The kernel of the generalized Clifford-Fourier transform and its generating function

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Abstract

In this paper, we study the generalized Clifford-Fourier transform introduced in [7] using the Laplace transform technique. We give explicit expressions in the even dimensional case, we obtain polynomial bounds for the kernel functions and establish a generating function.

Keywords: Clifford-Fourier transform, Laplace transform, Bessel function, generalized Fourier transform

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Contents

1 Introduction

In recent years, quite some attention has been devoted to the study of hypercomplex Fourier transforms. For the historical development of quaternion and Clifford-Fourier transforms we refer to [4]. In the present paper, we consider the Clifford-Fourier transform first established in [2, 3]. This is a genuinely non-scalar

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generalization of the Fourier transform, developed within the framework of Clifford analysis [9]. Indeed, it can be written as

$$F_{-}(f)(y) = (2\pi)^{-m/2} \int_{\mathbb{R}^m} K_m(x,y) f(x) dx$$

with

$$K_m(x,y) = e^{i\frac{\pi}{2}\Gamma_y} e^{-i(x,y)}$$

with Γ_y the spherical Dirac operator (see equation (7)).

It turned out to be a difficult problem to determine the kernel $K_m(x,y)$ explicitly. This was first achieved in [8] using plane wave decompositions. Later, in [6] a different method using wave equations was established. In [5], a short proof was obtained by considering the Clifford-Fourier kernel in the Laplace domain, where it takes on a much simpler form.

Our aim in the present paper is to develop the Laplace transform method for a much wider class of generalized Fourier transforms. According to investigations in [7] using the representation theory for the Lie superalgebra $\mathfrak{osp}(1|2)$, the following expression

$$e^{i\frac{\pi}{2}G(\Gamma_y)} e^{-i(x,y)} \tag{1}$$

where G is an integer-valued polynomial can be used as the kernel for a generalized Fourier transform that still satisfies properties very close to that of the classical transform. The extension of the Laplace transform technique to kernels of type (1) will allow us to find explicit expressions for the kernel. We will moreover determine which polynomials G give rise to polynomially bounded kernels and we will determine the generating function corresponding to a fixed polynomial G.

The paper is organized as follows. In order to make the exposition self-contained, in Section 2, we recall basic facts of the Laplace transform, Clifford analysis and the generalized Clifford-Fourier transform. Section 3 is devoted to establishing the connection between the kernel of the fractional Clifford-Fourier transform [5] and the generalized Clifford-Fourier transform. We first compute a special case in Section 3.1. Then the method is generalized to the case in which the polynomial has integer coefficients in Section 3.2. The kernel and the generating function in the even dimensional case are given. We also discuss which kernels are polynomially bounded.

2 Preliminaries

2.1 The Laplace transform

The Laplace transform of a real or complex valued function f which has exponential order α, i.e. $|f(t)| \leq Ce^{\alpha t}, t \geq t_0$ is defined as

$$F(s) = \mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t) dt.$$

By Lerch's theorem [16], the inverse transform

$$\mathcal{L}^{-1}(F(s)) = f(t)$$

is uniquely defined when we restrict to functions which are continuous on $[0,\infty)$. Usually, we can use integral transform tables (see e.g. [11]) and the partial fraction expansion to compute the Laplace transform and its inverse. We list some which will be used in this paper:

$$\mathcal{L}(e^{-\alpha t}) = \frac{1}{s+\alpha}; \tag{2}$$

$$\mathcal{L}(t^{k-1}e^{-\alpha t}) = \frac{\Gamma(k)}{(s+\alpha)^k}, \quad k > 0. \tag{3}$$

We also need the convolution formula and the inverse Laplace transform. Denote by $r = (s^2 + a^2)^{1/2}$, $R = s + r$, $G(s) = \mathcal{L}(g(t))$ and $F(s) = \mathcal{L}(f(t))$. We have

$$G(s)F(s) = \mathcal{L}(\int_0^t g(t-\tau)f(\tau)d\tau); \tag{4}$$

$$\mathcal{L}^{-1}(a^\nu r^{-2\nu-1}) = 2^\nu \pi^{-1/2}\Gamma(\nu + \frac{1}{2})t^\nu J_\nu(at), \qquad \mathrm{Re}(\nu) > -1/2, \mathrm{Re}(s) > |\mathrm{Im}(a)|. \tag{5}$$

2.2 Clifford analysis and generalized Fourier transforms

In this section, we give a quick review of the basic concepts in Clifford analysis and generalized Fourier transforms. Denoting by $\{e_1, e_2, \dots, e_m\}$ the orthonormal basis of $\mathbb{R}^m$, the Clifford algebra $\mathcal{C}\ell_{0,m}$ over $\mathbb{R}^m$ is spanned by the reduced products

$$\bigcup_{j=1}^m \{e_\alpha = e_{i_1}e_{i_2}\dots e_{i_j} : \alpha = \{i_1, i_2, \dots, i_j\}, \quad 1 \le i_1 < i_2 < \dots < i_j \le m\}$$

with the relations $e_ie_j + e_je_i = -2\delta_{ij}$. We identify the point $x = (x_1, \dots, x_m)$ in $\mathbb{R}^m$ with the vector variable $x = \sum_{j=1}^m e_jx_j$. The inner product and the wedge product of two vectors $x, y \in \mathbb{R}^m$ can be defined by the Clifford product:

$$(x, y) := \sum_{j=1}^m x_jy_j = -\frac{1}{2}(xy + yx);$$

$$x \wedge y := \sum_{j<k} e_je_k(x_jy_k - x_ky_j) = \frac{1}{2}(xy - yx).$$

We can find the Clifford product $xy = -(x, y) + x \wedge y$, and $(x \wedge y)^2 = -|x|^2|y|^2 + (x, y)^2$ (see [8]). The complexified Clifford algebra $\mathcal{C}\ell_{0,m}^c$ is defined as $\mathbb{C} \otimes \mathcal{C}\ell_{0,m}$.

