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Modelling Non-Commuting Decision Situations: A Quantumtechnical Approach.

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It's a strange world.
Let's keep it that way.

WARREN ELLIS

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Acknowledgments

There are too many people in the last six years that helped me complete this PhD in some way, so the following will always be incomplete.

First, I would like to thank all the artists and thinkers that, although we never met, inspired me along the way. They provided the soundtrack, literature[1] and science that lights the way. Standing on the shoulders of giants indeed.

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[1] My abundance of footnotes is a thinly veiled ode to the late Terry Pratchett

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Even though not all family members present at the start six years ago are present physically at the end, they are in spirit. Even last words were those of encouragement. This thesis is dedicated to them.

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I would like to end with the most profound conclusion these six years of research have resulted in, which is an affirmation of a decade old axiom in swing dancing. There can never be a Jack without a Jill.

Jacob,
Januari 2017

1 Introduction

1.1 Quantum & Cognition?

'Quantum' and 'cognition[1]' are two words one would not expect in collocation, and yet, their concatenation is the topic of this doctoral dissertation. While both terms belong to fields (subatomic physics and social sciences) that could hardly be further apart, we will show that the latter can learn a lot from the former[2]. In quantum cognition we will use the mathematical framework from quantum mechanics minus the physics. In the beginning of the twentieth century physicists were confronted with experimental data that seemed paradoxical in classical physics. As a result, a new mathematical framework, with new notions for concepts such as 'measurement' and 'state of the system', was constructed. Likewise, experimental situations in cognitive science that seem paradoxical from a traditional rational point of view occur frequently. The most common expression of such a paradoxical situation is a violation of a classical probabilistic rule. Quantum cognition uses the mathematical tools constructed for the paradoxes in physics to model the paradoxes in cognition. While the connection between the two fields seems surprising at first, this idea dates back to the origins of quantum mechanics. Bohr himself drew inspiration from psychological phenomena, such as ambiguous images. To quote directly from Bohr (1948):

> [...] in psychology, where the conditions for analysis and synthesis of experience exhibit striking analogy with the situation in atomic physics.

We believe, agreeing with Bohr, that quantum theory does not only have the mathematical means of exhibiting statistical paradoxes of interest, but can help explain this perceived 'irrational' behavior. While there are multiple examples of classical models exhibiting paradoxes, the difference is that in quantum theory they are not considered paradoxes anymore, but are now naturally implied behavior. At the core of the quantum-like approach lies the possibility of being in an *indeterminate* state. Classical cognitive models assume us to be in a definite state concerning decision situations. The stochasticity enters the model because this definite state is unknown. Performing the relevant measurement simply records this unknown state. In quantum theory the state can be inde-

[1] By cognition we mean the wide field of cognitive science, examining the human mind and behavior.

[2] The question if modern quantum mechanics can learn something from cognitive science, should be asked to a quantum scientist.

terminate, exhibiting the potential of certain states to be realized. This idea intuitively seems to reflect psychological reality, with somebody being conflicted, confused or uncertain, better than physical reality. This concept of an indeterminate state has profound consequences. It means that performing a measurement constructs rather than records an outcome. It makes the system realize the potential of one of its states. As pointed out in Busemeyer & Bruza (2012), this is in line with modern psychological theories of emotion and decision making. For example, in Schachter & Singer (1962) it is shown that a person can be ambiguous about his feelings, but after being asked about them, the ambiguity is lost and the state of emotion becomes definite. Likewise, Payne et al. (1992) show that beliefs and preferences are not necessarily directly recorded, but are constructed opportunistically. This points to (measurement of) human experimental behavior being *contextual*, depending on the specific experimental setup and measurement performed, rather than absolute. The same concept of contextuality lies at the heart of "the magic of quantum computation" (Howard et al., 2014). So while the remainder of this thesis might focus more on mathematical adequacy of quantum models in social sciences, we wish to express the belief that the connection between the two goes deeper than well-fitting numbers and formulas. This is reflected in the fact that the observed classical paradoxes in human behavior cease to be paradoxes in the quantum light. While classical models might explain the inconsistencies, with a possible nice fit to experimental data, in quantum cognition these inconsistencies are natural consequences of the way human cognition is assumed to work.

We also wish to stress that this dissertation is not about the 'quantum mind' in the vein of, for instance, Hameroff (2007). We do not talk about quantum mechanical interactions in the brain, we only use the mathematical framework. One of the striking successes of quantum cognition is the wide variety of fields in which it has been applied. In this thesis we will venture into human recollection and game theory, but successful applications have been constructed in, e.g., decision making (Lambert-Mogiliansky et al., 2009), concept combinations (Aerts et al., 2012), similarity judgments (Barque-Duran et al., 2016) and semantic representations (Widdows & Cohen, 2015).

1.2 The Quantum Formalism

We will give a short introduction in the mathematics behind the quantum approach in social sciences. The limited space reserved for this makes it impossible to be complete. However, the ground covered should suffice to understand the coming chapters. For a meticulous explanation of quantum theory we refer to Busemeyer & Bruza (2012) for its use in social sciences and to Nielsen & Chuang (2010) for its use in physics.

As the quantum toolbox is concerned with assigning probabilities to events, we will need a way to represent events and rules to calculate probabilities from these representations. In a classical approach, events are represented by sets of elements of a sample space. A probability function assigns a probability to these events, adhering to the Kolmogorov axioms. In quantum theory all these notions are redefined. As such, events are represented by subspaces in a Hilbert space. A Hilbert space is a vector space with a well-defined inner product, allowing the concepts of length and angle to be defined and measured. In this thesis we will only work with real vectors, even though in quantum mechanics Hilbert spaces are considered complex. We will also work with Dirac's bra-ket notation, in which $\langle V|$ is a row vector , $|W\rangle$ is a column vector, and $\langle V|W\rangle$ denotes their inner product. Subspaces larger than vectors can also represent outcomes. For example, the disjunction[3] of events V and W, represented by $|V\rangle$ and $|W\rangle$, is represented by the subspace spanned by $|V\rangle$ and $|W\rangle$.

The state of the system is represented by the state vector, which plays the role of the probability function in classical theory. The state vector is a normalized vector defined in the Hilbert space spanned by the vectors and subspaces representing the relevant events. Intuitively, the closer the state vector is to a subspace representing an event, the more likely the occurrence of that event is. To formalize this, we will use the projector onto the subspace representing an event. The probability of an event E, with associated subspace $\boldsymbol{E}$ and projector matrix P_E, given the state vector S, is defined as:

$$P(E) = \langle S|P_E|S\rangle.$$

After an event has occurred, a state revision takes place. The state vector is orthogonally projected onto the subspace $\boldsymbol{E}$ representing the occurred event and is normalized. This effectively changes the post-measurement state of the system , transforming the state vector $|S\rangle$ into a new normal-

[3] Assuming both events are compatible, see later.

ized state vector $|S'\rangle$:

$$|S'\rangle = \frac{P_E|S\rangle}{||P_E|S\rangle||}.$$

This is called the *collapse* of the state.

This collapse is one of the key non-classical features of quantum theory. It represents the evolution of the system from an indeterminate to a definite, recordable state.

1.2.1 One Measurement

Let us now apply this to a simple system in which a measurement A has two possible outcomes A_1 and A_2. Outcomes of a single measurement are represented by orthogonal subspaces. Thus, the two outcomes are represented by the orthogonal vectors $|A_1\rangle$ and $|A_2\rangle$. These two outcome vectors form an orthonormal basis, spanning the Hilbert space $\mathcal{H}_A$, in which we define the state vector $|S\rangle$. As $|A_1\rangle$ and $|A_2\rangle$ form a basis, we can express $|S\rangle$ as a linear combination of $|A_1\rangle$ and $|A_2\rangle$, with respective coordinates a_1 and a_2:

$$|S\rangle = a_1|A_1\rangle + a_2|A_2\rangle.$$

The system is said to be in *superposition* between the outcomes A_1 and A_2. This means that the system is not in a definite recordable state A_1 or A_2, as is supposed in classical theory. The system is in neither of the two states. However, when measurement A, which can only result in A_1 or A_2 is performed, the system is forced into either A_1 or A_2, changing the system. This is what the collapse of the state vector represents. In cognition, this superposition is used to model, for instance, uncertainty, doubt or ambiguity, as will become evident in future examples.

To calculate the probabilities associated with the outcomes, we construct the relevant projectors P_{A_1} and P_{A_2}:

$$P_{A_1} = |A_1\rangle\langle A_1| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$P_{A_2} = |A_2\rangle\langle A_2| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

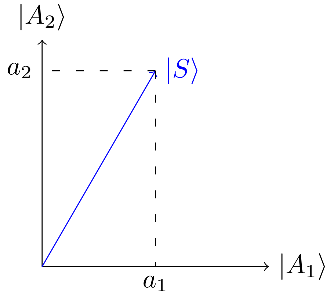

Figure 1.1 Outcome A_1, represented by the vector $|A_1\rangle$, has probability a_1^2 of being observed. Outcome A_2, represented by the vector $|A_2\rangle$, has probability a_2^2 of being observed. The state vector being normalized makes the probabilities sum to one.

This results in the probabilities $P(A_1)$ and $P(A_2)$, as shown in figure 1.1:

$$P(A_1) = \langle S|P_{A_1}|S\rangle = \langle A_1|S\rangle^2 = a_1^2 \quad (1.1)$$

$$P(A_2) = \langle S|P_{A_2}|S\rangle = \langle A_2|S\rangle^2 = a_2^2. \quad (1.2)$$

The simple form $\langle A_i|S\rangle^2$ always holds when the relevant outcomes are represented by a vector.

The fact that the state vector is normalized ensures that the probabilities sum to one. Suppose now that outcome A_1 was observed. To transform the state vector $|S\rangle$ into the new post-measurement state vector $|S'\rangle$, we project orthogonally (using projector P_{A_1}) onto the subspace spanned by the observed outcome vector $|A_1\rangle$ (step (i) in figure 1.2). We then normalize the projected vector (step (ii) in figure 1.2), resulting in the new normalized state vector $|S'\rangle$. Note that in the simple case of outcomes being represented by vectors (as opposed to bigger subspaces), the state vector is transformed into the vector that is associated with the observed outcome. This can easily be seen in figure 1.2, where

$$|S'\rangle = |A_1\rangle. \quad (1.3)$$

One can easily see that after the collapse of the state vector on $|A_1\rangle$, a repetition of measurement A will again yield the previously obtained A_1,

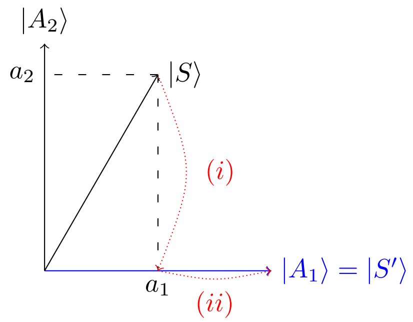

Figure 1.2 Outcome A_1 is observed. This transforms the starting state vector $|S\rangle$, after projecting (i) and normalizing (ii) into the new state vector $|S'\rangle$, which lies in the subspace representing the observed outcome.

as, due to the orthogonality of $|A_1\rangle$ and $|A_2\rangle$,

$$P(A_1|A_1) = \langle A_1|A_1\rangle^2 = 1 \quad (1.4)$$
$$P(A_2|A_1) = \langle A_1|A_2\rangle^2 = 0. \quad (1.5)$$

This property is called *repeatability* or *first kindness*.

Let us now consider a slightly more complex measurement situation: a measurement A', with three possible outcomes A'_1, A'_2 and A'_3. The relevant Hilbert space $\mathcal{H}_{\mathcal{A}'}$ is now 3-dimensional, with the outcome vectors $|A'_1\rangle, |A'_2\rangle$ and $|A'_3\rangle$ forming an orthonormal basis. The event of obtaining outcome 'A_1 or A_2' is represented by the plane $\boldsymbol{A'_{1,2}}$ spanned by the vectors $|A_1\rangle$ and $|A_2\rangle$. The projector $P'_{A'_{1,2}}$ associated with this event projects orthogonally onto the plane $\boldsymbol{A'_{1,2}}$:

$$P'_{A'_{1,2}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The resulting probability, given a state vector $|S\rangle$,

$$|S\rangle = a'_1|A'_1\rangle + a'_2|A'_2\rangle + a'_3|A'_3\rangle,$$

is therefore:

$$P(A_1 \cup A_2) = \langle S|P'_{A'_{1,2}}|S\rangle = a'^2_1 + a'^2_2.$$

This can be seen in Figure 1.3. If $A'_1 \cup A'_2$ is observed, the state vector

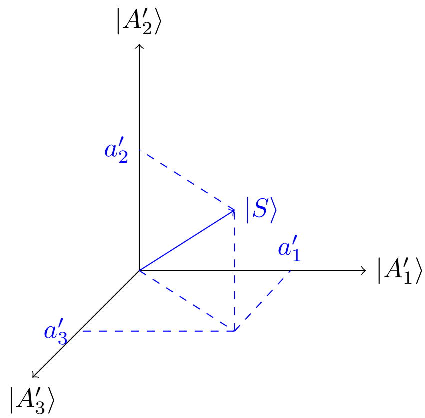

Figure 1.3 The probability of observing $A'_1 \cup A'_2$, represented by the plane $\boldsymbol{A'_{1,2}}$ spanned by $|A'_1\rangle$ and $|A'_2\rangle$, given the state vector $|S\rangle$, is $a'^2_1 + a'^2_2$.

$|S\rangle$ is projected and normalized into the new state vector $|S'\rangle$, which lies in the plane $\boldsymbol{A'_{1,2}}$:

$$
\begin{aligned}
|S'\rangle &= \frac{P'_{A'_{1,2}}|S\rangle}{||P'_{A'_{1,2}}|S\rangle||} \\
&= \frac{a'_1|A_1\rangle + a'_2|A_2\rangle}{\sqrt{a'^2_1 + a'^2_2}}.
\end{aligned}
$$

Note that the resulting probabilities do not differ from a simple classical system. This means that so far the quantumness has not added anything interesting in terms of modeling. We will come back to this in Chapter 2.

1.2.2 Multiple Measurements

We will now look at how we can model different measurements performed in one system. In these examples, the difference between the quantum and classical approaches will be more clear and interesting than the single measurement case discussed previously. At the core of this non-classicality lies the existence of *incompatible* measurements. A pair of measurements is incompatible when there is at least one pair of outcomes (one from each measurement) that is incompatible. Incompatible outcomes cannot be observed simultaneously, because the order in which the relevant measurements are performed, influences the outcome. We will construct small examples showing how to model both compatible and incompatible measurements. When two measurements are completely incompatible, the measurements are said to be *complementary*. This means that no outcome of one measurement can be observed together with any outcome of the other measurement. The most famous example of complementary properties in physics is the position and the momentum of a subatomic particle. Determining the position of a particle alters the information about the particle's momentum[4] and vice versa.

Compatible Measurements

Suppose we have measurement A, with possible outcomes A_1 and A_2, and measurement B, with possible outcomes B_1 and B_2. If these measurements are compatible and therefore can be performed at the same time, the system can be in a state that is definite over the two measurements. As such, the event of, for instance, observing A_1 and B_1 together has to be represented by a subspace in a Hilbert space, which we define as $\mathcal{H}_{A,B}$. As the same holds for the other possible combinations of outcomes, it should be clear that $\mathcal{H}_{A,B}$ has dimension 4, with $|A_1B_1\rangle, |A_1B_2\rangle, |A_2B_1\rangle$ and $|A_2B_2\rangle$ forming an orthonormal basis. The state of the system is represented by the state vector $|S\rangle$

$$|S\rangle = s_{1,1}|A_1B_1\rangle + s_{1,2}|A_1B_2\rangle + s_{2,1}|A_2B_1\rangle + s_{2,2}|A_2B_2\rangle.$$

Note that we changed the notation of the coordinates to $s_{i,j}$, as they do not refer to a single specific measurement. We will now discuss three different measurement situations with these two measurements. The situations not discussed are similar.

The event of obtaining A_1 for measurement A and B_1 for measurement B is associated with the vector $|A_1B_1\rangle$ and projector $P_{1,1}$

$$P_{A_1,B_1} = |A_1B_1\rangle\langle A_1B_1| = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

[4]The notion of complementarity in physics that we briefly discuss here is closely associated with the Copenhagen interpretation, which is the most widely accepted interpretation of quantum mechanics. We do not wish to go into detail and/or discussion of other interpretations of quantum mechanics, as this is not relevant for our applications in social sciences.

The associated probability is

$$P(A_1 \cap B_1) = \langle S|P_{A_1,B_1}|S\rangle = s_{1,1}^2 \tag{1.6}$$

If this outcome is obtained, the state vector $|S\rangle$ is projected and normalized into the new state vector $|S'\rangle$

$$|S'\rangle = \frac{P_{A_1,B_1}|S\rangle}{||P_{A_1,B_1}|S\rangle||} = |A_1B_1\rangle.$$

Now suppose that only measurement A is performed. The subspace associated with outcome A_1 is the plane $\boldsymbol{A_1}$ spanned by $|A_1B_1\rangle$ and $|A_1B_2\rangle$. This makes the associated projector $P_{A_1,\cdot}$.

$$P_{A_1,\cdot} = |A_1B_1\rangle\langle A_1B_1| + |A_1B_2\rangle\langle A_1B_2| = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The associated probability is

$$P(A_1) = \langle S|P_{A_1,\cdot}|S\rangle = s_{1,1}^2 + s_{1,2}^2 \tag{1.7}$$

If this outcome is obtained, the state vector $|S\rangle$ is projected and normalized into the new state vector $|S'\rangle$, which lies in the plane $\boldsymbol{A_1}$.

$$|S'\rangle = \frac{P_{A_1,\cdot}|S\rangle}{||P_{A_1,\cdot}|S\rangle||} = \frac{s_{1,1}|A_1B_1\rangle + s_{1,2}|A_1B_2\rangle}{\sqrt{s_{1,1}^2 + s_{1,2}^2}}. \tag{1.8}$$

Finally, we will consider the case in which measurement B is performed after measurement A has yielded A_1. The projector associated with outcome B_1 is

$$P_{\cdot,B_1} = |A_1B_1\rangle\langle A_1B_1| + |A_2B_1\rangle\langle A_2B_1| = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Note that, since outcome A_1 was obtained, the state vector $|S'\rangle$ now has the form 1.8. The associated probability is therefore:

$$\begin{aligned} P(B_1|A_1) &= \langle S'|P_{\cdot,B_1}|S'\rangle \\ &= \frac{\begin{pmatrix} s_{1,1} & s_{1,2} & 0 & 0 \end{pmatrix}}{\sqrt{s_{1,1}^2 + s_{1,2}^2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \frac{\begin{pmatrix} s_{1,1} & s_{1,2} & 0 & 0 \end{pmatrix}^T}{\sqrt{s_{1,1}^2 + s_{1,2}^2}}. \\ &= \frac{s_{1,1}^2}{s_{1,1}^2 + s_{1,2}^2}. \end{aligned} \tag{1.9}$$

Note that equations 1.6, 1.7 and 1.9 follow Bayes' rule. This is an example of the fact that if all outcomes of a system are compatible, the resulting probabilities are classical (see Busemeyer & Bruza (2012, Chapter 2.1.1.4).

The space, subspaces, projectors and probabilities we have just defined, can also be constructed starting from the smaller Hilbert spaces representing the individual measurements (as constructed in Section 1.2.1). To do so, we will tensor the relevant spaces and matrices. First define $\mathcal{H}_\mathcal{A}$ as the Hilbert space associated with measurement A, with a state vector $|S_A\rangle$

$$|S_A\rangle = a_1|A_1\rangle + a_2|A_2\rangle$$

and $\mathcal{H}_\mathcal{B}$ as the Hilbert space associated with measurement B, with a state vector $|S_B\rangle$

$$|S_B\rangle = b_1|B_1\rangle + a_b|B_2\rangle.$$

We can now define $\mathcal{H}_{A,B}$ as

$$\mathcal{H}_{A,B} = \mathcal{H}_\mathcal{A} \otimes \mathcal{H}_\mathcal{B},$$

in which $\otimes$ denotes the tensorproduct. The state vector $|S\rangle$ now looks like

$$|S\rangle = \sum_{i,j} s_{i,j} \left(|A_i\rangle \otimes |B_j\rangle\right).$$

The projector P_{A_1,B_1}, associated with obtaining A_1 and B_1, is defined as

$$P_{A_1,B_1} = P_{A_1} \otimes P_{B_1},$$

with P_{A_1} and P_{B_1} the relevant projectors in $\mathcal{H}_\mathcal{A}$ and $\mathcal{H}_\mathcal{B}$. The projector $P_{A_1,\cdot}$, associated with obtaining A_1 and not performing measurement B

is defined as

$$P_{A_1,\cdot} = P_{A_1} \otimes I_2{}^5.$$

The other constructions are similar. It should be clear that the construction starting from the 4-dimensional space and the construction tensoring the two 2-dimensional spaces result in an identical model.

Incompatible Measurements

The more interesting case is when we model incompatible measurements. Suppose we still have measurement A, with outcomes A_1 and A_2 and measurement B, with outcomes B_1 and B_2. Now, however, no outcome of measurement A can be observed simultaneously with an outcome of measurement B, as performing one measurement would influence the outcome of the other measurement. This means that there is no subspace representing, for instance, the event 'A_1 and B_1'. This is modeled by having the orthonormal basis, $|A_1\rangle$ and $|A_2\rangle$ representing measurement A, and the orthonormal basis, $|B_1\rangle$ and $|B_2\rangle$ representing measurement B, defined in the same 2-dimensional Hilbert Space. We will use coordinates relative to the basis associated with measurement A, making the state vector $|S\rangle$

$$|S\rangle = a_1|A_1\rangle + a_2|A_2\rangle.$$

Note that this choice is arbitrary. Using coordinates relative to the basis associated with measurement B would yield identical results. Suppose that the angle between $|A_1\rangle$ and $|B_1\rangle$ is θ. This gives

$$\begin{aligned} |B_1\rangle &= \cos\theta|A_1\rangle + \sin\theta|A_2\rangle, \\ |B_2\rangle &= -\sin\theta|A_1\rangle + \cos\theta|A_2\rangle, \end{aligned}$$

with their respective projectors

$$\begin{aligned} P_{B_1} &= |B_1\rangle\langle B_1| = \begin{pmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{pmatrix}, \\ P_{B_2} &= |B_2\rangle\langle B_2| = \begin{pmatrix} \sin^2\theta & -\sin\theta\cos\theta \\ -\sin\theta\cos\theta & \cos^2\theta \end{pmatrix}. \end{aligned}$$

This can be viewed in figure 1.4. Note that there is indeed no vector representing anything of the form 'A_i and B_j'.

[5] Throughout this thesis we define I_n as the $n \times n$ identity matrix.

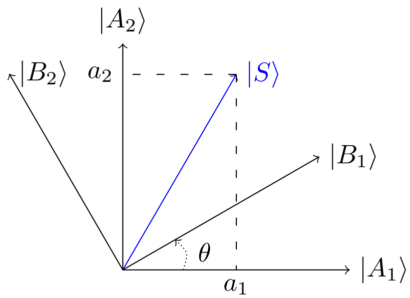

Figure 1.4 The incompatible measurements A and B are each represented by an orthonormal basis in the same Hilbert space. The state of the system is represented by state vector $|S\rangle$, described in coordinates relative to the basis associated with measurement A.

The probability of obtaining outcome B_1 (without having performed measurement A), given a state vector $|S\rangle$, can now be calculated directly:

$$\begin{aligned} P(B_1) &= \langle S|P_{B_1}|S\rangle \\ &= a_1^2 \cos^2\theta + a_2^2 \sin^2\theta + 2a_1 a_2 \cos\theta \sin\theta \\ &= (a_1\cos\theta + a_2 \sin\theta)^2. \end{aligned} \tag{1.10}$$

Or, using the simple forms from 1.1 and 1.2:

$$\begin{aligned} P(B_1) &= \langle B_1|S\rangle^2 \\ &= (a_1\cos\theta + a_2 \sin\theta)^2. \end{aligned} \tag{1.11}$$

If we now wish to calculate the probability of obtaining outcome B_1 after having obtained A_1 (making the state vector $|S'\rangle = |A_1\rangle$, see result 1.3), we get:

$$\begin{aligned} P(B_1|A_1) &= \langle A_1|P_{B_1}|A_1\rangle \\ &= \langle A_1|B_1\rangle^2 \qquad (1.12) \\ &= \cos^2\theta. \qquad (1.13) \end{aligned}$$

Note that this has a different form from equation 1.9. Likewise, the probability of obtaining outcome B_1 after having obtained A_2 is:

$$\begin{aligned} P(B_1|A_2) &= \langle A_2|P_{B_1}|A_2\rangle \\ &= \langle A_2|B_1\rangle^2 \\ &= \sin^2\theta. \end{aligned} \tag{1.14}$$

This leads to an interesting observation. Combining 1.1 with 1.13, 1.2 with 1.14 and comparing to 1.10, we get the surprising result

$$\begin{aligned} P(B_1|A_1)P(A_1) + P(B_1|A_2)P(A_2) &= a_1^2\cos^2\theta + a_2^2\sin^2\theta \\ &\neq a_1^2\cos^2\theta + a_2^2\sin^2\theta + 2a_1a_2\cos\theta\sin\theta \\ &= (a_1\cos\theta + a_2\sin\theta)^2 \\ &= P(B_1). \end{aligned}$$

This is a clear violation of classical probability rules. The extra term $2a_1a_2\cos\theta\sin\theta$ is called the *interference* term. The *act* of performing measurement A, regardless of the outcome, influences measurement B. The mathematical reason for this violation is that the projectors P_{A_i} and P_{B_j} do not commute, meaning that the order in which the measurements are performed influences the outcomes. Naturally emerging violations are the prime reason why quantum models are considered in social sciences, as in human behavior many similar violations are observed.

1.3 Two Examples

We will briefly discuss two applications of the quantum framework in cognition, to show the strength of the formalism. We will give no in-depth background or statistical fit of these paradigms, but briefly sketch how a certain classical statistical violation arises in human behavior and how quantum theory explains it. This should provide context to the previous section. As these two examples are well-known examples in quantum cognition literature, more background can easily be found in the references provided.

1.3.1 The Trustworthiness of Clinton and Gore - Order Effects

This example is discussed thoroughly, with statistical fit to data in Wang & Busemeyer (2013). In a Gallup poll, conducted September 6-7, 1997, participants were asked two separate yes-no questions: if they thought Clinton was honest and trustworthy and if they thought Gore was honest and trustworthy. The data show that the order of these question, influences the outcome. If the Clinton question was the first to be asked, the trustworthy-rating of Clinton was 53%, when second it increased to 59%. If the Gore question was the first, the trustworthy-rating of Gore was 76%, when second it decreased to 67%. This is explained in the original article (Moore, 2002) as a consistency effect: when the second question is linked to first, the difference between the two becomes smaller. As such, Gore's trustworthiness diminishes when measured after Clinton's trustworthiness, while Clinton's trustworthiness increases when measured after Gore's trustworthiness. We will view the two questions as incompatible, as answering one of them, influences the outcome of the other. This shows how both questions are contextual and not absolute. They depend on the context of the other question being asked.

The Hilbert space we will use will have coordinates relative to the Clinton question and is thus spanned by the orthonormal basis consisting of the vector $|C_+\rangle$, representing that Clinton is trustworthy and the vector $|C_-\rangle$, representing that Clinton is not trustworthy. As the questions are considered incompatible, we will define a second orthonormal basis, consisting of the vectors $|G_+\rangle$ and $|G_-\rangle$ which represent the Gore question similarly, as

$$\begin{aligned} |G_+\rangle &= \cos\theta|C_+\rangle + \sin\theta|C_-\rangle \\ |G_-\rangle &= -\sin\theta|C_+\rangle + \cos\theta|C_-\rangle. \end{aligned}$$

We define $\theta = 0.1566$[6] and the state vector $|S\rangle$ as

$$|S\rangle = 0.7798|C_+\rangle + 0.6261|C_-\rangle.$$

[6]The parameter $\theta = 0.1566$ and all following parameters are obtained by minimizing an appropriate χ^2-statistic. This process fits the derived probabilities to the observed proportions reported previously. As this is an introductory example, we will not go into further details. A thoroughly discussed statistical fit, using a complex Hilbert space, can be found in Wang & Busemeyer (2013).

Performing the Clinton question first results in the probabilities

$$\begin{aligned} P(C_+) &= \langle S|C_+\rangle^2 = 0.6080 \\ P(C_-) &= \langle S|C_-\rangle^2 = 0.3920. \end{aligned}$$

Performing the Gore question first results in the probabilities

$$\begin{aligned} P(G_+) &= \langle S|G_+\rangle^2 = 0.7532 \\ P(G_-) &= \langle S|G_-\rangle^2 = 0.2468 \end{aligned} \quad (1.15)$$

However, performing the Gore question after the Clinton question gives us

$$\begin{aligned} P(G_+|C_+) &= \langle C_+|G_+\rangle^2 = 0.9757 \\ P(G_+|C_-) &= \langle C_-|G_+\rangle^2 = 0.0243, \end{aligned}$$

as the state vector now has first collapsed on $|C_+\rangle$ or $|C_-\rangle$. So the probability of thinking Gore is trustworthy after replying to the Clinton question (regardless of the outcome) is

$$P(G_+|C_+)P(C_+) + P(G_+|C_-)P(C_-) = 0.6028. \quad (1.16)$$

Comparing results 1.15 and 1.16 shows that the quantum-like model indeed predicts a diminishing trustworthiness rating of Gore, when measured as a second question.

1.3.2 Linda, the Bank Teller - Conjunction Fallacy

There is an in-depth discussion of this example in Busemeyer et al. (2011), Franco (2009) and Busemeyer & Bruza (2012). It concerns the famous "Linda" problem, a conjunction fallacy discussed first in Tversky & Kahneman (1983). In the experiment, participants are provided a brief story about a woman named Linda:

> Linda is 31 years old, single, outspoken, and very bright. She majored in philosophy. As a student, she was deeply concerned with issues of discrimination and social justice, and also participated in anti-nuclear demonstrations.

The participants are then asked to rank the likelihood of a list of events. Two of these are

(a) Linda is a bank teller

(b) Linda is active in the feminist movement and is a bank teller.

