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Exploring $\mathbb{F}_1$-Geometry:

Deitmar schemes, loose graphs and motives.

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A mi hermana,

"Si hubiera elegido no meter la cabeza en este gran cuento, nunca hubiese sabido lo que me iba a perder."

– Jostein Gaarder, *La joven de las naranjas.*

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Preface

It was back in July 2012 in Sevilla, while I was benefiting from a research Summer fellowship, that Professor Luis Narváez sent me an email about a PhD position to work on Absolute Geometry and "the field with one element." I had never heard about such a thing before but, as it happens with many other topics in research, you only need few hours looking for relating information to see if it draws your attention. Almost 5 years later, this PhD is the result of that informative email.

I started my PhD in September 2013 under the supervision of Professor Koen Thas, who introduced me to the interesting theory of $\mathbb{F}_1$, the field with one element. The notion of $\mathbb{F}_1$ was introduced in 1957 by Jacques Tits, but it was not until 2006 that Anton Deitmar presented the very first definition of a scheme over $\mathbb{F}_1$. Other approaches to define $\mathbb{F}_1$-schemes soon appeared but in one way or another they are all related to Deitmar's first definition. Some years later, Koen Thas established a relation between the $\mathbb{F}_1$-schemes defined by Deitmar (also called Deitmar schemes) and a combinatorial generalization of graphs, which he called "loose graphs." In that paper, he defined a functor $\mathcal{S}$ from the category `LGraph` of loose graphs to the category of Deitmar schemes as follows:

Consider a loose graph Γ and embed it in a combinatorial projective space over $\mathbb{F}_1$, denoted by $\mathbf{P}_c(\Gamma)$. Let $\mathbf{P}(\Gamma)$ be the associated $\mathbb{F}_1$-projective space on the level of schemes. $\mathcal{S}(\Gamma)$ is then the subscheme of $\mathbf{P}(\Gamma)$ roughly defined by leaving out all multiplicative groups on $\mathbf{P}(\Gamma)$ that are defined by edges of $\mathbf{P}_c(\Gamma) \setminus \Gamma$.

This functor gave rise to a new way of understanding geometrical structure through combinatorial tools. In fact, Koen proved in the same paper that the automorphism group of a loose graph Γ is isomorphic to the automorphism group of the scheme $\mathcal{S}(\Gamma)$ associated to it.

Soon after I started my PhD, Koen Thas pointed out the idea of defining a new functor that extends the aforementioned relation into a bigger category of Deitmar schemes while keeping some analogies with the functor $\mathcal{S}$. For instance, a complete graph on $n+1$ vertices should always correspond to an n-dimensional projective space in order to satisfy the initial combinatorial motivation of $\mathbb{F}_1$ introduced by Tits. However, there was an essential feature missing in the definition of $\mathcal{S}$, which was related to the definition of combinatorial $\mathbb{F}_1$-affine spaces. If we consider, for example, the complete graph K_4 on 4 vertices, it will define a 3-dimensional projective space at the level of schemes but if we take now the loose graph Γ resulting by leaving out from K_4 the edges of a complete subgraph K_3, the functor $\mathcal{S}$ will associate a 3-dimensional projective space without 3 multiplicative groups to it. However, the natural association should be a 3-dimensional affine space (with some points at infinity) since it corresponds to the decomposition of a projective space into the disjoint union of an affine space of the same dimension and a projective space of one dimension lower.

After defining the new functor, which we call $\mathcal{F}$, and thanks to Deitmar's base extension of $\mathbb{F}_1$-schemes to $\mathbb{Z}$-schemes, we were able to define a functor $\mathcal{F}_k$ from the category $\texttt{LGraph}$ to the category $\texttt{CS}_k$ of k-constructible sets , for k a finite field, $\mathbb{F}_1$ or $\mathbb{Z}$. The definition of all the functors and the proof of them being indeed functors are described in detail in chapter 2.

Once the definition of a good functor was achieved, natural questions arose regarding what information about the constructible sets could be obtained by studying the loose graphs that defined them. We started to analyze some basic examples and we soon realized that in some cases (mainly in the case of loose trees), when fixing a finite field k, it was possible to define a polynomial function on the loose trees that counts the number of rational points of the constructible sets which arise by applying the functor $\mathcal{F}_k$. We further developed this idea and found a process, that we called "surgery," to inductively calculate the number of rational points on a constructible set $\mathcal{F}_k(\Gamma)$ for Γ coming from a loose graph Γ.

Several results came after defining the surgery process. Indeed, one of the main properties of this procedure is that for a given loose graph Γ, the function counting the number of rational points of the constructible set $\mathcal{F}_k(\Gamma)$ is polynomial, called the Grothendieck polynomial of Γ, and this polynomial is independent of the chosen finite field k. This property immediately implies that the constructible sets $\mathcal{F}_k(\Gamma)$ are polynomial-count; a result which pointed out a possible connection to motive theory.

According to a corollary of one of the Tate conjectures, a scheme being polynomial-count for all but finitely many primes is equivalent to the scheme having a mixed Tate motive. We were able to prove that for any finite field k and any loose graph Γ, the constructible set $\mathcal{F}_k(\Gamma)$ has a virtual mixed Tate motive. The procedure of obtaining the Grothendieck polynomial of a constructible set $\mathcal{F}_k(\Gamma)$ as well as the aforementioned results are fully explained in chapter 3.

In chapter 4, we describe a connection with Kurokawa's work about $\mathbb{Z}$-schemes being "defined over $\mathbb{F}_1$." In fact we prove that the constructible sets $\mathcal{F}_{\mathbb{Z}}(\Gamma)$ are defined over $\mathbb{F}_1$ in Kurokawa's sense for any loose graph Γ. This result allows us to define a new zeta function on the category of loose graphs that carries some information about the associated constructible sets.

Leaving apart the algebraic geometrical side of the constructible sets $\mathcal{F}_k(\Gamma)$ and following the spirit of Koen's work for the functor $\mathcal{S}$, we also studied the relation between loose graphs and the constructible sets associated to them by $\mathcal{F}_k$, where k is a finite field.

In chapter 5, given a loose tree Γ, we consider three different types of automorphism groups (topological, projective and combinatorial) for $\mathcal{F}_k(\Gamma)$ and describe how they are related to each other as well as related to the automorphism group of the loose tree Γ. We finish the chapter by giving some ideas that might be useful in order to prove similar results for the whole category of loose graphs.

At the end of the thesis we include two appendices concerning the surgery process. In appendix A we present a full study of how the surgery process works for the particular case of $\mathcal{F}_k(K_5)$, where K_5 is the complete graph on 5 vertices while appendix B contains the explanation of a computational code done in Magma to compute the Grothendieck polynomial described in chapter 3 for any loose graph.

Gent, May 2017.

Publications

There are four publications related to the work presented in this thesis [37, 38, 39, 40]. At the time of writing this thesis, one of these [40] has been published in *Journal of Geometry and Physics* and the other three are submitted.

- Chapter 2 is based on [40, 37].
- Chapter 3 is based on [40, 38].
- Chapter 4 is based on [40].
- Chapter 5 is based on [37].

These references will not be repeated in individual chapters.

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1 Preliminaries

The main goal of this chapter is to present some motivation for "the theory of the field with one element," $\mathbb{F}_1$, and introduce basic notions that will help the reader to understand the work presented in this PhD. We assume some familiarity with Linear Algebra, Commutative Algebra and Projective Geometry.

Before entering in the world of $\mathbb{F}_1$, one should remark that, according to the definition of a field, *there is no field with one element.* Nevertheless, Projective Geometry, Linear Algebra or Algebraic Geometry can be interpreted to have an $\mathbb{F}_1$-version. We shall see that in the course of this chapter.

1.1 Looking for $\mathbb{F}_1$

The birth of the field with one element was given by Jacques Tits [48] in 1957, where he suggested the existence of a "field of characteristic one," $\mathbb{F}_1$, over which one could interpret symmetric groups $\texttt{Sym}(n)$ as Chevalley groups, seen as limits of projective general linear groups over finite fields. After this analogy was made, the question on the existence of a theory of geometries which behave as if they were defined over $\mathbb{F}_1$ arose.

Already in [48], Tits described the buildings over $\mathbb{F}_1$ as the apartments of a spherical building of the same type and rank, and the corresponding Chevalley groups then become the Weyl groups of the associated BN-pair.

One of the main examples of the aforementioned question is the notion of a projective space over the field with one element. Let us recall the axiomatic definition of a combinatorial projective space starting with its point-line geometry, that can be found in [2]. It is a point-line geometry $\mathbf{P} = (\mathcal{P}, \mathcal{L}, \mathbf{I})$, in which $\mathcal{P}$ is the point set, $\mathcal{L}$ is the line set, and $\mathbf{I}$ is a symmetric relation on $\mathcal{P} \cup \mathcal{L}$, disjoint from $\mathcal{P} \times \mathcal{P}$ and $\mathcal{L} \times \mathcal{L}$, satisfying the following 3 axioms:

- Two different points are exactly incident with one line.

- *Thickness.* Each line has at least three points.
- *Veblen's axiom.* If a, b, c and d are different points and the lines ab and cd meet, then so do the lines ac and bd.

A *subspace* of a projective space $\mathbf{P}$ is a subgeometry X such that any line containing two different points of X is a subset of X (where we see lines as point sets) and this line then is a line of X. All projective spaces defined over fields satisfy also these axioms.

What is more, a projective space over a finite field $\mathbb{F}_q$, $\mathbf{PG}(n, q)$, has exactly $q + 1$ points on each line and a total number of $q^n + q^{n-1} + \cdots + q + 1$ points. If we imagine the limit of $\mathbf{PG}(n, q)$ when $q \to 1$, then one obtains the following axioms for a projective space over $\mathbb{F}_1$:

- Two different points are in exactly one line.
- Each line has exactly 2 points.
- *Veblen's axiom — $\mathbb{F}_1$-version.* Empty.

The total number of points for an n-dimensional projective space over $\mathbb{F}_1$ then becomes $n + 1$, by considering the limit when q tends to 1 on the number of points of $\mathbf{PG}(n, q)$. We denote this projective space by $\mathbf{PG}(n, 1)$. With this set of axioms, it is not difficult to see which combinatorial object one can naturally associate to $\mathbf{PG}(n, 1)$; a set of $n + 1$ points of which each pair is joined by one line is indeed the description of the complete graph on $n + 1$ vertices, K_{n+1}. Besides, it is clear that in that case the automorphism group of the projective space is isomorphic to the symmetric group $\mathtt{Sym}(n + 1)$.

This interpretation of a projective space over $\mathbb{F}_1$ and its automorphism group gave the motivation to develop a theory of Linear Algebra over $\mathbb{F}_1$ where addition was not allowed. An overview of what is called *Absolute Linear Algebra* will be introduced in the next section.

1.2 Absolute Linear Algebra

Even if the idea of a field with one element was introduced by Tits in 1957, it took more than 35 years until the notion of $\mathbb{F}_1$ was mentioned again in the mathematical literature. In some unpublished notes [23], Kapranov and Smirnov set the basis for a theory of Linear Algebra over the field with one element (following the aforementioned idea that excludes addition).

We will describe several aspects of Absolute Linear Algebra, considering mainly finite dimensional vector spaces, which are the ones involved in the work of chapters 2—5. References for this section are mainly [23] and [47, 46].

We depict the field with one element, $\mathbb{F}_1$, as the set $\{0,1\}$ in which we have the following operation:

$$1 \cdot 0 = 0 = 0 \cdot 0 \text{ and } 1 \cdot 1 = 1, \tag{1.1}$$

i.e., $\mathbb{F}_1$ is the trivial monoid together with a zero element, usually called *absorbing element*. Notice that by considering this way of describing $\mathbb{F}_1$ one keeps the idea of no addition.

Definition 1.2.1. A *vector space* V over $\mathbb{F}_1$ is a pair $V = (0, X)$, where X is a set of elements and 0 is a distinguished point with the property that $0 \notin X$. We define the *dimension of* V to be the cardinality of the set X.

Definition 1.2.2 ($\mathbb{F}_1$-rings)**.** A *commutative ring* A over $\mathbb{F}_1$ is a commutative multiplicative monoid — that is, a set with an associative binary operation that has a unit element— together with an absorbing element 0.

Since $\mathbb{F}_1$ plays the role of a "field," we also would like, as an analogy to finite fields, that for every $n \in \mathbb{N}$ there exist a field extension of degree n. Denote it by $\mathbb{F}_{1^n}$. We define *the extension field* $\mathbb{F}_{1^n}$ of $\mathbb{F}_1$ to be the monoid $\{0\} \cup \mu_n$, where μ_n is the (multiplicative) group of n-th roots of unity over $\mathbb{C}$ and 0 is the absorbing element. Notice that $\mathbb{F}_{1^n}$ can also be seen as vector space over $\mathbb{F}_1$ of dimension n since the cardinality of the set μ_n is n.

After defining field extensions of $\mathbb{F}_1$, one can extend the notion of a vector space over $\mathbb{F}_1$ to a vector space over $\mathbb{F}_{1^n}$.

Definition 1.2.3. A *vector space* V over the field $\mathbb{F}_{1^n}$, $n \geq 1$, is a triple $V = (0, X, \mu_n)$ where 0 is a distinguished point, X is a set and μ_n is the group of n-th roots of unity acting freely on the set X. Each μ_n-orbit corresponds to a direction.

A *basis* of a vector space over $\mathbb{F}_{1^n}$ is a subset B of X such that every μ_n-orbit contains a unique element of B, i.e., B is a set of representatives of the μ_n-action. We define the *dimension of* V to be the cardinality of any basis, that is, the number of μ_n-orbits. Since the action of μ_n on X is free, the dimension of V is $d := \mathrm{card}(V)/n$ and X is then a set of dn elements.

Remark 1.2.4. In the previous definition, when $n = 1$, one gets the notion of a vector space over the field $\mathbb{F}_1$ where X would be a set of d elements and the basis B is the whole set X.

Observe that once a choice of a basis $B = \{b_i \mid i \in I\}$ of a vector space X over $\mathbb{F}_{1^n}$ is made, any element can be uniquely written as $b_j^{\alpha^u}$, for unique $j \in I$ and $\alpha^u \in \mu_n = \langle \alpha \rangle$. Hence, we can define a *linear automorphism* f of a vector space V over $\mathbb{F}_{1^n}$ with basis B to be a map sending each element of the basis to

$$f(b_i) = b_{\sigma(i)}^{\beta_i}, \tag{1.2}$$

where σ is an element of $\mathtt{Sym}(d)$ and β_i is a power of a primitive n-th root of unity. Notice that indeed, the element σ sends the element b_i to an element of another orbit of the action of μ_n and β_i chooses the correct representant of the orbit of $b_{\sigma(i)}$. Then, we have that

$$\mathbf{GL}_d(\mathbb{F}_{1^n}) \cong \mu_n \wr \mathtt{Sym}(d), \tag{1.3}$$

where $\wr$ denotes the wreath product. As a consequence, we obtain that the general linear group of a vector space over $\mathbb{F}_1$ of dimension d is the symmetric group $\mathtt{Sym}(d)$.

1.3 Absolute Algebraic Geometry

Although Absolute Linear Algebra was introduced in a unique guise, this is not the case for Algebraic Geometry. Indeed there are different versions of schemes theory over the field with one element, some of which we will discuss in detail later in this chapter.

Let us set up the motivation for a scheme theory over the field with one element as it is explained in [29]. In the early 90's, Christopher Deninger published his studies [14, 15, 16] on motives and regularized determinants. In [15], he gave a description of conditions on a category of motives that would admit a similar proof to Weil's proof of the Riemann hypothesis for function fields of projective curves over finite fields $\mathbb{F}_q$ to the hypothetical curve $\overline{\mathtt{Spec}(\mathbb{Z})}$. In particular, he showed that the following formula would hold:

$$2^{-1/2}\pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s) =$$

$$\frac{\det_\infty(\frac{1}{2\pi}(s-\Theta)\Big|H^1(\overline{\mathtt{Spec}(\mathbb{Z})}, \mathcal{O}_{\mathcal{J}}))}{\det_\infty(\frac{1}{2\pi}(s-\Theta)\Big|H^0(\overline{\mathtt{Spec}(\mathbb{Z})}, \mathcal{O}_{\mathcal{J}}))\det_\infty(\frac{1}{2\pi}(s-\Theta)\Big|H^2(\overline{\mathtt{Spec}(\mathbb{Z})}, \mathcal{O}_{\mathcal{J}}))}$$

where $\det_\infty$ denotes the regularized determinant, Θ is an endofunctor that comes with the category of motives and $H^i(\overline{\mathtt{Spec}(\mathbb{Z})}, \mathcal{O}_{\mathcal{J}})$ are certain proposed cohomology groups. This description combines with the work of Kurokawa on multiple zeta functions [25] from 1992 to the hope that there are motives h^0, h^1 and h^2 with zeta functions:

$$\zeta_{h^w}(s) = \det_\infty(\frac{1}{2\pi}(s-\Theta)\Big|H^w(\overline{\mathtt{Spec}(\mathbb{Z})}, \mathcal{O}_{\mathcal{J}})),$$

for $w = 0, 1, 2$. It was Yuri Manin in [36] who suggested the interpretation of h^0 as $\mathtt{Spec}(\mathbb{F}_1)$ and h^2 as the affine line over $\mathbb{F}_1$. The quest for a proof of the Riemann hypothesis was from now on a main motivation to look for a geometry over $\mathbb{F}_1$.

1.3.1 Schemes over commutative rings

Before starting to describe the state of the art of Absolute Algebraic Geometry and give a detailed construction of certain schemes over $\mathbb{F}_1$, we will give a brief summary about schemes over commutative rings, which were introduced by Grothendieck in the 60's. For more details on this theory, we refer to [18, 28].

For the definition of a scheme, which is a generalization of the notion of algebraic variety, we first need to introduce some concepts.

Definition 1.3.1. Let X be a topological space. A *presheaf* $\mathscr{F}$ *of rings* on X consists of the data

i) for every open set $U \subseteq X$, a ring $\mathscr{F}(U)$.

ii) for every inclusion $V \subseteq U$ of open sets of X, a morphism of rings $\rho_{UV} : \mathscr{F}(U) \to \mathscr{F}(V)$,

subject to the conditions

a) $\mathscr{F}(\emptyset) = 0$, where 0 is the zero ring,

b) ρ_{UU} is the identity map $\mathscr{F}(U) \to \mathscr{F}(U)$ and

c) if $W \subseteq V \subseteq U$ are three open subsets of X, then $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$.

Remark 1.3.2. In the language of categories, the definition of a presheaf of rings corresponds to a contravariant functor from the category $\mathtt{Top}(X)$ of open subsets of X whose morphisms are the inclusion maps to the category of rings. Although a presheaf of rings is the type of presheaf that we will consider when defining a scheme, presheaves of other categories such us groups, sets or $\mathbb{R}$-algebras are equally defined.

If $\mathscr{F}$ is a presheaf of rings on X, we call the elements of $\mathscr{F}(U)$ the *sections* of the presheaf over the open set U. A section over X is called a *global section* and the maps ρ_{UV} are referred to as *the restriction maps*. We sometimes also use the notation $\Gamma(U, \mathscr{F})$ or $\Gamma(U)$ for the sections over U and $s|_V$ for the image $\rho_{UV}(s)$ if $s \in \mathscr{F}(U)$.

Examples 1.3.3. 1) Let X be a topological space. The functor

$$U \longmapsto C^0(U, \mathbb{R}),$$

sending an open set U to the set of continuous functions from U to $\mathbb{R}$ is a presheaf of $\mathbb{R}$-algebras on X.

2) Let X be a topological space and let E be a set. The functor $U \mapsto E$ is a presheaf of sets on X called *the constant presheaf associated to* E.

3) Let X be a topological space. The functor

$$U \longmapsto \begin{cases} \mathbb{Z} \text{ if } U = X \\ 0 \text{ else} \end{cases},$$

is a presheaf of rings on X.