The conjugation is defined by $\overline{(e_{j_1}\dots e_{j_l})} = (-1)^l e_{j_l}\dots e_{j_1}$ as a linear mapping. For $x, y \in \mathcal{C}\ell_{0,m}^c$, we have $\overline{(xy)} = \overline{y}\overline{x}, \overline{\overline{x}} = x$, and $\overline{i} = i$ which is not the usual complex conjugation. We define the Clifford norm of x by $|x|^2 = x\bar{x}, x \in \mathcal{C}\ell_{0,m}^c$.

The Dirac operator is given by $D = \sum_{j=1}^m e_j\partial_{x_j}$. Together with the vector variable x, they satisfy the relations

$$D^2 = -\Delta, \qquad x^2 = -|x|^2, \qquad \{x, D\} = -2\mathbb{E} - m,$$

where $\{a, b\} = ab + ba$ and $\mathbb{E} = \sum_{j=1}^m x_j\partial_{x_j}$ is the Euler operator and hence they generate a realization of the Lie superalgebra $\mathfrak{osp}(1|2)$, which contains the Lie algebra $\mathfrak{sl}_2 = \mathrm{span}\{\Delta, |x|^2, [\Delta, |x|^2]\}$ as its even part. A function $u(x)$ is called monogenic if $Du = 0$. An important example of monogenic functions is the generalized Cauchy kernel

$$G(x) = \frac{1}{\omega_m}\frac{\bar{x}}{|x|^m}$$

where ω_m is the surface area of the unit ball in $\mathbb{R}^m$. It is the fundamental solution of the Dirac operator [9]. Note that the norm here is $|x| = (\sum_{i=1}^m x_i^2)^{1/2}$ and coincides with Clifford norm.

Denote by $\mathcal{P}$ the space of polynomials taking values in $\mathcal{C}\ell_{0,m}$, i.e. $\mathcal{P} := \mathbb{R}[x_1, \dots, x_m] \otimes \mathcal{C}\ell_{0,m}$. The space of homogeneous polynomials of degree k is then denoted by $\mathcal{P}_k$. The space $\mathcal{M}_k := (\ker D) \cap \mathcal{P}_k$, is called the space of homogeneous monogenic polynomials of degree k. An arbitrary element of it is called a spherical monogenic of degree k [9].

The local behaviour of a monogenic function near a point can be investigated by the polynomials introduced above. The following theorem is the analogue of the Taylor series in complex analysis.

Theorem 1. *[9] Suppose f is monogenic in an open set Ω containing the origin. Then there exists an open neighbourhood Λ of the origin in which f can be developed into a normally convergent series of spherical monogenics $M_kf(x)$, i.e.*

$$f(x) = \sum_{k=0}^\infty M_kf(x),$$

with $M_k f(x) \in \mathcal{M}_k$.

The classical Fourier transform

$$\mathcal{F}(f)(y) = (2\pi)^{-m/2} \int_{\mathbb{R}^m} e^{-i(x,y)} f(x) dx,$$

with (x, y) the usual inner product can be represented by the operator exponential [14], [15]

$$\mathcal{F} = e^{-i\frac{\pi}{4}(\Delta - |x|^2 - m)}.$$

The Clifford-Hermite functions

$$\psi_{2p,k,l}(x) := 2^p p! L_p^{\frac{m}{2}+k-1}(|x|^2) M_k^l e^{-|x|^2/2},$$

$$\psi_{2p+1,k,l}(x) := 2^p p! L_p^{\frac{m}{2}+k}(|x|^2) x M_k^l e^{-|x|^2/2},$$

where $p, k \in \mathbb{Z}_{\geq 0}$ and $\{M_k^l | l = 1, \ldots, \dim(\mathcal{M}_k)\}$ form a basis for $\mathcal{M}_k$, the space of spherical monogenics of degree k. They moreover realize the complete decomposition of the rapidly decreasing functions $\mathcal{S}(\mathbb{R}^m) \otimes \mathcal{C}\ell_m \subset L^2(\mathbb{R}^m) \otimes \mathcal{C}\ell_m$ in irreducible subspaces under the action of the dual pair $(Spin(m), \mathfrak{osp}(1|2))$. The action of the regular Fourier transform on this basis is given by

$$\mathcal{F}\psi_{j,k,l} = e^{-i\frac{\pi}{2}(j+k)} \psi_{j,k,l} = (-i)^{j+k} \psi_{j,k,l}. \tag{6}$$

We further introduce the Gamma operator or the angular Dirac operator (see [9])

$$\Gamma_x := -\sum_{j<k} e_j e_k (x_j \partial_{x_k} - x_k \partial_{x_j}) = -x D_x - \mathbb{E}_x = -x \wedge D_x, \tag{7}$$

here $\mathbb{E}_x = \sum_{i=1}^m x_i \partial_{x_i}$ is the Euler operator. Note that Γ_x commutes with scalar radial functions. The operator Γ_x has two important eigenspaces:

$$\Gamma_x \mathcal{M}_k = -k \mathcal{M}_k, \tag{8}$$

$$\Gamma_x (x \mathcal{M}_{k-1}) = (k + m - 2) x \mathcal{M}_{k-1} \tag{9}$$

which follows from the definition of Γ_x. The Scasimir S in our operator realization of $\mathfrak{osp}(1|2)$ is related to the angular Dirac operator by $S = -\Gamma_x + \frac{m-1}{2}$, see [12]. The Casimir element $C = S^2$ acts on the Clifford-Hermite function by

$$C\psi_{j,k,l} = (k + \frac{m-1}{2})^2 \psi_{j,k,l}.$$

In [7], the authors studied the full class of integral transforms which satisfy the conditions stated in the following theorem.

