Three-term recurrence relations for systems of Clifford algebra-valued orthogonal polynomials

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Abstract. Recently, systems of Clifford algebra-valued orthogonal polynomials have been studied from different points of view. We prove in this paper that for their building blocks there exist some three-term recurrence relations, similar to that for orthogonal polynomials of one real variable. As a surprising byproduct of own interest we found out that the whole construction process of Clifford algebra-valued orthogonal polynomials via Gelfand-Tsetlin basis or otherwise relies only on one and the same basic Appell sequence of polynomials.

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1. Introduction

During the last decade, orthonormal polynomial systems in the space of square integrable monogenic Clifford algebra-valued functions in the unit ball of $\mathbb{R}^{n+1}$ have been constructed by different methods. In particular, we mention [6–8, 13, 15]), where for practical applications the case $n = 2$ was studied in great detail, by different approaches and from different points of view.
In [3], the authors succeeded to find a system of Appell sequences as orthogonal basis. Their construction method relied on monogenic primitivation and its connection to orthogonality gave their results particular relevance. The

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paper [25] confirmed these results from a representation theoretical point of view and in the joint work [4] the reader can find a more general and extensive explanation including the role of Gelfand-Tsetlin bases for spherical monogenics in dimension 3. This work was generalized in [26] for the case of spherical monogenics in any dimension.
A matrix approach to paravector valued Appell sequences can also advantageously be used, as it was shown in [11]. As one of its application a matrix recurrence for the paravector valued building blocks of those Gelfand-Tsetlin bases can be found in [12].
A central question like the construction of generating functions for spherical harmonics and spherical monogenics was answered in the paper [16]. Another central problem, well known from the general theory of orthogonal polynomials, is the existence of three-term recurrence relations. In [11], Theorem 6 proved such a three-term recurrence relation for a special Appell sequence of homogeneous monogenic polynomials. As continuation of these investigations we aim to derive here an analogue for the system of Clifford algebra-valued orthogonal polynomials obtained as result of the Gelfand-Tsetlin theory.
In this paper we give new insights on the reformulation of the construction process [26] of the general orthogonal Gelfand-Tsetlin bases with respect to the Dirac operator, based on the use of a generalized Cauchy-Riemann operator, which has been recently proposed in [12]. In this last work, the readers can find details about the common aspects and the differences of each approach.
The paper is structured as follows. Section 2 contains the fundamental concepts of monogenic function theory used through the paper. Section 3 provides the reader with some facts about the origin of the system of Clifford algebra-valued orthogonal polynomials that we will study. Therefore it refers results of [26] and [12]. Before coming to the main part of the paper, the second subsection serves to prove (as a byproduct of the chosen representation by Gegenbauer polynomials) a formula that allows an interpretation as a generalized De Moivre's formula in $\mathbb{R}^3$. Section 4 shows the existence of three-term recurrence relations for hypercomplex orthogonal polynomials and, as straightforward consequences, that orthogonal polynomials can also be obtained as solutions of second order differential equations. Furthermore their construction by an operational approach is shown. All the obtained results are in a rather surprisingly way connected through the application of a so-called Standard Appell Sequence. Final remarks refer to differences in the existing approaches to Clifford algebra-valued orthogonal polynomials, namely the analytical approach that we follow and the representation theoretical approach of other authors.

2. Basics of hypercomplex function theory

We recall some necessary basic notations and definitions following mainly [12]. Like all approaches to multivariate polynomial sequences by methods of Hypercomplex Function Theory, the present one relies on the following facts (see e. g. [5, 24]). Let $\{e_1, e_2, \ldots, e_n\}$ be an orthonormal basis of the Euclidean vector space $\mathbb{R}^n$ with a non-commutative product according to the multiplication rules

$$e_k e_l + e_l e_k = -2\delta_{kl}, \quad k, l = 1, \ldots, n,$$

where δ_{kl} is the Kronecker symbol. The set $\{e_A : A \subseteq \{1, \ldots, n\}\}$ with

$$e_A = e_{h_1} e_{h_2} \cdots e_{h_r}, \quad 1 \le h_1 < \cdots < h_r \le n, \quad e_\emptyset = e_0 = 1,$$

forms a basis of the 2^n-dimensional Clifford algebra $\mathcal{C}\ell_{0,n}$ over $\mathbb{R}$. The main involution in $\mathcal{C}\ell_{0,n}$, the conjugation, is defined by

$$\bar{e}_A = \bar{e}_{h_r} \bar{e}_{h_{r-1}} \cdots \bar{e}_{h_1},$$

where $\bar{e}_{h_j} = -e_{h_j}, j = 1, \ldots, r$. Let $\mathbb{R}^{n+1}$ be embedded in $\mathcal{C}\ell_{0,n}$ by identifying $(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1}$ with

$$x = x_0 + \underline{x} \in \mathcal{A}_n := \operatorname{span}_{\mathbb{R}}\{1, e_1, \ldots, e_n\} \subset \mathcal{C}\ell_{0,n}.$$

Here, $x_0 = \mathrm{Sc}(x)$ and $\underline{x} = \mathrm{Vec}(x) = e_1 x_1 + \cdots + e_n x_n$ are the scalar and vector parts of the so-called paravector $x \in \mathcal{A}_n$. The conjugate of x is given by $\bar{x} = x_0 - \underline{x}$ and its norm by $|x| = (x\bar{x})^{\frac{1}{2}} = (x_0^2 + x_1^2 + \cdots + x_n^2)^{\frac{1}{2}}$.

