^3 + c_5\,t^5 + \dots . \tag{4.8}$$

By adding (4.7) and (4.8), we obtain

$$\left(1+\frac{1}{t}\right)\frac{1}{(1-t^2)^{1/2}} = 1 + \frac{1}{t} + c_1\,t + c_2\,t^2 + c_3\,t^3 + c_4\,t^4 + c_5\,t^5 + c_6\,t^6 + \dots .$$

or, equivalently,

$$\left(1+\frac{1}{t}\right)\frac{1}{(1-t^2)^{1/2}} - \frac{1}{t} = c_0 + c_1\,t + c_2\,t^2 + c_3\,t^3 + c_4\,t^4 + c_5\,t^5 + c_6\,t^6 + \dots .$$

Hence

$$F(t) := \sum_{k=0}^{\infty} c_k\,t^k = \left(1+\frac{1}{t}\right)\frac{1}{\sqrt{1-t^2}} - \frac{1}{t},\ 0 < |t| < 1$$

and the result follows immediately. □

Corollary 5. *Let $(c_k)_{k\geq 0}$ be the Vietoris' number sequence defined by (4.1). Then for $0 < \alpha < \pi$,*

$$\sum_{k=0}^{\infty} c_k\,\cos^k\alpha = \frac{2}{1-\cos\alpha+\sin\alpha}$$

and

$$\sum_{k=0}^{\infty} c_k\,\sin^k\alpha = \frac{2}{1-\sin\alpha+\cos\alpha}.$$

Proof. Since (4.6) can be written as

$$F(t) = \frac{2}{1-t+\sqrt{1-t^2}},\quad 0 < |t| < 1,$$

substituting $t = \cos\alpha$ and $t = \sin\alpha$, respectively, the result follows immediately. □

Remark 2. *We underline the resemblance of these results and Theorem 3 where in the trigonometric series, multiple-angle appear related to the sequence $(a_k)_{k\geq 0}$.*

5. The minimal exponent integer sequence

As we saw in Section 4, the elements of the Vietoris' number sequence have been obtained as the alternating sum of

$$T_s^k := T_s^k(2) = \frac{\binom{k}{s}}{(k+1)!}\left(\frac{3}{2}\right)_{k-s}\left(\frac{1}{2}\right)_s,\ \ s = 0,\dots,k. \tag{5.1}$$

This formula can easily be used to show that for each fixed k, the $k+1$ elements in (5.1) can also be linked to the Vietoris' number sequence through

$$T_s^k = c_{2s} c_{2k-2s} \frac{2(k-s)+1}{k+1}, \quad s = 0, \ldots, k. \tag{5.2}$$

Table 1 contains the first rows of the triangular table T_s^k. Analyzing the table we observe that the first column shows a particular behavior, since it is constituted by rational numbers whose numerators are odd numbers and denominators are strictly increasing powers of 2. This behavior is not immediately visible neither in the expression of T_0^k in (5.1) nor in (5.2). Moreover, the corresponding sequence of powers of 2,

$$0, 2, 3, 6, 7, 9, 10, 14, 15, 17, 18, \ldots$$

seems not to follow a regular pattern and it is not listed in the On-Line Encyclopedia of Integer Sequences® [23]. This observation motivated us to look for a general expression of the above sequence. From now on we call this sequence the *minimal exponent integer sequence* with respect to 2, since it represents the *least non-negative integer* m_k such that $2^{m_k} T_0^k \in \mathbb{N}$.

TABLE 1. The first 7 rows of T_s^k

k \ s						
1						
$\frac{3}{4}$	$\frac{1}{4}$					
$\frac{5}{8}$	$\frac{1}{4}$	$\frac{1}{8}$				
$\frac{35}{64}$	$\frac{15}{64}$	$\frac{9}{64}$	$\frac{5}{64}$			
$\frac{63}{128}$	$\frac{7}{32}$	$\frac{9}{64}$	$\frac{3}{32}$	$\frac{7}{128}$		
$\frac{231}{512}$	$\frac{105}{512}$	$\frac{35}{256}$	$\frac{25}{256}$	$\frac{35}{512}$	$\frac{21}{512}$	
$\frac{429}{1024}$	$\frac{99}{512}$	$\frac{135}{1024}$	$\frac{25}{256}$	$\frac{75}{1024}$	$\frac{27}{512}$	$\frac{33}{1024}$

We can prove the following

Theorem 6. *The irreducible representation of* T_0^k $(k = 0, 1, \ldots)$ *is of the form*

$$T_0^k = \frac{A_k}{2^{m_k}},$$

where A_k *is an odd integer and*

$$m_k = k + \left\lfloor \frac{k+1}{2} \right\rfloor + \left\lfloor \frac{k+1}{2^2} \right\rfloor + \cdots + \left\lfloor \frac{k+1}{2^m} \right\rfloor, \tag{5.3}$$

with $m \leq \log_2(k+1)$.