Theorem 2. *The properties*

(1) the Clifford-Helmholtz relations

$$T \circ D_x = -iy \circ T,$$

$$T \circ x = -i D_y \circ T,$$

(2) $T\psi_{j,k,l} = \mu_{j,k} \psi_{j,k,l}$ *with* $\mu_{j,k} \in \mathbb{C}$,

(3) $T^4 = id$

are satisfied by the operators T *of the form*

$$T = e^{i\frac{\pi}{2}F(C)} e^{i\frac{\pi}{4}(\Delta - |x|^2 - m)} \in e^{i\frac{\pi}{2}\bar{\mathcal{U}}(\mathfrak{osp}(1|2)))}$$

where $F(C)$ *is a power series in* C *that takes integer values when evaluated in the eigenvalues of* C *and* $\bar{\mathcal{U}}(\mathfrak{osp}(1|2)$ *is the extension of the universal enveloping algebra that allows infinite power series in the elements of* $\mathfrak{sl}_2$.

The integral kernel of the generalized Fourier transform T can be expressed as $e^{i\frac{\pi}{2}F(C)}e^{-i(x,y)}$. We are in particular interested in the case where $F(C)$ reduces to a polynomial $G(\Gamma_y)$ with integer coefficients.

Remark 1. *In general, when $G(x) \neq 0$, the generalized Fourier transform T and the Clifford fractional Fourier transform in [1] are two different classes of transforms because their eigenvalues on the Clifford-Hermite functions are different.*

Remark 2. *The Clifford Fourier transform in $C\ell_{(3,0)}$ can also been expressed by operator exponential, see e.g. [10].*

3 Generalized kernel in the Laplace domain

3.1 Closed expression for $e^{i\frac{\pi}{2}\Gamma_y^2}e^{-i(x,y)}$

In this subsection, we use the Laplace transform method to compute $e^{i\frac{\pi}{2}\Gamma_y^2}e^{-i(x,y)}$. The trick here will be used to compute the more general case in next subsection. We use the notation $\sqrt{+} := \sqrt{s^2+|x|^2|y|^2}$. The following lemma was obtained in [5].

Lemma 1. *The Laplace transform of $t^{m/2-1}e^{-it(x,y)}$ can be expressed as*

$$\mathcal{L}(t^{m/2-1}e^{-it(x,y)}) = \frac{2^{m/2-1}\Gamma(m/2)}{\sqrt{+}(s+\sqrt{+})^{m/2-1}} \frac{1-\dfrac{iyx}{s+\sqrt{+}}+\dfrac{iy(1-\frac{iyx}{s+\sqrt{+}})x}{s+\sqrt{+}}}{\left|1-\dfrac{iyx}{s+\sqrt{+}}\right|^m}. \tag{10}$$

In the following, we will act with $e^{i\frac{\pi}{2}\Gamma_y^2}$ on both sides of (10) to obtain the integral kernel in the Laplace domain. Denote by

$$f(y) = \frac{2^{\frac{m}{2}}}{\sqrt{+}(s+\sqrt{+})^{m/2-1}} \frac{1-\dfrac{iyx}{s+\sqrt{+}}}{\left|1-\dfrac{iyx}{s+\sqrt{+}}\right|^m} = \frac{s+\sqrt{+}-iyx}{\sqrt{+}(s+i(x,y))^{m/2}},$$

and

$$g(y) = \frac{2^{\frac{m}{2}}}{\sqrt{+}(s+\sqrt{+})^{m/2-1}} \frac{\dfrac{iy(1-\frac{iyx}{s+\sqrt{+}})x}{s+\sqrt{+}}}{\left|1-\dfrac{iyx}{s+\sqrt{+}}\right|^m} = \frac{iy}{s+\sqrt{+}}f(y)x = \frac{\sqrt{+}-s+iyx}{\sqrt{+}(s+i(x,y))^{m/2}}.$$

In [5], it has been proved that $f(y)$ has a series expansion as

$$f(y) = \frac{2^{\frac{m}{2}}}{\sqrt{+}(s+\sqrt{+})^{m/2-1}} \sum_{k=0}^{\infty} \frac{M_k(y)}{(s+\sqrt{+})^k}.$$

Here we rewrite

$$f(y) = f_0(y)+f_1(y)+f_2(y)+f_3(y),$$

with

$$f_k(y) = \frac{2^{\frac{m}{2}}}{\sqrt{+}(s+\sqrt{+})^{m/2-1}} \sum_{n=0}^{\infty} \frac{M_{4n+k}(y)}{(s+\sqrt{+})^{4n+k}}, \quad k=0,1,2,3. \tag{11}$$

Each f_k is an eigenfunction of the operator $e^{i\frac{\pi}{2}\Gamma^2}$. In fact, by (8), we have

$$e^{i\frac{\pi}{2}\Gamma_y^2}M_k(y) = e^{i\frac{\pi}{2}(-k)^2}M_k(y),$$

so

$$\begin{aligned} e^{i\frac{\pi}{2}\Gamma_y^2}M_{4n}(y) &= M_{4n}(y);\\ e^{i\frac{\pi}{2}\Gamma_y^2}M_{4n+1}(y) &= iM_{4n+1}(y);\\ e^{i\frac{\pi}{2}\Gamma_y^2}M_{4n+2}(y) &= M_{4n+2}(y);\\ e^{i\frac{\pi}{2}\Gamma_y^2}M_{4n+3}(y) &= iM_{4n+3}(y), \end{aligned} \tag{12}$$

here $n = 0, 1, 2, \cdots$. Since the operator Γ commutes with radial functions, we know that each f_k is an eigenfunction of $e^{i\frac{\pi}{2}\Gamma^2}$ and the eigenvalues are given in (12). In the following, we denote

$$f_\alpha(y) = \frac{2^{\frac{m}{2}}}{\sqrt{+}(s+\sqrt{+})^{m/2-1}}\sum_{k=0}^{\infty}\frac{M_k(iy)}{(s+\sqrt{+})^k} = \frac{s+\sqrt{+}+yx}{\sqrt{+}(\sqrt{+}-(x,y))^{m/2}},$$

$$f_\beta(y) = \frac{2^{\frac{m}{2}}}{\sqrt{+}(s+\sqrt{+})^{m/2-1}}\sum_{k=0}^{\infty}\frac{M_k(-y)}{(s+\sqrt{+})^k} = \frac{s+\sqrt{+}+iyx}{\sqrt{+}(s-i(x,y))^{m/2}},$$

