Projective embeddings of dual polar spaces of mixed type

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Abstract

The classification of polar spaces was completed by Tits in his monumental work [19] who showed as an intermediate step that almost all of them were embeddable in a projective space. The current paper is part of a project to classify all full projective embeddings of the duals of polar spaces. A complete classification is available for rank 2 by the work of Tits [19], Buekenhout-Lefèvre [4] and Dienst [13]. Recently, significant process was made in the rank 3 case by De Bruyn and Van Maldeghem [12], who showed, among other things, that the members of five families of dual polar spaces of rank 3 related to alternative division rings have full projective embeddings. For two of these families, the quaternionic dual polar spaces and the dual polar spaces of mixed type, it was not yet known whether full projective embeddings exist for rank at least four. In the present paper, we prove that any "mixed dual polar space" of rank at least 2 has a full projective embedding.

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1 Introduction

Tits classified spherical buildings of rank at least 3 in his monumental work [19]. In particular, he treated the buildings of type C_n, also known as polar spaces. Tits's work on polar spaces extended earlier work of Veldkamp [21], who gave an axiomatic description of the geometry of isotropic or singular subspaces of a sesquilinear or quadratic form. Tits simplified Veldkamp's list of axioms, but in order to accommodate all examples, he had to enlarge the class of polar spaces with new objects, namely with polar spaces associated with new forms (which he coined pseudo-quadratic forms) and with polar spaces associated with so-called Cayley-Dickson division algebras.

An important part in Veldkamp's and Tits' work was to show that almost all these polar spaces have full projective embeddings. The notable exceptions here are the polar spaces associated with Cayley-Dickson division algebras. Projective embeddings have been studied for other classes of geometries and in many cases, it can be shown that a given geometry has a "largest full embedding" from which all other full embeddings can be derived by projecting. This embedding is called the *universal embedding.* Classifying all full projective embeddings of a given geometry then boils down to determining this universal embedding.

This paper is part of a project to classify all full projective embeddings of thick dual polar spaces of rank at least 3. This classification problem is open since the early nineties when the first results on this topic were published, see e.g. Shult [18, p. 229]. By results of Kasikova and Shult [14, Section 4.6], we know that if a thick dual polar space has a full projective embedding, then it also has a universal embedding. We note here that dual polar spaces of rank 2 are precisely the generalized quadrangles, and a classification of all full projective embeddings of these geometries is known by the work of Tits [19], Buekenhout & Lefèvre [4] (finite case) and Dienst [13] (general case).

Suppose Π is a thick polar space of rank at least 3 that is not associated with a Cayley-Dickson division algebra such that the dual polar space Δ corresponding to Π has a full projective embedding. If Q is a quad of Δ (i.e. a convex subgeometry that is a generalized quadrangle), then the embeddability of Δ implies that Q is embeddable as well. On the other hand, as the polar space Π is embeddable, the point-line dual of Q should also be embeddable. By Tits [19, Proposition 10.10], we then know that Q or its dual Q^D should be of four possible types, and related to an alternative division ring that is quadratic over a subfield of its center. An alternative division ring $\mathbb{O}$ is called *quadratic* over a subfield $\mathbb{F}$ of its center $Z(\mathbb{O})$ if there exist (necessarily unique) maps $T : \mathbb{O} \to \mathbb{F}$ and $N : \mathbb{O} \to \mathbb{F}$ such that

- $a^2 - T(a) \cdot a + N(a) = 0$ for all $a \in \mathbb{O}$;
- $T(a) = 2a$ and $N(a) = a^2$ for all $a \in \mathbb{F}$.

The following proposition taken from Tits and Weiss [20, Theorem 20.3] describes the possibilities.

Proposition 1.1 ([20]) *Suppose $\mathbb{O}$ is an alternative division ring that is quadratic over some subfield $\mathbb{F}$ of its center $Z(\mathbb{O})$. Then one of the following five cases occurs:*

(*a*) $\mathbb{O} = \mathbb{F}$ *is a field;*

(*b*) $\mathbb{O}$ *and* $\mathbb{F}$ *are fields such that* $\mathbb{O}$ *is a separable quadratic extension of* $\mathbb{F}$*;*

(*c*) $\mathbb{O}$ *is a field of characteristic* 2 *and* $\mathbb{O}^2 \subseteq \mathbb{F} \neq \mathbb{O}$*;*

(*d*) $\mathbb{O}$ *is a quaternion division algebra over* $\mathbb{F} = Z(\mathbb{O})$*;*

(*e*) $\mathbb{O}$ *is a Cayley-Dickson division algebra over* $\mathbb{F} = Z(\mathbb{O})$*.*

De Bruyn and Van Maldeghem [11] constructed in a uniform way polar spaces from pairs $(\mathbb{O}, \mathbb{F})$ as in Proposition 1.1. In the cases (a), (b), (c) and (d), such polar spaces can be defined for arbitrary rank $n \geq 2$, while in case (e) they can only be defined for ranks 2 and 3. All these polar spaces, except for those corresponding to case (e), are fully embeddable in a projective space. The polar spaces corresponding to cases (a), (b), (c) and (d) are respectively called *symplectic polar spaces*, *Hermitian polar spaces*, *polar spaces of mixed type* and *quaternionic polar spaces*.

With each polar space Π, there is associated a dual polar space in the sense of Cameron [5]. This is the point-line geometry whose points and lines are the maximal and next-to-maximal singular subspaces of Π, with incidence being reverse containment. The symplectic and Hermitian dual polar spaces are embeddable and their universal embeddings are known, see [1, 6, 7, 10, 15, 16]. The universal embeddings of all rank 3 dual polar spaces corresponding to a pair $(\mathbb{O}, \mathbb{F})$ as in Proposition 1.1 have been determined in [12] using a uniform treatment. For rank $n \geq 4$, the classification of full projective embeddings of quaternionic dual polar spaces and dual polar spaces of mixed type remained open. In fact, it was not yet known whether such dual polar spaces have full projective embeddings at all.

In the present paper, we show that all dual polar spaces of mixed type have full projective embeddings. Specifically, we prove the following.

Theorem 1.2 *Suppose $\mathbb{F}$ and $\mathbb{F}'$ are two fields of characteristic 2 such that $(\mathbb{F}')^2 = \{\lambda^2 \,|\, \lambda \in \mathbb{F}'\} \subseteq \mathbb{F} \neq \mathbb{F}'$. Let d denote the dimension of $\mathbb{F}'$ regarded as a vector space over its subfield $\mathbb{F}$. If Δ denotes the dual polar space of mixed type of rank n associated with $(\mathbb{F}', \mathbb{F})$, then Δ has a full projective embedding in a projective space $\mathrm{PG}(U)$, where U is a vector space of dimension $2^n + d \cdot \left[\binom{2n}{n} - \binom{2n}{n-2} - 2^n\right]$ over $\mathbb{F}$.*

Note that the number d can be infinite and so Δ can have full projective embeddings in an infinite dimensional projective space.

In Section 2, we give a construction for the polar spaces of mixed type. In Section 8, we show that the corresponding dual polar spaces have full projective embeddings and in Section 9, we determine the dimensions of these embeddings. We do not know whether the constructed projective embeddings are universal if the rank is at least 4. In Sections 3 till 7, we discuss the machinery that will be used in the proof. Along our way, we also explain how a basis of the vector space W can be explicitly determined.

There are several similarities between Hermitian dual polar spaces and dual polar spaces of mixed type. In fact, the existence proof for full projective embeddings of dual polar spaces of mixed type can be adapted so that it also includes the case of Hermitian dual polar spaces. That is why in the present paper we have opted to give a treatment that includes both these families of (dual) polar spaces.

2 Construction of the polar space $\Pi(\mathbb{F}', \mathbb{F}, \sigma, n)$

Suppose $\mathbb{F}$ and $\mathbb{F}'$ are two fields and σ is an automorphism of $\mathbb{F}'$ such that one of the following two cases occurs:

(I) $\mathbb{F}'$ is a separable quadratic extension of $\mathbb{F}$ and σ denotes the unique nontrivial automorphism of $\mathbb{F}'$ fixing each element of $\mathbb{F}$.

(II) $\mathbb{F}'$ is a field of characteristic 2, σ is the trivial automorphism of $\mathbb{F}'$ and $(\mathbb{F}')^2 = \{\lambda^2 \,|\, \lambda \in \mathbb{F}'\} \subseteq \mathbb{F} \neq \mathbb{F}'$.

If case (I) occurs, then $\sigma^2 = 1$ and $\mathbb{F}$ is the fixfield of the automorphism σ. Notice that if case (II) occurs, then we also have $\sigma^2 = 1$. Regardless of whether case (I) or (II) occurs, the following always holds.

Lemma 2.1 *For any $\lambda \in \mathbb{F}'$, the elements $\lambda + \lambda^\sigma$ and $\lambda^{\sigma+1} := \lambda^\sigma \cdot \lambda$ belong to $\mathbb{F}$.*

Throughout this paper, V will be a vector space of even dimension $2n \geq 2$ over $\mathbb{F}'$ and $B^* := (\bar{b}_1^*, \bar{b}_2^*, \ldots, \bar{b}_{2n}^*)$ will be a fixed ordered basis of V. For every vector $\bar{v} = X_1\bar{b}_1^* + X_2\bar{b}_2^* + \cdots + X_{2n}\bar{b}_{2n}^*$ of V, we define

$$f(\bar{v}) := X_1^\sigma X_2 + X_3^\sigma X_4 + \cdots + X_{2n-1}^\sigma X_{2n}.$$

Then $f(\lambda\bar{v}) = \lambda^{\sigma+1} f(\bar{v})$ for all $\bar{v} \in V$ and all $\lambda \in \mathbb{F}'$.

For all vectors $\bar{v} = X_1\bar{b}_1^* + X_2\bar{b}_2^* + \cdots + X_{2n}\bar{b}_{2n}^*$ and $\bar{w} = Y_1\bar{b}_1^* + Y_2\bar{b}_2^* + \cdots + Y_{2n}\bar{b}_{2n}^*$ of V, we define

$$\begin{aligned} g(\bar{v}, \bar{w}) &:= f(\bar{v} + \bar{w}) - f(\bar{v}) - f(\bar{w}) \\ &= (X_1^\sigma Y_2 + Y_1^\sigma X_2) + \cdots + (X_{2n-1}^\sigma Y_{2n} + Y_{2n-1}^\sigma X_{2n}) \end{aligned}$$

and

$$h(\bar{v}, \bar{w}) := (X_1^\sigma Y_2 - X_2^\sigma Y_1) + \cdots + (X_{2n-1}^\sigma Y_{2n} - X_{2n}^\sigma Y_{2n-1}).$$

If case (I) occurs, then h is a nondegenerate anti-Hermitian form on V, and if case (II) occurs, then h is a nondegenerate alternating bilinear form on V. Observe that $h(\bar{v}, \bar{v}) = f(\bar{v}) - f(\bar{v})^\sigma$ for every $\bar{v} \in V$. The verification of the following two lemmas is straightforward.

Lemma 2.2 *Suppose case (I) occurs. Then for every $\lambda \in \mathbb{F}' \setminus \mathbb{F}$ and for all $\bar{v}, \bar{w} \in V$, we have*

$$h(\bar{v}, \bar{w}) = \frac{\lambda\Big(g(\bar{v}, \bar{w})^\sigma - g(\bar{v}, \bar{w})\Big) - \Big(g(\lambda\bar{v}, \bar{w})^\sigma - g(\lambda\bar{v}, \bar{w})\Big)}{\lambda^\sigma - \lambda}.$$

Lemma 2.3 *If case (II) occurs, then $h = g$.*

By relying on Lemmas 2.1 and 2.3, the following can be proved.

Lemma 2.4 *Let $\bar{v}, \bar{w} \in V$ and $\lambda \in \mathbb{F}'$.*

- *If case (I) occurs, then* $g(\lambda v, \bar{w}) - \lambda^\sigma \cdot h(\bar{v}, \bar{w}) \in \mathbb{F}$.
- *If case (II) occurs, then* $g(\lambda v, \bar{w}) - \lambda^\sigma \cdot h(\bar{v}, \bar{w}) = 0 \in \mathbb{F}$.

Lemma 2.5 *Let* $\bar{v}_1, \bar{v}_2 \in V$ *and* $\lambda_1, \lambda_2 \in \mathbb{F}'$. *Then* $g(\lambda_1 \bar{v}_1, \lambda_2 \bar{v}_2) - \lambda_1^\sigma \lambda_2 \cdot h(\bar{v}_1, \bar{v}_2) \in \mathbb{F}$.

Proof. By Lemma 2.4, we know that $g(\lambda_1 \bar{v}_1, \lambda_2 \bar{v}_2) - \lambda_1^\sigma \cdot h(\bar{v}_1, \lambda_2 \bar{v}_2) = g(\lambda_1 \bar{v}_1, \lambda_2 \bar{v}_2) - \lambda_1^\sigma \lambda_2 \cdot h(\bar{v}_1, \bar{v}_2)$ belongs to $\mathbb{F}$. ∎

The following is an immediate consequence of Lemmas 2.4 and 2.5.

Corollary 2.6
- *Let* $\bar{v}, \bar{w} \in V$. *Then* $g(\lambda \bar{v}, \bar{w}) \in \mathbb{F}$ *for all* $\lambda \in \mathbb{F}'$ *if and only if* $h(\bar{v}, \bar{w}) = 0$.
- *Let* $\bar{v}_1, \bar{v}_2 \in V$ *and* $\lambda_1, \lambda_2 \in \mathbb{F}'$. *If* $h(\bar{v}_1, \bar{v}_2) = 0$, *then* $g(\lambda_1 \bar{v}_1, \lambda_2 \bar{v}_2) \in \mathbb{F}$.

Since $f(\lambda \bar{v}) = \lambda^{\sigma+1} f(\bar{v})$ and $\lambda^{\sigma+1} \in \mathbb{F}$ for all $\bar{v} \in V$ and all $\lambda \in \mathbb{F}'$, the (nonempty) set $\mathcal{P}$ consisting of all points $\langle \bar{v} \rangle$ of $\mathrm{PG}(V)$ satisfying $f(\bar{v}) \in \mathbb{F}$ is well-defined. Let $\mathcal{L}$ denote the set of all lines of $\mathrm{PG}(V)$ that have all their points in $\mathcal{P}$, and let $\Pi(\mathbb{F}', \mathbb{F}, \sigma, n)$ be the point-line geometry with point set $\mathcal{P}$, line set $\mathcal{L}$ and incidence derived from $\mathrm{PG}(V)$.

Lemma 2.7 *If* $p_1 = \langle \bar{v}_1 \rangle$ *and* $p_2 = \langle \bar{v}_2 \rangle$ *are two distinct points of* $\mathcal{P}$, *then* p_1 *and* p_2 *are collinear in* $\Pi(\mathbb{F}', \mathbb{F}, \sigma, n)$ *if and only if* $h(\bar{v}_1, \bar{v}_2) = 0$.

