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OBLIGATION AS WEAKEST PERMISSION: A STRONGLY COMPLETE AXIOMATIZATION

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Abstract. In (Anglberger *et al.*, 2015, Section 4.1), a deontic logic is proposed which explicates the idea that a formula φ is obligatory if and only if it is (semantically speaking) the weakest permission. We give a sound and strongly complete, Hilbert style axiomatization for this logic. As a corollary, it is compact, contradicting earlier claims from Anglberger *et al.* (2015). In addition, we prove that our axiomatization is equivalent to Anglberger et al.'s infinitary proof system, and show that our results are robust w.r.t. certain changes in the underlying semantics.

§1. Intro. In Roy *et al.* (2014, 2012) and Anglberger *et al.* (2015), a logic is developed for "obligation as weakest permission".[1] The semantics proposed in Anglberger *et al.* (2015) is meant to capture the deontic aspects of reasoning in strategic games, where we speak about properties of the best actions available to a given agent. Whereas usually in formal models of such games, actions and/or agents are modeled explicitly at the object level, the present logic only speaks about action tokens (which correspond to states in a Kripke-model) and action types (sets of action tokens). Let us explain this briefly—we refer to the cited works for a more elaborate discussion.

Consider a situation in which an agent can choose from a number of distinct action tokens, where at least some of these are optimal. Whereas the agent is permitted to perform one of those optimal action tokens, his sole obligation (if there is one at all – mind this important caveat) is to perform one of the optimal action tokens. This means that the deontic operators O and P can be read as follows, where φ refers to an arbitrary action type:

$O\varphi$: "φ is the (only) action type that is obligatory", or more elaborately: "an action token is optimal if and only if it is of type φ"
$P\varphi$: "if an action is of type φ, then it is optimal"

Anglberger et al. moreover introduce an alethic modality $\Box$, which they interpret as a universal modality. $\Box\varphi$ thus means that all available action tokens are of type φ.

They then propose what they call a "minimal logic" **5HD** for these three operators. However, as they argue, **5HD** only captures one half of the notion of "obligation as weakest permission". That is, if φ is obligatory, then the logic stipulates that φ is the weakest permitted action type. The converse does not hold: something can be the weakest permitted action type without being obligatory.

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[1] In more recent work Dong and Roy (2015); Van De Putte (2015), the logic is compared to other constructions in deontic logic.

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In the fourth section of Anglberger *et al.* (2015), a brief discussion of this converse direction is given, and it is shown how this translates to the semantics of **5HD**. Let us call the resulting logic **5HD***; it will be defined in Section 2. It is argued in Anglberger *et al.* (2015) that **5HD*** is not compact, and a proof system with an infinitary rule (R-Conv) is shown to be (weakly) sound and complete w.r.t. **5HD***.

The main aim of the present paper is to give a sound and strongly complete, Hilbert-style axiomatization for **5HD*** (Section 3). As a corollary, this consequence relation *is* compact, contradicting the claims mentioned in the previous paragraph. We prove in addition that the proof system proposed by Anglberger et al. is equivalent to **5HD*** (Section 4). Finally, we show that these results can be generalized to other, similar logics for "obligation as weakest permission" (Section 5).

§2. Definitions.

This section is meant to fix notation; it contains no new material. See Anglberger *et al.* (2015) for the original definitions and notation.

We work with a modal propositional language, obtained by closing the set of propositional letters $\mathcal{S} = \{p_1, p_2, \ldots\}$ and $\bot, \top$ under boolean connectives $\neg, \vee, \wedge, \supset, \equiv$ and the unary operators $\Box, O, P$. Call the resulting set of formulas $\mathcal{W}$. We treat only $\neg, \vee, \bot, O, P, \Box$ as primitive; $\wedge, \supset, \equiv$ are defined in the usual way. In the remainder, let the metavariables $\varphi, \psi, \ldots$ range over arbitrary members of $\mathcal{W}$ and $\Gamma, \Delta, \ldots$ over arbitrary subsets of $\mathcal{W}$.

DEFINITION 2.1. *A* strict deontic frame *F is a quadruple $\langle W, R_\Box, n_P, n_O\rangle$, where W is a non-empty set (the* domain *of F), $R_\Box = W \times W$, and $n_P : W \to \wp(\wp(W))$ and $n_O : W \to \wp(\wp(W))$ satisfy the following conditions*

(OR)	*If $X \cup Y \in n_P(w)$, then $X \in n_P(w)$ and $Y \in n_P(w)$*
(WP)	*If $X \in n_O(w)$ and $Y \in n_P(w)$, then $Y \subseteq X$*
(OP)	*If $X \in n_O(w)$ then $X \in n_P(w)$*
(OC)	*If $X \in n_O(w)$, then $X \neq \emptyset$*
(Conv)	*If $X \in n_P(w)$ and for all $Y \in n_P(w)$, $Y \subseteq X$, then $X \in n_O(w)$*

If a frame obeys all the above conditions except (possibly) (Conv), it is just a deontic frame.