Classical probability theory says that the probability of (a) is always higher than (or equal to) the probability of (b), due to the conjunction form of (b). However, participants regularly deem (b) more probable than (a). We will show that a simple quantum-like model can account for this conjunction fallacy, by assuming that the feminist assessment and the bank-teller assessment are incompatible. We will work in a two-dimensional Hilbert space, spanned by the basis $|F_+\rangle$ and $|F_-\rangle$, representing the participant thinking Linda is a feminist (F_+) or is not a feminist (F_-). The participant is represented by the state vector $|S\rangle$

$$|S\rangle = f_+|F_+\rangle + f_-|F_-\rangle.$$

The bank-teller assessment is represented by another basis in the same Hilbert space, in which the assumption that Linda is a bank teller (B_+) or is not a bank teller (B_-) is respectively represented by

$$\begin{aligned} |B_+\rangle &= \cos\theta|F_+\rangle + \sin\theta|F_-\rangle \\ |B_-\rangle &= \sin\theta|F_+\rangle - \cos\theta|F_-\rangle. \end{aligned}$$

Therefore, the probability associated with (a) is

$$P(a) = P(B_+) = \langle B_+|S\rangle^2 = (f_+\cos\theta + f_-\sin\theta)^2. \tag{1.17}$$

When resolving the likelihood of (b), the participant is supposed to first assess Linda's 'feminism' and then her 'bank tellerism'.

$$P(b) = P(F_+)P(B_+|F_+) = \langle F_+|S\rangle^2\langle B_+|F_+\rangle^2 = f_+^2\cos^2\theta. \tag{1.18}$$

Note that after assessing Linda as a feminist, the state vector transforms into $|F_+\rangle$. It is perfectly possible for $P(b)$ (1.18) to be greater than $P(a)$

(1.17). For instance, using the following values of f_+, f_- and θ[7]

$$\begin{aligned} f_+ &= 0.987 \\ f_- &= -0.1564 \\ \theta &= \frac{2}{5}\pi, \end{aligned}$$

gives us

$$\begin{aligned} P(a) &= (f_+ \cos\theta + f_- \sin\theta)^2 = 0.0254 \\ P(b) &= f_+^2 \cos^2\theta = 0.0932. \end{aligned}$$

1.4 Aim and Outline

We believe that the quantum formalism has already shown its worth in social sciences. The previous two short examples are only the tip of the iceberg of what has been done in this field. Therefore, the focus of this thesis is not to show that quantum cognition works, but to investigate what its strengths and weaknesses are. We wish to determine when a quantum-like model yields interesting predictions or when there is a clear natural classical equivalent, making the quantum nature of model not the core of a models strength. We also wish to investigate the boundaries of this approach. When will the quantum approach not suffice to model certain situations? These questions, and the answers we provide, are small steps towards a (utopian) 'general quantum recipe' for making it evident both when to use a quantum framework and how to implement it. We opted for an application-driven approach. We investigate two paradigms, each from a distinct field in social sciences, where quantum-like models are constructed and we take a critical look at their successes and shortcomings. In both cases, we propose improvements to remedy the identified shortcomings. This approach has two clear advantages. First, the examples are valuable on their own. The resulting models yield interesting outcomes in their respective fields, even when not framed in the larger discussion on the use of quantum cognition. Second, we avoid an all too theoretical discussion, that might cloud any prospect of applicability. The core of this thesis consists of three chapters. As Chapters 2 and 3 are published arti-

[7]The high coordinate f_+ signifies the description of Linda reflecting a feminist stereotype. The fact that θ puts $|F_-\rangle$ and $|B_+\rangle$ close, signifies the participants' difficulty of blending 'the feminist' and 'the bank teller' into one mental picture. These values are taken from Busemeyer & Bruza (2012).

cles and Chapter 4 is an extension of a published paper, they can be read separately. This means that overlap in content will occur. However, we do recommend reading them consecutively, as Chapter 4 continues ideas first explored in Chapter 3.

In Chapter 2 quantum-like models for a memory experiment are constructed and commented upon. The memory experiment itself concerns the human episodic memory. Participants were asked to memorize and recollect word lists. During this recollection, they exhibited a disjunction fallacy, pointing to a quantum-like treatment. A first attempt at capturing this experiment and explaining the disjunction fallacy using a quantum-like model was performed in Brainerd et al. (2013). After introducing this first model, we show that it has interpretational problems and we construct a classical equivalent showing that the explanation of the fallacy provided by Brainerd et al. is not quantum-like in nature. We also construct an alternative quantum-like model, by assuming complementarity between certain memory types. This way we are able to capture the intrinsic relationship between the two memory types in a satisfactory way, taking into account their mutual interference. We prove that our new model also exhibits the fallacy of interest. We show that our alternative, while having a slightly better statistical fit than model of Brainerd et al., has no clear classical equivalent.

Chapter 3 concerns a sequential prisoner's dilemma experiment, discussed originally in Blanco et al. (2014). In this experiment the strategic moves of participants are recorded. However, in a subgroup of the participant, the beliefs concerning opponents' behavior are also elicited. The obtained data show a violation of the sure-thing principle, which we explain by assuming the incompatibility of participants' moves and beliefs. We show that a straightforward translation of the paradigm into a quantum-like setting results in an unsatisfactory, overparametrized model. We solve this problem by going beyond the standard quantum-like approach and use projectors similar to a POVM approach, which makes use of the ordered nature of the outcomes of the belief measurement. POVMs are an extension of the typical PVM approach and are frequently used in physics to model noise in a measurement. This noise in a measurement is now reflected in an unsharp measurement, where it is possible for a participant to not completely distinguish between two possible answers to a question.

Chapter 4 aims at placing the alternative solution constructed in Chapter 3 in a more general light. We show that the problems arising from the construction the straightforward quantum-like model in Chapter 3 also emerge in other settings. We then investigate if the solution to these problems in Chapter 3 also works more generally. This results in a new type of quantum-like measurements, outside the realm of standard quantum theory, which can take into account the ordered structure of outcomes.

These three chapters together shed a light on what exactly propels this formalism to its success. By contrasting what works and what is problematic, we aim to further this new field as it shows in what fundamental ways (both mathematical and interpretational) the quantum-like approach differs from classical approaches. This should clarify the role of concepts such as 'superposition' and 'complementarity' in these models and showcase why they are so adequate at explaining human behavior.

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2 Bohr complementarity in memory retrieval

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Abstract. We comment on the use of the mathematical formalism of Quantum Mechanics in the analysis of the documented subadditivity phenomenon in human episodic memory. This approach was first proposed in Brainerd et al. (2013). The subadditivity of probability in focus arises as a violation of the disjunction rule of Boolean algebra. This phenomenon is viewed as a consequence of the co-existence of two types of memory traces: verbatim and gist. Instead of assuming that verbatim and gist trace can combine into a coherent memory state of superposition as is done in the QEM model, we propose to model gist and verbatim traces as Bohr complementary properties of memory. In mathematical terms, we represent the two types of memory as alternative bases of one and the same Hilbert Space. We argue that, in contrast with the QEM model, our model appeals to the one essential distinction between classical and quantum models of reality namely the existence of incompatible but complementary properties of a system. This feature is also at the heart of the quantum cognition approach to mental phenomena. We sketch an experiment that could separate the two models. We next test our model with data from the same word list experiment as the one used by Brainerd et al.. While our model entails significantly less degrees of freedom it yields a good fit to the experimental data.

2.1 Introduction

In this article we will extend the work done in Brainerd et al. (2013) in using the quantum formalism to explain phenomena in human memory. In Brainerd et al. (2013), a memory analogue to the superposition principle of quantum mechanics is proposed and formally tested. The phenomenon that is studied concerns a two step experiment dealing with human episodic memory, where autobiographical memories are stored. In the first step participants memorize various word lists. In the second step participants are asked to accept or decline statements about these memorized word lists. These can be specific statements, asking the agent if they remember a word being part of a specific list or be more general statements, regarding the presence of a word on any of the remembered lists. Participants are shown to exhibit episodic subadditivity, a violation of the classical disjunction rule, which is attributed to the episodic memory consisting of two distinct memory types: verbatim memory and gist memory. We will discuss these two memories types more extensively in Section 2. The authors of Brainerd et al. (2013) view this experiment as a memory analogue to the classic double slit experiment in Physics. We will summarize and discuss this approach in Section three and use it as an example to introduce the quantum formalism.

In Section 2.4 we propose an alternative view on subadditivity, where we view different types of human episodic memory as complementary properties of human memory. This idea was first proposed in Lambert-Mogiliansky (2014) and chapter 6 of Busemeyer & Bruza (2012) and was used as an example of the importance of non-orthogonal vectors as the distinction between quantum and classical models presented in Denolf (2015). Here we will flesh out this view in the form of a new model, called the Complementary Memory Types (CMT) model. We will fit this model to the data of an experiment discussed in Section 2. In our view, this CMT model elegantly models the overdistribution. We also claim that the CMT model is easily adjustable to be applied to other datasets, which might express different forms of additivity in their disjunction rule. We briefly suggest an extension of the previously discussed experiment, where we include the possibility of measuring order effects. These order effects are viewed as an expression of the non-classical nature of human memory and are naturally modeled within the CMT model.

2.2 The Source Memory Experiment and Overdistribution

Experiments and literature concerning human episodic memory are classically divided in two types, item memory and source memory. The former deals with the ability to remember previously acquired information, e.g., if a word was previously seen, the latter also deals with contextual information, e.g., where a word was previously seen. In these episodic memory experiments participants are asked to memorize different sets of words and recollect these afterward. Doing so, two types of memory distortions are exhibited, false memories and overdistribution.

To define these two memory distortions, we will expand on an example by Brainerd et al. concerning item memory. Suppose participants memorized a list of target words containing, amongst others, the words *Pepsi, 7up* and *Sprite* and are presented the test word *Coke*. They are then asked to categorize the given test word as a target word, where a target word denotes a word that was studied, a related distractor or an unrelated distractor. Since *Coke* was not on the list of target words, but shares semantic features with target words, it should be categorized as a related distractor. When a participant wrongly remembers *Coke* as a target word but not as a related distractor, we denote this distortion as false memories.

In addition to false memory, it can occur that participants remember *Coke* as both a target word and a related distractor. Here, memory retrieval is distorted by past experience, which are in this case, other memorized words. This form of memory distortion is denoted as overdistribution.

These two forms of memory errors are fundamentally different since the total error can be divided in these two types of mistakes, as shown in Brainerd et al. (2010).

Since participants know that a word cannot be both a target word and a related distractor, overdistribution can not be directly observed. We have to rely on the classic disjunction rule to measure the amount of overdistribution participants exhibit. Therefore, after presenting the participant a test word, the participant is also presented with one of three possible recognition statements. The participant is then asked to either accept or reject the statement they received. The three possible statements are: (a) the test word is a target word, (b) the test word is a related distractor and (c) the test word is a target word or related distractor. This way, the following quantities can be defined and measured for each test word: $P_w(T)$ as the proportion of participants remembering the test word w as a target word, $P_w(R)$ as the proportion of participants remembering the test word w as a related distractor and $P_w(T \cup R)$ as the proportion of participants remembering the test word w as a target word or a related distractor, without specifying which of the two. This way the probability that a participant would remember the test word w as both a target word and a related distractor can be defined as:

$$P_w(T \cap R) = P_w(T) + P_w(R) - P_w(T \cup R). \tag{2.1}$$

With this definition, the overdistribution phenomenon can be mathematically expressed as a violation of the disjunction rule, since participants with perfect memory would exhibit $P_w(T) + P_w(R) - P_w(T \cup R) = 0$ for each test word w. Viewing overdistribution as a disjunction fallacy, it is shown in Brainerd & Reyna (2008) and Brainerd et al. (2010) that overdistribution can be seen as a consequence of dual-trace distinctions from Fuzzy-Trace Theory developed in Reyna & Brainerd (1995). This theory postulates that human episodic memories are stored in two different types of memory. The first memory type is referred to as verbatim memory, encompassing the presentation and phonology of a memorized word. The second memory type is referred to as gist memory, encompassing the semantic meaning of a memorized word. Target words and related distractors can share the same gist trace (e.g. *coke* and *sprite* are both soft drinks). Since both verbatim and gist traces are used in deciding if a word is a target word, these gist traces account for words being viewed as both target words and related distractors, resulting in episodic overdistribution. For a more complete overview of episodic distribution, including the implementation of other theories than the Fuzzy-Trace theory, see Kellen et al. (2014).

In this paper we will focus on an experiment reported in Brainerd & Reyna (2008) and extended in Brainerd et al. (2012), concerning the overdistribution of the source memory. As this experiment concerned source memory, participants were tasked not only with remembering if a word was studied, but also with remembering where (e.g. which list) the word was first presented on.

Seventy participants were asked to memorize three distinct word lists, containing different words. Each of these lists contain 36 words (2-word starting and ending buffers, 32 target words), a different background color

and a different font in which the words were printed, to ensure that each list was distinctive. Each of these participants was then presented a list of 192 test items. A test item comprises a combination of a test word and a recognition statement. These test words originated from 1 out of 4 different sources: one of the three memorized lists or a non-memorized list of unrelated distractors. The four possible recognition statements were, (a) the test word is on list 1, (b) the test word is on list 2, (c) the test word is on list 3 or (d) the test word is on one of the lists. Each of these test words was presented with 1 out of these 4 recognition statements, such that, across all participants, each test word had probability .25 of being presented with each of the recognition statements. The experiment also varied the test words between word concreteness (abstract/concrete) and word frequency (high/low frequency use in common language), resulting in 4 different word types. These manipulations were done for theoretical reasons, since it was predicted that abstract and low frequency words create weaker verbatim traces than concrete high frequency words, resulting in a clearer overdistribution for abstract low frequency words, see Brainerd et al. (2012) and Brainerd & Reyna (2005) for more details. This gives us 16 experimental conditions (4 word types $\times$ 4 possible sources), each with four possible measurements (the four recognition statements).
For the participant responses, the following proportions were calculated, for each type of test word: p_1, p_2, p_3 which were the proportions of accepted statements of resp. type (a), type (b) and type (c) and p_{123} which was the proportion of accepted statements of type (d). These proportions are seen as the probability of the event that an agent thinks that the test word is on a certain list for proportion p_i (similar to $P(T)$ and $P(R)$ from the item version of overdistribution) or the probability of the event that the agent thinks that the test word is on any of the lists, for p_{123} (similar to $P(T \cup R)$ from the item version of overdistribution). These results can be found in table 2.1. Because of the structure of the recognition statements, the event associated with p_{123} can be seen as the disjunction of the events associated with p_i.

Now we can express the overdistribution in the source memory as a violation of the disjunction rule, with the conjunction part equal to 0, since agents know that none of the words appear on more than one list :

$$p_1 + p_2 + p_3 = p_{123}. \tag{2.2}$$

From this expression we define the subadditivity effect as:

$$S = p_1 + p_2 + p_3 - p_{123}. \tag{2.3}$$

This is shown in Brainerd et al. (2012) to be significantly differing from 0.

2.3 Quantum Episodic Memory

2.3.1 Introducing Quantum Models

In order to define a cognitive model based on the quantum formalism, the notion of state, measurement, outcome and probability need to be defined. These notions differ markedly with the corresponding notions commonly employed within cognitive science. Recently, different aspects of the quantum formalism have had encouraging success in producing models in areas such as game theory (Martínez-Martínez, 2014), decision theory (Lambert-Mogiliansky et al., 2009) and models for the human mental lexicon (Bruza et al., 2009), domains within cognition having links to human memory. In addition, the quantum formalism, has proven useful in modeling logical fallacies, such as the inverse fallacy (Franco, 2007). As the experiment previously described reveals the disjunction fallacy within human memory, it seems a feasible candidate for a quantum model. For an overview of the use of the quantum formalism in social sciences, see Busemeyer & Bruza (2012). Note that different interpretations can be given to what exactly happens in a quantum system on a subatomic scale. We are agnostic in this discussion and just borrow the mathematical framework devised for describing said events. The defining difference between our use of quantum techniques and classical models is that the subject, e.g., a particle in physics or in our case, a human, is not in a definite state. Take a simple system where a measurement A has two possible outcome states A_+ and A_-, e.g. the spin of a subatomic particle. In a non-quantum model, the system has a definite state, which can be measured by the observer. This state can evolve over time. Using quantum techniques, it is possible to model a system that can be in an indefinite state between different definite states or outcomes. This phenomenon is referred to as the superposition principle. When an observer performs a measurement, the act of measuring itself changes the system fundamentally and forces it to leave the superposition and become a possible outcome, or in quantum

Table 2.1 Proportions of accepted statements of the three word list experiment

	High-frequency/ Concrete	High-frequency/ Abstract	Low-frequency/ Concrete	Low-frequency/ Abstract
List 1 Test Word				
p_1	0.5211	0.5214	0.5929	0.5786
p_2	0.3286	0.3643	0.4643	0.6143
p_3	0.3786	0.3714	0.4143	0.5286
p_{123}	0.5571	0.5429	0.6357	0.6643
List 2 Test Word				
p_1	0.3143	0.3214	0.7508	0.3143
p_2	0.35	0.1320	0.5429	0.4643
p_3	0.35	0.1445	0.3786	0.3429
p_{123}	0.5357	0.1445	0.6357	0.4857
List 3 Test Word				
p_1	0.2972	0.4	0.3429	0.4286
p_2	0.4286	0.4357	0.35	0.5786
p_3	0.4214	0.4786	0.4935	0.519
p_{123}	0.6	0.5286	0.5643	0.5857
Unrelated Distractor				
p_1	0.1476	0.2524	0.1143	0.1857
p_2	0.1714	0.2405	0.1095	0.2071
p_3	0.2119	0.2429	0.1262	0.1715
p_{123}	0.2167	0.2596	0.1265	0.1976

jargon, collapse onto a definite state. This collapse is probabilistic in nature. This notion of measurement fundamentally changes the role of the observer of a system, which can not be seen as a separate entity, but is an intrinsic part of the system.

This is mathematically modeled by replacing the classical subsets of the sample space containing all the possible outcomes of the measurement by a Hilbert Space $\mathcal{H}$[1], which is spanned by normalized vectors representing the possible outcomes. The probability function, which maps outcomes to their associated probabilities, is replaced by a normalized state vector $|S\rangle$ within $\mathcal{H}$. Here we introduce Dirac's bra-ket notation, where $\langle V|$ is a row vector and $|V\rangle$ is a column vector. To continue our previous example, we have a Hilbert Space $\mathcal{H}^A = \langle |A_+\rangle, |A_-\rangle\rangle$ and a state vector $|S\rangle = a_+|A_+\rangle + a_-|A_-\rangle$, before measurement. This represents that, before measurement, the system is between two possible outcomes $A+$ and A_-. When the measurement is performed, the system has to collapse on a possible outcome, so the state vector transforms into either $|S\rangle = |A_+\rangle$ or $|S\rangle = |A_-\rangle$. Generally speaking, when an event is observed, the state vector gets projected orthogonally onto a subspace representing the observed outcome and is then normalized. This subspace is spanned by the vectors representing the events, which form the disjunction of the observed event. Here, in the most simple case, the event of observing outcome A_i is associated with the projector $P_{A_i} = |A_i\rangle\langle A_i|$, which projects the state vector orthogonally onto the vector associated with the observed outcome, after normalizing:

$$\frac{P_{A_i}|S\rangle}{||P_{A_i}|S\rangle||} = |A_i\rangle. \tag{2.4}$$

If, for example, the disjunction of events $A_1, \ldots, A_k$ is observed, the state vector is projected onto the subspace spanned by $\{|A_1\rangle, \ldots, |A_k\rangle\}$, with $|A_i\rangle$ representing event A_i. Collapse is probabilistic. The closer a state vector is to an outcome vector, the higher the probability that the outcome will be observed. Expressing this idea of distance as a probability, we define the probability p_i of $|S\rangle$ collapsing on $|A_i\rangle$ as $p_i = ||P_{A_i}|S\rangle||^2 = \langle S|P_{A_i}|S\rangle$. In our example, we have $p(A_+) = p_+ = a_+^2$ and $p(A_-) = p_- = a_-^2$, as can be seen in figure 1. The normalization restriction of the state vector makes the sum of the probabilities across all possible outcomes of measurement A equal to 1 [2].

[1] A Hilbert Space is a vector space with a inner product defined on its vectors. Here we will only consider real vector spaces with the Euclidean inner product.

[2] This is a direct consequence of the Hilbert Space being spanned by an orthonormal base.

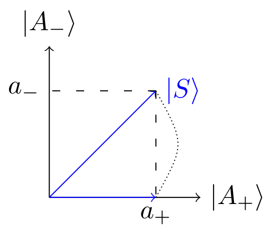

Figure 2.1 Observing outcome A_+, with probability a_+^2, projects the state vector $|S\rangle$ onto $|A_+\rangle$

2.3.2 The QEM Model

We will illustrate this formalism by constructing the QEM model of Brainerd et al. (2013), where it is fitted with the data from the three list experiment described previously. Here the authors describe the state of memory as being in superposition between different memory traces, represented by orthonormal basis vectors, spanning the Hilbert Space. These traces consist of a verbatim trace for each of memorized lists, represented by $|V_1\rangle, |V_2\rangle$ and $|V_3\rangle$, a gist trace for the semantic features represented by $|G\rangle$ and an unrelated distractor trace, represented by $|U\rangle$. We have the memory state represented by:

$$|S\rangle = v_1|V_1\rangle + v_2|V_2\rangle + v_3|V_3\rangle + g|G\rangle + u|U\rangle. \tag{2.5}$$

Since a state vector is always normalized, we can consider u as a function of v_1, v_2, v_3 and g. Here, the QEM model has 4 parameters for each of the 16 experimental conditions. Since at least two types of statement do not match the test word, the associated coordinates of the verbatim traces of lists not containing the test word are considered equal and will be denoted as v_{nt}. Likewise, we will denote the coordinate associated with the verbatim trace of the list the test word is found on as v_t. This way when a test word from list 1 (similarly for list 2 and 3) is presented, the memory state is represented by the state vector:

$$S_1 = v_t|V_1\rangle + v_{nt}|V_2\rangle + v_{nt}|V_3\rangle + g|G\rangle + u|U\rangle \tag{2.6}$$

and when a unrelated distractor word is presented the memory state is represented by the state vector:

$$S_4 = v_{nt}|V_1\rangle + v_{nt}|V_2\rangle + v_{nt}|V_3\rangle + g|G\rangle + u|U\rangle. \tag{2.7}$$

Now we need to define the projectors associated with each of the possible outcomes of each of the possible measurements. Since the dual trace distinction theorizes that agents use both gist and verbatim traces when recognizing test words as being on a memorized list, accepting a statement that a test word was on a list will project the memory state vector on a subspace spanned by the vectors associated with both the relevant verbatim trace and the gist trace. Accepting a statement of type (a), that the test word was on list 1, will therefore be represented by the projection on the plane spanned by $|V_1\rangle$ and $|G\rangle$. Since these vectors are basis vectors, spanning the Hilbert Space, the projector matrix is $M_1 = \text{diag}(1, 0, 0, 1, 0)$. The projector matrices associated with accepting statements of type (b) and (c) are calculated similarly and are respectively $M_2 = \text{diag}(0, 1, 0, 1, 0)$ and $M_3 = \text{diag}(0, 0, 1, 1, 0)$. The projectors associated with rejecting a statement of type (a), (b) or (c) are defined as $M_{\bar{i}} = I^5 - M_i$, with I^5 the 5×5 identity matrix.

The projector associated with accepting a statement of type (d), the OR statement, is constructed by viewing this acceptance as a decline of the conjunction that the test word is not on list i, for $i = 1, 2, 3$. Straightforward calculation gives us $M_{123} = I^5 - (I^5 - M_3)(I^5 - M_2)(I^5 - M_1) = \text{diag}(1, 1, 1, 1, 0)$. We also define the projector associated with rejecting a statement op type (d) as $M_{\overline{123}} = I^5 - M_{123}$. We define the probability of accepting a statement as p_i, with $i = 1 \ldots 3$ for statements of type (a), (b) or (c) respectively and, abusing notation, $i = 123$ for statements of type (d) . This gives us $\hat{p}_i = \langle S_j|M_i|S_j\rangle$, for a test word from list j, with $j = 4$ for words not on any of the lists.

For the 12 out of 16 experimental conditions where a test word from a list is presented, the QEM has 3 parameters: v_t, v_{nt} and g. For the 4 experimental conditions where an unrelated distractor is presented, the QEM model has 2 parameters: v_{nt} and g. This leads to a total of 44 parameters. The fit of this model to the observed proportions has been established by Brainerd et.al. in Brainerd et al. (2013), by calculating a G^2 statistic we will define in Section 4.4. This G^2 statistic compares the QEM model to a saturated model. A saturated model is a model with as many parameters as data points, therefore having a perfect fit of the data. This way, the G^2 statistic calculates how close the proportions estimated by the QEM model are to the observed proportions. Here, the QEM model does not predict significantly worse than a saturated model in 13 out of 16 experimental conditions at the $\alpha = .05$ level. The QEM model also does not significantly differ from the saturated model summed across all experimental conditions (p-value equal to .14), giving strong evidence that the QEM model fits the data well.

2.3.3 Discussion of the QEM Model

In this paper we only use the QEM model to compare its fit of the experimental data to the CMT model, which is defined in the next session. For this reason we will only summarize the two major criticisms leveled at this model. For a detailed critical analysis of the QEM model, see Denolf (2015),

The first criticism relates to the chosen representation. When modeling the three alternative verbatim traces and the gist trace as basis vectors in one and the same basis, by force of the mathematics they are defined as mutually exclusive. Theoretically, after the respondent would retrieve V_1, the probability for G is zero because the measurement (with V_1, V_2, V_3, U and G as a response) has already been performed and the outcome was V_1. But in our view retrieving purely orthographical memory is not inconsistent with retrieving semantic memory: verbatim and gist are not mutually exclusive even if they cannot be simultaneously retrieved. In what follows, we assume that after theoretically obtaining response V_1 you should have a non-zero probability in obtaining G in a next following measurement.

The second criticism relates to a technical issue: the orthogonality of the verbatim and gist vectors. It can be shown (as in Denolf (2015)) that performing one measurement, with all relevant vectors orthogonal, leads to a distribution which always has a classical equivalent. This classical equivalent has identical resulting predictions and fit, with it's parameters having a one-to-one connection to the parameters of the QEM model. Here, the classical distribution has a sample space of 5 discrete events, each with a probability as denoted in Table 2.2. The agent accepting a statement of type (a) is now represented by the event $V_1^c \cup G$. The corresponding parameters can also be easily calculated: $v_1^c = v_1^2$ and $g^c = g^2$. Statements of type (b), (c) and (d) have a similar representation. It can also be shown that all relevant matrices in the QEM model commute, so even extending the model to incorporate multiple measurements, would not lead to a model without a classical equivalent. Since these matrices commute, it is impossible to model order effects, which we consider a prime example of the non-classical nature of quantum measurements. We will briefly discuss the role of these order effect at the end of Section 4. This

Event	Probability
V_1^c	v_1^c
V_2^c	v_2^c
V_3^c	v_3^c
G^c	g^c
U^c	u^c

Table 2.2 A classical equivalent to the QEM model

reasoning holds for all quantum models with all relevant vectors orthogonal: one can always easily define a simple equivalent classical distribution, with the respective probabilities being the square of the coordinates of the state vector.

The QEM model, while having a good fit, does not seem to fully utilize the advantages the quantum formalism has. On both the interpretational level as the mathematical level, concerns can be raised. All of these concerns seem to have root in the fact that all relevant vectors are orthogonal. In the next section, we will propose a new model, where the introduction of one vector non-orthogonal to the other relevant vectors, will significantly enhance the use of the quantum formalism.

2.4 Complementary Memory Types

2.4.1 Complementarity

The term complementarity was introduced in quantum mechanics by Niels Bohr. It is also referred to as Bohr complementarity to distinguish it from the common notion of complementarity. Two properties of a system are said to be Bohr complementary if they cannot be measured simultaneously, that is the system cannot have a definite value with respect to both of the properties at the same time. Yet the properties are not mutually exclusive in the sense that they capture aspects that complement each other in the description of the system. The most well-known such pair is position and momentum. This feature central to quantum mechanics has other expressions. In particular, since complementary properties cannot have definite value simultaneously, measurements affect the system implying that the order of measurements matters to the outcome. This in turn leads to violations of the classical law of probability and generates phenomena of sub(super)additivity of the kind exhibited in the experiment under consideration in this paper.

The idea that mental phenomena exhibit Bohr complementarity is at the basis of most works within quantum cognition and consistent with the intuition of the grounding fathers of quantum mechanics, including Niels Bohr himself. This hypothesis has also shown itself very successful in explaining a wide range of psychological and behavioral phenomena (see for example Franco (2007) and Lambert-Mogiliansky et al. (2009)). The psychological interpretation is that the human mind cannot be fully decomposed into separated pieces but tends to function as a whole piece. As a consequence it exhibits "cognitive limitation"[3]. In particular, it cannot aggregate/combine all relevant perspectives on a phenomenon into a single synthetic mental picture, which is not unlike notions from dual trace theory. When we are determined with respect to one perspective another might get blurred. A stark illustration is provided by ambiguous pictures: two images can be true but you cannot see them simultaneously. When it comes to memory, the mind may be in the gist trace perspective and switch to the verbatim trace. But it is difficult to simultaneously retrieve a clear gist value and a clear verbatim trace value. The memory of a stimulus is more like one single system that cannot be addressed from one perspective without being affected i.e., without that operation affecting the value of future retrievals from other complementary perspectives. As a consequence the order of retrieval matters and the laws of classical probability can be violated. The idea of verbatim and gist traces being complementary has already been applied to item memory in Busemeyer & Bruza (2012) and Busemeyer & Trueblood (2010)[4]. It has been proposed in Lambert-Mogiliansky (2014) and Denolf (2015) to use complementarity between a verbatim and a gist trace in source memory.

2.4.2 The CMT Model

Two complementary measurements are represented by different orthonormal bases in the Hilbert Space. Applied here, the verbatim and gist traces

[3]It is limited in the sense that human cognition produces a mental picture that is not necessarily correctly reflecting the actual object in the outside world. In particular properties of an object that are fully compatible in nature may not be compatible in the mind. So as the mind creates a mental picture by processing information, it needs not converge to a single complete picture but can keep on oscillating as in the perception of ambiguous pictures.