Definition 1.3.4. If $\mathscr{F}$ is a presheaf on X and if p is a point of X, we define the *stalk* $\mathscr{F}_p$ *of* $\mathscr{F}$ *at* p to be the direct limit of the rings $\mathscr{F}(U)$ for all open sets U containing p via the restriction maps ρ.

An element of $\mathscr{F}_p$ is represented by a pair $\langle U, s\rangle$ where U is an open neighborhood of p and s an element of $\mathscr{F}(U)$, under the relation: $\langle U, s\rangle \sim \langle V, t\rangle$ if and only if there exist an open neighborhood $W \subseteq U \cap V$ of p such that $s|_W = t|_W$.

Example 1.3.5. If $\mathscr{F}$ is the constant presheaf of previous example 2), $\mathscr{F}_p \cong E$ $\forall p \in X$.

Definition 1.3.6. A presheaf $\mathscr{F}$ on a topological space X is a *sheaf* if it satisfies the following conditions:

i) If U is an open set, if $\{V_i\}_{i\in I}$ is an open covering of U and if $s \in \mathscr{F}(U)$ is an element such that $s|_{V_i} = 0$ for all i, then $s = 0$.

ii) If U is an open set, if $\{V_i\}_{i\in I}$ is an open covering of U and if we have elements $s_i \in \mathscr{F}(V_i)$ $\forall i$ such that $s_i|_{V_i\cap V_j} = s_j|_{V_j\cap V_i}$ $\forall i, j \in I$, then there exists a unique element $s \in \mathscr{F}(U)$ such that $s|_{V_i} = s_i$ for each i. (Note that condition i) implies uniqueness of s.)

Examples 1.3.7. 1) The sheaf of continuous real-valued functions on a topological space.

2) Let X be the real line; then the presheaf

$$\mathscr{F} : U \longmapsto \mathscr{F}(U) = \{\text{bounded continuous functions on } U\},$$

is a presheaf but not a sheaf.

Definition 1.3.8. Let $\mathscr{F}$ and $\mathscr{G}$ be two presheaves of rings on X. A *morphism* $\varphi : \mathscr{F} \to \mathscr{G}$ *of presheaves* consists of a morphism of rings $\varphi(U) : \mathscr{F}(U) \to \mathscr{G}(U)$ for each open set U, such that whenever $V \subseteq U$ is an inclusion of open sets, the diagram

$$\begin{tikzcd}
\mathscr{F}(U) \arrow[r, "\varphi(U)"] \arrow[d, "\rho_{UV}"] & \mathscr{G}(U) \arrow[d, "\rho'_{UV}"] \\
\mathscr{F}(V) \arrow[r, "\varphi(V)"] & \mathscr{G}(V)
\end{tikzcd}$$

is commutative, where ρ and ρ' are the restriction maps in $\mathscr{F}$ and $\mathscr{G}$, respectively. A *morphism of sheaves* is a morphism of presheaves.

Consider now a commutative ring A. The *spectrum* $\mathtt{Spec}(A)$ of A is the set of all prime ideals of A endowed with the Zariski topology. The closed sets defining the Zariski topology for $\mathtt{Spec}(A)$ are all of the form

$$V(I) = \{\mathfrak{p} \in \mathtt{Spec}(A) \mid I \subseteq \mathfrak{p}\},$$

where I is an ideal of A. For any element $f \in A$, we call the set $V(f) = \{\mathfrak{p} \in \mathtt{Spec}(A) \mid f \in \mathfrak{p}\}$ a *principal closed set* and the set $D(f) = \mathtt{Spec}(A) \setminus V(f) = \{\mathfrak{p} \in \mathtt{Spec}(A) \mid f \notin \mathfrak{p}\}$ a *principal open set.*

We define a sheaf of rings $\mathscr{O}_A$ on the topological space $\mathtt{Spec}(A)$ as follows: for any open set $U \subseteq \mathtt{Spec}(A)$, $\mathscr{O}_A(U)$ is the set of functions

$$s : U \longmapsto \coprod_{\mathfrak{p} \in U} A_{\mathfrak{p}},$$

where $s(\mathfrak{p}) \in A_{\mathfrak{p}}$ for each $\mathfrak{p}$, and such that there exist a neighborhood V of $\mathfrak{p}$ contained in U and elements $a, f \in A$ for which $f \notin \mathfrak{q}$ for every $\mathfrak{q} \in V$ and $s(\mathfrak{q}) = \frac{a}{f} \in A_{\mathfrak{q}}$. The sheaf $\mathscr{O}_A$ is called *the structure sheaf of* $\mathtt{Spec}(A)$.

Proposition 1.3.9. *Let A be a commutative ring and consider the pair $(\mathtt{Spec}(A), \mathscr{O}_A)$. Then:*

a) *For any $\mathfrak{p} \in \mathtt{Spec}(A)$, the stalk $\mathscr{O}_{A,\mathfrak{p}}$ of the structure sheaf $\mathscr{O}_A$ is isomorphic to the local ring $A_{\mathfrak{p}}$.*

b) *For any $f \in A$, the ring $\mathscr{O}_A(D(f))$ is isomorphic to the localized ring A_f.*

c) *$\mathscr{O}_A(\mathtt{Spec}(A)) = A$.*

We are about to be able to define schemes, by using the two concepts described above: the spectrum of a ring and a sheaf of rings on a topological space.

Let R and S be two local rings, we define a *local homomorphism of local rings* to be a ring homomorphism $\varphi : R \to S$ such that $\varphi(\mathfrak{m}_R) \subset \mathfrak{m}_S$.

Definition 1.3.10. A *ringed space* is a pair $(X, \mathscr{O}_X)$ consisting of a topological space and a sheaf of rings $\mathscr{O}_X$ on X. A *morphism of ringed spaces* from $(X, \mathscr{O}_X)$ to $(Y, \mathscr{O}_Y)$ is a pair $(f, f^{\#})$ consisting of a continuous map $f : X \to Y$ and a morphism $f^{\#} : \mathscr{O}_Y \to f_*\mathscr{O}_X$ of sheaves of rings on Y, where $f_*\mathscr{O}_X$ is the *direct image sheaf* on Y defined by $f_*\mathscr{O}_X(U) = \mathscr{O}_X(f^{-1}(U))$ for any open set $U \subseteq Y$.

A *locally ringed space* is a ringed space $(X, \mathscr{O}_X)$ in which for each point $p \in X$, the stalk $\mathscr{O}_{X,p}$ is a local ring. A morphism of locally ringed spaces is a morphism of ringed spaces such that for every $p \in X$, the induced map on the stalks $f^{\#}_p : \mathscr{O}_{Y,f(p)} \to \mathscr{O}_{X,p}$ is a local homomorphism of local rings.

Proposition 1.3.11. a) *If A is a ring, then $(\mathtt{Spec}(A), \mathscr{O}_A)$ is a locally ringed space.*

b) *If $\varphi : A \to B$ is a homomorphism of rings, then φ induces a natural morphism of locally ringed spaces*

$$(f, f^{\#}) : (\mathtt{Spec}(B), \mathscr{O}_B) \longrightarrow (\mathtt{Spec}(A), \mathscr{O}_A).$$

c) *If A, B are rings, then any morphism of locally ringed spaces from* $(\mathtt{Spec}(B), \mathscr{O}_B)$ *to* $(\mathtt{Spec}(A), \mathscr{O}_A)$ *is induced by a homomorphism of rings* $\varphi : A \to B$.

We can finally arrive at the main objective of this subsection.

Definition 1.3.12. An *affine scheme* is a locally ringed space $(X, \mathscr{O}_X)$ which is isomorphic (as locally ringed space) to $(\mathtt{Spec}(A), \mathscr{O}_A)$ for some ring A. A *scheme* is a locally ringed space $(X, \mathscr{O}_X)$ in which every point has an open neighborhood U such that the topological space U together with the restricted sheaf $\mathscr{O}_X|_U$ is an affine scheme.

We call X the *underlying topological space* of the scheme $(X, \mathscr{O}_X)$ and $\mathscr{O}_X$ its *structure sheaf.*

We finish this subsection by giving two examples of schemes.

Examples 1.3.13. 1) If k is a field, $\mathtt{Spec}(k)$ is an affine scheme whose topological space is only one point and whose structure sheaf consist of the field k.

2) If k is a field, we define the *affine line* over k to be $\mathtt{Spec}(k[X])$, which as a set of points consists of the zero ideal and all maximal ideals bijectively corresponding to non-constant monic irreducible polynomials in $k[X]$. In particular, if k is algebraically closed, there is a bijection between the closed points of $\mathtt{Spec}(k[X])$ and the elements of k.

1.3.2 Constructible sets

Constructible sets are sets inside schemes that have a particular interest of study. We will describe here the main properties of such objects.

Definition 1.3.14. Let X be a scheme. A set E is a *locally closed* set of X if it is the intersection of an open set and a closed set of the underlying topological space of X. We say that a set E is a *constructible set* of X if it is a finite union of locally closed sets.

Example 1.3.15. Projective spaces and affine subspaces of an arbitrary projective space are constructible sets.

Proposition 1.3.16. *Constructible sets are closed under finite unions, finite intersections and complements.*

Proof. This follows immediately by using the properties of union, intersection and complements of sets. ∎

Proposition 1.3.17. *Let X be a Noetherian scheme and E be a constructible set of X. Then E can be expressed as a finite disjoint union of locally closed sets.*

Proof. Suppose $E = U \cap V$ is a locally closed set, with U an open set and V a closed set of X. We can then write

$$E^c = (U \cap V)^c = U^c \cup V^c = U^c \coprod (V^c \cap U),$$

and hence, the complement of a locally closed set is a disjoint union of locally closed sets. We proceed now by induction on the number of components in a union. Suppose $E = \cup_{i=1}^{n} A_i$, with A_i locally closed sets. Denote now $A = \cup_{i=1}^{n-1} A_i$ and $B = A_n$. By induction, we asssume that A is a finite disjoint union of locally closed sets. Then we have that

$$A \cup B = B \coprod (A \setminus B) = B \coprod (A \cap B^c).$$

By the above reasoning, B^c is a disjoint union of locally closed sets and so is $A \cap B^c$. Hence the theorem follows. ∎

1.3.3 Different versions of schemes over $\mathbb{F}_1$

In 2004, Soulé [43] proposed the first definition of what an algebraic variety over $\mathbb{F}_1$ would be. Shortly after that, many different approaches to $\mathbb{F}_1$-geometry arose in the attempt of finding the right definition of schemes "over $\mathbb{F}_1$." In [8] Deitmar made the first attempt on defining such schemes. In fact, based on the idea stated in [27] by Kurokawa, Ochiai and Wakayama in which objects over $\mathbb{Z}$ have a notion of $\mathbb{Z}$-linearity, i.e., additivity, and on the idea that the forgetful functor to $\mathbb{F}_1$-objects must forget about that addition, he defined the schemes following the ideas of Kato [24] of mimicking classical scheme theory but considering the category of commutative monoids instead of rings. In the next section you can find a full description of this type of schemes.

After Deitmar's definition and due to the basic construction that considers absolutely no addition on the structure, other scheme approaches appeared to extend the notion of Deitmar schemes and allowed some more complex base extensions of the objects to $\mathbb{Z}$-schemes. However, it is believed that any "good" scheme theory over $\mathbb{F}_1$ should contain Deitmar's schemes in one way or another.

In [5] Connes and Consani merged the ideas of Soulé's varieties and Deitmar's schemes into a more subtle scheme theory. They define a scheme, called *CC-scheme*, to be a triple $(\tilde{X}, X, \mathrm{ev}_X)$ where $\tilde{X}$ is a monoidal scheme (with a zero element), X is a $\mathbb{Z}$-scheme and $\mathrm{ev}_X : \tilde{X}_{\mathbb{Z}} \to X$ is a certain well-behaved morphism of schemes. In fact, the base extension of $(\tilde{X}, X, \mathrm{ev}_X)$ to $\mathbb{Z}$ is X. Other different approaches for defining schemes over $\mathbb{F}_1$ were developed by Toën and Vaquié, by Borger (λ-schemes), by Lorscheid (blueprints) or by Deitmar (congruence schemes). Despite all the different attempts to define the same geometric structure over the field with one element, all theories are not completely independent from each other. One can find in [29] a brief introduction on some of the aformentioned schemes and in figure 1.1 (taking from [29]), one can see a diagram that relates the different schemes theories.

We will define in the next section Deitmar's first version of $\mathbb{F}_1$-schemes, which is based on monoids, and two important generalizations of them: congruence schemes and blueprints. The latter two categories of schemes are generalizations of schemes that include both the category of monoidal schemes and the one of classical schemes

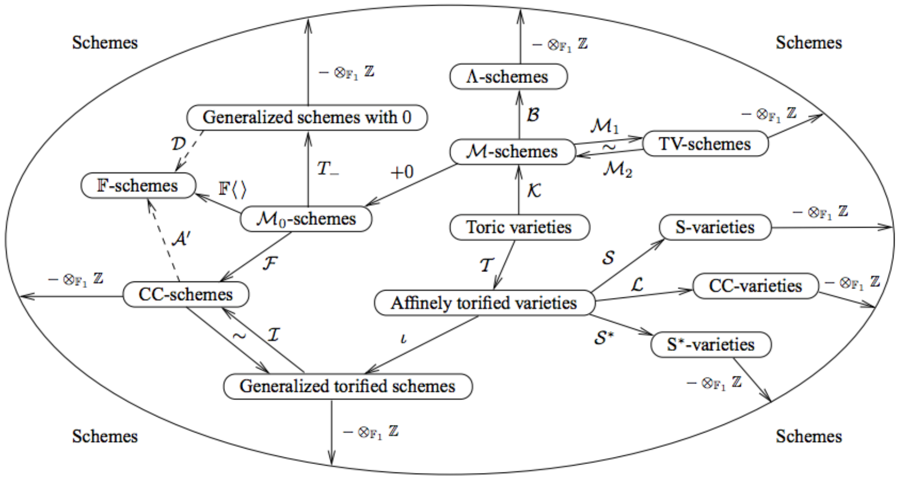


Figure 1.1: In this diagram $\mathcal{M}_0$-schemes stands for Deitmar schemes, CC-schemes are the Connes-Consani schemes and S-varieties are Soulé varieties.

(defined over commutative rings). Both schemes defined by Deitmar (monoidal and congruence schemes) play an essential role for the purpose of this work.

1.4 Monoidal schemes

As we mentioned in the previous subsection, the schemes defined by Anton Deitmar in [8] are based on the idea that commutative monoids are the $\mathbb{F}_1$-versions of commutative rings over "real fields." Recall that a monoid A is a set with an associative multiplication that has a unit, and that we consider monoids also to have an absorbing element 0.

We will give a full description of these monoidal schemes including some properties and theorems regarding their construction. More details on the definition as well as the proofs of all results can be found in [8]. For the rest of the thesis, a monoid will always mean a commutative monoid with an absorbing element.

Localization on monoids

Let A be a monoid, let $A^\times$ denote the group of invertible elements of A and let S be a submonoid of A. Define the *localization of A by S*, denoted by $S^{-1}A$, to be the set $A \times S$ modulo the equivalence relation

$$(m, s) \sim (m', s') \Leftrightarrow \exists s'' \in S \text{ such that } s''s'm = s''sm',$$

where the multiplication is given by $(m, s)(m', s') = (mm', ss')$. We also write $\frac{m}{s}$ for the class of the element (m, s).

Spectrum of a monoid and the structure sheaf

For the definition of a scheme over a commutative ring we endow the set of prime ideals of a ring with the Zariski topology. We do a similar construction for monoids.

Definition 1.4.1. Let A be a monoid. An *ideal* is a subset I of A such that $IA \subseteq I$. An ideal $\mathfrak{p} \neq A$ is called *prime* if $xy \in \mathfrak{p}$ implies that $x \in \mathfrak{p}$ or $y \in \mathfrak{p}$. Equivalently, an ideal $\mathfrak{p}$ is prime if and only if $A \setminus \mathfrak{p}$ is a submonoid.

Remark 1.4.2. As in the case of rings, if $\mathfrak{p}$ is a prime ideal of A, then we denote by $A_{\mathfrak{p}}$ the localization of A by the submonoid $S_{\mathfrak{p}} = A \setminus \mathfrak{p}$ and we call it the *localization of A at $\mathfrak{p}$*.

Definition 1.4.3. Let A be a monoid. We say that A is *integral* if for any element $a \neq 0$, $ab = ac$ implies $b = c$.

Definition 1.4.4. Let M be a monoid and I and ideal of M. We define the *monoidal quotient* M/I to be the set $\{[m] \mid m \in M\}/([m] = [0] \text{ if } m \in I)$.

Definition 1.4.5. Let A be a monoid. We define the *spectrum* $\mathtt{Spec}(A)$ of A to be the set of all prime ideals. We endow this set with the topology whose closed subsets are the empty set and all sets of the form

$$V(I) = \{\mathfrak{p} \in \mathtt{Spec}(A) \mid I \subseteq \mathfrak{p}\}.$$

We call this topology the *Zariski topology for* $\mathtt{Spec}(A)$.

To define a scheme there is only one thing left to define, namely the concept of a *structure sheaf* for $\mathtt{Spec}(A)$ in the case of monoids. As a consequence of the analogy of all the previous definitions with the ring case, one can define the *structure sheaf $\mathscr{O}_A$ of* $\mathtt{Spec}(A)$ in a similar way. For any open set $U \subseteq \mathtt{Spec}(A)$, $\mathscr{O}_A(U)$ is the set of functions

$$s : U \longmapsto \coprod_{\mathfrak{p} \in U} A_{\mathfrak{p}},$$

such that $s(\mathfrak{p}) \in A_{\mathfrak{p}}$ for each $\mathfrak{p}$, and such that there exist a neighborhood V of $\mathfrak{p}$ contained in U and elements $a, f \in A$ for which $f \notin \mathfrak{q}$ for every $\mathfrak{q} \in V$ and $s(\mathfrak{q}) = \frac{a}{f} \in A_{\mathfrak{q}}$.

Proposition 1.4.6. a) *For each $\mathfrak{p} \in \mathtt{Spec}(A)$ the stalk $\mathscr{O}_{A,p}$ of the structure sheaf is isomorphic to the localization $A_{\mathfrak{p}}$.*

b) $\mathscr{O}_A(\mathtt{Spec}(A)) \cong A$.

Monoidal spaces

A morphism $\varphi : A \to B$ of monoids is a multiplicative map that preserves the unit and the zero element. We say that a morphism of monoids is local if $\varphi^{-1}(B^\times) = A^\times$.

A *monoidal space* is a pair $(X, \mathscr{O}_X)$ consisting of a topological space X and a sheaf $\mathscr{O}_X$ of monoids. A *morphism of monoidal spaces* from $(X, \mathscr{O}_X)$ to $(Y, \mathscr{O}_Y)$ is a pair $(f, f^\#)$ with $f : X \to Y$ a continuous function and $f^\# : \mathscr{O}_Y \to f_*\mathscr{O}_X$ a morphism of sheaves, where $f_*\mathscr{O}_X$ is defined as in definition 1.3.10. The morphism is *local* if for each $x \in X$, the induced morphism $f_x^\# : \mathscr{O}_{Y,f(x)} \to f_*\mathscr{O}_{X,x}$ is local.

Proposition 1.4.7. a) *For an $\mathbb{F}_1$-ring A the pair $(\texttt{Spec}(A), \mathscr{O}_A)$ is a monoidal space.*

b) *If $\varphi : A \to B$ is a morphism of monoids, then φ induces a morphism of monoidal spaces*

$$(f, f^\#) : (\texttt{Spec}(B), \mathscr{O}_B) \longrightarrow (\texttt{Spec}(A), \mathscr{O}_A),$$

thus giving a functorial bijection

$$\texttt{Hom}(A, B) \cong \texttt{Hom}(\texttt{Spec}(B), \texttt{Spec}(A)),$$

where on the right-hand side one only admits local morphisms.

Deitmar's $\mathbb{F}_1$-schemes

An *affine scheme over* $\mathbb{F}_1$ is a monoidal space which is isomorphic to $\texttt{Spec}(A)$, for some monoid A. A monoidal space $(X, \mathscr{O}_X)$ is a *scheme over* $\mathbb{F}_1$ if for every point $x \in X$ there is an open neighborhood $U \subset X$ such that $(U, \mathscr{O}_X|_U)$ is an affine scheme over $\mathbb{F}_1$. A *morphism* of schemes over $\mathbb{F}_1$ is a local morphism of monoidal spaces.