A (strict) deontic model *is a (strict) deontic frame F together with a valuation v that maps every propositional atom to a subset of the domain of F.*

DEFINITION 2.2. *Let $M = \langle W, R_\Box, n_O, n_P, v\rangle$ be a (strict) deontic model and $w \in W$.*

$M, w \not\models \bot$
$M, w \models p$ *iff* $w \in v(p)$
$M, w \models \neg\varphi$ *iff* $M, w \not\models \varphi$
$M, w \models \varphi \vee \psi$ *iff* $M, w \models \varphi$ *or* $M, w \models \psi$
$M, w \models \Box\varphi$ *iff* $M, w' \models \varphi$ *for all* $w' \in R_\Box(w)$
$M, w \models O\varphi$ *iff* $\|\varphi\|^M \in n_O(w)$
$M, w \models P\varphi$ *iff* $\|\varphi\|^M \in n_P(w)$,

where $\|\varphi\|^M = \{u \in W \mid M, u \models \varphi\}$.

DEFINITION 2.3. $\Gamma \Vdash_{\mathbf{5HD}^*} \varphi$ *iff for all strict deontic models M: if $M, w \models \psi$ for all $\psi \in \Gamma$, then $M, w \models \varphi$.*

§3. Axiomatization of 5HD*.

DEFINITION 3.1. *The set of* **5HD***-*theorems is the closure of the set of all instances of the following axiom schemas*

(CL)	*All tautologies of classical propositional logic*
(S5$_\Box$)	**S5** *for* $\Box$
(EQ$_O$)	$\Box(\varphi \equiv \psi) \supset (O\varphi \equiv O\psi)$
(EQ$_P$)	$\Box(\varphi \equiv \psi) \supset (P\varphi \equiv P\psi)$
(FCP)	$P(\psi \vee \varphi) \supset (P\psi \wedge P\varphi)$
(Ought-Perm)	$O\varphi \supset P\varphi$
(Ought-Can)	$O\varphi \supset \Diamond\varphi$
(Weakest-Perm)	$O\varphi \supset (P\psi \supset \Box(\psi \supset \varphi))$
(Taut-Perm)	$P\top \supset O\top$

under the following rules:

$$\text{(MP)}\ \frac{\varphi, \varphi \supset \psi}{\psi} \qquad\qquad \text{(NEC)}\ \frac{\vdash \varphi}{\vdash \Box\varphi}$$

$\Gamma \vdash_{\mathbf{5HD}^*} \varphi$ *iff there are* $\psi_1, \ldots, \psi_n \in \Gamma$ *such that* $(\psi_1 \wedge \ldots \wedge \psi_n) \supset \varphi$ *is a* **5HD***-*theorem.*

This axiomatization is obtained by adding the axiom (Taut-Perm) to the axiomatization of the logic **5HD** from (Anglberger *et al.*, 2015, Section 3). In the remainder of this section, we establish the following:

THEOREM 3.2. $\Gamma \vdash_{\mathbf{5HD}^*} \varphi$ *iff* $\Gamma \Vdash_{\mathbf{5HD}^*} \varphi$.

Before we prove this theorem, let us note one property of $\vdash_{\mathbf{5HD}^*}$:

LEMMA 3.3. $P\varphi, \Box(\psi \supset \varphi) \vdash_{\mathbf{5HD}^*} P\psi$.

Proof. Suppose $P\varphi, \Box(\psi \supset \varphi)$. Since $\Box$ is a normal modal operator and by the second premise, we can infer $\Box(\varphi \equiv (\psi \vee \varphi))$. Hence by (EQ$_P$) and the first premise, $P(\psi \vee \varphi)$. But then by (FCP) and classical logic, $P\psi$. $\Box$

Soundness. For soundness, we refer to Section 3.3 of Anglberger *et al.* (2015), where the soundness of all the axioms except (Taut-Perm) is shown with respect to the set of all deontic models. So we are left with checking that (Taut-Perm) is valid in view of the additional condition (Conv). Suppose that $M, w \models P\top$. It follows that $\|\top\|^M \in n_P(w)$. Hence, $W \in n_P(w)$. Clearly, for all $X \in n_P(w)$, $X \subseteq W$, and hence by condition (Conv), $W \in n_O(w)$ so that $M, w \models O\top$.

Completeness, part 1. For (strong) completeness, we need a more elaborate proof. The main complication in the proof consists in applying a "copy-and-merge" technique to the completeness proof from Section 3.3 of Anglberger *et al.* (2015). This technique was originally developed in the 1980s by Passy, Tinchev, and Gargov for the completeness proof of modal logics for necessity and sufficiency; see e.g., Gargov and Passy (1990); Passy and Tinchev (1991).[2] There are very close links between **5HD*** and the notions

[2] The authors of Gargov and Passy (1990) refer to Vakarelov as the inventor of this technique.

of modal necessity and sufficiency—a discussion of this relationship can be found in Van De Putte (2015).