$$f_\gamma(y) = \frac{2^{\frac{m}{2}}}{\sqrt{+}(s+\sqrt{+})^{m/2-1}}\sum_{k=0}^{\infty}\frac{M_k(-iy)}{(s+\sqrt{+})^k} = \frac{s+\sqrt{+}-yx}{\sqrt{+}(\sqrt{+}+(x,y))^{m/2}}$$

as well as

$$\begin{aligned} g_\alpha(y) &= \frac{iy}{s+\sqrt{+}}f_\alpha(y)x = \frac{i(\sqrt{+}-s)+iyx}{\sqrt{+}(\sqrt{+}-(x,y))^{m/2}},\\ g_\beta(y) &= \frac{iy}{s+\sqrt{+}}f_\beta(y)x = \frac{s-\sqrt{+}+iyx}{\sqrt{+}(s-i(x,y))^{m/2}},\\ g_\gamma(y) &= \frac{iy}{s+\sqrt{+}}f_\gamma(y)x = \frac{i(s-\sqrt{+})+iyx}{\sqrt{+}(\sqrt{+}+(x,y))^{m/2}}. \end{aligned}$$

Remark 3. *Comparing with Theorem 3 in [5],* $\frac{\Gamma(m/2)}{2}(f_\gamma + g_\alpha)$ *is the Clifford-Fourier kernel of dimension* $m = 4n+1, n \in \mathbb{N}$ *in the Laplace domain. Denote the first part of the fractional Clifford-Fourier kernel as*

$$F_p(x,y) = \frac{s+\sqrt{+}-ie^{-ip}yx}{\sqrt{+}(e^{-ip}(s\cos p + i\sqrt{+}\sin p + i(x,y)))^{m/2}}$$

and the second part of the kernel as

$$G_p(x,y) = -e^{ip}\frac{s-\sqrt{+}-ie^{-ip}yx}{\sqrt{+}(e^{ip}(s\cos p - i\sqrt{+}\sin p + i(x,y)))^{m/2}}.$$

We find that $f(y) = F_0(x,y)$, $f_\alpha(y) = F_{-\frac{\pi}{2}}(x,y)$, $f_\beta(y) = F_\pi(x,y)$, $f_\gamma(y) = F_{\frac{\pi}{2}}(x,y)$, $g(y) = G_0(x,y)$, $g_\alpha(y) = G_{\frac{\pi}{2}}(x,y)$, $g_\beta(y) = G_\pi(x,y)$ *and* $g_\gamma(y) = G_{-\frac{\pi}{2}}(x,y)$. *We could get the plane wave expansion and integral expression of* $f, f_\alpha, f_\beta, f_\gamma$ *and* $g, g_\alpha, g_\beta, g_\gamma$ *from [5].*

As M_k is a polynomial of degree k, we have the following relations,

$$\begin{cases} f(y) = f_0(y) + f_1(y) + f_2(y) + f_3(y);\\ f_\alpha(y) = f_0(y) + if_1(y) - f_2(y) - if_3(y);\\ f_\beta(y) = f_0(y) - f_1(y) + f_2(y) - f_3(y);\\ f_\gamma(y) = f_0(y) - if_1(y) - f_2(y) + if_3(y). \end{cases}$$

Each $f_k(y)$ can be obtained as follows:

$$\left\{\begin{array}{l} 4f_0(y)=f(y)+f_\alpha(y)+f_\beta(y)+f_\gamma(y); \\ 4f_1(y)=f(y)-if_\alpha(y)-f_\beta(y)+if_\gamma(y); \\ 4f_2(y)=f(y)-f_\alpha(y)+f_\beta(y)-f_\gamma(y); \\ 4f_3(y)=f(y)+if_\alpha(y)-f_\beta(y)-if_\gamma(y). \end{array}\right. \tag{13}$$

Now the action of $e^{i\frac{\pi}{2}\Gamma_y^2}$ on $f(y)$ is known through its eigenfunctions,

$$\begin{aligned} e^{i\frac{\pi}{2}\Gamma_y^2}f(y) &= e^{i\frac{\pi}{2}\Gamma_y^2}\Big(f_0(y)+f_1(y)+f_2(y)+f_3(y)\Big) \\ &= f_0(y)+if_1(y)+f_2(y)+if_3(y) \\ &= \frac{1}{2}\Big(f(y)+f_\beta(y)+if(y)-if_\beta(y)\Big). \end{aligned}$$

The case $e^{i\frac{\pi}{2}\Gamma_y^2}g(y)$ can be treated similarly, using (12) and

$$\begin{aligned} e^{i\frac{\pi}{2}\Gamma_y^2}(yM_k(y)) &= e^{i\frac{\pi}{2}(m-1+k)^2}(yM_k(y)) \\ &= e^{i\frac{\pi}{2}(m-1)^2}e^{i\frac{\pi}{2}k^2}(yM_k(e^{i\pi(m-1)}y)) \\ &= e^{i\frac{\pi}{2}(m-1)^2}ye^{i\frac{\pi}{2}k^2}(M_k(e^{i\pi(m-1)}y)). \end{aligned}$$

Collecting everything, we have

Theorem 3. *The kernel $t^{m/2-1}e^{i\frac{\pi}{2}\Gamma_y^2}e^{-i(x,y)}$ in the Laplace domain is*

$$\begin{aligned} &\mathcal{L}(t^{m/2-1}e^{i\frac{\pi}{2}\Gamma_y^2}e^{-it(x,y)}) \\ =\ &\frac{\Gamma(m/2)}{4\sqrt{+}}\Big((1+i)U_m^1+(1-i)U_m^2+e^{i\frac{\pi}{2}(m-1)^2}((1+i)U_m^3+(1-i)U_m^4)\Big), \end{aligned}$$

with

$U_m^1=\dfrac{s+\sqrt{+}-iyx}{(s+i(x,y))^{m/2}}$; $\qquad U_m^2=\dfrac{s+\sqrt{+}+iyx}{(s-i(x,y))^{m/2}}$;

$U_m^3=\dfrac{(-1)^{m-1}(\sqrt{+}-s)+iyx}{(s+(-1)^{m-1}i(x,y))^{m/2}}$; $\qquad U_m^4=\dfrac{(-1)^{m-1}(s-\sqrt{+})+iyx}{(s-(-1)^{m-1}i(x,y))^{m/2}}$,

where $\sqrt{+}=\sqrt{s^2+|x|^2|y|^2}$.