We pay attention to its relation to the complex Wirtinger derivatives, by using the following notation for a generalized Cauchy-Riemann operator in $\mathbb{R}^{n+1}$, $n \ge 1$,

$$\overline{\partial} := \frac{1}{2}(\partial_0 + \partial_{\underline{x}})$$

and its conjugate

$$\partial := \frac{1}{2}(\partial_0 - \partial_{\underline{x}})$$

with

$$\partial_0 := \frac{\partial}{\partial x_0} \quad \text{and} \quad \partial_{\underline{x}} := e_1 \frac{\partial}{\partial x_1} + \cdots + e_n \frac{\partial}{\partial x_n}.$$

As usual, $\mathscr{C}^1$-functions f satisfying a generalized Cauchy-Riemann equation in its hypercomplex form given by

$$\overline{\partial} f = 0$$

(resp. $f\overline{\partial} = 0$), are called *left monogenic* (resp. *right monogenic*). We suppose that f is hypercomplex-differentiable in Ω in the sense of [23, 27], that is, it has a uniquely defined areolar derivative f' in each point of Ω (see also [28]). Then, f is real-differentiable and f' can be expressed by the conjugate generalized Cauchy-Riemann operator as $f' = \partial f$. Since a hypercomplex differentiable function belongs to the kernel of $\overline{\partial}$, it follows that, in fact, $f' = \partial_0 f = -\partial_{\underline{x}} f$ which is similar to the complex case.

Functions defined in some open subset $\Omega \subset \mathbb{R}^{n+1}$ with values in the Clifford algebra $\mathcal{C}\ell_{0,n}$ are of the form $f(z) = \sum_A f_A(z) e_A$, with real valued $f_A(z)$. As usual, we denote by $\mathcal{M}_k(\mathbb{R}^{n+1}, \mathcal{C}\ell_{0,n})$ the space of homogeneous monogenic polynomials of degree k with values in $\mathcal{C}\ell_{0,n}$.

We use the classical definition of sequences of Appell polynomials [1] adapted to the hypercomplex case.

Definition 2.1. A sequence of homogeneous monogenic polynomials $\left(\mathcal{F}^{(k)}\right)_{k\geq 0}$ of exact degree k is called a *generalized Appell sequence with respect to* ∂ if

1. $\mathcal{F}^{(0)}(x) \equiv 1$,
2. $\partial \mathcal{F}^{(k)} = k\, \mathcal{F}^{(k-1)},\ k = 1, 2, \dots$

The second condition, sometimes mentioned as *Appell property*, is the essential one while the first condition is the usually applied normalization condition which can be changed to any real or hypercomplex constant different from zero or even to a *monogenic constant* (cf. [30]). As already used in [3, 14, 15] a *monogenic constant* is a monogenic function whose hypercomplex derivative is zero. In dimension three ($n = 2$) a generalized constant is isomorphic to an anti-holomorphic complex function (cf. [3] or [14]).

Due to our main goal of establishing three-term recurrence relations for Clifford algebra-valued orthogonal polynomials, we need the Clifford algebra-valued inner product

$$(f, g)_{\mathcal{C}\ell_{0,n}} = \int_{B^{n+1}} \bar{f}\, g\, d\lambda^{n+1}, \tag{2.1}$$

where λ^{n+1} is the Lebesgue measure in $\mathbb{R}^{n+1}$ and $\bar{a}$ the conjugate of $a \in \mathcal{C}\ell_{0,n}$.

3. A Clifford algebra-valued orthogonal system

3.1. Cauchy-Kovalevskaya extension of normalized polynomials and the special case of Standard Appell Polynomials

The importance and central role of homogeneous monogenic polynomials, their different representations, properties and applications, are already manifest in the book [5], but chapter III of its sequel [20] is particularly dedicated to this question. Therein the authors constructed the Cauchy-Kovalevskaya extension of the vector-valued polynomials

$$\underline{x}^{k-j}\, P_j(\underline{x}),\ j = 0, \dots, k,$$

for arbitrarily fixed $P_j \in \mathcal{M}_j(\mathbb{R}^n, \mathcal{C}\ell_{0,n})$, $n \geq 2$, leading to monogenic paravector-valued polynomials $P_k(x_0, \underline{x})$ of degree k in $\mathbb{R}^{n+1}$.

For our goal and as was already done in [12], we will continue to work with the Cauchy-Kovalevskaya extension of the normalized polynomials

$$c_{k,j}(n) \binom{k}{j} \underline{x}^{k-j}\, P_j(\underline{x}),$$

where $c_{k,j}(n)$, for $k \geq 1$ and $j = 0, \dots, k$ are defined by

$$c_{k,j}(n) := \begin{cases} \dfrac{(k-j)!!(n+2j-2)!!}{(n+k+j-1)!!}, & \text{if } k, j \text{ have different parities} \\ c_{k-1,j}(n), & \text{if } k, j \text{ have the same parity} \end{cases} \tag{3.1}$$

and $c_{0,0}(n) := 1$.