Recall that to prove strong completeness, it suffices to establish that for all consistent $\Lambda \subseteq \mathcal{W}$, there is a model M and a state w in this model such that all the members of Λ are true at this state. So let in the remainder Λ be an arbitrary consistent subset of $\mathcal{W}$, and let Λ' be a maximally **5HD***-consistent extension of Λ.[3] Let W be the set of all maximally **5HD***-consistent sets $\Delta \subseteq \mathcal{W}$ such that $\{\varphi \mid \Box\varphi \in \Lambda'\} \subseteq \Delta$. Let $|\varphi| = \{\Delta \in W \mid \varphi \in \Delta\}$.

Let $M_\Lambda = \langle W, R_\Box, n_O, n_P, V\rangle$, where[4]

(1) $R_\Box = W \times W$
(2) for all $\Delta \in W$, $n_O(\Delta) = \{|\psi| \mid O\psi \in \Delta\}$
(3) for all $\Delta \in W$, $n_P(\Delta) = \downarrow\{|\psi| \mid P\psi \in \Delta\}$
(4) for all $\varphi \in \mathcal{S}$, $V(\varphi) = |\varphi|$,

where for any set of sets $\mathcal{X}$, $\downarrow\mathcal{X}$ is the set of all subsets of the members of $\mathcal{X}$ (also called the downset of X).

We now prove a number of lemmas about M_Λ – (variants of) these can be found in the completeness proof for **5HD** from Anglberger *et al.* (2015). Since the present model is defined in terms of **5HD***, we need to prove them here from scratch.

LEMMA 3.4 (Anglberger *et al.* (2015), Lemma 3.12). *$|\varphi| \subseteq |\psi|$ iff $\Box(\varphi \supset \psi) \in \Delta$ for all $\Delta \in W$.*

Proof. ($\Rightarrow$) Suppose the antecedent holds. Hence, every maximal consistent extension of $\{\tau \mid \Box\tau \in \Lambda'\}$ that contains φ, also contains ψ. By a standard proof (relying on **K**-properties of $\Box$) we can infer that $\Lambda' \vdash_{\mathbf{5HD}^*} \Box(\varphi \supset \psi)$. By the (4)-axiom for $\Box$, $\Box\Box(\varphi \supset \psi) \in \Lambda'$. And hence by the definition of W, $\Box(\varphi \supset \psi) \in \Delta$ for all $\Delta \in W$.

($\Leftarrow$) Suppose the consequent holds. By the T-axiom for $\Box$, $\varphi \supset \psi \in \Delta$ for all $\Delta \in W$. Hence, for all $\Delta \in W$ such that $\varphi \in \Delta$, also $\psi \in \Delta$. It follows that $|\varphi| \subseteq |\psi|$. □

LEMMA 3.5. *For all $\varphi \in \mathcal{W}$, $|\varphi| = \|\varphi\|^{M_\Lambda}$.*

Proof. By a standard induction on the complexity of φ. The inductive base is trivial in view of (4). For the inductive step, the case where $\varphi = \Box\tau$ is standard. So we are left with two cases:

CASE1: $\varphi = O\tau$. ($\Rightarrow$) Suppose that $O\tau \in \Delta$. Hence by (2), $|\tau| \in n_O(\Delta)$ and hence by the induction hypothesis (IH) and the semantic clause for O $M_\Lambda, \Delta \models O\tau$. ($\Leftarrow$) Suppose that $M_\Lambda, \Delta \models O\tau$. Hence by (IH), $|\tau| \in n_O(\Delta)$. By (2), there is a τ' such that $O\tau' \in \Delta$ and $|\tau'| = |\tau|$. By Lemma 3.4, $\Box(\tau \equiv \tau') \in \Delta$. But then by (EQ$_O$), $O\tau \in \Delta$.

CASE 2: $\varphi = P\tau$. ($\Rightarrow$) Suppose that $P\tau \in \Delta$. Hence by (3), $|\tau| \in n_P(\Delta)$ and hence by (IH) and the semantic clause for P, $M_\Lambda, \Delta \models P\tau$. ($\Leftarrow$) Suppose that $M_\Lambda, \Delta \models P\tau$. By the semantic clause for P and (IH), $|\tau| \in n_P(\Delta)$. By (3), there is a τ' such that $P\tau' \in \Delta$ and $|\tau| \subseteq |\tau'|$. By Lemma 3.4, $\Box(\tau \supset \tau') \in \Delta$. By Lemma 3.3, $P\tau \in \Delta$. □

LEMMA 3.6 (Anglberger *et al.* (2015), Claim 3.15). *M_Λ is a deontic model.*

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3 Here and below, we freely rely on Lindenbaum's lemma: every consistent $\Delta \subseteq \mathcal{W}$ has a maximally **5HD***-consistent extension $\Delta' \subseteq \mathcal{W}$.