When $m=2$,

$$\mathcal{L}(e^{i\frac{\pi}{2}\Gamma_y^2}e^{-it(x,y)})=\frac{1}{2\sqrt{+}}\left(\frac{\sqrt{+}}{s-i(x,y)}+\frac{s-iyx}{s+i(x,y)}\right).$$

By formula (2), (5), and the convolution formula (4), the kernel equals, putting $t=1$,

$$K_{2,\Gamma^2}(x,y)=e^{i(x,y)}+J_0(|x||y|)+ix\wedge y\int_0^1 e^{-i(x,y)(1-\tau)}J_0(|x||y|\tau)d\tau.$$

In the following, we analyze each term in Theorem 3 in detail. By formula (3), (4) and (5), letting

$t=1$, we get $U_m^1, U_m^2, U_m^3, U_m^4$ in the time domain as

$$\begin{aligned}
K_{U_m^1} &= \frac{e^{-i(x,y)}}{\Gamma(m/2)} + \frac{1}{\Gamma(m/2-1)}\int_0^1 \tau^{m/2-2} e^{-i(x,y)\tau} J_0(|x||y|(1-\tau))d\tau \\
&\quad + \frac{ix\wedge y}{\Gamma(m/2)}\int_0^1 e^{-i(x,y)} J_0(|x||y|(1-\tau))d\tau, \\
K_{U_m^2} &= \frac{e^{i(x,y)}}{\Gamma(m/2)} + \frac{1}{\Gamma(m/2-1)}\int_0^1 \tau^{m/2-2} e^{i(x,y)\tau} J_0(|x||y|(1-\tau))d\tau \\
&\quad - \frac{ix\wedge y}{\Gamma(m/2)}\int_0^1 e^{i(x,y)} J_0(|x||y|(1-\tau))d\tau, \\
K_{U_m^3} &= (-1)^{m-1}\Big(\frac{1}{\Gamma(m/2)} e^{i(-1)^m(x,y)} \\
&\quad - \frac{1}{\Gamma(m/2-1)}\int_0^1 \tau^{m/2-2} e^{i(-1)^m(x,y)\tau} J_0(|x||y|(1-\tau))d\tau\Big) \\
&\quad - \frac{ix\wedge y}{\Gamma(m/2)}\int_0^1 e^{i(-1)^m(x,y)} J_0(|x||y|(1-\tau))d\tau,
\end{aligned}$$

$$\begin{aligned}
K_{U_m^4} &= (-1)^{m-1}\Big(-\frac{1}{\Gamma(m/2)} e^{i(-1)^{m-1}(x,y)} \\
&\quad + \frac{1}{\Gamma(m/2-1)}\int_0^1 \tau^{m/2-2} e^{i(-1)^{m-1}(x,y)\tau} J_0(|x||y|(1-\tau))d\tau\Big) \\
&\quad - \frac{ix\wedge y}{\Gamma(m/2)}\int_0^1 e^{i(-1)^{m-1}(x,y)} J_0(|x||y|(1-\tau))d\tau.
\end{aligned}$$

Theorem 4. *Let $m \geq 2$. For $x, y \in \mathbb{R}^m$, the generalized Fourier kernel is given by*

$$\begin{aligned}
K_{m,\Gamma^2}(x,y) &= \frac{\Gamma(m/2)}{4}\Bigg((1+i)K_{U_m^1} + (1-i)K_{U_m^2} \\
&\quad + e^{i\frac{\pi}{2}(m-1)^2}((1+i)K_{U_m^3} + (1-i)K_{U_m^4})\Bigg).
\end{aligned}$$

There exists a constant c such that

$$|K_{m,\Gamma^2}(x,y)| \leq c(1+|x||y|).$$

Proof. This follows from the fact that $J_0(y)$ and $e^{i(x,y)}$ are bounded functions and $|x\wedge y| \leq |x||y|$. □

3.2 Closed expression for $e^{i\frac{\pi}{2}G(\Gamma_y)}e^{-i(x,y)}$

In this subsection, we consider the more general case. We act with $G(\Gamma_y)$ on the Fourier kernel. Here $G(x)$ is a polynomial with integer coefficients,

$$G(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0, \qquad a_k \in \mathbb{Z}.$$

Using the fact that $e^{i\frac{\pi}{2}j}$ is 4-periodic in j,

$$e^{i\frac{\pi}{2}G(\Gamma_y)}M_k(y) = e^{i\frac{\pi}{2}G(-k)}M_k(y)$$

and

$$G(4n+k) \equiv G(k) (\mathrm{mod} 4),$$

we have

$$e^{i\frac{\pi}{2}G(\Gamma_y)}f(y) = e^{i\frac{\pi}{2}G(0)}f_0 + e^{i\frac{\pi}{2}G(-1)}f_1 + e^{i\frac{\pi}{2}G(-2)}f_2 + e^{i\frac{\pi}{2}G(-3)}f_3$$
$$= i^{G(0)}f_0 + i^{G(-1)}f_1 + i^{G(-2)}f_2 + i^{G(-3)}f_3,$$

with each f_k defined in (11). By

$$e^{i\frac{\pi}{2}G(\Gamma_y)}(yM_k(y)) = e^{i\frac{\pi}{2}G(m-1+k)}(yM_k)$$

and

$$G(4n+k+m-1) \equiv G(k+m-1)(\text{mod}4),$$

we have

$$\begin{aligned} & e^{i\frac{\pi}{2}G(\Gamma_y)}g(y) \\ = \; & \frac{iy}{s+\sqrt{+}}\left(e^{i\frac{\pi}{2}G(m-1)}f_0 + e^{i\frac{\pi}{2}G(m)}f_1 + e^{i\frac{\pi}{2}G(m+1)}f_2 + e^{i\frac{\pi}{2}G(m+2)}f_3\right)x \\ = \; & \frac{iy}{s+\sqrt{+}}\left(i^{G(m-1)}f_0 + i^{G(m)}f_1 + i^{G(m+1)}f_2 + i^{G(m+2)}f_3\right)x. \end{aligned}$$