[4]Note that it would be hard to do a formal comparison between item memory models and source memory models. We list these as inspiration, since these models also assume complementarity between a verbatim and a gist trace.

are now considered to be represented by different orthonormal bases in the same Hilbert Space. Their relationship is represented by a base change matrix. We will keep $|V_1'\rangle, |V_2'\rangle, |V_3'\rangle$ and $|U'\rangle$ as basis vectors, giving us a four dimensional Hilbert space, one dimension less than the QEM model. We shall call $|V_1'\rangle, |V_2'\rangle, |V_3'\rangle$ and $|U'\rangle$ the verbatim base. To build the rotation that shifts the basis, we need to define the basis vectors of the gist base in terms of the verbatim base. This seems like an empirical question, giving us 4 orthonormal vectors to fit to the experimental data. However, we opt for a more theoretical approach in which we will build the gist base by discussing some constraints we wish to put on these 4 gist basis vectors. In this way, the CMT model is more compact as it involves less parameters. These restrictions also let us build the new model as similar to the QEM model as possible. We will only change the form of the gist vector, by making it non-orthogonal to the verbatim vectors. This allows us to investigate the role of this non-orthogonality, as any difference between the QEM model and the new model can be attributed to this change. The resulting theoretical gist base will then be tested, next to other features of CMT model in Section 2.4.3.

Most importantly, we want a gist base that gives us predictions exhibiting subadditivity, this will be verified later in this section after a particular form of the gist base is constructed. We also want to retain the idea that gist traces are represented by only one vector $|G'\rangle = v_1^g|V_1'\rangle + v_2^g|V_2'\rangle + v_3^g|V_3'\rangle + u^g|U'\rangle$, like in the QEM model. The resulting coordinates v_i^g and u^g are functions that might depend on word frequency/concreteness (4 possibilities), test word type (4 possibilities) and the verbatim traces (3 possibilities). As such, the coordinates for the gist vector needs to be, in the worst case, calculated for $4 \times 4 \times 3 = 48$ different conditions, this inflates the number of parameters in a dramatic way. This also complicates the construction and form of the relevant projectors and resulting probabilities. We will therefore explore a simple case of this construction as an exploratory first step. As we will show in Section 2.4.3, even this simplest form will result in an acceptable fit, while we still retain ample room for improvement.

To construct this simplest form, we will first take a look at the dependency of the gist vector on word concreteness/frequency. When certain words attribute differently to the gist trace due to their frequency in everyday use or their concreteness, we assume that this effect is word dependent and does not say anything about the relationship between verbatim and gist traces in general. This effect will therefore be incorporated in the state vector coordinates (and not in the form of the gist vector), as we will fit a different state vector for each of the 4 different frequency/concreteness combinations. We keep the same reasoning for the form of the gist vector depending on the type of test word: we assume that different attributions to the gist trace for different test words are word dependent (again not saying anything about the verbatim/gist relation in general) and will likewise fit a different state vector for each of the test word types. This gives us together 16 state vectors to fit to the data. The last possible dependency we will discuss is the idea that the 3 different (verbatim) word lists attribute differently to the gist trace. To construct the most simple form, we will assume that the three word lists from the experiment play a symmetrical role, as there is no experimental reason to assume there are key differences between the word lists. Therefore, we want the coordinates of $|G'\rangle$ associated with the verbatim traces to be equal, giving us $v_1^g = v_2^g = v_3^g$. We will call this the symmetry assumption. Next to this restriction, we also argue that $u^g = 0$, since unrelated distractor traces should not leave any gist traces. Keeping in mind that, since $|G'\rangle$ is a basis vector, it needs to be normalized, we get:

$$|G'\rangle = \frac{1}{\sqrt{3}}|V_1'\rangle + \frac{1}{\sqrt{3}}|V_2'\rangle + \frac{1}{\sqrt{3}}|V_3'\rangle + 0|U'\rangle. \tag{2.8}$$

This leaves us with defining the three remaining basis vectors to complete the gist base. These should all represent the participant not retrieving any gist traces. Since we won't be using these vectors in the following, we will just define them as three random orthonormal vectors $|NG_1\rangle$, $|NG_2\rangle$ and $|NG_3\rangle$, all orthogonal to $|G'\rangle$ and to each other. The observation that a participant does not exhibit any gist traces is seen as degenerate and is represented by the 3 dimensional hyperspace spanned by $\langle|NG_1\rangle, |NG_2\rangle, |NG_3\rangle\rangle = |G'\rangle^{\perp}$. This effectively gives us the most simple gist base possible, as it is parameter-free. This clearly is a rough estimation of how gist and verbatim truly relate, but will suffice as a first step and comparison with the QEM model, given the data. This also leaves the door open for more complex models, possibly with new experimental data.

Now we will combine the idea of complementary measurements, represented by the different bases, with the way the projectors are defined in the QEM model. This way we can adjust the model for the item version of episodic overdistribution from Busemeyer & Bruza (2012) to fit the data from the source version of episodic overdistribution. Note that we work with the gist base proposed previously, while the dimensionality of the gist trace subspace and the symmetry assumption still needs to be tested. When a statement of type (a), (b) or (c) is accepted, we will still project on the subspace spanned by the relevant verbatim trace vector $|V_i'\rangle$ and the gist vector $|G'\rangle$ (this in contrast with sequential projection as done in the item versions in chapter 6 of Busemeyer & Bruza (2012) and Busemeyer & Trueblood (2010)). Since we redefined the gist vector, the projectors will be different from the projectors of the QEM model. The projector associated with accepting a statement of type (a) will project the state vector on the subspace spanned by $\{|V_1'\rangle, |G'\rangle\}$, where $|G'\rangle = (1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}, 0)$. We will now derive the form of this projector using basic algebraic geometry: An orthonormal base for the plane spanned by $\{|V_1'\rangle, |G'\rangle\}$ is $\{|V_1'\rangle, \frac{1}{\sqrt{2}}|V_2'\rangle + \frac{1}{\sqrt{2}}|V_3'\rangle\}$. So, the orthogonal projection on the subspace spanned by this orthonormal base is:

$$M_1' = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \tag{2.9}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{2.10}$$

The projected state vector now does not have a unique decomposition in terms of verbatim and gist vectors. This reflects that when a participant accepts that a word was on list 1, we can not determine how much of this decision can be attributed to verbatim memory and how much to gist memory.

Similarly, the projector matrices associated with accepting statements of type (b) and (c) are equal to:

$$M'_2 = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.11}$$

$$M'_3 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{2.12}$$

We will also retain the idea from the QEM model that the projector associated with accepting a statement of type (d) is constructed by viewing this acceptance as a decline of the test word being an unrelated distractor, not on any list i, for $i = 1, 2, 3$. This means that the state vector gets projected on the orthogonal complement of $|U'\rangle$. This gives us:

$$M'_{123} = I^4 - \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.13}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{2.14}$$

With the relevant projectors now defined, we can calculate the probabilities of all possible outcomes. Using the same notation as the QEM model for these probabilities, we get for a starting vector $|S'\rangle = v'_1|V'_1\rangle + v'_2|V'_2\rangle + v'_3|V'_3\rangle + u'|U'\rangle$:

$$\hat{p}'_1 = ||M'_1|S'\rangle||^2 = (v'_1)^2 + (v'_2 + v'_3)^2/2 \tag{2.15}$$

$$\hat{p}'_2 = ||M'_2|S'\rangle||^2 = (v'_2)^2 + (v'_1 + v'_3)^2/2 \tag{2.16}$$

$$\hat{p}'_3 = ||M'_3|S'\rangle||^2 = (v'_3)^2 + (v'_1 + v'_2)^2/2 \tag{2.17}$$

$$\hat{p}'_{123} = ||M'_{123}|S'\rangle||^2 = (v'_1)^2 + (v'_2)^2 + (v'_3)^2. \tag{2.18}$$

While we argued that a complementarity approach suits the description of the human episodic memory better for interpretational reasons and the CMT has less parameters, we still need to check whether the CMT model allows for the subadditivity in its resulting probabilities (as this was the incentive for constructing these quantum models). Straightforward calculations easily give us that:

$$\hat{p}_1' + \hat{p}_2' + \hat{p}_3' = (v_1')^2 + (v_2')^2 + (v_3')^2 \quad (2.19)$$
$$+(v_2' + v_3')^2/2 + (v_1' + v_3')^2/2 + (v_1' + v_2')^2/2 \quad (2.20)$$
$$= 2(v_1')^2 + 2(v_2')^2 + 2(v_3')^2 \quad (2.21)$$
$$+v_1'v_2' + v_1'v_3' + v_1'v_2' \quad (2.22)$$
$$\geq (v_1')^2 + (v_2')^2 + (v_3')^2 \quad (2.23)$$
$$= \hat{p}_{123}', \quad (2.24)$$

showing that the CMT model exhibits subadditivity in its disjunction rule. Note that this inequality is derived under the symmetry assumption. Relaxing this assumption will make it possible to construct alternative models not expressing subadditivity. This shows that subadditivity is not a property inherently present within the complementarity approach, but merely a phenomenon that can be modeled within this approach. Subadditivity is however predicted by the symmetry assumption, as we demanded in the construction of our gist base. Situations were agents express additivity, or even superadditivity, could be modeled with a differently defined gist base. In the QEM model, the only case were subadditivity is not an inherent mathematical property, is the limit case where a subject does not express any gist traces. Here the QEM model does adhere to the classical disjunction rule.

As is well known, one of the expressions of the complementarity of properties, is order effects revealing the impact of measurements on the state of the system. In our context it means that the questions put to the participants affect their state of memory. A series of questions that only differ in the order in which they are put would yield different answers. In both Kellen et al. (2014) and Busemeyer & Bruza (2012), it is argued for the item memory case that agents first use their verbatim memory, before consulting their gist memory. So, it is reasonable to predict likewise behavior for our source memory version. This suggests an experiment that could distinguish the CMT, naturally modeling order effects, from the QEM, not naturally modeling order effects, by generating different predictions which could be confronted with the actual behavior. This experiment would be a variation on the one described before, but where, in a subset of the sample, a question only involving gist memory is posed before the actual list recollection is performed. This gist measurement manipulation might, for example (with the idea taken from Stahl & Klauer (2008)), be along the lines of confronting the participant with a test word of which they need to decide if it relates to a studied word. These actions are assumed to rely on gist memory, again, see Stahl & Klauer (2008). The CMT model predicts different outcomes in this subset in comparison with the rest of the sample. Detecting these differences would be a formal comparison of the CMT and QEM, next to the statistical one we perform in the next section. Conducting new experiments, however, falls outside of the scope of this paper.

2.4.3 Results and Discussion

We will now fit the data of the word list experiment, summarized in table 1, to the CMT model. These are the same data to which the QEM model was fitted in Brainerd et al. (2013), allowing for a comparison between the fit of the QEM model and that of the CMT model.

As was done in Section 3.2 for the QEM model, we will compare the CMT model to a saturated model. If the CMT does not significantly differ from a saturated model, it can be considered to have a good fit. We calculated a G^2 statistic, representing the difference between the proportions calculated from the data and the proportions estimated from the model, for each of the 16 experimental conditions: 4 types of test words for each of the 4 word types. Each of the experimental conditions had 4 possible statements, leading to a total of 64 obtained probabilities of accepting the presented statement. Therefore the saturated model has 4 degrees of freedom for each of the possible experimental conditions, leading to 64 degrees of freedom in total. As in the QEM model, we will consider the coordinates v'_{nt} associated with verbatim traces of lists not containing the test word as equal. The CMT model estimates 2 parameters, v'_t and v'_{nt} for the experimental conditions where a non-distractor word was presented, leading, e.g., to a state vector of the form $|S'\rangle = v'_t|V'_1\rangle + v'_{nt}|V'_2\rangle + v'_{nt}|V'_3\rangle + u'|U'\rangle$, in the experimental conditions where a list 1 test word was presented. Since the state vector has to be normalized, we have $u' = \sqrt{1' - (v')^2_t + 2(v')^2_{nt}}$.

Since we don't have a word list playing a special role in the experimental conditions where an unrelated distractor is presented, we lose the v'_t parameter, leaving only 1 parameter v'_{nt} to be estimated in the state vector

$|S'\rangle = v'_{nt}|V'_1\rangle + v'_{nt}|V_2\rangle + v'_{nt}|V'_3\rangle + u'|U'\rangle$. We now have $u' = \sqrt{1-3(v')^2_{nt}}$ to normalize $|S'\rangle$. This leads to a total of 28 parameters to be estimated in the CMT model. These parameters were estimated by minimizing the G^2 statistic using R.

The calculated G^2 statistic is:

$$G^2 = 2\left(m \times n \sum_{i=1}^{4}\left(O_i \ln \frac{O_i}{E_i} + (1-O_i)\ln \frac{1-O_i}{1-E_i}\right)\right). \tag{2.25}$$

With m being the number of observations per participant in each experimental condition and n being the number of participants. Here, the experiment consisted, for each experimental condition, of n= 70 participants each accepting or rejecting $m = 2$ statements, with O_i the observed proportion of accepted statements of type (a), (b), (c) and (d) for respectively $i = 1, 2, 3$ and 4; and E_i the estimated proportion by the CMT model of accepted statements of type (a), (b), (c) and (d) for respectively $i = 1, 2, 3$ and 4. The critical value for the three experimental conditions where target words are presented is 5.99 (χ^2 distribution, d.f.=2) at the $\alpha = .05$ level. For each $G^2 < 5.99$, there is no significant difference in prediction between the CMT model and the saturated model, with perfect prediction. For the experimental conditions where an unrelated distractor is presented, we have a critical value of 7.81 (χ^2 distribution, d.f.=3) at the $\alpha = .05$ level. Table 2.3 shows that in three experimental conditions we see a significant difference from the saturated model, the same amount as the QEM model. The total G^2 statistic, summed across all experimental conditions, is 50.4029, which is just smaller than the critical value 51 (with 64-28 = 36 degrees of freedom) at the $\alpha = .05$ level, with the p-value equal to .06, showing an acceptable fit.

The fact that the CMT model fits the same number of experimental conditions well as the QEM model (both differ significantly from a saturated model in three experimental conditions), but its overall fit is slightly worse, can be attributed to one very problematic experimental condition (list 3 test word, high frequency and concrete words). The G^2 statistic in this condition (14.2251) inflates the resulting G^2 statistic dramatically. Leaving out only this experimental condition gives us a total G^2 statistic of 36.1778 (critical value is 48.062, df=34) with a p-value of 0.36. As the fit of this condition so vastly differs from the other conditions, we can suspect this experimental condition to be an anomaly within the data. Note that this statistical analysis is done under the symmetry assump-

Table 2.3 Estimated parameters and G^2 statistics of the CMT model

	High-frequency/ Concrete	High-frequency/ Abstract	Low-frequency/ Concrete	Low-frequency/ Abstract
List 1 Test Word				
v'_t	0.7181	0.7052	0.7508	0.6408
v'_{nt}	0.1087	0.1320	0.1591	0.3246
G^2	1.1256	0.1445	1.2536	4.2925
List 2 Test Word				
v'_t	0.6274	0.7613	0.6662	0.7249
v'_{nt}	0.1569	0.0695	0.1250	0.1517
G^2	10.2486*	3.5036	0.3880	3.0504
List 3 Test Word				
v'_t	0.6858	0.6363	0.7113	0.6113
v'_{nt}	0.1421	0.2222	0.1072	0.2989
G^2	14.2251*	1.0675	1.422	7.5834*
Unrelated Distractor				
v'_{nt}	0.2496	0.2880	0.1993	0.2520
G^2	1.5382	0.0874	0.1475	0.3250

* Significant deviation at the $\alpha = 0.05$ level (critical value is 5.99, df =2), for list i target words.

tion, with the gist memory vector defined as $|G'\rangle = \frac{1}{\sqrt{3}}(|V_1'\rangle + |V_2'\rangle + |V_3'\rangle)$. Relaxing this assumption might improve the fit even more. To formally compare the QEM and CMT model, with both having a different number of parameters, we calculated the Bayesian information criterion (BIC) for both models. The BIC adds a penalty term for the number of parameters to the G^2 statistic. The resulting BIC for the QEM model is 244.17, while the BIC for the CMT model is 188.72. The CMT model is clearly favored.

2.5 Comparison and Conclusion

The QEM model proved to be an interesting and promising foray into the use of the quantum formalism, when modeling human episodic memory. We attempted to improve these findings by borrowing additional insights and techniques from quantum mechanics into this field. As such we retained the idea of the human episodic source memory to consist of two parallel distinct memory traces, as illustrated by an experiment concerning memorized words. The first, called the verbatim trace, encompasses the lexical and phonological components of these memorized words. The second, called the gist trace, encompasses semantic features of these memorized words. This approach stems from Fuzzy Trace Theory which posits that people form two types of mental representations about a past event. We also take from the QEM model the notion that a memory measurement is an intrusive act, influencing the agent. This makes the quantum formalism a prime candidate for this paradigm, as this role of measurement is the defining difference between quantum and other models, leading to the superposition principle.

However, we have two major issues with the QEM model. Firstly, there is a simple classical equivalent, as shown in Denolf (2015). As such, it seems that the QEM model does not fully utilize the possibilities that a quantum model offers. Next to this, we also argue that the structure of the relationship between verbatim and gist traces, each being represented by different basis vectors of the same base, does not accurately represent their relationship within the discussed experiment. As both traces are represented by vectors within the same base, it seems as they can not be activated at the same time. This way, expressing verbatim traces automatically leads to an impossibility of expressing gist traces. This is in contrast with the notion that gist and verbatim are parallel traces, both possibly expressed when an agent recollects a memorized word.

Considering both traces as complementary and representing these as different bases within the same Hilbert Space, resulted in a model which expresses the complex relationship between traces in a, to us, more elegant way. This CMT model kept the method of how the different memory traces were represented in the QEM model. Verbatim traces were still represented by one vector for each of the memorized lists and by one vector for unrelated distractors. Gist traces were still represented by one vector. This way, an agent can express both types of traces, while the expression of one of the two traces will influence the other. This leads to a view where both traces are present but can not be measured at the same time, as the measurement of one trace influences the other trace. This idea, together with the fact that both traces are needed for a full description of the memory state of the agent, fits perfectly the concept of complementarity. The CMT model shows, next to these interpretative arguments, the ability to model the subadditivity the human episodic memory exhibits. A formal fit to the experimental data shows that the CMT model does not predict significantly worse than the saturated model, while having 28 degrees of freedom, i.e., 16 less than the 44 degrees of freedom of the QEM model. The ability of the CMT model to model order effects, showcases its non-classicality. These order effects might prove an interesting subject for future research, as similar item versions of human episodic memory do incorporate these order effects.

Next to the these order effects, further research might shed light on the role of the symmetry assumption within the CMT model. As this is the most simple form within the complementarity approach, relaxing this assumption might lead to a better statistical (at the cost of more parameters) fit, more insight in human memory by investigating and interpreting different forms of the gist memory vector and applications of this approach to other datasets.

As these improvements were realized by just making one vector non-orthogonal to the other relevant vectors, this complementarity approach to human memory seems to more fully incorporate the distinct features of quantum techniques. As such, we believe that complementarity might prove successful in a vast array of domains within cognition and makes this notion of complementary measurements one of the main advantages of using the quantum formalism. Specifically other applications of the Fuzzy-Trace Theory, which is also used in, e.g., decision theory and the modeling of beliefs, seem prime candidates for this approach, as the idea of two types of mental representations seems to fit the notion of complementary measurements well.

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3 A Quantum-like Model for Complementarity of Preferences and Beliefs in Dilemma Games

This chapter will be published in Journal of Mathematical Psychology. Denolf, J., Martínez-Martínez, I., Josephy, H., and Barque-Duran, A. (in press). A quantum-like model for complementarity of preferences and beliefs in dilemma games. *Journal of Mathematical Psychology.* Special Issue on Quantum Probability.

Abstract. We propose a formal model to explain the mutual influence between observed behavior and subjects' elicited beliefs in an experimental sequential prisoner's dilemma. Three channels of interaction can be identified in the data set and we argue that two of these effects have a non-classical nature as shown, for example, by a violation of the sure-thing principle. Our model explains the three effects by assuming preferences and beliefs in the game to be complementary. We employ non-orthogonal subspaces of beliefs in line with the literature on positive-operator valued measure. Statistical fit of the model reveals successful predictions.

3.1 Introduction

During the recent decade, there is an increasing interest in decision-making and cognitive models that employ a quantum probabilistic (*QP*) framework. In fact, the application of quantum-like concepts to portray human information processing was considered since the early development of quantum mechanics. For example, Bohr (1948) defended the idea that some aspects of quantum theory could provide an understanding of cognitive processes but never provided a formal cognitive model in light of a QP hypothesis. The so called quantum cognitive theories have only begun to emerge as of late. This development encompasses publications in major journals (Deutsch, 1999; Pothos & Busemeyer, 2013; Wang et al., 2014; Yearsley & Pothos, 2014), special issues, and dedicated workshops, as well as several comprehensive books (Busemeyer & Bruza, 2012; Khrennikov, 2010; Haven & Khrennikov, 2013).

QP is defined as the set of mathematical rules used to assign probabilities to events from quantum mechanics (Hughes, 1989; Isham, 1989), but without any of the physics. As it is derived from a different sets of axioms than classical probability theory, it is subject to alternative constraints and has the potential to be relevant in any area of science where a need to formalize uncertainty arises. Since encoding uncertainty is a major aspect of cognitive functions in psychology, QP shows potential for cognitive modeling. These studies are not about the use of quantum physics in brain physiology, which is a disputable issue (Litt et al., 2006; Hameroff, 2007) about which we are skeptical. Rather, we are interested in QP theory as a mathematical framework for cognitive modeling.

Applications of QP theory have been presented in decision-making (White et al., 2014; Busemeyer et al., 2006, 2011; Bordley, 1998; Lambert-Mogiliansky et al., 2009; Pothos & Busemeyer, 2009; Trueblood & Busemeyer, 2011; Yukalov & Sornette, 2011), conceptual combination (Aerts, 2009; Aerts & Gabora, 2005; Blutner, 2009), memory (Bruza, 2010; Bruza et al., 2009), and perception (Atmanspacher et al., 2004). For a detailed study on the potential use of quantum modeling in cognition, see Busemeyer & Bruza (2012) and Pothos & Busemeyer (2013). The majority of models presented in the quantum cognition literature addresses standard aspects of decision-making processes: similarity judgments (Barque-Duran et al., 2016; Pothos et al., 2015; Yearsley et al., 2014), the constructive role of articulating impressions (White et al., 2014, 2016), and order effects in belief updating (Trueblood & Busemeyer, 2011) among numerous other applications.

Little literature has focused on strategic decision-making or game theory. Whenever two or more agents interact, one agent is not only reacting to the information that he receives, but is likewise generating information towards other players. These strategic environments are unique in relation to standard decision-making scenarios under uncertainty, since every agent needs to reason on two parts of the problem: his own actions and his expectations on the opponent's actions. Few studies applying QP instruments to model the way agents process the information in a game have been published with regards to this particular matter (Pothos & Busemeyer, 2009; Pothos et al., 2011; Busemeyer & Pothos, 2012; Martínez-Martínez & Sánchez-Burillo, 2016). Other approaches in which the quantumness enters through an extension of the classical space of strategies and/or signals have also been discussed, e.g., by La Mura (2005), Brandenburger (2005), and Brunner & Linden (2013); as well as a model to analyze games with agents exhibiting contextual preferences (Lambert-Mogiliansky & Martínez-Martínez, 2015).

In this paper, we describe the application of QP theory to modeling the mutual influence between preferences and beliefs in sequential social dilemmas. This idea was first explored in Martínez-Martínez et al. (2015). We present a quantum-like model for preferences and beliefs (QP&B) that replicates the experimental results from Blanco et al. (2014) while providing a novel theoretical approach on cognitive dynamics in strategic interactions. Our model asserts that the relationship between a player's beliefs and his preferences is inherently non-classical and continues the work done in Pothos & Busemeyer (2009) exploiting the ideas of measurement utilized in quantum theory. We redefine these two properties as complementary. In that capacity, they cannot be measured at the same time, as the act of measuring one property alters the state of the other property. The non-classical nature of such a relationship and its application in cognition has already been discussed in, e.g., Denolf & Lambert-Mogiliansky (2016).

3.2 Experimental Design

The data set that our QP&B model deals with is provided by Blanco et al. (2014). Their experiment was designed for explicitly testing different channels through which preferences and beliefs of an agent immersed in a social dilemma may influence each other. As the authors motivate, this experimental evidence is novel and its main interest stems from the fact that previous analyses of strategic interactions considered preferences and beliefs to be independent. This fact implies that the choice of actions in environments with uncertainty can be rationalized as just a best-response to some particular form of belief about the possible states of the world or about the action that is expected to be played by an opponent.

3.2.1 Standard Version of the Prisoner's Dilemma Game

The symmetric prisoner's dilemma game is a game involving two players, player I and player II, who can choose among two actions: cooperate (C) or defect (D). The normal form of this game is defined by the following 2×2 payoff matrix

Player II

		C	D
Player I	C	(π_c, π_c)	(π_b, π_a)
	D	(π_a, π_b)	(π_d, π_d)

(3.1)

where the payoff entries satisfy the inequalities $\pi_a > \pi_c > \pi_d > \pi_b$.

The scheme of possible results of payoffs is as follows. If player I decides to cooperate, I can receive the second best possible outcome if the opponent II also cooperates, but I's attempt to cooperate is exposed to being exploited by II if II decides to defect. In the latter scenario, II would collect the best outcome of value π_a while leaving I with the lowest payoff π_b. If player I decides to defect, then this player is guaranteed not to obtain the lowest payoff, but at least an amount π_d if player II defects as well. If player II decided to cooperate, then I is taking advantage of the situation and obtaining the maximum benefit π_a.

Technically, we say that mutual defection is the Nash equilibrium of this game because there is no unilateral deviation that could make the deviating player earn more, while mutual cooperation is the Pareto optimal situation. Therefore, this game represents a social dilemma for the players: the individual choice of defection dominates the attempt to cooperate for any given choice of the opponent, which is not socially optimal. Why is this a dilemma? Because this game formalizes a conflict between the individual (the Nash equilibrium) and the collective (Pareto optimal) level of reasoning: if both players actually choose to defect, both of them generate a total payoff of $2 \times \pi_d$, which is by definition lower than the aggregate payoff if both of them coordinated in full cooperation, $2 \times \pi_c$.

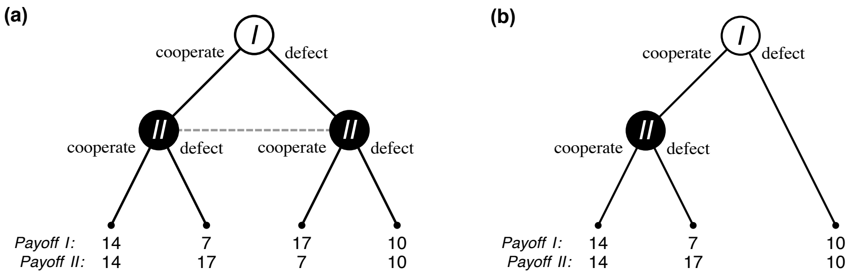

Figure 3.1 (a) Standard (simultaneous) Prisoner's Dilemma. (b) Sequential Prisoner's Dilemma.

The standard version of the prisoner's dilemma game is a one-shot strategic interaction with simultaneous moves by the opponents. This implies that both players make their own individual decision (whether to cooperate or not) without knowing what the opponent is choosing. Once both players have chosen their strategy, both actions become public and the payoffs are generated.

Each player reacts to his own belief or expectation on the opponent's intention, and as a consequence, the preferred action in the dilemma crucially depends on the way players form their beliefs about the opponent moves. Therefore, it is important to understand how beliefs and preferences do (or do not) influence each other in this decision-making process.[1]

3.2.2 Sequential Prisoner's Dilemma

The experiment conducted by Blanco et al. (2014) focuses on a variation of the Prisoner's Dilemma game discussed above: a sequential one. In Fig. 3.1 we show the game tree of the game played in this sequential experiment (b), and compare it to its standard (simultaneous) counterpart with equivalent payoffs (a). In the sequential version, the solution concept required is the Subgame Perfect Nash Equilibrium (SPNE), a usual refinement of the Nash Equilibrium (NE) when turning to sequential games. Solving by backwards induction, we see that it is in the best interest of Player *II* to defect if given the chance to move, which would leave Player *I* with a payoff of 7, and therefore *I* should choose defect at the beginning of the tree, because 10 is a better outcome. Thus, the sequential game

[1] See Blanco et al. (2014, Section 1) about possible correlations between preferences and beliefs in dilemmas with models of social preferences such as inequality aversion and reciprocal preferences.

Treatment	*Baseline*	*Elicit_Beliefs*	*True_Distribution*
Task 1	2nd move (*II*)	2nd move (*II*)	2nd move (*II*)
Feedback on *II*	No	No	**Yes**
Task 2	1st move (*I*)	**beliefs** (about *II*)	1st move (*I*)
Task 3	beliefs (about *I*)	1st move (*I*)	beliefs (about *I*)
# Participants	40	60	60

Table 3.1 Experimental treatments in Blanco et al. (2014, Table 1).

maintains the content of the social dilemma because the SPNE implies that both players' incentives drive them towards mutual defection, even though they could obtain a higher social payoff if they coordinated on full cooperation.

On the one hand, one can see how in the sequential variation, only the player *I* is bearing the risk of her cooperative choice being exploited by a selfish decision of player *II*. In order to restore the symmetry between the players, all participants in the experiment play the game twice. Once in role I and once in role II. After all decisions have been made, the players are randomly matched into pairs, with the assignment of roles being random as well. Subsequently, they earn the payoffs determined by the relevant decisions, given their roles.

On the other hand, this procedural 'complication' is a small price to pay if we compare it to the advantages it provides: because of the sequential structure in the decision-making, each choice can be observed (measured) at a time. The authors design three treatments that intersperse a belief-elicitation task with the choices of actions.[2] As we discuss now, the treatments differ in the order in which each task is performed and this allows to measure different correlations between actions (which are supposed to proxy the preferences of the players) and beliefs. We now briefly explain the three different treatments, which are also summarized in Table 3.1.

3.2.3 Experimental Treatments

Ten subjects participate in each session. For each of the following treatments, several sessions were conducted. The total numbers of participants are displayed in Table 3.1.

[2]In the belief-elicitation task, the players were asked how many of the other participants (potential rivals for the play of the game) cooperate in the role of Player II. This task is incentivized with a quadratic scoring rule rewarding the accuracy of the stated beliefs: players earn more the closer their prediction is to the actual rivals' cooperation rate (Blanco et al., 2014, Equation 3).

Treatment	*Baseline*	*Elicit_Beliefs*	*True_Distribution*	*Total*
First mover (Player *I*)	27.5%	55.0%	56.7%	48.8%
Second mover (Player *II*)	55.0%	53.3%	55.0%	54.4%

Table 3.2 Average cooperation rates by treatment in the experiment by Blanco et al. (2014), also labeled as Table 2 in their original paper.