4 Our definition of M_Λ is essentially the same as in (Anglberger *et al.*, 2015, Section 3.3), the only difference being that here we work with **5HD*** rather than **5HD**.

Proof. We need to check 4 conditions:
(OR) Immediate in view of the construction, item (3).
(WP) Suppose that $X \in n_O(\Delta)$ and $Y \in n_P(\Delta)$. By items (2) and (3) of the construction, there are φ, ψ such that $X = |\varphi|$ and $O\varphi \in \Delta$, and $Y \subseteq |\psi|$ and $P\psi \in \Delta$. By (Weakest-Perm), $\Box(\psi \supset \varphi) \in \Delta$. Hence by Lemma 3.4, $|\psi| \subseteq |\varphi|$. It follows that $Y \subseteq X$.
(OP) Suppose that $X \in n_O(\Delta)$. By item (2) of the construction, there is a φ such that $O\varphi \in \Delta$ and $X = |\varphi|$. By (Ought-Perm), $P\varphi \in \Delta$. Hence, by item (3) of the construction, $X \in n_P(\Delta)$.
(OC) Suppose that $X \in n_O(\Delta)$. By item (2) of the construction, there is a φ such that $O\varphi \in \Delta$ and $X = |\varphi|$. By (Ought-Can), $\Diamond\varphi \in \Delta$. Hence by Lemma 3.5, $M, \Delta \models \Diamond\varphi$. So there is a $\Theta \in W$ such that $M, \Theta \models \varphi$. Again by Lemma 3.5, $\varphi \in \Theta$ and hence $X = |\varphi| \neq \emptyset$. □

However, M_Λ will not (in general) be a *strict* deontic frame – in other words, (Conv) may not hold for M_Λ. To get this condition, we transform M_Λ into a more complex model M_Λ^+. Informally speaking, M_Λ^+ is obtained by first making two disjoint copies of M_Λ, and then merging the two resulting models. The merging is done in such a way that the truth lemma is preserved, and yet condition (Conv) is obeyed. We return to this point after giving the exact definition. For the sake of readibility, we will denote the copies of the members $\Delta \in W$ by Δ^1, Δ^2 rather than $\langle \Delta, 1\rangle, \langle \Delta, 2\rangle$.

Let $M_\Lambda^+ =_{\mathsf{df}} \langle W^+, R_\Box^+, n_O^+, n_P^+, v^+\rangle$, where

- (i) $W^+ = \{\Delta^1, \Delta^2 \mid \Delta \in W\}$
- (ii) $R_\Box^+ = W^+ \times W^+$
- (iii) for all $\Delta^i \in W^+$:
  - (iii.1) if there is a φ s.t. $O\varphi \in \Delta$, then $n_O^+(\Delta^i) = \{\{\Theta^1, \Theta^2 \mid \Theta \in X\} \mid X \in n_O(\Delta)\}$
  - (iii.2) otherwise,
    - (iii.2a) if there is no $X \in n_P(\Delta)$ such that $Y \subseteq X$ for all $Y \in n_P(\Delta)$, let $n_O^+(\Delta^i) = \emptyset$
    - (iii.2b) otherwise, let $X_w \in n_P(\Delta)$ be such that $Y \subseteq X_w$ for all $Y \in n_P(\Delta)$. Define $n_O^+(\Delta^i) = \{\{\Theta^i \mid \Theta \in X_w\} \cup \{\Lambda^j \in W^+ \mid j \neq i\}\}$
- (iv) for all $\Delta^i \in W^+$:
  - (iv.1) if $n_O^+(\Delta^i) = \emptyset$, let $n_P^+(\Delta^i) = \downarrow\{\{\Theta^1, \Theta^2 \mid \Theta \in X\} \mid X \in n_P(\Delta)\}$
  - (iv.2) otherwise, let $n_P^+(\Delta^i) = \downarrow n_O^+(\Delta^i)$.

Let in the remainder $|\varphi|^+ = \{\Delta^1, \Delta^2 \mid \varphi \in \Delta\} = \{\Delta^1, \Delta^2 \mid \Delta \in |\varphi|\}$. As usual, $\|\varphi\|^{M_\Lambda^+} = \{\Delta^i \in W^+ \mid M_\Lambda^+ \models \varphi\}$.