For the rest of this work, we will call such schemes *Deitmar schemes*. One of the most important types of schemes for the purpose of this work are the "schemes of finite type." Let X be a scheme over $\mathbb{F}_1$; we say that X is of *finite type* if it has a finite covering by affine Deitmar schemes $U_i = \texttt{Spec}(A_i)$ such that each A_i is finitely generated. The same definition happens in the case of schemes over a commutative ring, with the only difference that each A_i is finitely generated as algebra over the ring.

In fact, Deitmar proved in [9] the following result:

Proposition 1.4.8. *X is of finite type over $\mathbb{F}_1$ if and only if $X \otimes_{\mathbb{F}_1} \mathbb{Z}$ is a scheme of finite type over $\mathbb{Z}$. (See subsection 2.4.1 for the definition of $X \otimes_{\mathbb{F}_1} \mathbb{Z}$.)*

We mentioned in the beginning of the previous section that schemes are a generalization of varieties. We will therefore finish the study of monoidal schemes by constructing the affine spaces and projective spaces in the sense of scheme theory over the field with one element. (A similar construction carries over to schemes over commutative rings.)

Affine spaces

Define the polynomial ring on n variables $X_1, \ldots, X_n$ over $\mathbb{F}_1$ to be

$$\mathbb{F}_1[X_1, \ldots, X_n] := \{0\} \cup \{X_1^{u_1} \ldots X_n^{u_n} \mid u_j \in \mathbb{N}\}, \tag{1.4}$$

that is, the union of $\{0\}$ and the (free abelian) monoid generated by the X_j.

Let $A = \mathbb{F}_1[X_1, \ldots, X_n]$. Denote $\texttt{Spec}(\mathbb{F}_1[X_1, \ldots, X_n])$ by $\mathbb{A}^n_{\mathbb{F}_1}$ and call it the *n-dimensional affine space over* $\mathbb{F}_1$. The non-zero prime ideals of A are of the form $\mathfrak{p}_I = \bigcup_{i \in I}(X_i)$, where I is a subset of $\{1, \ldots, n\}$ and $(X_i) = X_i A = \{X_i a \mid a \in A\}$.

In the case of $n = 0$, we call the Deitmar scheme $\texttt{Spec}(\mathbb{F}_1)$ the *absolute point*. The spectrum of $\mathbb{F}_1$ consists of precisely one point, namely, the unique prime ideal $\{0\}$. In addition, the $\texttt{Spec}(\mathbb{F}_1)$ is a terminal object in the category of Deitmar schemes.

Proj-schemes and the Proj-construction

The Proj-construction is a way of defining a more general object than projective varieties, called "projective scheme." In [7, section 7] and [30, section 2] the authors introduced the Proj-construction for both graded monoids and blueprints, respectively. For graded monoids, the Proj-construction was independently found by Thas in [44]. We will describe the procedure for Deitmar schemes here.

Consider the $\mathbb{F}_1$-polynomial ring $\mathbb{F}_1[X_0, X_1, \ldots, X_m]$, where $m \in \mathbb{N}$. Since any polynomial is homogeneous in this ring, we have a natural grading

$$\mathbb{F}_1[X_0, \ldots, X_m] = \bigoplus_{i \geq 0} R_i = \coprod_{i \geq 0} R_i,$$

where R_i consists of the elements of $\mathbb{F}_1[X_0, X_1, \ldots, X_m]$ of total degree i, for $i \in \mathbb{N}$. The *irrelevant ideal* is the ideal defined as

$$\mathrm{Irr} = \{0\} \cup \coprod_{i \geq 1} R_i.$$

(It is just the monoid minus the element 1.) Define $\texttt{Proj}(\mathbb{F}_1[\mathbf{X}]) := \texttt{Proj}(\mathbb{F}_1[X_0, \ldots, X_m])$ to be the set consisting of the prime ideals of $\mathbb{F}_1[X_0, X_1, \ldots, X_m]$ which do not contain the irrelevant ideal Irr (so only Irr is left out of the complete set of prime ideals). We can endow this set with a Zariski topology whose closed sets are defined as usual: for any ideal I of $\mathbb{F}_1[X_0, X_1, \ldots, X_m]$, we define

$$V(I) := \{\mathfrak{p} \mid \mathfrak{p} \in \texttt{Proj}(\mathbb{F}_1[\mathbf{X}]), \quad I \subseteq \mathfrak{p}\},$$

where $V(I) = \emptyset$ if $I = \mathrm{Irr}$ and $V(\{0\}) = \texttt{Proj}(\mathbb{F}_1(\mathbf{X}))$. Its open sets are then of the form

$$D(I) := \{\mathfrak{p} \mid \mathfrak{p} \in \texttt{Proj}(\mathbb{F}_1[\mathbf{X}], \quad I \not\subseteq \mathfrak{p}\}.$$

It is obvious that $\texttt{Proj}(\mathbb{F}_1[\mathbf{X}])$ is a Deitmar scheme in a natural way. Each ideal (X_i) defines an open set $D((X_i))$ such that the restriction of the scheme $\texttt{Proj}(\mathbb{F}_1[\mathbf{X}])$ to this set is isomorphic to $\texttt{Spec}(\mathbb{F}_1[\mathbf{X}_{(i)}])$, where $\mathbf{X}_{(i)}$ is $X_0, X_1, \ldots, X_m$ with X_i left out. Indeed, every prime ideal of $\texttt{Spec}(\mathbb{F}_1[\mathbf{X}_{(i)}])$ does not contain the irrelevant ideal nor the ideal (X_i) and any ideal of the set $D(X_i)$ is of the form $\bigcup_{0\leq j\leq m}(X_j)$ with $j \neq i$, so it belongs to $\texttt{Spec}(\mathbb{F}_1[\mathbf{X}_{(i)}])$.

We define the scheme $\texttt{Proj}(\mathbb{F}_1[X_0, \ldots, X_m])$ to be the *projective m-space over* $\mathbb{F}_1$.

Remark 1.4.9. Notice that there is an immediate analogy with projective spaces as we know them in Projective Geometry, since when one considers an m-dimensional projective space defined over a field k and restrict it to the set of points where the coordinate $X_i \neq 0$, then one gets a space isomorphic to an m-dimensional affine space.

More generally, suppose M is any commutative unital monoid (with 0) with a grading

$$M = \coprod_{i\geq 0} M_i,$$

where the M_i are the sets with elements of total degree i (for $i \in \mathbb{N}$), and let, as above, the *irrelevant ideal* be $\mathrm{Irr} = \{0\} \cup \coprod_{i\geq 1} M_i$. Define the topology $\texttt{Proj}(M)$ as before (noting that homogeneous (prime) ideals are the same as ordinary monoidal (prime) ideals here). For an open U, define $\mathcal{O}_M(U)$ as consisting of all functions

$$f : U \longrightarrow \coprod_{\mathfrak{p}\in U} M_{(\mathfrak{p})},$$

where $M_{(\mathfrak{p})}$ is the subset of $M_{\mathfrak{p}}$ of fractions of elements with the same degree, for which $f(\mathfrak{p}) \in M_{(\mathfrak{p})}$ for each $\mathfrak{p} \in U$, and such that there exists a neighborhood V of $\mathfrak{p}$ in U, and elements $u, v \in M$, for which $v \notin \mathfrak{q}$ for every $\mathfrak{q} \in V$, and $f(\mathfrak{q}) = \frac{u}{v}$ in $M_{(\mathfrak{q})}$.

In this way we obtain a sheaf of $\mathbb{F}_1$-rings on $\texttt{Proj}(M)$ making it a Deitmar scheme.

Remark 1.4.10. The same construction can be done for any graded ring. For instance, if we consider the polynomial ring $k[X_0, \ldots, X_m]$ over a field k, one defines the projective space of dimension m as the scheme $\texttt{Proj}(k[X_0, \ldots, X_m])$ and the affine space of dimension m as the scheme $\texttt{Spec}(k[X_1, \ldots, X_m])$.

Remark 1.4.11. Notice that for the schemes defined with a $\texttt{Proj}$-construction, we use homogeneous prime ideals and the points have, as a consequence, homogeneous coordinates (in an analogy with the classical projective varieties). Recall that a point having *homogeneous coordinates* with respect to a basis means that if we multiply the coordinates by a non-zero scalar, the resulting coordinates then represent the same point.

Toric varieties

One of the most important results in the theory of Deitmar schemes is the one proved by Deitmar in [10] that relates certain schemes over $\mathbb{F}_1$ with toric varieties.

Definition 1.4.12. A (complex) *torus* is an affine variety isomorphic to $(\mathbb{C}^*)^n$ for some n.

In fact, $(\mathbb{C}^*)^n \simeq \mathbb{C}^n \setminus V(x_1x_2\cdots x_n)$. It is an affine variety whose coordinate ring is $\mathbb{C}[X_1^{\pm}, \dots, X_n^{\pm}]$.

Example 1.4.13. Let $V = V(x^2 - y) \subset \mathbb{C}^2$, and consider $V_{xy} = V \cap (\mathbb{C}^*)^2$. Since V_{xy} is the graph of the map $\mathbb{C}^* \to \mathbb{C}$ given by $t \to t^2$, the morphism $\mathbb{C}^* \to V_{xy}$ given by $t \to (t, t^2)$ is indeed bijective, and this isomorphism gives V_{xy} the action of $\mathbb{C}^*$ by $(a, a^2) \cdot (b, b^2) = (ab, (ab)^2)$.

Definition 1.4.14. A *toric variety* X over $\mathbb{C}$ is an irreducible variety such that for some $n \in \mathbb{N} \setminus \{0\}$

(1) $(\mathbb{C}^*)^n$ is a Zariski open subset of X, and

(2) the action of $(\mathbb{C}^*)^n$ on itself extends to an action of $(\mathbb{C}^*)^n$ on X.

Examples 1.4.15. 1) $(\mathbb{C}^*)^n$ and $\mathbb{C}^n$ are clearly toric varieties.

2) $\mathbb{P}^n_{\mathbb{C}}$ is a toric variety. The map

$$\begin{array}{cccc} \varphi : (\mathbb{C}^*)^n & \longrightarrow & \mathbb{P}^n_{\mathbb{C}} \\ (t_1, \dots, t_n) & \longmapsto & (1, t_1, \dots, t_n), \end{array}$$

identifies $(\mathbb{C}^*)^n$ as an open subset of $\mathbb{P}^n_{\mathbb{C}}$.

The theorem proved by Deitmar relating toric varieties with schemes over $\mathbb{F}_1$ is the following:

Theorem 1.4.16 ([10])**.** *Let X be a connected integral $\mathbb{F}_1$-scheme (defined as expected, cf. [10]) of finite type. Then every irreducible component of $X_{\mathbb{C}}$ is a toric variety. The components of $X_{\mathbb{C}}$ are mutually isomorphic as toric varieties.*

On the other hand, since toric varieties can be constructed from lattices (see [17, chapter 1]) it follows that every toric variety is a lift $X_{\mathbb{C}}$ of an $\mathbb{F}_1$-scheme (see [24, section 9] for a combinatorial description of the category of toric varieties in terms of monoids). Hence, integral $\mathbb{F}_1$-schemes of finite type are essentially toric varieties.

1.4.1 $\mathbb{F}_1$-Constructible sets

In an analogous way as in subsection 1.3.2, one can define constructible sets for Deitmar schemes.

Definition 1.4.17. Let X be a Deitmar scheme. We say that a subset E of X is *locally closed* if it is the intersection of an open set and a closed set of X. We say that E is a *Deitmar constructible set* or $\mathbb{F}_1$-*constructible set* if it is the finite union of locally closed sets.

Affine spaces and projective spaces over $\mathbb{F}_1$ are Deitmar constructible sets. Notice as well that the propositions mentioned in subsection 1.3.2 are also satisfied by Deitmar constructible sets.

1.4.2 Congruence schemes

A new category of schemes called *congruence schemes* was introduced by Deitmar in [11]. This category contains both the category of monoidal schemes, or schemes over $\mathbb{F}_1$, and the category of Grothendieck schemes (or schemes over commutative rings). The definition of congruence schemes, as in the previous versions of $\mathbb{F}_1$-schemes, follows also a similar guideline as the one of classical schemes, but using the category of *sesquiads.* We refer to [11] for more details about congruence schemes.

Definition 1.4.18. An *addition* or a $+$-*structure* on a monoid A is a family $(D_k, \sum_k)_{k\in\mathbb{Z}^n}$, where $D_k \subset A^n$, $n \geq 2$ and $\sum_k : D_k \to A$ is a map such that there exists an injective morphism $\varphi : A \hookrightarrow R$ to the multiplicative monoid of a ring R which satisfies $\varphi(0) = 0$ and $\varphi(\sum_k(a)) = \sum_{j=1}^n k_j\varphi(a_j)$ for every $a = (a_1 \ldots, a_n) \in D_k$, with $k = (k_1, \ldots, k_n) \in \mathbb{Z}^n$. We further insist that D_k is maximal in the sense

$$D_k = \Big\{a \in A^n \mid \sum_{j=1}^n k_j\varphi(a_j) \in \mathrm{Im}(\varphi)\Big\}.$$

This implies that the addition is associative and distributive when defined and respects zero, i.e., $a + 0 = a$ holds for every $a \in A$. It further implies that the addition is cancellative.

Definition 1.4.19. A *sesquiad* is a monoid A together with an addition $(D, \sum)$. By a *sesquiad morphism* φ we mean a morphism of monoids $A \to B$ such that $\varphi(\sum_k(a)) = \sum_k(\varphi(a))$ for all $k \in \mathbb{Z}^n$, $a \in D_k$.

Remark 1.4.20. A ring is a sesquiad and a monoid endowed with a *trivial addition*, where one can only add zero to any element, is also a sesquiad. Besides, every monoid morphism becomes a sesquiad morphism with the trivial addition. Hence, the category of sesquiads contains the category of rings and monoids as full subcategories.

In previous constructions, we use the set of prime ideals of a ring or a monoid to construct a topological space so that we could define a scheme. In the sense of congruence schemes, one does not use ideals but "congruences" on sesquiads. A *congruence* on a sesquiad A is an equivalence relation $\mathcal{C} \subseteq A \times A$ such that there is a sesquiad structure on $A/\mathcal{C}$ making the projection $A \to A/\mathcal{C}$ a morphism of sesquiads. This condition implies that $x \sim_{\mathcal{C}} y \Rightarrow xz \sim_{\mathcal{C}} yz$ for all $x, y, z \in A$ and $x + z \sim_{\mathcal{C}} y + z$ if $x + z$ is defined. We use the notation $x \sim_{\mathcal{C}} y$ to denote that x and y are equivalent with respect to the congruence $\mathcal{C}$.

We say that a sesquiad is *integral* if $1 \neq 0$ and

$$af = bf \Rightarrow (a = b \text{ or } f = 0).$$

If $A/\mathcal{C}$ is integral, the congruence $\mathcal{C}$ is called *prime*. This is equivalent to saying that $1 \not\sim_{\mathcal{C}} 0$ and

$$(af, bf) \in \mathcal{C} \Leftrightarrow (a, b) \in \mathcal{C} \text{ or } (f, 0) \in \mathcal{C}.$$

Deitmar proved in [11] that every congruence $\mathcal{C} \neq A \times A$ is contained in a prime congruence. This allows to define the $\mathtt{Spec}_c(A)$ as the set of all prime congruences on the sesquiad A with the topology generated by all sets of the form

$$D(a, b) = \{\mathcal{C} \in \mathtt{Spec}_c(A) \mid (a, b) \notin \mathcal{C}\}, \qquad a, b \in A.$$

In a similar way as for monoids, one can define a *structure sheaf* $\mathscr{O}_A$ *of sesquiads* for a given $\mathtt{Spec}_c(A)$, with A a sesquiad. A *sesquiaded space* is then a topological space X together with a sheaf $\mathscr{O}_X$ of sesquiads and a *morphism of sesquiaded spaces* $(X, \mathscr{O}_x) \to (Y, \mathscr{O}_y)$ is a pair $(f, f^\#)$, where $f : X \to Y$ is a continuous map and $f^\# : \mathscr{O}_Y \to f_*\mathscr{O}_X$ is a morphism of sheaves on Y. Such a morphism $(f, f^\#)$ is called *local* if for each $x \in X$, the induced morphism $f^\#_x : \mathscr{O}_{Y,f(x)} \to f_*\mathscr{O}_{X,x}$ is local, i.e., satisfies

$$(f^\#)^{-1} : (\mathscr{O}^\times_{X,x}) = \mathscr{O}^\times_{Y,f(x)}.$$

We saw in Propositions 1.3.11 and 1.4.7 that there is a functorial bijection between the morphisms of locally ringed spaces (local morphisms of monoidal spaces, respectively) and morphisms of rings (monoids, respectively). However, this relation is not bijective in the case of sesquiaded spaces.

Theorem 1.4.21 ([11]). a) *Let A be a sesquiad and let $\mathscr{O}_A$ be its structure sheaf. Then the pair $(\mathtt{Spec}_c(A), \mathscr{O}_A)$ is a sesquiaded space.*

b) *Let A, B be sesquiads. If $\varphi : A \to B$ is a morphism of sesquiads, then φ induces a local morphism of sesquiaded spaces*

$$(f, f^\#) : (\mathtt{Spec}_c(B), \mathscr{O}_B) \longrightarrow (\mathtt{Spec}_c(A), \mathscr{O}_A),$$

thus giving a functorial map

$$L : \mathtt{Hom}(A, B) \longrightarrow \mathtt{Hom}(\mathtt{Spec}_c(B), \mathtt{Spec}_c(A)),$$

where on the right-hand side one only admits local morphisms.

The main reason why the latter map L is not bijective lies in the fact that for a given sesquiad A, the global sections $\mathscr{O}_A(\mathtt{Spec}_c(A))$ contain A as a subsesquiad ([11]) while in the case of rings or monoids this containment is indeed an equality. We can now proceed to define the concept of a "congruence scheme."

Definition 1.4.22. An *affine congruence scheme* is a sesquiaded space that is of the form $(\mathtt{Spec}_c(A), \mathcal{O}_A)$, for A a sesquiad and $\mathcal{O}_A$ its corresponding structure sheaf. A *congruence scheme* is a sesquiaded space X which locally looks like an affine congruence scheme.

The $\mathtt{Proj}_c$-construction

Consider the monoid $\mathbb{F}_1[X_0, X_1, \dots, X_m]$, where $m \in \mathbb{N}$ and see it as a sesquiad together with the *trivial addition*. Since any polynomial is homogeneous in this sesquiad, we have a natural grading

$$\mathbb{F}_1[X_0, \dots, X_m] = \bigoplus_{i \geq 0} R_i = \coprod_{i \geq 0} R_i,$$

where R_i consists of the elements of $\mathbb{F}_1[X_0, \dots, X_m]$ of total degree i, for $i \in \mathbb{N}$. We define the *irrelevant congruence* as

$$\mathrm{Irr}_c = \langle X_0 \sim 0, \dots, X_m \sim 0 \rangle.$$

Now we can proceed with the usual $\mathtt{Proj}$-construction of projective schemes. We define $\mathtt{Proj}_c(\mathbb{F}_1[X_0, \dots, X_m])$ as the set of prime congruences of the sesquiad $\mathbb{F}_1[X_0, \dots, X_m]$ which do not contain Irr_c. The closed sets of the topology on this set are defined as usual: for any (a, b) pair of elements of $\mathbb{F}_1[X_0, \dots, X_m]$, we define

$$V(a, b) := \{\mathcal{C} \mid \mathcal{C} \in \mathtt{Proj}_c(\mathbb{F}_1[X_0, \dots, X_m]), \quad a \sim_{\mathcal{C}} b\},$$

and these sets form a basis for the closed set topology. Defining the structure sheaf as in [11], one obtains that $\mathtt{Proj}_c(\mathbb{F}_1[X_0, \dots, X_m])$ is a *projective* congruence scheme. Its closed points naturally correspond to the $\mathbb{F}_2$-rational points of the projective space $\mathbb{P}^m_{\mathbb{F}_2}$ (but the latter has a finer subspace structure, and also a different algebraic structure).