Proof. The points p_1 and p_2 are collinear points of $\Pi(\mathbb{F}', \mathbb{F}, \sigma, n)$ if and only if $f(\bar{v}_1 + \lambda \bar{v}_2) \in \mathbb{F}$ for every $\lambda \in \mathbb{F}'$. Now, $f(\bar{v}_1 + \lambda \bar{v}_2) = f(\bar{v}_1) + f(\lambda \bar{v}_2) + g(\lambda \bar{v}_2, \bar{v}_1) = f(\bar{v}_1) + \lambda^{\sigma+1} f(\bar{v}_2) + g(\lambda \bar{v}_2, \bar{v}_1)$. Since $f(\bar{v}_1)$ and $\lambda^{\sigma+1} f(\bar{v}_2)$ belong to $\mathbb{F}$, we have $f(\bar{v}_1 + \lambda \bar{v}_2) \in \mathbb{F}$ if and only if $g(\lambda \bar{v}_2, \bar{v}_1) \in \mathbb{F}$. By Corollary 2.6, $g(\lambda \bar{v}_2, \bar{v}_1) \in \mathbb{F}$ for all $\lambda \in \mathbb{F}'$ if and only if $h(\bar{v}_1, \bar{v}_2) = 0$. ∎

Lemma 2.8 (*a*) *If case (I) occurs, then a point* $p = \langle \bar{v} \rangle$ *of* $\mathrm{PG}(V)$ *belongs to* $\mathcal{P}$ *if and only if* $h(\bar{v}, \bar{v}) = 0$.

(*b*) *If case (II) occurs, then* $h(\bar{v}, \bar{v}) = 0$ *for every* $\bar{v} \in V$.

Proof. If case (II) occurs, then the fact that h is an alternating bilinear form implies that $h(\bar{v}, \bar{v}) = 0$ for every $\bar{v} \in V$. Suppose therefore that case (I) occurs. Then $p = \langle \bar{v} \rangle$ is an element of $\mathcal{P}$ if and only if $f(\bar{v}) \in \mathbb{F}$, i.e. if and only if $h(\bar{v}, \bar{v}) = f(\bar{v}) - f(\bar{v})^\sigma = 0$. ∎

Proposition 2.9 *The point-line geometry* $\Pi(\mathbb{F}', \mathbb{F}, \sigma, n)$ *is a (non-degenerate) polar space of rank* n.

Proof. We show that $\Pi(\mathbb{F}', \mathbb{F}, \sigma, n)$ is a polar space. Note that if $p_1 = \langle \bar{v}_1 \rangle$ and $p_2 = \langle \bar{v}_2 \rangle$ are two distinct points of $\Pi(\mathbb{F}', \mathbb{F}, \sigma, n)$, then by Lemma 2.7, we know that $p_1 p_2 \in \mathcal{L}$ if and only if $h(\bar{v}_1, \bar{v}_2) = 0$.

We show that $\Pi(\mathbb{F}', \mathbb{F}, \sigma, n)$ satisfies the one or all axiom. Let (x, L) be an anti-flag of $\Pi(\mathbb{F}', \mathbb{F}, \sigma, n)$. Choose $\bar{v}, \bar{w}_1, \bar{w}_2 \in V$ such that $x = \langle \bar{v} \rangle$ and $L = \langle \bar{w}_1, \bar{w}_2 \rangle$. Then $f(\bar{v}) \in \mathbb{F}$

and $f(\bar{w}) \in \mathbb{F}$ for all $\bar{w} \in \langle \bar{w}_1, \bar{w}_2 \rangle$. By Lemma 2.7, the points of L collinear with x are precisely the points $\langle \bar{w} \rangle$ of L for which $h(\bar{v}, \bar{w}) = 0$. Since h is an alternating bilinear or an anti-Hermitian form, either one or all points of L are collinear with x.

We show that for every point $x = \langle \bar{v} \rangle$ of $\Pi(\mathbb{F}', \mathbb{F}, \sigma, n)$, there exists a point y of $\Pi(\mathbb{F}', \mathbb{F}, \sigma, n)$ noncollinear with x. Since h is nondegenerate, there exists a vector $\bar{w}$ such that $h(\bar{v}, \bar{w}) \neq 0$. By Lemma 2.8, we have $h(\bar{v}, \bar{v}) = 0$ and so the vectors $\bar{v}$ and $\bar{w}$ are linearly independent. We show that there exists a $\lambda \in \mathbb{F}$ such that $f(\bar{w} + \lambda \bar{v}) \in \mathbb{F}$. This follows from the fact that $f(\bar{w} + \lambda \bar{v}) = f(\bar{w}) + f(\lambda \bar{v}) + g(\lambda \bar{v}, \bar{w}) = f(\bar{w}) + \lambda^{\sigma+1} \cdot f(\bar{v}) + \lambda^\sigma \cdot h(\bar{v}, \bar{w}) + \Big(g(\lambda \bar{v}, \bar{w}) - \lambda^\sigma \cdot h(\bar{v}, \bar{w}) \Big)$. Since $g(\lambda \bar{v}, \bar{w}) - \lambda^\sigma \cdot h(\bar{v}, \bar{w}) \in \mathbb{F}$ (Lemma 2.4), $\lambda^{\sigma+1} f(\bar{v}) \in \mathbb{F}$ and $h(\bar{v}, \bar{w}) \neq 0$, there exists precisely $|\mathbb{F}|$ values $\lambda \in \mathbb{F}'$ for which $f(\bar{w} + \lambda \bar{v}) \in \mathbb{F}$. For each such λ, $y := \langle \bar{w} + \lambda \bar{v} \rangle$ will be a point of $\Pi(\mathbb{F}', \mathbb{F}, \sigma, n)$ which is noncollinear with x. Indeed, if it were collinear with x, then $xy = xz \subseteq \mathcal{P}$, where $z := \langle \bar{w} \rangle$, and Lemma 2.7 would imply that $h(\bar{v}, \bar{w}) = 0$, which is not the case.

We now show that the maximal projective dimension of a singular subspace of $\Pi(\mathbb{F}', \mathbb{F}, \sigma, n)$ is equal to $n-1$. Obviously, $PG(\langle \bar{b}_1^*, \bar{b}_3^*, \ldots, \bar{b}_{2n-1}^* \rangle)$ is a singular subspace of $\Pi(\mathbb{F}', \mathbb{F}, \sigma, n)$ of dimension $n-1$. Conversely, if $PG(Z)$ is a singular subspace of $\Pi(\mathbb{F}', \mathbb{F}, \sigma, n)$, then by Lemmas 2.7 and 2.8 the subspace Z of V should be totally isotropic with respect to the form h. As h is nondegenerate and $\dim(V) = 2n$, we should have $\dim(Z) \leq n$. ∎

In the following proposition we determine the type of the polar space $\Pi(\mathbb{F}', \mathbb{F}, \sigma, n)$.

Proposition 2.10 *If case (I) occurs, then $\Pi(\mathbb{F}', \mathbb{F}, \sigma, n)$ is the polar space arising from a nonsingular Hermitian variety of Witt index n of a projective space of dimension $2n-1$ over $\mathbb{F}'$ (for which σ is the corresponding field automorphism). If case (II) occurs, then the polar space $\Pi(\mathbb{F}', \mathbb{F}, \sigma, n)$ is isomorphic to the polar space arising from a nonsingular quadric of Witt index n of a projective space of dimension $2n-1+\tilde{d}$ over $\mathbb{F}'$, where $\tilde{d} = [\mathbb{F} : (\mathbb{F}')^2]$ is the dimension of $\mathbb{F}$ regarded as a vector space over its subfield $(\mathbb{F}')^2$.*

Proof. If case (I) occurs, then the claim follows from Lemma 2.8(a) and the fact that h is a nondegenerate anti-Hermitian form on V.

Suppose case (II) occurs. The field $\mathbb{F}$ can be regarded as a vector space over its subfield $(\mathbb{F}')^2$. Let $\{\lambda_j \,|\, j \in J\}$ be a basis of this vector space for some index set J disjoint from $\{1, 2, \ldots, 2n\}$. Let Z be a vector space of dimension $|J| + 2n$ over $\mathbb{F}'$, and consider the nonsingular quadric Q of $PG(Z)$ containing all points $\langle \sum_{i=1}^{2n} X_i \bar{b}_i + \sum_{j \in J} X_j \bar{f}_j \rangle$ of $PG(Z)$ satisfying

$$X_1 X_2 + X_3 X_4 + \cdots + X_{2n-1} X_{2n} + \sum_{j \in J} \lambda_j X_j^2 = 0.$$

Here, $\{\bar{b}_1, \bar{b}_2, \ldots, \bar{b}_{2n}, \bar{f}_j \,|\, j \in J\}$ denotes some basis of Z.

We construct a bijection θ between $\mathcal{P}$ and Q. If $p = \langle X_1 \bar{b}_1^* + X_2 \bar{b}_2^* + \cdots + X_{2n} \bar{b}_{2n}^* \rangle \in \mathcal{P}$, then $X_1 X_2 + X_3 X_4 + \cdots + X_{2n-1} X_{2n} \in \mathbb{F}$ and hence there exist unique $X_j \in \mathbb{F}'$, $j \in J$, such that

$$X_1 X_2 + X_3 X_4 + \cdots + X_{2n-1} X_{2n} + \sum_{j \in J} \lambda_j X_j^2 = 0.$$

This implies that

$$\theta(p) := \langle \sum_{i=1}^{2n} X_i \bar{b}_i + \sum_{j \in J} X_j \bar{f}_j \rangle$$

is a point of Q. Obviously, if $p' = \langle \sum_{i=1}^{2n} X_i \bar{b}_i + \sum_{j \in J} X_j \bar{f}_j \rangle$ is a point of Q, then $p' = \theta(p)$ for a unique point $p \in \mathcal{P}$, namely $p = \langle \sum_{i=1}^{2n} X_i \bar{b}_i^* \rangle$.

Two points $p = \langle \sum_{i=1}^{2n} X_i \bar{b}_i + \sum_{j \in J} X_j \bar{f}_j \rangle$ and $p' = \langle \sum_{i=1}^{2n} X_i' \bar{b}_i + \sum_{j \in J} X_j' \bar{f}_j \rangle$ of Q are collinear if and only if $(X_1 X_2' + X_2 X_1') + \cdots + (X_{2n-1} X_{2n}' + X_{2n-1}' X_{2n}) = 0$. This is by Lemma 2.7 precisely the condition for the points $\theta^{-1}(p)$ and $\theta^{-1}(p')$ of $\Pi(\mathbb{F}', \mathbb{F}, \sigma, n)$ to be collinear.

We conclude that $\Pi(\mathbb{F}', \mathbb{F}, \sigma, n)$ is isomorphic to the polar space arising from Q. ■

If case (I) occurs, then $\Pi(\mathbb{F}', \mathbb{F}, \sigma, n)$ is called a *Hermitian polar space*. If case (II) occurs, then $\Pi(\mathbb{F}', \mathbb{F}, \sigma, n)$ is called a *polar space of mixed type*.

3 Admissible bases

We continue with the notation of Section 2. In particular, $\mathbb{F}$ and $\mathbb{F}'$ are two fields and σ is an automorphism of $\mathbb{F}'$ such that one of the cases (I), (II) of Section 2 occurs.

An ordered basis $(\bar{e}_1, \bar{f}_1, \ldots, \bar{e}_n, \bar{f}_n)$ of V is called an *admissible basis of* (V, f) whenever

(a) $f(\bar{e}_i)$ and $f(\bar{f}_i)$ belong to $\mathbb{F}$ for every $i \in \{1, 2, \ldots, n\}$;

(b) $h(\bar{e}_i, \bar{f}_i) = 1$ for every $i \in \{1, 2, \ldots, n\}$;

(c) $h(\bar{e}_i, \bar{f}_j) = 0$ for all $i, j \in \{1, 2, \ldots, n\}$ with $i \neq j$;

(d) $h(\bar{e}_i, \bar{e}_j) = h(\bar{f}_i, \bar{f}_j) = 0$ for all $i, j \in \{1, 2, \ldots, n\}$.

In particular, the fixed ordered basis $B^* = (\bar{b}_1^*, \bar{b}_2^*, \ldots, \bar{b}_{2n}^*)$ considered in Section 2 is an admissible basis of (V, f).

Lemma 3.1 *If $(\bar{g}_1, \bar{g}_2, \ldots, \bar{g}_{2n})$ is an admissible basis of (V, f), then $f(X_1 \bar{g}_1 + X_2 \bar{g}_2 + \cdots + X_{2n} \bar{g}_{2n}) - X_1^\sigma X_2 - X_3^\sigma X_4 - \cdots - X_{2n-1}^\sigma X_{2n} \in \mathbb{F}$.*

Proof. We prove this by induction on $n \geq 1$.

Suppose first that $n = 1$. Then $f(X_1 \bar{g}_1 + X_2 \bar{g}_2) = f(X_1 \bar{g}_1) + f(X_2 \bar{g}_2) + g(X_1 \bar{g}_1, X_2 \bar{g}_2) = X_1^{\sigma+1} f(\bar{g}_1) + X_2^{\sigma+1} f(\bar{g}_2) + g(X_1 \bar{g}_1, X_2 \bar{g}_2)$. Now, $X_1^{\sigma+1} f(g_1)$ and $X_2^{\sigma+1} f(g_2)$ belong to $\mathbb{F}$. By Lemma 2.5, also $g(X_1 \bar{g}_1, X_2 \bar{g}_2) - X_1^\sigma X_2 \cdot h(\bar{g}_1, \bar{g}_2) = g(X_1 \bar{g}_1, X_2 \bar{g}_2) - X_1^\sigma X_2 \in \mathbb{F}$. Hence, $f(X_1 \bar{g}_1 + X_2 \bar{g}_2) - X_1^\sigma X_2 \in \mathbb{F}$.

Suppose next that $n \geq 2$. Then $f(X_1 \bar{g}_1 + X_2 \bar{g}_2 + \cdots + X_{2n} \bar{g}_{2n}) = f(X_1 \bar{g}_1 + X_2 \bar{g}_2 + \cdots + X_{2n-2} \bar{g}_{2n-2}) + f(X_{2n-1} \bar{g}_{2n-1} + X_{2n} \bar{g}_{2n}) + g(X_1 \bar{g}_1 + X_2 \bar{g}_2 + \cdots + X_{2n-2} \bar{g}_{2n-2}, X_{2n-1} \bar{g}_{2n-1} + X_{2n} \bar{g}_{2n})$. By the induction hypothesis, we know that

(a) $f(X_1 \bar{g}_1 + X_2 \bar{g}_2 + \cdots + X_{2n-2} \bar{g}_{2n-2}) - X_1^\sigma X_2 - \cdots - X_{2n-3}^\sigma X_{2n-2} \in \mathbb{F}$,

(b) $f(X_{2n-1}\bar{g}_{2n-1} + X_{2n}\bar{g}_{2n}) - X_{2n-1}^{\sigma} X_{2n} \in \mathbb{F}$.

Since $h(X_1\bar{g}_1 + X_2\bar{g}_2 + \cdots + X_{2n-2}\bar{g}_{2n-2}, X_{2n-1}\bar{g}_{2n-1} + X_{2n}\bar{g}_{2n}) = 0$, Corollary 2.6 implies that

(c) $g(X_1\bar{g}_1 + X_2\bar{g}_2 + \cdots + X_{2n-2}\bar{g}_{2n-2}, X_{2n-1}\bar{g}_{2n-1} + X_{2n}\bar{g}_{2n}) \in \mathbb{F}$.

By (a), (b) and (c), it now follows that $f(X_1\bar{g}_1 + X_2\bar{g}_2 + \cdots + X_{2n}\bar{g}_{2n}) - X_1^{\sigma}X_2 - X_3^{\sigma}X_4 - \cdots - X_{2n-1}^{\sigma}X_{2n} \in \mathbb{F}$. ■

The following is a consequence of Lemma 3.1.

