Regular partitions of half-spin geometries

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Abstract

We describe several families of regular partitions of half-spin geometries and determine their associated parameters and eigenvalues. We also give a general method for computing the eigenvalues of regular partitions of half-spin geometries.

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1 Introduction

Let q be a prime power and $n \in \mathbb{N} \setminus \{0,1\}$. Let $Q^+(2n-1,q)$ be a hyperbolic quadric of $\mathrm{PG}(2n-1,q)$ and denote by $\mathcal{M}$ the set of generators of $Q^+(2n-1,q)$, i.e. the set of all subspaces of $Q^+(2n-1,q)$ of maximal (projective) dimension $n-1$. On the set $\mathcal{M}$, an equivalence relation can be defined by calling two generators equivalent whenever they intersect in a subspace of even co-dimension. There are two equivalence classes which we will denote by $\mathcal{M}^+$ and $\mathcal{M}^-$.

For every $\epsilon \in \{+,-\}$, the following point-line geometry $HS^\epsilon(2n-1,q)$ can be defined:

- the points of $HS^\epsilon(2n-1,q)$ are the elements of $\mathcal{M}^\epsilon$;
- the lines of $HS^\epsilon(2n-1,q)$ are the subspaces of dimension $n-3$ of $Q^+(2n-1,q)$;
- incidence is reverse containment.

The isomorphic geometries $HS^+(2n-1,q)$ and $HS^-(2n-1,q)$ are called the *half-spin geometries* of $Q^+(2n-1,q)$. We denote any of these geometries by $HS(2n-1,q)$.

The half-spin geometry $HS(3,q)$ is a line containing $q+1$ points, $HS(5,q)$ is isomorphic to $\mathrm{PG}(3,q)$ (regarded as a linear space) and $HS(7,q)$ is isomorphic to the geometry of the points and lines of $Q^+(7,q)$.

In this paper, we study *regular partitions* of half-spin geometries. These are partitions $\mathcal{P} = \{X_1, X_2, \dots, X_{|\mathcal{P}|}\}$ of the point sets for which there are constants a_{ij}, $i,j \in \{1,2,\dots,|\mathcal{P}|\}$, such that every point $x \in X_i$ is collinear with precisely a_{ij} points of $X_j \setminus \{x\}$. The eigenvalues of the matrix $A_{\mathcal{P}} = (a_{ij})$ are called the *eigenvalues of the* *regular partition*, and each of these eigenvalues is also an eigenvalue of the collinearity graph of the geometry.

In this paper, we describe several families of regular partitions of half-spin geometries and determine their parameters and eigenvalues. Regular partitions have already been studied for other families of point-line geometries, like generalized polygons [8] and dual polar spaces [3]. Many regular partitions are associated with nice substructures, and the eigenvalues of these regular partitions might yield information about the structures from which they are derived. Indeed, Proposition 2.1 below shows that these eigenvalues are a helpful tool for determining intersections sizes of combinatorial structures.

This paper is organized as follows. Section 3 contains the main results of this paper. In this section, we describe several families of regular partitions of half-spin geometries, many of which are related to nice geometrical substructures, and mention their parameters and eigenvalues. In Section 4, we describe a general method for computing eigenvalues of regular partitions of half-spin geometries, and in Section 5 we apply this method to compute the eigenvalues of all families of regular partitions described in Section 3. Before we do that, we recall some basic facts about regular partitions, half-spin geometries and hyperbolic dual polar spaces in the next section.

2 Preliminaries

2.1 Regular partitions of point-line geometries

Let X be a nonempty finite set and $R \subseteq X \times X$ a symmetric relation on X.

A partition $\mathcal{P} = \{X_1, X_2, \ldots, X_k\}$ of size k of X is called *R-regular* if there exist constants a_{ij}, $i, j \in \{1, 2, \ldots, k\}$, such that for every $x \in X_i$, there are precisely a_{ij} elements $y \in X_j$ for which $(x, y) \in R$. The numbers a_{ij} are called the *parameters* of the R-regular partition. We denote the matrix $(a_{ij})_{1 \leq i,j \leq k}$ by $A_{\mathcal{P}}$ and $E_{\mathcal{P}}$ denotes the multiset of the complex eigenvalues of $A_{\mathcal{P}}$.

Given two finite multisets $M_1 = \{\lambda_1, \lambda_2, \ldots, \lambda_k\}$ and $M_2 = \{\mu_1, \mu_2, \ldots, \mu_l\}$ whose elements are complex numbers, we denote by $O(M_1, M_2)$ the multiplicity of 0 as an element of the multiset $\{\lambda_i - \mu_j \,|\, 1 \leq i \leq k, 1 \leq j \leq l\}$.

For a proof of the following proposition, see e.g. De Wispelaere and Van Maldeghem [8, Lemma 3.3] or De Bruyn [5, Theorem 1.1].

Proposition 2.1 *Let $R \subseteq X \times X$ be a symmetric relation on a nonempty finite set X and assume there exists a constant $\mu \in \mathbb{N} \setminus \{0\}$ such that for every $x \in X$, there are precisely μ elements $y \in X$ for which $(x, y) \in R$. Let $\mathcal{P} = \{X_1, X_2, \ldots, X_k\}$ and $\mathcal{P}' = \{X'_1, X'_2, \ldots, X'_{k'}\}$ be two R-regular partitions with $O(E_{\mathcal{P}}, E_{\mathcal{P}'}) = 1$. Then $|X_i \cap X'_j| = \frac{|X_i| \cdot |X'_j|}{|X|}$ for any $(i, j) \in \{1, 2, \ldots, k\} \times \{1, 2, \ldots, k'\}$.*

Suppose now that X is the point set of a point-line geometry $\mathcal{S}$ and R is the collinearity relation on X, i.e. if $p_1, p_2 \in X$, then $(p_1, p_2) \in R$ if and only if $p_1 \neq p_2$ and there is a line containing p_1 and p_2. An R-regular partition of X will then also be called a *a regular* or *equitable partition* of $\mathcal{S}$. Put $X = \{p_1, p_2, \ldots, p_v\}$ where $v = |X|$ and let A be the $(v \times v)$-matrix whose (i,j)th entry is equal to 1 if $(p_i, p_j) \in R$ and equal to 0 otherwise. The eigenvalues of the collinearity matrix A are independent of the ordering of the points of X and are called the *eigenvalues of* $\mathcal{S}$. The following lemma and proposition are known, see e.g. Godsil [9, Section 5.2], Godsil & Royle [10, Theorem 9.3.3] and Cardinali & De Bruyn [3, Lemmas 1.2 and 1.3].

Lemma 2.2 *Suppose A can be partitioned as*

$$A = \begin{bmatrix} A_{11} & \cdots & A_{1k} \\ \vdots & \ddots & \vdots \\ A_{k1} & \cdots & A_{kk} \end{bmatrix},$$

where each A_{ii}, $i \in \{1, 2, \ldots, k\}$, is square and each A_{ij}, $i, j \in \{1, 2, \ldots, k\}$, has constant row sum b_{ij}. Then the matrix $B = (b_{ij})_{1 \leq i,j \leq k}$ is diagonalizable and any eigenvalue of B is also an eigenvalue of A.

Proposition 2.3 *If $\mathcal{P} = \{X_1, X_2, \ldots, X_k\}$ is a regular partition of $\mathcal{S}$, then every eigenvalue of $A_{\mathcal{P}}$ is also an eigenvalue of A (or of $\mathcal{S}$).*

Proof. Without loss of generality we may suppose that we have ordered the points of X in such a way that if $p_i \in X_l$ and $p_j \in X_m$ with $l < m$, then $i < j$. If we partition the matrix A in submatrices A_{ij}, $i, j \in \{1, 2, \ldots, k\}$, such that A_{ij} has dimension $|X_i| \times |X_j|$, then each of these submatrices has constant row sum. Lemma 2.2 then implies that every eigenvalue of B (as defined there) is also an eigenvalue of A. The proposition then follows from the fact that $B = A_{\mathcal{P}}$. ■

2.2 Half-spin geometries

A connected finite graph of diameter $d \geq 2$ is called *distance-regular* if there exist constants a_i, b_i, c_i ($i \in \{0, 1, \ldots, d\}$) such that the following hold for any two vertices x and y of the graph at distance i from each other:

- there are precisely a_i vertices adjacent to y at distance i from x;
- there are precisely b_i vertices adjacent to y at distance $i+1$ from x;
- there are precisely c_i vertices adjacent to y at distance $i-1$ from x.

Consider again the hyperbolic quadric $Q^+(2n-1, q)$ in $\mathrm{PG}(2n-1, q)$, where $n \in \mathbb{N} \setminus \{0, 1\}$ and q is a prime power. Let $HS(2n-1, q)$ denote one of the two half-spin geometries associated with $Q^+(2n-1, q)$. The total number of points of $HS(2n-1, q)$ is equal to $(q+1)(q^2+1) \cdots (q^{n-1}+1)$. Every line of $HS(2n-1, q)$ is incident with precisely $q+1$ points and every point of $HS(2n-1, q)$ is incident with exactly $\begin{bmatrix} n \\ 2 \end{bmatrix} := \begin{bmatrix} n \\ 2 \end{bmatrix}_q$ lines (a Gaussian binomial coefficient with respect to q). Distances between points of $HS(2n-1,q)$ will be measured in the collinearity graph Γ of $HS(2n-1,q)$. If α_1 and α_2 are two points of $HS(2n-1,q)$ at distance $i \in \mathbb{N}$ from each other, then $\dim(\alpha_1 \cap \alpha_2) = n-1-2i$. The graph Γ has diameter $d := \lfloor \frac{n}{2} \rfloor$ and is regular with valency $k = q \cdot \begin{bmatrix} n \\ 2 \end{bmatrix}$. If $n \geq 4$, then by [2, Theorem 9.4.8] we know that Γ is distance-regular with parameters

$$b_i = q^{4i+1} \cdot \begin{bmatrix} n-2i \\ 2 \end{bmatrix}, \qquad c_i = \begin{bmatrix} 2i \\ 2 \end{bmatrix}, \qquad a_i = k - b_i - c_i,$$

where $i \in \{0,1,\dots,d\}$. Also by [2, Theorem 9.4.8], the eigenvalues of Γ are

$$\theta_i = q^{2i+1} \cdot \begin{bmatrix} n-2i \\ 2 \end{bmatrix} - \frac{q^{2i}-1}{q^2-1} = \frac{q^{2n-2i}+q^{2i}-q^{n+1}-q^n+q-1}{(q-1)(q^2-1)}, \quad i \in \{0,1,\dots,d\},$$

and the corresponding multiplicities are

$$f_i = \gamma_i q^i \cdot \begin{bmatrix} n \\ i \end{bmatrix} \cdot \frac{1+q^{n-2i}}{1+q^{n-i}} \cdot \prod_{j=1}^{i} \frac{1+q^{n-j}}{1+q^j}, \quad i \in \{0,1,\dots,d\}.$$

Here, $\gamma_i = 1$ if $i < \frac{n}{2}$ and $\gamma_i = \frac{1}{2}$ if n is even and $i = \frac{n}{2}$.