Collecting everything and applying (13), we get

Theorem 5. *For $G(x) \in \mathbb{Z}[x]$, the Laplace transform of $t^{m/2-1}e^{i\frac{\pi}{2}G(\Gamma_y)}e^{-it(x,y)}$ is given by*

$$\mathcal{L}(t^{m/2-1}e^{i\frac{\pi}{2}G(\Gamma_y)}e^{-it(x,y)}) = \frac{\Gamma(m/2)}{8}\left(A_m^1 B C_m^T + \frac{iy}{s+\sqrt{+}}A_m^2 B C_m^T x\right)$$

with A_m^1, A_m^2, B, C_m the matrices given by

$$\begin{aligned} A_m^1 &= \begin{pmatrix} i^{G(0)} & i^{G(-1)} & i^{G(-2)} & i^{G(-3)} \end{pmatrix}, \\ A_m^2 &= \begin{pmatrix} i^{G(m-1)} & i^{G(m)} & i^{G(m+1)} & i^{G(m+2)} \end{pmatrix}, \\ B &= \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix}, \\ C_m &= \begin{pmatrix} f(y) & f_\alpha(y) & f_\beta(y) & f_\gamma(y) \end{pmatrix}. \end{aligned}$$

Remark 4. *We could get the regular Fourier kernel $e^{-i(x,y)}$ by setting $G(x) = 0$ or $4x$ for dimension $m \geq 2$. When $G = 2x^2$, we get the inverse Fourier kernel $e^{i(x,y)}$ for even dimension. When $G(x) = \pm x$, it is the Clifford-Fourier transform [8].*

As the constant term of the polynomial will only contribute a constant factor to the integral kernel, in the following we only consider polynomials without constant term

$$G(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x, \qquad a_k \in \mathbb{Z}.$$

By

$$G(4n+k) \equiv G(k)(\text{mod}4),$$

it reduces to four cases $G(k)(\text{mod}4)$, $k = 0,1,2,3$. The set $\{x^m\} \cup \{1\}, m \in \mathbb{N}$ is a basis for polynomials over the ring of integers. We consider the four cases on this basis

$$\begin{aligned} x^j &= 0, \qquad \text{when} \quad x = 0; \\ x^j &= 1, \qquad \text{when} \quad x = 1; \\ x^j &\equiv \begin{cases} 2(\text{mod}4), & \text{when } j = 1 \text{ and } x = 2; \\ 0(\text{mod}4), & \text{when } j \geq 2 \text{ and } x = 2; \end{cases} \\ x^j &\equiv \begin{cases} 1(\text{mod}4), & \text{when } j \text{ is even and } x = 3; \\ 3(\text{mod}4), & \text{when } j \text{ is odd and } x = 3. \end{cases} \end{aligned}$$

For each $G(x)$, we denote $\frac{G(1)+G(-1)}{2} = s_0 = \sum_{j=0}^{\lfloor n/2 \rfloor} a_{2j}$ and $\frac{G(1)-G(-1)}{2} = s_1 = \sum_{j=0}^{\lfloor n/2 \rfloor} a_{2j+1}$ with n the degree of $G(x)$. We have

$$\begin{aligned}
&G(0) = 0,\\
&G(1) = s_0 + s_1,\\
&G(2) \equiv 2a_1 (\mathrm{mod} 4),\\
&G(3) \equiv G(-1) \equiv s_0 - s_1 (\mathrm{mod} 4).
\end{aligned}$$

Therefore

$$\begin{aligned}
&i^{G(0)} = 1, \qquad i^{G(-1)} = i^{G(3)} = i^{s_0+3s_1},\\
&i^{G(-2)} = i^{G(2)} = (-1)^{a_1}, \qquad i^{G(-3)} = i^{G(1)} = i^{s_0+s_1}.
\end{aligned}$$

The class of integral transforms with polynomially bounded kernel is of great interest. For example, new uncertainty principles have been given for this kind of integral transforms in [13]. As we can see in Theorem 5, the generalized Fourier kernel is a linear combination of $f_\alpha, f_\beta, f_\gamma, f, g_\alpha, g_\beta, g_\gamma, g$. At present, very few of $f_\alpha, f_\gamma, g_\alpha, g_\gamma$ are known explicitly. The integral representations of $f_\alpha, f_\gamma, g_\alpha, g_\gamma$ are obtained in [5] but without the bound. Only in even dimensions, special linear combinations of $f_\alpha, f_\gamma, g_\alpha, g_\gamma$ are known to be polynomially bounded which is exactly the Clifford-Fourier kernel [8].

We have showed in Theorem 4 that f, f_β, g, g_β with polynomial bounds behaves better than $f_\alpha, f_\gamma, g_\alpha, g_\gamma$. So it is interesting to consider the generalized Fourier transform whose kernel only consists of f, f_β, g, g_β. It also provides ways to define hypercomplex Fourier transforms with polynomially bounded kernel in odd dimensions. We will hence characterize polynomials such that $e^{i\frac{\pi}{2}G(\Gamma_y)}e^{-i(x,y)}$ are only linear combination of f, f_β, g, g_β.