1.4.3 The multiplicative group $\mathbb{G}_m$

Let G be the abelian group defined on one generator X and put $A = \mathbb{F}_1[G] := \{0\} \cup G$. Then $A := \mathbb{F}_1[X, X^{-1}]$. Consider k a field (or $\mathbb{Z}$); then we define the *multiplicative group* $\mathbb{G}_m$ *over the field* k as $\texttt{Spec}(k[X, X^{-1}])$ (or $\texttt{Spec}(\mathbb{Z}[X, X^{-1}])$ over $\mathbb{Z}$).

In the case of $\mathbb{F}_1$, and seeing A as a monoid, we define *the multiplicative group over* $\mathbb{F}_1$ as the Deitmar affine scheme $\texttt{Spec}(A) = \texttt{Spec}(\mathbb{F}_1[X, X^{-1}])$. Finally, if we consider A as a sesquiad with an addition structure, then one defines the "congruence" multiplicative group as the congruence affine scheme $\texttt{Spec}_c(A)$.

1.4.4 Blueprints

Another remarkable scheme theory over the field with one element was introduced by Lorscheid. He develops the theory of blueprints and blue schemes in [33], [32], [30] (with López Peña) as well as in [34] and [31]. We will follow his review in [35] to define blue schemes.

Definition 1.4.23. A *blueprint* B is a monoid with zero A together with a *pre-addition* $\mathcal{R}$, i.e., $\mathcal{R}$ is an equivalence relation on the semiring

$$\mathbb{N}[A] = \{\sum a_i \mid a_i \in A\}$$

of finite formal sums of elements of A that satisfies the following axioms (where we write $\sum a_i \equiv \sum b_j$) whenever $(\sum a_i, \sum b_j \in \mathcal{R})$:

1) The relation $\mathcal{R}$ is additive and multiplicative, namely if $\sum a_i \equiv \sum b_j$ and $\sum c_k \equiv \sum d_l$, then $\sum a_i + \sum c_k \equiv \sum b_j + \sum d_l$ and $\sum a_i c_k \equiv \sum b_j d_l$.

2) The absorbing element 0 of A is in relation with the zero of $\mathbb{N}[A]$, i.e., $0 \equiv$ (empty sum).

3) If $a \equiv b$, then $a = b$ (as elements in A).

A *morphism of blueprints* $f : B_1 \to B_2$ is a multiplicative map $f : A_1 \to A_2$ between the underlying monoids of B_1 and B_2 with $f(0) = 0$ and $f(1) = 1$ such that for every relation $\sum a_i \equiv \sum b_j$ in the pre-addition $\mathcal{R}_1$ of B_1, the pre-addition $\mathcal{R}_2$ of B_2 contains the relation

$$\sum f(a_i) \equiv \sum f(b_j).$$

Let $\texttt{Blpr}$ be the category of blueprints.

The concept of blueprints generalizes both rings and monoids. Indeed, there are full embeddings from the category of monoids and the category of rings into the category $\texttt{Blpr}$ of blueprints. The definition of a *blue scheme* follows the same guideline as schemes defined over commutative rings.

Definition 1.4.24. A *blueprinted space* is a topological space X together with a sheaf $\mathscr{O}_X$ in Blpr. A *morphism of blueprinted spaces* is a continuous map together with a sheaf morphism that induces morphisms between the stalks $\mathscr{O}_{X,x}$ at points $x \in X$. A *locally blueprinted space* is a blueprinted space whose stalks $\mathscr{O}_{X,x}$ are local blueprints and a *local morphism* between locally blueprinted spaces is a morphism of blueprinted spaces that induces local morphisms of blueprints between all stalks, where local blueprints and local morphisms are defined in the usual way.

The *spectrum of a blueprint* B is defined analogously as in the case of rings. We denote it by $\mathtt{Spec}(B)$ and it is a locally blueprinted space. A *blue scheme* is a locally blueprinted space that is locally isomorphic to the spectra of blueprints. For more details and examples, see [33].

Since the definition of a blue scheme is formally the same as the definition of a usual scheme, the full embedding of the category of rings (monoids) into the category of blueprints defines as well a full embedding of the category of schemes over rings (Deitmar schemes) into the category of blue schemes, which implies that blue schemes are a generalization of both monoidal and usual schemes.

2 The Functor $\mathcal{F}_k$

In this chapter we will study the relation between Combinatorics and Algebraic Geometry over the field with one element through the definition of a functor $\mathcal{F}$. We will take this relation a step further and extend it to scheme theory over any finite field and over $\mathbb{Z}$ using base extensions of $\mathbb{F}_1$.

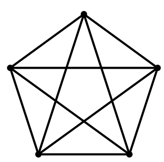
2.1 Combinatorial realization of $\mathbb{F}_1$

Let us recall the construction of a projective space over $\mathbb{F}_1$ done in chapter 1. After considering the limit of $\mathbf{PG}(n,q)$ when $q \to 1$, the axioms for a projective space over $\mathbb{F}_1$ are as follows:

- Two different points are in exactly one line.
- Each line has exactly 2 points.
- *Veblen's axiom — $\mathbb{F}_1$-version.* Empty.

We also saw in chapter 1 that the total number of points for an n-dimensional projective space $\mathbf{PG}(n,1)$ over $\mathbb{F}_1$ is $n+1$ and that its combinatorial representation is the complete graph on $n+1$ vertices, K_{n+1}.

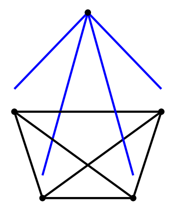
Let us consider now, for instance $\mathbf{PG}(4,1)$; then its combinatorial representation will be the graph K_5:



But, if we consider the decomposition of $\mathbf{PG}(4,1)$ as it would happen over finite fields, one could write

$$\mathbf{PG}(4,1) = \mathbf{AG}(4,1) \coprod \mathbf{PG}(3,1) \tag{2.1}$$

and, again, $\mathbf{PG}(3,1)$ would be represented by the complete graph K_4. Hence, in terms of graphs, this decomposition could be depicted as



where the blue part corresponds to the combinatorial representation of an affine space of dimension 4 over $\mathbb{F}_1$. Nevertheless, this combinatorial object is not a graph since no edges have two vertices, so if we want to express affine spaces over $\mathbb{F}_1$ in a combinatorial way, we need to introduce a new category of combinatorial objects that includes such an object.

Besides, considering the Algebraic Geometry side of $\mathbb{F}_1$ and the language of schemes, one easily sees that there is a relation between complete graphs and schemes of the form $\texttt{Proj}(\mathbb{F}_1[X_0, \ldots, X_m])$, and between a new object and affine schemes of the form $\texttt{Spec}(\mathbb{F}_1[X_1, , \ldots, X_m])$.

The motivation of exploring this relation and extending it to a larger category of schemes or related objects leads us to the definition of a functor between the category of generalized versions of graphs (that we will call *loose graphs*) and the category of constructible sets over $\mathbb{F}_1$ "in the sense of Deitmar."

2.2 Loose graphs and the functor $\mathcal{S}$

We have seen in the previous section that there is a combinatorial correspondence between the projective and affine spaces over $\mathbb{F}_1$ and some objects that we previously called loose graphs. In [45], the author introduces a functor $\mathcal{S}$ that associates a Deitmar scheme over $\mathbb{F}_1$ to any loose graph. Generalizing this correspondence is one of the main motivations for the work in this PhD. We will start by introducing the category of loose graphs before describing how the functor $\mathcal{S}$ works.

2.2.1 Loose graphs

A loose graph is a combinatorial object that generalizes the notion of a graph.

Definition 2.2.1. A loose graph Γ is a point-line geometry $\Gamma = (V, E, \mathbf{I})$, where V is a set of *vertices*, E is a set of *edges* and $\mathbf{I}$ is a symmetric relation on $V \cup E$, disjoint from $V \times V$ and $E \times E$, which indicates when a vertex and an edge are incident, with the property that *each edge is incident with at most two distinct vertices.* We call the edges having less than two vertices *loose edges* and the edges with two vertices *proper edges.*

As a matter of convenience, we will assume that for any loose graph the *empty edge*, denoted by $e_\emptyset$, is an element of the edge set that has no incidence relation with any vertex and which is different from the loose graph consisting of one edge with no incident vertices. For examples of loose graphs, see figure 2.1.

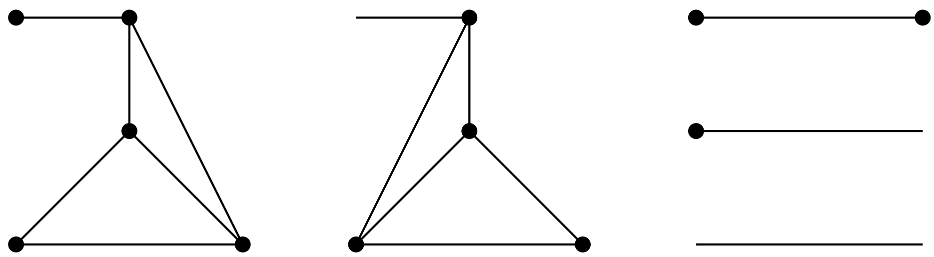


Figure 2.1: Different examples of loose graphs.

Definition 2.2.2. Let Γ be a loose graph. We call the *minimal graph of* Γ, and we denote it by $\overline{\Gamma}$, to be the minimal graph containing Γ as a loose subgraph, i.e., the graph obtained by adding extra vertices to every loose edge different from the empty edge so as to obtain proper edges (see an example in figure 2.2). The *reduced graph of* Γ, denoted as $\widetilde{\Gamma}$, is the subgraph of Γ obtained after deleting all loose edges.

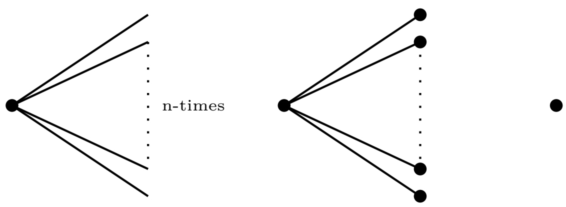


Figure 2.2: A loose graph Γ, its minimal graph $\overline{\Gamma}$ and its reduced graph $\widetilde{\Gamma}$.

Remark 2.2.3. Notice that the definition of loose graphs relaxes the definition of graphs, in that an edge can now also have one, or even no, point(s). In fact, the category of graphs can be embedded in the category of loose graphs by adding the empty edge to the set of edges for each graph. We say that a loose graph is *connected* if its minimal graph is connected. For the purpose of this work we do not allow loops, the geometry is always undirected and we consider connected and simple loose graphs.

Definition 2.2.4. Let $\Gamma = (V, E, \mathbf{I})$ be a loose graph. A *loose subgraph* of Γ is a loose graph $\Gamma' = (V', E', \mathbf{I}')$, where $V' \subseteq V$ and $E' \subseteq E$ with the property that if a vertex $v \in V'$ and an edge $e \in E'$ are incident in Γ, they are also incident in Γ'. We say that a loose subgraph Γ' of Γ is *proper* (or that the inclusion $\Gamma' \subset \Gamma$ is *strict*) if there exists at least one edge or vertex of Γ that is not in Γ'.

Definition 2.2.5. Let $\Gamma_1 = (V_1, E_1, \mathbf{I}_1)$ and $\Gamma_2 = (V_2, E_2, \mathbf{I}_2)$ be two loose graphs. We say that a map $f : \Gamma_1 \to \Gamma_2$ is a *morphism of loose graphs* from Γ_1 to Γ_2 if:

i) f sends vertices to vertices and edges to edges, i.e., $f|_{V_1} : V_1 \to V_2$ and $f|_{E_1} : E_1 \to E_2$.

ii) $f(e_{1,\emptyset}) = e_{2,\emptyset}$.

iii) If $e \in E$ is a proper edge, $f(e) = e_{2,\emptyset}$ if and only if the two vertices incident with e in Γ_1 have the same image by f.

iv) If a vertex $v \in V_1$ is incident with an edge $e \in E_1$, then $f(v)$ and $f(e)$ are also incident if $f(e) \neq e_{2,\emptyset}$.

A morphism $f : \Gamma_1 \to \Gamma_2$ of loose graphs is an *isomorphism of loose graphs* if

a) f is bijective in both the set of vertices and the set of edges.

b) A vertex v is incident with an edge e in Γ_1 if and only if $f(v)$ is incident with $f(e)$ in Γ_2.

This definition implies that f is also bijective in the set of loose edges and in the set of proper edges. An *automorphism* of a loose graph Γ is then an isomorphism from Γ to itself.

Example 2.2.6. Morphisms of loose graphs

1. If we consider the two loose graphs of figure 2.3, the morphism ι sending the vertex v to w and each loose edge to a proper edge is a morphism of loose graphs. We will call ι *an inclusion morphism of loose graphs*, since the graph on the left is a loose subgraph of the one on the right.

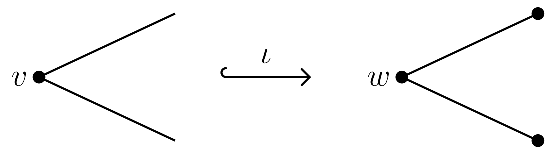


Figure 2.3: Inclusion morphism of loose graphs

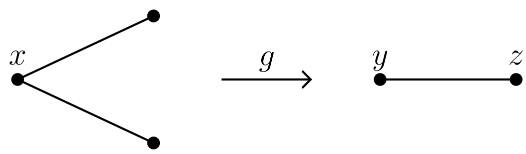


Figure 2.4: Morphism of graph

2. A morphism of graphs is also a morphism of loose graphs. Suppose we have the graph on 3 vertices: 1 of degree 2, called x, and the other two of degree 1; and the complete graph K_2 on two vertices called y and z (see figure 2.4). Then the morphism g sending x to y and the other two vertices to z is also a morphism of loose graphs.

3. Contractions are allowed. For instance, the morphism h (figure 2.5) sending the two vertices of the complete graph K_2 to the vertex of K_1 is a morphism of loose graphs. In this case, h sends the edge of K_2 to the empty edge of K_1.

Figure 2.5: Contraction of loose graphs

Proposition 2.2.7. *Morphisms of loose graphs are stable under compositions.*

Proof. Let Γ_1, Γ_2 and Γ_3 be three loose graphs and consider two morphisms $f : \Gamma_1 \to \Gamma_2$ and $g : \Gamma_2 \to \Gamma_3$ of loose graphs. It is easy to check that the loose graph morphism $g \circ f$ satisfies the conditions i), ii) and iv) from definition 2.2.5. We will prove that it also satisfies iii).

Suppose that there is a proper edge e of Γ_1 incident with vertices v_1 and v_2 such that $g(f(e)) = e_{3,\emptyset}$. Then, there are two possibilities:

- $f(e) = e_{2,\emptyset}$ and then, since f is a morphism of loose graphs, $f(v_1) = f(v_2)$. So, $g(f(v_1)) = g(f(v_2))$.
- $f(e) \neq e_{2,\emptyset}$. In this case, using condition iii) in g, the two vertices incident with $f(e)$ in Γ_2 have the same image by g. It is easy to check that the vertices incident with $f(e)$ are indeed $f(v_1)$ and $f(v_2)$, so $g(f(v_1)) = g(f(v_2))$.

Let us now prove the other direction. Suppose $g(f(v_1)) = g(f(v_2))$; then we have again two possibilities:

- $f(v_1) = f(v_2)$, in which case $f(e) = e_{2,\emptyset}$ and then $g(f(e)) = e_{3,\emptyset}$.
- $f(v_1) \neq f(v_2)$. But then, we have that $f(e)$ is a proper edge of Γ_2 and the incident vertices have the same image by g, so $g(f(e)) = e_{3,\emptyset}$. ∎

We define the *category of loose graphs*, denoted as `LGraph`, to be the category whose objects are loose graphs and whose morphisms are morphisms of loose graphs. A very important object in this category is the *loose star*.

Definition 2.2.8 (Loose star). A loose star S_n is a loose graph consisting of a single vertex together with n edges incident with it. If Γ is a loose graph and v a vertex of Γ, we will denote by S_v the loose star formed by v and all the edges of Γ incident with it. See figure 2.2 for an example of a loose star.

One of the main theorems that allows the definition of both the functor $\mathcal{S}$ and its new version $\mathcal{F}$ is the embedding theorem (see [45] for more details).

Theorem 2.2.9 (Embedding theorem). *Let Γ be a loose graph and $\overline{\Gamma}$ its minimal graph. Then Γ can be seen as a loose subgraph of the combinatorial projective-space $\mathbf{P}_c(\Gamma)$, i.e, the projective $\mathbb{F}_1$-space defined by the complete graph on the set of vertices of $\overline{\Gamma}$.*

In the next sections we will use the notation $\mathbf{P}(\Gamma)$ for the projective space, as scheme, associated to $\mathbf{P}_c(\Gamma)$. As a quick remark, remember that $\mathbf{P}(\Gamma)$ is the projective scheme $\texttt{Proj}(\mathbb{F}_1[X_i]_{i\in V})$, where $V = \{0, \dots, m\}$ and $m+1$ is the number of vertices of $\overline{\Gamma}$.

2.2.2 The functor $\mathcal{S}$

Let us describe how the functor $\mathcal{S}$ works. The reader can find more details about it in [45]. Let Γ be a not necessarily finite graph, consider $\mathbf{P} = \mathbf{P}_c(\Gamma)$ and note that, since Γ is a graph, $\mathbf{P} \setminus \Gamma$ is just a set S of edges. Let μ be arbitrary in S, and let z be one of the two (closed) points on μ in $\mathbf{P}(\Gamma) = \texttt{Proj}(\mathbb{F}_1[X_i]_{i\in V})$). Suppose that in the projective space $\mathbf{P}(\Gamma)$, z is defined by the ideal generated by the polynomials

$$X_i, \quad i \in V, i \neq j = j(z). \tag{2.2}$$

Let $\mathbf{P}(z)$ be the complement in $\mathbf{P}$ of z; it is a hyperplane defined by $X_j = 0$ (and it forms a complete graph on all the points but z). Denote the corresponding closed subset of $\texttt{Proj}(\mathbb{F}_1[X_i]_{i\in V})$ by $C(z)$. Let $z' \neq z$ be the other point of the edge μ corresponding to the index $j' = j(z') \in V$. Define the subset $\mathbf{P}(z') = \mathbf{P} \setminus \{z'\}$ of V, and denote the corresponding closed subset by $C(z')$. Finally, define

$$C(\mu) = C(z) \cup C(z'). \tag{2.3}$$

It is also closed in $\texttt{Proj}(\mathbb{F}_1[X_i]_{i\in V})$, and the corresponding closed subscheme is the projective space $\mathbf{P}(\Gamma)$ "without the edge μ;" the projective coordinate ring is $\mathbb{F}_1[X_i]_{i\in V}/I_\mu$ (where $(X_jX_l) =: I_\mu$) and its scheme is the `Proj`-scheme defined by this ring. Now introduce the closed subset

$$C(\Gamma) = \bigcap_{\mu \in S} C(\mu). \tag{2.4}$$

Then $C(\Gamma)$ defines a closed subscheme $S(\Gamma)$ which corresponds to the graph Γ. We have

$$S(\Gamma) = \mathtt{Proj}(\mathbb{F}_1[X_i]_{i\in V} / \bigcup_{\mu\in S} I_\mu). \tag{2.5}$$

Remark 2.2.10 (Edges and relations)**.** In this presentation, an edge corresponds to a relation, and we construct a coordinate ring for $S(\Gamma)$ by deleting all relations of the ambient space $\mathbf{P}(\Gamma)$ which are defined by edges in the complement of Γ.

A similar construction can be done for loose graphs, [44].

Some properties of the functor $\mathcal{S}$

In [45], it is shown that certain properties of Deitmar schemes arising from loose graphs can be easily verified on the loose graphs. We mention some results without the proofs, which can be found in [45].