Baseline. This treatment can be considered as a mere control group, such that the subjects play the game in its natural structure, with no attention paid to observing their beliefs. The players first choose what their action *II* will be and no information is revealed to them so that the participants' beliefs are not exogenously influenced. Subsequently, they choose what their action for the role of I will be, and finally they are given a meaningless question about their beliefs on the global rate of cooperation in the group of first movers. The informational gain of this last task is void because its only use is to balance the different treatments making their length comparable (both in time and the number of tasks).
Elicit_Beliefs. In this treatment, the players first choose what their action *II* will be, and then they have to reveal their belief about the rate of cooperation that they will receive from the second movers. Finally, they have to choose their action *I*. Thus, this treatment introduces a belief-measurement between the two choices of actions. This allows us to explore the effect of a measurement of the beliefs about the move by opponent *II* on the choice of action *I*.
True_Distribution. This treatment presents a somewhat 'similar' sequence of tasks for the players compared to the previous treatment *Elicit_Beliefs*. The players begin by choosing their action *II*. Then, they are told what the true cooperation rate for action *II* was in their group. They finish by choosing the action *I*. This treatment differs from the previous one in that this time, the forecast of the opponents' move is not a belief generated by the players themselves, but true information being released to them exogenously.

3.3 Aggregate Behavior and Basic Modeling

Table 3.2 presents the aggregate results of the three experimental treatments. First off, we cannot observe any significant difference in the cooperation rates as a second mover between treatments. This is to be expected as the question (measurement) regarding the choice of action in the role of player II is identical in all aspects over all treatments.[3] The small variation in the proportion of cooperation reported for the *Elicit_Beliefs* treatment (53.3% vs. 55% in the others) can be attributed to sample variance.

The cooperation rates in the role of first mover (player I) show meaningful differences. A chi square test across all three treatments yields a p-value of 0.007886 ($\chi^2 = 9.6853$, df=2). Starting with the first move cooperation rates of the *Baseline* treatment (27.5%) and the *Elicit_Beliefs* treatment (55.0%), the null hypothesis of no difference between these two proportions yields a p-value of 0.007 ($\chi^2 = 7.3661$, df=1), clearly indicating a significant difference. There is only one procedural variation between these two treatments: *Elicit_Beliefs* includes the elicitation of beliefs about the cooperation rate expected from the rivals *II* before the agents choose their action in the role of I. Thus, we can attribute the difference in the player *I* cooperation rate to the effect that measuring a subject's beliefs about the opponent *II* may have on his attitude toward the actions as first mover.

A similar result can be found for the first move cooperation rates of the *Baseline* treatment (27.5%) and the *True_Distribution* treatment (56.7%). The null hypothesis claiming no difference between these two proportions can be rejected, as it gives us a p-value of 0.004 ($\chi^2 = 8.2674$, df=1). For the first move cooperation rates (role I) of the *Elicit_Beliefs* treatment (55.0%) and the *True_Distribution* treatment (56.7%), the null hypothesis of no difference between these proportions yields a p-value of 0.85 ($\chi^2 = 0.0351$, df=1), indicating no significant difference between the result in the two treatments. In this sense, the incentivized elicitation of beliefs impacts the state of the subjects participating in the experiment similarly to an update of beliefs via the acquisition of true information revealed exogenously.

3.3.1 Violation of the Sure-Thing Principle

The differences in first move cooperation rates reveal the presence of a violation of the sure-thing principle in the data, as

$$27.5\% = p(C_I) \neq \sum_i p(C_I|B_i) = 55\%,$$

[3]Note especially that it is the first measurement performed in all treatments and therefore, it is not subject to the effects targeted by this experimental design.

with C_I the event of the player cooperating on the first move and B_i the event of the player answering that he thinks i opponents cooperate during the belief elicitation. This in turn points out the interest in using a quantum-like model to describe the behavior of the participants in this experiment, since classical statistics cannot account for them in a simple manner, while quantum-like easily do.

3.3.2 The Simplest Quantum-like Model

In the remaining of Section 3, we illustrate the basic mechanics of quantum-like toy models designed to address the issue of measurement as well as construct different building blocks that will be fully developed later. As the reader will see, Section 3.4 integrates them in a unified model. Now, we only show which aspects of quantum-like modeling can account for the empirical effects observed in the dataset, without taking into account how they correlate to form the proper model.

We introduce the most basic quantum-like model to represent concepts such as actions, preferences and beliefs in quantum-like terms (observables, measurements and orthonormal basis of their outcomes) and use projective measurements (with their resulting probabilities) to explain the first results observed in the data from Blanco et al. (2014). We consider the preferences of an agent as the individual's attitude toward the different elements of a set of outcomes, to be reflected in the choices observed along the sequence of decisions (Lichtenstein & Slovic, 2006). In this case, and because of the strategic nature of this decision-making process, the outcomes (possible payoffs to be obtained) depend on the actions (cooperate or defect) a players chooses, but also on the choices made by a rival.

The actions of a player can be represented by two orthogonal vectors $|C\rangle$ (for cooperation) and $|D\rangle$ (for defection). The two vectors form an orthonormal basis and span the Hilbert space $\mathcal{H}_i \equiv \mathbb{R}^2$, with $i \in \{I, II\}$ denoting the role in the game as player I or II for which such action is chosen.[4] The player is considered to be in a superposition over these

[4]For the finite dimensional case, a Hilbert space $\mathcal{H}$ is a linear space endowed with a scalar product $\langle\psi_1|\psi_2\rangle \in \mathbb{R}$. Its elements (or states) are denoted by $|\psi\rangle \in \mathcal{H}$. If the state of the system is $|\psi\rangle$ we say it is in a *pure state*. The projector $P_\psi = |\psi\rangle\langle\psi|$, an operator acting on $\mathcal{H}$ as $P_\psi|\phi\rangle = \langle\psi|\phi\rangle|\psi\rangle$, has a bijective relation with $|\psi\rangle$, and we can describe the state $|\psi\rangle$ in terms of P_ψ. Any element or vector of the space of states is called a *ket*-vector and represented by $|\cdot\rangle$, and we have the dual space of the *bra*-vectors, symbolized by $\langle\cdot|$. Hilbert spaces are generally defined over the field of complex numbers, but in this paper it is enough to work only with reals. Note that

actions, being represented by a normalized state vector $|S\rangle$. The projection of the state vector onto the elements of the orthonormal basis defines the probability that the player chooses each of the actions, as a proxy of her preferences.

We consider the beliefs as the subjective distribution with which the agents judge the likelihood of realization of each possible relevant state of the world. The possible states in this setting concern the possible cooperation of opponents, as this, together with one's own actions, determines the outcome of the game. These beliefs are also represented by a set of mutually orthogonal vectors $\{|B_j\rangle\}$, with the index j running from 0 to 9. This j represents how many of the opponents (maximum 9) are believed to cooperate. This orthonormal basis also spans a Hilbert space, $\mathcal{H}_B$, with the player's beliefs being represented by a normalized state vector: a superposition over the orthonormal basis of beliefs. Straightforwardly, $j/9$ is the expected share of cooperation among the opponents, and $1-j/9$ is the expected rate of defection.

3.3.3 Projective Measurement

Quantum-like models use projective measurements to represent measurements being performed on the system of interest.[5] Here, we apply this to model the observed behavior in the choice of action as player II in the data from Blanco et al. (2014). The state of the player is represent by a normalized state vector $|S_{II}\rangle$ in the two-dimensional Hilbert space $\mathcal{H}_{II}$:

$$|S_{II}\rangle = c_{II}|C_{II}\rangle + d_{II}|D_{II}\rangle. \tag{3.2}$$

given a state $|\psi\rangle$ associated to a vector $\psi \in \mathbb{R}^N$, we obtain $\langle\psi|$ associated to ψ^T, where ^{T} is the operation of vector transposition. The name of *bra-ket* (or Dirac's) notation comes from splitting the *bracket* $\langle\cdot|\cdot\rangle$ representing the scalar product, which is the crucial operation to compute probabilities in this framework.

[5]The probability of observing an outcome is calculated as the square of the norm of the projection of the state vector onto the subspace spanned by the vectors representing the outcome. When the outcome is represented by only one vector (simplest case), this calculation reduces to the square of the inner product of the state vector and the outcome vector. The act of measurement changes the state vector of the system from an initial state to a post-measurement state, by projecting (and normalizing) the state vector onto the subspace spanned by the outcome vectors. Projective measurements deal naturally with incompatible measurements, and note also that when they are performed on a density matrix diagonal in a particular basis, they are equivalent to Bayesian updates.

The probability $p(C_{II})$ of the player choosing to cooperate is therefore:

$$p(C_{II}) = ||P_{C_{II}}|S_{II}\rangle||^2 = \langle C_{II}|S_{II}\rangle^2 = c_{II}^2, \tag{3.3}$$

with $P_{C_{II}} = |C_{II}\rangle\langle C_{II}| = \text{diag}(1,0)$ the projector on $|C_{II}\rangle$. This outcome would project the state vector unto its post-measurement state $|S'_{II}\rangle = |C_{II}\rangle$. The probability of the player defecting as second mover is:

$$p(D_{II}) = ||P_{D_{II}}|S_{II}\rangle||^2 = \langle D_{II}|S_{II}\rangle^2 = d_{II}^2, \tag{3.4}$$

with $P_{D_{II}} = |D_{II}\rangle\langle D_{II}| = \text{diag}(0,1)$ the projector on $|D_{II}\rangle$. This outcome would likewise project the state vector unto its post-measurement state $|S'_{II}\rangle = |D_{II}\rangle$. The normalization restriction on the state vector implies that total probabilities add up to one, $c_{II}^2 + d_{II}^2 = 1$. From the cooperation rates as player *II* reported in Table 3.2, we can estimate these through our sample as:

$$\hat{c}_{II}^2 = 0.544 \text{ and } \hat{d}_{II}^2 = 0.456. \tag{3.5}$$

Note that we estimate by taking the average cooperation rates across the treatments, because we have justified above that they are not significantly different from one another.

We can model the choice of the players for their action as player *I* in the *Baseline* condition in a Hilbert space $\mathcal{H}_I \equiv \mathbb{R}^2$, with the basis $\{|C_I\rangle, |D_I\rangle\}$. The state vector is now

$$|S_I\rangle = c_I|C_I\rangle + d_I|D_I\rangle, \tag{3.6}$$

and we can infer from the data (Table 3.2, column 1) that

$$\hat{c}_I^2 = 0.275, \text{ and } \hat{d}_I^2 = 0.725. \tag{3.7}$$

In this case, we only consider the cooperation and defection rates in the *Baseline* treatment. Because of the significant difference in the cooperation rate as player *I* across treatments, considering the average is not sensible (see discussion in Section 3.3).

Finally, we model the beliefs of the players in the Hilbert space $\mathcal{H}_B$, (spanned by $\{|B_j\rangle\}$). The normalized state vector is

$$|S_B\rangle = \sum_{j=0}^{9} b_j|B_j\rangle. \tag{3.8}$$

#Cooperators (Belief)	0	1	2	3	4	5	6	7	8	9
Abs. frequency (out of 60 subjects)	5	2	5	5	12	9	9	6	4	3

Table 3.3 Number of players in treatment *Elicit_Beliefs* expecting each possible number of cooperators in their session.

From the data regarding the *Elicit_Beliefs* treatment (see Table 3.3), we get that

$$\begin{aligned} \hat{b}_0^2 = 5/60,\ \hat{b}_1^2 = 2/60,\ \hat{b}_2^2 = 5/60,\ \hat{b}_3^2 = 5/60,\ \hat{b}_4^2 = 12/60, \\ \hat{b}_5^2 = 9/60,\ \hat{b}_6^2 = 9/60,\ \hat{b}_7^2 = 6/60,\ \hat{b}_8^2 = 4/60,\ \hat{b}_9^2 = 3/60. \end{aligned} \tag{3.9}$$

3.4 Building Blocks

3.4.1 Three Effects

Effect 1 (Consensus effect). Proof of and an extensive commentary on the presence of this effect are presented in Blanco et al. (2014) where it is shown that players' beliefs are biased towards their own actions. As such, a player who cooperates as second mover will expect a higher second-mover cooperation rate amongst the other players. A visualization of this effect can be found in Fig. 3.2. Viewing this in light of the performed measurements, the consensus effect denotes the influence of second mover action measurements on the beliefs of the same participant.

Effect 2 (Reasoned player). The second effect is the influence that belief measurements have on action measurements. As these actions are driven by one's preferences, this effect encompasses the influence of the belief measurements on the preferences of the same player. We claim that the act of eliciting the beliefs of the player fundamentally changes this player even when disregarding the exact outcome of this belief measurement. When the player is asked to form an opinion about the cooperation rate of his opponents, this changes him into a more reasoned state about the opponent, in opposition to a more intuitive state when not explicitly asked to form this opinion. In the data, this can be viewed in the violation of the sure-thing principle discussed in Section 3.3.1. The average first move cooperation rate of players, after forming explicitly their beliefs about the cooperation of the opponent (*Elicit_Beliefs*), is twice as large

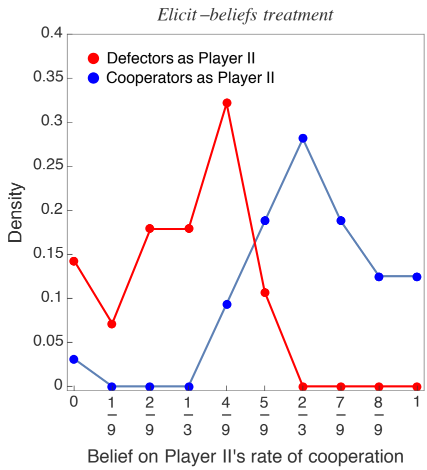

Figure 3.2 Second move defecting players (red line) believe that less opponents will cooperate. Second move cooperating players (blue line) believe more opponents will cooperate. The second move action was measured before the beliefs.

as the average first move cooperation rate of players, in which beliefs were not elicited (*Baseline*) (see Table 3.2). Nevertheless, this cooperation rate in the *Elicit_Beliefs* group is not differing significantly from the cooperation rate in the *True_Distribution* group. In this group, participants received full information about the cooperation rate of the opponents and are therefore assumed to make a more deliberate decision. Since these cooperation rates are similar, we can assume that players are in a similar reasoned state in the *Elicit_Beliefs* group.

Effect 3 (Classical correlation). The third effect we discuss is the correlation between a player's first and second move. This is observed in all three conditions, as noted in Results 1, 2 and 3 from Blanco et al. (2014). That is, first move cooperators are likely to also cooperate on the second move and vice versa. We concur with Blanco et al. that this correlation is exhibited mostly through an indirect belief-based channel. This way, we attempt to include the observed correlation as a logical consequence of our previously described effects. The second move action measurement influences the first move action measurement through a player's beliefs. We assume this correlation to be classical in nature, as opposed to the two other effects.

3.4.2 Compatible and Incompatible Measurements

Roughly speaking, two measurements M_1 and M_2 are considered incompatible if the order in which the measurements are done changes the outcome, as the act of performing one measurement influences the other measurements regardless of the outcome. Mathematically speaking, this means that one or more projector matrices associated with outcomes of measurement M_1 do not commute with one or more projector matrices associated with outcomes of measurement M_2. If two measurements are maximally incompatible, no projector matrix associated with an outcome of measurement M_1 commutes with a projector matrix associated with an outcome of measurement M_2, and they are called complementary. As such, both measurements M_1 and M_2 cannot be performed together, as the *act* of performing one of the measurements (without specifying its outcome), influences the other measurement. These concepts elegantly deal with situations where violations of the sure-thing principle emerge.

We will consider the belief elicitation to be complementary with the action measurements, as this explains both the consensus effect and the reasoned player effect. This approach should not come as a surprise. First, using complementarity as an explanation for the consensus effect is argued in Busemeyer & Pothos (2012) where the consensus effect is seen as a form of social projection. Second, the idea of the player being more reasoned can be seen as a violation of the sure-thing principle. These violations are a prime indicator of measurements not commuting which is the definition of incompatible measurements. We will now show how the projective measurement formalism deals with our hypothetically compatible (first and second move actions) and incompatible (actions and beliefs) measurements.

When two measurements are considered compatible, the Hilbert spaces representing the outcomes of these measurement can be tensored to construct a larger Hilbert space spanned by vectors that now represent joint outcomes. As argued before, we consider the first move action and second move action to be compatible, as they are considered to be measurable at the same time. Therefore, the Hilbert space which models the relationship between both is $\mathcal{H}_I \otimes \mathcal{H}_{II}$, spanned by $\{|CC\rangle, |CD\rangle, |DC\rangle, |DD\rangle\}$, with $|CD\rangle = |C_I\rangle \otimes |D_{II}\rangle$ (other vectors defined similarly). The player is represented by a normalized state vector:

$$|S\rangle = s_{CC}|CC\rangle + s_{CD}|CD\rangle + s_{DC}|DC\rangle + s_{DD}|DD\rangle. \qquad (3.10)$$

We now provide two examples of how probabilities are calculated within this Hilbert space. The other relevant probabilities are calculated in a similar way. The projector and probability associated with a player defecting on the role of I, but cooperating on the role of II is

$$P_{DC} = P_{D_I} \otimes P_{C_{II}} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{3.11}$$

so

$$p(DC) = ||P_{DC}|S\rangle||^2 = s_{DC}^2. \tag{3.12}$$

The projector and probability associated with the player cooperating on the second move (without specifying a choice as player I), are:

$$P_{.C} = I^2 \otimes P_{C_{II}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{3.13}$$

and

$$p(.C) = ||P_{.C}|S\rangle||^2 = s_{CC}^2 + s_{DC}^2. \tag{3.14}$$

Directly from the data (for the *Baseline* treatment), we derive

$$\hat{s}_{CC}^2 = 0.25,\ \hat{s}_{CD}^2 = 0.025,\ \hat{s}_{DC}^2 = 0.3, \text{ and } \hat{s}_{DD}^2 = 0.425. \tag{3.15}$$

This models the (classical) correlation between first and second move, as noted above in Effect 3.

Incompatible measurements are represented by different bases in the same Hilbert space (as opposed to one tensored basis for compatible measurements). To model the relationship between the choice of action in the role of player I and the beliefs that a player holds, we could use a Hilbert space $\mathcal{H}_{\mathrm{I},B}$ of large enough dimensionality to present 10 orthogonal subspaces, each one representing one belief. As such, we would need at least a 10-dimensional space, with 10 orthonormal vectors forming the belief basis. In such 10-dimensional Hilbert space, the 2 possible outcomes of the first movement action are each represented by orthogonal 5-dimensional subspaces.

The Hilbert space $\mathcal{H}_{II,B}$, which models the relationship between the belief measurement and the second movement action would be similarly spanned by 10 orthonormal basisvectors, each one representing an outcome of the belief measurement. The outcomes of the second movement action are also represented by 5-dimensional subspaces. The rules for projection and calculating probabilities remain the same. The probability of an outcome of a measurement is still the square of the norm of the projection of the state vector on the relevant subspace. The act of measuring still changes the superposition of the state vector, projecting and normalizing it onto the relevant subspace.

In summary, the relationship between the belief and action measurement is represented by the description of the action subspaces in terms of the belief basis. In such setting, the consensus effect would be represented by the form of the 5-dimensional action subspaces in $\mathcal{H}_{II,B}$, while the effect of the player becoming more reasoned would be represented by the form of the 5-dimensional action subspaces in $\mathcal{H}_{\mathrm{I},B}$.

3.4.3 A Very Basic Model

We can now attempt to construct a model which successfully incorporates all three effects, by combining how we modeled the compatible action measurements, with how we could model the incompatible belief and action measurements. The standard procedure from quantum-like measurement theory tells us to construct the Hilbert space $\mathcal{H}^{orth} = \mathcal{H}_{I,B} \otimes \mathcal{H}_{II,B}$. This is a 100-dimensional Hilbert space, with 2 orthogonal 50-dimensional subspaces representing the actions in role I, 2 orthogonal 50-dimensional subspaces representing the actions in role II, and 10 orthogonal 10-dimensional subspaces representing the possible beliefs. As the first and second move actions are considered compatible, they can be measured at the same time. As such, the 4 possible joint outcomes of the action measurements are represented by four 25-dimensional subspaces.

The player would be represented by a normalized state vector in this 100-dimensional Hilbert space, from which the relevant probabilities can be calculated. From a statistical point of view this state vector already provides us with 99 degrees of freedom (we lose 1 as the state vector is normalized), without even delving into how many degrees of freedom pop up due to the different 10-, 25- and 50-dimensional subspaces used in this construction. As we have 160 data points, this simple model would be by no means elegant, and a statistical fit is not feasible because of being greatly overparametrized. One solution is to impose further restrictions on the state vector and/or the different outcome subspaces, for example, by allowing only state vectors within a certain subspace or assuming a certain distribution over the resulting probabilities. The form of these restrictions is, however, an open question at this point.

In the following section we show how a small deviation from the most common quantum-like approach, using a less structured set of planes to represent beliefs, allows us to reduce the complexity of the total Hilbert space to only four dimensions and provides a truly intuitive connection between the different elements of the model.

3.5 The Model for Quantum-like Preferences and Beliefs

3.5.1 A New Belief Base.

To diminish the problematic dimensionality of $\mathcal{H}_B$ we need to allow the vectors $|B_i\rangle$ (the outcomes of the belief elicitation) to be non-orthogonal because otherwise, the 10 orthogonal vectors would span a 10-dimensional Hilbert space. Next to making the dimension of $\mathcal{H}_B$ sufficiently small, this modification will allow us to model some implicit structure between the different outcomes and will link the construction of these beliefs directly to the approach of Pothos & Busemeyer (2009) to the classical prisoner dilemma. Roughly speaking, in Pothos & Busemeyer (2009), the emergence and evolution of the player's beliefs about his opponent's behavior is represented by a rotation of the state vector in the Hilbert space. While in Pothos & Busemeyer (2009) this rotation is defined by a Hamiltonian with a parameter γ, we now have the means to explicitly incorporate the elicited beliefs into our model. To do so, we redefine the belief vectors $|B_i\rangle$ in a 2-dimensional Hilbert space, with $|B_0\rangle$ and $|B_9\rangle$ orthogonal and the other $|B_i\rangle$ in between them. For simplicity, we will assume the distribution of the $|B_i\rangle$ $(i \neq 0, 9)$, to be uniform between $|B_0\rangle$ and $|B_9\rangle$, yielding an angle $\pi/(2 \times 9)$ between all $|B_i\rangle$ and $|B_{i+1}\rangle$. This provides us with an elegant, parameter free (as the '9' is endogenous to the game) form of the vectors representing the outcomes of the belief elicitation. This is a simple first approach to the exact distribution of the $|B_i\rangle$, which can be adjusted or made more complex if necessary. This effectively makes the players development of her explicit beliefs to be represented by a rotation of the state vector, as in Pothos & Busemeyer (2009). Our view differs from Pothos & Busemeyer (2009) in the sense that we still want to make predictions and derive probabilities from this rotation, using the standard

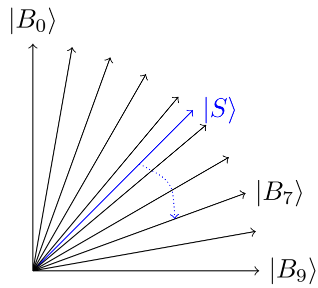

Figure 3.3 The redefined $|B_i\rangle$. A player thinking 7 out of 9 opponents cooperate projects the state vector onto $|B_7\rangle$.

rules of projective measurements: an outcome and its probability as defined by a projector on the relevant subspace. Note that this approach also models the implicit order between the different outcome vectors (e.g. $|B_i\rangle$ being 'in between' $|B_{i-1}\rangle$ and $|B_{i+1}\rangle$), something lacking in the previous approach with all belief vectors orthogonal. This idea is depicted in figure 3.3.

With the redefined 2-dimensional Hilbert Space $\mathcal{H}_B$, we rebuild the Hilbert space $\mathcal{H}_{QP\&B}$ which contains the representations of all measurements, as well as their correlations. As we redefined our vectors $|B_i\rangle$ representing the elicited beliefs, the projectors onto these vectors will also have a new form:

$$|B_i\rangle\langle B_i| = \begin{pmatrix} \cos^2\left(\frac{i\pi}{18}\right) & \cos\left(\frac{i\pi}{18}\right)\sin\left(\frac{i\pi}{18}\right) \\ \cos\left(\frac{i\pi}{18}\right)\sin\left(\frac{i\pi}{18}\right) & \sin^2\left(\frac{i\pi}{18}\right) \end{pmatrix}, \qquad (3.16)$$

with $i = 0, 1, \ldots 9$.

To incorporate a measurement with outcome vectors not orthogonal, we will go beyond the basic procedure of quantum measurement as done in Section 3.3.2. To do so, we present two options, one favoring quantum theoretic consistency and one favoring a simpler experimental interpretation. Note that the resulting model and probabilities in these two options are identical. Readers not interested in the derivation and discussion of these options can skip to the last paragraph of this section.

In the first option we use positive-operator valued measures (POVMs), a well known measurement framework within quantum theory, in which non orthogonal outcome vectors can be used. These POVMs allow us to easily build our smaller model with our newly defined belief space. For an introduction to these POVMs, as well as the mathematical details and recipe on how to construct them, we refer to Yearsley (2016, Section 4). The following derivations rely on the derived probabilities given in Yearsley (2016, Equation 56). In short, when using the POVM framework, the measurement outcome is still represented by an outcome vector (and its associated projector). If an outcome is observed, the state vector is still projected onto the relevant subspace; however, the probability of obtaining this outcome is calculated slightly differently. Assume that the player is represented by a state vector $|S\rangle$, the probability of the player thinking that i opponents have cooperated is now:

$$P'(B_i) = \frac{\langle B_i|S\rangle^2}{\sum_{j=0}^{9} \langle B_j|S\rangle^2}. \tag{3.17}$$

This form deviates from the probabilities derived in Section 3.3.2 only in the factor $\sum_{j=0}^{9} \langle B_j|S\rangle^2$. This extra factor finds root in the fact that projectors P_j forming a POVM need to adhere to *completeness*:

$$\sum P_j = \mathbb{I},$$

with $\mathbb{I}$ the identity matrix. Equation 3.16 shows that the projectors onto our belief vectors can never sum to the identity matrix, as the off-diagonal elements can never sum to zero. To make sure that the relevant projectors still form a POVM, a new projector (and outcome) is added to the formalism. This projector is associated with the outcome 'measurement failed'. When this outcome is obtained the measurement is redone, ensuring *completeness*. For details, see again Yearsley (2016).

The second option dismisses the idea of an extra 'measurement fails' outcome and allows the set of projectors $|B_i\rangle\langle B_i|$ to violate the *completeness* criteria. This violation makes the probabilities of our possible belief outcomes not sum to one:

$$\sum_{j=0}^{9} P(B_i) = \langle B_i|S\rangle^2 \neq 1. \tag{3.18}$$

As this is clearly problematic from a modeling point of view, we now introduce a scaling factor. This scaling factor makes sure that the total sum of probabilities does sum to one, *after* the standard quantum measurement (calculating probabilities and projecting the state vector) is done. This scaling factor is defined as:

$$C = \sum_{j=0}^{9} \langle B_j|S\rangle^2, \tag{3.19}$$

making the probability of eliciting belief i, given the state vector $|S\rangle$:

$$P'(B_i) = \langle B_i|S\rangle^2/C. \tag{3.20}$$

It is vital to note that the end result of both approaches is identical. We have ten outcome vectors, representing the ten possible beliefs, in a two dimensional Hilbert Space $\mathcal{H}_B$. The probability of eliciting the belief that i opponents have cooperated, given the state vector $|S\rangle$ is:

$$P'(B_i) = \frac{\langle B_i|S\rangle^2}{\sum_{j=0}^{9} \langle B_j|S\rangle^2}. \tag{3.21}$$

If the result was i, the state vector gets projected onto $|B_i\rangle$. The difference between the two options lies in the difference between an approach where we remain firmly within the quantum theoretic setting at the cost of adding an ad hoc new outcome (actually not present in the experimental setting) and an approach slightly departing from the quantum sphere by redefining the probabilities with an ad hoc scaling factor, but having a clear interpretation of all the elements of its machinery regarding the experiment. The choice between the options has no effect on the rest of the paper.

3.5.2 The QP&B Model

With our belief measurement now adequately defined in the two dimensional $\mathcal{H}_B$, we can redefine $\mathcal{H}_{QP\&B}$. We still assume the second move action and the belief elicitation to be complementary, representing them by different bases in the redefined 2 dimensional Hilbert Space $\mathcal{H}_{\mathrm{II},B}$. Additionally, we define the angle between $|C_{II}\rangle$ and $|B_9\rangle$ as β_{II} (see figure 3.4). This allows us to derive estimated probabilities for a player replying that he thinks i opponents cooperate, after the player has cooperated or defected on his second move. As such, this models the consensus effect. We assume β_{II} to be close to 0, as the consensus effect tells us that people who cooperate are more likely to assume that opponents cooperate as well.

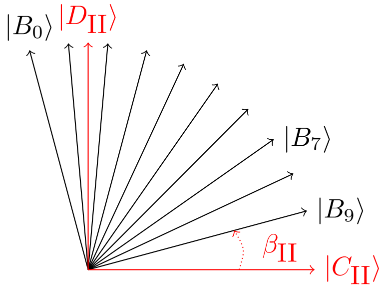

Figure 3.4 The redefined $\mathcal{H}_{\mathrm{II},B}$ with both an action basis and the new belief basis.

Now we can derive the estimated probabilities for the beliefs of a player defecting on his second move (making the state vector $|S\rangle = |D_{II}\rangle$):

$$P(B_i|D) = \langle B_i|D_{II}\rangle^2 / \sum_{j=0}^{9} \langle B_j|D_{II}\rangle^2 \tag{3.22}$$

$$= \cos^2\left(\beta_{\mathrm{II}} + \frac{i}{9}\frac{\pi}{2}\right) / \sum_{j=0}^{9} \langle B_j|D_{II}\rangle^2 \tag{3.23}$$

and for the beliefs of a player cooperating on his second move (making the state vector $|S\rangle = |C_{II}\rangle$):

$$P(B_i|C) = \langle B_i|C_{II}\rangle^2 / \sum_{j=0}^{9} \langle B_j|C_{II}\rangle^2 \tag{3.24}$$

$$= \sin^2\left(\beta_{\mathrm{II}} + \frac{i}{9}\frac{\pi}{2}\right) / \sum_{j=0}^{9} \langle B_j|C_{II}\rangle^2, \tag{3.25}$$

with $i \in \{0, \ldots, 9\}$.