Corollary 3.2 *If $(\bar{g}_1, \bar{g}_2, \ldots, \bar{g}_{2n})$ is an admissible basis of (V, f), then the points of the polar space $\Pi(\mathbb{F}', \mathbb{F}, \sigma, n)$ are precisely the points $\langle X_1\bar{g}_1 + X_2\bar{g}_2 + \cdots + X_{2n}\bar{g}_{2n}\rangle$ of* PG(V) *for which $X_1^{\sigma}X_2 + X_3^{\sigma}X_4 + \cdots + X_{2n-1}^{\sigma}X_{2n} \in \mathbb{F}$.*

Lemma 3.3 *Suppose $(\bar{g}_1, \bar{g}_2, \ldots, \bar{g}_{2n})$ is an ordered basis of V such that the points of $\Pi(\mathbb{F}', \mathbb{F}, \sigma, n)$ are precisely the points $\langle X_1\bar{g}_1 + X_2\bar{g}_2 + \cdots + X_{2n}\bar{g}_{2n}\rangle$ of* PG(V) *satisfying $X_1^{\sigma}X_2 + \cdots + X_{2n-1}^{\sigma}X_{2n} \in \mathbb{F}$. Then there exists a $\lambda \in \mathbb{F}\setminus\{0\}$ such that $(\bar{g}_1, \lambda\bar{g}_2, \bar{g}_3, \lambda\bar{g}_4, \ldots, \bar{g}_{2n-1}, \lambda\bar{g}_{2n})$ is an admissible basis of (V, f).*

Proof. Since $\langle\bar{g}_i\rangle$ with $i \in \{1, 2, \ldots, 2n\}$ is a point of $\Pi(\mathbb{F}', \mathbb{F}, \sigma, n)$, we have $f(\bar{g}_i) \in \mathbb{F}$ and hence $f(\lambda\bar{g}_i) \in \mathbb{F}$ for all $\lambda \in \mathbb{F}'$. By Lemma 2.8, $h(\lambda\bar{g}_i, \lambda\bar{g}_i) = 0$ for all $i \in \{1, 2, \ldots, 2n\}$ and all $\lambda \in \mathbb{F}'$.

Suppose i and j are two distinct elements of $\{1, 2, \ldots, 2n\}$ such that $\{i, j\} \notin \{\{1, 2\}, \{3, 4\}, \ldots, \{2n-1, 2n\}\}$. Then $\langle\lambda\bar{g}_i + \bar{g}_j\rangle$ is a point of $\Pi(\mathbb{F}', \mathbb{F}, \sigma, n)$ for all $\lambda \in \mathbb{F}'$. Lemma 2.7 then implies that $h(\bar{g}_i, \bar{g}_j) = 0$.

Now, put $\mu_1 := h(\bar{g}_1, \bar{g}_2)$, $\mu_2 := h(\bar{g}_3, \bar{g}_4)$, ..., $\mu_n := h(\bar{g}_{2n-1}, \bar{g}_{2n})$. Since $\langle\bar{g}_{2i-1} + \bar{g}_{2i}\rangle$ is a point of $\Pi(\mathbb{F}', \mathbb{F}, \sigma, n)$ for every $i \in \{1, 2, \ldots, n\}$, we have $f(\bar{g}_{2i-1} + \bar{g}_{2i}) = f(\bar{g}_{2i-1}) + f(\bar{g}_{2i}) + g(\bar{g}_{2i-1}, \bar{g}_{2i}) = f(\bar{g}_{2i-1}) + f(\bar{g}_{2i}) + (g(\bar{g}_{2i-1}, \bar{g}_{2i}) - h(\bar{g}_{2i-1}, \bar{g}_{2i})) + \mu_i \in \mathbb{F}$ for all $i \in \{1, 2, \ldots, n\}$. By Lemma 2.4, $g(\bar{g}_{2i-1}, \bar{g}_{2i}) - h(\bar{g}_{2i-1}, \bar{g}_{2i}) \in \mathbb{F}$. Since also $f(\bar{g}_{2i-1})$ and $f(\bar{g}_{2i})$ are elements of $\mathbb{F}$, we have $\mu_i \in \mathbb{F}$ for all $i \in \{1, 2, \ldots, n\}$.

We show that $\mu_i = \mu_j$ for all $\{1, 2, \ldots, n\}$ with $i \neq j$. For every $\lambda \in \mathbb{F}'$, we have that $\langle\bar{g}_{2i-1} + \lambda\bar{g}_{2i} - \bar{g}_{2j-1} + \lambda\bar{g}_{2j}\rangle$ is a point of $\Pi(\mathbb{F}', \mathbb{F}, \sigma, n)$, implying that $f(\bar{g}_{2i-1} + \lambda\bar{g}_{2i} - \bar{g}_{2j-1} + \lambda\bar{g}_{2j}) = f(\bar{g}_{2i-1} + \lambda\bar{g}_{2i}) + f(-\bar{g}_{2j-1} + \lambda\bar{g}_{2j}) + g(\bar{g}_{2i-1} + \lambda\bar{g}_{2i}, -\bar{g}_{2j-1} + \lambda\bar{g}_{2j}) = f(\bar{g}_{2i-1}) + f(\lambda\bar{g}_{2i}) + f(-\bar{g}_{2j-1}) + f(\lambda\bar{g}_{2j}) + g(\bar{g}_{2i-1}, \lambda\bar{g}_{2i}) + g(-\bar{g}_{2j-1}, \lambda\bar{g}_{2j}) + g(\bar{g}_{2i-1} + \lambda\bar{g}_{2i}, -\bar{g}_{2j-1} + \lambda\bar{g}_{2j}) \in \mathbb{F}$. Now:

(a) By the above, $f(\bar{g}_{2i-1})$, $f(\lambda\bar{g}_{2i})$, $f(-\bar{g}_{2j-1})$ and $f(\lambda\bar{g}_{2j})$ are elements of $\mathbb{F}$.

(b) Since $h(\bar{g}_{2i-1} + \lambda\bar{g}_{2i}, -\bar{g}_{2j-1} + \lambda\bar{g}_{2j}) = 0$, we have $g(\bar{g}_{2i-1} + \lambda\bar{g}_{2i}, -\bar{g}_{2j-1} + \lambda\bar{g}_{2j}) \in \mathbb{F}$ by Corollary 2.6.

(c) By Lemma 2.5, we know that $g(\bar{g}_{2i-1}, \lambda\bar{g}_{2i}) - \lambda\mu_i$ and $g(-\bar{g}_{2j-1}, \lambda\bar{g}_j) + \lambda\mu_j$ are elements of $\mathbb{F}$.

We can conclude that $\lambda(\mu_i - \mu_j) \in \mathbb{F}$ for all $\lambda \in \mathbb{F}'$. Hence, $\mu_i = \mu_j$ as we needed to prove.

Now, put $\mu := \mu_1 = \mu_2 = \ldots = \mu_n \in \mathbb{F}$. As h is nondegenerate, we have $\mu \neq 0$ and so we can define $\lambda := \mu^{-1} \in \mathbb{F}$. By the above, we know that $(\bar{g}_1, \lambda\bar{g}_2, \ldots, \bar{g}_{2n-1}, \lambda\bar{g}_{2n})$ is an admissible basis of (V, f). ■

If $B = (\bar{e}_1, \bar{f}_1, \ldots, \bar{e}_n, \bar{f}_n)$ is an admissible basis of (V, f), then by Corollary 3.2 $E_B := \langle \bar{e}_1, \bar{e}_2, \ldots, \bar{e}_n \rangle$ and $F_B := \langle \bar{f}_1, \bar{f}_2, \ldots, \bar{f}_n \rangle$ are two disjoint maximal subspaces of $\Pi := \Pi(\mathbb{F}', \mathbb{F}, \sigma, n)$, i.e. two opposite points of the dual polar space Δ associated with Π.

Lemma 3.4 *If E and F are two opposite points of Δ, then there exists an admissible basis $B = (\bar{e}_1, \bar{f}_1, \ldots, \bar{e}_n, \bar{f}_n)$ of (V, f) such that $E = E_B$ and $F = F_B$.*

Proof. Let $\{p_1, p_2, \ldots, p_n\}$ be a basis of E, and let $\bar{e}_i$ with $i \in \{1, 2, \ldots, n\}$ be a vector of V such that $p_i = \langle \bar{e}_i \rangle$. For every $i \in \{1, 2, \ldots, n\}$, let q_i be the unique point of F collinear with every p_j with $j \in \{1, 2, \ldots, n\} \setminus \{i\}$ and noncollinear with p_i. By Lemma 2.7, there exists a unique vector $\bar{f}_i$ such that $q_i = \langle \bar{f}_i \rangle$ and $h(\bar{e}_i, \bar{f}_i) = 1$. The points $q_1, q_2, \ldots, q_n$ generate the subspace F, implying that $B = (\bar{e}_1, \bar{f}_1, \ldots, \bar{e}_n, \bar{f}_n)$ is an ordered basis of V. We now verify that B is an admissible basis.

Since $p_1, p_2, \ldots, p_n, q_1, q_2, \ldots, q_n$ are points of the polar space Π, we have that $f(\bar{e}_i) \in \mathbb{F}$ and $f(\bar{f}_i) \in \mathbb{F}$ for every $i \in \{1, 2, \ldots, n\}$. By construction, $h(\bar{e}_i, \bar{f}_i) = 1$ for every $i \in \{1, 2, \ldots, n\}$. If i and j are two distinct elements of $\{1, 2, \ldots, n\}$, then p_i and q_j are collinear points of Π, implying that $h(\bar{e}_i, \bar{f}_j) = 0$ by Lemma 2.7. Since $p_1, p_2, \ldots, p_n$ are points of the singular subspace E, Lemmas 2.7 and 2.8 imply that $h(\bar{e}_i, \bar{e}_j) = 0$ for all $i, j \in \{1, 2, \ldots, n\}$. Similarly, since $q_1, q_2, \ldots, q_n$ are points of the singular subspace F, we must have that $h(\bar{f}_i, \bar{f}_j) = 0$ for all $i, j \in \{1, 2, \ldots, n\}$. ■

Proposition 3.5 *If $B = (\bar{e}_1, \bar{f}_1, \ldots, \bar{e}_n, \bar{f}_n)$ is an admissible basis of (V, f), then*

(1) *for every permutation τ of $\{1, 2, \ldots, n\}$, also $(\bar{e}'_1, \bar{f}'_1, \ldots, \bar{e}'_n, \bar{f}'_n) = (\bar{e}_{\tau(1)}, \bar{f}_{\tau(1)}, \ldots, \bar{e}_{\tau(n)}, \bar{f}_{\tau(n)})$ is an admissible basis of (V, f);*

(2) *for every $\lambda \in \mathbb{F}' \setminus \{0\}$, also $(\bar{e}'_1, \bar{f}'_1, \ldots, \bar{e}'_n, \bar{f}'_n) = (\bar{e}_1 + \lambda\bar{e}_2, \bar{f}_1, \bar{e}_2, -\lambda^\sigma \bar{f}_1 + \bar{f}_2, \bar{e}_3, \bar{f}_3, \ldots, \bar{e}_n, \bar{f}_n)$ is an admissible basis of (V, f);*

(3) *for every $\lambda \in \mathbb{F}' \setminus \{0\}$, also $(\bar{e}'_1, \bar{f}'_1, \ldots, \bar{e}'_n, \bar{f}'_n) = (\frac{\bar{e}_1}{\lambda}, \lambda^\sigma \bar{f}_1, \bar{e}_2, \bar{f}_2, \ldots, \bar{e}_n, \bar{f}_n)$ is an admissible basis of (V, f);*

(4) *for every $\lambda \in \mathbb{F} \setminus \{0\}$, also $(\bar{e}'_1, \bar{f}'_1, \ldots, \bar{e}'_n, \bar{f}'_n) = (\bar{e}_1, \bar{f}_1, \ldots, \bar{e}_{n-1}, \bar{f}_{n-1}, \bar{e}_n, \bar{f}_n + \lambda\bar{e}_n)$ is an admissible basis of (V, f);*

(5) *for every $\lambda \in \mathbb{F} \setminus \{0\}$, also $(\bar{e}'_1, \bar{f}'_1, \ldots, \bar{e}'_n, \bar{f}'_n) = (\bar{e}_1, \bar{f}_1, \ldots, \bar{e}_{n-1}, \bar{f}_{n-1}, \bar{e}_n + \lambda\bar{f}_n, \bar{f}_n)$ is an admissible basis of (V, f).*

Proof. Using the fact that h is either anti-Hermitian (case (I)) or alternating (case (II)), it is straightforward to verify that $h(\bar{e}'_i, \bar{e}'_j) = h(\bar{f}'_i, \bar{f}'_j) = 0$ and $h(\bar{e}'_i, \bar{f}'_i) = 1$ for all $i, j \in \{1, 2, \ldots, n\}$, and that $h(\bar{e}'_i, \bar{f}'_j) = 0$ for all $i, j \in \{1, 2, \ldots, n\}$ with $i \neq j$. In order to verify that $(\bar{e}'_1, \bar{f}'_1, \ldots, \bar{e}'_n, \bar{f}'_n)$ is an admissible basis, we need to show that $f(\bar{e}'_i)$ and $f(\bar{f}'_i)$ belong to $\mathbb{F}$ for all $i \in \{1, 2, \ldots, n\}$. This obviously holds for case (1).

In case (2), we need to verify that $f(\bar{e}_1 + \lambda\bar{e}_2) = f(\bar{e}_1) + f(\lambda\bar{e}_2) + g(\bar{e}_1, \lambda\bar{e}_2)$ and $f(-\lambda^\sigma \bar{f}_1 + \bar{f}_2) = f(-\lambda^\sigma \bar{f}_1) + f(\bar{f}_2) + g(-\lambda^\sigma \bar{f}_1, \bar{f}_2)$ belong to $\mathbb{F}$. This is indeed the case. We know that each of $f(\bar{e}_1)$, $f(\lambda\bar{e}_2) = \lambda^{\sigma+1} f(\bar{e}_2)$, $f(-\lambda^\sigma \bar{f}_1) = \lambda^{\sigma+1} f(\bar{f}_1)$, $f(\bar{f}_2)$ belongs to $\mathbb{F}$, and since $h(\bar{e}_1, \bar{e}_2) = h(\bar{f}_1, \bar{f}_2) = 0$, Corollary 2.6 also implies that $g(\bar{e}_1, \lambda\bar{e}_2) \in \mathbb{F}$ and $g(-\lambda^\sigma \bar{f}_1, \bar{f}_2) \in \mathbb{F}$.

In case (3), we need to verify that $f(\frac{\bar{e}_1}{\lambda}) = \frac{1}{\lambda^{\sigma+1}} f(\bar{e}_1)$ and $f(\lambda^\sigma \bar{f}_1) = \lambda^{\sigma+1} f(\bar{f}_1)$ belong to $\mathbb{F}$. This is indeed the case since each of $\lambda^{\sigma+1}$, $f(\bar{e}_1)$, $f(\bar{f}_1)$ belongs to $\mathbb{F}$.

In case (4), we need to verify that $f(\bar{f}_n + \lambda\bar{e}_n) = f(\bar{f}_n) + f(\lambda\bar{e}_n) - \lambda + \Big(g(\bar{f}_n, \lambda\bar{e}_n) + \lambda\Big) \in \mathbb{F}$. This is indeed the case since each of the elements $f(\bar{f}_n)$, $f(\lambda\bar{e}_n) = \lambda^{\sigma+1} f(\bar{e}_n)$, λ, $g(\bar{f}_n, \lambda\bar{e}_n) + \lambda$ belongs to $\mathbb{F}$ (the latter because of Lemma 2.5).