The next theorem shows that the automorphism group of projective spaces from the incidence geometrical point of view, which we denote by $\mathrm{Aut}_{\mathrm{synth}}(.)$, coincides with the automorphism group from the point of view of $\mathbb{F}_1$-schemes, denoted $\mathrm{Aut}_{\mathrm{sch}}(.)$.

Remark 2.2.11. Remember that at the level of $\mathbb{F}_1$-rings, the possible automorphisms of schemes are the ones given by permutations on the indices i, there are no translations or any other kind of automorphisms.

Theorem 2.2.12 ([45])**.** *Let $\mathbf{P}_c$ be a combinatorial projective space over $\mathbb{F}_1$, i.e., a complete graph K_m for some $m \in \mathbb{N}$; and let $\mathtt{Proj}(\mathbb{F}_1[X_i]_{i\in V})$ be the corresponding projective scheme. Then we have*

$$\mathrm{Aut}_{\mathrm{synth}}(\mathbf{P}_c) \cong \mathrm{Aut}_{\mathrm{sch}}(\mathtt{Proj}(\mathbb{F}_1[X_i]_{i\in V})). \tag{2.6}$$

A similar proof (considering the action on the ideals that correspond to the "directions" instead of the closed points) leads to the same theorem for affine spaces.

Corollary 2.2.13 ([45])**.** *Each group H is the full automorphism group of some Deitmar scheme.*

Theorem 2.2.14 ([45])**.** *For any element $\Gamma \in$ `LGraph`, we have that*

$$\mathrm{Aut}_{\mathrm{synth}}(\Gamma) \cong \mathrm{Aut}_{\mathrm{sch}}(S(\Gamma)). \tag{2.7}$$

Theorem 2.2.15 ([45])**.** *A loose scheme $S(\Gamma)$ is connected if and only if the loose graph Γ is connected.*

2.3 Modifying the functor $\mathcal{S}$

In this section, we introduce a new map $\mathcal{F}$ which, similarly as $\mathcal{S}$, associates Deitmar schemes and Deitmar constructible sets to loose graphs. As a main feature we want that the degree of a vertex reflects the local dimension of the constructible set "at a close point," i.e., each point will locally look like an affine space of dimension the degree of the associated vertex. As we can easily notice, this map $\mathcal{F}$ differs already from $\mathcal{S}$. It will have some common properties as well; it will for instance associate a projective space to a complete graph. A loose edge without vertices will also be associated to a multiplicative group $\mathbb{G}_m$ for both $\mathcal{S}$ and $\mathcal{F}$, but $\mathcal{S}$ is constructed by deleting several multiplicative groups from the ambient projective space, so the property described above is not satisfied by $\mathcal{S}$.

We will first describe some examples to make the idea of $\mathcal{F}$ clear enough.

Example 2.3.1. A projective plane without a multiplicative group

Figure 2.6: The inclusion $A \hookrightarrow B$.

Let B be the complete graph on three vertices minus one edge (see figure 2.6): by the embedding theorem, it embeds in a combinatorial projective 2-space $\mathbf{P}_c(B)$. Call the vertex of degree two x, and call the others y and z. The graph B contains the loose graph A, which should correspond to the affine plane $\texttt{Spec}(\mathbb{F}_1[X_1, X_2])$ (x has degree two). On the level of Deitmar constructible sets what we would like to have is $\mathcal{F}(B)$ to be the projective plane over $\mathbb{F}_1$ minus the multiplicative group (since we delete a projective line minus two points). Besides, we also want that since $A \subset B$ is a strict inclusion of loose graphs, $\mathcal{F}(A)$ also is a proper constructible subset of $\mathcal{F}(B)$. In fact, we will introduce this property as one of our axioms to define the modified map:

CV If $\Gamma \subset \widetilde{\Gamma}$ is a strict inclusion of loose graphs, $\mathcal{F}(\Gamma)$ also is a proper constructible subset of $\mathcal{F}(\widetilde{\Gamma})$.

L-D If x is a vertex of degree $m \in \mathbb{N}^\times$ in Γ, then there is an affine space of dimension m containing x and contained in $\mathcal{F}(\Gamma)$.

Let us remark that by "proper constructible subset" in the (COV) property we mean proper in the set theorical point of view. Now consider an open cover B_1, B_2, B_3 of $\mathcal{F}(B)$, where B_1 contains y and is defined as the affine line defined by xz without the point x, B_2 contains x and is defined by taking out the closed points y and z, and B_3 contains z and is similarly defined as B_1. With the same indices, we obtain affine schemes $\texttt{Spec}(A_i)$ which in case of $i \in \{1,3\}$ it corresponds to the absolute line and in case $i = 2$ to the affine plane over $\mathbb{F}_1$.

As in [45], we also want the following two properties to hold:

C any vertex of a loose graph Γ defines a closed point of $\mathcal{F}(\Gamma)$, and vice versa;

PL any edge of Γ which contains two distinct vertices corresponds to a closed subprojective line of $\mathcal{F}(\Gamma)$.

Both are in fact instances of the following more general property:

CO If K_m is a complete subgraph on m vertices in Γ, then $\mathcal{F}(K_m)$ is a closed subprojective space of dimension $m-1$ in $\mathcal{F}(\Gamma)$.

Example 2.3.2. Open and closed sets — Caution!

Let Γ be any finite loose graph, and let L and L' be two different complete subgraphs on two vertices. After passing to $\mathcal{F}(\Gamma)$, we obtain different closed projective lines $\mathcal{F}(L)$ and $\mathcal{F}(L')$. Nevertheless the subgraph of Γ defined by the union $L \cup L'$ does not necessarily define a closed subset of $\mathcal{F}(\Gamma)$ (even if it looks rather closed in the loose graph). In the graph B of the previous example the union defines a proper constructible set. (We will see the proof of this fact in section 2.4.2.) The reason of this possible confusion is that although, for instance in the aforementioned example, $L \cup L'$ looks like the union of two projective lines meeting in a point, it is in fact a projective plane minus a multiplicative group. Unfortunately, we only see this after the scheme has acquired enough flesh — that is, after a base change to "real" fields, cf. Remark 2.4.5. It is essentially a corollary of Axiom (COV): the union $L \cup L'$ (with L, L' still meeting in a point) contains the loose graph of the $\mathbb{F}_1$-affine plane, so it must define something 2-dimensional.

On the other hand, if L and L' would not meet in some loose graph Γ, then the union *does* define a closed set (by simply multiplying equations). In general, the same holds for a finite number of subprojective lines in general position.

Similarly, one has to be careful with finite unions of complete subgraphs. But finite unions of vertices always define closed sets — we will call this property "FUCP" for further reference.

2.3.1 The new functor $\mathcal{F}$

In conclusion, the map $\mathcal{F}$ must obey the following set of rules:

CV If $\Gamma \subset \widetilde{\Gamma}$ is a strict inclusion of loose graphs, $\mathcal{F}(\Gamma)$ also is a proper constructible subset of $\mathcal{F}(\widetilde{\Gamma})$.

L-D If x is a vertex of degree $m \in \mathbb{N}^\times$ in Γ, then there is an affine space of dimension m containing x and contained in $\mathcal{F}(\Gamma)$.

FIN The constructible set $\mathcal{F}(\Gamma)$ equals the union of the affine spaces described in the previous rule.

CO If K_m is a complete subgraph on m vertices in Γ, then $\mathcal{F}(K_m)$ is a closed subprojective space of dimension $m-1$ in $\mathcal{F}(\Gamma)$.

MG An edge without vertices should correspond to a multiplicative group.

There is now a transparent way to define $\mathcal{F}$, as follows:

(F1) For any loose star S_n, $\mathcal{F}(S_n)$ is the affine $\mathbb{F}_1$-space of dimension n.

(F2) Let Γ be any connected loose graph, and let $\overline{\Gamma}$ be its minimal graph. Say that $\overline{\Gamma}$ has $m+1$ vertices. Let $\mathbf{P}(\overline{\Gamma})$ be the projective $\mathbb{F}_1$-space of dimension m defined by these vertices; then $\mathcal{F}(\Gamma)$ is the union in $\mathbf{P}(\overline{\Gamma})$ of the affine $\mathbb{F}_1$-spaces defined by the stars which are defined by each vertex, without the closed points which correspond to the vertices which were added to obtain $\overline{\Gamma}$.

Remark 2.3.3. If we choose homogeneous coordinates (in $\mathbf{P}(\overline{\Gamma})$) such that each such vertex has as coordinates a vector in $\{0,1\}^{m+1}$ with precisely one nonzero entry, $\mathcal{F}(\Gamma)$ can be described explicitly analytically. The disconnected case is easily derived from the connected case.

2.3.2 $\mathcal{F}(\Gamma)$ seen as a congruence scheme

Although we defined $\mathcal{F}$ as a functor between the categories of loose graphs and Deitmar constructible sets, some schemes might not be defined correctly in the latter category. To solve this problem, one can consider the schemes among these constructible sets inside the category of congruence schemes, since monoids are a full subcategory of sesquiads (cf. section 1.4.2).

For instance, in the classical scheme theory over a field k, the multiplicative group $\mathbb{G}_m$ is isomorphic to the affine scheme $\texttt{Spec}(k[X,Y]/(XY-1))$ which has $k[X,Y]/(XY-1)$ as its coordinate ring. As a scheme over $\mathbb{F}_1$, seeing that $\mathbb{F}_1$-rings only have one operation, the equation $XY-1=0$ "only" acquires some meaning if the schemes are considered in the category of congruence schemes. Hence, we define the multiplicative group $\mathbb{G}_m$ over $\mathbb{F}_1$ to be isomorphic to the scheme $\texttt{Spec}(\mathbb{F}_1[X,Y]/(XY \sim 1))$.

For the next chapters, all the constructions and proofs regarding the functor $\mathcal{F}$ will be made in the category of Deitmar constructible sets. If needed, the use of congruence schemes will be explicitly mentioned.

2.3.3 Gluing the affine schemes?

Let Γ be a loose graph and $\mathcal{F}(\Gamma)$ be the Deitmar constructible set associated to it. By definition of the functor $\mathcal{F}$ and the embedding theorem (see Theorem 2.2.9), for every vertex of Γ we have an affine scheme over $\mathbb{F}_1$ defined from the loose star corresponding to the said vertex. Let us call $v_1, \ldots, v_k$ the vertices of Γ and $\mathtt{Spec}(A_i)$ the affine schemes associated to v_i, $1 \leq i \leq k$. We will now study the intersection of these affine schemes.

Lemma 2.3.4. *For all* $1 \leq r, s, \leq k$, $\mathtt{Spec}(A_r) \cap \mathtt{Spec}(A_s) \neq \emptyset$ *if and only if* v_r *and* v_s *are adjacent vertices.*

Proof. Suppose that v_r and v_s are adjacent and let e be the edge having these vertices as end points. Then, e belongs to the loose stars associated to v_r and v_s and is used to define their corresponding schemes $\mathtt{Spec}(A_r)$ and $\mathtt{Spec}(A_s)$. Thus, due to the property CV of the functor $\mathcal{F}$, the subscheme defined by e is contained in the intersection.

The converse is similar. Suppose v_r and v_s are not adjacent. Then their respective loose stars do not have any edge in common. But the edges of the loose stars correspond bijectively to the elements of some bases of the affine schemes $\mathtt{Spec}(A_r)$ and $\mathtt{Spec}(A_s)$, which implies that there are no relations between the bases of both schemes. Thus their intersection must be empty. ■

This lemma implies that the intersections in the constructible set $\mathcal{F}(\Gamma)$ of the subschemes corresponding to loose stars coming from vertices of Γ are completely determined by the graph Γ, happening locally on the affine schemes corresponding to adjacent vertices. Let us now describe how these affine schemes are (not) "glued" together.

Consider first the case where Γ is a tree and take two adjacent vertices v_r and v_s in Γ, with degrees r and s respectively. Let $\mathtt{Spec}(A_r)$ and $\mathtt{Spec}(A_s)$ be their corresponding affine schemes and e the edge joining both vertices. Each of these two schemes is isomorphic to an affine space (see definition of $\mathcal{F}$ in the previous subsection); therefore we can write $\mathtt{Spec}(A_r) \simeq \mathtt{Spec}(\mathbb{F}_1[X_1, \ldots, X_r])$ and $\mathtt{Spec}(A_s) \simeq \mathtt{Spec}(\mathbb{F}_1[Y_1, \ldots, Y_s])$. Since the "gluing" happens only at the level of the two vertices and the common edge, we can restrict ourselves to the schemes $\mathcal{F}(\{v_r\}, \{e\})$ embedded in $\mathtt{Spec}(A_r)$ and $\mathcal{F}(\{v_s\}, \{e\})$ embedded in $\mathtt{Spec}(A_s)$, both schemes being isomorphic to the affine line over $\mathbb{F}_1$.

Represent by X the scheme $\mathcal{F}(\{v_r\}, \{e\})$ and by Y the scheme $\mathcal{F}(\{v_s\}, \{e\})$. Then, w.l.o.g. we can assume that $X = \mathtt{Spec}(\mathbb{F}_1[X_1])$ and $Y = \mathtt{Spec}(\mathbb{F}_1[Y_1])$. We denote by U the open set $D(X_1)$ of X and by V the open set $D(Y_1)$ of Y. One can easily check that the intersection $X \cap Y$ is equal to U as an open set of X and equal to V as an open set of Y. Hence, to determine the "gluing" of X and Y along their intersection, we only need to define an appropriate isomorphism between the open sets U and V (such that the gluing of X and Y gives a projective line).

We know that $U \simeq \texttt{Spec}(\mathbb{F}_1[X_1, X_1^{-1}])$ and $V \simeq \texttt{Spec}(\mathbb{F}_1[Y_1, Y_1^{-1}])$ and then, the isomorphism of $\mathbb{F}_1$-rings

$$\begin{array}{ccc} \mathbb{F}_1[X_1, X_1^{-1}] & \longrightarrow & \mathbb{F}_1[Y_1, Y_1^{-1}] \\ X_1 & \longmapsto & Y_1^{-1} \\ X_1^{-1} & \longmapsto & Y_1 \end{array}$$

induces an isomorphism ψ of Deitmar schemes between X and Y.

Consider now the loose tree Γ, its minimal tree $\overline{\Gamma}$ and $\mathbf{P}(\overline{\Gamma})$ the projective space in which $\mathcal{F}(\Gamma)$ is embedded. Coordinatize $\mathbf{P}(\overline{\Gamma})$ such that vertices of $\overline{\Gamma}$ correspond to canonical vectors of $\mathbf{P}(\overline{\Gamma})$. This choice of coordinates induces bases on the affine schemes $\texttt{Spec}(A_i)$ corresponding to the vertices of Γ. Denoting by ι the embedding of X in $\texttt{Spec}(A_r)$, by ι_1 the embedding of U in X and by j, j_1 the embeddings of Y into $\texttt{Spec}(A_s)$ and V in Y, respectively, we could try to"glue" $\texttt{Spec}(A_r)$ to $\texttt{Spec}(A_s)$ according to the following diagram

$$\begin{tikzcd} \mathbf{P}(\overline{\Gamma}) & \mathbf{P}(\overline{\Gamma}) \\ \texttt{Spec}(A_r) \arrow[u, hook, "\iota'"] & \texttt{Spec}(A_s) \arrow[u, hook, "j'"] \\ \texttt{Spec}(\mathbb{F}_1[X_1]) \arrow[u, hook, "\iota"] & \texttt{Spec}(\mathbb{F}_1[Y_1]) \arrow[u, hook, "j"] \\ U \arrow[u, hook, "\iota_1"] \arrow[r, "\simeq"] & V, \arrow[u, hook, "j_1"] \end{tikzcd}$$

where ι' and j' are the embeddings given by the coordinatization of $\mathbf{P}(\overline{\Gamma})$. The problem here is that U and V are not necessarily *open* anymore in $\texttt{Spec}(A_r)$ and $\texttt{Spec}(A_s)$, so that at the end one would not end up with a Deitmar scheme. So on the level of trees (but clearly also in general), gluing the affine spaces to a scheme cannot work. (In fact, even the example 2.3.1 is not a scheme!) We thank Lieven Le Bruyn for kindly noting these facts to us.

What we *do* know is that each $\mathcal{F}(\Gamma)$ is a constructible set in the ambient projective space. We will show this after having introduced the maps $\mathcal{F}_k$.

Finally, as a direct consequence of Lemma 2.3.4, we deduce a similar result as [45, Theorem 4.7] for the functor $\mathcal{F}$.

Corollary 2.3.5. *Let Γ be a loose graph and $\mathcal{F}(\Gamma)$ its constructible set. Then, $\mathcal{F}(\Gamma)$ is connected if and only if Γ is connected.*

From this moment on, we will keep calling $\mathcal{F}$ (and its liftings $\mathcal{F}_k$ to fields k) a *functor*, although this will be proved in detail in section 2.5. First, we need to define $\mathcal{F}_k$, with k a finite field (or $\mathbb{Z}$).

2.4 From $\mathcal{F}$ to $\mathcal{F}_k$

As we mentioned in example 2.3.2, studying only the structure of a constructible set defined over $\mathbb{F}_1$ might lead to mistakes that are only seen when one pulls the structures to "real" fields. Remember that the $\mathbb{F}_1$-structure of schemes defined over a field k is usually interpreted as the skeleton of the field. To have a consistent idea of the Deitmar constructible sets defined by the functor $\mathcal{F}$, one should also keep track of the corresponding constructible sets obtained after base extension.

2.4.1 Base extension of Deitmar schemes

Let us consider A to be a ring over $\mathbb{F}_1$, i.e., a commutative monoid with an absorbing element 0, and R to be a ring. We define the *base extension of A to R* as the monoidal ring

$$R[A] = A \otimes_{\mathbb{F}_1} R = \bigoplus_{a \in A} Ra \Big/ R0_A,$$

where 0_A is the absorbing element of the monoid A. The multiplication structure is essentially given by the multiplication in A, considering that, by definition, elements of R commute with elements of A.

Although this construction is possible for any ring R, throughout this thesis, we will only consider the cases where R is either a finite field k or $\mathbb{Z}$. In the rest of the section we will assume $R = \mathbb{Z}$ and we will point out when extra conditions are needed.

With the definition of base extension, we obtain a functor from the category of $\mathbb{F}_1$-rings to the category of rings over $\mathbb{Z}$

$$\begin{array}{ccc} \{\mathbb{F}_1\text{-rings}\} & \xrightarrow{\cdot \otimes_{\mathbb{F}_1} \mathbb{Z}} & \{\mathbb{Z}\text{-rings}\} \\ A & \longmapsto & \mathbb{Z}[A] \end{array}$$

called the *base change functor*. It is left adjoint to the *forgetful functor F*, i.e., the functor sending a commutative ring to its multiplicative monoid (while keeping 0)

$$\begin{array}{ccc} \{\mathbb{Z}\text{-rings}\} & \xrightarrow{F} & \{\mathbb{F}_1\text{-rings}\} \\ R & \longmapsto & (R, \times) \end{array}$$

Theorem 2.4.1 ([8])**.** *The functor of base extension* $\cdot \otimes_{\mathbb{F}_1} \mathbb{Z}$ *is left adjoint to* F*, i.e., for every ring* R *and every* $\mathbb{F}_1$*-ring* A*, we have*

$$\mathtt{Hom}_{\mathbb{Z}}(A \otimes_{\mathbb{F}_1} \mathbb{Z}, R) \cong \mathtt{Hom}_{\mathbb{F}_1}(A, F(R)).$$

Proof. Consider a $\mathbb{Z}$-ring homomorphism ϕ from $\mathbb{Z}[A]$ to R and define $F(\phi)$ to be its restriction to the monoid A. This map is injective, since such a ring homomorphism is uniquely determined by its action on A. Besides, it is also surjective because a morphism of monoids from A to $(R, \times)$ extends uniquely to a ring homomorphism from $\mathbb{Z}[A]$ to R. ∎

Using the two aforementioned functors, Deitmar explained in [8] how one can extend schemes over $\mathbb{F}_1$ to schemes over $\mathbb{Z}$ as well as how one can go in the other direction and descend schemes to their $\mathbb{F}_1$-version. Let us consider first the forgetful functor F. Using the fact that every scheme over $\mathbb{Z}$ can be written as the union of affine schemes, F "extends" to a functor $F_{\mathtt{Sch}}$

$$\begin{array}{ccc} \{\text{Schemes}/\mathbb{Z}\} & \xrightarrow{F_{\mathtt{Sch}}} & \{\text{Schemes}/\mathbb{F}_1\} \\ X = \bigcup_{i\in I} \mathtt{Spec}(A_i) & \longmapsto & \bigcup_{i\in I} \mathtt{Spec}(A_i, \times), \end{array}$$

where $F(X)$ is glued via the gluing maps of X. In the same way, the base change functor "extends" to the functor

$$\begin{array}{ccc} \{\text{Schemes}/\mathbb{F}_1\} & \xrightarrow{\cdot \otimes_{\mathbb{F}_1} \mathbb{Z}} & \{\text{Schemes}/\mathbb{Z}\} \\ Y = \bigcup_{j\in J} \mathtt{Spec}(A_j) & \longmapsto & \bigcup_{j\in J} \mathtt{Spec}(A_j \otimes_{\mathbb{F}_1} \mathbb{Z}), \end{array}$$

where we also use the gluing maps coming from Y. The fact that these constructions do not depend on the choices of affine coverings follows from Proposition 1.4.7 and the fact that every scheme is a union of affine schemes.