Intermezzo. Cases (iii.2b) and (iv.2) are the interesting ones. We need these to ensure that the additional condition (Conv) is satisfied but that nevertheless, the truth lemma is preserved. That is, consider the following (**5HD***-consistent!) set of formulas:

$$\Gamma_{\mathsf{ex}} = \{P\varphi \supset \Box(\varphi \supset p_1) \mid \varphi \in \mathcal{W}\} \cup \{Pp_1\} \cup \{\neg Op_1\}$$

This set is satisfiable in a strict deontic model. The reason is that "the weakest permission" can mean two different things: it can refer to an object-level formula, but it can also refer to a semantic entity, viz. a set of states in our model. It may well be that in our model, the "weakest permission" $X \subseteq W$ is such that it cannot be expressed at the object-level.

Now if (iii.2b) applies, then this means that under the object-level interpretation, our weakest permission can be expressed by some formula ψ, even though $O\psi$ is not a member of the set Δ. To make sure that $O\psi$ is false in the model at Δ^i, we add (at least) one weaker permission to $n_P(\Delta^i)$, which is not expressible at the object level. That it is not expressible at the object level (and more generally, that no additional formulas of the form $P\tau$ become valid), follows from the Truth Lemma and Lemma 3.9 below. The main point is that in this symmetric construction, only sets of the type $\{|\varphi|^1, |\varphi|^2 \mid \varphi \in \mathcal{W}\}$ are expressible in the object language.

Completeness, part 2. We now prove the main lemmas that allow us to obtain strong completeness for **5HD***.

LEMMA 3.7. *For all $\Delta^i \in W^+$: if $n_O^+(\Delta^i) \neq \emptyset$, then $n_O^+(\Delta^i)$ is a singleton set.*

Proof. If (iii.2b) applies, then this is immediate in view of the construction. If (iii.1) applies, then it suffices to check that $n_O(\Delta)$ is a singleton. This follows from the fact that $(O\varphi \wedge O\psi) \supset \Box(\varphi \equiv \psi)$ is a theorem in **5HD*** (see Observation 3.5 from Anglberger *et al.* (2015)), and Lemma 3.4. □

LEMMA 3.8. *M_Λ^+ is a strict deontic model.*

Proof. It suffices to check that all conditions of Definition 2.1 are satisfied:
Ad (OR). Trivial in view of the construction, item (iv.1) and (iv.2).
Ad (OP). Trivial in view of item (iv) of the construction.
Ad (OC). Let $X^+ \in n_O^+(\Delta^i)$. If (iii.1) applies, then $X^+ = \{\Theta^1, \Theta^2 \mid \Theta \in X\}$ where $X \in n_O(\Delta)$. By Lemma 3.6, $X \neq \emptyset$ and hence also $X^+ \neq \emptyset$.

If (iii.2b) applies, then by the construction, every set $\Theta^j \in W^+$ with $j \neq i$ is a member of X^+, and hence $X^+ \neq \emptyset$.
Ad (WP). Trivial in view of Lemma 3.7 and item (iv) of the construction.
Ad (Conv). Suppose that $X \in n_P^+(\Delta^i)$ and for all $Y \in n_P^+(\Delta^i)$, $Y \subseteq X$.

CASE 1: $n_O^+(\Delta^i) \neq \emptyset$. Then in view of the construction, X is the only member of $n_O^+(\Delta^i)$ and we are done.

CASE 2: $n_O^+(\Delta^i) = \emptyset$. Note that by the construction, and since X is a maximal member of $n_P^+(\Delta^i)$, $X = \{\Theta^1, \Theta^2 \mid \Theta \in X'\}$, with $X' \in n_P(\Delta)$. Let $Y' \in n_P(\Delta)$ be arbitrary and let $Y = \{\Psi^1, \Psi^2 \mid \Psi \in Y'\}$. By item (iv.1) of the construction, $Y \in n_P^+(\Delta^i)$. By the supposition, $Y \subseteq X$. Hence, $Y' \subseteq X'$. So we have shown that for all $Y' \in n_P(\Delta)$, $Y' \subseteq X'$. But this means that the condition of (iii.2a) is false, and hence $n_O^+(\Delta^i) \neq \emptyset$ — a contradiction. So we have shown that case 2 cannot apply given our supposition. □

LEMMA 3.9. *Let $X, Y \subseteq W$ and $X \neq Y$. Let $Z = \{\Theta^1, \Psi^2 \mid \Theta \in X, \Psi \in Y\}$. Then there is no φ such that $Z = |\varphi|^+$.*

Proof. Immediate in view of the construction and the definition of $|\varphi|^+$. □

LEMMA 3.10. *Where $\psi \in \mathcal{W}$ and $i \in \{1, 2\}$: $M_\Lambda^+, \Delta^i \models \psi$ iff $\psi \in \Delta$.*

Proof. By a standard induction on the complexity of ψ, henceforth denoted by $\mathsf{c}(\psi)$. Note that our inductive hypothesis is equivalent to

(IH) for all $\psi \in \mathcal{W}$ with $\mathsf{c}(\psi) \leq n$, $\|\psi\|^{M_\Lambda^+} = |\psi|^+$.