In case (5), we need to verify that $f(\bar{e}_n + \lambda\bar{f}_n) = f(\bar{e}_n) + f(\lambda\bar{f}_n) + \lambda + \Big(g(\bar{e}_n, \lambda\bar{f}_n) - \lambda\Big) \in \mathbb{F}$. This is indeed the case since each of the elements $f(\bar{e}_n)$, $f(\lambda\bar{f}_n) = \lambda^{\sigma+1} f(\bar{f}_n)$, λ, $g(\bar{e}_n, \lambda\bar{f}_n) - \lambda$ belongs to $\mathbb{F}$ (the latter because of Lemma 2.5). ■

For every $i \in \{1, 2, 3, 4, 5\}$, let Ω_i denote the set of all ordered pairs (B_1, B_2) of admissible bases of (V, f) such that B_2 can be obtained from B_1 as described in (i) of Proposition 3.5. Also, put $\widetilde{\Omega_1} := \Omega_1$, $\widetilde{\Omega_4} := \Omega_4$ and $\widetilde{\Omega_5} := \Omega_5$. For every $i \in \{2, 3\}$, let $\widetilde{\Omega_i}$ denote the set of all pairs (B_1, B_2) of admissible bases of (V, f) such that B_2 can be obtained from B_1 as described in (i) of Proposition 3.5 with λ belonging to $\mathbb{F} \setminus \{0\}$.

A proof of the following lemma was given in [8] (part 3 of the proof of Lemma 2.1) in the special case that $\Delta' = DW(2n-1, \mathbb{F})$, but as this proof directly extends to arbitrary dual polar spaces, we will omit it here. If x_1 and x_2 are two distinct points of a dual polar space Δ', then we write $x_1 \sim x_2$ if these points are collinear.

Lemma 3.6 *Let Δ' be a dual polar space of rank n with the property that every line is incident with at least three points. Let Γ be the graph whose vertices are the ordered pairs (x, y) of opposite vertices of Δ', where two distinct vertices (x_1, y_1) and (x_2, y_2) are adjacent whenever one of the following cases occurs:*

(1) $x_1 = x_2$ *and* $y_1 \sim y_2$*;*

(2) $x_1 \sim x_2$ *and* $y_1 = y_2$*.*

Then Γ is connected.

Proposition 3.7 *If B_1 and B_2 are two admissible bases of (V, f), then there exist admissible bases $B'_0, B'_1, \ldots, B'_k$ of (V, f) for some $k \in \mathbb{N}$ such that $B'_0 = B_1$, $B'_k = B_2$ and $(B'_{i-1}, B'_i) \in \Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \Omega_4 \cup \Omega_5$ for every $i \in \{1, 2, \ldots, k\}$.*

Proof. By Lemma 3.6, it suffices to prove the proposition for one of the following cases:

(I) $E_{B_1} = E_{B_2}$ and $F_{B_1} = F_{B_2}$;

(II) $E_{B_1} = E_{B_2}$ and $\dim(F_{B_1} \cap F_{B_2}) = n-1$;

(III) $\dim(E_{B_1} \cap E_{B_2}) = n-1$ and $F_{B_1} = F_{B_2}$.

(I) Suppose $E := E_{B_1} = E_{B_2}$ and $F := F_{B_1} = F_{B_2}$. Put $B_2 = (\bar{e}_1, \bar{f}_1, \bar{e}_2, \bar{f}_2, \dots, \bar{e}_n, \bar{f}_n)$. Since the maps $(\bar{h}_1, \bar{h}_2, \dots, \bar{h}_n) \mapsto (\bar{h}_{\tau(1)}, \bar{h}_{\tau(2)}, \dots, \bar{h}_{\tau(n)})$ for $\tau \in S_n$, $(\bar{h}_1, \bar{h}_2, \dots, \bar{h}_n) \mapsto (\frac{\bar{h}_1}{\lambda}, \bar{h}_2, \dots, \bar{h}_n)$ for $\lambda \in \mathbb{F}' \setminus \{0\}$ and $(\bar{h}_1, \bar{h}_2, \dots, \bar{h}_n) \mapsto (\bar{h}_1 + \lambda \bar{h}_2, \bar{h}_2, \dots, \bar{h}_n)$ for $\lambda \in \mathbb{F}' \setminus \{0\}$ allow us to transform any basis of E to any other basis of E, there exist admissible bases $B_0', B_1', \dots, B_k'$ of (V, f) for some $k \geq 0$ such that (i) $B_0' = B_1$, (ii) $(B_{i-1}', B_i') \in \Omega_1 \cup \Omega_2 \cup \Omega_3$ for every $i \in \{1, 2, \dots, k\}$, and (iii) B_k' is of the form $(\bar{e}_1, \bar{f}_1', \dots, \bar{e}_n, \bar{f}_n')$ with $F = \langle \bar{f}_1', \bar{f}_2', \dots, \bar{f}_n' \rangle$. The vector $\bar{f}_i'$, $i \in \{1, 2, \dots, n\}$, is uniquely determined by the vectors $\bar{e}_1, \bar{e}_2, \dots, \bar{e}_n$: it is the unique vector of F satisfying $h(\bar{e}_i, \bar{f}_i') = 1$ and $h(\bar{e}_j, \bar{f}_i') = 0$ for all $j \in \{1, 2, \dots, n\} \setminus \{i\}$. Hence, $\bar{f}_i' = \bar{f}_i$ for every $i \in \{1, 2, \dots, n\}$, i.e. $B_k' = B_2$.

(II) + (III) We will give a proof in the case where $E_{B_1} = E_{B_2}$ and $\dim(F_{B_1} \cap F_{B_2}) = n-1$. The proof in the case $\dim(E_{B_1} \cap E_{B_2}) = n-1$ and $F_{B_1} = F_{B_2}$ is completely similar. By Part (I), the proposition will hold for such a (B_1, B_2) as soon as it holds for some pair (C_1, C_2) of admissible bases satisfying $E_{B_1} = E_{C_1} = E_{B_2} = E_{C_2}$, $F_{C_1} = F_{B_1}$ and $F_{C_2} = F_{B_2}$. So, without loss of generality we may assume that $B_1 = (\bar{e}_1, \bar{f}_1, \dots, \bar{e}_n, \bar{f}_n)$ and $B_2 = (\bar{e}_1', \bar{f}_1', \dots, \bar{e}_n', \bar{f}_n')$ are such that $F_{B_1} \cap F_{B_2} = \langle \bar{f}_1, \bar{f}_2, \dots, \bar{f}_{n-1} \rangle$, $\bar{e}_i' = \bar{e}_i$ for every $i \in \{1, 2, \dots, n\}$ and $\bar{f}_i' = \bar{f}_i$ for every $i \in \{1, 2, \dots, n-1\}$. The fact that $h(\bar{f}_i', \bar{f}_n') = h(\bar{e}_i', \bar{f}_n') = 0$ for every $i \in \{1, 2, \dots, n-1\}$ and that $h(\bar{e}_n', \bar{f}_n') = 1$ then implies that $\bar{f}_n' = \bar{f}_n + \lambda \bar{e}_n$ for some $\lambda \in \mathbb{F}'$. Since $f(\bar{f}_n') = f(\bar{f}_n + \lambda \bar{e}_n) = f(\bar{f}_n) + f(\lambda \bar{e}_n) + g(\lambda \bar{e}_n, \bar{f}_n) = f(\bar{f}_n) + \lambda^{\sigma+1} f(\bar{e}_n) + g(\lambda \bar{e}_n, \bar{f}_n) \in \mathbb{F}$, we must have that $g(\lambda \bar{e}_n, \bar{f}_n) \in \mathbb{F}$. By Lemma 2.4, $\lambda \in \mathbb{F}$ and hence $(B_1, B_2) \in \Omega_4$. ■

4 The subgroup G_f of $GL(V)$

Let G_f denote the set of all $\theta \in GL(V)$ such that $f(\bar{v}^\theta) - f(\bar{v}) \in \mathbb{F}$ for all $\bar{v} \in V$. If $\theta, \theta_1, \theta_2 \in G_f$, then θ^{-1} and $\theta_1 \theta_2$ also belong to G_f and so G_f should be a subgroup of $GL(V)$.

Lemma 4.1 *G_f leaves the form h invariant.*

Proof. Let $\theta \in G_f$. Since $g(\bar{v}, \bar{w}) = f(\bar{v} + \bar{w}) - f(\bar{v}) - f(\bar{w})$, we have $g(\bar{v}^\theta, \bar{w}^\theta) - g(\bar{v}, \bar{w}) \in \mathbb{F}$ for all $\bar{v}, \bar{w} \in V$.

Suppose case (I) occurs. Lemma 2.2 then implies that $h(\bar{v}^\theta, \bar{w}^\theta) = h(\bar{v}, \bar{w})$ for all $\bar{v}, \bar{w} \in V$.

Suppose case (II) occurs. Then $g = h$ is an alternating bilinear form on V. From $\lambda \cdot \Big(g(\bar{v}^\theta, \bar{w}^\theta) - g(\bar{v}, \bar{w}) \Big) = g((\lambda \bar{v})^\theta, \bar{w}^\theta) - g(\lambda \bar{v}, \bar{w}) \in \mathbb{F}$ for all $\lambda \in \mathbb{F}'$ and all $\bar{v}, \bar{w} \in V$, it follows that $g(\bar{v}^\theta, \bar{w}^\theta) = g(\bar{v}, \bar{w})$ for all $\bar{v}, \bar{w} \in V$. So, θ leaves $h = g$ invariant. ■

The following is an immediate consequence of Lemma 4.1.

Corollary 4.2 *If $B = (\bar{e}_1, \bar{f}_1, \ldots, \bar{e}_n, \bar{f}_n)$ is an admissible basis of (V, f) and $\theta \in G_f$, then $B^\theta := (\bar{e}_1^\theta, \bar{f}_1^\theta, \ldots, \bar{e}_n^\theta, \bar{f}_n^\theta)$ is also an admissible basis of (V, f).*

Lemma 4.3 *Suppose B_1 and B_2 are two admissible bases of (V, f) and let θ be the unique element of $GL(V)$ mapping B_1 to B_2. Then $\theta \in G_f$.*

Proof. Put $B_1 = (\bar{e}_1, \bar{f}_1, \ldots, \bar{e}_n, \bar{f}_n)$. Then $B_2 = (\bar{e}_1^\theta, \bar{f}_1^\theta, \ldots, \bar{e}_n^\theta, \bar{f}_n^\theta)$. Let $\bar{v} = X_1\bar{e}_1 + X_2\bar{f}_1 + \cdots + X_{2n-1}\bar{e}_n + X_{2n}\bar{f}_n$ be an arbitrary vector of V. Then $\bar{v}^\theta = X_1\bar{e}_1^\theta + X_2\bar{f}_1^\theta + \cdots + X_{2n-1}\bar{e}_n^\theta + X_{2n}\bar{f}_n^\theta$. By Lemma 3.1, $f(\bar{v}^\theta) - f(\bar{v}) \in \mathbb{F}$, implying that $\theta \in G_f$. ∎

The following is an immediate consequence of Corollary 4.2 and Lemma 4.3.

Corollary 4.4 *The group G_f consists of those elements of $GL(V)$ that map admissible bases of (V, f) to admissible bases of (V, f).*

We now describe generators for the group G_f. Such generators will be described using the fixed admissible basis $(\bar{e}_1^*, \bar{f}_1^*, \ldots, \bar{e}_n^*, \bar{f}_n^*) := B^* = (\bar{b}_1^*, \bar{b}_2^*, \ldots, \bar{b}_{2n}^*)$ of (V, f).

- For every $i \in \{2, 3, \ldots, n\}$, let $\theta_1^{(i)}$ denote the element of $GL(V)$ mapping the ordered basis B^* to the ordered basis $(\bar{e}_{\tau(1)}^*, \bar{f}_{\tau(1)}^*, \ldots, \bar{e}_{\tau(n)}^*, \bar{f}_{\tau(n)}^*)$ of V, where τ is the transposition $(1, i)$.
- For every $\lambda \in \mathbb{F}' \setminus \{0\}$, let $\theta_2(\lambda)$ denote the element of $GL(V)$ mapping the ordered basis B^* to the ordered basis $(\bar{e}_1^* + \lambda\bar{e}_2^*, \bar{f}_1^*, \bar{e}_2^*, -\lambda^\sigma\bar{f}_1^* + \bar{f}_2^*, \bar{e}_3^*, \bar{f}_3^*, \ldots, \bar{e}_n^*, \bar{f}_n^*)$ of V.
- For every $\lambda \in \mathbb{F}' \setminus \{0\}$, let $\theta_3(\lambda)$ denote the element of $GL(V)$ mapping the ordered basis B^* to the ordered basis $(\frac{\bar{e}_1^*}{\lambda}, \lambda^\sigma\bar{f}_1^*, \bar{e}_2^*, \bar{f}_2^*, \ldots, \bar{e}_n^*, \bar{f}_n^*)$ of V.
- For every $\lambda \in \mathbb{F} \setminus \{0\}$, let $\theta_4(\lambda)$ denote the element of $GL(V)$ mapping the ordered basis B^* to the ordered basis $(\bar{e}_1^*, \bar{f}_1^*, \ldots, \bar{e}_{n-1}^*, \bar{f}_{n-1}^*, \bar{e}_n^*, \bar{f}_n^* + \lambda\bar{e}_n^*)$ of V.
- For every $\lambda \in \mathbb{F} \setminus \{0\}$, let $\theta_5(\lambda)$ denote the element of $GL(V)$ mapping the ordered basis B^* to the ordered basis $(\bar{e}_1^*, \bar{f}_1^*, \ldots, \bar{e}_{n-1}^*, \bar{f}_{n-1}^*, \bar{e}_n^* + \lambda\bar{f}_n^*, \bar{f}_n^*)$ of V.

Observe that the transpositions $(1, i)$, $i \in \{2, 3, \ldots, n\}$, generate the whole symmetric group S_n.

Proposition 4.5 *The group G_f coincides with the subgroup G of $GL(V)$ that is generated by the elements $\theta_1^{(i)}$ ($i \in \{2, 3, \ldots, n\}$), $\theta_2(\lambda)$ ($\lambda \in \mathbb{F}' \setminus \{0\}$), $\theta_3(\lambda)$ ($\lambda \in \mathbb{F}' \setminus \{0\}$), $\theta_4(\lambda)$ ($\lambda \in \mathbb{F} \setminus \{0\}$) and $\theta_5(\lambda)$ ($\lambda \in \mathbb{F} \setminus \{0\}$).*

Proof. The listed elements all belong to G_f by Proposition 3.5 and Lemma 4.3 since each of them maps the admissible basis B^* to another admissible basis of (V, f). Hence, $G \subseteq G_f$.

Conversely, suppose θ is an arbitrary element of G_f. Then $(B^*)^\theta$ is an admissible basis of (V, f) by Corollary 4.2. By Proposition 3.7, we know that there exist admissible bases $B'_0, B'_1, \ldots, B'_k$ of (V, f) for some $k \in \mathbb{N}$ such that $B'_0 = B^*$, $B'_k = (B^*)^\theta$ and $(B'_{i-1}, B'_i) \in \Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \Omega_4 \cup \Omega_5$ for every $i \in \{1, 2, \ldots, k\}$. We prove by induction on k that $\theta \in G$. As this is the case for $k \in \{0, 1\}$, we assume that $k \geq 2$ and that the claim is valid for smaller values of k.