Definition 2.4.2 (The functor $\mathcal{F}_k$)**.** Let $\mathcal{F}$ be the functor defined in subsection 2.3.1 and let k be a finite field (or $\mathbb{Z}$). We define the functor $\mathcal{F}_k$ to be the composition of $\mathcal{F}$ with the base change functor, i.e., it is a functor from the category `LGraph` to the category of constructible sets over k, $\mathtt{CS}_k$, that associates to any loose graph Γ the constructible set

$$\mathcal{F}_k(\Gamma) = \mathcal{F}(\Gamma) \otimes_{\mathbb{F}_1} k = \bigcup_{v\in\Gamma} \mathtt{Spec}(A_v \otimes_{\mathbb{F}_1} k),$$

where $\mathtt{Spec}(A_v)$ denotes the local $\mathbb{F}_1$-affine space associated by $\mathcal{F}$ to the vertex v of Γ. Thus, for every finite field k (or $\mathbb{Z}$), $\mathcal{F}_k(\Gamma)$ is also a union of a finite number of affine spaces over k.

Corollary 2.4.3. *Let* Γ *be a loose graph. The intersections between the local affine spaces in* $\mathcal{F}(\Gamma) \otimes_{\mathbb{F}_1} k$ *are determined by* Γ*.*

Proof. This lemma is a direct consequence of Lemma 2.3.4. ■

Theorem 2.4.4. *For each loose graph Γ and each field k (including $\mathbb{F}_1$), $\mathcal{F}_k(\Gamma)$ is a constructible set in the ambient projective space $\mathbf{P}(\overline{\Gamma})$.*

Proof. This follows from the fact that each affine space is a constructible set and finite unions of constructible sets are constructible. ■

2.4.2 Equations of some liftings

Describing schemes over k with concrete equations (as in the case of varieties) is not always easy or even possible. Nevertheless, there are some examples of schemes coming from loose graphs for which it is feasible to find the right equations. We will discuss here how one can describe the base extension of Deitmar schemes coming from loose stars.

Lifting a projective plane without a multiplicative group

Let B be the graph of example 2.3.1. For any finite field $k = \mathbb{F}_q$, we obtain a constructible set $\mathcal{F}_k(B) \hookrightarrow \mathbf{P}^2(\mathbb{F}_q)$ which is covered by three affine spaces, in which two affine lines and one affine plane are induced, and the intersections *are defined by the graph* (cf. subsection 2.3.3), that is, $\mathcal{F}_k(B)$ is $\mathbf{P}^2(\mathbb{F}_q)$ without a projective line minus two points. Choosing homogeneous coordinates X, Y, Z in $\mathbf{P}^2(\mathbb{F}_q)$ such that the aforementioned projective line is defined by $Z = 0$, we obtain the space $V(k)$ of which the complement is defined by the property

$$Z = 0 \quad \Longrightarrow \quad XY \neq 0. \tag{2.8}$$

Whence $V(k)$ is given by the constructible set

$$(Z \neq 0) \quad \vee \quad (XY = 0), \tag{2.9}$$

that is, the union of the $X-$ and $Y-$coordinate axes and the affine plane with $Z = 0$ as line at infinity.

Remark 2.4.5. Although, going from A to B, we add the points on the edges through x, this does not imply, as we have seen, that once we have lifted $\mathcal{F}(A)$ and $\mathcal{F}(B)$ to $\mathbb{Z}$ or $\mathbb{F}_q$, all the points of the lines in the affine plane corresponding to A get an extra point. In fact, an affine plane over $\mathbb{F}_1$ is just an absolute point with two directions, so these (two) lines correspond to canonical coordinate axes. Once lifted to $\mathbb{Z}$ or $\mathbb{F}_q$, only the (two) lines corresponding to these axes get an extra point.

Lifting of general loose stars

After the analysis on the previous example, and taken the main axioms of $\mathcal{F}$ into account, it is now easy to write down the k-constructible set corresponding to a general "loose star graph." Let Γ have a vertex v which is incident with $m \geq 2$ edges, of which $\ell \leq m$, $\ell \geq 0$, have a second vertex (and let Γ contain no other vertices and edges). Then after choosing homogeneous coordinates in $\mathbf{P}^m(k)$, the corresponding constructible set in $k[X, X_1, \ldots, X_m]$ is

$$(X \neq 0) \quad \vee \quad \Big(\cup_{1 \leq i \leq l} \{X_j = 0 \mid j \neq i;\ 1 \leq j \leq m\}\Big) \tag{2.10}$$

that is, the union of the (projective) X_1–,..., X_ℓ–coordinate axes and the affine space with as hyperplane at infinity $X = 0$.

Remark 2.4.6. Although these examples are constructed considering k as a finite field, it is also possible to do them in case $k = \mathbb{Z}$ since the functor $\mathcal{F}_k$ considers both possibilities in its definition.

2.5 The functors $\mathcal{F}_k$

In this section we will prove in detail that the functors $\mathcal{F}_k$, with k a finite field, $\mathbb{F}_1$ or $\mathbb{Z}$, are indeed functors.

2.5.1 Local action

Let Γ be a finite loose graph and f a loose graph automorphism of Γ. Remember that $\mathcal{F}(\Gamma)$ is the union of finite dimensional affine schemes defined from the vertices of Γ, i.e.,

$$\mathcal{X} = \mathcal{F}(\Gamma) = \bigcup_{v \in V(\Gamma)} \mathtt{Spec}(A_v)$$

where A_v is a finite $\mathbb{F}_1$-ring isomorphic to $\mathbb{F}_1[X_1, \ldots, X_{\deg(v)}]$.

Let us consider $v_i \in V(\Gamma)$, a vertex of degree n_i. We denote by $\mathtt{Adj}(v_i)$ the set of adjacent vertices of v_i, with cardinality s_i, by $E(v_i)$ the set of edges incident with v_i and by $LE(v_i)$ the set of loose edges incident with v_i. Note that $s_i \leq n_i$ and that $LE(v_i) \subseteq E(v_i)$. As f is a loose graph automorphism, $f(v_i)$ is also a vertex v_j of Γ with degree n_i and $f(S_{v_i}) = S_{v_j}$. Using the same terminology for v_j, we consider the sets $E(v_j)$, $LE(v_j)$ and $\mathtt{Adj}(v_j)$ (also with cardinality s_i). Then, f induces a bijection between $E(v_i)$ and $E(v_j)$ and a bijection between $LE(v_i)$ and $LE(v_j)$. Note that both $\mathtt{Spec}(A_{v_i})$ and $\mathtt{Spec}(A_{v_j})$ are isomorphic to an n_i-dimensional affine space. Since each edge incident with a vertex corresponds bijectively to an element of a basis of the associated affine space, f induces a bijection between the corresponding bases. We will call $\overline{f_i}$ the induced map

$$\overline{f_i} : \mathtt{Spec}(A_{v_i}) \longrightarrow \mathtt{Spec}(A_{v_j})$$

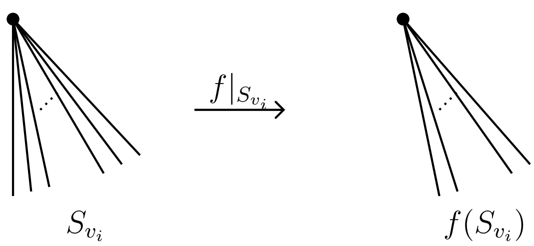
between the affine schemes associated to the vertices v_i and $f(v_i) = v_j$. Hence, for any vertex $v_k \in V(\Gamma)$ we define a homeomorphism $\overline{f_k}$ between affine spaces and we can then define the map $\mathcal{F}(f)$ to be the map $\widetilde{f}$ resulting from pasting together all morphisms $\overline{f_k}$. This pasting of morphisms of schemes is well defined thanks to the fact that $\mathcal{F}(\Gamma)$ is the union of such affine spaces, $\texttt{Spec}(A_k)$, that all morphisms $\overline{f_k}$ are induced by one and the same automorphism of Γ and that Γ completely determines the intersections of the subspaces $\texttt{Spec}(A_k)$ in $\mathcal{F}(\Gamma)$.

Once we have shown how to construct the homeomorphism $\mathcal{F}(f)$ of constructible sets from an automorphism f of a loose graph, we have to generalize the construction of $\widetilde{f}$ starting from a *general* morphism between two loose graphs. Let Γ_1 and Γ_2 be two loose graphs and let $\texttt{Hom}(\Gamma_1, \Gamma_2)$ be the set of loose graph homomorphims from Γ_1 to Γ_2. Let us take $f \in \texttt{Hom}(\Gamma_1, \Gamma_2)$ and use the same notation as above.

Notice that we can assume f to be surjective. Otherwise we restrict to the image $f(\Gamma_1) \subset \Gamma_2$ and, using the (CV) property, we obtain the required map (composition with an embedding):

$$\begin{tikzcd} \mathcal{F}(\Gamma_1) \arrow[r, "\mathcal{F}(f)"] \arrow[dr, dotted] & \mathcal{F}(f(\Gamma_1)) \arrow[d, hook, "i"] \\ & \mathcal{F}(\Gamma_2) \end{tikzcd}$$

For general morphisms of loose graphs, the degree of vertices does not have to be preserved. When the degrees of v_i and $f(v_i) = v_j$ are equal, we define the morphism $\overline{f_i}$ as before. For the remaining case we consider f restricted to the loose star S_{v_i}, which corresponds to a morphism between two affine spaces on the scheme level.



We have that $f(S_{v_i})$ is a proper loose subgraph of S_{v_j}, hence the following diagram gives us the desired morphism:

$$\begin{tikzcd} \texttt{Spec}(A_{v_i}) \arrow[r, "\mathcal{F}(f|_{S_{v_i}})"] \arrow[dr, dotted, "\widetilde{f_i}"] & \mathcal{F}(f(S_{v_i})) \arrow[d, hook, "i"] \\ & \texttt{Spec}(A_{v_j}) \end{tikzcd}$$

In this way, we reduce the study to the local restriction of f to loose stars. Let us describe this situation in detail. Suppose v_i is a vertex of Γ_1 of degree m, and suppose the vertex of the loose star $f(S_{v_i})$ has degree $n \leq m$ (this is always the case by assumption); then the morphism $f|_{S_{v_i}}$ is a loose graph morphism between the two loose stars S_{v_i} and $f(S_{v_i})$.

We will call $\overline{f_i}$ the morphism induced between the corresponding affine spaces

$$\overline{f_i} : \mathcal{F}(S_{v_i}) \longrightarrow \mathcal{F}(f(S_{v_i})).$$

The morphism $\overline{f_i}$ is by definition a linear morphism between affine spaces with dimension the number of edges incident with the vertex of the corresponding loose star. Hence, we choose a basis $\{e_1, \ldots, e_m\}$ for $\mathcal{F}(S_{v_i}) \cong \mathbb{A}^m_{\mathbb{F}_1}$, where by definition e_i is represented by the n-tuple with 1 on the i-th coordinate and 0 elsewhere such that each element of the basis corresponds bijectively to an edge of the loose star S_{v_i}, and v_i corresponds to the point $(0, \ldots, 0)$. We do the same for the affine space $\mathcal{F}(f(S_{v_i})) \cong \mathbb{A}^n_{\mathbb{F}_1}$ and so we choose a basis $\{e'_1, \ldots, e'_n\}$ corresponding to the edges of $f(S_{v_i})$.

Now that since we have chosen a basis, we can easily describe the morphism $\overline{f_i}$ in terms of matrices. For each element e_t of the basis of $\mathcal{F}(S_{v_i})$, we consider the corresponding edge g_t in S_{v_i} and we set $\overline{f_i}(e_t) = e'_k$, where e'_k is the element of the basis of $\mathcal{F}(f(S_{v_i}))$ associated to the edge $f(g_t)$. In the definition of loose graph morphism we allow contractions of edges having two end points, i.e., one edge with two end points might be contracted into the graph with one vertex. So it may happen that $f(g_t)$ is a vertex. But the only vertex existing in the loose star $f(S_{v_i})$ is $f(v_i)$ so, in this case, we choose $\overline{f_i}(e_t)$ to be the zero vector. In fact, what we do in the congruence setting is adding an extra point on each affine line corresponding to an edge on a given vertex v_i, $f(v_i)$.

Notice that the zero vector is possible as an image because the monoids we are working with contain the element 0 as an absorbing element. Allowing contractions to be morphisms of loose graphs allows us to have projections on the level of $\mathbb{F}_1$-constructible sets since, for instance, a projection of a projective line onto a point (figure 2.7) will be induced by the graph morphism sending the complete graph K_2 onto one vertex.

Figure 2.7: Projection of $\mathbb{P}^1_{\mathbb{F}_1}$ on one point P.

So locally $\overline{f_i}$ can be expressed by a matrix of size $n \times m$ whose columns are either the zero vector or a canonical vector, i.e., vector with only one non-zero entry. Reordering the basis $\{e_1, \ldots, e_m\}$, we obtain a block matrix of the form

$$A_{f_i} := \left(\begin{array}{ccc|c|c|ccc|c}
0 & \cdots & 0 & A_1 & 0 & \cdots & \cdots & 0 & 0 \\ \hline
0 & \cdots & 0 & 0 & A_2 & 0 & \cdots & 0 & 0 \\ \hline
\vdots & & & \vdots & 0 & & & & \vdots \\
\vdots & & & & \vdots & & & & \vdots \\ \hline
0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & A_n
\end{array}\right)$$

where the blocks A_r are of size $1 \times s_r$, with s_r the number of vectors from $\{e_1, \ldots, e_m\}$ whose image is the vector e'_r, and have all 1-entries. Let us remark that $\sum_{r=1}^{n} s_r + s_0 = m$, where s_0 is the number of columns where all entries are 0. Note as well that if all edges are sent to edges by the morphism $f|S_{v_i}$, then A_{f_i} has no zero part and if $n = m$, then A_{f_i} is a nonsingular matrix.

Example 2.5.1. If f is the projection of $\mathbb{P}^1_{\mathbb{F}_1}$ on one point (figure 2.7), then the matrices defining f_1 and f_2 will correspond to the (1×1)-zero matrix.

These matrices A_{f_i} are well defined over $\mathbb{F}_1$ since every column is a vector with at most one coordinate different from 0, and is the image of some point by the morphism f_i. Hence, the image of a point is well defined over $\mathbb{F}_1$. What is more, the composition of such morphisms corresponds to a product of matrices. It is easy to verify that the product of two matrices of this form also gives a matrix with maximum one non-zero entry in each column, this entry being 1, so the composition of two morphisms is well defined. Finally, after constructing the morphisms $\overline{f_i}$ for each vertex of Γ_1, you can glue all of them using the relations given by the graphs Γ_1 and Γ_2 to finally obtain the morphism $\mathcal{F}(f) : \mathcal{F}(\Gamma_1) \to \mathcal{F}(\Gamma_2)$.

Although the process before has been made for Deitmar constructible sets, and in particular monoidal schemes, the above construction equally works if we consider $\mathcal{F}(\Gamma_1)$ and $\mathcal{F}(\Gamma_2)$ to be in the category of congruence constructible sets instead (by this we mean that the local affine spaces are considered to be congruence schemes). While working in the latter category, one should consider the matrices A_{f_i} defined over $\mathbb{F}_2$. The reason why this consideration is also possible relies on the fact that points of a vector space over $\mathbb{F}_1$ (in the congruence setting) are exactly the same as points of the same vector space over $\mathbb{F}_2$. The difference between, e.g., projective spaces over $\mathbb{F}_1$ and over $\mathbb{F}_2$ can be seen geometrically on subvarieties of dim ≥ 1 (and on the level of polynomial rings).

2.5.2 Global action

In the previous subsection we have locally constructed a morphism of constructible sets $\mathcal{F}(f)$ considering the loose graph morphisms that $f : \Gamma_1 \to \Gamma_2$ induces between the loose stars. There is, nevertheless, another way of obtaining $\mathcal{F}(f)$ starting from the global action induced by f on the projective spaces in which the graphs Γ_1 and Γ_2 are embedded. We will start this process by defining a morphism between their respective ambient spaces, $\mathbf{P}(\Gamma_1)$ and $\mathbf{P}(\Gamma_2)$, such that the restriction to the constructible sets $\mathcal{F}(\Gamma_1)$ and $\mathcal{F}(\Gamma_2)$ naturally induces a morphism between $\mathbb{F}_1$-constructible sets. This morphism will also induce the local mappings described in the previous subsection (just by considering their local action on the loose stars of vertices).

Consider the completion $\overline{\Gamma_1}$, with $m_1 + 1$ vertices, and $\overline{\Gamma_2}$, with $m_2 + 1$ vertices, together with the embedding in their minimal projective space $\mathbf{PG}(m_1, \mathbb{F}_1)$ and $\mathbf{PG}(m_2, \mathbb{F}_1)$, respectively. Coordinatize $\mathbf{PG}(m_1, \mathbb{F}_1)$ and $\mathbf{PG}(m_2, \mathbb{F}_1)$ such that the vertices of $\overline{\Gamma_1}$ correspond to the canonical vectors $e_0, \ldots, e_{m_1}$ of $\mathbf{PG}(m_1, \mathbb{F}_1)$, and the vertices of $\overline{\Gamma_2}$ to the canonical vectors $e'_0, \ldots, e'_{m_2}$, and define h and h' as having coordinates $[1 : \cdots : 1]$ w.r.t. the corresponding basis. Now put $\mathcal{R}_1 = \{e_0, \ldots, e_{m_1}, h\}$, $\mathcal{R}_2 = \{e'_0, \ldots, e'_{m_2}, h'\}$, $\mathcal{B}_1 := \mathcal{R}_1 \setminus \{h\}$ and $\mathcal{B}_2 := \mathcal{R}_2 \setminus \{h'\}$. Notice that when considering the extension of $\mathbb{F}_1$-constructible sets to k-constructible sets, the sets $\mathcal{R}_1$ and $\mathcal{R}_2$ would be skeletons of the projective spaces $\mathbf{PG}(m_1, k)$ and $\mathbf{PG}(m_2, k)$, respectively.

