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Maria Antónia Forjaz, António Mário Almeida, T. de Lacerda–Arôso, and Jorge Pamplona

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Special Matrices for Visco-elastic Systems

Maria Antónia Forjaz*, António Mário Almeida †, T. de Lacerda–Arôso† and Jorge Pamplona**

*Centre of Mathematics, University of Minho
†Centre of Physics, University of Minho
**Institute of Earth Sciences (ICT), University of Minho Pole

Abstract. In this work the evolution of visco-elastic systems under external stress is addressed. An approach as a mixed complementary eigenvalue problem to model the geological folding and asymmetric boudinage in the same direction is considered. A matricial dynamics equation that comprehends elasticity and viscosity matrices is presented. An algorithm to connect material points and to build the adjacency matrix has been developed. Numerical results for a set of 16 nodes are shown.

Keywords: complementary eigenvalue problem, visco-elastic systems, physical model, adjacency matrix
PACS: 02.60.Cb

INTRODUCTION

The study of instabilities and bifurcations in systems with viscosity has been motivated by many experimental observations related to technological problems or industrial processes. In [1] the mathematical problem of the mixed complementary eigenproblem (MEiCP) is motivated by the study of divergence instabilities of static equilibrium states of finite dimensional mechanical systems with unilateral frictional contact.

This kind of approach is suitable to tackle geological problems, namely the prediction of the evolution less ductile layers embedded in a more ductile matrix under simple progressive shear, i.e. rotation and non-coaxial strain. Under these conditions it is expected that only one kind of geological structure (e.g. in compressive field) happens in the same direction and strain field [2]. However, the observation of simultaneous occurrence of folding and asymmetric boudinage in the same direction opens a new trend [3] that opposes the current state of the art in structural geology.

An attempt to formulate this type of problem has been started earlier in [4]. In this work we briefly address the problem of MEiCP by stating the dynamics equation. A 2D model is proposed for a visco-elastic system, for which we show the adjacency matrix for a reduced set of points with connections, followed by final remarks on future work.

DYNAMICS EQUATION FOR VISCO-ELASTIC SYSTEMS

Under constant applied forces F^0 the dynamics of the system is governed by the momentum balance equation (Newton second law), considering elasticity and viscosity

$$M\ddot{u}(t) = F^0 + R(t) - Ku(t) - B\dot{u}(t) \tag{1}$$

where M is the mass matrix (symmetric and positive definite), K is the elasticity matrix (positive definite), B is the viscosity matrix (symmetric), R is the reaction force at time t ($t \geq 0$), and $u(t)$ and $\dot{u}(t)$ stand for the individual point masses displacements and velocities, respectively.

In [5] and [6] a particular type of dynamic instability of an equilibrium state of the system is considered. Under the same applied forces F^0 for a constant displacement rate ($\ddot{u} = 0$), and in the absence of viscosity B, equation (1) assumes the form $Ku^0 = F^0 + R^0$.

It is shown that for some t there are dynamic solutions of the form

$$u(t) = u^0 + \alpha(t)v$$
$$R(t) = R^0 + \beta(t)w$$

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if and only if there exists a number $\lambda \geq 0$, and two vectors v and w ($v \neq 0$), are such that

$$\begin{aligned}(\lambda^2 M + K)v &= w \\ w_j &= 0 \\ v_d &= 0\end{aligned}$$

which can be treated as mixed complementary eigenproblem (MEiCP).

2D PHYSICAL MODEL

In geological processes (e.g. layering, folding and boudinage) materials with contrasting physical properties are often in contact. The understanding of the inter and intra layer contact dynamics is the first step to solve relevant and more complex geological problems.

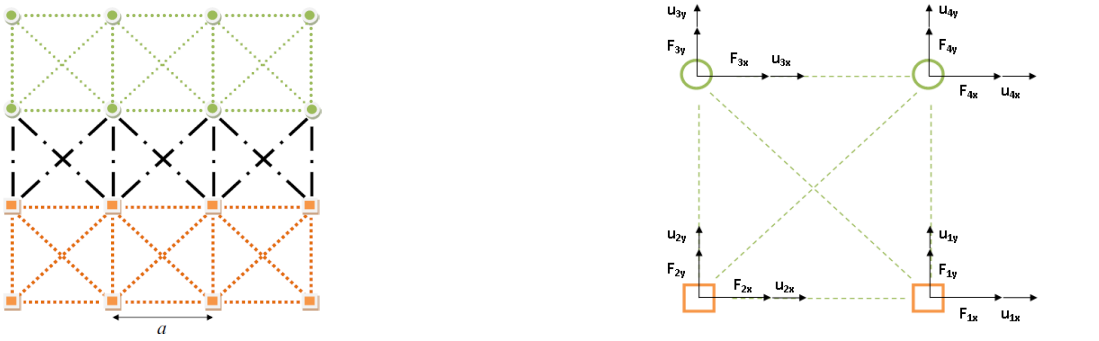
We propose to model a physical material by a 2D domain of points, each characterised by its mass (preserving material density), viscosity, and elasticity, defining a mesh of nodes.