The choice of the normalized polynomials allows in the following some simplification and better accordance between different formulas.

As a result of the Cauchy-Kovalevskaya extension described in [20], we obtain for each $k \in \mathbb{N}$ and $j = 0, \ldots, k$, a monogenic polynomial $\widetilde{X}^{(k)}_{n+1,j}$ of degree k and index j as a product of a, in general, non-monogenic polynomial of degree[1] $(k-j)$, denoted by $X^{(k-j)}_{n+1,j}$, with $P_j(\underline{x})$. More explicitly,

$$\widetilde{X}^{(k)}_{n+1,j}(x) := X^{(k-j)}_{n+1,j}(x)\, P_j(\underline{x}), \quad x \in \mathcal{A}_n, \tag{3.2}$$

where, as mentioned before, $P_j \in \mathcal{M}_j(\mathbb{R}^n, \mathcal{C}\ell_{0,n})$ and

$$X^{(k-j)}_{n+1,j}(x) = F^{(k-j)}_{n+1,j}(x) + \frac{j+1}{n+2j} F^{(k-j-1)}_{n+1,j+1}(x)\, \underline{x}. \tag{3.3}$$

Here,

$$F^{(k-j)}_{n+1,j}(x) = \frac{(j+1)_{k-j}}{(n-1+2j)_{k-j}} |x|^{k-j} C^{\frac{n-1}{2}+j}_{k-j}\left(\frac{x_0}{|x|}\right),$$

with $F^{(-1)}_{n+1,k+1} \equiv 0$, $x = (x_0, \ldots, x_n) \in \mathbb{R}^{n+1}$, $|\,.\,|$ is the usual Euclidean norm in $\mathbb{R}^{n+1}$, $(\mu)_m = \mu(\mu+1)(\mu+2)\ldots(\mu+m-1)$, and C^{ν}_m is the Gegenbauer polynomial of degree m and parameter $\nu \neq 0$, given by

$$C^{\nu}_m(t) = (2\nu)_m \sum_{l=0}^{\lfloor m/2 \rfloor} \frac{\left(\frac{1}{2}\right)_l t^{m-2l}(t^2-1)^l}{(2l)!(m-2l)!\left(\nu+\frac{1}{2}\right)_l}. \tag{3.4}$$

Remark 3.1. It follows from the construction that the value of the index j of each obtained polynomial $\widetilde{X}^{(k)}_{n+1,j}$ is ranging from 0 to the degree of homogeneity k. For $j > k$, we consider that $\widetilde{X}^{(k)}_{n+1,j} \equiv 0$, as usual.

As already mentioned in the introduction, the homogeneous monogenic polynomials $\widetilde{X}^{(k)}_{n+1,j}$ $(j = 0, \ldots, k)$ are orthogonal with respect to the Clifford-valued inner product (2.1). Moreover, for each fixed j $(j = 0, \ldots, k)$ they form an Appell sequence, i.e.,

$$\partial \widetilde{X}^{(k)}_{n+1,j}(x) = k\, \widetilde{X}^{(k-1)}_{n+1,j}(x), x \in \mathcal{A}_n,\ k \geq 1,$$

as a consequence of

$$\partial_{x_0} X^{(k-j)}_{n+1,j}(x) = (k-j)\, X^{(k-1-j)}_{n+1,j}(x), x \in \mathcal{A}_n,\ k \geq 1, \tag{3.5}$$

(see [12] for details).

The standard choice of 1 as the constant P_0 in (3.2) according to the classical definition of Appell sequences leads to $\widetilde{X}^{(k)}_{n+1,0} \equiv X^{(k)}_{n+1,0} \equiv \mathcal{P}^n_k$ for all $k \in \mathbb{N}_0$ and $j = 0$, where $\mathcal{P}^n_k$ is the monogenic polynomial of degree k, constructed in [21] and represented by

$$\mathcal{P}^n_k(x) = \sum_{s=0}^{k} \binom{k}{s} c_s(n)\, x_0^{k-s}\, \underline{x}^s. \tag{3.6}$$

[1]Due to the introduction of a second index j, the degree of a polynomial is written as upper index, except in appropriately identified cases.

Here $c_s(n) = c_{s,0}(n)$ $(s = 0, \dots, k)$ are given by (3.1). More explicitly,

$$c_s(n) = c_{s,0}(n) = \begin{cases} \frac{s!!(n-2)!!}{(n+s-1)!!}, & \text{if } s \text{ is odd} \\ c_{s-1,0}(n), & \text{if } s \text{ is even} \end{cases} \tag{3.7}$$

and $c_0(n) = 1$. For convenience, in [19] the $(\mathcal{P}_k^n)_{k\geq 0}$ has been called *Standard Appell Sequence* compared with other types of hypercomplex Appell sequences considered therein. As we will show in the sequel, the coefficients (3.7) of the $\mathcal{P}_k^n$ are dominating the construction process of the system of Clifford algebra-valued orthogonal polynomials, thereby justifying the special role of $(\mathcal{P}_k^n)_{k\geq 0}$ as some kind of a standard Appell sequence.