For every integer $i > d$, we define $\theta_i := \frac{q^{2n-2i}+q^{2i}-q^{n+1}-q^n+q-1}{(q-1)(q^2-1)}$. Then

$$\theta_{n-i} = \theta_i \tag{1}$$

for every $i \in \{0,1,\dots,n\}$, and

$$\sum_{i=0}^{m} \theta_i = -\frac{(q^{n+1}+q^n-q+1)(m+1)}{(q-1)(q^2-1)} + \frac{q^{2n-2m}+1}{(q-1)(q^2-1)} \cdot \frac{q^{2m+2}-1}{q^2-1}, \quad \forall m \in \mathbb{N}. \tag{2}$$

2.3 The dual polar space $DQ^+(2n-1,q)$

It will often be very useful to reason in the dual polar space $DQ^+(2n-1,q)$ associated with the hyperbolic quadric $Q^+(2n-1,q)$. We recall that $DQ^+(2n-1,q)$ is the point-line geometry whose points, respectively lines, are the $(n-1)$-dimensional, respectively $(n-2)$-dimensional, subspaces of $Q^+(2n-1,q)$, with incidence being reverse containment. The collinearity graph of $DQ^+(2n-1,q)$ is a bipartite graph. If α_1 and α_2 are two points of $DQ^+(2n-1,q)$, then the distance $\mathrm{d}(\alpha_1,\alpha_2)$ between them in the collinearity graph of $DQ^+(2n-1,q)$ is equal to $n-1-\dim(\alpha_1 \cap \alpha_2)$. The dual polar space $DQ^+(2n-1,q)$ is an example of a *near polygon*, meaning that for every point x and every line L, there exists a unique point on L nearest to x (with respect to the distance in the collinearity graph). Since the maximal distance between two points of $DQ^+(2n-1,q)$ is equal to n, $DQ^+(2n-1,q)$ is a near $2n$-gon. There exists a bijective correspondence between the possibly empty subspaces of $Q^+(2n-1,q)$ and the nonempty convex subspaces of $DQ^+(2n-1,q)$. Indeed, if α is a subspace of dimension $n-1-k$ ($k \in \{0,1,\dots,n\}$) of

$Q^+(2n-1,q)$, then the set F_α of all maximal subspaces of $Q^+(2n-1,q)$ containing α is a convex subspace of diameter k of $DQ^+(2n-1,q)$. If F is a convex subspace and x is a point of $DQ^+(2n-1,q)$, then x is *classical with respect to* F, meaning that F contains a unique point $x' = \pi_F(x)$ such that $\mathrm{d}(x,y) = \mathrm{d}(x,x') + \mathrm{d}(x',y)$ for every $y \in F$. If F is a convex subspace of diameter k, then the maximal distance from a point of $DQ^+(2n-1,q)$ to F is equal to $n-k$. If $*_1$ and $*_2$ are two objects of $DQ^+(2n-1,q)$ (like points or nonempty sets of points), then $\langle *_1, *_2 \rangle$ denotes the smallest convex subspace of $DQ^+(2n-1,q)$ containing $*_1$ and $*_2$. The convex subspaces of $DQ^+(2n-1,q)$ through a given point x of $DQ^+(2n-1,q)$ define a projective space $Res(x)$ isomorphic to $\mathrm{PG}(n-1,q)$. Every two points x and y of $DQ^+(2n-1,q)$ at distance $\delta \in \{0,1,\ldots,n\}$ from each other are contained in a unique convex subspace of diameter δ. These convex subspaces are called *quads* if $\delta = 2$ and they are isomorphic to dual grids of type $DQ^+(3,q)$. Note that any line of such a dual grid has two points and through every point there are precisely $q+1$ lines.

Every point of the half-spin geometry $HS(2n-1,q)$ is also a point of $DQ^+(2n-1,q)$ and two points x and y of $HS(2n-1,q)$ are collinear in $HS(2n-1,q)$ if they lie at distance 2 as points of $DQ^+(2n-1,q)$, or equivalently, if their convex closure $\langle x,y \rangle$ is a quad. This point of view will be very useful when we will compute the parameters of the various regular partitions we are going to describe.

For more information on dual polar spaces and half-spin geometries (including proofs of the above-mentioned facts), we refer to Chapter 8 of [7].

3 Some families of regular partitions of half-spin geometries

In this section, we list several families of regular partitions of half-spin geometries, many of which are related to nice geometrical substructures, and mention for each of them the eigenvalues. Proofs will be postponed till Section 5 after we have described a method in Section 4 that allows to compute eigenvalues. Note that by Proposition 2.3, the eigenvalues of any regular partition of $HS(2n-1,q)$ are also eigenvalues of the collinearity graph Γ of $HS(2n-1,q)$. Recall that Γ has diameter $d = \lfloor \frac{n}{2} \rfloor$ and is a distance-regular graph if $n \geq 4$.

If $\mathcal{P}$ is the point set of $HS(2n-1,q)$, then $\{\mathcal{P}\}$ is a regular partition with eigenvalue $k = \theta_0$ and $\{\{p\} \,|\, p \in \mathcal{P}\}$ is a regular partition for which each θ_i, $i \in \{0,1,\ldots,d\}$, is an eigenvalue with multiplicity f_i. In general, a regular partition of $HS(2n-1,q)$ can be constructed from any group G of automorphisms of the geometry by collecting all orbits of the action of G on $\mathcal{P}$. If $n \in \{2,3\}$, then $HS(2n-1,q)$ is a linear space and every partition in $k \geq 2$ classes is a regular partition with eigenvalues $\theta_0 = k$ (multiplicity 1) and $\theta_1 = -1$ (with multiplicity $k-1$). In the sequel, we list nontrivial examples of regular partitions of $HS(2n-1,q)$, hereby assuming that $n \geq 4$ so that $HS(2n-1,q)$ is not a linear space. We also note that if $n \in \{4,5\}$, then Γ is a strongly regular graph (the collinearity graph of $Q^+(7,q)$ if $n = 4$) and regular partitions with only two parts were already studied for these geometries, see [1] and [4].

Class 1. (a) Take a fixed point M^* of $HS(2n-1,q)$. For every $i \in \{0,1,\ldots,d\}$, let X_i denote the set of all points M of $HS(2n-1,q)$ at distance i from M^*, i.e. the set of all points M of $HS(2n-1,q)$ for which $\dim(M \cap M^*) = n-1-2i$. We will show in Section 5.1 that $\{X_0, X_1, \ldots, X_d\}$ is a regular partition of $HS(2n-1,q)$ with eigenvalues $\theta_0, \theta_1, \ldots, \theta_d$.

(b) Take a fixed element $M^* \in \mathcal{M}$ that is not a point of $HS(2n-1,q)$. Then every point of $HS(2n-1,q)$ intersects M^* in a subspace whose dimension is $n-2-2i$ for some $i \in \{0,1,\ldots,\lfloor\frac{n-1}{2}\rfloor\}$. For every $i \in \{0,1,\ldots,d'\}$, where $d' := \lfloor\frac{n-1}{2}\rfloor$, let X_i denote the set of all points M of $HS(2n-1,q)$ that intersect M^* in a subspace whose dimension is $n-2-2i$. We will show in Section 5.2 that $\{X_0, X_1, \ldots, X_{d'}\}$ is a regular partition of $HS(2n-1,q)$ with eigenvalues $\theta_0, \theta_1, \ldots, \theta_{d'}$.

(c) Take a subspace α of $Q^+(2n-1,q)$ of dimension $k \in \{0,1,\ldots,n-2\}$. For every $i \in \{0,1,\ldots,k+1\}$, let X_i denote the set of all points M of $HS(2n-1,q)$ that intersect α in a subspace of dimension $k-i$. We will show in Section 5.3 that $\{X_0, X_1, \ldots, X_{k+1}\}$ is a regular partition of $HS(2n-1,q)$. We also show the following.

- If $k+1 \leq d = \lfloor\frac{n}{2}\rfloor$, then the eigenvalues are $\theta_0, \theta_1, \ldots, \theta_{k+1}$.
- If n is even and $k+1 > d = \frac{n}{2}$, then the eigenvalues are $\theta_0, \theta_1, \ldots, \theta_{n-k-2}$ and θ_d with multiplicity 1, and $\theta_{n-k-1}, \theta_{n-k}, \ldots, \theta_{d-1}$ with multiplicity 2.
- If n is odd and $k+1 > d = \frac{n-1}{2}$, then the eigenvalues are $\theta_0, \theta_1, \ldots, \theta_{n-k-2}$ with multiplicity 1 and $\theta_{n-k-1}, \theta_{n-k}, \ldots, \theta_d$ with multiplicity 2.

Class 2. Consider a subspace Π that intersects $Q^+(2n-1,q)$ in either a $Q^+(2m-1,q)$, $Q(2m,q)$ or $Q^-(2m+1,q)$ with $m \in \{2,3,\ldots,n-\varepsilon\}$, where $\varepsilon = 0$ if $\Pi \cap Q^+(2n-1,q) = Q^+(2m-1,q)$, $\varepsilon = 1$ if $\Pi \cap Q^+(2n-1,q) = Q(2m,q)$ and $\varepsilon = 2$ if $\Pi \cap Q^+(2n-1,q) = Q^-(2m+1,q)$. For every $i \in \{0,1,\ldots,\widetilde{m}\}$, where $\widetilde{m} := \min(m, n-m-\varepsilon)$, let X_i denote the set of all points M of $HS(2n-1,q)$ that intersect $Q^+(2n-1,q) \cap \Pi$ in a subspace of dimension $m-1-i$. We show in Section 5.4 that $\{X_0, X_1, \ldots, X_{\widetilde{m}}\}$ is a regular partition of $HS(2n-1,q)$ with eigenvalues $\theta_0, \theta_1, \ldots, \theta_{\widetilde{m}}$.