At first, each node is linked to every other within a predefined range (Figure 1A). Afterwards, by defining the relevant bonds among the material points, such links are reduced so that each node becomes connected to no more than 8 adjacent nodes (Figure 1B). In average, each node is apart from its nearest neighbour by a distance a, and to each one is assigned a fraction of the total mass ($density \times a^3$).

(A) Complete set of one-to-one material points bonding. (B) Post-reduction set of one-to-one material points bonding.

FIGURE 1. Bidimensional domain of 16 material points randomly distributed.

In Figure 2A two layers of point masses are shown: round dots for material A and squares for material B. Dashed lines depict the interactions amid them. The particular connections between neighbouring point masses belonging to either one or the other material are coloured in black. A 4-mass cell is outlined in Figure 2B with nodal displacements and the forces applied therein.



(A) A model for materials A and B in contact. (B) Applied forces and displacements at each node.

FIGURE 2. 2D model for two materials in contact.

Considering the elastic binding between every two point masses, the elastic force F_e acting on each individual point mass related to its respective displacement u is given by the Hooke's law

$$F_e = -ku, \tag{2}$$

where k is the elastic constant.

For the frictional contact between two point masses, assuming Coulomb friction (proportional to the velocity $\dot{u}$), the viscous force is given by

$$F_b = -\mu\dot{u}, \tag{3}$$

where μ is the viscosity coefficient.

THE ADJACENCY MATRIX

The formulation of equation (1) as a complementarity eigenproblem (EiCP) requires the construction of elasticity and viscosity matrices, K and B.

Starting by the simplest case of 4 nodes forming a quadrilateral, equations (2) and (3) may be written in matricial form for two dimensions

$$\begin{pmatrix} F_{e\,1x} \\ F_{e\,1y} \\ F_{e\,2x} \\ F_{e\,2y} \\ F_{e\,3x} \\ F_{e\,3y} \\ F_{e\,4x} \\ F_{e\,4y} \end{pmatrix} = \mathrm{K} \begin{pmatrix} u_{1x} \\ u_{1y} \\ u_{2x} \\ u_{2y} \\ u_{3x} \\ u_{3y} \\ u_{4x} \\ u_{4y} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} F_{b\,1x} \\ F_{b\,1y} \\ F_{b\,2x} \\ F_{b\,2y} \\ F_{b\,3x} \\ F_{b\,3y} \\ F_{b\,4x} \\ F_{b\,4y} \end{pmatrix} = \mathrm{B} \begin{pmatrix} \dot{u}_{1x} \\ \dot{u}_{1y} \\ \dot{u}_{2x} \\ \dot{u}_{2y} \\ \dot{u}_{3x} \\ \dot{u}_{3y} \\ \dot{u}_{4x} \\ \dot{u}_{4y} \end{pmatrix}.$$

Both K and B matrices are obtained from the projections of the forces and either the displacements or velocities along the directions X and Y, according to the angles $\alpha, \beta, \gamma, \theta, \varphi$ and δ (Figure 3).

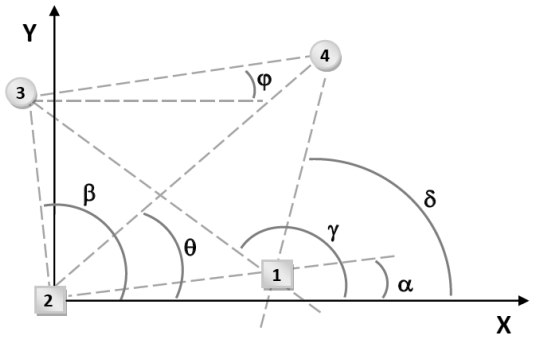


FIGURE 3. Angles defined by the lines linking two nodes relative to the X axis.