Remark 3.2. It is also worth to notice that the process of construction of the polynomials (3.2) is valid for $n \geq 2$. However the polynomials (3.6) are also defined for $n = 1$ and in this case they are isomorphic to the holomorphic powers z^k, with $z = x_0 + e_1 x_1 \in \mathcal{A}_1 \simeq \mathbb{C}$. In fact, $c_s(1) = 1$, for all $s = 0, \dots, k$ and

$$\mathcal{P}_k^1 = \sum_{s=0}^{k} \binom{k}{s} x_0^{k-s}\, \underline{x}^s = (x_0 + e_1 x_1)^k.$$

3.2. The generalized De Moivre's formula

Before dealing with three-term recurrences, the main goal of this paper, we will have a closer look to the monogenic polynomials (3.2) and study them in detail in the case of the lowest dimension $n = 2$, i.e. the *first* non-complex case. We also focus our attention only on the index $j = 0$ and the choice of $P_0 = 1$. As we have seen $\widetilde{X}_{3,0}^{(k)} \equiv X_{3,0}^{(k)} \equiv \mathcal{P}_k^2$ and therefore these polynomials represent the generalization of the holomorphic powers z^k to higher dimensions.

For the case $n = 2$, $x \in \mathcal{A}_2 \simeq \mathbb{R}^3$ and $\mathcal{A}_2 \subset \mathcal{C}\ell_{0,2}$. The Clifford algebra $\mathcal{C}\ell_{0,2}$ is 4-dimensional with basis $\{1, e_1, e_2, e_{12}\}$ and can be identified with the Hamilton's quaternion algebra $\mathbb{H}$, with

$$e_1 \simeq i, \qquad e_2 \simeq j, \qquad e_{12} \simeq k.$$

In this context, the paravectors $x = x_0 + x_1 e_1 + x_2 e_2 \in \mathcal{A}_2$ are usually also called reduced quaternions.

We start by recalling the representation stated in [12] of the Clifford-valued homogeneous polynomials (3.3), $X_{n+1,j}^{(k-j)}$ $(j = 0, \dots, k)$, namely

$$X_{n+1,j}^{(k-j)}(t, \underline{\boldsymbol{\omega}}) = A_{k,j}^n\, |x|^{k-j} \Big[(k + j + n - 1) + \underline{\boldsymbol{\omega}}\, \sqrt{1-t^2}\, \frac{d}{dt}\Big] C_{k-j}^{\frac{n-1}{2}+j}(t), \tag{3.8}$$

where

$$A_{k,j}^n := \frac{k!}{j!(n-1+2j)_{k+1-j}}, \quad t := \frac{x_0}{|x|} \in [-1,1] \quad \text{and} \quad \underline{\boldsymbol{\omega}} := \frac{\underline{x}}{|\underline{x}|} \in S^{n-1}.$$

For $j = 0$ and $n = 2$ this expression becomes

$$X_{3,0}^{(k)}(t, \underline{\boldsymbol{\omega}}) = \mathcal{P}_k^2(t, \underline{\boldsymbol{\omega}}) = |x|^k \frac{1}{k+1} \Big[k + 1 + \underline{\boldsymbol{\omega}}\, \sqrt{1-t^2}\, \frac{d}{dt}\Big] C_k^{1/2}(t).$$

As it is well-known, $C_k^{1/2}$ coincides with the Legendre polynomial L_k and $\sqrt{1-t^2}\,\frac{d}{dt}L_k(t) = L_k^1(t)$, being L_k^1 the associated Legendre function of degree k and order 1. Thus

$$\mathcal{P}_k^2(t,\underline{\boldsymbol{\omega}}) = |x|^k \frac{1}{k+1}\left[(k+1)\,L_k(t) + \underline{\boldsymbol{\omega}}\,L_k^1(t)\right].$$

Introducing spherical coordinates, we have $x = |x|\,(\cos\theta + \underline{\boldsymbol{\omega}}\sin\theta)$ ($\theta \in [0,\pi[$) with $\underline{\boldsymbol{\omega}} = \cos\varphi\, e_1 + \sin\varphi\, e_2 \in S^2$ ($\varphi \in [0,2\pi[$). It follows that

$$\mathcal{P}_k^2(t,\underline{\boldsymbol{\omega}}) = |x|^k \frac{1}{k+1}\left[(k+1)\,L_k(\cos\theta) + \underline{\boldsymbol{\omega}}\,L_k^1(\cos\theta)\right]. \tag{3.9}$$

Notice that for quaternions $q = q_0+q_1 i+q_2 j+q_3 k$, $q_s \in \mathbb{R}$ ($s=0,1,2,3$), the De Moivre's formula is given by

$$q^k = |q|^k\left[\cos(k\theta) + \underline{\boldsymbol{\omega}}\sin(k\theta)\right],$$

when $q = |q|\,(\cos\theta + \underline{\boldsymbol{\omega}}\sin\theta)$, being $\underline{\boldsymbol{\omega}} \in S^2$ (see [18]). However the powers q^k ($k \in \mathbb{N}$) are not, in general, monogenic.

Despite their coefficients are different, the structure of the trigonometric polynomials $L_k(\cos\theta)$ and $L_k^1(\cos\theta)$ is exactly the same as the polynomials $\cos(k\theta)$ and $\sin(k\theta)$, respectively for each $k \in \mathbb{N}$. In this way, we can interpret the formula (3.9) as the analogue of the De Moivre's formula for the reduced quaternionic monogenic *powers* $\widetilde{X}_{3,0}^{(k)} \equiv X_{3,0}^{(k)} \equiv \mathcal{P}_k^2$ ($k \in \mathbb{N}$).