Class 3. A *(partial) ovoid* of $Q^+(2n-1,q)$ is a set of points of $Q^+(2n-1,q)$ intersecting every maximal subspace in (at most) one point. Suppose now that $\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_k$ is a collection of $k \geq 1$ non-empty mutually disjoint sets of points of $Q^+(2n-1,q)$ such that $\mathcal{O}_1 \cup \mathcal{O}_2 \cup \cdots \cup \mathcal{O}_k$ is a partial ovoid of $Q^+(2n-1,q)$, but not an ovoid. For every $i \in \{1,2,\ldots,k\}$, let X_i denote the set of all points M of $HS(2n-1,q)$ having nonempty intersection with $\mathcal{O}_i$. Denote by $X_{k+1} \neq \emptyset$ the set of all points of $HS(2n-1,q)$ not contained in $X_1 \cup X_2 \cup \cdots \cup X_k$. We show in Section 5.5 that $\{X_1, X_2, \ldots, X_{k+1}\}$ is a regular partition of $HS(2n-1,q)$ with eigenvalues θ_0 (multiplicity 1) and θ_1 (multiplicity k).

Class 4. Put $\overline{x} = (x_0, x_1, \ldots, x_{2n-1})$ and let $\phi(\overline{x})$ be a nondegenerate quadratic form over $\mathbb{F}_q$ defining an elliptic quadric $Q^-(2n-1,q)$ in $\mathrm{PG}(2n-1,q)$ and a hyperbolic quadric $Q^+(2n-1,q^2)$ in $\mathrm{PG}(2n-1,q^2)$. We regard $\mathrm{PG}(2n-1,q)$ as (naturally) embedded into $\mathrm{PG}(2n-1,q^2)$. Let θ be the automorphism of $\mathrm{PG}(2n-1,q^2)$ defined by $(x_0, x_1, \ldots, x_{2n-1}) \mapsto (x_0^q, x_1^q, \ldots, x_{2n-1}^q)$. Then the fixpoints of θ are precisely the points of $\mathrm{PG}(2n-1,q)$. Let $\mathcal{M}^+$ and $\mathcal{M}^-$ denote the two families of generators of $Q^+(2n-1,q^2)$ and let $HS(2n-1,q^2)$ be one of the two half-spin geometries of $Q^+(2n-1,q^2)$. By [6, Lemma 2.2], $(\mathcal{M}^+)^\theta = \mathcal{M}^-$ and $(\mathcal{M}^-)^\theta = \mathcal{M}^+$. Hence, for every point M of $HS(2n-1,q^2)$, we have $\dim(M \cap M^\theta) = n-2-2i$ for some $i \in \{0,1,\ldots,\lfloor\frac{n-1}{2}\rfloor\}$. For every $i \in \{0,1,\ldots,d'\}$, where $d' := \lfloor\frac{n-1}{2}\rfloor$, let X_i denote the set of all points M of $HS(2n-1,q^2)$ for which $\dim(M \cap M^\theta) = n-2-2i$. In Section 5.6, we show that $\{X_0, X_1, \ldots, X_{d'}\}$ is a regular partition of $HS(2n-1,q^2)$ with corresponding eigenvalues $\theta_0, \theta_1, \ldots, \theta_{d'}$.

4 General method to determine eigenvalues of regular partitions of half-spin geometries

If n is even, then for every $i \in \{0,1,\ldots,\frac{n}{2}\}$, consider the polynomial

$$\lambda_i(x) := x^{2i}(1+x^2+\cdots+x^{n-2i-2})\cdot(x+x^2+\cdots+x^{n-2i-1}) - (1+x^2+\cdots+x^{2i-2})$$

of $\mathbb{Z}[x]$. If n is odd, then for every $i \in \{0,1,\ldots,\frac{n-1}{2}\}$, consider the polynomial

$$\lambda_i(x) := x^{2i}(1+x+\cdots+x^{n-2i-1})\cdot(x+x^3+\cdots+x^{n-2i-2}) - (1+x^2+\cdots+x^{2i-2})$$

of $\mathbb{Z}[x]$. For every $i \in \{0,1,\ldots,\lfloor\frac{n}{2}\rfloor\}$, put $\lambda_i'(x) := x^{2n-2i} + x^{2i} \in \mathbb{Z}[x]$. Clearly, $\deg(\lambda_i'(x)) = 2n-2i$.

Lemma 4.1 *For every $i \in \{0,1,\ldots,\lfloor\frac{n}{2}\rfloor\}$, we have $\theta_i = \lambda_i(q)$.*

Proof. We have

$$\begin{aligned}
\lambda_i(q) &= q^{2i}\cdot\frac{(q^{n-2i}-1)(q^{n-2i}-q)}{(q-1)(q^2-1)} - \frac{q^{2i}-1}{q^2-1} \\
&= q^{2i+1}\cdot\frac{(q^{n-2i}-1)(q^{n-2i}-q)}{(q^2-1)(q^2-q)} - \frac{q^{2i}-1}{q^2-1} \\
&= q^{2i+1}\cdot\begin{bmatrix} n-2i \\ 2 \end{bmatrix} - \frac{q^{2i}-1}{q^2-1} \\
&= \theta_i.
\end{aligned}$$

■

Lemma 4.2 *For every $i \in \{0,1,\ldots,\lfloor\frac{n}{2}\rfloor\}$, we have $\lambda_i'(x) = (x-1)(x^2-1)\cdot\lambda_i(x) + x^{n+1} + x^n - x + 1$.*

Proof. We have

$$\begin{aligned}(x-1)(x^2-1)\cdot\lambda_i(x) &= x^{2i}(x^{n-2i}-1)(x^{n-2i}-x)-(x-1)(x^{2i}-1)\\ &= x^{2i}(x^{2n-4i}-x^{n-2i}-x^{n-2i+1}+x)-(x^{2i+1}-x^{2i}-x+1)\\ &= x^{2n-2i}-x^n-x^{n+1}+x^{2i}+x-1.\end{aligned}$$

■

The proof of the following proposition is similar to the proofs of Lemmas 2.1 and 2.3 of [3].

Proposition 4.3 *Let $k\geq 1$ and let $a_{ij}(x)\in\mathbb{Z}[x]$ for all $i,j\in\{1,2,\ldots,k\}$. Let $\mathcal{Q}$ be an infinite set of prime powers. Suppose that for every $q\in\mathcal{Q}$, all eigenvalues of the matrix $A(q):=(a_{ij}(q))_{1\leq i,j\leq k}$ belong to the set $\{\lambda_0(q),\lambda_1(q),\ldots,\lambda_d(q)\}$. Suppose also that $i_1,i_2,\ldots,i_k$ are k not necessarily distinct elements of $\{0,1,\ldots,d\}$ such that*

$$\lambda_{i_1}(q)+\lambda_{i_2}(q)+\cdots+\lambda_{i_k}(q)=a_{11}(q)+a_{22}(q)+\cdots+a_{kk}(q)$$

for every $q\in\mathcal{Q}$. Then for every $q\in\mathcal{Q}$, $\lambda_{i_1}(q),\lambda_{i_2}(q),\ldots,\lambda_{i_k}(q)$ are all the k not necessarily distinct eigenvalues of the matrix $A(q)$.

Proof. Denote by I_k the $k\times k$ identity matrix. We need to prove that

$$\det(X\cdot I_k-A(q))=(X-\lambda_{i_1}(q))(X-\lambda_{i_2}(q))\cdots(X-\lambda_{i_k}(q)) \tag{3}$$

for every $q\in\mathcal{Q}$. The right-hand and left-hand sides of (3) can be regarded as polynomials of degree k in the variable X whose coefficients are polynomials in q. So, in order to prove (3) for every $q\in\mathcal{Q}$, it suffices to prove that (3) holds for an infinite number of elements of $\mathcal{Q}$. We shall prove that equation (3) holds for all prime powers $q\in\mathcal{Q}$ which are bigger than a certain number K.

Since $\deg(\lambda'_0(x))>\deg(\lambda'_1(x))>\cdots>\deg(\lambda'_d(x))$, we know that there exists a $K>0$ such that $\lambda'_i(q)>k\cdot\lambda'_{i+1}(q)>0$ for every $i\in\{0,1,\ldots,d-1\}$ and every element $q>K$ of the set $\mathcal{Q}$.

Now, take an arbitrary prime power $q\in\mathcal{Q}$ bigger than K. We know that there exist k not necessarily distinct elements $j_1,j_2,\ldots,j_k$ of $\{0,1,\ldots,d\}$ such that

$$\det(X\cdot I_k-A(q))=(X-\lambda_{j_1}(q))(X-\lambda_{j_2}(q))\cdots(X-\lambda_{j_k}(q)). \tag{4}$$

Equating the coefficients of X^{k-1} in the left-hand and right-hand sides of (4), we find that $\lambda_{j_1}(q)+\lambda_{j_2}(q)+\cdots+\lambda_{j_k}(q)=a_{11}(q)+a_{22}(q)+\cdots+a_{kk}(q)$. Hence,

$$\lambda_{i_1}(q)+\lambda_{i_2}(q)+\cdots+\lambda_{i_k}(q)=\lambda_{j_1}(q)+\lambda_{j_2}(q)+\cdots+\lambda_{j_k}(q). \tag{5}$$

Now, equation (5) and Lemma 4.2 imply that

$$\lambda'_{i_1}(q)+\lambda'_{i_2}(q)+\cdots+\lambda'_{i_k}(q)=\lambda'_{j_1}(q)+\lambda'_{j_2}(q)+\cdots+\lambda'_{j_k}(q). \tag{6}$$

Since $\lambda'_i(q)>k\cdot\lambda'_{i+1}(q)>0$ for every $i\in\{0,1,\ldots,d-1\}$, there exist by equation (6) constants M_i, $i\in\{0,1,\ldots,d\}$, such that:

(1) among the numbers $\lambda'_{i_1}(q), \lambda'_{i_2}(q), \ldots, \lambda'_{i_k}(q)$, there are precisely M_i that are equal to $\lambda'_i(q)$;

(2) among the numbers $\lambda'_{j_1}(q), \lambda'_{j_2}(q), \ldots, \lambda'_{j_k}(q)$, there are precisely M_i that are equal to $\lambda'_i(q)$.