Let θ_1 be the element of $GL(V)$ mapping B^* to B'_{k-1}. By Lemma 4.3, $\theta_1 \in G_f$ and by the induction hypothesis, we then also know that $\theta_1 \in G$. Since $(B'_{k-1}, B'_k) \in \Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \Omega_4 \cup \Omega_5$, we also have $(B^*, (B'_k)^{\theta_1^{-1}}) \in \Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \Omega_4 \cup \Omega_5$ and hence there exists a $\theta_2 \in G$ such that $(B^*)^{\theta_2} = (B'_k)^{\theta_1^{-1}}$. Hence, $(B^*)^\theta = B'_k = (B^*)^{\theta_2 \theta_1}$, implying that $\theta = \theta_2 \theta_1 \in G$. ■

Lemma 4.6 *For each $\theta \in G_f$, there exists an $\eta_\theta \in \mathbb{F}'$ such that* $\det(\theta) = \frac{\eta_\theta^\sigma}{\eta_\theta}$.

Proof. It suffices to prove this for each of the elements of the generating set of G_f mentioned in Proposition 4.5. The verification has been done in the following table:

θ	η_θ
θ_1^i with $i \in \{2, 3, \ldots, n\}$	1
$\theta_2(\lambda)$ with $\lambda \in \mathbb{F}' \setminus \{0\}$	1
$\theta_3(\lambda)$ with $\lambda \in \mathbb{F}' \setminus \{0\}$	λ
$\theta_4(\lambda)$ with $\lambda \in \mathbb{F} \setminus \{0\}$	1
$\theta_5(\lambda)$ with $\lambda \in \mathbb{F} \setminus \{0\}$	1

■

The following is a special case of Lemma 4.6.

Corollary 4.7 *If case (II) occurs, then each element of G_f has determinant* 1.

5 The subspace W_k of $\bigwedge^k V$

Throughout this section, we suppose that $\mathbb{F}'$ is a field of characteristic 2, σ is the trivial automorphism of $\mathbb{F}'$ and $(\mathbb{F}')^2 = \{\lambda^2 \,|\, \lambda \in \mathbb{F}'\} \subseteq \mathbb{F} \neq \mathbb{F}'$ (case (II)).

Then h is a nondegenerate alternating bilinear form on the $2n$-dimensional vector space V. We denote the group of isometries of the symplectic space (V, h) by $Sp(V, h)$. Then $Sp(V, h) \cong Sp(2n, \mathbb{F}')$ consists of all $\theta \in GL(V)$ such that $h(\bar{v}_1^\theta, \bar{v}_2^\theta) = h(\bar{v}_1, \bar{v}_2)$ for all $\bar{v}_1, \bar{v}_2 \in V$. An ordered basis $B = (\bar{e}_1, \bar{f}_1, \bar{e}_2, \bar{f}_2, \ldots, \bar{e}_n, \bar{f}_n)$ of V is called a *hyperbolic basis* of (V, h) if $h(\bar{e}_i, \bar{e}_j) = h(\bar{f}_i, \bar{f}_j) = 0$ and $h(\bar{e}_i, \bar{f}_j) = \delta_{ij}$ for all $i, j \in \{1, 2, \ldots, n\}$. The elements of $Sp(V, h)$ are precisely the elements of $GL(V)$ that map hyperbolic bases of (V, h) to hyperbolic bases of (V, h).

For every $k \in \{1, 2, \ldots, n\}$, we denote by $\bigwedge^k V$ the k-th exterior power of V and by W_k the subspace of $\bigwedge^k V$ generated by all vectors of the form $\bar{v}_1 \wedge \bar{v}_2 \wedge \cdots \wedge \bar{v}_k$, where $\langle \bar{v}_1, \bar{v}_2, \ldots, \bar{v}_k \rangle$ is a k-dimensional subspace of V that is totally isotropic with respect to h. If $\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_k$ is a collection of vectors of V, then $\langle \bar{v}_1, \bar{v}_2, \ldots, \bar{v}_k \rangle_{\mathbb{F}}$ denotes the set of all vectors that can be written as a linear combination of $\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_k$ with all coefficients belonging to $\mathbb{F}$. If $\chi \in \bigwedge^k V$, then we define $\langle \chi \rangle_{\mathbb{F}} := \{k \cdot \chi \,|\, k \in \mathbb{F}\}$.

Every $\theta \in GL(V)$ has a natural induced action on $\bigwedge^k V$ such that $(\bar{v}_1 \wedge \bar{v}_2 \wedge \cdots \wedge \bar{v}_k)^\theta = \bar{v}_1^\theta \wedge \bar{v}_2^\theta \wedge \cdots \wedge \bar{v}_k^\theta$ for all $\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_k \in V$. If $\theta \in Sp(V, h)$, then θ leaves the subspace W_k of $\bigwedge^k V$ invariant.

For any hyperbolic basis $B = (\bar{e}_1, \bar{f}_1, \ldots, \bar{e}_n, \bar{f}_n)$ of (V, h), we denote by $\mathcal{G}_{B,k}$ the set of all vectors $\chi \in \bigwedge^k V$ of the form[1]

$$(\bar{e}_{\tau(1)} \wedge \bar{f}_{\tau(1)} - \bar{e}_{\tau(2)} \wedge \bar{f}_{\tau(2)}) \wedge \cdots \wedge (\bar{e}_{\tau(2m-1)} \wedge \bar{f}_{\tau(2m-1)} - \bar{e}_{\tau(2m)} \wedge \bar{f}_{\tau(2m)}) \wedge$$

$$\bar{g}_{\tau(2m+1)} \wedge g_{\tau(2m+2)} \wedge \cdots \wedge \bar{g}_{\tau(k)},$$

where $m \in \{0, 1, \ldots, \lfloor \frac{k}{2} \rfloor\}$, τ is a permutation of $\{1, 2, \ldots, n\}$ and $\bar{g}_{\tau(i)} \in \{\bar{e}_{\tau(i)}, \bar{f}_{\tau(i)}\}$ for every $i \in \{2m+1, 2m+2, \ldots, k\}$. Any such vector χ is called a *standard vector* of $\bigwedge^k V$ with respect to B. If χ is as above then we define $m := m(\chi)$ and call $\{\tau(1), \tau(2), \ldots, \tau(k)\}$ the *support* of χ.

We extend the above definitions to $k = 0$. Put $W_0 = \bigwedge^0 V = \mathbb{F}'$ and $\mathcal{G}_{B,0} = \{1\}$. We define $m(1) := 0 \in \mathbb{N}$ and call $1 \in \mathbb{F}'$ a *standard vector* with respect to B. The *support* of 1 is defined to be the empty set.

Proofs of the following result can be found in [2, §13.3], [3, 8, 17] (some of which do not cover fields of arbitrary characteristic).

Proposition 5.1 *Let B be a hyperbolic basis of (V, h) and $k \in \{0, 1, \ldots, n\}$. Then:*

(1) *We have* $\dim(W_k) = \binom{2n}{k} - \binom{2n}{k-2}$.

(2) *All vectors of $\mathcal{G}_{B,k}$ belong to W_k.*

(3) *There exists a basis $\mathcal{B}_{B,k}$ of W_k that entirely consists of vectors of $\mathcal{G}_{B,k}$.*

In Proposition 5.1(1), the convention has been used that $\binom{i}{j} = 0$ for every $i \in \mathbb{N}$ and every $j \in \mathbb{Z} \setminus \{0, 1, \ldots, i\}$. We will follow this convention also in the remainder of the paper. We will now prove the following extension of Property (3) in Proposition 5.1.

Proposition 5.2 *For every hyperbolic basis $B = (\bar{e}_1, \bar{f}_1, \ldots, \bar{e}_n, \bar{f}_n)$ of (V, h) and every $k \in \{0, 1, \ldots, n\}$, there exists a basis $\mathcal{B}_{B,k}$ of W_k that satisfies the following:*

(1) *all $\binom{2n}{k} - \binom{2n}{k-2}$ vectors of $\mathcal{B}_{B,k}$ belong to $\mathcal{G}_{B,k}$;*

(2) *for every $m \in \{0, 1, \ldots, \lfloor \frac{k}{2} \rfloor\}$, there are precisely*

$$\alpha(n, k, m) := \binom{n}{k-2m} \cdot 2^{k-2m} \cdot \binom{n-k+2m}{m} \cdot \frac{n-k+1}{n-k+m+1}$$

vectors $\chi \in \mathcal{B}_{B,k}$ for which $m(\chi) = m$.

[1]The vector should be interpreted as $\bar{g}_{\tau(2m+1)} \wedge g_{\tau(2m+2)} \wedge \cdots \wedge \bar{g}_{\tau(k)}$ if $m = 0$ and as $(\bar{e}_{\tau(1)} \wedge \bar{f}_{\tau(1)} - \bar{e}_{\tau(2)} \wedge \bar{f}_{\tau(2)}) \wedge \cdots \wedge (\bar{e}_{\tau(2m-1)} \wedge \bar{f}_{\tau(2m-1)} - \bar{e}_{\tau(2m)} \wedge \bar{f}_{\tau(2m)})$ if $k = 2m$.

The rest of this section is devoted to the proof of Proposition 5.2. During that proof, we will make use of the following combinatorial identities, whose verification is straightforward.

Lemma 5.3 (1) *(**Pascal's rule**) For every $i \in \mathbb{N} \setminus \{0\}$ and every $j \in \mathbb{Z}$, we have*

$$\binom{i}{j} = \binom{i-1}{j} + \binom{i-1}{j-1}.$$

(2) *For all $i, j \in \mathbb{Z}$ with $i + j \geq 0$ and $j \geq 0$, we have*

$$\binom{i+2j}{j} \cdot \frac{i+1}{i+j+1} = \binom{i+2j}{j} - \binom{i+2j}{j-1}.$$

In [8], it was explained how a basis $\mathcal{B}_{B,k}$ of the subspace W_k of $\bigwedge^k V$ can be constructed in a recursive way (see proof of [8, Lemma 3.2]). We will recall this recursive construction here and show that the basis that arises in this way satisfies the conditions mentioned in Proposition 5.2.

Suppose first that $n = 1$. Then we put $\mathcal{B}_{B,0} = \{1\}$ and $\mathcal{B}_{B,1} = \{\bar{e}_1, \bar{f}_1\}$. Obviously, $|\mathcal{B}_{B,0}| = \binom{2}{0} - \binom{2}{-2} = 1$ and $|\mathcal{B}_{B,1}| = \binom{2}{1} - \binom{2}{-1} = 2$.

Suppose next that $n \geq 2$. Let V' be the vector space $\langle \bar{e}_2, \bar{f}_2, \ldots, \bar{e}_n, \bar{f}_n \rangle$ and h' the restriction of h to $V' \times V'$. Then $B' := (\bar{e}_2, \bar{f}_2, \ldots, \bar{e}_n, \bar{f}_n)$ is a hyperbolic basis of the symplectic space (V', h'). For every $i \in \{1, 2, \ldots, n-1\}$, let W'_i denote the subspace of $\bigwedge^i V'$ generated by all vectors $\bar{v}_1 \wedge \bar{v}_2 \wedge \cdots \wedge \bar{v}_i$, where $\langle \bar{v}_1, \bar{v}_2, \ldots, \bar{v}_i \rangle$ is an i-dimensional subspace of V' that is totally isotropic with respect to h'. If $i = 0$, then we define $W'_i = \mathbb{F}'$.

If $k = 0$, then we define $\mathcal{B}_{B,0} = \{1\}$. We have $|\mathcal{B}_{B,0}| = \binom{2n}{0} - \binom{2n}{-2} = 1$. Suppose therefore that $n \geq 2$ and $1 \leq k \leq n$.

If $k = n$, then we define $\mathcal{B}^{(1)}_{B,k} := \emptyset$. In this case, we have $|\mathcal{B}^{(1)}_{B,k}| = \binom{2n-2}{k} - \binom{2n-2}{k-2} = \binom{2n-2}{n} - \binom{2n-2}{n-2} = 0$. If $k < n$, then by our recursive construction, we know that there exists a basis $\mathcal{B}^{(1)}_{B,k}$ of W'_k that entirely consist of vectors that are standard with respect to B', and for such a basis we have $|\mathcal{B}^{(1)}_{B,k}| = \binom{2n-2}{k} - \binom{2n-2}{k-2}$.

By our recursive construction, there exists a basis $\mathcal{B}'$ of W'_{k-1} that entirely consists of vectors that are standard with respect to B'. Moreover, the size of $\mathcal{B}'$ is $\binom{2n-2}{k-1} - \binom{2n-2}{k-3}$. Now, put $\mathcal{B}^{(2)}_{B,k} = \{\bar{e}_1 \wedge \chi \,|\, \chi \in \mathcal{B}'\}$ and $\mathcal{B}^{(3)}_{B,k} = \{\bar{f}_1 \wedge \chi \,|\, \chi \in \mathcal{B}'\}$. Then $|\mathcal{B}^{(2)}_{B,k}| = |\mathcal{B}^{(3)}_{B,k}| = \binom{2n-2}{k-1} - \binom{2n-2}{k-3}$.

If $k \geq 2$, then by our recursive construction, we know that there exists a basis $\mathcal{B}''$ of W'_{k-2} that entirely consists of vectors that are standard with respect to B'. Moreover, the size of $\mathcal{B}''$ is equal to $\binom{2n-2}{k-2} - \binom{2n-2}{k-4}$. If $k = 1$, then we put $\mathcal{B}'' = \emptyset$. Now, as $k - 2 < n - 1$, there exists for every $\chi \in \mathcal{B}''$ an $i_\chi \in \{2, 3, \ldots, n\}$ not belonging to the support of χ. We choose only one such i_χ for each $\chi \in \mathcal{B}''$. We define

$$\mathcal{B}^{(4)}_{B,k} = \{(\bar{e}_1 \wedge \bar{f}_1 - \bar{e}_{i_\chi} \wedge \bar{f}_{i_\chi}) \wedge \chi \,|\, \chi \in \mathcal{B}''\}.$$

Then $|\mathcal{B}_{B,k}^{(4)}| = \binom{2n-2}{k-2} - \binom{2n-2}{k-4}$ regardless of whether $k = 1$ or $k \geq 2$. Now, we put

$$\mathcal{B}_{B,k} = \bigcup_{i=1}^{4} \mathcal{B}_{B,k}^{(i)}.$$

Using Pascal's rule a number of times, we find

$$\begin{aligned} |\mathcal{B}_{B,k}| &= |\mathcal{B}_{B,k}^{(1)}| + |\mathcal{B}_{B,k}^{(2)}| + |\mathcal{B}_{B,k}^{(3)}| + |\mathcal{B}_{B,k}^{(4)}| \\ &= \binom{2n-2}{k} - \binom{2n-2}{k-2} + 2 \cdot \binom{2n-2}{k-1} - 2 \cdot \binom{2n-2}{k-3} + \binom{2n-2}{k-2} - \binom{2n-2}{k-4} \\ &= \binom{2n}{k} - \binom{2n}{k-2}. \end{aligned}$$

For every $n \in \mathbb{N} \setminus \{0\}$, for every $k \in \{0, 1, \ldots, n\}$ and every $m \in \mathbb{Z}$, let $\gamma(n, k, m)$ denote the number of vectors $\chi \in \mathcal{B}_{B,k}$ for which $m = m(\chi)$. Obviously, we have $\gamma(n, k, m) = 0$ if $m \notin \{0, 1, \ldots, \lfloor \frac{k}{2} \rfloor\}$. For every $n \in \mathbb{N} \setminus \{0\}$, for every $k \in \mathbb{Z} \setminus \{0, 1, \ldots, n\}$ and every $m \in \mathbb{Z}$, we put $\gamma(n, k, m) = 0$.