4. Three-term recurrence relations and related properties

4.1. The structural Appell sequence

As we have seen in subsection 2.1, the Cauchy-Kovalevskaya extension of normalized polynomials includes the special case of Standard Appell Polynomials $\mathcal{P}_k^n$ obtained for arbitrary $k \geq 1$ and $j = 0$ from $c_k(n)\underline{x}^k\,P_0(\underline{x}) = c_k(n)\underline{x}^k$. But in the general case of $0 < j \leq k$ the essential contribution to the Cauchy-Kovalevskaya extension of normalized polynomials is the factor $X_{n+1,j}^{(k-j)}(x)$ defining the structure of $\widetilde{X}_{n+1,j}^{(k)}$ (cf. (3.2)). The surprising relation of the structural Appell sequence $\big(X_{n+1,j}^{(k-j)}\big)_{k\geq 0}$ with respect to ∂_{x_0} (cf. (3.5)), with the Standard Appell Polynomials $\mathcal{P}_k^n$ is the subject of the next theorem. That connection is described by shifting their coefficients, one shift decreases the degree k by j whereas the other increases the number n in the expression of the coefficients $c_k(n)$ by $2j$. This means that here the parameter n of the coefficient $c_k(n)$ in $\mathcal{P}_k^n$ is untied from its connection with the number n of components in the vector part $\underline{x}$ of $x = x_0 + \underline{x}$.

The described situation is the content of the following theorem.

Theorem 4.1. *For all $k \in \mathbb{N}_0$ and each fixed j ($j = 0,\dots,k$), it holds*

$$X_{n+1,j}^{(k-j)}(x) = \binom{k}{j}\mathcal{P}_{k-j}^{n+2j}(x), \quad x \in \mathcal{A}_n.$$

Proof. From its explicit expression (3.6), it follows

$$\mathcal{P}^{n+2j}_{k-j}(x) = \sum_{s=0}^{\lfloor\frac{k-j}{2}\rfloor} (-1)^s \binom{k-j}{2s} c_{2s}(n+2j) x_0^{k-j-2s} |\underline{x}|^{2s} + \sum_{s=0}^{\lfloor\frac{k-j-1}{2}\rfloor} (-1)^s \binom{k-j}{2s+1} c_{2s+1}(n+2j) x_0^{k-j-2s-1} |\underline{x}|^{2s}\, \underline{x}.$$

Introducing the real variable $t := \frac{x_0}{|x|}$ and $\underline{\boldsymbol{\omega}} \in S^{n-1}$, already used in (3.8), we obtain

$$\mathcal{P}^{n+2j}_{k-j}(t,\underline{\boldsymbol{\omega}}) = |x|^{k-j} \left[\sum_{s=0}^{\lfloor\frac{k-j}{2}\rfloor} (-1)^s \binom{k-j}{2s} c_{2s}(n+2j)\, t^{k-j-2s}(1-t^2)^s \right.$$
$$\left. +\ \underline{\boldsymbol{\omega}}\sqrt{1-t^2} \sum_{s=0}^{\lfloor\frac{k-j-1}{2}\rfloor} (-1)^s \binom{k-j}{2s+1} c_{2s+1}(n+2j) t^{k-j-2s-1}(1-t^2)^s \right].$$

Defining $\nu := \frac{n-1}{2} + j$, the coefficients (3.7) can also be written in the form

$$c_{2s}\,(2\nu+1) = c_{2s-1}\,(2\nu+1) = \frac{\left(\frac{1}{2}\right)_s}{\left(\nu+\frac{1}{2}\right)_s}.$$

Therefore, writing $m := k-j$, we obtain after some simplifications

$$\mathcal{P}^{2\nu+1}_{m}(t,\underline{\boldsymbol{\omega}}) = |x|^m m! \left[\sum_{s=0}^{\lfloor\frac{m}{2}\rfloor} \frac{\left(\frac{1}{2}\right)_s t^{m-2s}(t^2-1)^s}{(2s)!(m-2s)!\left(\nu+\frac{1}{2}\right)_s} \right.$$
$$\left. +\ \underline{\boldsymbol{\omega}}\sqrt{1-t^2} \sum_{s=0}^{\lfloor\frac{m-1}{2}\rfloor} \frac{\left(\frac{1}{2}\right)_s t^{m-2s-1}(t^2-1)^s}{(2s)!(m-2s-1)!\left(\nu+\frac{3}{2}\right)_s(2\nu+1)} \right].$$

Recalling now the explicit formula (3.4) for the Gegenbauer polynomials, we have

$$\mathcal{P}^{2\nu+1}_{m}(t,\underline{\boldsymbol{\omega}}) = |x|^m m! \left[\frac{C^{\nu}_m(t)}{(2\nu)_m} + \underline{\boldsymbol{\omega}}\sqrt{1-t^2}\,\frac{C^{\nu+1}_{m-1}(t)}{(2\nu+1)_m} \right]$$

and the result follows from the well-known property $\frac{d}{dt}C^{\nu}_m(t) = 2\nu C^{\nu+1}_{m-1}(t)$ and the representation (3.8) of the polynomials $X^{(m)}_{n+1,j}$ $(j = 0,\ldots,k)$. □