Let's describe the global construction. In this case we can also consider f to be a surjective morphism without loss of generality. The morphism f obviously induces a morphism $\overline{f}$ from $\overline{\Gamma_1}$ to $\overline{\Gamma_2}$. This morphism $\overline{f}$ sends every vertex of $\overline{\Gamma_1}$ to a vertex of $\overline{\Gamma_2}$. Besides, every element of $\mathcal{B}_1$ corresponds bijectively to a vertex of $\overline{\Gamma_1}$ and the same holds for elements of $\mathcal{B}_2$ and the vertices of $\overline{\Gamma_2}$, so reasoning as in the affine case and reordering the basis $\mathcal{B}_1$, we get an $(m_2 + 1) \times (m_1 + 1)$-matrix of the form

$$P_f := \left(\begin{array}{c|c|ccc|c} P_0 & 0 & \cdots & & \cdots\ 0 & 0 \\ \hline 0 & P_1 & 0 & & \cdots\ 0 & 0 \\ \hline \vdots & 0 & & & & \vdots \\ & \vdots & & & & \vdots \\ \hline 0 & 0 & 0 & & \cdots\ 0 & P_{m_2} \end{array}\right)$$

Note that in this case, the matrix P_f has no zero columns since all vertices are sent to vertices, i.e., every canonical vector of $\mathcal{B}_1$ is sent to a canonical vector of $\mathcal{B}_2$. It may also happen that an edge $e \in \overline{\Gamma_1}$ is contracted but this will imply that the two elements of $\mathcal{B}_1$ corresponding to the vertices of e are sent to the same element of the basis $\mathcal{B}_2$. As it happens for the affine case, the blocks P_i are all of size $1 \times n_i$, with all entries equal to 1 and n_i being the number of elements of $\mathcal{B}_1$ whose image is the element e'_i. The identity $\sum_{j=0}^{m_2} n_j = m_1 + 1$ is also satisfied.

As in the affine case, those matrices are well defined over $\mathbb{F}_1$ and the composition of morphisms, which is translated into a product of matrices, is also well defined. Notice that when you compose two morphisms, it is not possible to reorder the frames so that both matrices are of the above form. Nevertheless, the product of two matrices having columns with only one non-zero element is also a matrix satisfying the same condition. Hence, we obtain that for two loose graph morphisms $f : \Gamma_1 \to \Gamma_2$ and $g : \Gamma_2 \to \Gamma_3$, the following property is satisfied:

$$\mathcal{F}(g \circ f) = \mathcal{F}(g) \circ \mathcal{F}(f).$$

Remark that when we consider the constructible sets $\mathcal{X}_{k,1} = \mathcal{F}_k(\Gamma_1)$ and $\mathcal{X}_{k,2} = \mathcal{F}_k(\Gamma_2)$ over a finite field k (or $\mathbb{Z}$), the morphism defined by the matrix P_f also induces an action on the level of constructible sets over k (or $\mathbb{Z}$). Hence the maps $\mathcal{F}_k$, with k a finite field, $\mathbb{F}_1$ or $\mathbb{Z}$, are functors from the category `LGraph` of loose graphs to the category of constructible sets over k.

Remark 2.5.2. If we *first* consider the extension of $\mathbb{F}_1$-constructible sets to constructible sets over a field k (or $\mathbb{Z}$) and then define a matrix $P_{f,k}$ in the same way as we did with P_f, one realizes that there exist many choices for $P_{f,k}$ inducing the same action on the basis vectors. So, in general, the following diagram

$$\begin{tikzcd}
\mathcal{F}(\Gamma_1) \arrow[r, "P_f"] \arrow[d, "\otimes k"] & \mathcal{F}(\Gamma_2) \arrow[d, "\otimes k"] \\
\mathcal{F}_k(\Gamma_1) \arrow[r, "P_{f,k}"] & \mathcal{F}_k(\Gamma_2)
\end{tikzcd}$$

is not commutative. One way of dealing with this problem is to define a morphism on the level of $k/\mathbb{Z}$-constructible sets as the class $[P_{f,k}]$ of morphisms having the same action on the basis vectors. As such, we obtain a well-defined functor from the category of loose graphs to the category of $k/\mathbb{Z}$-constructible sets making the previous diagram commutative.

2.5.3 Different categories for projective spaces

After the previous construction one could realize that morphisms might not be injective on the level of projective spaces, so we need to choose the concrete category of projective spaces that we want to work with. We will see in this subsection how one can interpret the morphisms $\mathcal{F}(f)$ when a category of projective spaces with different homomorphisms is chosen. We only work over $\mathbb{F}_1$; similar considerations over "real fields" follow easily (and are in fact *easier*).

Category with injective linear maps

Consider the category of projective spaces whose morphisms are injective linear maps. In this case P_f defines an injective linear map if and only if $m_2 \geq m_1$ and the rank of P_f equals $m_1 + 1$ over $\mathbb{F}_2$. These conditions are equivalent to saying that every element of the basis $\mathcal{B}_1$ is sent to a different element of the basis $\mathcal{B}_2$ and, by the bijection described above between bases and vertices of the graphs, we have that P_f is induced from an injective morphism of graphs (so injective on the level of each of the loose stars).

So in this case, our functor $\mathcal{F}$ will be a functor between the category of graphs with injective morphisms and the category of (congruence) constructible sets in projective spaces with injective linear morphisms.

Category with rational maps

In the second case we consider the category of projective spaces whose morphisms are *rational maps.* A *rational map* $f : V \to W$ between two varieties is an equivalence class of pairs (f_U, U) in which f_U is a morphism of varieties defined from an open set U of V to W, and two pairs (f_U, U) and $(f_{U'}, U')$ are equivalent if f_U and $f_{U'}$ coincide on $U \cap U'$. We adapt the same nomenclature for constructible sets, such as Deitmar constructible sets (with possible congruences).

Now consider the linear map P_f defined in the previous subsection and suppose it has a nontrivial kernel. One should remember that the map P_f is defined on the minimal projective spaces in which $\mathcal{F}(\Gamma_1)$ and $\mathcal{F}(\Gamma_2)$ are embedded (denoted $\mathbf{PG}(m_1, \mathbb{F}_1)$ and $\mathbf{PG}(m_2, \mathbb{F}_1)$, respectively). That implies we only have to consider the case where the kernel of P_f intersects with the constructible set $\mathcal{F}(\Gamma_1)$, since otherwise the induced map on the constructible sets will be injective.

We will then prove that the kernel of P_f is a closed subset of $\mathbf{PG}(m_1, \mathbb{F}_1)$. By relative topology its intersection with $\mathcal{F}(\Gamma_1)$ will be closed in $\mathcal{F}(\Gamma_1)$. We define the kernel of P_f over $\mathbb{F}_1$, which we denote by $\ker(P_f)_{\mathbb{F}_1}$, as

$$\ker(P_f)_{\mathbb{F}_1} := \{x \in \mathcal{F}(\Gamma_1) \otimes \mathbb{F}_2 \mid x \in \ker(P_f)\}.$$

Notice that in order to define the kernel of map P_f one has to consider the constructible sets $\mathcal{F}(\Gamma_1)$ and $\mathcal{F}(\Gamma_2)$ in the category of congruence constructible sets and so, the matrix P_f defined over $\mathbb{F}_2$. This consideration does not pose any problems in terms of points, since there is a bijective correspondence between the points of the constructible set $\mathcal{F}(\Gamma_i) \otimes_{\mathbb{F}_1} \mathbb{F}_2$ and the points of the congruence constructible set $\mathcal{F}(\Gamma_i)$, $i = 1, 2$. So in this way $\ker(P_f)_{\mathbb{F}_1}$ is well defined.

To prove that $\ker(P_f)_{\mathbb{F}_1}$ is a closed subset of $\mathcal{F}(\Gamma_1)$ we will verify that every point of the set $\ker(P_f)_{\mathbb{F}_1}$ is indeed closed in the congruence constructible set $\mathcal{F}(\Gamma_1)$. For, take a point $x \in \ker(P_f)_{\mathbb{F}_1}$. Considering x as a point in the projective space gives us its coordinates; let us write $x = [a_0 : \cdots : a_{m_1}]$ (not all entries 0) and let a_{i_0} be the first coordinate equal to 1. Then x defines a congruence $\mathcal{C}_x$ in the projective congruence scheme corresponding to $\mathbf{PG}(m_1, \mathbb{F}_1)$, given by $\langle x_i \sim 0 \text{ if } a_i = 0,\ x_i \sim x_{i_0} \text{ if } a_i = 1\rangle$. It is a homogeneous maximal congruence in the Zariski topology (remember that a maximal congruence in projective schemes is maximal w.r.t. "not containing the irreducible congruence"). [1] We have proved that a point $x \in \ker(P_f)_{\mathbb{F}_1}$ is a closed point in the projective scheme $\mathbf{PG}(m_1, \mathbb{F}_1)$, so the set $\ker(P_f)_{\mathbb{F}_1}$ is closed as well since it is a finite union of closed sets. Hence, if f is a morphism of loose graphs, $\mathcal{F}(f) = P_f$ is a rational map defined on $U = \mathcal{F}(\Gamma_1) \setminus \ker(P_f)_{\mathbb{F}_1}$. We proved then that, in this case, the

[1] To see the construction of congruence projective schemes, we refer to subsection 1.4.2.

functor $\mathcal{F}$ is a functor between the category of loose graphs with morphisms of loose graphs and the category of congruence constructible sets with rational maps.

2.6 In conclusion

We summarize in this last section how we construct morphisms on the level of k-constructible sets using the functor $\mathcal{F}$. Suppose we have a morphism $f : \Gamma \to \Gamma'$ of loose graphs. We follow the following procedure:

i) The morphism f induces a morphism f' between the corresponding ambient projective spaces of Γ and Γ', respectively. This morphism is indeed continuous.

ii) The morphism f' induces morphisms of affine schemes

$$f'_i : \mathbb{A}_v \cong \texttt{Spec}(A) \longrightarrow \texttt{Spec}(A') \cong \mathbb{A}_{f(v)},$$

between the affine spaces associated to the vertices of the loose graphs.

iii) Since the previous morphisms are induced by the same morphisms of loose graphs, f' also induces a map $\mathcal{F}_k(f) : \mathcal{F}_k(\Gamma_1) \to \mathcal{F}_k(\Gamma_2)$ on the level of constructible sets.

Notice that in the case where $\mathcal{F}_k(\Gamma)$ and $\mathcal{F}_k(\Gamma')$ happen to be schemes, the morphism $\mathcal{F}(f)$ would be a scheme-theoretical morphism.

Observation 2.6.1. At the moment we do not bother to try to define morphisms for general constructible sets of projective spaces. Later in chapter 5 we will study different kinds of automorphism groups of the constructible sets $\mathcal{F}_k(\Gamma)$ to get a deeper understanding on this matter.

3 Counting Polynomial and Zeta Equivalence

As we mentioned in subsection 1.4, Deitmar schemes are based on considering commutative multiplicative monoids (with an absorbing element) as commutative rings over $\mathbb{F}_1$. Besides, the $\texttt{Spec}$-construction allows us to have a whole scheme theory over $\mathbb{F}_1$ defined in an analogous way to the classical scheme theory over $\mathbb{Z}$. In this chapter we will define the Grothendieck ring of schemes over $\mathbb{F}_1$ and develop some techniques to count rational points (see its definition in section 3.2.1) of constructible sets that come from loose graphs. We will finally prove that the class of those constructible sets in the Grothendieck ring is polynomial; meaning that their class can be expressed as a linear combination of powers of the class of the affine line.

3.1 Grothendieck ring of schemes

We start this section by defining the Grothendieck ring of varieties over a field k.

Definition 3.1.1. Let k be a field and let $\widehat{\texttt{Var}(k)}$ be the category of varieties over k. We define the *Grothendieck ring of varieties over* k, $K_0(\widehat{\texttt{Var}(k)})$, to be the ring generated by the isomorphism classes of the variety X, $[X]_k$, with the relations

$$[X]_k = [X \setminus Y]_k + [Y]_k$$

for any closed subvariety Y of X and with the product structure given by

$$[X]_k \cdot [Y]_k = [X \times_{\texttt{Spec}(k)} Y]_k.$$

As we can see the classical definition of the Grothendieck ring only takes into account varieties over a field k, i.e., separated integral k-schemes of finite type, but it is indeed possible to define the Grothendieck ring in the category of schemes of finite type over k in a similar way (see [4, section 2]). For our study of constructible sets over $\mathbb{F}_1$, we will directly define the Grothendieck ring for Deitmar schemes of finite type.

In the course of this chapter, we change the notation for affine and projective spaces. We denote by $\mathbb{A}_k^n$ and $\mathbb{P}_k^n$ the n-dimensional affine and projective spaces over k, respectively, where k can be a finite field, $\mathbb{Z}$ or $\mathbb{F}_1$. In the cases when the field is clear, we will use the notation $\mathbb{A}^n$ and $\mathbb{P}^n$.

Definition 3.1.2. The *Grothendieck ring of schemes of finite type over* $\mathbb{F}_1$, denoted as $K_0(\mathtt{Sch}_{\mathbb{F}_1})$, is the ring (not $\mathbb{F}_1$-ring) generated by the isomorphism classes of schemes X of finite type over $\mathbb{F}_1$, $[X]_{\mathbb{F}_1}$, with the relation

$$[X]_{\mathbb{F}_1} = [X \setminus Y]_{\mathbb{F}_1} + [Y]_{\mathbb{F}_1} \tag{3.1}$$

for any closed subscheme Y of X and with the product structure given by

$$[X]_{\mathbb{F}_1} \cdot [Y]_{\mathbb{F}_1} = [X \times_{\mathtt{Spec}(\mathbb{F}_1)} Y]_{\mathbb{F}_1}. \tag{3.2}$$

Taking into account the definition of our functors $\mathcal{F}_k$, for k a finite field or $\mathbb{F}_1$, it is natural to ask whether constructible sets (who do not carry a sheaf structure) have a class inside the Grothendieck ring of schemes. The answer to this questions is *yes* and it is proved in the following theorem.

Theorem 3.1.3 ([42])**.** *Let X be a scheme over a finite field k.*

i) *Let U and V be two locally closed sets in X (note that these define subschemes); then*
$$[U \cup V]_k + [U \cap V]_k = [U]_k + [V]_k. \tag{3.3}$$

ii) *Let E be a constructible set of X; then E has a well-defined class in $K_0(\mathtt{Sch}_k)$.*

Remark 3.1.4. For obtaining ii), one uses the decomposition of a constructible set in a finite disjoint union of locally closed sets (see Proposition 1.3.17). Note also that this theorem remains correct for the case of Deitmar constructible sets.

3.1.1 Connection to motives

“Contrairement à ce qui se passait en topologie ordinaire, on se trouve donc placé là devant une abondance déconcertante de théories cohomologiques différentes. On avait l’impression très nette qu’en un sens qui restait d’abord assez flou, toutes ces théories devaient "revenir au même", qu’elles "donnaient les mêmes résultats". C’est pour parvenir à exprimer cette intuition de "parenté" entre théories cohomologiques différentes, que j’ai dégagé la notion de "motif" associé à une variété algébrique. Par ce terme, j’entends suggérer qu’il s’agit du "motif commun" (ou de la "raison commune") sous-jacent à cette multitude d’invariants cohomologiques différents associés à la variété, à l’aide de la multitude des toutes les théories cohomologiques possibles à priori.”

A.Grothendieck. *Récoltes et Semailles*, 1986

The notion of “motif” was originally introduced by Alexander Grothendieck in 1964 in a letter to Serre as an attempt to find a universal cohomology theory that carries the invariants and properties of varieties obtained by the different cohomology theories. For instance, if we consider X a manifold of dimension $2n$, one can define cohomology groups, which are finite-dimensional $\mathbb{Q}$-vector spaces, in many different ways (singular cohomology, Čech cohomology, De Rham cohomology, etc.). However, under some conditions, they all define the same or isomorphic groups. In Algebraic Geometry one also has the same similarities with cohomology theories defined on varieties over a field k.

We will briefly explain how one can construct motives, following the notes [41]. In the attempt to find such a theory, Grothendieck took into account the definition of a cohomology as a contravariant functor from the category of varieties to a graded abelian category where groups of morphisms have the structure of a $\mathbb{Q}$-vector space. Grothendieck's idea was to start with the category of projective varieties and replace the morphisms of that category by *equivalence classes of correspondences* (see [41] for details on correspondences).

Then, the category of motives, denoted by $\mathcal{M}(k)$, should satisfy the following two conditions:

i) $\mathcal{M}(k)$ should be an abelian category.

ii) The homomorphism groups should behave as $\mathbb{Q}$-vetor spaces.

With these two conditions, the universal cohomology theory should be a contravariant functor h from the category $\widehat{\mathtt{Var}(k)}$ of projective varieties over a field k to the category $\mathcal{M}(k)$ of motives such that the following properties are satisfied:

⋆ Disjoint unions of varieties are translated into direct sums, i.e.,

$$h(X \coprod Y) = h(X) \oplus h(Y).$$

⋆ The $\mathtt{Hom}$-sets are finite-dimensional $\mathbb{Q}$-vector spaces.

⋆ The Künneth formula holds, i.e.,

$$h(X \times Y) = h(X) \otimes h(Y).$$

3.1.2 Virtual Tate motives

Consider the projective line $\mathbb{P}^1_k$ over a field k. By its well-known decomposition $\mathbb{P}^1_k = \mathbb{A}^1_k \coprod \mathbb{A}^0_k$ and one of the properties of the category of motives, we have that

$$h(\mathbb{P}^1_k) = h(\mathbb{A}^1_k) \oplus h(\mathbb{A}^0_k), \tag{3.4}$$

where the second term of the right-hand side corresponds to the motive of a point, which we denote by $\mathbf{1}$. Define now $h(\mathbb{A}^1_k) := \mathbb{L}$; it is the so-called *Lefschetz motive*.

Similarly for an n-dimensional projective space $\mathbb{P}^n_k$ over k (using the identity $\mathbb{P}^n_k = \mathbb{A}^n_k \coprod \mathbb{P}^{n-1}_k$) one obtains the following decomposition in the category of motives:

$$h(\mathbb{P}^n_k) = \mathbf{1} \oplus h(\mathbb{A}^1_k) \oplus h(\mathbb{A}^2_k) \oplus \cdots \oplus h(\mathbb{A}^n_k), \tag{3.5}$$

which transforms into

$$h(\mathbb{P}^n_k) = \mathbf{1} \oplus \mathbb{L} \oplus \mathbb{L}^2 \oplus \cdots \oplus \mathbb{L}^n \tag{3.6}$$

by using the Künneth formula for each affine space.

We define the Grothendieck ring $K_0(\mathcal{M}(k))$ of k-motives in a similar way as the Grothendieck ring $K_0(\widehat{\mathtt{Var}(k)})$ of varieties over k (see section 3.1). In both rings we have indeed a similar decomposition for the classes $[\mathbb{P}^1]_k$ and $[h(\mathbb{P}^1)]_k$, that is:

$$[\mathbb{P}^1]_k = \mathbf{1} + [\mathbb{A}^1]_k,$$

$$[h(\mathbb{P}^1)]_k = \mathbf{1} + [h(\mathbb{A}^1)]_k = \mathbf{1} + \mathbb{L}. \tag{3.7}$$

Following this analogy, we denote the class $[\mathbb{A}_1]_k$ of the affine line in the Grothendieck ring $K_0(\widehat{\mathtt{Var}(k)})$ also by $\mathbb{L}$ and we call it *virtual Lefschetz motive*. The generators of this ring are usually called *virtual motives* and the subring $\mathbb{Z}[\mathbb{L}] \subset K_0(\widehat{\mathtt{Var}(k)})$ is called the subring of *virtual mixed Tate motives*.

Definition 3.1.5. Let X be a variety over k. We say that X has a *mixed Tate motive* if its class $[X]_k \in \mathbb{Z}[\mathbb{L}]$.

Let us remark that the multiplicative group $\mathbb{G}_m$ (cf. subsection 1.4.3) will satisfy, as a consequence, the equality $[\mathbb{G}_m]_k = \mathbb{L} - 1$ since it can be identified with the affine line minus one point. To make a distinction, in the case of $\mathbb{F}_1$, we will denote by $\underline{\mathbb{L}} = [\mathbb{A}^1]_{\mathbb{F}_1}$ the class of the affine line over $\mathbb{F}_1$ in $K_0(\mathtt{Sch}_{\mathbb{F}_1})$ and, consequently, $[\mathbb{G}_m]_{\mathbb{F}_1} = \underline{\mathbb{L}} - 1$.

From now on, we will work in the category $\mathtt{Sch}_k$ of schemes of finite type over k instead of in the category of varieties. As mentioned in section 3.1, one can without a problem adapt the definition 3.1.5 for schemes of finite type.