So, the multisets $\{\lambda'_{i_1}(q), \lambda'_{i_2}(q), \ldots, \lambda'_{i_k}(q)\}$ and $\{\lambda'_{j_1}(q), \lambda'_{j_2}(q), \ldots, \lambda'_{j_k}(q)\}$ are equal, implying by Lemma 4.2 that also the multisets $\{\lambda_{i_1}(q), \lambda_{i_2}(q), \ldots, \lambda_{i_k}(q)\}$ and $\{\lambda_{j_1}(q), \lambda_{j_2}(q), \ldots, \lambda_{j_k}(q)\}$ are equal. Hence,

$$\begin{aligned}\det(X \cdot I_k - A(q)) &= (X - \lambda_{j_1}(q))(X - \lambda_{j_2}(q)) \cdots (X - \lambda_{j_k}(q)) \\ &= (X - \lambda_{i_1}(q))(X - \lambda_{i_2}(q)) \cdots (X - \lambda_{i_k}(q)),\end{aligned}$$

as we needed to show. ∎

5 Determination of the parameters and eigenvalues

5.1 Treatment of Class 1(a)

Take a fixed point M^* of $HS(2n-1,q)$. For every $i \in \{0,1,\ldots,d\}$ with $d = \lfloor \frac{n}{2} \rfloor$, let X_i denote the set of all points of $HS(2n-1,q)$ at distance i from M^*. The fact that Γ is a distance-regular graph implies that $\{X_0, X_1, \ldots, X_d\}$ is a regular partition. If $A = (a_{ij})$ denotes the coefficient matrix of this regular partition, then

- $a_{ij} = 0$ for all $i, j \in \{0,1,\ldots,d\}$ with $|i-j| \geq 2$,
- $a_{i,i-1} = c_i = \frac{q^{4i-1}-q^{2i}-q^{2i-1}+1}{(q-1)(q^2-1)}$ for all $i \in \{1,2,\ldots,d\}$,
- $a_{ii} = a_i = k - b_i - c_i$ for all $i \in \{0,1,\ldots,d\}$,
- $a_{i,i+1} = b_i = \frac{q^{2n}-q^{n+2i+1}-q^{n+2i}+q^{4i+1}}{(q-1)(q^2-1)}$ for all $i \in \{0,1,\ldots,d-1\}$,

where a_i, b_i, c_i are the parameters of the distance-regular graph Γ. We show that $\theta_0, \theta_1, \ldots, \theta_d$ are the eigenvalues of this regular partition. In fact this follows from [2, §4.1.B], but we can also use Proposition 4.3 to prove that. By that proposition, it suffices to prove that

$$\sum_{i=0}^{d} \theta_i = \sum_{i=0}^{d} a_{ii} = \sum_{i=0}^{d} a_i$$

for all prime powers q. Taking into account that $k = \frac{q^{2n}-q^{n+1}-q^n+q}{(q-1)(q^2-1)}$, we compute that $a_i = a_{ii} = k - b_i - c_i$ is equal to

$$-\frac{q^2+1}{q(q-1)(q^2-1)} \cdot q^{4i} + \frac{(q^{n+1}+1)}{q(q-1)^2} \cdot q^{2i} - \frac{q^{n+1}+q^n-q+1}{(q-1)(q^2-1)}$$

for every $i \in \{0, 1, \ldots, d\}$. Hence,

$$\sum_{i=0}^{d} a_i = -\frac{q^{4d+4}-1}{q(q-1)(q^2-1)^2} + \frac{(q^{n+1}+1)(q^{2d+2}-1)}{q(q-1)^2(q^2-1)} - \frac{(q^{n+1}+q^n-q+1)(d+1)}{(q-1)(q^2-1)}.$$

On the other hand, from equation (2) we know that

$$\sum_{i=0}^{d} \theta_i = -\frac{(q^{n+1}+q^n-q+1)(d+1)}{(q-1)(q^2-1)} + \frac{q^{2n-2d}+1}{(q-1)(q^2-1)} \cdot \frac{q^{2d+2}-1}{q^2-1}.$$

Using the fact that $d = \frac{n}{2}$ if n is even and $d = \frac{n-1}{2}$ if n is odd, it is now straightforward to verify that $\sum_{i=0}^{d} \theta_i = \sum_{i=0}^{d} a_i$.

5.2 Treatment of Class 1(b)

Let M^* be a maximal subspace of $Q^+(2n-1,q)$ that is not a point of $HS(2n-1,q)$. Put $d' := \lfloor \frac{n-1}{2} \rfloor$. For every $i \in \{0, 1, \ldots, d'\}$, let $X_i \neq \emptyset$ denote the set of all points of $HS(2n-1,q)$ that intersect M^* in a subspace of dimension $n-2-2i$. If $M \in X_i$, then the distance between M and M^* in the dual polar space $DQ^+(2n-1,q)$ is equal to $2i+1$.

Lemma 5.1 *Let x and y be two points of $DQ^+(2n-1,q)$ at distance $2i+1$ from each other, where $i \in \{0, 1, \ldots, \lfloor \frac{n-1}{2} \rfloor\}$. Let F denote the unique convex subspace of $DQ^+(2n-1,q)$ of diameter $2i+1$ through x and y. For every quad Q through y, let A_Q denote the set of q points of Q at distance 2 from y. Then one of the following three cases occurs for such a quad.*

(1) *The quad Q is contained in F. Then one point of A_Q lies at distance $2i-1$ from x and the other $q-1$ lie at distance $2i+1$ from x.*

(2) *The quad Q intersects F in a line. Then all q points of A_Q lie at distance $2i+1$ from x.*

(3) *The quad Q intersects F in the singleton $\{y\}$. Then all q points of A_Q lie at distance $2i+3$ from x.*

Proof. The collinearity graph of $DQ^+(2n-1,q)$ is bipartite and so every point of A_Q has distance $2i-1$, $2i+1$ or $2i+3$ from x, and every point of $Q \setminus A_Q$ lies at distance $2i$ or $2i+2$ from x.

(1) If the quad Q is contained in F, then the maximal distance that a point of F can have to Q is $2i-1$. As $\mathrm{d}(x,y) = 2i+1$ and x is classical with respect to Q, the quad Q contains a unique point x' at distance $2i-1$ from x. This point x' is the unique point of A_Q at distance $2i-1$ from x and any other point of A_Q has distance $2i+1$ from x.

(2) Suppose Q intersects F in a line $\{y, z\}$. Then $A_Q \cap F = \emptyset$, $\mathrm{d}(x,z) = 2i$ and every point of A_Q is collinear with z. Since every point of A_Q is classical with respect to F, the distance of such a point to x is equal to $1 + \mathrm{d}(z,x) = 2i+1$.

(3) Suppose Q intersects F in the singleton $\{y\}$. If $z \in A_Q$, then the unique point $z' \in F$ nearest to z lies on a shortest path from z to y, implying that $z' = y$ and $\mathrm{d}(z,x) = \mathrm{d}(z,z') + \mathrm{d}(z',x) = \mathrm{d}(z,y) + \mathrm{d}(y,z) = 2i+3$. ■

For every $j \in \{1,2,3\}$, let $N_i^{(j)}$ denote the number of quads Q as in (j) of Lemma 5.1. Using the fact that $Res(y) \cong \mathrm{PG}(n-1,q)$, we see that

$$N_i^{(1)} = \frac{(q^{2i+1}-1)(q^{2i}-1)}{(q-1)(q^2-1)}, \qquad N_i^{(2)} = \frac{(q^{2i+1}-1)(q^{n-2i-1}-1)}{(q-1)^2} q^{2i},$$

$$N_i^{(3)} = \frac{(q^n-1)(q^{n-1}-1)}{(q-1)(q^2-1)} - N_i^{(1)} - N_i^{(2)} = \frac{(q^n-q^{2i+1})(q^{n-1}-q^{2i+1})}{(q-1)(q^2-1)}.$$

The following is an immediate consequence of Lemma 5.1, taking into account the following two facts from Section 2.3:

- two points of $HS(2n-1,q)$ lie at distance 1 from each other if and only if they lie at distance 2 regarded as points of $DQ^+(2n-1,q)$;
- two points of $DQ^+(2n-1,q)$ that lie at distance 2 from each other are contained in a unique quad.

Corollary 5.2 *The set $\{X_0, X_1, \ldots, X_{d'}\}$ is a regular partition of $DQ^+(2n-1,q)$. If $A = (a_{ij})$ denotes the coefficient matrix, then*

- *$a_{ij} = 0$ for all $i,j \in \{0,1,\ldots,d'\}$ with $|i-j| \geq 2$;*
- *$a_{i,i-1} = N_i^{(1)}$ for all $i \in \{1,2,\ldots,d'\}$;*
- *$a_{ii} = (q-1)N_i^{(1)} + qN_i^{(2)}$ for all $i \in \{0,1,\ldots,d'\}$;*
- *$a_{i,i+1} = qN_i^{(3)}$ for all $i \in \{0,1,\ldots,d'-1\}$.*

We show that $\theta_0, \theta_1, \ldots, \theta_{d'}$ are the eigenvalues of this regular partition. By Proposition 4.3, it suffices to prove that

$$\sum_{i=0}^{d'} \theta_i = \sum_{i=0}^{d'} a_{ii}$$

for all prime powers q. We compute that

$$\begin{aligned} a_{ii} &= \frac{(q^{2i+1}-1)(q^{2i}-1)}{q^2-1} + \frac{(q^{2i+1}-1)(q^n-q^{2i+1})}{(q-1)(q-1)} \\ &= -\frac{q(q^2+1)}{(q-1)^2(q+1)} q^{4i} + \frac{q^{n+1}+1}{(q-1)^2} q^{2i} - \frac{q^{n+1}+q^n-q+1}{(q-1)(q^2-1)}. \end{aligned}$$

Hence,

$$\sum_{i=0}^{d'} a_{ii} = -\frac{q(q^{4d'+4}-1)}{(q-1)(q^2-1)^2} + \frac{(q^{n+1}+1)(q^{2d'+2}-1)}{(q-1)^2(q^2-1)} - \frac{(q^{n+1}+q^n-q+1)(d'+1)}{(q-1)(q^2-1)}.$$

On the other hand, by equation (2) we know that

$$\sum_{i=0}^{d'} \theta_i = -\frac{(q^{n+1}+q^n-q+1)(d'+1)}{(q-1)(q^2-1)} + \frac{q^{2n-2d'}+1}{(q-1)(q^2-1)} \cdot \frac{q^{2d'+2}-1}{q^2-1}.$$

Using the fact that $d' = \frac{n-2}{2}$ if n is even and $d = \frac{n-1}{2}$ if n is odd, it is now straightforward to verify that $\sum_{i=0}^{d'} \theta_i = \sum_{i=0}^{d'} a_{ii}$.