Using the relation stated in the previous theorem, we can rewrite the homogeneous polynomials $X^{(k-j)}_{n+1,j}$ in terms of the variables x_0 and $\underline{x}$ as

$$X^{(k-j)}_{n+1,j}(x_0,\underline{x}) = \binom{k}{j} \sum_{s=0}^{k-j} \binom{k-j}{s} c_s(n+2j)\, x_0^{k-j-s}\, \underline{x}^s, \quad x \in \mathcal{A}_n.$$

4.2. A three-term type recurrence

Using the results of the previous subsections, which emphasized the role of Standard Appel Sequence $(\mathcal{P}_k^n)_{k\geq 0}$ as a unique tool for the construction of all $\widetilde{X}_{n+1,j}^{(k)}$, we are now able to prove another important general property of hypercomplex orthogonal polynomials, namely the existence of three-term recurrence relations. One should be aware that the obtained three-term recurrence is a relation between homogeneous polynomials and therefore their structure is slightly different from the usual one. Nevertheless, it seems to us rightful to call them at least three-term type recurrence.

As we know from the real or complex case, according to Favards theorem (cf. [17, 22]) a three-term relation essentially characterizes the orthogonality of polynomials. Again, the starting point for the characterization of all elements of the considered systems of Clifford algebra-valued orthogonal polynomials by the validity of those three-term recurrence relations will be the corresponding property of the Standard Appel Sequence $(\mathcal{P}_k^n)_{k\geq 0}$.

The $\mathcal{A}_n$-valued monogenic Appell polynomials $\mathcal{P}_j^n$, here defined by (3.6) admit a variety of different representations and, even more, they satisfy the following recurrence relation (see [11]):

$$(n+k+1)\mathcal{P}_{k+2}^n(x)-((2k+n+2)x_0+\underline{x})\mathcal{P}_{k+1}^n(x)+(k+1)|x|^2\mathcal{P}_k^n(x)=0, \quad (4.1)$$

$$\mathcal{P}_0^n(x)=1, \qquad \mathcal{P}_1^n(x)=x_0+\frac{1}{n}\underline{x},\ x\in\mathcal{A}_n. \quad (4.2)$$

This relation together with Theorem 4.1 allow to deduce a three-term type recurrence for the paravector valued polynomials (3.8), which in turn can be used to obtain in Subsection 4.3 several properties of the monogenic polynomials $\widetilde{X}_{n+1,j}^{(k)}$ defined in (3.2).

Theorem 4.2. *For all $k\in\mathbb{N}_0$ and each fixed j $(j=0,\dots,k)$, the paravector-valued polynomials $X_{n+1,j}^{(k-j)}(x)$, $x\in\mathcal{A}_n$, satisfy the three-term type recurrence*

$$\begin{aligned}(n+k+1+j)(k+2-j)X_{n+1,j}^{(k+2-j)} &- \left[(n+2k+2)x_0+\underline{x}\right](k+2)X_{n+1,j}^{(k+1-j)}\\ &+(k+2)(k+1)\,|x|^2\,X_{n+1,j}^{(k-j)}=0, \quad (4.3)\end{aligned}$$

$$X_{n+1,j}^{(0)}=1,\ \ X_{n+1,j}^{(1)}=(j+1)\Big(x_0+\frac{1}{n+2j}\underline{x}\Big). \quad (4.4)$$

Proof. From the recurrence (4.1)-(4.2), it follows at once that for all $x\in\mathcal{A}_n$, all $k\in\mathbb{N}_0$ and each fixed j $(j=0,\dots,k)$,

$$\begin{aligned}(n+j+k+1)\mathcal{P}_{k-j+2}^{n+2j}(x) &- ((2k+n+2)x_0+\underline{x})\mathcal{P}_{k-j+1}^{n+2j}(x)\\ &+(k-j+1)|x|^2\mathcal{P}_{k-j}^{n+2j}(x)=0.\end{aligned}$$

$$\mathcal{P}_0^{n+2j}(x)=1, \qquad \mathcal{P}_1^{n+2j}(x)=x_0+\frac{1}{n+2j}\underline{x}.$$

Using now Theorem 4.1 the result follows immediately. $\square$

Corollary 4.3. *For all $k \in \mathbb{N}_0$ and each fixed j $(j = 0, \ldots, k)$, the monogenic polynomials $\widetilde{X}^{(k)}_{n+1,j}(x)$, $x \in \mathcal{A}_n$ satisfy the three-term type recurrence*

$$(n+k+1+j)(k+2-j)\widetilde{X}^{(k+2)}_{n+1,j} - [(n+2k+2)x_0 + \underline{x}]\,(k+2)\widetilde{X}^{(k+1)}_{n+1,j} + (k+2)(k+1)\,|x|^2\,\widetilde{X}^{(k)}_{n+1,j} = 0, \quad (4.5)$$

$$\widetilde{X}^{(j)}_{n+1,j} = P_j(\underline{x}), \quad \widetilde{X}^{(j+1)}_{n+1,j} = (j+1)(x_0 + \frac{1}{n+2j}\underline{x})P_j(\underline{x}). \quad (4.6)$$

Proof. The result follows from the multiplication of the relations (4.3) and (4.4) by an arbitrary chosen monogenic constant, i. e. by $P_j \in \mathcal{M}_j(\mathbb{R}^n, \mathcal{C}\ell_{0,n})$, for some fixed j $(j = 0, \ldots, k)$, with $k \in \mathbb{N}_0$. □

Figure 1 illustrates the recursive construction procedure of $\widetilde{X}^{(k)}_{n+1,j}$ in terms of the Standard Appell polynomials with shifted coefficients.