Proof. Recalling (5.1) we have

$$T_0^k = \frac{1}{(k+1)!} \left(\frac{3}{2}\right)_k = \frac{1}{2^k} \frac{(2k+1)!!}{(k+1)!}.$$

Applying the well-known Legendre formula, first published in [18], for the prime decomposition of the factorial $(k+1)!$, namely

$$(k+1)! = \prod_{\text{prime } p \le k+1} p^{\varepsilon_p(k+1)},$$

where

$$\varepsilon_p(k+1) = \sum_{j=1}^{\lfloor \log_p (k+1) \rfloor} \left\lfloor \frac{k+1}{p^j} \right\rfloor,$$

we obtain

$$T_0^k = \frac{1}{2^{k+\varepsilon_2(k+1)}} \frac{(2k+1)!!}{\prod\limits_{2<\text{prime } p \le k+1} p^{\varepsilon_p(k+1)}}.$$

Therefore

$$T_0^k = \frac{1}{2^{m_k}} A_k, \tag{5.4}$$

where m_k is given by (5.3) and

$$A_k = \frac{(2k+1)!!}{\prod\limits_{2<\text{prime } p \le k+1} p^{\varepsilon_p(k+1)}}, \tag{5.5}$$

is the quotient of odd numbers which we will prove to be an integer. To this purpose we consider now a second representation of T_0^k, obtained from (5.2) as

$$T_0^k = c_0 c_{2k} \frac{2k+1}{k+1} = \frac{1}{2^{2k}} \binom{2k}{k} \frac{2k+1}{k+1} = \frac{1}{2^{2k}} \binom{2k+1}{k}. \tag{5.6}$$

The integer $\binom{2k+1}{k}$ can be written as the product of a power of 2 with multiplicity α ($\alpha \ge 0$) by an odd number, say n_1. Substituting this in (5.6) and using (5.4) yields the identity of two expressions for

$$T_0^k = \frac{1}{2^{2k-\alpha}} n_1 = \frac{1}{2^{m_k}} A_k. \tag{5.7}$$

This is equivalent to a representation of A_k in the form

$$A_k = 2^{\alpha - 2k + m_k} n_1. \tag{5.8}$$

Using the fact that A_k is a quotient of odd numbers it is evident that this implies

$$\alpha - 2k + m_k = 0,$$

which in turn leads to the conclusion that $A_k = n_1 \in \mathbb{N}$, i.e. A_k is really an odd integer. □

We have just proved that for T_0^k the minimal exponent integer sequence with respect to 2 is exactly given by (5.3). Analyzing the first elements of the integer sequence $(2^{m_k} T_0^k)_{k \ge 0}$,

$$1, 3, 5, 35, 63, 231, \ldots,$$

we can recognize the numerators of the subsequence $(c_{2k+1})_{k \ge 0}$ (cf. (4.2)), as the next result states.

Corollary 7. *The irreducible representation of Vietoris' numbers c_{2k+1} is of the form*

$$c_{2k+1} = \frac{A_k}{2^{m_k+1}}, \ k = 0, 1, \ldots,$$

where m_k and A_k are defined as in Theorem 6.

Proof. The second representation (5.6) of T_0^k can be rewritten as

$$T_0^k = \frac{1}{2^{2k}}\binom{2k+1}{k} = \frac{2}{2^{2k+1}}\binom{2k+1}{k} = 2c_{2k+1}, \ k = 0, 1, \ldots. \qquad (5.9)$$

and the final result follows at once by using (5.4). □

Corollary 7 allows now to determine the exact power of 2 in the prime-power factorization of an arbitrary central binomial coefficient.