Similarly, we redefine $\mathcal{H}_{\mathrm{I},B}$ as 2-dimensional with both a first move action basis and a belief basis, with β_{I} the angle between $|C_{DM}\rangle$ and $|B_9\rangle$. We once again assume β_{I} close to zero, as players who explicitly think their opponent will defect, are assumed to be more reasoned and will defect as well. We can now derive the estimated probabilities of a player cooperating or defecting on his first moves, after replying that he thinks i opponents cooperated on their second move, which made the state vector $|S\rangle = |B_i\rangle$. Note that this first move measurement once again uses the more simple derived probabilities as defined in 3.3.2, as this measurement has both outcome vectors orthogonal.

$$
\begin{aligned}
P(D|B_i) &= \langle D_I|B_i\rangle^2 && (3.26)\\
&= \cos^2\left(\beta_\mathrm{I} + \frac{i}{9}\frac{\pi}{2}\right). && (3.27)
\end{aligned}
$$

The first and second moves are still considered to be compatible, allowing for a tensoring of their respective Hilbert spaces to represent their correlation. The projectors and probabilities associated with these measurements are identical to the ones defined in the quantum-like model from Section 3.3.2. This gives us a final model $\mathcal{H}_{QP\&B} = \mathcal{H}_{\mathrm{I},B} \otimes \mathcal{H}_{\mathrm{II},B}$. In $\mathcal{H}_{QP\&B}$, the belief that all opponents cooperate is represented by a plane $\boldsymbol{B_9}$. The angle between $\boldsymbol{B_9}$ and the plane representing second move cooperation is β_{II}. The angle between $\boldsymbol{B_9}$ and the plane representing first move cooperation is β_I. This also defines the plane $\boldsymbol{B_0}$, which is orthogonal to $\boldsymbol{B_9}$, naturally representing the belief of all opponents defecting and the planes $\boldsymbol{B_i}$ between $\boldsymbol{B_9}$ and $\boldsymbol{B_0}$. This incorporates the representation of all 3 measurements and their relationships (compatible or complementary) into one 4 dimensional Hilbert space, with clear estimated probabilities resulting from this representation.

3.5.3 Fitting the Data

We fit the experimental data of the three measurements to our model. Note that the proportions of the second move actions are already incorporated in the starting state vector (Eqs. 3.15). Since we have derived concrete dependencies of the beliefs on the second moves, and of the first moves on the beliefs, we can formally fit the experimental data of the *Elicit_Beliefs* group to our model. To do so, we shall estimate an optimal β-value for the beliefs on the second moves, as well as for the first moves on the beliefs. This can be achieved by minimizing the distance between the counts observed in our data set and the expected frequencies based on the equations derived above. Since a chi-squared test is typically used to check whether or not an observed set of proportions sufficiently matches the expected set, we will focus on minimizing this statistic.

Let us first focus on the two contingency tables representing the dependencies of the beliefs on the second moves (see Tables 3.4 and 3.5). When

i	0	1	2	3	4	5	6	7	8	9
Observed counts	1	0	0	0	3	6	9	6	4	3
Observed proportions	0.031	0.000	0.000	0.000	0.094	0.188	0.281	0.188	0.125	0.094
Expected proportions	0.011	0.000	0.005	0.025	0.058	0.099	0.144	0.187	0.223	0.248

Table 3.4 The observed counts, as well as the observed and expected proportions of the beliefs of second move cooperators

a specific β_{II}-value is provided, we can estimate the expected probabilities $P(B_i|D)$ and $P(B_i|C)$ based on Eqs. (3.23) and (3.25), respectively, and subsequently evaluate a chi-squared statistic for each of the two tables. In order to estimate an appropriate β_{II}, we optimize an algorithm in which the sum of the two chi-squared statistics (one for the II defectors and one for II cooperators) is minimized over a range of possible values for β_{II} (ranging from $-\pi/2$ to $\pi/2$). The β_{II}-value for which this sum reaches its lowest point equals -0.2048, corresponding to chi-squared statistics of 14.13 and 14.24 for the two contingency tables (one concerning the second move cooperators and one concerning the second move defectors), respectively. As expected, our estimated β_{II} is indeed close to 0.

Under normal circumstances, these chi-squared statistics can be translated into p-values, by relying on their asymptotic approximation of a chi-squared distribution with $I-1$ degrees of freedom (with $I=10$ the number of possible beliefs). For our data set, however, this asymptotic procedure can be problematic because several of the expected frequencies fall below five. As this induces concern about the accuracy of any p-value obtained through asymptotic approximation, we will resort to a more accurate estimation via Monte Carlo simulation. This technique simulates the sampling distribution of the test statistic (in this case, chi-squared) using Monte Carlo methods. In short, it will generate random contingency tables with the same marginal distribution as our data (i.e. the same sample size), and calculate their chi-squared statistic. Subsequently, it is determined how many of these random samples display a test-statistic which is larger than the one that was originally obtained. The resulting proportion of more extreme chi-squared statistics represents our new and more accurate p-value. Note that what can be calculated for one chi-squared statistic can also be achieved for a sum of chi-squared statistics: we can simulate a p-value corresponding to the proportion of summed test-statistics, which are larger than the original sum $(14.13+14.24=28.37)$. For our analyses, we chose to rely on 10000 simulated samples.

According to the reasoning in the previous paragraph, these two test-statistics allow us to calculate a p-value through Monte Carlo simulation:

i	0	1	2	3	4	5	6	7	8	9
Observed counts	4	2	5	5	9	3	0	0	0	0
Observed proportions	0.143	0.071	0.179	0.179	0.321	0.107	0.000	0.000	0.000	0.000
Expected proportions	0.156	0.163	0.160	0.147	0.127	0.101	0.072	0.045	0.022	0.007

Table 3.5 The observed counts, as well as the observed and expected proportions of the beliefs of second move defectors

i	Observed counts	Totals	Observed proportions	Expected proportions	χ^2
9	5	5	1.000	0.997	0.016
8	2	2	1.000	0.947	0.111
7	5	5	1.000	0.844	0.925
6	5	5	1.000	0.699	2.153
5	6	12	0.500	0.530	0.044
4	3	9	0.333	0.358	0.023
3	1	9	0.111	0.202	0.464
2	0	6	0.000	0.083	0.542
1	0	4	0.000	0.014	0.056
0	0	3	0.000	0.003	0.010

Table 3.6 The observed number of cooperators, total number of participants, and observed as well as expected proportions of first move cooperators. Note that the observed number of defectors (as well the the respective observed and expected frequencies) are not mentioned in this table since this information is redundant (the observed counts of cooperators and defectors sum to the totals and the observed/expected proportions of both defectors and cooperators sum to one).

we obtain one of 0.071 for both tables combined. Their observed counts, alongside the observed and expected frequencies, can be found in tables 3.4 and 3.5. The p-value testing the null hypothesis of no significant difference between our observed and expected proportions on the $\alpha = 0.05$ level indicates an acceptable fit. As this p-value is estimated using simulation, degrees of freedom are not taken into account, unlike a traditional p-value where the chi-square distribution is used. As such, this p-value does not take into account that 20 proportions are estimated using only 1 free parameter, making our estimated p-value more favorable to accepting the null hypothesis than the value suggests at first sight. See Tables 3.4 and 3.5.

When we aim to establish an optimal value of β_{I} for modeling the first move actions, we see that we have to deal with ten different contingency tables: one for each belief in the number of cooperators ($i = 0, \ldots, 9$). Since the observed and expected probabilities in each of these contingency tables sum to one, we only need to focus on the data counts and proportions for the cooperators $P(C|B_i)$. Similar to the beliefs of the second moves, we establish an optimal value of β_{I} for the first move cooperators by min-

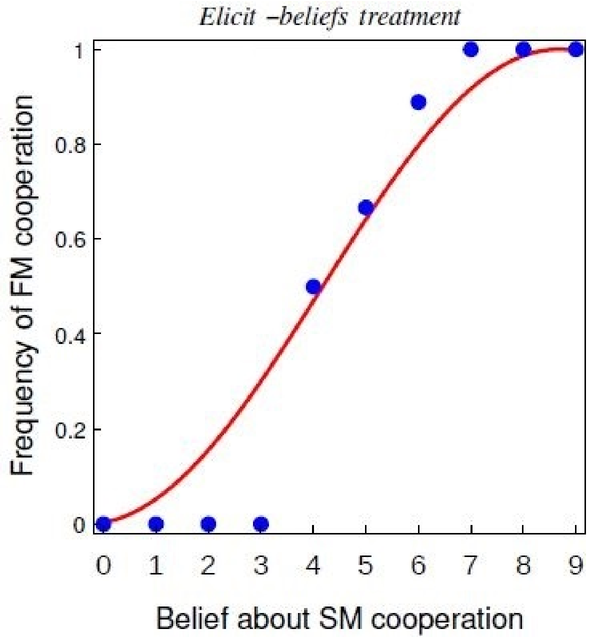

Figure 3.5 Observed frequency of first move cooperation versus elicited beliefs on second mover cooperation (blue dots) and fitted model (equation 3.27, red line).

imizing the sum of the ten chi-squared statistics using equation (3.27). The optimal value of β_{I} is 0.057 which is close to 0, as expected. Figure 3.5 plots the analytical prediction of the POVM model (equation 3.27) for the relationship between first move cooperation rates and stated beliefs about second mover cooperation with $\beta_{\mathrm{I}} = 0.057$, and compares it to the experimental observations.[6] The chi-squared statistics and expected proportions are displayed in Table 3.6; and the corresponding simulated p-value equals 0.715 indicating a very good fit.

[6]Blanco et al. (2014, Figure 3) explain the observed relationship between both experimental variables with a probit regression, obtaining a similar dependency. Nevertheless, our analytical curve has a deeper meaning because the functional form (equation 3.27) is a direct consequence of the geometrical structure of the POVM model.

3.6 Discussion

Our decision to abandon the restriction that outcome vectors coming from one measurement are orthogonal to each other has consequences. The most important one is the loss of the repeatability of outcomes.[7] Repeatability entails that when a measurement is performed twice (without any manipulations of the system between two measurements), the same outcome is observed twice. This is assured in a standard quantum-like model, as the projection of a state vector onto an orthogonal subspace gives the null vector. Repeatability seems a very logical and sensible restriction, but has been called into question, specifically when applied in quantum cognition. See for example Khrennikov et al. (2014) for a thorough discussion of this problem and Aliakbarzadeh & Kitto (2016) about the use of POVMs, which lack repeatability, in Social Sciences.

In our context, the loss of repeatability in the belief measurement means that when a player replies that, e.g., 6 opponents cooperate, he might reply 7 when the question would be posed again. To justify this, we consider the measurements to be *unsharp*. Unsharp measurements are measurements such that the outcome represents a bigger subset of a (possible non-discrete) set of outcomes. This is applicable in cases where a subject is asked to form a precise opinion or belief (e.g. a discrete number), but he is actually forming a more broad opinion or belief (e.g. 'much'). Applied to our dataset, we assume that when a player replies that, e.g., 6 out of 9 opponents cooperate, this indicates the player believing 'somewhere around 6 out of 9 opponents cooperate'. This implies that he would not necessarily disagree with the opinion that 7 out 9 opponents have cooperated. This structure can be viewed in the form of the belief vectors $|B_i\rangle$. The state vector collapsing on $|B_6\rangle$ does not preclude the outcome associated with $|B_7\rangle$, as they are close to each other, with the angle between them equaling $\pi/18$. The closer two vectors are to being orthogonal, the more the outcomes they represent do preclude each other. The vectors $|B_0\rangle$ and $|B_9\rangle$ are the limit case: being orthogonal makes the events associated with them (the opponent cooperating and defecting for sure) completely preclude each other.

The use of these non-orthogonal outcome vectors also opens up new research possibilities within quantum cognition. Inflated dimensionality is a common obstacle in elegant model building. Once multiple (compatible) measurements with more than two possible outcomes are taken into

[7] Also called first kindness in Danilov & Lambert-Mogiliansky (2008).

account, any standard quantum model would require high dimensionality. Next to the ease of reducing dimensionality, some extra structure can be incorporated in the model. As can be seen in our model, implicit relationships between different outcomes can be represented. E.g., our model allows to have the B_1 outcome be 'closer' to the B_2 outcome than it is to the B_8 outcome. In a standard quantum model, all outcome vectors are orthogonal, so all outcomes play a similar role towards each other. This works for all discrete examples in Physics, but in cognition there are numerous examples of ordinal scales, where a kind of structure is implied between the different outcomes. As there is, to our knowledge, no known way of simply incorporating ordinal scales into the quantum framework (preserving some sense of the notion of an order), constructing bases similar to the one in this paper seems to be an interesting road for future research. One obvious candidate for this treatment would be the quantum-like modeling of Likert scales. They allow for (mostly) 5 or 7 different ordered outcomes and are an ordinal scale used widely within cognition. Some first steps for Likert scales of this form are presented in Yearsley (2016).

3.7 Conclusion

In this paper we constructed a quantum-like model for beliefs and preferences in a social dilemma game. By taking a new look at data collected by Blanco et al. (2014) during a sequential prisoner's dilemma, we identified and discussed three distinct effects. These effects are all explained as a specific type of relationship between the measurements performed in the experiment. First, there is a direct positive correlation between the player's first and second move. As it is shown in Blanco et al. (2014), however, this does not provide a complete picture of the player's behavior because this correlation is also driven by an indirect belief-based channel. This indirect belief-based channel is made up of the two other effects: the influence of the second move on the beliefs of a player and the influence of the beliefs of a player on his first move. We called the former effect the consensus effect and attributed the latter effect to the player becoming more reasoned in his preferences.

The nature of these last two effects both pointed us towards a quantum-like model. The quantum-like nature of the consensus effect is already discussed in Busemeyer & Pothos (2012) where it is viewed as a form of social projection. In Busemeyer & Pothos (2012), this is modeled by representing the construction of a player's belief as a rotation of the state vector in a Hilbert Space. This made the belief construction and the second move action to be non-commuting (and thus incompatible) in nature. The effect of the player becoming more reasoned can be seen as a violation of the sure-thing principle. The act of belief elicitation significantly changes the cooperation rate of the first move action, regardless of the beliefs elicitation outcome. This also pointed us towards viewing the belief elicitation and the first move action as incompatible. Combining all these observations, we constructed a Hilbert Space in which the action measurements on the one hand and the belief measurements on the other hand were viewed as incompatible measurements by defining a different basis for each.

Following the more traditional recipe, we obtained a model within a 100-dimensional Hilbert Space, which was greatly overparametrized from a statistical point of view. As a solution to this problem, we proposed to redefine the belief base as two-dimensional. We draw inspiration from the rotation idea from Busemeyer & Pothos (2012) but still define probabilities and perform projections after measurements, as if dealing with projectors. To define this new belief base, two options were discussed. The first option constructed a POVM, which framed our model neatly into conventional quantum theory, at the cost of defining a new 'measurement failed' outcome. The second option dismissed this new outcome, staying closer to the actual experiment, at the cost of leaving the standard quantum-like framework. Both options resulted in identical models. This allowed us to diminish the problematic dimensionality, incorporate the three discussed effects, and yield elegant dependencies between actions and beliefs. The statistical fit was positive for the dependency of the beliefs on the second move action and satisfactory for the dependency of the first move action on the beliefs.

As not all vectors associated with outcomes of the belief measurement were orthogonal, we lose repeatability of outcomes: obtaining an outcome does not exclude obtaining a different outcome when the same measurement is performed again immediately. While this might have seemed problematic at first, there are other examples in which these techniques are successfully used, again, see Aliakbarzadeh & Kitto (2016) and Yearsley (2016). To this end, we defined unsharp measurements as measurements where forcing the player to pick one outcome does not mean he disagrees with another possible outcome. These unsharp measurements do not adhere to repeatability. By viewing our belief elicitation as an unsharp measurement, we showed that this loss of repeatability is not problematic despite the complexity brought in by the experimental design. For more on the need (or lack thereof) of repeatability in psychological measurements, see Khrennikov et al. (2014).

As for future plans, a new experiment in which the missing treatment with the sequence of measurements being ‘beliefs-second move-first move’ is also performed might shed new and conclusive light on the presumed incompatibility of the action and beliefs measurements because the order in which measurements are performed has an influence on the outcomes in the quantum-like framework. Next to this final type of experimental treatment, a more extended theoretical analysis could be given to the consequences and possible applications of our non-orthogonal basis. As mentioned previously, this approach could be a first step towards an implementation of ordinal scales in a quantum-like way, as an implicit order is present in how the outcome vectors were redefined.

3.8 Acknowledgments

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4 Towards Ordered Projective Measurement

This chapter is an extension of work presented at the Quantum Interaction 2016 conference in San Francisco which will published in the proceedings of said conference: Denolf, J. (in press). A First Attempt at Ordinal Projective Measurement, *Quantum Interaction*, Lecture Notes in Computer Science. Springer International Publishing.

Abstract. To our knowledge, all applications of the quantum framework in social sciences are used to model measurements done on a discrete nominal scale. However experiments in social sciences often produce data on an ordered scale, which implies some structure between the possible outcomes. We show that in quantum cognition, orthodox projection-valued measurement (PVM) lacks the tools and methods to properly deal with these ordered scales. We sketch out an attempt to incorporate the ordered structure of outcomes into the subspaces representing these outcomes. This will also allow us to reduce the dimensionality of the resulting Hilbert spaces, as these often become too high in more complex quantum-like models. To do so, we loosen restrictions placed upon the PVM (and even POVM) framework. We discuss the two major consequences of this generalization: scaling and the loss of repeatability. We contrast this new approach with more traditional structures by constructing different models for the data of game theoretic experiment.

4.1 Introduction

With the emerging success of applying the quantum probabilistic toolbox in social sciences, there is also an increasing focus on its limitations. In physics, the construction of the needed model is relatively straightforward. However, in quantum cognition, the quite rigid recipe sometimes shows its limits both mathematically and interpretationally (Khrennikov et al., 2014). So, it shouldn't come as a surprise that more recent work tries to expand the reach of these tools by looking at possibilities beyond the standard projective measurement (PVM) principles. The best known generalization beyond PVM is the use of Positive Operator Valued Measurement (POVMs) (Aliakbarzadeh & Kitto, 2016), but alternative, sometimes even more general, approaches also arise (e.g. Matvejchuk & Widdows (2015) and Aerts & de Bianchi (2015)). These ventures are mostly theoretical in nature, with applications using experimental data being rather sparse. None of these approaches, however, deals with the problem of representing situations where there is structure in the set of outcomes.

In this paper we present an idea which also goes beyond orthodox quantum-like techniques. This new technique was originally formulated for a specific setting in Martínez-Martínez et al. (2015) and further developed and tested in Denolf et al. (2016). In these two papers, a model is constructed which deals with the relationship of a participant's beliefs and preferences in a game theoretic setting, taken from Blanco et al. (2014) and also discussed in Chapter 3. During this process, problems concerning a too high dimension of a Hilbert space arose, which where solved by drawing inspiration from a rotational solution presented in Pothos & Busemeyer (2009) and (ab)using the ordered structure of the possible outcomes. To do so, we opted to loosen certain restrictions which lead to alternative types of projectors. While the solution to these problems served an ad hoc purpose, the question whether this new technique could be applied in different settings presented itself. This will deepen the discussion started in Chapter 3, where some of the details concerning, e.g., nature of scales were glossed over, as the focus of that chapter was the game theoretic experiment.

Here we argue that this generalization of P(O)VM can be used to model any situation where different outcomes of a measurement have some ordered structure. After defining this generalization, we discuss two consequences of using this new structure. Finally, we go back to game theoretic setting this method was originally devised for and formulate some alternative models within standard quantum theory. These other models are then contrasted with our new approach, resulting in an overview of its strengths and weaknesses.

4.2 Revisiting the Clinton/Gore Example

We take a fresh look at the quantum-like model concerning public opinion on Bill Clinton and Al Gore. This is one of the go-to introductory examples in quantum cognition, see for example Busemeyer & Bruza (2012). In a Gallup poll, conducted September 6-7, 1997, participants were asked two separate questions: if they think Clinton is trustworthy and if they think Gore is trustworthy. When the Clinton question is posed first, 53% of the participants consider him to be trustworthy and 73% consider Gore to be trustworthy. However, when the question order is reversed, 67% think Gore is trustworthy and 59% think Clinton is trustworthy. This change in attitude indicates an order effect, which suggests a quantum-like approach by considering the Clinton and Gore questions to be incompatible. In the resulting quantum-like model each question is represented by an orthogonal 2-dimensional basis, with each vector representing the relevant 'yes' or 'no' answer and by defining a 2-dimensional Hilbert space containing both bases. The resulting model has a good statistical fit, with only two parameters (one coordinate of the state vector, as the second coordinate is fixed due to the normalization restriction, and one angle between the two bases) to be estimated (see Section 1.3.1).

We now identify two properties of this experimental paradigm, which become problematic when we leave this relative simple example for more complex ones. First, the number of possible outcomes is low. Both questions only allow two possible replies, while trustworthiness of presidential candidates could be considered far more complex. This gives the resulting Hilbert space a manageable two dimensions. Note that as all measurements are considered incompatible, no tensoring is required, which would increase dimensionality exponentially. Second, there is no structure in the outcomes. The yes and no outcomes are on a discrete nominal scale, with no implicit relationship between them.

Let us make the situation a bit more complex. First, suppose we want to add some more nuance to the questions and allow for more replies: very trustworthy/quite trustworthy/somewhat trustworthy/neutral/somewhat untrustworthy/quite untrustworthy/very untrustworthy. These outcomes clearly have some structure, as they are ordered. This extension makes the resulting Hilbert space 7-dimensional. Second, suppose that, for whatever research reasons, a third similar measurement is performed, which also allows for a similar set of 7 outcomes, that does not produce order effects. Even though the situation is not extreme from an experimental point of view, the Hilbert space needed to model this situation would be (at least) 49-dimensional. This would increase the amount of parameters needed to fit the state vectors and subspaces dramatically, resulting in an inoperable model. Next to this unwieldy dimensionality, this approach lacks the tools to incorporate the ordered structure of the outcomes. When we extended the discrete 'yes or no' Clinton and Gore questions, no new modeling concepts were introduced, so it is unclear what element of our model should represent this ordered structure[1].

In what follows, we propose a first attempt at modeling ordered outcomes, within the quantum-like approach. This attempt also reduces the problematic dimensionality that arises when measurements with more than two outcomes are performed and tensoring is needed, when constructing the relevant bases.

4.3 Modeling an Ordered Structure

4.3.1 A New Type of Outcome Projector

Paraphrasing Kirsty Kitto in her QI15 talk (see Aliakbarzadeh & Kitto (2016)), a quantum(-like) measurement $\boldsymbol{M}$, with its set of possible outcomes $\{\boldsymbol{M}_i\}$, is represented by a set of subspaces $\{\mathcal{M}_i\}$, where $\mathcal{M}_i$ represents outcome $\boldsymbol{M}_i$. These subspaces $\mathcal{M}_i$ each define a projector P_i, which projects any vector $|S\rangle$ on the relevant subspace $\mathcal{M}_i$. The state of a system (e.g. a participant in a psychological experiment) is represented by a normalized state vector $|S\rangle$. Now, the mathematical rules are quite straightforward:

(i) The probability of obtaining outcome $\boldsymbol{M}_i$ is $\langle S|P_i|S\rangle$ or, intuitively, the closer the state vector is to the relevant subspace, the higher the probability of obtaining that outcome.

[1]When in physics, e.g., ratio scales (which also have an order) are used, the eigenvalues of the projectors on outcome subspaces induce an ordering on the outcomes. As eigenvalues, to our knowledge, are not used in quantum cognition, we will not introduce this technique here. Note that our Clinton/Gore example is not on a ratio scale. It is therefore unclear what the relevant eigenvalues, if introduced, would mean.

(ii) After obtaining outcome $\boldsymbol{M}_i$, the state after measurement becomes $\frac{P_i|S\rangle}{\sqrt{\langle S|P_i|S\rangle}}$ or, intuitively, when obtaining an outcome, the state vector becomes a normalized vector in the relevant subspace.

PVM structure. As is widely known, the orthodox quantum measurement paradigm (Projection-valued measurement or PVM) demands that all subspaces associated with one measurement are orthogonal and, perhaps trivially, that these subspaces span the entire Hilbert space. This ensures that probabilities sum to one and that when a measurement is performed twice, without any manipulation between both measurements, the same outcome is obtained twice. We call this last property *repeatability.*

POVM structure. Perhaps less widely known, when we weaken the demand that all subspaces associated with one measurement are orthogonal but still ensure that all probabilities sum to one by demanding that all relevant projector matrices[2] sum to the identity matrix:

$$\sum_i P_i = I, \tag{4.1}$$

we obtain a more general class of measurements which we call Positive Operator-Valued Measurement (POVMs). This generalization is commonly used in physics to model noise, resulting in a measurement that is not precise. Note that POVMs do not adhere to repeatability. This first generalization gives us freedom to incorporate structure in the outcomes, while reducing the dimensionality. However, this solution is still more restrictive then one might think at first, as Restriction 4.1 is still quite strong. More concrete, when a set of outcome vectors is defined, typically an extra outcome vector has to be introduced to ensure all projectors sum to the identity matrix. Take, as an example, a simple two dimensional case. When two non-orthogonal vectors $|M_1\rangle = (1, 0)$ and $|M_2\rangle = (\cos\theta, \sin\theta)$, with respective projectors

$$P_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } P_2 = \begin{pmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{pmatrix},$$

[2]This implies that all elements of the POVM are projectors, which theoretically does not need to be the case. However, if outcomes are associated with operators that are not projectors, we lose the elegant geometric interpretation, as this means that are no subspaces associated with the outcomes. As we will later define an order on subspaces, non-projector operators would also become problematic there.

are needed to model an experimental situation, their projectors sum to:

$$P_1 + P_2 = \begin{pmatrix} 1 + \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{pmatrix}.$$

Having the diagonal elements equal to one can easily be achieved by appropriate scaling. However, to have the off-diagonal elements equal to zero, a third outcome vector $|M_3\rangle = (\pm\cos\theta, \mp\sin\theta)$ or $|M_3\rangle = (\pm\sin\theta, \mp\cos\theta)$ must be introduced, even when there is no third possible experimental outcome! This is visualized in figure 4.1.

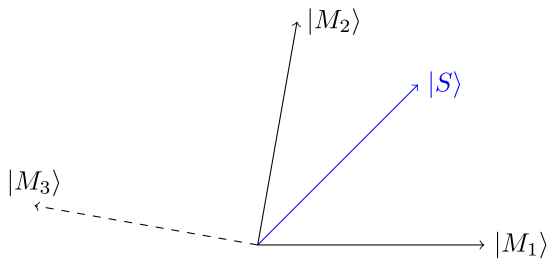

Figure 4.1 A two dimensional POVM structure. Note the necessary extra $|M_3\rangle$ outcome vector.

Ordered scales. To solve this need for an extra outcome vector, we propose to omit the demand that all projectors sum to the identity matrix, effectively losing almost all structure, but use this freedom to add new structure which reflects our ordered scale, while still adhering to our basic quantum-like rules (i) and (ii). The necessity of generalizing measurement beyond POVMs is not a new idea, as remarked in Khrennikov et al. (2014) and discussed in chapter 8 of Nielsen & Chuang (2010).

As we only have two mathematical entities at hand (a state vector $|S\rangle$ and a set of subspaces $\{\mathcal{M}_i\}$ representing outcomes), this structure has to be incorporated in these two. On the one hand, as the state vector is supposed to represent the particular state of the system, the type of scale of the measurement should not impact this state vector. On the other hand, as the set of subspaces is representing the outcomes, any structure between these outcomes, should be reflected in a structure between the subspaces. This is why we allow subspaces associated with outcomes of the same measurement to be non-orthogonal to each other. Now we can define the notion of a subspace $\mathcal{M}_i$ being *closer* to a subspace $\mathcal{M}_j$ then to a subspace $\mathcal{M}_k$, when $\widehat{\mathcal{M}_i\mathcal{M}_j}$, the angle[3] between $\mathcal{M}_i$ and $\mathcal{M}_j$, is smaller then $\widehat{\mathcal{M}_i\mathcal{M}_k}$ the angle between $\mathcal{M}_i$ and $\mathcal{M}_k$. This gives us a natural way of representing an ordered scale with outcomes $\boldsymbol{M}_i$ (admitting to a well defined order $\prec$) by demanding that :

Definition 1. *if* $\boldsymbol{M}_i \prec \boldsymbol{M}_j \prec \boldsymbol{M}_k$*, then* $\widehat{\mathcal{M}_i\mathcal{M}_j} \leq \widehat{\mathcal{M}_i\mathcal{M}_k}$ *&* $\widehat{\mathcal{M}_j\mathcal{M}_k} \leq \widehat{\mathcal{M}_i\mathcal{M}_k}$*.*

Note that the maximum angle between two subspaces is $\pi/2$, so orthogonal subspaces are considered to be the farthest away possible from each other.

The exact value of these angles is an empirical question, which we discuss later. When all relevant subspaces are orthogonal, which is the case with a PVM structure, each subspace adds its own dimension to the total dimension of the encompassing Hilbert space, which is the reason of the exploding dimensionality in the introductory example. As the need for orthogonality is now omitted, the resulting dimensionality can be greatly reduced as compared to the traditional PVM approach. This makes the dimension of the final Hilbert space also an empirical question and/or a deliberate choice, taking into account, e.g., the number of data points or certain demands for elegance or simplicity of the resulting model. The concepts for calculating probabilities (i) and post-measurements states (ii) remain identical to the ones used with PVMs and POVMs. Note that as all considered P_i are projectors, they are still Hermitian positive semi-definite, so $\langle S|P_i|S\rangle$ is real and positive. Because the state vector still gets projected on the subspace representing the obtained outcome, this approach keeps the quantum-like nature. As a result, all concepts (order effects, contextuality, entanglement...) used in quantum cognition are still a part of this approach because the regular PVM structure is now a specific case of our more general framework.