If $k = 0$, then $\mathcal{B}_{B,0} = \{1\}$ and hence $\gamma(n, 0, 0) = 1$. If $n = 1$, then since $\mathcal{B}_{B,0} = \{1\}$ and $\mathcal{B}_{B,1} = \{\bar{e}_1, \bar{f}_1\}$, we have $\gamma(1, 0, 0) = 1$ and $\gamma(1, 1, 0) = 2$.

Lemma 5.4 *For every $n \in \mathbb{N} \setminus \{0, 1\}$, for every $k \in \{0, 1, \ldots, n\}$ and for every $m \in \{0, 1, \ldots, \lfloor \frac{k}{2} \rfloor\}$, we have*

$$\gamma(n, k, m) = \gamma(n-1, k, m) + 2 \cdot \gamma(n-1, k-1, m) + \gamma(n-1, k-2, m-1).$$

Proof. The formula is true if $k = 0$. We show that it is also valid if $k \geq 1$. The proof is based on the recursive construction of the basis $\mathcal{B}_{B,k}$ that we discussed above. An element $\chi \in \mathcal{B}_{B,k}$ for which $m = m(\chi)$ belongs to either $\mathcal{B}_{B,k}^{(1)}$, $\mathcal{B}_{B,k}^{(2)}$, $\mathcal{B}_{B,k}^{(3)}$ or $\mathcal{B}_{B,k}^{(4)}$.

If χ is an element of $\mathcal{B}_{B,k}^{(1)}$, then $k < n$ and χ is an element of W'_k for which $m(\chi) = m$. There are precisely $\gamma(n-1, k, m)$ such elements.

If χ is an element of $\mathcal{B}_{B,k}^{(2)}$, then it arises from an element χ' of W'_{k-1} for which $m(\chi') = m$. There are precisely $\gamma(n-1, k-1, m)$ such elements.

If χ is an element of $\mathcal{B}_{B,k}^{(3)}$, then it arises from an element χ' of W'_{k-1} for which $m(\chi') = m$. There are precisely $\gamma(n-1, k-1, m)$ such elements.

If χ is an element of $\mathcal{B}_{B,k}^{(4)}$, then $k \geq 2$ and χ arises from an element χ' of W'_{k-2} for which $m(\chi') = m - 1$. There are precisely $\gamma(n-1, k-2, m-1)$ such elements. ∎

Lemma 5.5 *For every $n \in \mathbb{N} \setminus \{0\}$ and for every $k \in \{0, 1, \ldots, n\}$, we have $\gamma(n, k, 0) = \binom{n}{k} \cdot 2^k$.*

Proof. We will prove this by induction on n.

Suppose first that $n = 1$. Then $(n, k) \in \{(1, 0), (1, 1)\}$. If $(n, k) = (1, 0)$, then $\gamma(n, k, 0) = 1 = \binom{n}{k} \cdot 2^k$. If $(n, k) = (1, 1)$, then $\gamma(n, k, 0) = 2 = \binom{n}{k} \cdot 2^k$.

Suppose next that $n \geq 2$ and that the lemma holds when the first argument of γ is smaller that n. By the induction hypothesis, we know that

$$\gamma(n-1,k,0) = \binom{n-1}{k} \cdot 2^k, \tag{1}$$

$$\gamma(n-1,k-1,0) = \binom{n-1}{k-1} \cdot 2^{k-1}. \tag{2}$$

We wish to note that (1) remains valid for $n = k$ (since the right and left hand sides are then equal to 0). Also, (2) remains valid for $k = 0$ (since right and left hand sides are then equal to 0). From Lemma 5.4, we then know that

$$\gamma(n,k,0) = \gamma(n-1,k,0)+2\cdot\gamma(n-1,k-1,0) = \binom{n-1}{k}\cdot 2^k+2\cdot\binom{n-1}{k-1}\cdot 2^{k-1} = \binom{n}{k}\cdot 2^k.$$

■

Lemma 5.6 *For every $n \in \mathbb{N}\backslash\{0\}$, for every $k \in \{0,1,\ldots,n\}$ and every $m \in \{0,1,\ldots,\lfloor\frac{k}{2}\rfloor\}$, we have*

$$\gamma(n,k,m) = \binom{n}{k-2m} \cdot 2^{k-2m} \cdot \binom{n-k+2m}{m} \cdot \frac{n-k+1}{n-k+m+1}.$$

Proof. By Lemma 5.5, we know that the lemma is valid if $m = 0$ and hence also if $k \in \{0,1\}$.

We will prove the lemma by induction on $n \in \mathbb{N} \setminus \{0\}$. By the above, we know that the lemma is valid of $n = 1$ (since $k \in \{0,1\}$ in this case). We will therefore suppose that $n \geq 2$ and that the lemma holds when the first argument of γ is less than n (induction hypothesis). As the lemma is valid if $m = 0$, we may assume that $m \geq 1$ and $k \geq 2$. By the induction hypothesis, we then know that

$$\begin{aligned}
\gamma(n-1,k,m) &= \binom{n-1}{k-2m} \cdot 2^{k-2m} \cdot \binom{n-1-k+2m}{m} \cdot \frac{n-k}{n-k+m}, \\
\gamma(n-1,k-1,m) &= \binom{n-1}{k-1-2m} \cdot 2^{k-1-2m} \cdot \binom{n-k+2m}{m} \cdot \frac{n-k+1}{n-k+m+1}, \\
\gamma(n-1,k-2,m-1) &= \binom{n-1}{k-2m} \cdot 2^{k-2m} \cdot \binom{n-1-k+2m}{m-1} \cdot \frac{n-k+2}{n-k+m+1}.
\end{aligned}$$

Observe that the first formula remains valid if $k = n$ and that the second formula remains valid if $k = 2m$, since both sides are 0 in each of the two cases. Invoking Lemmas 5.3 and 5.4, we find

$$\frac{1}{2^{k-2m}} \cdot \gamma(n,k,m) = \frac{1}{2^{k-2m}}\Big(\gamma(n-1,k,m) + 2\cdot\gamma(n-1,k-1,m) + \gamma(n-1,k-2,m-1)\Big)$$

$$= \binom{n-1}{k-1-2m}\cdot\binom{n-k+2m}{m}\cdot\frac{n-k+1}{n-k+m+1}+\binom{n-1}{k-2m}\cdot\binom{n-1-k+2m}{m}\cdot\frac{n-k}{n-k+m}+\binom{n-1}{k-2m}\cdot\binom{n-k-1+2m}{m-1}\cdot\frac{n-k+2}{n-k+m+1}$$

$$= \binom{n}{k-2m}\cdot\binom{n-k+2m}{m}\cdot\frac{n-k+1}{n-k+m+1} - \binom{n-1}{k-2m}\cdot\binom{n-k+2m}{m}\cdot\frac{n-k+1}{n-k+m+1}$$

$$+\binom{n-1}{k-2m}\cdot\left(\binom{n-1-k+2m}{m}-\binom{n-1-k+2m}{m-1}\right)+\binom{n-1}{k-2m}\cdot\left(\binom{n-1-k+2m}{m-1}-\binom{n-1-k+2m}{m-2}\right)$$

$$=\binom{n}{k-2m}\cdot\binom{n-k+2m}{m}\cdot\frac{n-k+1}{n-k+m+1}-\binom{n-1}{k-2m}\cdot\left(\binom{n-k+2m}{m}-\binom{n-k+2m}{m-1}+\binom{n-1-k+2m}{m-2}-\binom{n-1-k+2m}{m}\right)$$

$$=\binom{n}{k-2m}\cdot\binom{n-k+2m}{m}\cdot\frac{n-k+1}{n-k+m+1}-\binom{n-1}{k-2m}\cdot\left(\binom{n-k+2m-1}{m-1}-\binom{n-k+2m-1}{m-1}\right)$$

$$=\binom{n}{k-2m}\cdot\binom{n-k+2m}{m}\cdot\frac{n-k+1}{n-k+m+1}.$$

■

6 The symplectic space $(\widetilde{V},\widetilde{h})$

Put $\widetilde{V}:=\langle\bar{b}_1^*,\bar{b}_2^*,\dots,\bar{b}_{2n}^*\rangle_{\mathbb{F}}$ and denote by $\widetilde{h}$ the restriction of h to $\widetilde{V}\times\widetilde{V}$. For all vectors $\widetilde{v}=X_1\bar{b}_1^*+X_2\bar{b}_2^*+\cdots+X_{2n}\bar{b}_{2n}^*$ and $\widetilde{w}=Y_1\bar{b}_1^*+Y_2\bar{b}_2^*+\cdots+Y_{2n}\bar{b}_{2n}^*$ of $\widetilde{V}$, we have

$$\begin{aligned}\widetilde{h}(\widetilde{v},\widetilde{w}) &= (X_1^\sigma Y_2-X_2^\sigma Y_1)+\cdots+(X_{2n-1}^\sigma Y_{2n}-X_{2n}^\sigma Y_{2n-1})\\ &= (X_1Y_2-X_2Y_1)+\cdots+(X_{2n-1}Y_{2n}-X_{2n}Y_{2n-1}).\end{aligned}$$

and hence $\widetilde{h}$ is a nondegenerate alternating bilinear form on $\widetilde{V}$.

Obviously, $(\bar{b}_1^*,\bar{b}_2^*,\dots,\bar{b}_{2n}^*)$ is a hyperbolic basis of $(\widetilde{V},\widetilde{h})$. As $f(\widetilde{v})\in\mathbb{F}$ for any $\widetilde{v}\in\widetilde{V}$, we see that any hyperbolic basis of $(\widetilde{V},\widetilde{h})$ is also an admissible basis of (V,f).

In [8, §2], we proved the following results.

Proposition 6.1 ([8]) (1) *If B_1 and B_2 are two ordered bases of $\widetilde{V}$ such that B_1 is a hyperbolic basis of $(\widetilde{V},\widetilde{h})$ and $(B_1,B_2)\in\widetilde{\Omega_1}\cup\widetilde{\Omega_2}\cup\widetilde{\Omega_3}\cup\widetilde{\Omega_4}\cup\widetilde{\Omega_5}$, then B_2 is also a hyperbolic basis of $(\widetilde{V},\widetilde{h})$.*

(2) *If B_1 and B_2 are two hyperbolic bases of $(\widetilde{V},\widetilde{h})$, then there exist hyperbolic bases $B_0',B_1',\dots,B_k'$ of $(\widetilde{V},\widetilde{h})$ for some $k\in\mathbb{N}$ such that $B_0'=B_1$, $B_k'=B_2$ and $(B_{i-1}',B_i')\in\widetilde{\Omega_1}\cup\widetilde{\Omega_2}\cup\widetilde{\Omega_3}\cup\widetilde{\Omega_4}\cup\widetilde{\Omega_5}$ for every $i\in\{1,2,\dots,k\}$.*

Observe that Proposition 6.1 is similar to Propositions 3.5 and 3.7. In fact, their proofs (as given here and in [8]) are also similar.

We denote the group of isometries of $(\widetilde{V},\widetilde{h})$ by $Sp(\widetilde{V},\widetilde{h})$. The elements of $Sp(\widetilde{V},\widetilde{h})\cong Sp(2n,\mathbb{F})$ are precisely the elements of $GL(\widetilde{V})$ that map hyperbolic bases of $(\widetilde{V},\widetilde{h})$ to hyperbolic bases of $(\widetilde{V},\widetilde{h})$.

Every element $\theta\in GL(V)$ that stabilizes $\widetilde{V}$ determines an element $\widetilde{\theta}\in GL(\widetilde{V})$. Conversely, every element $\phi\in GL(\widetilde{V})$ is of the form $\widetilde{\theta}$ for some $\theta\in GL(V)$ stabilizing $\widetilde{V}$.

For every $k\in\{1,2,\dots,n\}$, let $\widetilde{W_k}$ denote the subspace of $\bigwedge^k\widetilde{V}$ generated by all vectors of the form $\widetilde{v_1}\wedge\widetilde{v_2}\wedge\cdots\wedge\widetilde{v_k}$, where $\langle\widetilde{v_1},\widetilde{v_2},\dots,\widetilde{v_k}\rangle$ is a k-dimensional subspace of $\widetilde{V}$ that is totally isotropic with respect to $\widetilde{h}$. Put $\widetilde{W_0}:=\mathbb{F}$.

The following result is the equivalent version of Proposition 5.1. Proofs can also be found in [2, §13.3], [3, 8, 17].

Proposition 6.2 *Let B be a hyperbolic basis of $(\widetilde{V}, \widetilde{h})$ and let $k \in \{0, 1, \ldots, n\}$. Then:*

(1) *We have* $\dim(\widetilde{W_k}) = \binom{2n}{k} - \binom{2n}{k-2}$.

(2) *All vectors of $\mathcal{G}_{B,k}$ belong to $\widetilde{W_k}$.*

Lemma 6.3 *Suppose case (II) occurs. If B is a hyperbolic basis of $(\widetilde{V}, \widetilde{h})$, then the subsets of $\mathcal{G}_{B,k}$ that are bases of W_k are precisely the subsets of $\mathcal{G}_{B,k}$ that are bases of $\widetilde{W_k}$.*

Proof. The subsets of $\mathcal{G}_{B,k}$ that are bases of W_k [resp. $\widetilde{W_k}$] are precisely the subsets of size $\binom{2n}{k} - \binom{2n}{k-2}$ of $\mathcal{G}_{B,k}$ that are linearly independent over $\mathbb{F}'$ [resp. $\mathbb{F}$]. The claim then follows from the fact that subsets of $\mathcal{G}_{B,k}$ are linearly independent over $\mathbb{F}$ if and only if they are linearly independent over $\mathbb{F}'$. ∎

7 The $\mathbb{F}$-vector spaces W_B

We continue with the notation introduced in Section 2. For every admissible basis $B = (\bar{e}_1, \bar{f}_1, \ldots, \bar{e}_n, \bar{f}_n)$ of (V, f), let W_B denote the subset of $\bigwedge^n V$ whose elements consist of all linear combinations with coefficients in $\mathbb{F}$ of all vectors of the form[2]

$$\Big(\bar{g}_{\tau(1)} \wedge \bar{g}_{\tau(2)} \wedge \cdots \wedge \bar{g}_{\tau(k)}\Big) \wedge \Big(\lambda \cdot \bar{e}_{\tau(k+1)} \wedge \bar{f}_{\tau(k+1)} \wedge \cdots \wedge \bar{e}_{\tau(k+l)} \wedge \bar{f}_{\tau(k+l)}$$

$$+(-1)^l \lambda^\sigma \cdot \bar{e}_{\tau(k+l+1)} \wedge \bar{f}_{\tau(k+l+1)} \wedge \cdots \wedge \bar{e}_{\tau(n)} \wedge \bar{f}_{\tau(n)}\Big),$$

(1) $k, l \in \{0, 1, \ldots, n\}$ such that $k + 2l = n$;

(2) $\lambda \in \mathbb{F}'$;

(3) τ is a permutation of $\{1, 2, \ldots, n\}$ satisfying:

  (i) $\tau(1) < \tau(2) < \cdots < \tau(k)$,

  (ii) $\tau(k+1) < \tau(k+2) < \cdots < \tau(k+l)$,

  (iii) $\tau(k+l+1) < \tau(k+l+2) < \cdots < \tau(n)$,

  (iv) $\tau(k+1) < \tau(k+l+1)$;

(4) $\bar{g}_i \in \{\bar{e}_i, \bar{f}_i\}$ for every $i \in \{\tau(1), \tau(2), \ldots, \tau(k)\}$.