Theorem 8. *Let $r \geq 1$ be a positive integer, then the corresponding central binomial coefficient C_r can be factorized in the form*

$$C_r := \binom{2r}{r} = 2^{2r-1-m_{r-1}}B, \qquad (5.10)$$

where B is a product of primes $p > 2$ and m_{r-1} is an element of the minimal exponent integer sequence with respect to 2 of the form

$$m_{r-1} = (r-1) + \lfloor \frac{r}{2} \rfloor + \lfloor \frac{r}{2^2} \rfloor + \cdots + \lfloor \frac{r}{2^m} \rfloor,$$

with $m \leq \log_2(r)$.

Proof. First of all we notice that $c_{2k} = c_{2k-1}, \ k \geq 1$. This implies, using Corollary 7 and Theorem 6

$$c_{2(k+1)} = \frac{1}{2^{2(k+1)}}\binom{2(k+1)}{k+1} = c_{2k+1} = \frac{A_k}{2^{m_k}+1}$$

with m_k defined in (5.3). Writing now $r := k+1$, we get

$$c_{2r} = \frac{1}{2^{2r}}\binom{2r}{r} = \frac{A_{r-1}}{2^{m_{r-1}}+1},$$

where A_{r-1} is an odd number. Formula (5.10) follows now immediately, taking also into account the definition of A_k and m_k.

□

Examples To illustrate the statement of Theorem 8 we calculate explicitly the power of 2 in the prime-power representation of C_8 and C_{11}.

1. If $r = 8$ then

$$C_8 = \binom{16}{8} = 12\ 870 = 2 \cdot 3^2 \cdot 5 \cdot 11 \cdot 13.$$

In this case we have

$$m_7 = 7 + \lfloor \frac{8}{2} \rfloor + \lfloor \frac{8}{2^2} \rfloor + \lfloor \frac{8}{2^3} \rfloor = 7 + 4 + 2 + 1 = 14$$

and

$$C_8 = \binom{16}{8} = 2^{16-1-14} \cdot B = 2^1 \cdot 3^2 \cdot 5 \cdot 11 \cdot 13.$$

2. If $r = 11$ then

$$C_{11} = \binom{22}{11} = 705\ 432 = 2^3 \cdot 3 \cdot 7 \cdot 13 \cdot 17 \cdot 19.$$

In this case we have

$$m_{10} = 10 + \lfloor \frac{11}{2} \rfloor + \lfloor \frac{11}{2^2} \rfloor + \lfloor \frac{11}{2^3} \rfloor = 10 + 5 + 2 + 1 = 18,$$

and

$$C_{11} = \binom{22}{11} = 2^{22-1-18} \cdot B = 2^3 \cdot 3 \cdot 7 \cdot 13 \cdot 17 \cdot 19.$$

6. Final remarks

About ten years ago, the concept of hypercomplex Appell polynomial sequences was for the first time discussed [13]. The aim was to solve the problem of dealing with a suitable form of polynomials as substitute for the non-monogenic powers of a general paravector $x^k \in \mathbb{R}^{n+1}; k \geq 1$, without loosing too much of the particular important properties of the real or complex Appell sequences $\left(x^k\right)_{k\geq 1}$ resp. $\left(z^k\right)_{k\geq 1}$ which are the simplest examples of Appell polynomials. A way out of this dilemma was found by following the classical approach of Appell [1] and employing the hypercomplex derivative as it was mentioned in Section 3. Meanwhile hypercomplex Appel polynomials have found their application in many fields of multivariate analysis.

The aim of this paper was twofold. First of all it was an attempt to explain in a more detailed way than had been previously done the role of a hypercomplex generalized geometric series for the construction of Appell polynomials in the most natural way, namely as suitable monogenic generalizations of $\left(z^k\right)_{k\geq 1}$ whose restriction to $\underline{x} = 0$ leads most naturally to the real power function $\left(x^k\right)_{k\geq 1}$. This has been done in Section 3. It seems that the fundamental role that a hypercomplex generalized geometric series could play (analogously to the geometric series in one complex variable) has been neglected so far. Secondly, our observation that a coefficient sequence in hypercomplex Appell polynomials arises also in other contexts like positive trigonometric sums and orthogonal polynomials, univalent functions or even number theory was for us justification enough to understand our work also as a contribution to more interdisciplinary research suggested by hypercomplex problems. The results of Sections 4 and 5 reflect only a first step in this direction.

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