Underlying these matrices there is the adjacency matrix $A_{8\times8}$ locally defined by

$$A_{8\times8} = \left(\begin{array}{cc|cc} P & O_\alpha & O_\gamma & O_\delta \\ O_\alpha & Q & O_\beta & O_\theta \\ \hline O_\gamma & O_\beta & R & O_\varphi \\ O_\delta & O_\theta & O_\varphi & S \end{array}\right),$$

where sub-matrices P, Q, R and S traduce the geometrical relations between each node with its neighbors, such that:

$$P=\begin{pmatrix} c_\alpha^2+c_\delta^2+c_\gamma^2 & c_\alpha s_\alpha+c_\delta s_\delta+c_\gamma s_\gamma \\ c_\alpha s_\alpha+c_\delta s_\delta+c_\gamma s_\gamma & s_\alpha^2+s_\delta^2+s_\gamma^2 \end{pmatrix}, \quad Q=\begin{pmatrix} c_\alpha^2+c_\beta^2+c_\theta^2 & c_\alpha s_\alpha+c_\beta s_\beta+c_\theta s_\theta \\ c_\alpha s_\alpha+c_\beta s_\beta+c_\theta s_\theta & s_\alpha^2+s_\beta^2+s_\theta^2 \end{pmatrix}$$

$$R=\begin{pmatrix} c_\varphi^2+c_\beta^2+c_\gamma^2 & c_\varphi s_\varphi+c\beta s_\beta+c_\gamma s_\gamma \\ c_\varphi s_\varphi+c_\beta s_\beta+c_\gamma s_\gamma & s_\varphi^2+s_\beta^2+s_\gamma^2 \end{pmatrix}, \quad S=\begin{pmatrix} c_\varphi^2+c_\delta^2+c_\theta^2 & c_\varphi s_\varphi+c_\delta s_\delta+c_\theta s_\theta \\ c_\varphi s_\varphi+c_\delta s_\delta+c_\theta s_\theta & s_\varphi^2+s_\delta^2+s_\theta^2 \end{pmatrix}$$

and the non-diagonal sub-matrices $O_i=\begin{pmatrix} c_i^2 & c_i s_i \\ c_i s_i & s_i^2 \end{pmatrix}$ reproduce the linkage between two bonded nodes. (It is used the short notation $c_i=\cos i$, $s_i=\sin i$, $c_i s_i=\cos i \sin i$.)

For the set of 16 nodes, depicted in Figure 1, we obtain the $A_{32\times 32}$ matrix from the cartesian coordinates (not shown). Due to its dimension it is only shown in (4) the first 8×8 block matrix.

$$A_{32\times 32}=\begin{pmatrix} \begin{pmatrix} \mathbf{2.06} & -0.30 \\ -0.30 & \mathbf{2.94} \end{pmatrix} & \begin{matrix} 0.99 & -0.10 \\ -0.10 & 0.01 \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \dots \\ \begin{matrix} 0.99 & -0.10 \\ -0.10 & 0.01 \end{matrix} & \begin{pmatrix} \mathbf{4.09} & 0.10 \\ 0.10 & \mathbf{3.91} \end{pmatrix} & \begin{matrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \dots \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{matrix} & \begin{pmatrix} \mathbf{3.08} & 0 \\ 0 & \mathbf{1.92} \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \dots \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{pmatrix} \mathbf{2.88} & 0.01 \\ 0.01 & \mathbf{2.12} \end{pmatrix} & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix} \tag{4}$$

As expected the $A_{32\times 32}$ matrix is symmetric. Along the diagonal, subsets of matrix elements (equivalent to P, Q, R and S, referred above) are highlighted. The trace of each subset equals the number of bonds for the respective node. For unconnected nodes the respective matrix elements are equal to zero. As can be seen in Figure 1B, the nodes 1 and 3 are not connected so the respective matrix elements a_{15}, a_{16}, a_{25} and a_{26} are null.

Generalizing for a 2D domain of n mass points, the adjacency matrix A, and consequently B and K matrices, will have $2n\times 2n$ dimension.

The algorithms for designing the mesh, for the calculation of the A matrix and for the graphical plots have been developed under MATLAB [7].

FUTURE WORK

The application of the algorithm developed to obtain the A matrix to an arbitrary large number of points is trivial.

It is necessary to evolve from the A matrix to obtain K and B matrices comprehending the elastic constant and the viscosity coefficient for domains comprising one or two kinds of materials.

To apply this model in systems with high viscosity material embedded in a low viscosity medium, equation (1) must be fully formulated as a MEiCP problem.

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