For fixed m, the kernel is a linear sum of f, f_β, g, g_β when the polynomial $G(x) \in \mathbb{Z}[x]$ satisfies the following conditions

$$\begin{cases}
i^{G(0)} - i i^{G(-1)} - i^{G(-2)} + i i^{G(-3)} = 0,\\
i^{G(0)} + i i^{G(-1)} - i^{G(-2)} - i i^{G(-3)} = 0,\\
i^{G(m-1)} - i i^{G(m)} - i^{G(m+1)} + i i^{G(m+2)} = 0,\\
i^{G(m-1)} + i i^{G(m)} - i i^{G(m+1)} - i i^{G(m+2)} = 0.
\end{cases} \tag{14}$$

We find that (14) is equivalent with

$$\begin{cases}
G(0) \equiv G(-2) (\mathrm{mod} 4),\\
G(-1) \equiv G(-3) (\mathrm{mod} 4),\\
G(m-1) \equiv G(m+1) (\mathrm{mod} 4),\\
G(m) \equiv G(m+2) (\mathrm{mod} 4).
\end{cases} \tag{15}$$

As $G(k)(\mathrm{mod} 4)$ is uniquely determined by $G(0), G(-1), G(-2)$ and $G(-3)$, the first two formulas in (15) imply the last two formulas for all $m \geq 2$ automatically. Now (15) becomes

$$\begin{cases}
i^{G(0)} = 1 = i^{G(-2)} = (-1)^{a_1},\\
i^{G(-1)} = i^{s_0+3s_1} = i^{G(-3)} = i^{s_0+s_1}.
\end{cases}$$

It follows that the kernel only consists of f, f_β, g, g_β if and only if a_1 and s_1 are even. We have the following

Theorem 6. *Let $m \geq 2$. For $x, y \in \mathbb{R}^m$ and a polynomial $G(x)$ with integer coefficients, the kernel $e^{i\frac{\pi}{2}G(\Gamma_y)}e^{-i(x,y)}$ is a linear combination of f, f_β, g, g_β in the Laplace domain if and only if a_1 and $\frac{G(1)-G(-1)}{2}$ are even. Furthermore, the generalized Fourier kernel is bounded and equals*

$$\frac{1+i^{G(1)}}{2}e^{-i(x,y)} + \frac{1-i^{G(1)}}{2}K^\pi(x,y),$$

with $K^\pi(x,y)$ the fractional Clifford-Fourier kernel in [5]. When $m \geq 2$ is even, the kernel is

$$\frac{1+i^{G(1)}}{2}e^{-i(x,y)} + \frac{1-i^{G(1)}}{2}e^{i(x,y)}.$$

When $m \geq 2$ is odd, there exists a constant c which is independent of m such that

$$|e^{i\frac{\pi}{2}G(\Gamma_y)}e^{-i(x,y)}| \leq c(1+|x||y|). \tag{16}$$

Proof. We only need to prove the generalized Fourier kernel is

$$\frac{1+i^{s_0+s_1}}{2}e^{-i(x,y)} + \frac{1-i^{s_0+s_1}}{2}K^{\pi}(x,y).$$

In fact, by verification, we have,

$$(e^{i0})^{m-1}A_m^1\begin{pmatrix}1\\1\\1\\1\end{pmatrix} = A_m^2\begin{pmatrix}1\\1\\1\\1\end{pmatrix}; \qquad (e^{i\pi})^{m-1}A_m^1\begin{pmatrix}1\\-1\\1\\-1\end{pmatrix} = A_m^2\begin{pmatrix}1\\-1\\1\\-1\end{pmatrix},$$

and

$$A_m^1\begin{pmatrix}1\\1\\1\\1\end{pmatrix} = 2+2i^{s_0+s_1}; \qquad A_m^1\begin{pmatrix}1\\-1\\1\\-1\end{pmatrix} = 2-2i^{s_0+s_1}.$$

By Remark 3, $f+(e^{i0})^{m-1}g$ is the kernel K_0 and $f_\beta+(e^{i\pi})^{m-1}g_\beta$ is the fractional Clifford-Fourier kernel K^π. The bound (16) follows from the integral expression of f, f_β, g, g_β in the time domain. □

Remark 5. *The case $G(x) = x^2$ is a special case of this theorem.*

In the following, we consider the generalized Fourier kernel which has polynomial bound and consists of $f_\alpha, f_\beta, f_\gamma, f, g_\alpha, g_\beta, g_\gamma, g$. For even dimension, we already know the Clifford-Fourier kernel has a polynomial bound. If the polynomial $G(x)$ satisfies

$$(-i)^{m-1}A_m^1\begin{pmatrix}1\\-i\\-1\\i\end{pmatrix} = A_m^2\begin{pmatrix}1\\i\\-1\\-i\end{pmatrix}; \qquad i^{m-1}A_m^1\begin{pmatrix}1\\i\\-1\\-i\end{pmatrix} = A_m^2\begin{pmatrix}1\\-i\\-1\\i\end{pmatrix}, \tag{17}$$

by Remark 3, $e^{i\frac{\pi}{2}G(\Gamma_y)}e^{-i(x,y)}$ is a linear combination of the Clifford-Fourier kernel and some function bounded by $c(1+|x||y|)$. Hence it has a polynomial bound as well. When $m=4j$, (17) becomes

$$i(1-i^{s_0+3s_1+1}-(-1)^{a_1}+i^{s_0+s_1+1}) = i^{s_0+3s_1}+i-i^{s_0+s_1}-i(-1)^{a_1}$$

and

$$-i(1+i^{s_0+3s_1+1}-(-1)^{a_1}-i^{s_0+s_1+1}) = i^{s_0+3s_1}-i-i^{s_0+s_1}+i(-1)^{a_1}.$$

It shows that (17) is true for any $G(x) \in \mathbb{Z}[x]$ when $m=4j$. When $m=4j+2$, (17) becomes

$$-i(1-i^{s_0+3s_1+1}-(-1)^{a_1}+i^{s_0+s_1+1}) = i^{s_0+s_1}+i(-1)^{a_1}-i^{s_0+3s_1}-i$$

and

$$i(1+i^{s_0+3s_1+1}-(-1)^{a_1}-i^{s_0+s_1+1}) = i^{s_0+s_1}-i(-1)^{a_1}-i^{s_0+3s_1}+i.$$