5.3 Treatment of Class 1(c)

Let α be a nonempty subspace of $Q^+(2n-1,q)$ for which $k := \dim(\alpha) \leq n-2$, let $X_i \neq \emptyset$ for $i \in \{0,1,\ldots,k+1\}$ denote the set of all points M of $HS(2n-1,q)$ that intersect α in a subspace of dimension $k-i$.

Let F_α denote the set of all maximal subspaces of $Q^+(2n-1,q)$ that contain α. Then F_α is a convex subspace of diameter $n-1-k$ of the dual polar space $DQ^+(2n-1,q)$. The generators of $Q^+(2n-1,q)$ intersecting α in subspaces of dimension $k-i$ are precisely the points of $DQ^+(2n-1,q)$ at distance i from F_α, see e.g. [7, Chapter 8].

Lemma 5.3 *Let x be a point of $DQ^+(2n-1,q)$ at distance $i \in \{0,1,\ldots,k+1\}$ from F_α. Then the following hold:*

- (a) *The number of points of $DQ^+(2n-1,q)$ collinear with x and at distance $i-1$ from F_α is equal to $N_i^- := \frac{q^i-1}{q-1}$.*
- (b) *The number of points of $DQ^+(2n-1,q)$ collinear with x and at distance i from F_α, not including the point x itself, is equal to $N_i := \frac{q^{i+n-1-k}-q^i}{q-1}$.*
- (c) *The number of points of $DQ^+(2n-1,q)$ collinear with x and at distance $i+1$ from F_α is equal to $N_i^+ := \frac{q^n-q^{i+n-1-k}}{q-1}$.*

Proof. Let x' be the unique point of F_α at distance i from x and let y' be a point of F_α at maximal distance $n-1-k$ from x'. Then $\langle x, x'\rangle$ is a convex subspace of diameter i. As there exists a shortest path from x to y' containing x', the smallest convex subspace containing x and F_α coincides with $\langle x, y'\rangle$ and has diameter $i+n-1-k$.

If L is a line through x containing a point u having distance $i-1$ from a point $u' \in F_\alpha$, then necessarily $u' = x'$ and so L must be one of the $\frac{q^i-1}{q-1}$ lines through x contained in $\langle x, x'\rangle$.

In the convex subspace $\langle x, y'\rangle$ of diameter $i+n-1-k$, every point has distance at most i from F_α. So, if L is one of the $\frac{q^{i+n-1-k}-q^i}{q-1}$ lines through x contained in $\langle x, y'\rangle$, but not in $\langle x, x'\rangle$, then the unique point in $L \setminus \{x\}$ has distance i from F_α.

Suppose L is one of the $\frac{q^n-q^{i+n-1-k}}{q-1}$ lines through x not contained in $\langle x,y'\rangle$. The unique point v in $L\setminus\{x\}$ is classical with respect to $\langle x,y'\rangle$ and so $\mathrm{d}(v,F_\alpha)=\mathrm{d}(v,x)+\mathrm{d}(x,F_\alpha)=i+1$. ■

If $j\in\mathbb{Z}\setminus\{0,1,\ldots,k+1\}$, then we define $N_j=N_j^+=N_j^-=0$.

As every two points of $DQ^+(2n-1,q)$ at distance 2 from each other have precisely $q+1$ common neighbours, we see from Lemma 5.3 that $\{X_0,X_1,\ldots,X_{k+1}\}$ is a regular partition whose coefficient matrix $A=(a_{ij})$ is as follows:

- $a_{ij}=0$ if $i,j\in\{0,1,\ldots,k+1\}$ with $|i-j|>2$;
- $(q+1)\cdot a_{i,i-2}=N_i^-\cdot N_{i-1}^-$ for every $i\in\{2,3,\ldots,k+1\}$;
- $(q+1)\cdot a_{i,i-1}=N_i\cdot N_i^-+N_i^-\cdot N_{i-1}$ for every $i\in\{1,2,\ldots,k+1\}$;
- $(q+1)\cdot a_{ii}=N_i\cdot(N_i-1)+N_i^+\cdot(N_{i+1}^--1)+N_i^-\cdot(N_{i-1}^+-1)$ for every $i\in\{0,1,\ldots,k+1\}$;
- $(q+1)\cdot a_{i,i+1}=N_i\cdot N_i^++N_i^+\cdot N_{i+1}$ for every $i\in\{0,1,\ldots,k\}$;
- $(q+1)\cdot a_{i,i+2}=N_i^+\cdot N_{i+1}^+$ for every $i\in\{0,1,\ldots,k-1\}$.

We compute that

$$(q+1)\cdot a_{ii}=\frac{q^i-1}{q-1}\cdot\Big(\frac{q^n-q^{i+n-k-2}}{q-1}-1\Big)+\frac{q^{i+n-1-k}-q^i}{q-1}\cdot\Big(\frac{q^{i+n-1-k}-q^i}{q-1}-1\Big)$$
$$+\frac{q^n-q^{i+n-1-k}}{q-1}\cdot\frac{q^{i+1}-q}{q-1}.$$

It is possible to rewrite a_{ii} in the following way

$$a_{ii}=\frac{1}{(q^2-1)(q-1)}(Aq^{2i}+Bq^i+C),$$

where

$$\begin{aligned}A&=q^{2n-2k-2}-q^{n-k-2}(q+1)^2+1,\\ B&=q^{n-k-2}(q+1)(q^{k+2}+1),\\ C&=-(q^{n+1}+q^n-q+1).\end{aligned}$$

It is straightforward to verify that

$$\sum_{i=0}^{k+1}a_{ii}=\frac{1}{(q^2-1)(q-1)}\Big(A\cdot\frac{q^{2k+4}-1}{q^2-1}+B\cdot\frac{q^{k+2}-1}{q-1}+C(k+2)\Big)$$

is equal to

$$\sum_{i=0}^{k+1}\theta_i=\frac{C(k+2)}{(q^2-1)(q-1)}+\frac{(q^{2n-2k-2}+1)(q^{2k+4}-1)}{(q-1)(q^2-1)^2}.$$

So, by Proposition 4.3, the eigenvalues of the regular partition are $\theta_0,\theta_1,\ldots,\theta_{k+1}$. Using equation (1) of Section 2.2, we can then conclude the following:

- If $k+1 \leq d = \lfloor \frac{n}{2} \rfloor$, then the eigenvalues are $\theta_0, \theta_1, \dots, \theta_{k+1}$.
- If n is even and $k+1 > d = \frac{n}{2}$, then the eigenvalues are $\theta_0, \theta_1, \dots, \theta_{n-k-2}$ and θ_d with multiplicity 1, and $\theta_{n-k-1}, \theta_{n-k}, \dots, \theta_{d-1}$ with multiplicity 2.
- If n is odd and $k+1 > d = \frac{n-1}{2}$, then the eigenvalues are $\theta_0, \theta_1, \dots, \theta_{n-k-2}$ with multiplicity 1 and $\theta_{n-k-1}, \theta_{n-k}, \dots, \theta_d$ with multiplicity 2.

5.4 Treatment of Class 2

Let $X_0, X_1, \dots, X_{\widetilde{m}}$ with $\widetilde{m} := \min(m, n-m-\varepsilon)$ be the sets of points of $HS(2n-1,q)$ as described in Section 3. We show that these sets determine a regular partition and compute the corresponding parameters and eigenvalues. We achieve these goals using a reasoning in the dual polar space $DQ^+(2n-1,q)$. The following lemma was proved in [3, Section 6] (consider the special case $e=0$).

Lemma 5.4 *Let x be a point of $DQ^+(2n-1,q)$ intersecting Π in a subspace of dimension $m-1-i$. Then every point of $DQ^+(2n-1,q)$ collinear with x intersects Π in a subspace of dimension $m-1-(i+\epsilon)$ where $\epsilon \in \{0,1,-1\}$. Moreover:*

(a) *The number of points of $DQ^+(2n-1,q)$ collinear with x that intersect Π in a subspace of dimension $m-1-(i-1)$ is equal to $N_i^- := \frac{(q^i-1)(q^{i+\varepsilon-1}+1)}{q-1}$.*

(b) *The number of points of $DQ^+(2n-1,q)$ collinear with x that intersect Π in a subspace of dimension $m-1-i$, not including the point x itself, is equal to $N_i := \frac{q^{n-m+i}-1}{q-1} - \frac{(q^i-1)(q^{i+\varepsilon-1}+1)}{q-1} + q^{2i+\varepsilon}\frac{q^{m-i}-1}{q-1}$.*

(c) *The number of points of $DQ^+(2n-1,q)$ collinear with x that intersect Π in a subspace of dimension $m-1-(i+1)$ is equal to $N_i^+ := q^{2i+\varepsilon}\frac{(q^{m-i}-1)(q^{n-m-\varepsilon-i}-1)}{q-1}$.*

For every $j \in \mathbb{Z} \setminus \{0,1,\dots,\widetilde{m}\}$, we put $N_j = N_j^+ = N_j^- = 0$.

Lemma 5.5 $\{X_0, X_1, \dots, X_{\widetilde{m}}\}$ *is a partition of the point set of $HS(2n-1,q)$.*

Proof. Note that $X_0 \neq \emptyset$. Since $DQ^+(2n-1,q)$ is connected and $N_{\widetilde{m}}^+ = 0$, we see that $X_0 \cup X_1 \cup \dots \cup X_{\widetilde{m}}$ coincides with the whole point set.