4.3. A second order differential equation

Three-term relations can be considered as some type of difference equations as it is pointed out in the work [22] on numerical problems connected with three-term relations. This makes it easy to understand why orthogonal polynomials can also be obtained as solutions of second order differential equations. Like in the real case the proof of this fact is straightforward but, for instance in [2], it is shown that the characterization of orthogonal polynomials by second order differential equations can lead to far reaching generalizations of orthogonal polynomials. Such argument seems to us sufficient to ask for the validity of a corresponding second order differential equation for Clifford algebra-valued orthogonal polynomials. This issue is addressed in the following theorem.

Theorem 4.4. *For all $k \in \mathbb{N}_0$ and each fixed j $(j = 0, \ldots, k)$, the monogenic polynomials $\widetilde{X}^{(k)}_{n+1,j}(x)$, $x \in \mathcal{A}_n$, satisfy the second order differential equation*

$$|x|^2\partial^2 y(x) - ((n+2k-2)x_0 + \underline{x})\partial y(x) + (n+k+j-1)(k-j)y(x) = 0. \quad (4.7)$$

Proof. The result follows from Corollary 4.3 and the fact that the polynomials $\widetilde{X}^{(k)}_{n+1,j}$ $(k \in \mathbb{N}_0,\ j = 0, \ldots, k)$ form an Appell system. □

Remark 4.5. The second-order differential equation (4.7) characterizes the inner structure of any family of monogenic Appell polynomials. In fact, consider a family $\left\{\mathcal{F}^{(k)}_j : j = 0, \ldots, k\right\}$ of monogenic Appell polynomials of degree k $(k \in \mathbb{N}_0)$ which are solutions of the differential equation (4.7). For each fixed $j, (j = 0, \ldots k)$ the Appell property of the polynomials $\mathcal{F}^{(k)}_j$ implies at once that they satisfy a three-term type recurrence of the form (4.5).
Observe that when $k = j$ the third term in (4.7) vanishes as well as the first and second order hypercomplex derivatives of $\mathcal{F}^{(j)}_j$ (cf. Remark 3.1) and therefore, (4.7) being trivially satisfied, does not reveal any new information

about the Appell family. On the other hand, when $k = j + 1$, the differential equation together with the Appell property applied to $\mathcal{F}_j^{(j+1)}$ gives again a vanishing first term. The result is the expression of $\mathcal{F}_j^{(j+1)}$ by $\mathcal{F}_j^{(j)}$ in the form

$$\mathcal{F}_j^{(j+1)} = (j+1)(x_0 + \frac{1}{n+2j}\underline{x})\mathcal{F}_j^{(j)}. \tag{4.8}$$

The choice of the polynomial $\mathcal{F}_j^{(j)} = P_j(\underline{x}) \in \mathcal{M}_j(\mathbb{R}^n, \mathcal{C}\ell_{0,n})$ in (4.8) corresponds to (4.6) and determines the structure of the Appell polynomials under consideration.

It is also worth to notice that the case $j = 0$ corresponds to the choice of an ordinary real or Clifford algebra-valued constant as an initial value. In particular, the choice of $\mathcal{F}_0^{(0)} = 1$ leads to the Standard Appell Polynomials $\mathcal{P}_k^n$, revisited in Section 3.1. On the other hand, the choice of an arbitrary fixed vector-valued polynomial $P_j(\underline{x})$ in (4.8) corresponds to the choice of a monogenic constant as initial value, taking into account the role of ∂ as the hypercomplex derivative of a monogenic function. This perspective coincides completely with the result obtained in the paper [30] for the construction of monogenic Appell sequences having a monogenic constant as initial value.

4.4. Ladder operators

Ladder operators for the sequence $\left\{\widetilde{X}_{n+1,j}^{(k)} : j = 0, \ldots, k\right\}_{k\in\mathbb{N}_0}$ can now be obtained in an easy way. It is clear that the Appell property of the polynomials $\widetilde{X}_{n+1,j}^{(k)}$ $(j = 0, \ldots, k)$ gives naturally a lower operator for that sequence. A raising operator can be obtained using the so-called Euler operator $\mathbb{E} := \sum_{i=0}^{n} x_i \frac{\partial}{\partial_{x_i}}$ in $\mathbb{R}^{n+1}$. Any homogeneous function is an eigenfunction of this operator and therefore it appears frequently in the context of homogeneous polynomials. The Euler operator played also a crucial role in the papers [9,10] where monogenic Laguerre polynomials and Laguerre-type exponential functions were constructed.