Note that W_B can be regarded as a vector space over $\mathbb{F}$.

Lemma 7.1 *If B is an admissible basis of (V, f) and $\eta \in \mathbb{F}' \setminus \{0\}$, then $\eta \cdot W_B = W_B$ if and only if $\eta \in \mathbb{F} \setminus \{0\}$.*

[2] If $l = 0$, then this vector should be interpreted as $\bar{g}_{\tau(1)} \wedge \bar{g}_{\tau(2)} \wedge \cdots \wedge \bar{g}_{\tau(k)}$.

Proof. Obviously, $\eta \cdot W_B = W_B$ if $\eta \in \mathbb{F} \setminus \{0\}$.

Conversely, suppose $\eta \cdot W_B = W_B$ for a certain $\eta \in \mathbb{F}' \setminus \{0\}$. If $B = (\bar{e}_1, \bar{f}_1, \ldots, \bar{e}_n, \bar{f}_n)$, then $\bar{e}_1 \wedge \bar{e}_2 \wedge \cdots \wedge \bar{e}_n \in W_B$ and $\eta \cdot \bar{e}_1 \wedge \bar{e}_2 \wedge \cdots \wedge \bar{e}_n \in W_B$ implies that $\eta \in \mathbb{F} \setminus \{0\}$. ∎

Lemma 7.2 *If B_1 and B_2 are two admissible bases of (V, f), then there exists an $\eta \in \mathbb{F}' \setminus \{0\}$ such that $W_{B_2} = \frac{1}{\eta} \cdot W_{B_1} = \{\frac{\alpha}{\eta} \,|\, \alpha \in W_{B_1}\}$. This η is uniquely determined, up to a nonzero factor in $\mathbb{F}$.*

Proof. We first prove that there exists such an η. By Proposition 3.7, it suffices to show this in the case that $(B_1, B_2) \in \Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \Omega_4 \cup \Omega_5$. If $(B_1, B_2) \in \Omega_1 \cup \Omega_2 \cup \Omega_4 \cup \Omega_5$, then one verifies that the claim of the lemma is valid if we take $\eta = 1$. If $(B_1, B_2) \in \Omega_3$, then $B_1 = (\bar{e}_1, \bar{f}_1, \ldots, \bar{e}_n, \bar{f}_n)$ and $B_2 = (\frac{\bar{e}_1}{\lambda}, \lambda^\sigma \bar{f}_1, \bar{e}_2, \bar{f}_2, \ldots, \bar{e}_n, \bar{f}_n)$ for some admissible basis $(\bar{e}_1, \bar{f}_1, \ldots, \bar{e}_n, \bar{f}_n)$ of (V, f) and some $\lambda \in \mathbb{F}' \setminus \{0\}$, and we can take $\eta = \lambda$ since $\lambda^\sigma \lambda \in \mathbb{F}$. (The explicit calculations can be found in [9, Lemma 4.1] for the Hermitian case, but these calculations immediately extend to the mixed case as well.)

The fact that η is uniquely determined up to a nonzero factor of $\mathbb{F}$ follows from Lemma 7.1. ∎

Lemma 7.3 *If B_1 and B_2 are two hyperbolic bases of $(\widetilde{V}, \widetilde{h})$, then $W_{B_1} = W_{B_2}$.*

Proof. By Proposition 6.1(2), it suffices to prove this in the case $(B_1, B_2) \in \widetilde{\Omega_1} \cup \widetilde{\Omega_2} \cup \widetilde{\Omega_3} \cup \widetilde{\Omega_4} \cup \widetilde{\Omega_5}$. The verification in each of these cases is similar as in Lemma 7.2. ∎

Lemma 7.4 *Let B, B_1, B_2 be three admissible bases of (V, f) and let θ be the unique element of G_f mapping B_1 to B_2. Then the following are equivalent:*

(1) $W_{B_1} = W_{B_2}$;

(2) *the induced action of θ on $\bigwedge^n V$ stabilizes W_B.*

Proof. For every $i \in \{1, 2\}$, let $\eta_i \in \mathbb{F}' \setminus \{0\}$ such that $W_B = \frac{1}{\eta_i} \cdot W_{B_i}$. Then $W_{B_2} = \frac{\eta_2}{\eta_1} \cdot W_{B_1}$. Since $W_{B_1}^\theta = W_{B_2}$, we have $W_B^\theta = \left(\frac{1}{\eta_1} \cdot W_{B_1}\right)^\theta = \frac{1}{\eta_1} \cdot W_{B_2} = \frac{\eta_2}{\eta_1} \cdot W_B$. By Lemma 7.1, both claims of the lemma are equivalent with the condition that $\frac{\eta_1}{\eta_2} \in \mathbb{F}$. ∎

8 A full embedding of the dual polar space Δ corresponding to $\Pi(\mathbb{F}', \mathbb{F}, \sigma, n)$

Let W denote the $\mathbb{F}$-vector space W_{B^*}, where B^* is the admissible basis $(\bar{b}_1^*, \bar{b}_2^*, \ldots, \bar{b}_{2n}^*) = (\bar{e}_1^*, \bar{f}_1^*, \ldots, \bar{e}_n^*, \bar{f}_n^*)$ of (V, f), and denote by $\mathrm{PG}(W)$ the corresponding projective space. If B is an admissible basis of (V, f), then by Lemma 7.2, there exists an $\eta_B \in \mathbb{F}' \setminus \{0\}$ such that $W = W_{B^*} = \frac{1}{\eta_B} \cdot W_B$.

Lemma 8.1 *Let $(\bar{e}_1, \bar{f}_1, \bar{e}_2, \bar{f}_2, \ldots, \bar{e}_n, \bar{f}_n)$ be an admissible basis of (V, f). Let $\mathcal{A}$ denote the set of all subspaces of V of the form $\langle \bar{g}_1, \bar{g}_2, \ldots, \bar{g}_n \rangle$, where $\bar{g}_i \in \{\bar{e}_i, \bar{f}_i\}$ for every $i \in \{1, 2, \ldots, n\}$. If Z is an n-dimensional subspace of V corresponding to an $(n-1)$-dimensional singular subspace of $\Pi(\mathbb{F}', \mathbb{F}, \sigma, n)$, then there exists an $A \in \mathcal{A}$ such that $\langle A, Z \rangle = V$.*

Proof. We will prove this by induction on n.

Suppose $n = 1$. Then Z is of the form $\langle a_1\bar{e}_1 + b_1\bar{f}_1 \rangle$, where $(a_1, b_1) \in (\mathbb{F}' \times \mathbb{F}') \setminus \{(0, 0)\}$. If $a_1 \neq 0$, then we take $A = \langle \bar{f}_1 \rangle$; otherwise we take $A = \langle \bar{e}_1 \rangle$. In any case, $V = \langle Z, A \rangle$.

Suppose next that $n \geq 2$. Since $h(\bar{e}_1, \bar{f}_1) = 1$, it is impossible that both $\bar{e}_1, \bar{f}_1$ belong to Z. Let $\bar{g}_1 \in \{\bar{e}_1, \bar{f}_1\}$ such that $\bar{g}_1 \notin Z$. Put $Z' := Z \cap \bar{g}_1^{\perp_h}$, and let $\bar{h}$ be any vector of $Z \setminus Z'$. Then $\bar{h}$ is a nonzero multiple of a vector of the form $(\bar{e}_1 + \bar{f}_1 - \bar{g}_1) + c_1\bar{g}_1 + a_2\bar{e}_2 + b_2\bar{f}_2 + \cdots + a_n\bar{e}_n + b_n\bar{f}_n$ for certain $c_1, a_2, b_2, \ldots, \bar{a}_n, \bar{b}_n \in \mathbb{F}'$. For every vector $\bar{v} \in Z'$, we define $\bar{v} := \bar{v}' + \bar{v}''$, where $\bar{v}' \in \langle \bar{g}_1 \rangle$ and $\bar{v}'' \in \langle \bar{e}_2, \bar{f}_2, \ldots, \bar{e}_n, \bar{f}_n \rangle$. Put $Z'' := \langle \bar{v}'' \,|\, \bar{v} \in Z' \rangle$. Then $\langle \bar{g}_1, Z'' \rangle = \langle \bar{g}_1, Z' \rangle$ is n-dimensional. So, Z'' has dimension $n-1$ and is totally isotropic with respect to h. Moreover, for every $\bar{v} \in Z'$, we have $f(\bar{v}'') = f(\bar{v} - \bar{v}') = f(\bar{v}) + f(-\bar{v}') + g(\bar{v}, -\bar{v}') = f(\bar{v}) + f(-\bar{v}') + g(\bar{v}, -\bar{v}') - h(\bar{v}, -\bar{v}') \in \mathbb{F}$, since $f(\bar{v})$, $f(-\bar{v}')$ and $g(\bar{v}, -\bar{v}') - h(\bar{v}, -\bar{v}')$ belong to $\mathbb{F}$. By the induction hypothesis, there exist $\bar{g}_2 \in \{\bar{e}_2, \bar{f}_2\}$, ..., $\bar{g}_n \in \{\bar{e}_n, \bar{f}_n\}$ such that $\langle \bar{g}_2, \ldots, \bar{g}_n, Z'' \rangle = \langle \bar{e}_2, \bar{f}_2, \ldots, \bar{e}_n, \bar{f}_n \rangle$. Then

$$\begin{aligned}\langle \bar{g}_1, \bar{g}_2, \ldots, \bar{g}_n, Z \rangle &= \langle \bar{g}_1, \bar{g}_2, \ldots, \bar{g}_n, Z', \bar{h} \rangle = \langle \bar{g}_1, \bar{g}_2, \ldots, \bar{g}_n, Z'', \bar{h} \rangle \\ &= \langle \bar{g}_1, \bar{h}, \bar{e}_2, \bar{f}_2, \bar{e}_2, \bar{f}_3, \ldots, \bar{e}_n, \bar{f}_n \rangle = \langle \bar{e}_1, \bar{f}_1, \ldots, \bar{e}_n, \bar{f}_n \rangle = V.\end{aligned}$$

■

Lemma 8.2 *If $E = \langle \bar{e}_1, \bar{e}_2, \ldots, \bar{e}_n \rangle$ is an n-dimensional subspace of V corresponding to an $(n-1)$-dimensional singular subspace of $\Pi(\mathbb{F}', \mathbb{F}, \sigma, n)$, then:*

(1) *$(\bar{e}_1, \bar{e}_2, \ldots, \bar{e}_n)$ can be extended to an admissible basis $B = (\bar{e}_1, \bar{f}_1, \ldots, \bar{e}_n, \bar{f}_n)$ of (V, f).*

(2) *There exists an $\eta \in \mathbb{F}' \setminus \{0\}$ such that*

$$\frac{1}{\eta} \cdot \bar{e}_1 \wedge \bar{e}_2 \wedge \cdots \wedge \bar{e}_n \in W. \qquad (*)$$

(3) *The η for which $(*)$ holds is uniquely determined, up to a nonzero factor in $\mathbb{F} \setminus \{0\}$.*

(4) *If $\eta = \eta_B$, then $(*)$ holds.*

Proof. Let F be an n-dimensional subspace of V corresponding to an $(n-1)$-dimensional singular subspace of $\Pi(\mathbb{F}', \mathbb{F}, \sigma, n)$ disjoint from E. For every $i \in \{1, 2, \ldots, n\}$, let $\bar{f}_i$ be the unique vector of F such that $h(\bar{e}_i, \bar{f}_i) = 1$ and $h(\bar{e}_j, \bar{f}_i) = 0$ for all $j \in \{1, 2, \ldots, n\} \setminus \{i\}$. Then $B = (\bar{e}_1, \bar{f}_1, \ldots, \bar{e}_n, \bar{f}_n)$ is an admissible basis of (V, f). Since $\bar{e}_1 \wedge \bar{e}_2 \wedge \cdots \wedge \bar{e}_n \in W_B$, we have $\frac{1}{\eta_B} \cdot \bar{e}_1 \wedge \bar{e}_2 \wedge \cdots \wedge \bar{e}_n \in W$. It remains to show the validity of (3).

By Lemma 8.1, there exist $\bar{g}_1 \in \{\bar{e}_1^*, \bar{f}_1^*\}$, $\bar{g}_2 \in \{\bar{e}_2^*, \bar{f}_2^*\}, \ldots,$ $\bar{g}_n \in \{\bar{e}_n^*, \bar{f}_n^*\}$ such that

$$\langle \bar{g}_1, \bar{g}_2, \ldots, \bar{g}_n, \bar{e}_1, \bar{e}_2, \ldots, \bar{e}_n \rangle = V,$$

or equivalently, such that

$$\bar{g}_1 \wedge \bar{g}_2 \wedge \cdots \wedge \bar{g}_n \wedge \bar{e}_1 \wedge \bar{e}_2 \wedge \cdots \wedge \bar{e}_n \neq 0.$$

For every $i \in \{1, 2, \ldots, n\}$, let $\bar{h}_i \in V$ such that $\{\bar{e}_i^*, \bar{f}_i^*\} = \{\bar{h}_i, \bar{g}_i\}$. Now, consider all vectors of the form $\bar{b}_{i_1}^* \wedge \bar{b}_{i_2}^* \wedge \cdots \wedge \bar{b}_{i_n}^*$, where $i_1, i_2, \ldots, i_n \in \{1, 2, \ldots, 2n\}$ are such that $i_1 < i_2 < \cdots < i_n$. If we write $\bar{e}_1 \wedge \bar{e}_2 \wedge \cdots \wedge \bar{e}_n$ as a linear combination with coefficients in $\mathbb{F}'$ of these vectors, then the fact that $\bar{g}_1 \wedge \bar{g}_2 \wedge \cdots \wedge \bar{g}_n \wedge \bar{e}_1 \wedge \bar{e}_2 \wedge \cdots \wedge \bar{e}_n \neq 0$ implies that in this expansion there is a nonzero term in $\bar{h}_1 \wedge \bar{h}_2 \wedge \cdots \wedge \bar{h}_n$, say $\lambda \cdot \bar{h}_1 \wedge \bar{h}_2 \wedge \cdots \wedge \bar{h}_n$ with $\lambda \in \mathbb{F}' \setminus \{0\}$. Then every suitable η necessarily belongs to the set $\{\lambda \cdot k \mid k \in \mathbb{F} \setminus \{0\}\}$, showing that also (3) is valid. ∎

Suppose $p = \langle \bar{e}_1, \bar{e}_2, \ldots, \bar{e}_n \rangle$ is an n-dimensional subspace of V corresponding to an $(n-1)$-dimensional singular subspace of $\Pi(\mathbb{F}', \mathbb{F}, \sigma, n)$. By Lemma 8.2(2), there exists an $\eta \in \mathbb{F}' \setminus \{0\}$ such that

$$\frac{1}{\eta} \cdot \bar{e}_1 \wedge \bar{e}_2 \wedge \cdots \wedge \bar{e}_n \in W.$$

We denote by $e(p)$ the point

$$\langle \frac{1}{\eta} \cdot \bar{e}_1 \wedge \bar{e}_2 \wedge \cdots \wedge \bar{e}_n \rangle_{\mathbb{F}}$$

of $\mathrm{PG}(W)$. By Lemma 8.2(3), we also know that the point $e(p)$ is well-defined. The map e is an injective mapping from the point set of Δ to the point set of $\mathrm{PG}(W)$.