Note that $N_0 \neq 0$ and that $N_i^+ \neq 0$ for every $i \in \{0,1,\dots,\widetilde{m}-1\}$. So, for every $j \in \{0,1,\dots,\widetilde{m}\}$, there exists a path of even length in $DQ^+(2n-1,q)$ connecting a point of X_0 with a point of X_j, implying that $X_j \neq \emptyset$. ■

As every two points of $DQ^+(2n-1,q)$ at distance 2 from each other have precisely $q+1$ common neighbours, we see from Lemmas 5.4 and 5.5 that $\{X_0, X_1, \dots, X_{\widetilde{m}}\}$ is a regular partition whose coefficient matrix $A = (a_{ij})$ is as follows:

- $a_{ij} = 0$ if $i, j \in \{0, 1, \ldots, \widetilde{m}\}$ with $|i - j| > 2$;
- $(q+1) \cdot a_{i,i-2} = N_i^- \cdot N_{i-1}^-$ for every $i \in \{2, 3, \ldots, \widetilde{m}\}$;
- $(q+1) \cdot a_{i,i-1} = N_i \cdot N_i^- + N_i^- \cdot N_{i-1}$ for every $i \in \{1, 2, \ldots, \widetilde{m}\}$;
- $(q+1) \cdot a_{ii} = N_i \cdot (N_i - 1) + N_i^+ \cdot (N_{i+1}^- - 1) + N_i^- \cdot (N_{i-1}^+ - 1)$ for every $i \in \{0, 1, \ldots, \widetilde{m}\}$;
- $(q+1) \cdot a_{i,i+1} = N_i \cdot N_i^+ + N_i^+ \cdot N_{i+1}$ for every $i \in \{0, 1, \ldots, \widetilde{m} - 1\}$;
- $(q+1) \cdot a_{i,i+2} = N_i^+ \cdot N_{i+1}^+$ for every $i \in \{0, 1, \ldots, \widetilde{m} - 2\}$.

From this we compute that

$$\begin{aligned} a_{ii} \; = \; & \frac{1}{q+1}\Big(\frac{(q^i-1)(q^{i+\varepsilon-1}+1)}{q-1} \cdot \Big(\frac{q^{2i-2+\varepsilon}(q^{m-i+1}-1)(q^{n-m-i+1-\varepsilon}-1)}{q-1} - 1\Big) \\ & + \frac{(q^{n-m+i}-1) - (q^i-1)(q^{i+\varepsilon-1}+1) + q^{2i+\varepsilon}(q^{m-i}-1)}{q-1} \cdot \\ & \Big(\frac{(q^{n-m+i}-1) - (q^i-1)(q^{i+\varepsilon-1}+1) + q^{2i+\varepsilon}(q^{m-i}-1)}{q-1} - 1\Big) \\ & + \frac{q^{2i+\varepsilon}(q^{m-i}-1)(q^{n-\varepsilon-m-i}-1)}{q-1} \cdot \Big(\frac{(q^{i+1}-1)(q^{i+\varepsilon}+1)}{q-1} - 1\Big)\Big). \end{aligned}$$

Theorem 5.6 *The eigenvalues of the regular partition $\{X_0, X_1, \ldots, X_{\widetilde{m}}\}$ are as follows.*

(1) *If $m \leq (n-\varepsilon)/2$, then there are $m+1$ eigenvalues with multiplicity 1. They are the first $m+1$ eigenvalues $\theta_0, \theta_1, \ldots, \theta_m$ of Γ.*

(2) *If $m > (n-\varepsilon)/2$, then there are $n-m-\varepsilon+1$ eigenvalues with multiplicity 1. They are the first $n-m-\varepsilon+1$ eigenvalues $\theta_0, \theta_1, \ldots, \theta_{n-m-\varepsilon}$ of Γ.*

Proof. It is possible to rewrite a_{ii} in the following way

$$a_{ii} = \frac{1}{(q^2-1)(q-1)}(Aq^{4i} + Bq^{3i} + Cq^{2i} + Dq^i + E), \tag{7}$$

where

$$\begin{aligned} A &= q^{2\varepsilon-3}(q^2+1)(q^2+q+1), \\ B &= -q^{-m+\varepsilon-3}(q+1)(q^2+q+1)(q^{n+1} + q^{2m+\varepsilon+1} + q^{m+\varepsilon} - q^{m+1}), \\ C &= q^{-2m-2}(q^{2n+2} + q^{n+2m+\varepsilon+3} + 2q^{n+2m+\varepsilon+2} + q^{n+2m+\varepsilon+1} + q^{n+m+\varepsilon+2} + \\ & \quad 2q^{n+m+\varepsilon+1} + q^{n+m+\varepsilon} - q^{n+m+3} - 2q^{n+m+2} - q^{n+m+1} + q^{4m+2\varepsilon+2} + q^{3m+2\varepsilon+2} + \\ & \quad 2q^{3m+2\varepsilon+1} + q^{3m+2\varepsilon} - q^{3m+\varepsilon+3} - 2q^{3m+\varepsilon+2} - q^{3m+\varepsilon+1} + \\ & \quad q^{2m+2\varepsilon} - q^{2m+\varepsilon+2} - 2q^{2m+\varepsilon+1} - q^{2m+\varepsilon} + q^{2m+2}), \\ D &= -q^{-m-1}(q+1)(q^{n+m+\varepsilon} - q^{n+m+1} - q^n - q^{2m+\varepsilon}), \\ E &= -q^{n+1} - q^n + q - 1. \end{aligned}$$

Suppose $m \leq (n-\varepsilon)/2$. Then $m \leq \lfloor \frac{n}{2} \rfloor$. By Proposition 4.3, we need to prove that

$$\sum_{i=0}^{m} a_{ii} = \sum_{i=0}^{m} \theta_i \tag{8}$$

for all prime powers q. By equations (2) and (7), equation (8) can be rewritten as

$$A \cdot \frac{q^{4(m+1)}-1}{q^4-1} + B \cdot \frac{q^{3(m+1)}-1}{q^3-1} + (C - q^{2n-2m}-1) \cdot \frac{q^{2(m+1)}-1}{q^2-1} + D \cdot \frac{q^{(m+1)}-1}{q-1} = 0.$$

Plugging in the previous equation the above values of A, B, C and D, we obtain an identity (after a lengthy, tedious, but straightforward computation). So, part (1) of the theorem is proved.

Suppose $m > (n-\varepsilon)/2$. Then $n-m-\varepsilon \leq \lfloor \frac{n}{2} \rfloor$. We need to prove that

$$\sum_{i=0}^{n-m-\varepsilon} a_{ii} = \sum_{i=0}^{n-m-\varepsilon} \theta_i \tag{9}$$

for all prime powers q. By equations (2) and (7), equation (9) can be rewritten as

$$A \cdot \frac{q^{4(n-m-\varepsilon+1)}-1}{q^4-1} + B \cdot \frac{q^{3(n-m-\varepsilon+1)}-1}{q^3-1} + (C - q^{2m+2\varepsilon}-1) \cdot \frac{q^{2(n-m-\varepsilon+1)}-1}{q^2-1} + D \cdot \frac{q^{(n-m-\varepsilon+1)}-1}{q-1} = 0.$$

Plugging in the previous equation the above values of A, B, C and D, we again obtain an identity. So, also part (2) of the theorem is proved. ∎

5.5 Treatment of Class 3

During the treatment of Class 3, we need to rely on the following lemma.

Lemma 5.7 *Let $k \in \mathbb{N} \setminus \{0\}$. Then for all $n_1, n_2, \ldots, n_k \in \mathbb{N} \setminus \{0\}$, there exists a $Q \in \mathbb{N}$ such that for every prime power $q > Q$ the quadric $Q^+(2n-1,q)$ has a collection of $n_1 + n_2 + \cdots + n_k$ mutually noncollinear points that is not an ovoid.*

Proof. The total number of generators of $Q^+(2n-1,q)$ is equal to $2(q+1)(q^2+1)\cdots(q^{n-1}+1)$. Put $Q := n_1 + n_2 + \cdots + n_k$. We show by induction on i that for every prime power $q > Q$ and every $i \in \{1, 2, \ldots, n_1 + n_2 + \cdots + n_k + 1\}$, there are i mutually noncollinear points on $Q^+(2n-1,q)$. Clearly, this holds if $i = 1$. So, suppose $i \in \{2, 3, \ldots, n_1 + n_2 + \cdots + n_k + 1\}$. By the induction hypothesis, there exists a collection $x_1, x_2, \ldots, x_{i-1}$ of $i-1$ mutually noncollinear points of $Q^+(2n-1,q)$. The number of generators of $Q^+(2n-1,q)$ containing one of these points is equal to $(i-1) \cdot 2(q+1)(q^2+1)\cdots(q^{n-2}+1) < 2(q+1)(q^2+1)\cdots(q^{n-2}+1)(Q+1) < 2(q+$

$1)(q^2+1)\cdots(q^{n-2}+1)(q^{n-1}+1)$, and so there exists a generator α of $Q^+(2n-1,q)$ not containing any of the points $x_1, x_2, \ldots, x_{i-1}$. The set of points of α collinear with a point $x_j \in \{x_1, x_2, \ldots, x_{i-1}\}$ is a hyperplane containing $q^{n-2}+q^{n-1}+\cdots+1$ points. These $i-1$ hyperplanes cover at most $(i-1)(q^{n-2}+q^{n-1}+\cdots+1) < q(q^{n-2}+q^{n-1}+\cdots+1)$ points, and so there exists a point $x_i \in \alpha$ collinear with neither of $x_1, x_2, \ldots, x_{i-1}$. ■

In the literature, one can find better lower bounds for q. However the above lemma will suffice for our purposes.

Let $\mathcal{O}_1$, $\mathcal{O}_2, \ldots, \mathcal{O}_k$ be a collection of $k \geq 1$ non-empty mutually disjoint sets of points of $Q^+(2n-1,q)$ such that $\mathcal{O}_1 \cup \mathcal{O}_2 \cup \cdots \cup \mathcal{O}_k$ is a partial ovoid of $Q^+(2n-1,q)$, but not an ovoid. For every $i \in \{1,2,\ldots,k\}$, let X_i denote the set of all points of $HS(2n-1,q)$ that have nonempty intersection with $\mathcal{O}_i$. Then $X_1, X_2, \ldots, X_k$ are mutually disjoint. Denote by $X_{k+1} \neq \emptyset$ the set of all points of $HS(2n-1,q)$ not contained in $\mathcal{O}_1 \cup \mathcal{O}_2 \cup \cdots \cup \mathcal{O}_k$. Put $n_i := |\mathcal{O}_i|$ for every $i \in \{1,2,\ldots,k\}$.

Lemma 5.8 *Let α be a generator of $Q^+(2n-1,q)$.*

(1) *If x is a point of $Q^+(2n-1,q)$ not contained in α, then x is contained in $\frac{q^{n-1}-1}{q-1}$ generators intersecting α in a subspace of co-dimension 2.*

(2) *If x is a point of α, then x is contained in $q \cdot \begin{bmatrix} n-1 \\ 2 \end{bmatrix}_q$ generators intersecting α in a subspace of co-dimension 2.*

Proof. (1) Let β denote the unique generator through x intersecting α in a subspace of dimension $n-2$. There are $\frac{q^{n-1}-1}{q-1}$ hyperplanes γ of $\alpha \cap \beta$, and for each such hyperplane, we denote by $\widetilde{\gamma}$ the unique generator of the same family as α that contains $\langle x, \gamma \rangle$. In this way, we find all $\frac{q^{n-1}-1}{q-1}$ generators through x that intersect α in a subspace of co-dimension 2.