That is, the truth set of ψ in M_Λ^+ is simply the set of all points Θ^1, Θ^2 where $\psi \in \Theta$.

The base case ($\mathsf{c}(\psi) = 0$, hence $\psi \in \mathcal{S}$) is trivial. Proving the inductive step for the connectives is a routine task, we safely leave this to the reader. So we are left with three cases:

CASE 1: $\psi = \Box\varphi$ with $\mathsf{c}(\varphi) \leq n$. We have: $M^+, \Delta^i \models \Box\varphi$ iff for all $\Theta^j \in W^+$, $M^+, \Theta^j \models \varphi$ iff [by the construction of W^+ and (IH)] for all $\Theta \in W$, $\varphi \in \Theta$ iff [by the construction of W] $\Box\varphi \in \Delta$.

CASE 2: $\psi = O\varphi$ with $\mathsf{c}(\varphi) \leq n$. ($\Rightarrow$) Suppose that $M^+_\Lambda, \Delta^i \models O\varphi$. Hence, there is an $X^+ \in n^+_O(\Delta^i)$ such that $X^+ = \|\varphi\|^{M^+_\Lambda}$. By (IH), $X^+ = |\varphi|^+$. We now distinguish two cases:
(iii.1) applies. Hence, $X^+ = \{\Theta^1, \Theta^2 \mid \Theta \in X\}$, where X is the only member of $n_O(\Delta)$. By the construction of M^Λ, $X = |\psi|$ for some ψ such that $O\psi \in \Delta$. Hence, $|\psi| = |\varphi|$. By Lemmas 3.5 and 3.4, $\Box(\varphi \equiv \psi) \in \Delta$. Hence by (EQ$_O$), $O\varphi \in \Delta$.
(iii.2) and (iii.2b) apply. Assume, first, that $X_w = W = \|\top\|^{M_\Lambda}$. It follows that $P\top \in \Delta$. Hence, by (Taut-Perm), $O\top \in \Delta$, which contradicts condition (iii.2). So $X_w \subset W$. But then, where Z is the only member of $n^+_O(\Delta_i)$, we can infer by Lemma 3.9 that there is no τ such that $Z = |\tau|^+$. Hence, by (IH), there is no τ such that $Z = \|\tau\|^{M^+_\Lambda}$. But then there is no τ such that $M^+_\Lambda, \Delta^i \models O\tau$, contradicting our original supposition. So we have shown that given the supposition, (iii.2b) cannot apply.

($\Leftarrow$) Suppose that $O\varphi \in \Delta$. Hence condition (iii.1) applies, and hence the only member of $n^+_O(\Delta^i)$ is $X^+ = \{\Theta^1, \Theta^2 \mid \Theta \in X\}$, where $X = |\varphi|$. By (IH), $X^+ = |\varphi|^+ = \|\varphi\|^{M^+_\Lambda}$ and hence $M^+_\Lambda, \Delta^i \models O\varphi$.

CASE 3: $\psi = P\varphi$ with $\mathsf{c}(\varphi) \leq n$. ($\Rightarrow$) Suppose that $M^+_\Lambda, \Delta^i \models P\varphi$. Hence, there is an $X^+ \in n^+_P(\Delta^i)$ such that $X^+ = \|\varphi\|^{M^+_\Lambda}$. By (IH), $X^+ = |\varphi|^+ = \{\Theta^1, \Theta^2 \mid \varphi \in \Theta\}$. We will prove that there is a τ such that $P\tau \in \Delta$ and $\Box(\varphi \supset \tau) \in \Delta$; applying Lemma 3.3 we obtain that $P\varphi \in \Delta$. To get there, we distinguish three cases:
(iii.1) applies. Hence, $X^+ \subseteq Y^+ = \{\Theta^1, \Theta^2 \mid \Theta \in Y\}$, where Y is the only member of $n_O(\Delta)$. Hence, $Y = |\tau|$ and $O\tau \in \Delta$. By (Ought-Perm), $P\tau \in \Delta$. Since $X^+ \subseteq Y^+$, also $X \subseteq Y$ and hence $|\varphi| \subseteq |\tau|$. By Lemma 3.4, $\Box(\varphi \supset \tau) \in \Delta$.
(iii.2a) applies. Hence, by item (iv.2) of the construction, $X^+ \subseteq Y^+ = \{\Theta^1, \Theta^2 \mid \Theta \in Y\}$, where $Y \in n_P(\Delta)$. By the construction of M_Λ, there is a τ such that $P\tau \in \Delta$ and $Y \subseteq |\tau|$. Hence, $|\varphi| \subseteq |\tau|$. By Lemma 3.4, $\Box(\varphi \supset \tau) \in \Delta$.
(iii.2b) applies. By item (iv.1) of the construction, X^+ does not contain any set Θ^i with $\Theta \notin X_w$. Since $X^+ = |\varphi|^+$ and by Lemma 3.9, it follows that $X^+ \subseteq \{\Theta^1, \Theta^2 \mid \Theta \in X_w\}$. Note that there is a τ such that $|\tau| = X_w$ and $P\tau \in \Delta$. So we can again apply the same reasoning to show that $\Box(\varphi \supset \tau) \in \Delta$.