A *full projective embedding* of a point-line geometry $\mathcal{S}$ is an injective map e' from the point set of $\mathcal{S}$ to the point set of a projective space Σ mapping lines of $\mathcal{S}$ to full lines of Σ such that the image of e' generates Σ. In the following proposition, we show that e determines a full projective embedding.

Proposition 8.3 *The map e defines a full embedding of Δ into a subspace* $\mathrm{PG}(U)$ *of* $\mathrm{PG}(W)$.

Proof. Suppose L is a line of Δ. Let p denote an arbitrary point of L. Then there exists an admissible basis $B = (\bar{e}_1, \bar{f}_1, \bar{e}_2, \bar{f}_2, \ldots, \bar{e}_n, \bar{f}_n)$ of (V, f) such that $p = \langle \bar{e}_1, \bar{e}_2, \ldots, \bar{e}_n \rangle$ and L corresponds to the $(n-2)$-dimensional subspace $\langle \bar{e}_1, \bar{e}_2, \ldots, \bar{e}_{n-1} \rangle$. Then L consists of the point $\langle \bar{e}_1, \bar{e}_2, \ldots, \bar{e}_n \rangle$ and all points of the form $\langle \bar{e}_1, \ldots, \bar{e}_{n-1}, \lambda \bar{e}_n + \bar{f}_n \rangle$, where $\lambda \in \mathbb{F}$. (Indeed, the fact that $f(\lambda \bar{e}_n + \bar{f}_n) \in \mathbb{F}$ readily implies that $\lambda \in \mathbb{F}$, see e.g. the end of the proof of Proposition 3.7.) If $B' = (\bar{e}_1, \bar{f}_1, \ldots, \bar{e}_{n-1}, \bar{f}_{n-1}, \bar{e}_n, \lambda \bar{e}_n + \bar{f}_n)$ with $\lambda \in \mathbb{F}$, then B' is an admissible basis of (V, f) by Proposition 3.5 and $W_B = W_{B'}$. So, by Lemma 7.2, there exists an $\eta \in \mathbb{F}' \setminus \{0\}$ such that

$$\frac{1}{\eta} \cdot \bar{e}_1 \wedge \bar{e}_2 \wedge \cdots \wedge \bar{e}_n \in W,$$

$$\frac{1}{\eta}\cdot\bar{e}_1\wedge\bar{e}_2\wedge\cdots\wedge\bar{e}_{n-1}\wedge(\lambda\bar{e}_n+\bar{f}_n)\in W,\quad\forall\lambda\in\mathbb{F}.$$

So, e maps the line L of Δ to the line of $\mathrm{PG}(W)$ determined by the 2-space

$$\langle\frac{1}{\eta}\cdot\bar{e}_1\wedge\bar{e}_2\wedge\cdots\wedge\bar{e}_n,\frac{1}{\eta}\cdot\bar{e}_1\wedge\cdots\wedge\bar{e}_{n-1}\wedge\bar{f}_n\rangle_{\mathbb{F}}.$$

■

The subspace $\mathrm{PG}(U)$ of $\mathrm{PG}(W)$ mentioned in Proposition 8.3 is generated by the image of e. By Lemma 8.2, we then know

Corollary 8.4 *The subspace U of the $\mathbb{F}$-vector space W is generated by all vectors of the form $\frac{1}{\eta_B}\cdot\bar{e}_1\wedge\bar{e}_2\wedge\cdots\wedge\bar{e}_n$, where $B=(\bar{e}_1,\bar{f}_1,\bar{e}_2,\bar{f}_2,\ldots,\bar{e}_n,\bar{f}_n)$ is an admissible basis of (V,f).*

Lemma 8.5 *Let B_1 and B_2 be two admissible bases of (V,f) such that $W_{B_1}=W_{B_2}$ and let θ be the unique element of G_f mapping B_1 to B_2. Then the induced action of θ on $\bigwedge^n V$ stabilizes U.*

Proof. By Corollary 8.4, we must prove that the induced action of θ maps every vector of the form $\frac{1}{\eta_B}\cdot\bar{e}_1\wedge\bar{e}_2\wedge\cdots\wedge\bar{e}_n$ with $B=(\bar{e}_1,\bar{f}_1,\bar{e}_2,\bar{f}_2,\ldots,\bar{e}_n,\bar{f}_n)$ an admissible basis of (V,f) to a vector of U. By Lemma 7.4, we know that the induced action of θ stabilizes W_B, implying that $\bar{e}_1^\theta\wedge\bar{e}_2^\theta\wedge\cdots\wedge\bar{e}_n^\theta\in W_B$. So, the vector $\frac{1}{\eta_B}\cdot\bar{e}_1^\theta\wedge\bar{e}_2^\theta\wedge\cdots\wedge\bar{e}_n^\theta$ belongs to W and hence also to U as $B^\theta=(\bar{e}_1^\theta,\bar{f}_1^\theta,\bar{e}_2^\theta,\bar{f}_2^\theta,\ldots,\bar{e}_n^\theta,\bar{f}_n^\theta)$ is also an admissible basis. ■

In the following section, we give a more detailed description for the subspace U of W that allows to compute its dimension.

9 Determination of U

In case (I), we know that the dual polar space Δ is isomorphic to $DH(2n-1,\mathbb{F}'/\mathbb{F})$ and the embedding e of Δ in $\mathrm{PG}(U)$ is isomorphic to the so-called Grassmann embedding of $DH(2n-1,\mathbb{F}'/\mathbb{F})$. In this case, it is also known that $U=W$ and $\dim(U)=\dim(W)=\binom{2n}{n}$, see e.g. Cooperstein [6]. We will therefore suppose that we are in case (II). Now, let $\mathcal{B}:=\mathcal{B}_{B^*,n}$ be a basis of W_n satisfying the conditions of Proposition 5.2 (with $B=B^*$).

Since $\alpha(n,n,0)=2^n$, we see that $\mathcal{B}$ contains the set $\mathcal{B}_1$ of all vectors of the form $\bar{g}_1\wedge\bar{g}_2\wedge\cdots\wedge\bar{g}_n$, where $\bar{g}_i\in\{\bar{e}_i^*,\bar{f}_i^*\}$ for every $i\in\{1,2,\ldots,n\}$. We put $\mathcal{B}_2=\mathcal{B}\setminus\mathcal{B}_1$. Then

$$|\mathcal{B}|=\binom{2n}{n}-\binom{2n}{n-2},\quad|\mathcal{B}_1|=2^n,\quad|\mathcal{B}_2|=\binom{2n}{n}-\binom{2n}{n-2}-2^n.$$

We define the following sets of vectors.

- U_1: the set consisting of all linear combinations of the elements of $\mathcal{B}$ with all coefficients belonging to $\mathbb{F}'$;
- U_2: the set consisting of all linear combinations of the elements of $\mathcal{B}$ with all coefficients belonging to $\mathbb{F}$;
- U_3: the set consisting of all linear combinations of the elements of $\mathcal{B}$ with all coefficients involving elements of $\mathcal{B}_1$ belonging to $\mathbb{F}$ and all coefficients involving elements of $\mathcal{B}_2$ belonging to $\mathbb{F}'$.

Then $U_2 \subseteq U_3 \subseteq U_1$. By Proposition 5.2, $U_1 = W_n$ and by Lemma 6.3, $U_2 = \widetilde{W_n}$.

Lemma 9.1 *We have $U_3 = U_1 \cap W$.*

Proof. We write $W := W' \oplus W''$, where W' [respectively, W''] consists of all linear combinations with coefficients in $\mathbb{F}$ of all vectors of the form $\Big(\bar{g}_{\tau(1)} \wedge \bar{g}_{\tau(2)} \wedge \cdots \wedge \bar{g}_{\tau(k)}\Big) \wedge \Big(\lambda \cdot \bar{e}_{\tau(k+1)} \wedge \bar{f}_{\tau(k+1)} \wedge \cdots \wedge \bar{e}_{\tau(k+l)} \wedge \bar{f}_{\tau(k+l)} + (-1)^l \lambda^\sigma \cdot \bar{e}_{\tau(k+l+1)} \wedge \bar{f}_{\tau(k+l+1)} \wedge \cdots \wedge \bar{e}_{\tau(n)} \wedge \bar{f}_{\tau(n)}\Big)$ satisfying the properties (1), (2), (3) and (4) of Section 7 with $l = 0$ [respectively, $l \geq 1$], $\sigma = 1$, $-1 = 1$ and $B = B^*$. After expansion of the vectors of $\mathcal{G}_{B^*,n}$, we see that all of them belong to W. In particular, all vectors of $\mathcal{B} = \mathcal{B}_{B^*,n}$ belong to W. In fact, we see that all vectors of $\mathcal{B}_1$ belong to W' and all vectors of $\mathcal{B}_2$ belong to W''. A linear combination of the vectors of $\mathcal{B}_2$ always belongs to W, and a linear combination of the vectors of $\mathcal{B}_1$ belongs to W if and only if all coefficients belong to $\mathbb{F}$. We conclude that $U_3 = U_1 \cap W$. ∎

The sets U_1, U_2 and U_3 can be regarded as vector spaces over the field $\mathbb{F}$, with following dimensions:

$$\begin{aligned}
\dim_{\mathbb{F}}(U_2) &= \binom{2n}{n} - \binom{2n}{n-2}, \\
\dim_{\mathbb{F}}(U_1) &= d \cdot \Big[\binom{2n}{n} - \binom{2n}{n-2}\Big], \\
\dim_{\mathbb{F}}(U_3) &= 2^n + d \cdot \Big[\binom{2n}{n} - \binom{2n}{n-2} - 2^n\Big],
\end{aligned}$$

where $d := [\mathbb{F}' : \mathbb{F}]$.

Lemma 9.2 *We have $U \subseteq U_1$.*

Proof. If $\chi \in U_1$, then $\eta \cdot \chi \in U_1$ for every $\eta \in \mathbb{F}'$. So, in view of Corollary 8.4, it suffices to show that $\bar{e}_1 \wedge \bar{e}_2 \wedge \cdots \wedge \bar{e}_n \in U_1$ for any admissible basis $B = (\bar{e}_1, \bar{f}_1, \ldots, \bar{e}_n, \bar{f}_n)$ of (V, f). Now, as case (II) occurs, h is a nondegenerate alternating bilinear form on V and B is a hyperbolic basis of (V, h). But then $\bar{e}_1 \wedge \bar{e}_2 \wedge \cdots \wedge \bar{e}_n \in W_n = U_1$. ∎

Since $U \subseteq W$ and $W \cap U_1 = U_3$, Lemma 9.2 implies the following.

Corollary 9.3 *We have $U \subseteq U_3$.*

Lemma 9.4 *We have $U_2 \subseteq U$.*

Proof. The $\mathbb{F}$-vector space $U_2 = \widetilde{W_n}$ is generated (with coefficients in $\mathbb{F}$) by all vectors of the form $\bar{e}_1 \wedge \bar{e}_2 \wedge \cdots \wedge \bar{e}_n$, where $\langle \bar{e}_1, \bar{e}_2, \ldots, \bar{e}_n \rangle$ is an n-dimensional subspace of $\widetilde{V}$ that is totally isotropic with respect to $\widetilde{h}$. So, it suffices to show that each such vector belongs to U. Now, extend $(\bar{e}_1, \bar{e}_2, \ldots, \bar{e}_n)$ to a hyperbolic basis $B = (\bar{e}_1, \bar{f}_1, \ldots, \bar{e}_n, \bar{f}_n)$ of $(\widetilde{V}, \widetilde{h})$. Then B is also an admissible basis of (V, f). By Lemma 7.3, $W_B = W$ and so we can take $\eta_B = 1$. Corollary 8.4 then implies that $\bar{e}_1 \wedge \bar{e}_2 \wedge \cdots \wedge \bar{e}_n \in U$, as we needed to show. ■

Proposition 9.5 *We have $U = U_3$.*

Proof. We already know that $U \subseteq U_3$. So, in order to show that $U = U_3$, it suffices to prove the following:

(1) all elements of $\mathcal{B}_1$ belong to U;

(2) if $\chi \in \mathcal{B}_2$ and $\lambda \in \mathbb{F}' \setminus \{0\}$, then $\lambda \cdot \chi \in U$.

As all elements of $\mathcal{B}_1$ belong to $\widetilde{W_n} = U_2$, they also belong to U as $U_2 \subseteq U$.

Now, let χ be an arbitrary element of $\mathcal{B}_2$ and $\lambda \in \mathbb{F}' \setminus \{0\}$. As all $\chi' \in \mathcal{B}$ with $m(\chi') = 0$ are contained in $\mathcal{B}_1$, we have $m(\chi) \geq 1$ and so there exist $i, j \in \{1, 2, \ldots, n\}$ with $i \neq j$ such that $\bar{e}_i^* \wedge \bar{f}_i^* - \bar{e}_j^* \wedge \bar{f}_j^*$ is a factor of χ. Then $\chi = \chi' \wedge (\bar{e}_i^* \wedge \bar{f}_i^* - \bar{e}_j^* \wedge \bar{f}_j^*)$ for a certain vector $\chi' \in \bigwedge^{n-2} V$. Since $\chi' \wedge \bar{e}_i^* \wedge \bar{f}_j^*$ belongs to $\mathcal{G}_{B^*,n} \subseteq \widetilde{W_n} = U_2$, it also belongs to U by Lemma 9.4. Consider now the element θ of $GL(V)$ determined by:

- $\bar{e}_i^* \mapsto \bar{e}_i^* + \lambda \bar{e}_j^*$;
- $\bar{f}_j^* \mapsto -\lambda \bar{f}_i^* + \bar{f}_j^*$;
- $\bar{g} \mapsto \bar{g}$ for all $\bar{g} \in \{\bar{e}_1^*, \bar{f}_1^*, \ldots, \bar{e}_n^*, \bar{f}_n^*\} \setminus \{\bar{e}_i^*, \bar{f}_j^*\}$.

By Proposition 3.5, $(B^*)^\theta$ is an admissible basis of (V, f) and hence $\theta \in G_f$ by Lemma 4.3. Since $W_{B^*} = W_{(B^*)^\theta}$ (see proof of Lemma 7.2), the element θ stabilizes the subspace U by Lemma 8.5. So, we have $\chi' \wedge (\bar{e}_i^* + \lambda \bar{e}_j^*) \wedge (-\lambda \bar{f}_i^* + \bar{f}_j^*) \in U$, i.e.

$$\chi' \wedge (\bar{e}_i^* \wedge \bar{f}_j^* - \lambda^2 \cdot \bar{e}_j^* \wedge \bar{f}_i^* + \lambda \cdot \bar{e}_j^* \wedge \bar{f}_j^* + \lambda \cdot \bar{e}_i^* \wedge \bar{f}_i^*) \in U.$$

The vectors $\chi' \wedge \bar{e}_i^* \wedge \bar{f}_j^*$ and $\chi' \wedge \bar{e}_j^* \wedge \bar{f}_i^*$ belong to $\mathcal{G}_{B^*,n} \subseteq \widetilde{W_n} = U_2$ and hence also to U. As $\lambda^2 \in \mathbb{F}$, we know that $\lambda^2 \cdot \chi' \wedge \bar{e}_j^* \wedge \bar{f}_i^*$ also belongs to U. We can conclude that $\lambda \cdot \chi' \wedge (\bar{e}_i^* \wedge \bar{f}_i^* + \bar{e}_j^* \wedge \bar{f}_j^*) = \lambda \cdot \chi \in U$, which is precisely what we needed to prove. ■

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