(2) Look at the quotient space at the point x. Then the required number is equal to the number of generators of $Q^+(2n-3,q)$ intersecting a given generator in a subspace of co-dimension 2. This number is equal to $q \cdot \begin{bmatrix} n-1 \\ 2 \end{bmatrix}_q$. ■

A consequence of Lemma 5.8 is that $\{X_1, X_2, \ldots, X_{k+1}\}$ is a regular partition with the following intersection numbers:

- $a_{ii} = (n_i - 1)\frac{q^{n-1}-1}{q-1} + q \cdot \begin{bmatrix} n-1 \\ 2 \end{bmatrix}_q$ for all $i \in \{1,2,\ldots,k\}$,
- $a_{ij} = n_j \cdot \frac{q^{n-1}-1}{q-1}$ for all $i,j \in \{1,2,\ldots,k\}$ with $i \neq j$,
- $a_{i,k+1} = q \cdot \begin{bmatrix} n \\ 2 \end{bmatrix}_q - a_{i1} - a_{i2} - \cdots - a_{ik} = (q^{n-1}+1-n_1-n_2-\cdots-n_k) \cdot \frac{q^{n-1}-1}{q-1}$ for all $i \in \{1,2,\ldots,k\}$,

- $a_{k+1,i} = n_i \cdot \frac{q^{n-1}-1}{q-1}$ for all $i \in \{1, 2, \ldots, k\}$,
- $a_{k+1,k+1} = q \cdot \begin{bmatrix} n \\ 2 \end{bmatrix}_q - (n_1 + n_2 + \cdots + n_k) \cdot \frac{q^{n-1}-1}{q-1}$.

We have

$$a_{11} + a_{22} + \cdots + a_{kk} + a_{k+1,k+1} = q \cdot \begin{bmatrix} n \\ 2 \end{bmatrix}_q + k\Big(q \cdot \begin{bmatrix} n-1 \\ 2 \end{bmatrix}_q - \frac{q^{n-1}-1}{q-1}\Big) = \theta_0 + k\theta_1.$$

By Proposition 4.3 and Lemma 5.7, the eigenvalues of the regular partition are θ_0 (with multiplicity 1) and θ_1 (with multiplicity k).

5.6 Treatment of Class 4

Put $\overline{x} = (x_0, x_1, \ldots, x_{2n-1})$ and let $\phi(\overline{x})$ be a nondegenerate quadratic form over $\mathbb{F}_q$ defining an elliptic quadric $Q^-(2n-1, q)$ in $\mathrm{PG}(2n-1, q)$ and a hyperbolic quadric $Q^+(2n-1, q^2)$ in $\mathrm{PG}(2n-1, q^2)$. We will regard $\mathrm{PG}(2n-1, q)$ as naturally embedded in $\mathrm{PG}(2n-1, q^2)$. Let θ be the automorphism $(x_0, x_1, \ldots, x_{2n-1}) \mapsto (x_0^\theta, x_1^\theta, \ldots, x_{2n-1}^\theta)$ of $\mathrm{PG}(2n-1, q^2)$. The points of $\mathrm{PG}(2n-1, q)$ are precisely the fixpoints of θ. For every subspace α of $\mathrm{PG}(2n-1, q)$, let $\overline{\alpha}$ denote the unique subspace of $\mathrm{PG}(2n-1, q^2)$ containing α and having the same dimension as α. The following hold:

(1) If γ is a subspace of $\mathrm{PG}(2n-1, q^2)$, then $\gamma \cap \gamma^\theta = \overline{\beta}$ for some subspace β of $\mathrm{PG}(2n-1, q)$.

(2) If γ is a subspace of $Q^+(2n-1, q^2)$, then $\gamma \cap \gamma^\theta = \overline{\beta}$ for some subspace β of $Q^-(2n-1, q)$.

Put $d' := \lfloor \frac{n-1}{2} \rfloor$. For every $i \in \{0, 1, \ldots, d'\}$, let X_i denote the set of all points x of $HS(2n-1, q^2)$ for which $\dim(x \cap x^\theta) = n-2-2i$.

Lemma 5.9 *Let x be a point of $DQ^+(2n-1, q^2)$ such that $x \cap x^\theta$ has dimension $n-2-2i$. Then for every point y of $DQ^+(2n-1, q^2)$ collinear with x, we have* $\dim(y \cap y^\theta) = \dim(x \cap x^\theta) + \epsilon$ *for some $\epsilon \in \{-2, 0, 2\}$. Moreover:*

(a) The number of points y of $DQ^+(2n-1, q^2)$ collinear with x such that $y \cap y^\theta$ has dimension $n-2-2(i-1)$ is equal to $N_i^- := \frac{(q^{2i}-1)(q^{2i+1}+1)}{(q-1)(q+1)}$.

(b) The number of points $y \neq x$ of $DQ^+(2n-1, q^2)$ collinear with x such that $y \cap y^\theta$ has dimension $n-2-2i$ is equal to $N_i := \frac{q^{n-1-2i}-1}{q-1} \cdot q^{4i+2} + \frac{q^{4i+2}-1}{q^2-1} - \frac{(q^{2i}-1)(q^{2i+1}+1)}{q^2-1}$.

(c) The number of points y of $DQ^+(2n-1, q^2)$ collinear with x such that $y \cap y^\theta$ has dimension $n-2-2(i+1)$ is equal to $N_i^+ := \frac{q^{4i+3}}{q^2-1}(q^{n-2i-1}-1)(q^{n-2i-2}-1)$.

Proof. Since $y \cap y^\theta \subseteq y$ and $x \cap y$ is a hyperplane of y, we have $\dim(x \cap y \cap y^\theta) \geq \dim(y \cap y^\theta) - 1$. Since $x \cap y \cap y^\theta \subseteq y^\theta$ and $x^\theta \cap y^\theta$ is a hyperplane of y^θ, we have $\dim(x \cap x^\theta \cap y \cap y^\theta) \geq \dim(x \cap y \cap y^\theta) - 1 \geq \dim(y \cap y^\theta) - 2$. It follows that $\dim(x \cap x^\theta) \geq \dim(y \cap y^\theta) - 2$. By symmetry, we then also know that $\dim(y \cap y^\theta) \geq \dim(x \cap x^\theta) - 2$. Since both $\dim(x \cap x^\theta)$ and $\dim(y \cap y^\theta)$ are congruent to $n-2$ modulo 2, we have that $\dim(y \cap y^\theta) = \dim(x \cap x^\theta) + \epsilon$ for some $\epsilon \in \{-2, 0, 2\}$.

(a) Suppose y is a point of $DQ^+(2n-1, q^2)$ collinear with x such that $y \cap y^\theta$ has dimension $n-2-2(i-1)$. By the above, we know that $\dim(x \cap x^\theta \cap y \cap y^\theta) \geq \dim(y \cap y^\theta) - 2 = \dim(x \cap x^\theta)$. It follows that $(x \cap y) \cap (x \cap y)^\theta = x \cap x^\theta \cap y \cap y^\theta = x \cap x^\theta$, implying that $x \cap x^\theta \subseteq y \cap x$.

The subspace y of $Q^+(2n-1, q^2)$ is therefore obtained as follows. Put $x \cap x^\theta = \overline{\beta}$ for some subspace β of $Q^-(2n-1, q)$ of dimension $n-2-2i$, let γ be a subspace of $Q^-(2n-1, q)$ for which β is a hyperplane, and let y be the unique generator of $Q^+(2n-1, q^2)$ through $\overline{\gamma}$ intersecting x in a subspace of dimension $n-2$. Note that such a γ cannot be contained in x, as otherwise it would be contained in $x \cap x^\theta = \overline{\beta}$. Also, since $\overline{\gamma} \subseteq y \cap y^\theta$ and $\dim(\overline{\gamma}) = n-2-2i+1$, we indeed have $\dim(y \cap y^\theta) = n-2-2(i-1)$.

The quotient polar space of $Q^-(2n+1, q)$ determined by β is a $Q^-(4i+1, q)$ containing $\frac{(q^{2i}-1)(q^{2i+1}+1)}{q-1}$ points. So, there are that many possibilities for γ. Each such γ gives rise to a y. However, for each such γ, we have $\dim(y \cap y^\theta) = (n-2-2i)+2$ and so each y arises from $q+1$ possible γ's. The number N_i^- is therefore equal to

$$N_i^- = \frac{(q^{2i}-1)(q^{2i+1}+1)}{(q-1)(q+1)}.$$

(b) Put $x \cap x^\theta = \overline{\beta}$ for some subspace β of $Q^-(2n-1, q)$ of dimension $n-2-2i$. Let y be a point of $DQ^+(2n-1, q^2)$ collinear with x.

Suppose there exists a hyperplane γ of β such that $\overline{\gamma}$ is contained in $x \cap y$. Then $y \cap y^\theta$ contains $\overline{\gamma}$. As $\overline{\gamma}$ has dimension $n-2-2i-1$, the dimension of $y \cap y^\theta$ should be at least $n-2-2i$.

Conversely, suppose that $y \cap y^\theta$ has dimension at least $n-2-2i$ and that there exists no hyperplane γ of β such that $\overline{\gamma}$ is contained in $x \cap y$. Then $\overline{\beta}$ is not contained in $x \cap y$ and so the subspace $x \cap y \cap \overline{\beta}$ is a hyperplane of $\overline{\beta}$. Since $\dim(y \cap y^\theta) \geq n-2-2i$ and $\dim(x \cap y \cap \overline{\beta}) = n-2-2i-1$, there exists a point u of $\mathrm{PG}(2n-1, q)$ contained in y, but not in $x \cap \overline{\beta} = \overline{\beta}$. Then u is not contained in x, as every point of $x \cap \mathrm{PG}(n-1, q)$ is also contained in $x \cap x^\theta = \overline{\beta}$. So, y is the unique generator of $Q^+(2n-1, q^2)$ through u meeting x in a subspace of dimension $n-2$. As β is a subspace of $Q^-(2n-1, q)$ and u is a point of $Q^-(2n-1, q)$, there exists a hyperplane η of β consisting only of points of $Q^-(2n-1, q)$ collinear with u. Then $\overline{\eta} \subseteq y$ and hence $\overline{\eta} = x \cap y$, a contradiction.