4.3.2 An Example: Likert Scales

A natural candidate for this treatment is the modeling of Likert scales (for an overview on Likert scales, see Spector (1992)). Likert scales are used in polling of opinions and consist of multiple Likert items. A Likert item consists of a statement, which the participant evaluates on a given scale. This scale should be *symmetric* (a neutral option and equal number

[3]The angle $\widehat{\mathcal{M}_i\mathcal{M}_j}$ between two subspaces $\mathcal{M}_i$ and $\mathcal{M}_j$ is classically defined as $\min(\widehat{V_iV_j})$, with $V_i \in \mathcal{M}_i$ and $V_j \in \mathcal{M}_j$.

of positive and negative options) and *balanced* (the perceived distance between following options is equal). The format of a typical five level Likert item looks like

strongly disagree (1) - disagree (2) - neutral (3) - agree (4) - strongly agree (5),

which is clearly on an ordered scale. These Likert scales are widely used in psychology in general and in opinion polling surveys in particular. The use of quantum-like techniques when dealing with these kind of surveys is already established, as, e.g., they are prone to order effects (Moore, 2002). Some work has already been done to use quantum-like techniques when dealing with Likert scales (Camparo, 2013). However, this approach suffers from the two problems flagged before. First, the dimension of the used Hilbert spaces very quickly increases and second, the implicit ordered structure of the outcomes is represented in the state vector (or density matrix), which should only represent the participant, and not in the outcome vectors. Our view opens up new possibilities to tackle these Likert scales. We construct one as an example, but keep in mind that this particular form has not been tested against any experimental data. We only wish to take some first steps to showcase the flexibility of our approach. When looking at the (1)-(5) scale presented above, we argue that 'strongly disagree (1)', 'neutral (3)' and 'strongly agree (5)' should exclude each other, as we consider them in our example as non-nuanced, very clear opinions. As such, they are represented by orthogonal vectors, called $|1\rangle, |3\rangle$ and $|5\rangle$ respectively, giving us a 3-dimensional Hilbert space $\mathcal{H}$. We also argue that picking options (2) or (4), represented by the vectors $|2\rangle$ and $|4\rangle$, does not necessarily mean that the participant disagrees with (1) and (3) or (3) and (5) respectively. Keeping in mind the *balanced* property of Likert scales, places $|2\rangle$ symmetrically between $|1\rangle$ and $|3\rangle$ and $|4\rangle$ symmetrically between $|3\rangle$ and $|5\rangle$. Note that we can easily incorporate assumptions (e.g. *balanced*) from Likert scale theory into our model. This naturally leads to the structure depicted in figure 4.2.

Our implied structure in the outcomes does not impose restrictions on the agents. We can still model a person who doubts between (1) and (5) but not (3), by having a state vector equal to, for example, $|S\rangle = (1/\sqrt{2}, 0, 1/\sqrt{2})$.

Our arguments about the (non)-excluding outcomes and resulting dimensions here are very superficial. One could, e.g., argue that option (3) should be symmetrical between (1) and (5), leading to a 2-dimensional

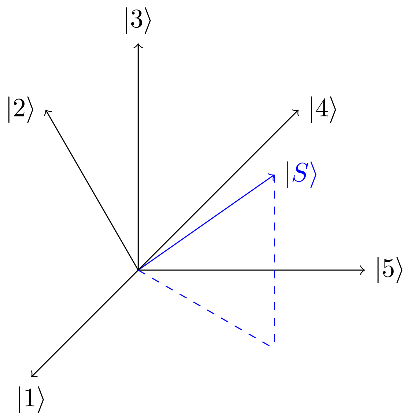

Figure 4.2 The outcome '(i)' is represented by $|i\rangle$. The participant is represented by state vector $|S\rangle$.

Hilbert space. A meticulous investigation of Likert scales in this paradigm falls outside the scope of this paper. We only wish to show that it is possible to represent inherent ordered structure in the outcomes, possibly combined with other theoretical assumptions or restrictions.

4.4 Consequences

Our loosening of restrictions used when defining P(O)VMs has significant consequences. Here, we discuss the major two.

4.4.1 Sum of Probabilities

As we do not require restriction 4.1 to hold, it is possible that the sum of the possibilities across all possible outcomes exceeds 1. While this seems problematic at first, two solutions naturally present themselves. First, a scaling factor can be introduced. This is the solution used in Denolf et al. (2016). Keeping the notations defined as in the previous section, for all $|S\rangle$ define C_M as:

$$C_M = \sum_j \langle S|P_j|S\rangle. \tag{4.2}$$

This allows us to scale appropriately. Now, we redefine the probability of obtaining outcome $\boldsymbol{M}_i$ as

$$P'(\boldsymbol{M}_i) = \frac{P(\boldsymbol{M}_i)}{C_M} \quad (4.3)$$

$$= \frac{\langle S|P_i|S\rangle}{\sum_j \langle S|P_j|S\rangle}. \quad (4.4)$$

This gives us

$$\sum_i P'(\boldsymbol{M}_i) = \frac{\sum_i \langle S|P_i|S\rangle}{\sum_j \langle S|P_j|S\rangle} \quad (4.5)$$

$$= 1. \quad (4.6)$$

While this approach lacks mathematical elegance, it effectively makes the probabilities sum to one.

A second, more elegant, solution is inspired by classical logistic regression. In logistic regression, a function $f(x_1 \ldots x_n)$ is derived, where, given a number of predictors $x_1 \ldots x_n$, the outcome of a binary variable (A or $\neg A$) is estimated. The natural way of predicting a binary outcome would be to estimate the probability of obtaining A. However, as there is no way to ensure that the image of the derived function $f(x_1 \ldots x_n)$ is a subset of $[0, 1]$ (the same problem as with our non-orthogonal subspaces) the odds $\frac{P(A)}{P(\neg A)}$ are modeled, instead of the probability $P(A)$. Since odds only have the restriction that they are positive, this approach can also be successfully introduced here:

$$\text{ODDS}(\boldsymbol{M}_i) = \frac{P(\boldsymbol{M}_i)}{P(\neg \boldsymbol{M}_i)} \quad (4.7)$$

$$= \frac{\langle S|P_i|S\rangle}{\langle S|I - P_i|S\rangle}. \quad (4.8)$$

Using odds does not introduce any new factors, making it more elegant mathematically. One can easily calculate standard probabilities from these odds since the scaling factor needed beforehand would disappear throughout the calculations. However, odds might be more difficult to interpret. To our knowledge, there are no quantum-like models where these odds are used. It can be easily shown by calculating the odds with the newly defined $P'(\boldsymbol{M}_i)$ that both solutions are identical from a modeling point of view.

4.4.2 Loss of Repeatability

As a consequence of allowing non-orthogonal subspaces to represent outcomes of the same measurement, we lose repeatability: when a measurement is performed twice, without any manipulation between both measurements, two different outcomes can be obtained. While repeatability seems a necessity at first, multiple instances where it is not required (or is even considered too strict) can be found in, among other fields, cognition. The best known approach lacking repeatability is the use of POVMs, which we defined in Section 4.3.1. For an in-depth discussion of the use of POVMs in cognition and the relationship to repeatability, we refer to Aliakbarzadeh & Kitto (2016) and Khrennikov et al. (2014). More on the application of POVMs in physics can be found in de Muynck (2007). Summarizing, models not adhering to repeatability are not only feasible, but also sometimes required within quantum cognition.

What could this loss of repeatability mean within our Clinton/Gore example and ordered scales in general? When we go back to our 7 outcome ordinal scale 'very trustworthy/quite trustworthy/somewhat trustworthy/neutral/somewhat untrustworthy/quite untrustworthy/very untrustworthy', we claim that some of these outcomes should not exclude each other. To justify this, we introduce the notion of *unsharp measurement*. This idea is already successfully implemented in Denolf et al. (2016). We claim that when participants are forced to pick one of these outcomes, their reply does not mean a complete dismissal of another option as these opinions are not completely distinguishable (see also the discussion of 'distinguishing quantum' states in 2.2.4 of Nielsen & Chuang (2010)). When, e.g., a participant replies that he thinks Gore is somewhat trustworthy, the participant does not necessarily disagree with the notion that Gore is quite trustworthy. The more probable it is that two options do not preclude each other, the closer their respective vector spaces should be. While the example might be too simple and underestimating the cognitive abilities of the participants, there is always a tipping point where outcomes do become psychologically indistinguishable. To construct an extreme example, suppose that the trustworthiness question allows for an ordinal scale ranging from 1 (untrustworthy) to 1000 (trustworthy). There is no participant that could successfully fathom the difference between, e.g., replying 503 and replying 504. This is similar to noise in a measurement, for which

POVMs are used in physics. Even though the noise in the measurement now comes from the cognitive abilities of the participant, it still results in the measurement not being precise. The structure we incorporated, ensures that if repeatability is violated in such cases, the possible outcomes of the repeated questions are neatly scattered around the original answer, as the closer two subspaces are, the more likely it is that the outcomes they represent are obtained after each other. The upper limit case of this is the original outcome, which has the highest probability of being obtained again. The lower limit case of this are outcome vectors orthogonal to the vector representing the original outcome. They can not be obtained in the repeated measurement. As such, the class of measurements where repeatability does occur, is a subclass of the one we propose, by having all relevant outcome vectors orthogonal.

Note that this idea of unsharp measurement can be empirically tested. To do so, simply confront the participant with a different option than the given reply and ask if the participant could agree with it. These ideas allow the model to be constructed in an empirical way. First, test or argue which outcomes are mutually exclusive and represent these by orthogonal subspaces (this also determines the dimension of the resulting Hilbert space). Second, observe which outcomes are not excluded and define their subspaces accordingly. We illustrate this type of reasoning in the second example of the next section. Moreover, this approach allows for statistical testing of certain cognitive hypotheses concerning cognitive abilities and/or ordered scaling by checking if allowing these 'close' subspaces results in (more) satisfying statistical fits of experimental data.

4.5 Revisiting the QP&B Model

4.5.1 QP&B Model

In this section, we will revisit the model of a player's beliefs and preferences in a game theoretic setting that was discussed in Chapter 3. We will construct different models, using the three structures (ordered scales, POVM and PVM) described previously. This will allow us to contrast these three measurement structures and list their uses and (dis)advantages. For all three structures, we will take a look at both the mathematical side ('Does it results into a nice model') and the interpretational side ('What does the model mean?'). As we aim to have all chapters to be readable independently, we will start with a short overview of the experiment. For an in-depth discussion of the experiment and data, we refer to Chapter 3.

In this experiment three measurement are performed. Two measurements are game theoretic actions in a sequential prisoner dilemma. We will call these measurements the first move (I) and the second move (II). Both measurements have two possible outcomes, cooperate and defect, which we will label accordingly as $C_{\mathrm{I}}, D_{\mathrm{I}}, C_{\mathrm{II}}$ and D_{II}. The third measurement is the belief measurement (B) in which players are asked how many out of nine opponents they think have cooperated. This measurement has 10 possible outcomes labeled $B_i, i \in 0 \ldots 9$. Clearly, these outcomes have an order. The crux of the experiment is that the belief measurement is only performed in a subgroup of participants. In this *Elicit_Beliefs* subgroup, the order of measurements is: II-B-I. In the *Baseline* subgroup, only two measurements are performed: II-I[4]. The comparison between these two groups of the average cooperation rate as a first mover I, shows that the act of performing the belief measurement B has a significant influence on the first mover I behavior. This can be seen in the data as a violation of the sure-thing principle. This influence of the act of measuring is the incentive for constructing a quantum-like model

This overview skipped over a lot of details of the experiment, but allows to summarize what this QP&B model should do: model two measurements (I & II) with two possible outcomes, that are considered compatible and a measurement (B) with ten possible outcomes, which have a natural order, and is considered incompatible with the two other measurements (I & II). This should be done in a way that has a satisfactory interpretation and is feasible from a modeling point of view.

4.5.2 Using Ordered Scales

Just as with the Likert scales from Section 4.3.2 we will use the ordered structure of the outcomes of the belief measurement to define the relevant outcome vectors. Again, we will only summarize the reasoning presented in Chapter 3. We argue that only two belief outcomes exclude each other: the belief that all opponents cooperate and the belief that all opponent defects. The main argument is that this information is used to form an opinion about the *direct* opponent a player will face during the game. It is this opinion that (partly) drives the decision making process when the player chooses his first move action. The belief that all opponents

[4]Some caution is advised as the terminology is somewhat confusing. The second move (II) is the first performed measurement in both groups. The first move (I) is the second performed measurement.

cooperate indicates that the player beliefs that his direct opponent will cooperate. Likewise, the belief that all opponents defect indicates that the player beliefs that his direct opponent will defect. Any belief between these extremes indicates an uncertainty about the behavior of the direct opponent, represented by a superposition between these extreme states. This makes the belief space two dimensional. We also argue that the 8 remaining belief outcomes should be ordered symmetrically between these two extreme outcomes:

$$|B_i\rangle = (\sin\frac{i\pi}{18}, \cos\frac{i\pi}{18}). \tag{4.9}$$

This leads naturally to the basis shown in figure 4.3. Note that this structure is completely parameter free. This also makes the belief elicitation a rotation (as we still project the state vector on the relevant outcome vector), which is identical as how the (non elicited) belief construction is represented in Pothos & Busemeyer (2009).

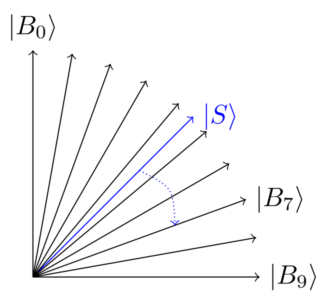

Figure 4.3 The outcome 'i' is represented by $|B_i\rangle$. The participant is represented by state vector $|S\rangle$. Here, the participant replies that he thinks 7 opponents have cooperated, projecting/rotating the state vector onto $|B_7\rangle$.

This construction allows us to define $\mathcal{H}_{I,B}$, the Hilbert space used to model the first move action, the belief measurement and the interaction of these two. As the belief measurement is considered incompatible with the first move action, both measurements are represented by a different basis in the Hilbert space $\mathcal{H}_{I,B}$, as depicted in figure 4.4. This construction comes with one parameter, β_I, the angle between both bases. An identical construction can be made for $\mathcal{H}_{II,B}$, the Hilbert space used to model the second move action, the belief measurement and the interaction of these two. The angle between both bases is now defined as β_{II}.

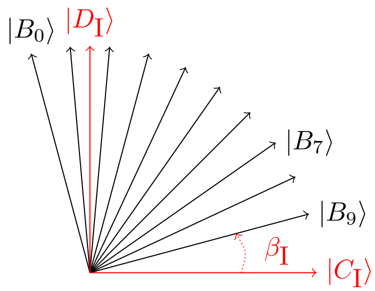

Figure 4.4 The Hilbert space $\mathcal{H}_{\text{I},B}$, with the first move basis and the (overcomplete) belief basis.

Since the first and second moves are considered compatible, we define the resulting Hilbert space, modeling the entire experiment, as $\mathcal{H} = \mathcal{H}_{\text{I},B} \otimes \mathcal{H}_{\text{II},B}$. As the dimension of $\mathcal{H}$ is only four, only three parameter are added when fitting the starting state vector

$$|S\rangle = (s_{CC}, s_{CD}, s_{DC}, s_{DD}) \tag{4.10}$$

to the data, as we have the restriction:

$$s_{CC}^2 + s_{CD}^2 + s_{DC}^2 + s_{DD}^2 = 1. \tag{4.11}$$

With $s_{i,j}$, we indicate decision i as a second move action and decision j as first move action. As a result, the final model only has 5 parameters, which is elegant from a modeling point of view. The resulting probabilities, which should be matched to the observed proportions can be calculated straightforwardly. We get for the second move measurement:

$$\begin{aligned} P(C_{\text{II}}) &= s_{CC}^2 + s_{CD}^2 \\ P(D_{\text{II}}) &= s_{DC}^2 + s_{DD}^2 \end{aligned}$$

and for the first move measurement, after the second move measurement (which is the order in *Baseline* subgroup):

$$\begin{aligned}
P(C_{\rm I}|C_{\rm II}) &= \frac{s_{CC}^2}{s_{CC}^2 + s_{CD}^2} \\
P(D_{\rm I}|C_{\rm II}) &= \frac{s_{CD}^2}{s_{CC}^2 + s_{CD}^2} \\
P(C_{\rm I}|D_{\rm II}) &= \frac{s_{DC}^2}{s_{DC}^2 + s_{DD}^2} \\
P(D_{\rm I}|D_{\rm II}) &= \frac{s_{DD}^2}{s_{DC}^2 + s_{DD}^2}.
\end{aligned}$$

In the *Elicit_Beliefs* subgroup, we will use the ordered scales structure. In this example we opt to use the scaling technique to deal with the fact that our probabilities do not sum to one. Modeling odds would have identical results. We obtain for the belief measurement, after the second move measurement:

$$\begin{aligned}
P(B_i|C_{\rm II}) &= \langle B_i|C_{\rm II}\rangle^2 / \sum_{j=0}^{9} \langle B_j|C_{\rm II}\rangle^2 \\
&= \sin^2\left(\beta_{\rm II} + \frac{i}{9}\frac{\pi}{2}\right) / \sum_{j=0}^{9} \langle B_j|C_{\rm II}\rangle^2 \qquad (4.12) \\
P(B_i|D_{\rm II}) &= \langle B_i|D_{\rm II}\rangle^2 / \sum_{j=0}^{9} \langle B_j|D_{\rm II}\rangle^2 \\
&= \cos^2\left(\beta_{\rm II} + \frac{i}{9}\frac{\pi}{2}\right) / \sum_{j=0}^{9} \langle B_j|D_{\rm II}\rangle^2, \qquad (4.13)
\end{aligned}$$

and for the first move measurement, after the belief measurement:

$$\begin{aligned}
P(C_I|B_i) &= \langle C_{\rm I}|B_i\rangle^2 \\
&= \sin^2\left(\beta_{\rm I} + \frac{i}{9}\frac{\pi}{2}\right) \qquad (4.14) \\
P(D_I|B_i) &= \langle D_{\rm I}|B_i\rangle^2 \\
&= \cos^2\left(\beta_{\rm I} + \frac{i}{9}\frac{\pi}{2}\right). \qquad (4.15)
\end{aligned}$$

As shown in Chapter 4.4, this model has a good fit, using only 5 parameters. Note that, as this structure uses non-orthogonal outcome vectors, it lacks repeatability.

4.5.3 Using a POVM Structure

To model the above example using a POVM structure, most of the construction of the previous section can be retained. We define the belief base in an identical way, with $|B_0\rangle$ and $|B_9\rangle$ orthogonal and the remaining $|B_i\rangle$ in between, with a constant angle between $|B_i\rangle$ and $|B_{i-1}\rangle$. The projectors unto these outcome vectors are:

$$\begin{aligned} P_i &= |B_i\rangle\langle B_i| \\ &= \begin{pmatrix} \sin\frac{i\pi}{18} \\ \cos\frac{i\pi}{18} \end{pmatrix} . \begin{pmatrix} \sin\frac{i\pi}{18} & \cos\frac{i\pi}{18} \end{pmatrix} \\ &= \begin{pmatrix} \sin^2\frac{i\pi}{18} & \sin\frac{i\pi}{18}\cos\frac{i\pi}{18} \\ \sin\frac{i\pi}{18}\cos\frac{i\pi}{18} & \cos^2\frac{i\pi}{18} \end{pmatrix} \end{aligned}$$

Straightforward calculation shows us that the *completeness* restriction, needed to form a POVM, does not hold:

$$\sum_i P_i \neq I_2.$$

The most straightforward way (taken from Yearsley (2016)) to save the construction, is to introduce a new operator P_F which will make the set of operators $\{\frac{P_0}{\mathcal{N}} \dots \frac{P_9}{\mathcal{N}}, \frac{P_F}{\mathcal{N}}\}$[5] satisfy completeness:

$$\frac{P_F}{\mathcal{N}} = I_2 - \sum_i \frac{P_i}{\mathcal{N}}, \tag{4.16}$$

where now the operator $\frac{P_i}{\mathcal{N}}$ represents the outcome B_i that the player thinks i opponents have cooperated.

This is a variation on the method used in Section 4.3.1, where a third $|M_3\rangle$ outcome was introduced. This effectively embeds our wanted outcome construction into a POVM structure, anchoring it firmly into 'standard quantum theory' and dismissing the need for scaling or odds modeling. However, doing this gives rise to a new question: What does the operator P_F represents? As the operator P_F does not have an associated

[5] The constant $\mathcal{N} = \frac{1}{2}\left(10 + \cot\left(\frac{\pi}{18}\right)\right)$ is needed during the calculation to ensure that P_F is a positive operator. As it will disappear during further calculations, we will just refer to Yearsley (2016) where this is proven.

outcome, it can be associated with the outcome B_F, which represents 'no outcome obtained', as suggested in Yearsley (2016). As this is not a valid option during the experiment, we assume the measurement is simply done again, when this (theoretically) would occur. This gives the following probability for obtaining outcome B_i, on the condition that the measurement didn't fail ($\overline{B}_F$), for a given state vector $|S\rangle$:

$$\begin{aligned} P'(B_i) &= P(B_i|\overline{B}_F) \\ &= \frac{P(B_i)}{1-P(B_F)} \\ &= \frac{P(B_i)}{\sum_{j=0}^{9} P(B_j)} \\ &= \frac{\langle S|\frac{P_i}{\mathcal{N}}|S\rangle}{\sum_{j=0}^{9} \langle S|\frac{P_j}{\mathcal{N}}|S\rangle} \\ &= \frac{\langle S|P_i|S\rangle}{\sum_{j=0}^{9} \langle S|P_j|S\rangle} \\ &= \frac{\langle S|B_i\rangle\langle B_i|S\rangle}{\sum_{j=0}^{9} \langle S|B_j\rangle\langle B_j|S\rangle} \\ &= \frac{\langle B_i|S\rangle^2}{\sum_{j=0}^{9} \langle B_j|S\rangle^2}. \end{aligned} \tag{4.17}$$

With this newly defined belief basis, we can redefine $\mathcal{H}_{\mathrm{I},B}$ and $\mathcal{H}_{\mathrm{II},B}$ in an identical way as in Section 4.5.2. Both $\mathcal{H}_{\mathrm{I},B}$ and $\mathcal{H}_{\mathrm{II},B}$ are again two dimensional Hilbert spaces. In $\mathcal{H}_{\mathrm{I},B}$ we have a basis associated with the first move measurement consisting of $|C_I\rangle$ and $|D_I\rangle$, together with the belief basis. The angle between $|B_9\rangle$ and $|C_I\rangle$ is again defined as β_{I}. The construction of $\mathcal{H}_{\mathrm{II},B}$ is identical, with the angle between $|B_9\rangle$ and $|C_{\mathrm{II}}\rangle$ defined as β_{II}. The resulting Hilbert space, modeling the entire experiment is again defined as $\mathcal{H} = \mathcal{H}_{\mathrm{I},B} \otimes \mathcal{H}_{\mathrm{II},B}$.

As we didn't change anything regarding the first and second move basis, only the probabilities of performing the belief measurement after the second move measurement, need to be recalculated, using result 4.17:

$$
\begin{aligned}
P(B_i|C_{\mathrm{II}}) &= \frac{\langle B_i|C_{\mathrm{II}}\rangle^2}{\sum_{j=0}^{9}\langle B_j|C_{\mathrm{II}}\rangle^2} \\
&= \frac{\sin^2\left(\beta_{\mathrm{II}}+\frac{i}{9}\frac{\pi}{2}\right)}{\sum_{j=0}^{9}\langle B_j|C_{\mathrm{II}}\rangle^2} \\
P(B_i|D_{\mathrm{II}}) &= \frac{\langle B_i|D_{\mathrm{II}}\rangle^2}{\sum_{j=0}^{9}\langle B_j|D_{\mathrm{II}}\rangle^2} \\
&= \frac{\cos^2\left(\beta_{\mathrm{II}}+\frac{i}{9}\frac{\pi}{2}\right)}{\sum_{j=0}^{9}\langle B_j|D_{\mathrm{II}}\rangle^2}.
\end{aligned}
$$

This is exactly the same result as in equations 4.12 and 4.13, where the ordered scales, not adhering to completeness are used! This makes the only difference between using a POVM structure or ordered scales the trade-off between a firm embedding in standard quantum theory, with no need for scaling or odds modeling versus no need of an extra operator P_F, representing that (theoretically) no outcome was obtained. In all other aspects (fit of the model, lack of repeatability,. . .) the two approaches are identical.

4.5.4 Using a PVM Structure

We will construct two PVM models, one which mimics the two models described in the previous sections and one alternative model, with different predictions. Note that this sort of construction naturally conflicts with our arguments about ordered scales in Section 4.3.1, as no order of the outcomes is in any way represented in the outcome vectors. Therefore, any sort of structure has to be represented in the state vector, something we argued against.

As the POVM model of Section 4.5.3 resulted in a model having the same predictions as the model of Section 4.5.2, at the cost of the extra outcome vector, the question whether the same results can be obtained using a PVM model presents itself naturally. Naimark's theorem (Gelfand & Naimark, 1943) indeed ensures this. In short, Naimark's theorem shows that any set of operators $\{O_1, \ldots, O_n\}$ forming a POVM in an m-dimensional Hilbert space ($m < n$) can be *lifted* into a PVM, defined in a n-dimensional Hilbert space (see Aliakbarzadeh & Kitto (2016) and Khrennikov & Basieva (2014)). This means that the construction performed in 4.5.3 can be done in a PVM setting. To do so, we lift the Hilbert spaces $\mathcal{H}_{\mathrm{I},B}$ and $\mathcal{H}_{\mathrm{II},B}$ into $\mathcal{H}^l_{\mathrm{I},B}$ and $\mathcal{H}^l_{\mathrm{II},B}$. Both $\mathcal{H}^l_{\mathrm{I},B}$ and $\mathcal{H}^l_{\mathrm{II},B}$ are 11-dimensional as they each contain a set of 11 operators forming a PVM. This makes the resulting Hilbert space $\mathcal{H}^l = \mathcal{H}^l_{\mathrm{I},B} \otimes \mathcal{H}^l_{\mathrm{II},B}$ 121-dimensional. This model does adhere to repeatability, as do all models using only PVM constructions. Note that this model also includes the 'measurement failed outcome. As the probabilities derived in Section 4.5.3 and Section 4.5.2 were shown to be identical, this PVM structure has also the same derived probabilities (and statistical fit) as the model constructed in Section 4.5.2, without the extra outcome. It is unclear if the construction without extra outcome can be lifted into an appropriate PVM setting, as Naimark's theorem is only proven for POVM structures. However, given the amount of free parameters in the 121 dimensional $\mathcal{H}^l$, this seems very plausible.

While the use of Naimark's theorem proves that there is a valid PVM model, the structure of this model is not entirely clear[6]. We will now build a PVM model, not starting from previous models, in the hope of having a clearer geometrical interpretation.

To perform a straightforward implementation of quantum techniques, we should again need to define a Hilbert space $\mathcal{H}_{\mathrm{I},B}$ and $\mathcal{H}_{\mathrm{II},B}$ to represent the incompatibility of the belief and actions measurements. We then should again tensor these two spaces into $\mathcal{H} = \mathcal{H}_{\mathrm{I},B} \otimes \mathcal{H}_{\mathrm{II},B}$, as both action measurements are considered compatible. However, when using a standard PVM structure, the outcomes associated with a single measurement must be represented by a set of orthogonal vectors. Therefore, to model the belief measurement a Hilbert space of dimension at least 10 is, by definition, required. This means that both $\mathcal{H}_{\mathrm{I},B}$ and $\mathcal{H}_{\mathrm{II},B}$ have to be at least 10 dimensional, making $\mathcal{H} = \mathcal{H}_{\mathrm{I},B} \otimes \mathcal{H}_{\mathrm{II},B}$ at least 100 dimensional. In the best case scenario of both $\mathcal{H}_{\mathrm{I},B}$ and $\mathcal{H}_{\mathrm{II},B}$ being 10 dimensional, both move measurements are each represented by two 5-dimensional subspaces[7]. This construction already results in twice 35 free parameters[8], as opposed to the two angles β_{I} and β_{II} in the previous approaches. The state vector $|S\rangle$ in the 100 dimensional $\mathcal{H}$ results in an additional 99 (!) free pa-

[6]When all operators of the POVM structure are projectors, the structure of the lifted PVM is more clear. Here, however, the extra operator B_F is not necessarily a projector.

[7]In what follows all subspaces will be considered Hilbert subspaces.

[8]We need two 5-dimensional subspaces in both $\mathcal{H}_{\mathrm{I},B}$ and $\mathcal{H}_{\mathrm{II},B}$. So, for each space, we have the following construction. A 5-dimensional subspace is spanned by 5 orthonormal vectors. The first vector gives us 9 free parameters, the second vector 8 free parameters,.... This gives us a total of 9+8+7+6+5=35 free parameters for one 5-dimensional subspace. These 35 parameters also fix the second 5-dimensional subspace, as this is the subspace orthogonal to the first 5-dimensional subspace.

rameters, as opposed to three free parameters in the previous approaches. Needless to say that such a model is completely overparametrized and, even when a huge amount of data points would be available, completely unwieldy[9]. As such, if one wishes to just employ PVM techniques, certain external restrictions have to be introduced. We will describe one such model to showcase this possibility. We will specifically aim for this alternative PVM model to have the same amount of parameters as the previous model: three free parameters to fit the starting state vector (like s_{CC}, s_{CD} and s_{DC} previously), one parameter to model the incompatibility between the first move action and the belief measurement (like β_{I} previously) and one parameter to model the incompatibility between the second move action and the belief measurement (like β_{II} previously). This way, this alternative is easy to compare to the previous approaches. It will become apparent that even these relatively simple restrictions will results in an overly complex model.

As mentioned before, since the belief measurement has ten different outcomes, represented by the vectors $|B_i\rangle$, this measurement alone spans a 10-dimensional Hilbert space. As our main concern is to keep the number of dimensions low, we opt to have our resulting Hilbert space $\mathcal{H}$ of this minimal dimensionality. Thus, our state vector has the form

$$|S\rangle = \sum_{i=0}^{9} b_i |B_i\rangle \tag{4.18}$$

Note that the coordinates used in $\mathcal{H}$ now refer to the belief outcome vectors, as opposed to the action outcome vectors previously. In this space, we want to use the ordered structure of the belief measurement to induce some restriction on the state vector[10]. Therefore, we also argue that the state vector should, for some $m \in \{1, \ldots, 8\}$, have the form:

$$|S\rangle = b_{m-1}|B_{m-1}\rangle + b_m|B_{m_i}\rangle + b_{m+1}|B_{m+1}\rangle$$

or

$$|S\rangle = (0, \ldots, 0, b_{m-1}, b_m, b_{m+1}, 0, \ldots, 0).$$

This means that the player can only be in superposition between three

[9] This too high dimensionality was the initial incentive of looking into bases with non-orthogonal vectors

[10] Restrictions on state vectors are more common in physics, as the particular form of the state vector can often be deduced from the experimental setting. This is a lot more difficult in quantum cognition.

consecutive beliefs. This induces an order on the coordinates of the state vector, which in turn induces an order on belief outcome vectors associated with said coordinates. Using these restrictions, makes the state vector result in only three free parameters: m, b_{m-1} and b_m, as b_{m+1} is fixed due to the normalization restriction.