($\Leftarrow$) Suppose that $P\varphi \in \Delta$. By (IH), it suffices that we prove that $|\varphi|^+ \in n^+_P(\Delta^i)$. We distinguish again three cases:
(iii.1) applies. Let $O\tau \in \Delta$. Hence by (Weakest-Perm), $\Box(\varphi \supset \tau) \in \Delta$. By Lemmas 3.5 and 3.4 it follows that $|\varphi| \subseteq |\tau|$. Hence, since $|\tau|^+ \in n^+_O(\Delta^i)$, and by item (iv.2) of the construction, also $|\varphi|^+ \in n^+_P(\Delta^i)$.
(iii.2a) applies. By the construction of M_Λ, $|\varphi| \in n_P(\Delta)$. By item (iv.1) of our construction, $|\varphi|^+ \in n^+_P(\Delta^i)$.
(iii.2b) applies. Hence, $|\varphi| \subseteq X_w$. By items (iii.2b) and (iv.2) of the construction, $X^+_w = \{\Theta^1, \Theta^2 \mid \Theta \in X_w\}$ is a member of $n^+_P(\Delta^i)$. Hence by the same construction, also $|\varphi|^+ \in n^+_P(\Delta^i)$. $\Box$

§4. The proof system from Anglberger et al. (2015). As mentioned in the introduction, Anglberger et al. present a nonstandard, infinitary proof system for **5HD***, which they show to be sound and weakly complete w.r.t. the semantics. This proof system is obtained by adding to the rules and axioms of **5HD** the following infinitary rule (here, $\{p_1, p_2, \ldots\}$ is a complete enumeration of the members of $\mathcal{S}$):

$$\text{(R-Conv)}\ \frac{\vdash Pp_1 \supset \Box(p_1 \supset \varphi), \vdash Pp_2 \supset \Box(p_2 \supset \varphi), \ldots}{\vdash P\varphi \supset O\varphi}$$

The logic obtained by adding (R-Conv) to **5HD** is called **5HD**$^+$ in Anglberger *et al.* (2015). We will now show that it is equivalent to **5HD***, and hence also strongly complete w.r.t. the **5HD***-semantics.

To prove that **5HD**$^+$ is at least as strong as **5HD***, it suffices to show that adding (R-Conv) to **5HD** yields (Taut-Perm). This is fairly straightforward: putting $\varphi = \top$, the premise of (R-Conv) holds trivially, and its conclusion is simply (Taut-Perm).[5]

To prove that **5HD*** is as strong as **5HD**$^+$, we first show that (R-Conv) is sound with respect to the **5HD***-semantics.[6] We prove this by contraposition. Suppose that for a given φ, the conclusion of (R-Conv) is not valid. Hence, there is a **5HD***-model $M = \langle W, R_\Box, n_O, n_P, v\rangle$ and a point $w \in W$ such that $M, w \models P\varphi$, $M, w \not\models O\varphi$. Let $p_i \in \mathcal{S}$ be such that it does not occur in φ. Define $v' : \mathcal{S} \to \wp(W)$ such that $v'(\tau) = v(\tau)$ for all $\tau \in \mathcal{S}\setminus\{p_i\}$ and $v'(p_i) = n_O(w)$. Let $M' = \langle W, R_\Box, n_O, n_P, v'\rangle$. It follows that $M', w \models Op_i$ and hence also $M', w \models Pp_i$. Since $\|\varphi\|^{M'} = \|\varphi\|^M$, $M', w \not\models O\varphi$ and $M', w \models P\varphi$. But then by (WP) and the semantic clause for O, $\|\varphi\|^{M'} \subset \|p_i\|^{M'}$ and hence $M', w \not\models \Box(p_i \supset \varphi)$. So we have shown that $\not\Vdash_{\mathbf{5HD}^*} Pp_i \supset \Box(p_i \supset \varphi)$.

Relying on our completeness result from the preceding section (Theorem 3.2), this means that (R-Conv) is also sound with respect to our axiomatization of **5HD***. This finishes our proof of the identity of $\vdash_{\mathbf{5HD}^*}$ and $\vdash_{\mathbf{5HD}^+}$.

§5. Generalizations and an open issue.

Generalizations of the result. Our completeness result can be easily generalized to weaker logics that are obtained by skipping some of the frame conditions such as (OR) and (OC), and leaving out the associated axioms. If we leave out (OC), no changes need to be made to the construction of M_Λ or M_Λ^+. If we leave out (OR), the construction of n_P and n_P^+ just needs to be simplified, so that they are no longer closed under subsets.