It also shows that (17) is true for any $G(x) \in \mathbb{Z}[x]$ when $m=4j+2$. Now we have

Theorem 7. *Let $m \geq 2$ be even. For $x, y \in \mathbb{R}^m$ and any polynomial $G(x)$ with integer coefficients, the kernel $e^{i\frac{\pi}{2}G(\Gamma_y)}e^{-i(x,y)}$ has a polynomial bound, i.e. there exists a constant c which is independent of $G(x)$ such that*

$$|e^{i\frac{\pi}{2}G(\Gamma_y)}e^{-i(x,y)}| \leq c(1+|x||y|)^{\frac{m-2}{2}}.$$

At the end of this section, we give the formal generating function of the even dimensional generalized Fourier kernels for a class of polynomials. We define

$$H(x,y,a,G)=\sum_{m=2,4,6,\cdots}\frac{K_{m,G}(x,y)a^{m/2-1}}{\Gamma(m/2)}.$$

Theorem 8. *Let $m\geq 2$ be even. For $x,y\in\mathbb{R}^m$ and any polynomial $G(x)$ with integer coefficients, the formal generating function of the even dimensional generalized Fourier kernel is given by*

$$\begin{aligned}
&H(x,y,a,G)\\
=\;&\frac{1-i^{G(-1)+1}-(-1)^{G'(0)}+i^{G(1)+1}}{2}\left(\cos(\sqrt{|x|^2|y|^2-((x,y)+a)^2})-(x\wedge y-a)\frac{\sin\sqrt{|x|^2|y|^2-((x,y)+a)^2}}{\sqrt{|x|^2|y|^2-((x,y)+a)^2}}\right)\\
&+\frac{1+i^{G(-1)+1}-(-1)^{G'(0)}-i^{G(1)+1}}{2}\left(\cos(\sqrt{|x|^2|y|^2-((x,y)-a)^2})+(x\wedge y+a)\frac{\sin\sqrt{|x|^2|y|^2-((x,y)-a)^2}}{\sqrt{|x|^2|y|^2-((x,y)-a)^2}}\right)\\
&+\frac{1+i^{G(-1)}+(-1)^{G'(0)}+i^{G(1)}}{2}e^{-(i(x,y)-a)}+\frac{1-i^{G(-1)}+(-1)^{G'(0)}-i^{G(1)}}{2}e^{i(x,y)+a}.
\end{aligned}$$

Proof. When m is even, the generalized Fourier kernel is

$$\begin{aligned}
e^{i\frac{\pi}{2}G(\Gamma_y)}e^{-i(x,y)} \;=\;& \frac{1}{2}\Big((1-i^{s_0+3s_1+1}-(-1)^{a_1}+i^{s_0+s_1+1})(f_\alpha+e^{i\frac{-\pi}{2}(m-1)}g_\gamma)\\
&+(1+i^{s_0+3s_1+1}-(-1)^{a_1}-i^{s_0+s_1+1})(f_\gamma+e^{i\frac{\pi}{2}(m-1)}g_\alpha)\\
&+(1+i^{s_0+3s_1}+(-1)^{a_1}+i^{s_0+s_1})e^{-i(x,y)}+(1-i^{s_0+3s_1}+(-1)^{a_1}-i^{s_0+s_1})e^{i(x,y)}\Big),
\end{aligned}$$

with $s_0=\sum_{j=0}^{\lfloor n/2\rfloor}a_{2j}$ and $s_1=\sum_{j=0}^{\lfloor n/2\rfloor}a_{2j+1}$.

By $s_0+3s_1\equiv s_0-s_1\equiv G(-1)(\text{mod}4)$, $s_0+s_1=G(1)$, $a_1=G'(0)$ and because $\frac{\Gamma(m/2)}{2}(f_\alpha+e^{i\frac{-\pi}{2}(m-1)}g_\gamma)$ and $\frac{\Gamma(m/2)}{2}(f_\gamma+e^{i\frac{\pi}{2}(m-1)}g_\alpha)$ are the Clifford-Fourier kernel $K^{\frac{-\pi}{2}}$ and $K^{\frac{\pi}{2}}$ in the Laplace domain, the result follows from the generating function of Clifford-Fourier kernel, see [5] Theorem 8. □

Remark 6. *When $G(x)=x$, we get the generating function of the Clifford-Fourier kernel.*

For the case that the coefficients of $G(x)$ are not integers but fractions, we write $G_1(x)=cG(x)$ in which c is the least common multiple of each denominator of $G(x)$. So $G_1(x)$ is a polynomial with integer coefficients. We only need to compute $e^{i\frac{\pi}{2c}G_1(\Gamma_y)}f(y)$ and $e^{i\frac{\pi}{2c}G_1(\Gamma_y)}g(y)$. The same method will also work but f and g split into $4c$ parts.

4 Conclusion

By working on the Laplace domain, we found explicit expressions for the generalized kernel. For even dimension, we obtained the closed expression. As the bound of the kernel is important to see in which function space the transform is well defined, we moreover determined which polynomials G give rise to polynomially bounded kernels for all dimensions and only even dimensions. When the kernel has polynomial bound, the transform can be proved to be well defined on the Schwartz function space and the transform is a continuous operator. Following a similar discussion in Section 6 of [7], we can get the existence of the inversion formula for this kind transform. Also we determined the generating function corresponding to a fixed polynomial G. We point out that the closed kernel of odd dimension is still an open problem since 2005 and deserves more study. We think it is interesting to develop the Laplace method on the hyperbolic space or sphere to compute the closed kernel of the hypercomplex Fourier transform.

First application of a quaternion Fourier transform to color images was reported in 1996 by Sangwine et al. The advantage of Fourier-type transforms of quaternionic signals over the classical Fourier transform is that the kernel through which they act are quaternion-valued and the transforms therefore "mix the channels" rather than acting on each channel separately. The generalized transform we studied here will also "mix the channels" and yet still has a simple inversion and Plancherel theorem.

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