If we consider both action measurements to be compatible, four subspaces $\boldsymbol{C}_{\mathrm{I}}\boldsymbol{C}_{\mathrm{II}}, \boldsymbol{C}_{\mathrm{I}}\boldsymbol{D}_{\mathrm{II}}, \boldsymbol{D}_{\mathrm{I}}\boldsymbol{C}_{\mathrm{II}}$ and $\boldsymbol{D}_{\mathrm{I}}\boldsymbol{D}_{\mathrm{II}}$ need to be defined, each representing a combination of outcomes of the action measurements. Due to symmetry reasons, it is impossible to define these 4 subspace adequately in a 10-dimensional space. We will therefore abandon the requirement that both action measurements are compatible. We will return to this point later. The action measurements are each represented by a pair of orthogonal 5-dimensional subspaces. This still leaves a lot of freedom, so further restrictions will need to be implemented.

To construct restrictions for the 5-dimensional Hilbert subspace representing the first and second move outcomes in $\mathcal{H}$, we infer from the data that players believing that a high number (6,. . . ,9) of opponents have cooperated are likely to cooperate themselves, while players believing that a high number (6,. . . ,9) of opponents have defected are likely to defect themselves. As such we argue that for the 5-dimensional subspace $\boldsymbol{C}_{\mathrm{I}}$, representing first move cooperation in $\mathcal{H}$:

$$|B_6\rangle, \ldots, |B_9\rangle \in \boldsymbol{C}_{\mathrm{I}} \tag{4.19}$$

and for the subspace $\boldsymbol{C}_{\mathrm{I}}^{\perp} = \boldsymbol{D_I}$, representing first move defection in $\mathcal{H}_{\mathrm{I},B}$:

$$|B_0\rangle, \ldots, |B_3\rangle \in \boldsymbol{D_I}. \tag{4.20}$$

This defines $\boldsymbol{C}_{\mathrm{I}}$ as:

$$\boldsymbol{C}_{\mathrm{I}} = \langle |B_9\rangle, |B_8\rangle, |B_7\rangle, |B_6\rangle, |B_{C_I}\rangle\rangle$$

and $\boldsymbol{D}_{\mathrm{I}}$ as:

$$\boldsymbol{D_I} = \langle |B_{D_I}\rangle, |B_3\rangle, |B_2\rangle, |B_1\rangle, |B_0\rangle\rangle\,,$$

where the vectors $|B_C\rangle$ and $|B_D\rangle$ are, by definition, orthogonal and normalized vectors in the plane $\langle |B_5\rangle, |B_4\rangle\rangle$. Therefore, they have the form:

$$\begin{aligned} |B_{C_I}\rangle &= (0,0,0,0,\cos\theta_{\mathrm{I}},\sin\theta_{\mathrm{I}},0,0,0,0) \\ |B_{D_I}\rangle &= (0,0,0,0,-\sin\theta_{\mathrm{I}},\cos\theta_{\mathrm{I}},0,0,0,0). \end{aligned}$$

The projector $P_{C_{\mathrm{I}}}$ associated with cooperating on the first move is the 10×10 matrix:

$$P_{C_{\mathrm{I}}} = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & B_{C_{\mathrm{I}}} & \vdots \\ 0 & \dots & I_4 \end{pmatrix} \text{ with } B_{C_{\mathrm{I}}} = \begin{pmatrix} \cos^2\theta_{\mathrm{I}} & \cos\theta_{\mathrm{I}}\sin\theta_{\mathrm{I}} \\ \cos\theta_{\mathrm{I}}\sin\theta_{\mathrm{I}} & \sin^2\theta_{\mathrm{I}} \end{pmatrix}$$

and the projector $P_{D_{\mathrm{I}}}$ associated with defecting on the first move:

$$P_{D_{\mathrm{I}}} = \begin{pmatrix} I_4 & \dots & 0 \\ \vdots & B_{D_{\mathrm{I}}} & \vdots \\ 0 & \dots & 0 \end{pmatrix} \text{ with } B_{D_{\mathrm{I}}} = \begin{pmatrix} \sin^2\theta_{\mathrm{I}} & -\cos\theta_{\mathrm{I}}\sin\theta_{\mathrm{I}} \\ -\cos\theta_{\mathrm{I}}\sin\theta_{\mathrm{I}} & \cos^2\theta_{\mathrm{I}} \end{pmatrix}.$$

An identical construction can be done to define the 5-dimensional subspaces $\boldsymbol{C}_{\mathrm{II}}$ and $\boldsymbol{D}_{\mathrm{II}}$, representing cooperation and defection on the second move in $\mathcal{H}$. Each has an associated projector $P_{C_{\mathrm{II}}}$ and $P_{D_{\mathrm{II}}}$, now with the free parameter θ_{II}. Note that the subspaces associated with the first move depend on one parameter, just as in the previous models. The same holds for the subspaces associated with the second move. While the action measurements are not completely compatible, the subspace spanned by $\{|B_0\rangle, \dots, |B_3\rangle\}$, for example, does represent a player defecting on both moves.

The restrictions on the starting state vector (resulting in the parameters m, b_{m-1} and b_m) and on the first and second move subspaces (resulting in the parameters θ_{I} and θ_{II}) define our alternative PVM model. The resulting probabilities for the second move measurement are:

$$\begin{aligned} P(C_{\mathrm{II}}) &= \langle S|P_{C_{\mathrm{II}}}|S\rangle \\ &= b_4^2\cos^2\theta_{\mathrm{II}} + b_5^2\sin^2\theta_{\mathrm{II}} \\ &\quad +2b_4b_5\cos\theta_{\mathrm{II}}\sin\theta_{\mathrm{II}} \\ &\quad +b_6^2+\dots+b_9^2 \\ &= (b_4\cos\theta_{\mathrm{II}} + b_5\sin\theta_{\mathrm{II}})^2 \\ &\quad +b_6^2+\dots+b_9^2. \end{aligned} \tag{4.21}$$

Note that this gives us

$$P(C_{\mathrm{II}}) = 1 \qquad \text{for } m = 7 \text{ or } 8 \tag{4.22}$$

$$P(C_{\mathrm{II}}) = 0 \qquad \text{for } m = 1 \text{ or } 2. \tag{4.23}$$

Likewise we get:

$$\begin{aligned} P(D_{\mathrm{II}}) &= \langle S|P_{D_{\mathrm{II}}}|S\rangle \\ &= b_0^2 + \ldots + b_3^2 \\ &\quad + b_4^2 \sin^2\theta_{\mathrm{II}} + b_5^2 \cos^2\theta_{\mathrm{II}} \\ &\quad - 2b_4 b_5 \cos\theta_{\mathrm{II}} \sin\theta_{\mathrm{II}} \\ &= b_0^2 + \ldots + b_3^2 \\ &\quad + (b_4 \sin\theta_{\mathrm{II}} - b_5 \cos\theta_{\mathrm{II}})^2, \end{aligned} \tag{4.24}$$

which now gives us

$$P(D_{\mathrm{II}}) = 0 \qquad \text{for } m = 7 \text{ or } 8 \tag{4.25}$$

$$P(D_{\mathrm{II}}) = 1 \qquad \text{for } m = 1 \text{ or } 2. \tag{4.26}$$

The probability of the player cooperating on both the second and first move in the *Baseline* subgroup are[11]:

$$P(C_{\mathrm{II}}).P(C_I|C_{\mathrm{II}}) = \langle S|P_{C_{\mathrm{II}}} P_{C_{\mathrm{I}}} P_{C_{\mathrm{II}}}|S\rangle \tag{4.27}$$

$$\begin{aligned} &= b_0^2 + \ldots + b_3^2 \\ &\quad + b_4^2 f_4(\theta_{\mathrm{I}}, \theta_{\mathrm{II}}) + b_5^2 f_5(\theta_{\mathrm{I}}, \theta_{\mathrm{II}}) \\ &\quad + 4b_4 b_5 f_{4,5}(\theta_{\mathrm{I}}, \theta_{\mathrm{II}}), \end{aligned} \tag{4.28}$$

[11]Note that we now model an event of the form 'A and then B' instead of 'B if A', as the $P(B)$ in the denominator of $P(B|A) = \frac{P(A \text{ and then } B)}{P(B)}$ would make the resulting formula even more unwieldy. If one wishes to derive the conditional probability, simply divide by the appropriate formula 4.21 or 4.24.

with

$$
\begin{aligned}
f_4(\theta_{\mathrm{I}},\theta_{\mathrm{II}}) &= \cos^2\theta_{\mathrm{I}}\cos^4\theta_{\mathrm{II}} + 4\sin^2\theta_{\mathrm{II}}\sin^2\theta_{\mathrm{I}}\cos^2\theta_{\mathrm{II}} \\
&\quad +8\sin\theta_{\mathrm{II}}\cos^3\theta_{\mathrm{II}}\sin\theta_{\mathrm{I}}\cos\theta_{\mathrm{I}} \qquad (4.29)\\
f_5(\theta_{\mathrm{I}},\theta_{\mathrm{II}}) &= 4\sin^4\theta_{\mathrm{II}}\sin^2\theta_{\mathrm{I}} + \sin^2\theta_{\mathrm{II}}\cos^2\theta_{\mathrm{II}}\cos^2\theta_{\mathrm{I}} \\
&\quad +8\sin^3\theta_{\mathrm{II}}\cos^2\theta_{\mathrm{II}}\cos\theta_{\mathrm{I}}\sin\theta_{\mathrm{I}} \qquad (4.30)\\
f_{4,5}(\theta_{\mathrm{I}},\theta_{\mathrm{II}}) &= \sin\theta_{\mathrm{II}}\cos^2\theta_{\mathrm{I}}\cos^3\theta_{\mathrm{II}} \\
&= +\sin^3\theta_{\mathrm{II}}\cos\theta_{\mathrm{II}}\sin^2\theta_{\mathrm{I}} \\
&\quad +5\sin^2\theta_{\mathrm{II}}\cos^2\theta_{\mathrm{II}}\cos\theta_{\mathrm{I}}\sin\theta_{\mathrm{I}}. \qquad (4.31)
\end{aligned}
$$

Likewise formulas can be derived for the other three cases. Luckily, these formulas never have to be formally fitted, as this model is overparametrized: only three proportions are observed in the *Baseline* subgroup and the above formulas have 5 free parameters[12]. This derivation, however, shows how quickly calculations increase in complexity when dealing with more than two measurements with at least two outcomes. The reason is that, even in the *Baseline* group, everything is described in terms of ten dimensional belief coordinates.

The more interesting case is the *Elicit_Beliefs* subgroup, as now the interactions between beliefs and actions come into play. When beliefs are elicited, projections will occur on vectors in $\mathcal{H}$ (as opposed to 5-dimensional subspaces). This will make calculations will also be more manageable. Note that the probabilities $P(C_{\mathrm{II}})$ and $P(D_{\mathrm{II}})$ derived in 4.21 and 4.24 also hold in *Elicit_Beliefs* subgroup. The resulting probabilities are

$$
P(B_i|C_{\mathrm{II}}) = \frac{\langle S|P_{C_{\mathrm{II}}}|B_i\rangle\langle B_i|P_{C_{\mathrm{II}}}|S\rangle}{P(C_{\mathrm{II}})}
$$

[12] In the non-PVM models, the similar formulas had only three parameters: s_{CC}, s_{CD} and s_{DC}.

which gives us

$$P(B_i|C_{\rm II}) = \frac{b_i^2}{P(C_{\rm II})} \qquad \text{for } i = 6, 7, 8 \text{ or } 9 \tag{4.32}$$

$$P(B_4|C_{\rm II}) = \cos^2\theta_{\rm II}\frac{b_4^2\cos^2\theta_{\rm II} + 2b_4b_5\cos\theta_{\rm II}\sin\theta_{\rm II} + b_5^2\sin^2\theta_{\rm II}}{P(C_{\rm II})} \tag{4.33}$$

$$= \cos^2\theta_{\rm II}\frac{(b_4\cos\theta_{\rm II} + b_5\sin\theta_{\rm II})^2}{P(C_{\rm II})} \tag{4.34}$$

$$P(B_5|C_{\rm II}) = \sin^2\theta_{\rm II}\frac{b_4^2\cos^2\theta_{\rm II} + 2b_4b_5\cos\theta_{\rm II}\sin\theta_{\rm II} + b_5^2\sin^2\theta_{\rm II}}{P(C_{\rm II})} \tag{4.35}$$

$$= \sin^2\theta_{\rm II}\frac{(b_4\cos\theta_{\rm II} + b_5\sin\theta_{\rm II})^2}{P(C_{\rm II})}$$

$$P(B_i|C_{\rm II}) = 0 \qquad \text{for } i = 0, 1, 2 \text{ or } 3. \tag{4.36}$$

Similar calculations give us:

$$P(B_i|D_{\rm II}) = 0 \qquad \text{for } i = 6, 7, 8 \text{ or } 9 \tag{4.37}$$

$$P(B_4|D_{\rm II}) = \sin^2\theta_{\rm II}\frac{b_4^2\sin^2\theta_{\rm II} - 2b_4b_5\cos\theta_{\rm II}\sin\theta_{\rm II} + b_5^2\cos^2\theta_{\rm II}}{P(D_{\rm II})} \tag{4.38}$$

$$= \sin^2\theta_{\rm II}\frac{(b_4\sin\theta_{\rm II} - b_5\cos\theta_{\rm II})^2}{P(D_{\rm II})} \tag{4.39}$$

$$P(B_5|D_{\rm II}) = \cos^2\theta_{\rm II}\frac{b_4^2\sin^2\theta_{\rm II} - 2b_4b_5\cos\theta_{\rm II}\sin\theta_{\rm II} + b_5^2\cos^2\theta_{\rm II}}{P(D_{\rm II})} \tag{4.40}$$

$$= \cos^2\theta_{\rm II}\frac{(b_4\sin\theta_{\rm II} - b_5\cos\theta_{\rm II})^2}{P(D_{\rm II})} \tag{4.41}$$

$$P(B_i|D_{\rm II}) = \frac{b_i^2}{P(D_{\rm II})} \qquad \text{for } i = 0, 1, 2 \text{ or } 3. \tag{4.42}$$

Lastly, the probabilities of cooperating as first mover, after the belief elicitation are

$$P(C_I|B_i) = \langle B_i|P_{C_{\rm I}}|B_i\rangle \tag{4.43}$$

$$P(C_I|B_i) = 0 \qquad \text{for } i = 0, 1, 2 \text{ or } 3 \tag{4.44}$$

$$P(C_I|B_i) = 1 \qquad \text{for } i = 6, 7, 8 \text{ or } 9 \tag{4.45}$$

$$P(C_I|B_4) = \cos^2\theta_{\rm I} \tag{4.46}$$

$$P(C_I|B_5) = \sin^2\theta_{\rm I} \tag{4.47}$$

and the probabilities of defecting as first mover, after the belief elicitation are

$$P(D_I|B_i) = \langle B_i|P_{D_\mathrm{I}}|B_i\rangle \tag{4.48}$$
$$P(D_I|B_i) = 1 \qquad (\text{for } i = 0,1,2 \text{ or } 3) \tag{4.49}$$
$$P(D_I|B_i) = 0 \qquad (\text{for } i = 6,7,8 \text{ or } 9) \tag{4.50}$$
$$P(D_I|B_4) = \sin^2\theta_\mathrm{I} \tag{4.51}$$
$$P(D_I|B_5) = \cos^2\theta_\mathrm{I}. \tag{4.52}$$

The particular form of the model makes it hard to perform a sensible complete fit to the data. Next to the very unwieldy formulas, the problem lies in the restriction we placed on the initial state vector, as this makes any player only in superposition between three consecutive beliefs. If we would fit this to the data at hand, the 'average player' would also have to be in a superposition between three consecutive beliefs, effectively discarding most of the results, as the model would predict that most of its outcomes can never be obtained, due to dependencies such as 4.22, 4.23, 4.25 and 4.26. The model only yields interesting results if a starting state can be attributed to certain (group of) players[13]. As we only have data consisting of proportions, we cannot infer this type of structure. If this would be possible, density matrices could be used to model the 'average player' and the entire model could be held against the data. This again shows the limits of placing restrictions on the state vector, as opposed to incorporating in it in the outcome vectors from both a mathematical and an interpretational point of view.

However, a discussion of the conditional probabilities is interesting, as this means that there is information about the state of the system. As such, results 4.32 to 4.42, tell us that second move cooperators never think that $0\ldots3$ opponents have cooperated and second move defectors never think that $6\ldots9$ opponents have cooperated. The data follows this same structure, with only one player violating this (Table 3.4 and Table 3.5). The same seemingly nice result can be seen when looking at the first move cooperation probabilities, given a belief. Results 4.43 to 4.52 tell us that players believing $0\ldots3$ opponents have cooperated, would always defect and that players believing $6\ldots9$ opponents have cooperated, would always cooperate. This again reflects the structure in the data very well,

[13]In physics, this means that a specific state preparations would be performed as the first step of the experiment. While there are numerous ways of doing this in physics, this is practically impossible in cognition: How could you make somebody certainly only doubt between three out of ten options? See also footnote 10.

with no violation (Table 3.6). It might be tempting to use this part of the data set to calculate the estimated value of θ_{I}, but as this parameters also appears elsewhere in the model, this cannot be considered a true fit to the data[14]. While these results seem interesting at first, these predictions do not flow naturally from the model, but were introduced when implementing Restrictions 4.19 and 4.20.

It is clear that even a ‘simple’ PVM model, with very rough, but very strict restrictions almost immediately results in a very complex model, even though the amount of parameters is low, while it does not result in any interesting new predictions. Even though this cannot strictly be held against this PVM approach from a modeling point of view, it does showcase the difficulties that arise when slightly more complex modeling situations are considered for a quantum-like treatment. Keep in mind that the restrictions used here (the particular form of both state vectors and action subspaces) are very ad hoc and only serve as an exploration of a possible alternative PVM model. This reservation, however, does not mean that a thorough researched PVM model will be less complex. On the contrary, well fleshed-out restrictions probably would increase the complexity.

4.5.5 Comparison

We now wish to compare our three approaches, contrasting their strengths and weaknesses. While we refer to the game theoretic models constructed in this paper, this list also indicates strengths and weaknesses of the different approaches in a more general setting. The different comparisons are listed in Table 4.1 Note that we have two different PVM models: one which is the lift of the POVM models into a large enough Hilbert space, using Naimark’s theorem (NaiPVM) and the alternative model, using self-constructed restrictions (altPVM). We refrain from a thorough comparison with classical models, as this falls outside the scope of the paper. We just wish compare techniques within the quantum realm. We do wish to mention that all models constructed here, explain the violation of the sure-thing principle observed in the data. As such, their ‘quantumness’ does what it was intended to do, as this violation was the incentive of looking in quantum-like models. This is commented on more profoundly in Chapter 3.

[14]In the non-PVM models, this was not a problem, as β_{I} only appeared in the probabilities 4.14 and 4.15.

	Ordered Scales	POVM	NaiPVM	altPVM
Explains the Sure-Thing violation	✓	✓	✓	✓
Represents ordered structure in outcome vectors	✓	✓	×	×
Adheres to repeatability	×	×	✓	✓
Scaling or odds modeling not needed	×	✓	✓	✓
Within standard quantum theory	×	✓	✓	✓
No extra theoretical outcome needed	✓	×	×/?[a]	✓
Clear geometrical interpretation	✓	✓	×	✓
Action measurements compatible	✓	✓	✓	×
Natural low number of parameters, without complex restrictions	✓	✓	×	×
Good statistical fit/interesting predictions	✓	✓	✓	×[b]

Table 4.1 Comparing all derived models for the sequential prisoner dilemma data.

[a] There might be a way to lift the Ordered Scales model into a PVM setting. This is an open question, as Naimark's theorem only is proven for POVM structures.

[b] For the specific model constructed previously.

4.6 Concluding Remarks

In this paper we propose some tentative first steps towards modeling ordered scales using quantum-like techniques. After loosening some of the restrictions used in the construction of projective measurements (and even loosening restrictions placed upon POVMs), we use this lack of structure to impose new structure, now originating from the structure that outcomes themselves exhibit. These techniques also allow for a reduction of the resulting dimensionality, as this can become problematic quickly in slightly more complex situations than the common examples seen in quantum cognition. We briefly mention the possibility to model Likert scales as an example. We discuss the two biggest consequences of this approach, the first one being the total sum across all probabilities exceeding one and the second one being the loss of repeatability of outcomes. Exceeding one when adding the probabilities makes scaling necessary or requires the modeling of odds of outcomes instead of probabilities. We argue that the loss of repeatability is not as problematic as it seems at first and provide a possible interpretation of this phenomenon.

Finally, we use data of a game theoretic experiment to investigate the merits of this new approach and contrast it with more traditional quantum-like structures. The contrast between using the new projectors and the POVM structure shows that they result in the same predictions. So the choice between which approach is preferred, lies in the hand of the modeler. The new projectors have a straightforward geometrical interpretation, with its vectors clearly reflecting the ordered structure of the experiment, at the cost of needing scaling or odds modeling. The POVM approach, while now clearly embedded within standard quantum theory, needed an extra operator, with its own theoretical outcome, thereby muddying the straightforward interpretation. Both these methods lack repeatability, but represent the order structure of outcomes appropriately.

If the modeler desires to adhere to repeatability, PVM structures need to be employed. There is a direct candidate for this, by lifting the POVM structure using Naimark's theorem, into a 121-dimensional Hilbert space. While this approach results in the same probabilities, the structure itself is unclear. This approach also does not model any ordered structures in the outcome spaces, it just mimics the results of another approach that does. We also construct an alternative PVM model to show that lifting a POVM structure is not the only solution. This example shows that, when trying to reduce the enormous amount of parameters, introducing even the simplest restrictions results in a very opaque model. This model only shows a nice structure when conditional probabilities are considered.

This contribution is only a first step into modeling ordered scales in a quantum-like way. The theoretical side of this story needs to be deepened, with a more thorough discussion of the concepts sketched out in Section 4.3.1, next to investigating structures similar in role to Naimark's Theorem for POVMs. Also, more data-driven applications than the one presented here need to be formulated and statistically tested to investigate the true merit of this new approach. Next to deepening the understanding of our proposed projectors, other ways of incorporating an ordering should be investigated. As mentioned in footnote 1, eigenvalues are used to represent ratio scales (and its ordering). While eigenvalues are not presently used in quantum cognition, it might be interesting to incorporate them to model ordered scales. To do so, outcomes would need an associated numerical value. This is not a trivial task (Krantz et al., 2006; Suppes et al., 2006; Luce et al., 2006), but might, e.g., be used to model utilities in decision making.

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5 General discussion

5.1 General Overview

In this thesis we examined what exactly makes the quantum-technical framework successful in cognition and when its implementation becomes problematic. We focus on two types of issues. The first type of issue concerns the question when and why a quantum-like model naturally predicts interesting non-classical results and when an easily deducible classical equivalent can be found, meaning that the quantumness of the approach doesn't necessarily add much relevancy. The second type of issue is the mathematical or modeling limits this approach in certain cases runs up against. An example of such limits is a lack of tools to deal with certain experimental phenomena (e.g. the outcomes having a certain relation to each other). Another example is the situation in which a naturally implied quantum-like model becomes too overly complex to result in sensible predictions. We hope this discussion will lead to a better understanding of how to construct successful quantum-like models, fully employing the strengths of this approach.

We claim that in the cases we discussed the (non)-orthogonality of basis vectors is a key feature, being both cause of and solution to mentioned issues. To argue this point, we opted for an applied and data-driven approach. As opposed to a more theoretical approach, we discuss two experimental paradigms with collected data, show why they benefit from certain quantum-like approaches and contrast these different quantum models to discuss which is better suited. Even though our two examples are very specific (from two very distinct fields), we think they can be used to comment on quantum cognition in general. To do so, we will now situate the previous chapters within the broader quantum cognition field.

5.2 CMT Model - Discussion

In Chapter 2 we took a critical look at a quantum-like model, viz. the QEM model, constructed in Brainerd et al. (2013) for an experiment about word recollection. In these types of experiments a violation of the classical disjunction rule is frequently observed. This phenomenon is commonly attributed to the existence of two distinct types of memory traces: gist and verbatim. The authors sought to explain this violation of a Kolmogorovian rule by considering a quantum approach. Such violations are commonly the first indicator of and the incentive to consider a quantum-like model.

However, we showed that a straightforward classical equivalent[1] can easily be constructed, because Brainerd et al. represent the different memory traces by orthogonal structures. This means that the disjunction violation can easily be explained by the equivalent classical model, not using any quantum-like structure. In this classical model the reason for this violation is clear: the conjunction part of the disjunction equation disappears (see 2.3.3 for details). As this model is equivalent to the QEM model, the similar disappearance of the conjunction part when defining the QEM model is also the reason why the disjunction fallacy is exhibited in the QEM model. While the quantum nature of the QEM model might obscure this peculiar disappearance, it is certainly not the reason for the violation itself[2].

We then argued that the relevance of the model might be saved by one simple change: make the subspaces representing the two distinct but interacting memory traces non-orthogonal. By using the implied complementarity between these traces, this intervention effectively made the quantum-like approach explain the violation, without a clear natural classical equivalent. The interpretational advantage over the QEM model is that the violation is now a natural consequence of our quantum-like structure, not an inconsistency explained by redefining the disjunction rule. Our alternative CMT model had a satisfying statistical fit, slightly outperforming the QEM model.

We claim that the type of reasoning adopted in this chapter, can be applied in a broader setting. We can construct a clear classical equivalent for quantum-like models with all outcome subspaces orthogonal and, for that reason, those ventures within the quantum-technical realm should be met with some skepticism. We do not claim that there are no classical models that can exhibit violations of statistical laws or, in this particular case, that the CMT model is 'wrong'. However, in the quantum formalism these violations are not considered 'inconsistencies that should be explained' (which is the case in a classical approach), but naturally implied consequences. To us, the natural rise of this behavior lies at the heart of the quantum approach. As such, quantum-like models with a clear classical equivalent lack one of the main draws of the quantum-like approach, as their exhibition of the violation of interest is not a natural, unparadoxical consequence.

[1]By 'equivalent' we mean resulting in identical predictions.

[2]By no means do we claim that quantum approaches fail to explain violations of classical statistical rules. We just wish to show that this is the case in this particular paradigm

Note that we do not mean that different incompatible measurements (associated with different, non-orthogonal bases) should be performed to observe any quantum-like, non-classical behavior. For example the Kochen-Specker (KS) theorem is a clearly non-classical result, with only one measurement performed. However, even in the KS theorem, some non-orthogonal outcome vectors are introduced, see for example Cabello (1997).

5.3 The QP&B Model - Discussion

In Chapter 3 we investigated a sequential prisoner dilemma experiment from Blanco et al. (2014), in which not only the moves of players were recorded, but, in a subgroup of the sample, the beliefs about the behavior of opponents were also elicited. We identified three effects, two of which pointed towards a quantum-like approach. It quickly became apparent that a straightforward quantum-like construction resulted in an unsatisfactory model. These problems were countered by allowing non-orthogonal subspaces to represent outcomes of one measurement. The end result was an elegant, statistically well-fitting model.

There are two points of interest about this construction in a broader view. First, it serves as a nice quantum-like venture into a game-theoretical setting. Most research concerning strategic decision-making, using the quantum toolbox is quite theoretical and/or does not go beyond the classical simultaneous prisoner dilemma setting. As such, this QP&B model is one of the first attempts at tackling a data set from a slightly more complicated game. The fact that the model performs statistically well, makes us hopeful about future research into other game-theoretical structures. As the projections on the non-orthogonal subspaces actually induce a rotation on the state vector, this model is firmly embedded into other quantum-like game-theoretical approaches. For example, for the standard prisoner dilemma Pothos & Busemeyer (2009) introduce a Hamiltonian which also rotates the state vector in a Hilbert space. This Hamiltonian is said “to produce the change in beliefs about the opponent towards defection or cooperation". It causes the model to violate the sure-thing principle. This is very similar to what our non-orthogonal belief vectors do: to rotate the state vector to represent belief construction, while explaining the sure-thing principle violation. The main difference between the two models is that in our QP&B model these beliefs are explicitly elicited. This means we wish to derive predictions which we then fit (successfully) to the experimental data.

Secondly, even though POVM structures, for instance, have been theoretically discussed, the QP&B model is the first application[3] in social sciences that uses quantum-like measurements beyond the PVM structure, even though they are used in physics to model, e.g., noise in measurement. The abandonment of the orthogonal subspaces was a necessity at first in order to avoid an overparametrized model, but was also used to incorporate the ordered structure of the outcomes in a natural way. This was explored further in Chapter 4.

This development mirrors the advances made in Chapter 2 and is the recurring theme in this thesis. In both paradigms, viz. the memory experiment and sequential prisoner dilemma, the possible use of a quantum-technical framework becomes clear quite quickly. The construction itself, however, runs into fundamental issues twice. Even though the problems at hand show a different nature, the proposed solutions follow the same pattern: by making previously orthogonal subspaces non-orthogonal, we twice obtained a well-fitting model, exhibiting the observed violations of classical statistics as a direct consequence of their quantum-like nature.

5.4 Ordered Scales - Discussion

In Chapter 4 we investigated a possible representation of ordered structure in outcomes. We loosened restrictions of the P(O)VM structures, using the freedom gained to add structure emerging from the experimental setting. The two major consequences, viz. loss of repeatability and need for scaling or odds modeling, were discussed. An overview of the (dis)advantages of all considered approaches was given. This idea was introduced and implemented in Chapter 3, but the introduction there was ad hoc, specifically meant to solve the issues emerging in that particular setting. Therefore, this structure deserved a more thorough look and an investigation in a broader sense.

This research can be placed in a broader movement within quantum cognition. Recently, an increasing amount of research papers have appeared, focusing on structures beyond the PVM framework. For example, an interesting foray into the use of POVMs can be found in Aliakbarzadeh & Kitto (2016) and Yearsley (2016). The former is a more theoretical work, discussing how Bell-type inequalities can be obtained using so called

[3] Or at least the first application known to us.

'non-ideal measurements' represented by POVMs. While the applications presented focus on the modeling of semantic memory, the discussion is interesting for other fields as well. The latter is part of a tutorial which aims at providing a relatively simple method of applying POVMs (among other quantum techniques) in social sciences.