Theorem 4.6. *For all $k \in \mathbb{N}_0$ and each fixed j $(j = 0, \ldots, k)$, the operators $\widehat{M} := \mathbb{E}\left[(2\mathbb{E} + n - 2)x_0 + \underline{x} - |x|^2\partial\right]$ and $\widehat{P} := \partial$ are such that*

$$\widehat{M}\widetilde{X}_{n+1,j}^{(k)} = (n+k+j)(k+1-j)\widetilde{X}_{n+1,j}^{(k+1)} \tag{4.9}$$

$$\widehat{P}\widetilde{X}_{n+1,j}^{(k)} = k\,\widetilde{X}_{n+1,j}^{(k-1)}. \tag{4.10}$$

Proof. The second identity follows from the fact that the considered polynomials form an Appell system, for each fixed j $(j = 0, \ldots, k)$ and $k \geq 1$. The first identity is easily obtained taking into account (4.5)-(4.6) (see also Remark 3.1), the Appell property and the homogeneity of the polynomials. □

Theorem 4.7. *For all $k \in \mathbb{N}_0$ and each fixed j $(j = 0, \ldots, k)$, the monogenic polynomials $\widetilde{X}_{n+1,j}^{(k)}$ are eigenvectors of the operator $\widehat{M}\widehat{P}$ with eigenvalues $\alpha_{k,j}^n := k(n+k+j-1)(k-j)$.*

Proof. Follows immediately from identities (4.9)-(4.10). □

5. Final remarks

The theory of orthogonal polynomials in one or several real or complex variables is well known and has a wide range of applications for solving mathematical and physical problems. At the first glance, their importance seems to be based on their analytical properties, particularly because of their role in almost all aspects of approximation theory. But subject of intensive research are also their algebraic properties which lay behind the treatment of Clifford algebra-valued orthogonal polynomials by representation theoretical methods as we mentioned in the introduction.

In the present paper we tried to put emphasis on *general analytical aspects* like three-term recurrence relations, the characterization as solutions of differential equations and the construction by an operational approach. As far as we know, contrary to frequent studies on generalized types of classical orthogonal polynomials, those general properties have not been studied so far in the context of hypercomplex homogeneous polynomial.

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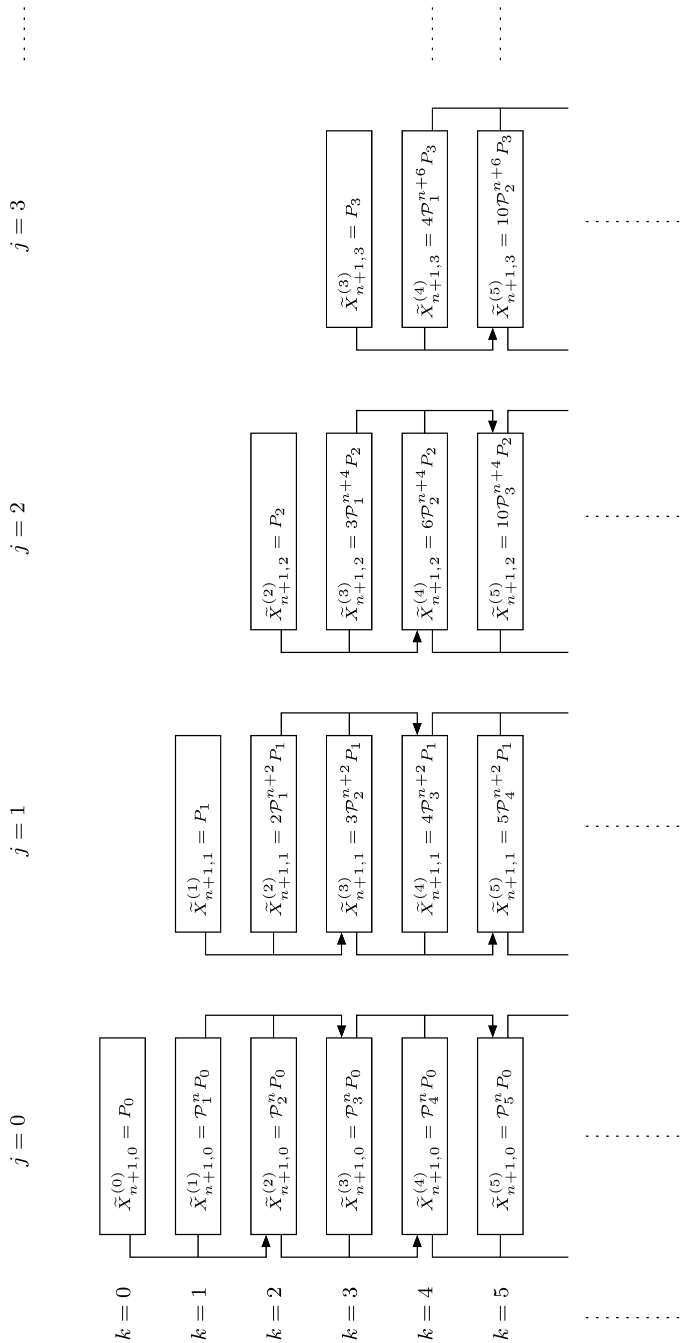

FIGURE 1. $\widetilde{X}^{(k)}_{n+1,j}$ - Recursive construction scheme