So, every point y of $DQ^+(2n-1, q^2)$ at distance 1 from x satisfying $\dim(y \cap y^\theta) \geq n-2-2i$ is obtained as follows. Take a hyperplane μ of x containing a subspace $\overline{\gamma}$ with γ a hyperplane of β, and let y be the unique generator through μ distinct from x. The number of such y is easily counted:

$$\frac{q^{n-1-2i}-1}{q-1} \cdot q^{4i+2} + \frac{q^{4i+2}-1}{q^2-1}.$$

Indeed, there are $\frac{q^{4i+2}-1}{q^2-1}$ hyperplanes in x that contain $\overline{\beta}$, and through each of the $\frac{q^{n-1-2i}-1}{q-1}$ hyperplanes of β there are $q^{4i+2} = \frac{q^{4i+4}-1}{q^2-1} - \frac{q^{4i+2}-1}{q^2-1}$ hyperplanes of x not containing $\overline{\beta}$.

Taking into account part (a), the number of points y of $DQ^+(2n-1, q^2)$ at distance 1 from x such that $\dim(y \cap y^\theta) = n - 2 - 2i$ is equal to

$$N_i = \frac{q^{n-1-2i}-1}{q-1} \cdot q^{4i+2} + \frac{q^{4i+2}-1}{q^2-1} - \frac{(q^{2i}-1)(q^{2i+1}+1)}{q^2-1}.$$

(c) Every point of the dual polar space $DQ^+(2n-1, q^2)$ is collinear with $\frac{q^{2n}-1}{q^2-1}$ other points. So, the number N_i^+ is equal to

$$\begin{aligned} N_i^+ = \frac{q^{2n}-1}{q^2-1} - N_i - N_i^- &= \frac{q^{2n}-1}{q^2-1} - \frac{q^{4i+2}-1}{q^2-1} - \frac{q^{n-1-2i}-1}{q-1} q^{4i+2} \\ &= \frac{q^{4i+3}(q^{n-2i-1}-1)(q^{n-2i-2}-1)}{q^2-1}. \end{aligned}$$

■

For every $j \in \mathbb{Z} \setminus \{0, 1, \ldots, d'\}$, we put $N_j = N_j^+ = N_j^- = 0$.

Lemma 5.10 *$\{X_0, X_1, \ldots, X_{d'}\}$ is a partition of the point set of $HS(2n-1, q^2)$.*

Proof. If β is a generator of $Q^-(2n-1, q)$ and γ is the unique point of $HS(2n-1, q^2)$ containing $\overline{\beta}$, then $\gamma \cap \overline{\gamma} = \overline{\beta}$. So, $\gamma \in X_0$ and the set X_0 is nonempty. Since $DQ^+(2n-1, q^2)$ is connected and $N_{d'}^+ = 0$, we see that $X_0 \cup X_1 \cup \cdots \cup X_{d'}$ coincides with the whole point set.

Note that $N_i \neq 0$ for every $i \in \{1, 2, \ldots, d'\}$ and that $N_i^+ \neq 0$ for every $i \in \{0, 1, \ldots, d'-1\}$. So, for every $j \in \{0, 1, \ldots, d'\}$, there exists a path of even length in $DQ^+(2n-1, q^2)$ connecting a point of X_0 with a point of X_j, implying that $X_j \neq \emptyset$. ■

As every two points of $DQ^+(2n-1, q^2)$ at distance 2 from each other have precisely q^2+1 common neighbours, we see from Lemma 5.9 that $\{X_0, X_1, \ldots, X_{d'}\}$ is a regular partition whose coefficient matrix $A = (a_{ij})$ is as follows:

- $a_{ij} = 0$ if $i, j \in \{0, 1, \ldots, d'\}$ with $|i - j| > 2$;
- $(q^2+1) \cdot a_{i,i-2} = N_i^- \cdot N_{i-1}^-$ for every $i \in \{2, 3, \ldots, d'\}$;
- $(q^2+1) \cdot a_{i,i-1} = N_i \cdot N_i^- + N_i^- \cdot N_{i-1}$ for every $i \in \{1, 2, \ldots, d'\}$;
- $(q^2+1) \cdot a_{i,i} = N_i \cdot (N_i - 1) + N_i^+ \cdot (N_{i+1}^- - 1) + N_i^- \cdot (N_{i-1}^+ - 1)$ for every $i \in \{0, 1, \ldots, d'\}$;
- $(q^2+1) \cdot a_{i,i+1} = N_i \cdot N_i^+ + N_i^+ \cdot N_{i+1}$ for every $i \in \{0, 1, \ldots, d'-1\}$;
- $(q^2+1) \cdot a_{i,i+2} = N_i^+ \cdot N_{i+1}^+$ for every $i \in \{0, 1, \ldots, d'-2\}$.

We compute that

$$(q^2+1)\cdot a_{ii} = \frac{(q^{2i}-1)(q^{2i+1}+1)}{q^2-1}\cdot\Big(\frac{q^{4i-1}}{q^2-1}\cdot(q^{n-2i+1}-1)(q^{n-2i}-1)-1\Big)$$

$$+\Big(q^{4i+2}\cdot\frac{q^{n-1-2i}-1}{q-1}+\frac{q^{4i+2}-1}{q^2-1}-\frac{(q^{2i}-1)(q^{2i+1}+1)}{q^2-1}\Big)\cdot$$

$$\Big(q^{4i+2}\cdot\frac{q^{n-1-2i}-1}{q-1}+\frac{q^{4i+2}-1}{q^2-1}-\frac{(q^{2i}-1)(q^{2i+1}+1)}{q^2-1}-1\Big)$$

$$+\frac{q^{4i+3}}{q^2-1}(q^{n-2i-1}-1)(q^{n-2i-2}-1)\Big(\frac{(q^{2i+2}-1)(q^{2i+3}+1)}{q^2-1}-1\Big).$$

Theorem 5.11 *The eigenvalues of the regular partition* $\{X_0, X_1, \ldots, X_{d'}\}$ *are as follows.*

(1) *If n is even, then there are $n/2$ eigenvalues with multiplicity 1. They are the first $n/2$ eigenvalues $\theta_0, \theta_1, \ldots, \theta_{n/2-1}$ of Γ.*

(2) *If n is odd, then there are $(n+1)/2$ eigenvalues with multiplicity* 1*. They are the first $(n+1)/2$ eigenvalues $\theta_0, \theta_1, \ldots, \theta_{(n-1)/2}$ of Γ.*

Proof. It is possible to rewrite a_{ii} in the following way

$$a_{ii} = \frac{1}{(q^2-1)^2(q^2+1)}(Aq^{8i}+Bq^{6i}+Cq^{4i}+Dq^{2i}+E), \tag{10}$$

where

$$\begin{aligned} A &= (q^2-q+1)(q^2+q+1)(q^4+1),\\ B &= -((q^2+1)(q^2-q+1)(q^2+q+1)(q^{n+2}+q^{n+1}+q-1))/q,\\ C &= ((q^2+1)(q^{2n+4}+q^{2n+3}+q^{2n+2}+q^{n+4}-q^n-q^2+q-1))/q,\\ D &= -q^{n-1}(q^2+1)(q^{n+2}-q^{n+1}-q-1),\\ E &= -q^{2n+2}-q^{2n}+q^2-1. \end{aligned}$$

Suppose n is even. By Proposition 4.3, we need to prove that

$$\sum_{i=0}^{(n-2)/2} a_{ii} = \sum_{i=0}^{(n-2)/2}\theta_i. \tag{11}$$

By equations[1] (2) and (10), equation (11) can be rewritten as

$$A\cdot\frac{q^{4n}-1}{q^8-1}+B\cdot\frac{q^{3n}-1}{q^6-1}+(C-q^{2n+4}-1)\cdot\frac{q^{2n}-1}{q^4-1}+D\cdot\frac{q^n-1}{q^2-1}=0.$$

[1]We have to substitute q with q^2.

Plugging in the previous equation the above values of A, B, C and D, we obtain an identity. Part (1) of the theorem is thus proved.

Suppose n is odd. By Proposition 4.3, we need to prove that

$$\sum_{i=0}^{(n-1)/2} a_{ii} = \sum_{i=0}^{(n-1)/2} \theta_i. \tag{12}$$

By equations (2) and (10), equation (12) can be rewritten as

$$A \cdot \frac{q^{4(n+1)}-1}{q^8-1} + B \cdot \frac{q^{3(n+1)}-1}{q^6-1} + (C - q^{2(n+1)} - 1) \cdot \frac{q^{2(n+1)}-1}{q^4-1} + D \cdot \frac{q^{n+1}-1}{q^2-1} = 0.$$

Plugging in the previous equation the above values of A, B, C and D, we obtain an identity. Part (2) of the theorem is thus also proved. ∎

References

[1] J. Bamberg, S. Kelly, M. Law and T. Penttila. Tight sets and m-ovoids of finite polar spaces. *J. Combin. Theory Ser. A* 114 (2007), 1293–1314.

[2] A. E. Brouwer, A. M. Cohen and A. Neumaier. *Distance-Regular Graphs*. Springer-Verlag, Berlin, 1989.

[3] I. Cardinali and B. De Bruyn. Regular partitions of dual polar spaces. *Linear Algebra Appl.* 432 (2010), 744–769.

[4] B. De Bruyn. Tight sets of points in the half-spin geometry related to $Q^+(9, q)$. *Linear Algebra Appl.* 424 (2007), 480–491.

[5] B. De Bruyn. On the intersection of distance-j-ovoids and subpolygons in generalized polygons. *Discrete Math.* 309 (2009), 3023–3031.

[6] B. De Bruyn. A note on the spin-embedding of the dual polar space $DQ^-(2n+1, \mathbb{K})$. *Ars Combin.* 99 (2011), 365–375.

[7] B. De Bruyn. *An introduction to Incidence Geometry*. Frontiers in Mathematics. Birkhäuser, to appear (2016).

[8] A. De Wispelaere and H. Van Maldeghem. Regular partitions of (weak) finite generalized polygons. *Des. Codes Cryptogr.* 47 (2008), 53–73.

[9] C. D. Godsil. *Algebraic combinatorics*. Chapman and Hall Mathematics Series. Chapman & Hall, 1993.

[10] C. D. Godsil and G. Royle. *Algebraic graph theory*. Graduate Texts in Mathematics 207. Springer-Verlag, 2001.

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