Likewise, the results can be generalized to the logic obtained by adding the following frame condition from Roy *et al.* (2012):

(UC) If $X \in n_P(w)$ and $Y \in n_P(w)$, then $X \cup Y \in n_P(w)$

and the associated axiom schema

(Union-Closure) $(P\varphi \wedge P\psi) \supset P(\varphi \vee \psi)$

---

5 Interestingly, there are also true instances of the premise of (R-Conv) in which φ is not equivalent to $\top$. For instance, $\Vdash_{\mathbf{5HD}^*} P(\Box Op \supset p) \supset O(\Box Op \supset p)$, alhough $\not\Vdash_{\mathbf{5HD}^*} \Box Op \supset p$. We are indebted to one of the referees for pointing this out, and thereby correcting an earlier mistake in the paper.

6 A similar proof is given in (Anglberger *et al.*, 2015, p. 15); we include ours for the sake of self-containedness. In principle, there should also be a direct syntactic proof of the derivability of (R-Conv) within **5HD***, but we were not able to find one.

Soundness for this extension is routine. For completeness of the logic with both (OR) and (UC), the only difficult case is the one where $n_O^+(w) = \emptyset$. If this is so, note that if $X \in n_P^+(\Delta^i)$ and $Y \in n_P^+(\Delta^i)$, then in view of item (iv.1) of the construction, there are propositions φ and ψ such that $P\varphi, P\psi \in \Delta$ and $X \subseteq |\varphi|, Y \subseteq |\psi|$. It follows that $X \cup Y \subseteq |\varphi| \cup |\psi|$. By (Union Closure), $P(\varphi \vee \psi) \in \Delta$ and hence $|\varphi| \cup |\psi| \in n_P(\Delta)$. Since $n_P(\Delta)$ is closed under subsets, $X \cup Y \in n_P(\Delta)$.

Additional frame conditions can be thought of. For instance, one may add the condition that every impossible action is permitted:

(IP) $\emptyset \in n_P(w)$

This condition is axiomatized by the axiom $P\bot$. Interestingly, if we add both (UC) and (IP) to the semantics of **5HD***, it can be rephrased in a much simpler fashion: all we need to do is pin down a set of "permitted" states $R(w)$. $P\varphi$ is then true at w in M iff $\|\varphi\|^M \subseteq R(w)$, and $O\varphi$ is true at w iff $\|\varphi\|^M = R(w)$. See also Van De Putte (2015) where this link is studied in more detail.

Likewise, the results generalize to the case where $\Box$ is a weaker modality. Of course, this requires a re-formulation of some of the semantic clauses. Their general form becomes this:

(WP') If $X \in n_O(w)$ and $Y \in n_P(w)$, then $Y \cap R_\Box(w) \subseteq X \cap R_\Box(w)$
(OP') If $X \in n_O(w)$ then $X \in n_P(w)$
(OC') If $X \in n_O(w)$, then $X \cap R_\Box(w) \neq \emptyset$
(Conv') If $X \in n_P(w)$ and for all $Y \in n_P(w)$, $Y \cap R_\Box(w) \subseteq X \cap R_\Box(w)$, then $X \in n_O(w)$

Is 5HD* what we are after? In view of the completeness result, one may ask whether the semantic consequence relation for **5HD*** was the intended logic of "obligation as weakest permission", or whether the authors of Anglberger *et al.* (2015) want a stronger consequence relation instead. This can be explained again in terms of the example Γ_{ex} (see page 374): perhaps they want this premise set to be trivial after all, even if none of its finite subsets is trivial.

Theorem 3.2 implies that such a stronger consequence relation can only be obtained if we impose additional conditions on our models. Let us suggest two such conditions, leaving their full study for a later occasion. Where w is an arbitrary point in an **5HD***-model M, let $\|w\|^M = \{\varphi \in \mathcal{W} \mid M, w \models \varphi\}$. The conditions are:

(C1) Where $X \in n_O(w)$: if $\|v\|^M = \|v'\|^M$, then $v \in X$ iff $v' \in X$
(C2) Where $X \in n_O(w)$: if $\|v\|^M \cap \mathcal{S} = \|v'\|^M \cap \mathcal{S}$, then $v \in X$ iff $v' \in X$

(C2) is clearly a (strictly) stronger condition than (C1). Arguably, neither of these can be characterized by a finitary rule, since the semantic consequence relation they yield is not compact. We conjecture that the following infinitary rules suffice to obtain a sound and (strongly) complete axiomatization:

(R1) $\{P\varphi \supset \Box(\psi \supset \varphi) \mid \varphi \in \mathcal{W}\} \vdash P\psi \supset O\psi$
(R2) $\{P\varphi \supset \Box(\psi \supset \varphi) \mid \varphi \in \mathcal{S}\} \vdash P\psi \supset O\psi$.

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