However, we propose a structure that generalizes beyond the POVM structure. While situations in which POVMs are too strict were known in, for instance, quantum computation (see Chapter 8 of Nielsen & Chuang (2010)), this need for generalization was first mentioned quite recently in the quantum cognition field in Khrennikov et al. (2014). In this paper the authors discuss the fact that it is impossible to use conventional techniques in order to construct a model adhering to both repeatability[4] and order effects. This situation didn't emerge before in known quantum cognition models, as the concept of repeating a measurement did not occur in any experimental setting. Whereas Khrennikov et al. (2014) provide a meticulous mathematical proof, we can briefly sketch the issue using a simple example, relying on conventional techniques. If we go back to the Clinton/Gore experiment, discussed in Section 1.3.1, we have a clear example of an experimental setting exhibiting question order effects. As such, if a participant was asked if he thought Clinton was trustworthy, it was assumed his position about the trustworthiness of Gore, is changed. However, a simple symmetry argument shows that the Gore question should consequently also change the position regarding Clinton's trustworthiness. So, if a third question, identical to the first, is asked, this framework does not predict that the first answer will be repeated. Khrennikov et al. prove that this cannot be solved using a POVM structure. Although our ordered scales lack first kindness repeatability (two identical measurements not yielding the same outcome when performed without any other measurement in between) and Khrennikov et al. discuss the lack of second kindness repeatability (two identical measurements not yielding the same outcome when performed with one other measurement in between), the discussion in Khrennikov et al. (2014) is still relevant for our analysis in Chapter 4. One possible solution to this is presented in Aerts & de Bianchi (2015), where the General Tension Reduction (GTR) model is proposed. An overview of this model falls outside the scope of this thesis, but we mention it to show that POVM generalizations are emerging elsewhere as well, next to other types of research beyond the standard quantum setting

[4]The type of repeatability is different from the one discussed in Chapter 4. See the example infra.

such as quantum pseudo-logics (Matvejchuk & Widdows, 2015) and the use of negative probabilities (de Barros & Oas, 2014).

So far, none of the ventures mentioned here have investigated other types of scales nor focused on structure within the set of outcomes. Even in the GTR model, which is the most generalized model known to us, all outcomes play an identical role towards each other. One area where some sort of internal structure is incorporated in a quantum-like model is semantic representations. In, for instance., Widdows & Cohen (2015), graded (and thus structure exhibiting) semantic quantities are represented by graded semantic vectors in a Hilbert space. While the fields of game-theory and semantic representation are far apart, some crossover of ideas for this problem might prove fruitful. It is clear that much research along these lines still needs to be done.

5.5 Conclusion and Future Research

This brings us back to our initial research question: what drives the power behind quantum cognition and how is this power best harnessed? It turns out that in both of the models we presented, the non-orthogonality of relevant outcome vectors transformed a problematic model into a well-functioning one. As pointed out in Chapter 2, non-orthogonal outcome vectors transform the QEM model into the CMT model, which was shown to be clearly non-classical. In Chapter 3, the non-orthogonality solved the overparametrization problem and incorporated the ordered structure of outcomes. Note that the use of the mentioned non-orthogonality is different in both examples. In Chapter 2 the non-orthogonality was between vectors from different measurements, signifying complementarity between these measurements. In Chapter 3 the non-orthogonality was not only used to model incompatible measurements, but was also between vectors from the same measurement, signifying an unsharp measurement leading to noise, as with a POVM structure. However, in both cases, the non-orthogonality (and its resulting non-commuting operators) plays a central role. When looking at formal tests for 'quantumness', as opposed to our example-driven approach, the same story emerges. For example, the q-test, from Busemeyer & Bruza (2012), essentially tests if two measurements commute. A second example, also containing non-commuting operators, are the grand reciprocity (GR) equations, discussed in Boyer-Kassem et al. (2016). Here, it is shown that most basic "quantum"[5] mod-

[5]The quotation marks are deliberate.

els, actually *fail* the GR equations. A suggested solution is to construct degenerate models[6], as these do not need to adhere to the GR equations. We will come back to this point later.

This leads us to the idea of the 'general quantum recipe'. It should be clear from the discussions in Section 3.4.3 and Section 4.5.1 that a naive straightforward implementation of an experimental setting into the standard PVM structure can run into trouble fast, where it concerns situations more complex than the standard examples. Due to quick overparametrization (the number of dimensions rises exponentially with more outcomes) extra structure needs somehow to be incorporated. How and which structure is an open question. This need for extra complexity becomes even more pressing if we consider the previously mentioned observation of Boyer-Kassem et al. (2016) that degenerate (and therefore larger) models are required to ensure good predictions, even for the simple models. This, combined with the already highlighted questions related to second kindness, makes it clear that the quantum cognition field is facing a big task.

It is evident that the use of quantum techniques in cognition leads to compelling, viable models. Both on a mathematical and an interpretational level, they offer insights and interesting predictions. This point has been proven abundantly. However, in our opinion, to evolve the field beyond these first successful steps, we presently need more models that go beyond the standard exemplary (toy) models. The issues mentioned in this thesis (quantumness, overparametrization, structure in outcomes, second kindness . . .) make it appear as if a technical hurdle needs to be taken. As a result, a lot of recent research has been focused on theoretical ideas, discussions and techniques, which seek to investigate or push the boundaries of the theoretical side of the quantum cognition field. On the other hand, (complex) applications in cognition seem rather sparse. To put this in perspective, during the recent tenth Quantum Interaction Conference in San Francisco only 5 out of 26 presented papers or posters included discussions of a tested quantum-like model in an experimental setting[7]. We do not mean to diminish the importance of any of these theoretical ventures (our Chapter 4 is also a rather theoretical discussion of techniques beyond the standard quantum theory). They seem currently necessary to forward the field.

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[6]By degenerate models, we mean models containing degenerate outcomes. A degenerate outcome is an outcome represented by a subspace of dimension more than one.

[7]An observation made by Sebastien Duchêne at said conference.

However, it is important that this theoretical research does not lose sight of the possible applications. In this thesis, we have advocated this by always trying to present the more theoretical discussions through an application. These applications were not toy models, but interesting in themselves. The drawback of this approach is that one successful application of a certain new technique does not mean that the technique is applicable in other cases. Therefore, more applications of our ordered scales structure in particular, other 'beyond PVM' structures in general and more sophisticated PVM applications are needed. We mention one such possible research program in our conclusion to Chapter 4. The introduction of the use of eigenvalues seems a promising idea, as this would allow more (ordered) structure to be represented in the model. This is again no small task as the assignment of numerical values to ordered scales is no trivial problem (Krantz et al., 2006; Suppes et al., 2006; Luce et al., 2006). These numerical values might, for instance, be used to represent utilities, opening up possibilities of deeper connections to decision making.

Let us conclude by voicing the firm belief that quantum cognition says something very fundamental about human behavior. It takes into account aspects of the contextuality of human decision making in a manner that revolutionizes the way we think about and deal with cognition. We are looking forward to see how the field deals with the challenges we discussed and continues to grow.

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6 Nederlandstalige Samenvatting

6.1 Inleiding

Kwantumcognitie is een relatief nieuw veld waarin gebruikt wordt gemaakt van het wiskundig formalisme uit kwantummechanica, om fenomenen uit menswetenschappen te modelleren. Dit formalisme herdefinieert concepten zoals 'meting' en 'gebeurtenis'. Door gebruik te maken van deze hergedefinieerde aspecten, worden statistische modellen gebouwd die niet per se de klassieke statistische regels (conjunctie regel, disjunctie regel, ...) hoeven te volgen. Hoewel we de klassieke statistische regels traditioneel associëren met 'rationeel denken' zijn er talloze experimentele voorbeelden die aantonen dat menselijk gedrag deze regels niet noodzakelijk volgt. Het kwantumformalisme slaagt er in dit gedrag te verklaren door te veronderstellen dat tijdens deze menswetenschappelijke experimenten concepten zoals 'meting' en 'gebeurtenis' meer gemeen hebben met de varianten uit kwantummechanica dan uit traditionele theorieën.

Het fundamenteel verschil tussen een traditioneel en een kwantumsysteem, is dat een systeem zich in een *superpositie* kan bevinden. Dit impliceert, in deeltjesfysica, dat, bijvoorbeeld, een subatomair deeltje zich niet noodzakelijk op één welomlijnde positie bevindt, maar tussen meerdere posities tegelijk[1]. Dit superpositie concept, van een systeem die balanceert tussen verschillende uitkomsten, wordt gebruikt om te modelleren dat een persoon kan twijfelen of onzeker zijn. Naast deze interpretationele verandering, resulteert het invoeren van het superpositie principe ook in wiskundige veranderingen, die succesvol zijn in het verklaren van menselijk gedrag die de klassieke probabilistische wetten niet volgt.

Om dit idee van superpositie mathematisch voor te stellen, wordt gebruik gemaakt van een Hilbertruimte[2]. Mogelijke (atomaire) uitkomsten van een meting worden in deze Hilbert ruimte voorgesteld als vectoren die een orthonormale basis van de Hilbertruimte vormen. Niet-atomaire gebeurtenissen worden gerepresenteerd door deelruimten opgespannen door de vectoren geassocieerd met de relevante atomaire gebeurtenissen. De toestand van de persoon die deelneemt aan het experiment is geassocieerd met een genormalizeerde toestandsvector in dezelfde Hilbertruimte. Aangezien de uitkomstvectoren een basis vormen van deze Hilbertruimte,

[1] Deze bewering is niet 100% accuraat, aangezien het deeltje zich eigenlijk op geen enkele positie bevindt. Het deeltje heeft de mogelijk om zich op meerdere plaatsen te manifesteren. Aangezien deze introductie erg summier is, dienen we erg oppervlakkig te gaan over de fundamentele concepten van kwantummechanica. Een gedetailleerde introductie kan gevonden worden in Nielsen & Chuang (2010) voor kwantummechanica en in Busemeyer & Bruza (2012) voor kwantumcognitie.

[2] Een Hilbertruimte is een vectorruimte met een inproduct.

kan de toestandsvector beschreven worden in coördinaten ten opzichte van deze uitkomstbasis. Deze coördinaten zijn de wiskundige uitdrukking van de superpositie, ze stellen voor hoe het systeem zich tussen verschillende uitkomsten bevindt. Met deze nieuwe manier van representeren van uitkomsten, komen ook een nieuwe regels voor het toekennen van probabiliteiten aan deze uitkomsten. Intuïtief komen deze nieuwe regels neer op kijken hoe dicht een toestandsvector ligt bij een relevante uikomstdeelruimte of -vector. Aangezien een systeem zich in superpositie tussen verschillende uitkomsten kan bevinden, maar een meting slechts één resultaat kan uitkomen, zorgt een meting ervoor dat het systeem de superpositie moet verlaten. Dit wordt mathematisch voorgesteld door het projecteren en normalizeren van de toestandsvectoren op de vector of deelruimte geassocieerd met de bekomen uitkomst. Dit concept maakt 'het meten' iets actief, die een invloed uitoefent op het systeem. Dit verandert fundamenteel de relatie tussen 'de observator' en 'het subject'. Het is deze nieuwe relatie, waarin het meten zelf het systeem beïnvloedt, die leidt tot schendingen van de klassieke probabiliteitsregels. De belangrijkste uitdrukking van deze contextualiteit van een meting is het concept van incompatibele metingen. Dit zijn metingen, waarvan het uitvoeren van de ene meting een invloed heeft op het resultaat van de andere meting. Op deze manier kunnen sommige uitkomsten van de ene meting niet tegelijk met sommige uitkomsten van de andere meting geobserveerd worden. Mathematisch wordt dit gemodelleerd door elke meting te associëren met een eigen basis in dezelfde Hilbertruimte. Het bekendste voorbeeld in fysica van dit fenomeen zijn de positie en de impuls van een subatomair deeltje.

Deze kwantumtechnieken zijn reeds succesvol toegepast in, o.a., beslissingstheorie (Lambert-Mogiliansky et al., 2009), semantische representaties (Widdows & Cohen, 2015), speltheorie (Martínez-Martínez, 2014), modelleren van menselijk geheugen (Bruza, 2010) en modelleren van perceptie (Atmanspacher et al., 2004).

Het doel van dit proefschrift is onderzoeken van het 'waarom' en het 'wanneer' van het succes van deze benadering. We kozen ervoor om dit aan de hand van uitgewerkte voorbeelden te doen, in tegenstelling tot een eerder theoretische benadering. Dit heeft als voordelen dat onze twee uitgewerkte voorbeelden op zichzelf staande onderzoekspistes zijn, die ook hun waarde hebben buiten het vermelde overkoepelend onderzoek. Daarnaast verliezen we op deze manier ook geen contact met de toegepaste, modellerende kant van het veld. Dit is een reëel gevaar bij erg theoretische onderzoeken, zoals we bespreken in Hoofdstuk 5.

6.2 Samenvatting Hoofdstuk 2

In Hoofdstuk 2 wordt een experiment over menselijke geheugen besproken en gemodelleerd. In dit experiment, oorspronkelijk uitgevoerd door Brainerd & Reyna (2008), werden proefpersonen gevraagd om drie verschillende woordlijsten te memoriseren. Vervolgens kregen de proefpersonen uitspraken voorgeschoteld waarmee ze akkoord moesten gaan of verwerpen. Deze uitspraken hadden twee mogelijke vormen. De eerste vorm was de bewering dat een bepaald woord op een specifieke lijst stond. De tweede vorm was dat een bepaald woord ergens op een (niet gespecifieerde) lijst stond. De verkregen data toonde dat de proefpersonen tijdens dit experiment een schending van de klassieke disjunctie regel vertoonden. Deze schending wordt traditioneel verklaard door het bestaan van twee types geheugen: een verbatim geheugen, geassocieerd met de vorm en fonologie van een woord, en een 'gist' geheugen, geassocieerd met de betekenis en associaties van een woord.

Brainerd et al. (2013) construeerden reeds een kwantumtechnisch model voor dit experiment, het QEM model. In dit model worden de proefpersonen verondersteld zich in een superpositie te bevinden tussen vijf orthogonale vectoren. Drie vectoren, die elk geassocieerd worden met verbatim herinneringen van een woordlijst, een vector geassocieerd met 'gist' herinneringen (over de drie lijsten heen) en een vector geassocieerd met het zich herinneren dat het woord zich niet op een lijst bevond. Dit model verklaart de schending van de disjunctie regel. We beargumenteren dat, hoewel het idee om dit probleem kwantumtechnisch te benaderen interessant is, het QEM model problematisch is. Zo zijn er interpretationele problemen met hoe de verschillende geheugentypes voorgesteld worden en tonen we aan dat we erg eenvoudig een klassiek equivalent kunnen construeren. Aangezien ons eenvoudig klassiek model exact dezelfde voorspellingen maakt, kan het verklaren van de schending van de disjunctie regel niet liggen aan de kwantumtechnische benadering van Brainerd et al.. Het blijkt dat het QEM model deze schending verklaart, omdat de conjunctie regel (ad hoc) anders gedefinieerd wordt. Dit is duidelijk in ons eenvoudig klassiek equivalent, maar is in het QEM model moeilijker te zien.

Aangezien we wel akkoord gingen met de argumentatie om een kwantummodel te gebruiken, enkel niet met de concrete uitwerking, construeren we een alternatief kwantumtechnisch model, het CMT model. In dit model worden de verschillende geheugen types niet voorgesteld door orthogonale vectoren, die samen één basis van de Hilbertruimte vormen, maar opteren we voor het beschouwen van het verbatim en 'gist' geheugen als complementair[3]. Het mathematisch gevolg is dat de drie vectoren geassocieerd met het verbatim geheugen enerzijds en de vector geassocieerd met het gist geheugen anderzijds niet meer orthogonaal zijn. Deze relatief kleine verandering zorgt ervoor dat de interpretationele problemen verdwijnen en er geen duidelijk eenvoudig klassiek equivalent is, terwijl de schending van de disjunctie regel nog steeds verklaard wordt. Ook heeft ons CMT model een betere statistische fit dan het QEM model, onder meer omdat het leidt tot minder vrije parameters. Op deze manier bereiken we in Hoofdstuk 2 onze twee beoogde doelstellingen. Ten eerste is het nieuwe CMT model een succesvol kwantumtechnisch model, die er in slaagt een paradox in het menselijk geheugen te verklaren. Ten tweede maken we ook een punt over de kwantumtechnische benadering in het algemeen, namelijk dat we voor kwantumtechnische modellen, met alle relevante vectoren orthogonaal, een eenvoudig klassiek alternatief kunnen construeren. Hiermee illustreren we dat het al dan niet orthogonaal zijn van uitkomstvectoren een belangrijke factor is in het succes van een kwantumtechnisch model.

6.3 Samenvatting Hoofdstuk 3

In Hoofdstuk 3 wordt een kwantumtechnisch model, het QP&B model, gemaakt gebaseerd op een speltheoretisch experiment die origineel besproken werd in Blanco et al. (2014). In dit experiment werden proefpersonen geconfronteerd met een sequentieel dilemma van de gevange. De proefpersonen werden gevraagd om het spel tweemaal te spelen, eerst als tweede speler (in de veronderstelling dat de eerste speler heeft meegewerkt) en vervolgens als eerste speler. Bij een deel van de proefpersonen werd ook gevraagd om, voor ze zelf hun zet als eerste speler deden, in te schatten hoeveel van hun negen mogelijke tegenstanders hadden samengewerkt als tweede speler. Deze extra meting veroorzaakte een significante verandering in het geobserveerd gedrag. De expliciete inschatting van het gedrag van de tegenstander zorgde voor een verdubbeling van het aantal samenwerkers. Mathematisch leidt dit tot een schending van het zogenaamde 'sure thing' principe. Deze invloed van het meten zelf wijst op de mogelijkheid van een interessant kwantumtechnisch model, waarbij de twee spelacties (zet als eerste speler en zet als tweede speler) incompatibel zijn met de inschatting van de tegenstanders.

[3] Complementaire metingen zijn maximaal incompatibel.

Wanneer deze situatie op een naïeve standaardmanier gegoten wordt in een kwantumtechnische model, is de resulterende Hilbertruimte (minstens) honderddimensionaal. Dit model is bijgevolg volledig overgeparametriseerd. De reden voor de hoge dimensionaliteit is dat de inschatting van de tegenstanders tien mogelijke antwoorden geeft. Het modelleren van enkel deze meting resulteert bijgevolg op zich al in een ruimte van dimensie tien. Door de relatie tussen de verschillende metingen (de spelacties compatibel met elkaar, maar incompatibel met de inschatting van de tegenstanders), dienen twee tiendimensionele ruimten getensord te worden, wat tot de honderd dimensies leidt.

Dit probleem wordt opgelost door toe te laten dat de vectoren die de inschatting van het aantal meewerkende tegenstanders voorstellen, niet orthogonaal hoeven te zijn. Aangezien deze vector eigenlijk de resultaten van één meting voorstellen, verlaten we hiermee de standaardvorm van het kwantumformalisme. Op deze manier kunnen we de inschattingsmeting representeren in een tweedimensionele ruimte. De uiteindelijke vorm van de overcomplete, niet orthogonale inschattingsbasis binnen de tweedimensionale ruimte wordt afgeleid uit de geordende structuur van de verschillende mogelijke uitkomsten (0 meewerkende tegenstanders, . . . , negen meewerkende tegenstanders). De drie metingen samen worden hierdoor gemodelleerd in een ruimte van (bevattelijke) dimensie vier. Deze oplossing is verwant met het model uit Busemeyer & Pothos (2012), waarin verondersteld wordt dat het vormen van overtuigingen over een tegenstander de toestandsvector roteert. Het verschil tussen ons model en het model van Busemeyer & Pothos (2012) is dat in ons gemodelleerd experiment deze overtuigingen expliciet gevraagd worden aan de spelers. De statistische fit van ons model is goed.

Het verlaten van het eenvoudige standaard kwantumtechnisch formalisme (het PVM formalisme) heeft gevolgen. Zo is de som van de probabiliteiten bij de inschattingsmeting niet meer noodzakelijk gelijk aan één. Hier worden twee oplossingen voor besproken. Bij de eerste oplossing wordt een schalingsfactor ingevoerd die de totale som naar één herleidt. Bij de tweede oplossing wordt een elfde operator toegevoegd aan de inschattingsprojectoren. Op deze manier krijgt het model een POVM structuur, een gekende uitbreiding van PVM structuren binnen kwantummechanica (Yearsley, 2016). Dit zorgt er voor dat de som van probabiliteiten één wordt, maar aangezien deze nieuwe operator ook iets binnen de meting voorstelt, moet een elfde (theoretische) uitkomst toegevoegd worden aan de mogelijke uitkomsten van de inschattingsmeting. We stellen voor (gebaseerd op een gelijkaardige constructie uit Yearsley (2016)) om deze vector te associëren met de uitkomst 'de meting faalde en wordt herdaan'. Deze oplossing, die nu wel binnen kwantumtheorie past, leidt, statistisch gezien, tot een identiek model als deze met de schalingsfactor, zonder extra uitkomst. Een tweede gevolg van het niet orthogonaal zijn van de inschattingsvectoren is dat voor twee identieke metingen, zonder manipulatie ertussen, niet voorspeld wordt dat ze noodzakelijk in eenzelfde uitkomst zullen resulteren. Deze eigenschap wordt gedefinieerd als herhaalbaarheid. In de bespreking van dit fenomeen voeren we het concept van onscherpe metingen in.

De opbouw van dit hoofdstuk volgt dezelfde structuur als van Hoofdstuk 2. In een dataset wordt een schending van een klassieke probabilistische wet geobserveerd. Een eerste poging tot een kwantumtechnisch model voldoet niet. Hierop worden bepaalde vectoren hergedefinieerd als niet orthogonaal. Deze aanpassing leidt wel tot een succesvol model. Het verschil tussen de aanpak in beide hoofdstukken ligt hem in dat in Hoofdstuk 2 de vectoren die niet orthogonaal werden gemaakt, horen bij verschillende metingen. In Hoofdstuk 3 wordt nog een stap verder gegaan en worden vectoren die horen bij één meting ook als niet orthogonaal gedefinieerd.

6.4 Samenvatting Hoofdstuk 4

In Hoofdstuk 4 wordt dieper ingegaan op de wiskundige techniek met niet orthogonale basis die ontwikkeld werd voor het modelleren van speltheoretisch experiment in Hoofdstuk 3. In Hoofdstuk 3 werd deze techniek ad hoc ontwikkeld om de te grote dimensionaliteit van de resulterende Hilbertruimte te verlagen en zo het aantal vrije parameter drastisch te reduceren. Hiervoor werd gebruikt gemaakt van de geordende structuur van de uikomstenverzameling van de inschattingsmeting. Aangezien deze techniek in dit paradigma erg succesvol werkt, stellen we ons de vraag of ze in andere experimentele modelleringssituaties (met geordende uitkomsten) even succesvol kan zijn. We tonen eerst aan dat POVM structuren, hoewel ze ook niet orthogonale uitkomstvectoren bevatten, nog steeds te restrictief zijn. Om dit op te lossen wordt een veralgemening van het POVM formalisme gedefinieerd, die ons toelaat om een orde te definiëren op vectoren (of deelruimten). Op deze manier wordt orde op uitkomsten, geassocieerd met een vector of deelruimte, wiskundig voorgesteld in de relevante Hilbertruimte. We beargumenteren dat dit in kwantumcognitie de meest natuurlijk representatie van orde in de uitkomstverzameling is, aangezien het enige alternatief is dat deze orde gerepresenteerd wordt in de vorm de toestandsvector. Dit is voor ons paradoxaal, aangezien de toestandsvector de proefpersoon voorstelt en de orde een eigenschap is van de meting, niet van de proefpersoon.

De twee gevolgen van deze veralgemening van kwantumtechnische metingen die in Hoofdstuk 3 naar voor kwamen, worden in een algemener kader besproken. Zo kan niet gegarandeerd worden dat de som van alle probabiliteiten geassocieerd met één meting gelijk is aan één. We stellen twee oplossingen voor. Als eerste oplossing construeren we algemene schalingsfactoren. Als tweede oplossing stellen we voor dat de odds op een bepaalde uitkomt gemodelleerd worden. Er wordt aangetoond dat beide oplossingen tot eenzelfde model leiden. Ook kan nog steeds herhaalbaarheid niet gegarandeerd worden zodat het model niet noodzakelijk voorspelt dat twee identieke metingen, die na elkaar uitgevoerd worden, dezelfde uitkomst zullen hebben. Ook hier wordt het concept van onscherpe metingen, ad hoc ingevoerd in Hoofdstuk 3 in een algemene setting besproken. Als voorbeeld wordt een voorstel voor het modelleren van Likert schalen gepresenteerd. Aangezien Likert schalen een duidelijke geordende structuur hebben en vaak meer dan vijf verschillende uitkomsten bezitten, zijn deze schalen een prima kandidaat voor het nieuwe formalisme.

Ten slotte wordt het speltheoretische experiment uit Hoofdstuk 3 gemodelleerd aan de hand van de verschillende besproken formalismen. Dit leidt tot vier verschillende modellen. Deze worden onderling vergeleken en we contrasteren hun verschillende voor- en nadelen. Ook al is deze vergelijking aan de hand van slechts één datatset, maakt ze het mogelijk om iets te zeggen over de verschillende technieken in het algemeen. Het eerste model is het model die gebruik maakt van de geordende schalen. Dit model heeft als grote voordelen dat ze een natuurlijk representatie is van het experiment, de geordende aard van de inschattingsmeting opneemt en in een laag aantal parameters resulteert. De belangrijkste nadelen zijn het verlaten van de orthodoxe kwantumtheorie en de hierboven vermelde twee gevolgen (som van probabiliteiten en verlies van herhaalbaarheid). Het tweede model verandert het eerste model in een model die gebruik maakt van een POVM formalisme. Om dit te bekomen, dient een extra theoretische uitkomst toegevoegd te worden. Deze uitkomst is de (fictieve) gebeurtenis dat de meting faalt en herdaan wordt. Dit model resulteert in dezelfde predicties (en bijgevolg dezelfde statistische fit) als het eerste model. Ook wordt de orde binnen de uitkomstverzameling gerepresenteerd in de respectievelijke vectoren en deelruimten. Dit model bevindt zich nu wel in de orthodoxe kwantumtheorie. De som van alle probabiliteiten van een meting is altijd één. Herhaalbaarheid kan nog steeds niet gegarandeerd worden. Als prijs hiervoor verschijnt de fictieve uitkomst, waardoor dit model een minder natuurlijke representatie van het experiment is. Het derde en vierde model maken beiden gebruik van een PVM formalisme. Hierdoor voldoen ze beiden aan herhaalbaarbeid en is de som van alle probabiliteiten van een meting één. Alle vectoren die geassocieerd worden met één meting zijn orthogonaal, wat geen orde kan modelleren. De orde van de uitkomstverzameling moet bijgevolg vervat zitten in de vorm van de toestandsvector, iets waar we tegen argumenteerden. Desalniettemin is een vergelijking van deze twee modelen met de eerste twee modelen interessant. Het derde model maakt gebruikt van de stelling van Naimark (Gelfand & Naimark, 1943), om de resultaten van het tweede model na te bootsen. Aangezien de resultaten van het tweede model identiek waren aan deze van het eerste model, zijn deze van het derde model ook identiek aan die van het eerste model. De stelling van Naimark garandeert het bestaan van het derde model, maar de precieze vorm ervan is onduidelijk. Dit ten gevolge van de vorm van theoretisch toegevoegde uitkomst in het tweede model. Wanneer alle operatoren van een POVM structuur projectoren zijn, is het resulturende model na het gebruik van de stelling van Naimark duidelijk. Dit is echter niet het geval, aangezien de operator van de theoretische uitkomst niet noodzakelijk een projector is. Het vierde en laatste model maakt gebruik van een PVM structuur zonder te vertrekken van de voorgaande modellen. Zonder het toevoegen van extra restricties resulteert dit in een model met een gigantisch aantal vrije parameters. Om de vergelijking met de voorgaande modellen mogelijk te maken voeren we enkel extra restricties in die leiden tot een model met hetzelfde aantal vrije parameters als de vorige. Een eerste verzameling van restricties zijn specifieke vormen van de deelruimten geassocieerd met de eerste en tweede acties van het spel, waardoor deze niet meer compatibel zijn. Een tweede verzameling restricties zijn een specifieke vorm van de toestandsvector, die het geordend karakter van de inschattingsmeting weerspiegelen. Ondanks deze (zware) restricties blijven de resulterende gemodelleerde restricties erg complex van vorm, waarvoor een statistische fit praktisch ondoenbaar blijkt.

6.5 Conclusie

In dit proefschrift werd onderzocht welke wiskundige technieken binnen kwantumcognitie de drijvende kracht zijn. In beide uitgewerkte paradigma's kwam hetzelfde verhaal naar boven, waarin de orthogonaliteit van uitkomstvectoren een cruciale rol speelt. Een geobserveerde schending van een klassieke statistische wet in experimentele data wees op de mogelijkheid om een kwantumtechnisch model te ontwikkelen. In beide gevallen werd eerst een naïef model voorgesteld, die op problemen van interpretationale en/of wiskundige aard stootte. Deze problemen werden telkens opgelost door uitkomstvectoren te herdefiniëren als niet ortogonaal. De nieuwe modellen waren telkens eleganter, interpretationeel duidelijker en hadden een betere statistische fit. Het verschil tussen beide modellen is dat bij het ene model vectoren geassocieerd met verschillende metingen als niet orthogonaal gedefinieerd werden, terwijl het ander model nog verder ging door ook vectoren geassocieerd met één meting als niet orthogonaal te definiëren. Deze laatste techniek slaagt er ook in om geordende schalen op een adequate manier voor te stellen binnen kwantumcognitie. Hoewel deze nieuwe technieken nog weinig getest zijn tegen experimentele data, denken we dat ze kunnen bijdragen aan de verdere evolutie van het veld.

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7 Data Storage Fact Sheets

Data Storage Fact Sheets Chapter 2

% Data Storage Fact Sheet

% Name/identifier study: PhD dissertation Jacob Denolf, Chapter 2
% Author: Jacob Denolf
% Date: 23/01/2017

1. Contact details
===

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Denolf, J. and Lambert-Mogiliansky, A. (2016). Bohr complementarity in memory retrieval. Journal of Mathematical Psychology, 73, 28-36.

* Which datasets in that publication does this sheet apply to?:

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raw data of modeled recollection experiment


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Data Storage Fact Sheets Chapter 3

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