3.2 Grothendieck polynomials

Let Γ be a loose graph, and let k be any finite field different from $\mathbb{F}_1$. Consider $\mathcal{F}_k(\Gamma) := \mathcal{F}(\Gamma) \otimes_{\mathbb{F}_1} k$. In the next sections we will compute a polynomial $P_\Gamma(X) \in \mathbb{Z}[X]$ for $\mathcal{F}_k(\Gamma)$ (which we sometimes also denote as $\mathbb{P}(\Gamma)$) such that $\left|\mathcal{F}_{\mathbb{F}_q}(\Gamma)\right|_{q^n} = P_\Gamma(q^n)$ for all $\mathbb{F}_q$ and all $n \geq 1$, where $\left|\mathcal{F}_{\mathbb{F}_q}(\Gamma)\right|_{q^n}$ denotes the number of $\mathbb{F}_{q^n}$-points of the constructible set $\mathcal{F}_{\mathbb{F}_q}(\Gamma)$. As a consequence of finding such a polynomial, we obtain that every constructible set $\mathcal{F}_k(\Gamma)$ is polynomial-count and that the polynomial counting the rational points is independent of q.

3.2.1 Zeta-equivalence and polynomial-count

Before introducing the notion of zeta-equivalence and polynomial-count for schemes and constructible sets over k, we will introduce the concept of rational points of a scheme defined over a field k.

Definition 3.2.1 (Rational points). Let X be a scheme over k. A *k-rational point* of X is a morphism of schemes $\mathtt{Spec}(k) \to X$.

Let us suppose now that X is a k-scheme of finite type and let x be a point of X. By definition, there exist a neighborhood U of x such that $U = \mathtt{Spec}(A)$, where A is a commutative ring. Then, x corresponds to a point of $\mathtt{Spec}(A)$, i.e., to a prime ideal $\mathfrak{p} \subset A$. For $\mathfrak{p}$, we know that the stalk $\mathcal{O}_{\mathfrak{p}}$ of the sheaf $\mathcal{O}_X$ is isomorphic to the local ring $A_{\mathfrak{p}}$ with maximal ideal $\mathfrak{p}A_{\mathfrak{p}}$ (see Proposition 1.3.9). We define the *residue field of the point x* to be:

$$k(x) := A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}.$$

One can easily prove that this definition is independent of the choice of the affine neighborhood U of x.

A morphism $\mathtt{Spec}(k) \to X$ is determined by the choice of a point x and its residue field $k(x)$. In the case where X is of finite type over k, one can prove that x is a rational point if and only if its residue field $k(x)$ is isomorphic to k. Besides, one can also prove that a point x of X is a *closed point* of X if and only if $k(x)$ is a finite extension of k.

For the rest of the section we will consider X to be a scheme of finite type over a finite field $\mathbb{F}_q$. Following [20], for any $n \geq 1$, one can define the following map

$$\begin{array}{ccc} \mathtt{Sch}_{\mathbb{F}_q} & \longrightarrow & \mathbb{Z} \\ X & \longmapsto & \Big|X\Big|_{q^n} \end{array}$$

sending a scheme X of finite type over $\mathbb{F}_q$ to its number of $\mathbb{F}_{q^n}$-points. We can also put all these functions together and define a function, that we will call *zeta series of X*, as follows

$$Q_X(T) = \sum_{n\geq 1} \Big|X\Big|_{q^n} T^n. \tag{3.8}$$

The definition of zeta serie can be extended to the elements of $K_0(\mathtt{Sch}_{\mathbb{F}_q})$ since isomorphic schemes have the same number of $\mathbb{F}_{q^n}$-points.

Definition 3.2.2. Let X and Y be elements of $K_0(\mathtt{Sch}_{\mathbb{F}_q})$. We say that X is *zeta-trivial* if $\Big|X\Big|_{q^n} = 0$ for all $n \geq 1$, i.e, if the zeta series of X is equal to zero. We say that X and Y are *zeta-equivalent* if they have the same zeta series, i.e., if their difference is zeta-trivial.

Definition 3.2.3. Let $\mathbb{F}_q$ be a finite field and X be a scheme of finite type over $\mathbb{F}_q$. We say that X is *polynomial-count* if there exists a (necessarily unique) polynomial $P_X(T) = \sum_{i=0}^{m} a_i T^i \in \mathbb{C}[T]$ such that for every finite extension $\mathbb{F}_{q^n}/\mathbb{F}_q$, we have

$$\left|X\right|_{q^n} = P_X(q^n). \tag{3.9}$$

In an analogous way, the definition of polynomial-count can also be extended to the elements of $K_0(\mathtt{Sch}_k)$.

Lemma 3.2.4 ([20]). *An element $\gamma \in K_0(\mathtt{Sch}_{\mathbb{F}_q})$ is polynomial-count if and only if it is zeta-equivalent to a $\mathbb{Z}$-linear combination of classes of affine spaces $[\mathbb{A}^i]_{\mathbb{F}_q}$.*

3.2.2 Tate conjecture and counting polynomial

Definition 3.2.5. Let R be a commutative ring. An *additive invariant* is a map $\chi : \widehat{\mathtt{Var}(k)} \to R$ satisfying the following properties:

- $\star$ Isomorphism Invariance

$$\chi(X) = \chi(Y) \text{ if } X \cong Y.$$

- $\star$ Multiplicativity

$$\chi(X \times Y) = \chi(X)\chi(Y).$$

- $\star$ Inclusion-Exclusion

$$\chi(X) = \chi(Y) + \chi(X \setminus Y) \text{ for } Y \text{ closed in } X.$$

Example 3.2.6. Topological Euler characteristic. For a variety Y over $\mathbb{C}$, one defines the *topological Euler characteristic* of Y as

$$\chi(Y) \quad := \quad \sum_{i=0}^{2\dim Y} (-1)^i b_i \tag{3.10}$$

where b_i are the Betti numbers of Y, i.e., b_i is the dimension of the i-th singular cohomology group $\mathrm{H}^i(Y_{\mathbb{C}}, \mathbb{C})$.

If we consider a $\mathbb{Z}$-variety X, we know that the number $\left|X\right|_{q^n}$ of $\mathbb{F}_{q^n}$-rational points of $X \times_{\mathtt{Spec}(\mathbb{Z})} \mathtt{Spec}(\mathbb{F}_{q^n})$ is an additive invariant. Indeed, if we consider that X has a mixed Tate motive (over $\mathbb{F}_{q^n}$), then it follows that up to a finite number of primes $\left|X\right|_{q^n}$ is a polynomial in q^n because $\left|\mathbb{A}^1\right|_{q^n} = q^n$.

In this sense, we can state a corollary of one of the Tate conjectures: *Let X be a variety; then X being polynomial-count for all but finitely many primes is equivalent to X having a mixed Tate motive.*

Consider now a constructible set $\mathcal{F}_{\mathbb{F}_q}(\Gamma)$ for a given loose graph Γ. We define a *rational point of* $\mathcal{F}_{\mathbb{F}_q}(\Gamma)$ to be an $\mathbb{F}_q$-rational point of a local affine space contained in $\mathcal{F}_{\mathbb{F}_q}(\Gamma)$. Note that one can also define a rational point of $\mathcal{F}_{\mathbb{F}_q}(\Gamma)$ as a rational point of the ambient projective space that is also contained in $\mathcal{F}_{\mathbb{F}_q}(\Gamma)$. We define *closed points of* $\mathcal{F}_{\mathbb{F}_q}(\Gamma)$ in a similar way.

In the next sections, we will describe an inductive procedure to compute, for each prime p, a polynomial that counts the number of rational points of $\mathcal{F}_{\mathbb{F}_p}(\Gamma)$ (with the additional property that the obtained polynomials for all p are one and the same). We call this polynomial the *Grothendieck polynomial* of Γ, or of $\mathcal{F}_k(\Gamma)$, or of $\mathcal{F}(\Gamma)$, or of $\mathcal{F}(\Gamma) \otimes_{\mathbb{F}_1} \mathbb{Z}$.

Suggestively, because of the corollary of one of the Tate conjectures, we will write $P_\Gamma(\mathbb{L})$ instead of $P_\Gamma(X)$. We will prove in section 3.7 that the class of each $\mathcal{F}_k(\Gamma)$ is indeed an element of $\mathbb{Z}[\mathbb{L}]$. In terms of [20], $\mathcal{F}_k(\Gamma)$ being polynomial-count translates into the constructible sets $\mathcal{F}_k(\Gamma)$ being zeta-equivalent to objects γ of $\mathtt{CS}_k$ for which $[\gamma] \in \mathbb{Z}[\mathbb{L}] \subset K_0(\mathtt{Sch}_k)$ (since for each such k, they have the same Grothendieck polynomial).

3.3 Grothendieck polynomial for trees

Let Γ be a *loose tree*, that is, a connected loose graph in which any two vertices are connected by a unique simple path. So, a loose tree is a connected loose graph without cycles. Note that a tree is also a loose tree. Let $\mathcal{F}(\Gamma)$ be the Deitmar constructible set associated to it. We will define in this section a new function associating a polynomial to Γ that gives us information about the class of $\mathcal{F}(\Gamma)$ in the Grothendieck ring of Deitmar schemes of finite type. For the sake of convenience we will denote this new function by $\left[\Gamma\right]_{\mathbb{F}_1}$, which represents the class of $\mathcal{F}(\Gamma)$ in $K_0(\mathtt{Sch}_{\mathbb{F}_1})$.

To find this new function, we start thinking about the most basic examples, the affine spaces $\mathbb{A}^n_{\mathbb{F}_1}$, whose corresponding loose graphs are the loose stars S_n, i.e., the loose graphs formed by one vertex and n edges incident to it, see figure 3.1. We know that the class of $\mathbb{A}^n_{\mathbb{F}_1}$ in the Grothendieck ring of $\mathbb{F}_1$-schemes of finite type is the n-th power of the Lefschetz motive $\underline{\mathbb{L}}$ and the number of $\mathbb{F}_1$-closed points of $\mathbb{A}^n_{\mathbb{F}_1}$ is one (the only closed point of $\mathtt{Spec}(\mathbb{F}_1[X_1, \ldots, X_n])$). So, our function must verify the following condition:

- $\star$ If we denote by $\Gamma_{\mathbb{A}^n}$ the loose graph corresponding to $\mathbb{A}^n_{\mathbb{F}_1}$, then $\left[\Gamma_{\mathbb{A}^n}\right]_{\mathbb{F}_1} = \underline{\mathbb{L}}^n$.

Besides, if $\Gamma_{\mathbb{P}^1}$ is the graph associated to the projective line $\mathbb{P}^1_{\mathbb{F}_1}$ (figure 3.2), our function should satisfy $\left[\Gamma_{\mathbb{P}^1}\right]_{\mathbb{F}_1} = \underline{\mathbb{L}} + 1$, which corresponds to the class of the projective line in $K_0(\mathtt{Sch}_{\mathbb{F}_1})$.

After making some analogies with other basic examples, such as loose stars with some of the edges being proper edges, and taking into account how the affine spaces

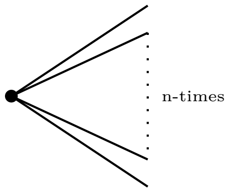


Figure 3.1: Affine space $\mathbb{A}^n_{\mathbb{F}_1}$.

Figure 3.2: Projective line $\mathbb{P}^1_{\mathbb{F}_1}$.

associated to vertices through the functor $\mathcal{F}$ intersect (cf. section 2.3.3), we can define our polynomial function for loose trees as follows:

Definition 3.3.1. Let Γ be a finite loose tree. Consider the following notation:

- Let D be the set of degrees $\{d_1, \ldots, d_k\}$ of $V(\Gamma)$ such that $1 < d_1 < d_2 < \ldots < d_k$.
- Let us call n_i the number of vertices of Γ with degree d_i, $1 \leq i \leq k$.
- We call $I = \sum_{i=1}^{k} n_i - 1$.
- We call E the number of vertices of Γ with degree 1, that is the *end points*.

We define the function "class of a loose tree," and we denote it as $\big[\ .\ \big]_{\mathbb{F}_1}$, as follows:

$$\begin{array}{rccc} \big[\,\cdot\,\big]_{\mathbb{F}_1}: & \{\text{Loose trees}\} & \longrightarrow & K_0(\mathtt{Sch}_{\mathbb{F}_1}) \\ & \Gamma & \longmapsto & \displaystyle\sum_{i=1}^{k} n_i \mathbb{L}^{d_i} - I \cdot \mathbb{L} + I + E. \end{array}$$

We will prove by induction on the number of vertices of the loose tree that the function counts the number of rational points of $\mathcal{F}(\Gamma)$ for every finite loose tree, i.e., $\big[\Gamma\big]_{\mathbb{F}_1} = \big[\mathcal{F}(\Gamma)\big]_{\mathbb{F}_1}$. We start by proving that for some basic cases, which will be used for proving the formula for a general loose tree.

The projective line $\mathbb{P}^1_{\mathbb{F}_1}$. As we mentioned before, for the projective line, the corresponding tree is one edge with two end points. That gives us $E = 2$, $I = -1$ and D is the empty set, so the formula will be:

$$\big[\Gamma_{\mathbb{P}^1}\big]_{\mathbb{F}_1} = \mathbb{L} + 1,$$

which corresponds with the desired class of the projective line in the Grothendieck ring $K_0(\mathtt{Sch}_{\mathbb{F}_1})$.

The affine space $\mathbb{A}^n_{\mathbb{F}_1}$. In this case we have one vertex of degree n, $E = 0$ and $I = 0$. Then,

$$\Big[\Gamma_{\mathbb{A}^n}\Big]_{\mathbb{F}_1} = \underline{\mathbb{L}}^n.$$

A (loose) star S_n^k. Suppose we have a (graph) star $\overline{S_n}$ (figure 3.3), that means, a complete bipartite graph $K_{1,n}$. We know that a vertex in the tree corresponds to a closed point of the constructible set. By property "FUCP" (see section 2.3), the disjoint union of all the vertices of degree 1 in the tree is also a closed subset of the constructible set. Then, according to the relations in the Grothendieck ring, we can express the class of the star $\overline{S_n}$ as follows.

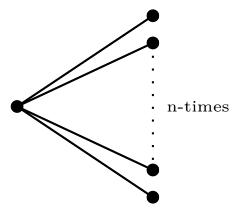


Figure 3.3: A star $\overline{S_n}$.

Let us call $v_1, \ldots, v_n$ the vertices with degree one. Then

$$\Big[S_n\Big]_{\mathbb{F}_1} = \Big[\coprod_{i=1}^n v_i\Big]_{\mathbb{F}_1} + \Big[\Gamma_{\mathbb{A}^n}\Big]_{\mathbb{F}_1} = \sum_{i=1}^n \Big[v_i\Big]_{\mathbb{F}_1} + \underline{\mathbb{L}}^n = n\Big[v_1\Big]_{\mathbb{F}_1} + \underline{\mathbb{L}}^n = n + \underline{\mathbb{L}}^n. \tag{3.11}$$

We now define a *loose star* with *parameters* (n, k), $k \leq n$, and denote it by S_n^k, as a loose star S_n with k vertices of degree 1 and $n - k$ loose edges. The same reasoning of above gives a similar formula for a loose star:

$$\Big[S_n^k\Big]_{\mathbb{F}_1} = k + \underline{\mathbb{L}}^n. \tag{3.12}$$

A general loose tree. For the case of a general loose tree we want to be able to "break" the tree into pieces in a certain way. We will w.l.o.g. assume that a loose tree Γ has an edge e such that $\Gamma \setminus \{e\}$ is a disjoint union of two loose trees, both with at least one edge. We will call this condition (D) (in case Γ does not satisfy this condition, we will be in one of the previous cases).

Lemma 3.3.2. *Let Γ be a loose tree, $\mathcal{F}(\Gamma)$ the corresponding Deitmar constructible set and e an edge making Γ satisfy the condition (D). Let Γ_1 and Γ_2 be the two disjoint loose trees obtained as above; then $\mathcal{F}(\Gamma_1)$ and $\mathcal{F}(\Gamma_2)$ are disjoint.*

Proof. Since Γ_1 and Γ_2 are both connected loose trees, from Corollary 2.3.5, we have that $\mathcal{F}(\Gamma_1)$ and $\mathcal{F}(\Gamma_2)$ are connected Deitmar constructible sets. Also, by definition of the $\mathcal{F}$ functor, $\mathcal{F}(\Gamma_1 \coprod \Gamma_2) = \mathcal{F}(\Gamma_1) \cup \mathcal{F}(\Gamma_2)$ and has two connected components (Corollary 2.3.5). So, it is clear that $\mathcal{F}(\Gamma_1)$ and $\mathcal{F}(\Gamma_2)$ are disjoint. ■

After this lemma, we are ready to prove the consistency of the formula for all trees. We will prove it using induction on the number N which is the sum of the number of edges and the number of vertices of a loose tree.

Start with Γ and let e be one of the vertices of condition (D). We will denote by $\bar{e}$ the subgraph having e as the only edge and having two end points v_1, v_2. Then, $\bar{e}$ defines a projective line $\mathbb{P}^1_{\mathbb{F}_1}$ which is a closed subscheme of the Deitmar constructible set associated to Γ. By the relations in the Grothendieck ring of schemes of finite type over $\mathbb{F}_1$, we have:

$$\Big[\Gamma\Big]_{\mathbb{F}_1} = \Big[\bar{e}\Big]_{\mathbb{F}_1} + \Big[\mathcal{F}(\Gamma) \setminus \mathcal{F}(\bar{e})\Big]_{\mathbb{F}_1}. \tag{3.13}$$

Remark 3.3.3. Remark that by $[\mathcal{F}(\Gamma) \setminus \mathcal{F}(\bar{e})]_{\mathbb{F}_1}$ we mean the class of the Deitmar constructible set defined by the loose graph Γ minus the projective line define by $\bar{e}$ *embedded in the constructible set defined by* Γ. Otherwise, if we just take the graph $\Gamma \setminus \bar{e}$, we will obtain a different constructible set in a projective space of higher dimension than the one in which $\mathcal{F}(\Gamma)$ is embedded!

Let's clarify this remark with the following example. Let Γ and $\Gamma \setminus \bar{e}$ be the graphs of figure 3.4. We can see that in this case $\Gamma \setminus \bar{e}$ is the disjoint union of two stars S_2 without their respective vertices of degree 2.

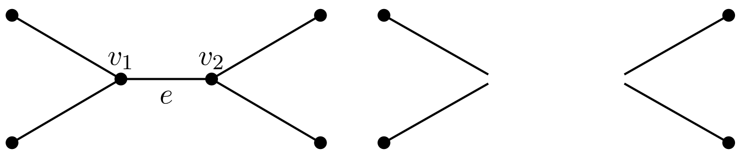


Figure 3.4: Grahps Γ and $\Gamma \setminus \bar{e}$.

By considering the constructible set $\mathcal{F}(\Gamma \setminus \bar{e})$ instead of $\mathcal{F}(\Gamma) \setminus \mathcal{F}(\bar{e})$ embedded in $\mathcal{F}(\Gamma)$ we obtain $2 \cdot (\underline{\mathbb{L}}^2 + 1)$ as the Grothendieck polynomial of $[\Gamma \setminus \bar{e}]_{\mathbb{F}_1}$, since it corresponds to two disjoint copies of the affine plane without a point and with two extra points at infinity. Nevertheless, since we eventually also want that the same polynomial counts the number of rational points of $\mathcal{F}_k(\Gamma)$ (using formula 3.13), this Grothendieck polynomial does *not* give the correct information about the number of $\mathbb{F}_q$-rational points of $\mathcal{F}_q(\Gamma) \setminus \mathcal{F}_q(\bar{e})$ in $\mathcal{F}_q(\Gamma)$. For instance, locally in $\mathcal{F}(\Gamma)$ the vertices v_1 and v_2 define 3-dimensional affine spaces, which have $\underline{\mathbb{L}}^3$ as Grothendieck polynomial; while in the graph $\Gamma \setminus \bar{e}$ the 3-dimensional affine spaces would become affine planes instead. The correct polynomial is, in fact, $2 \cdot (\underline{\mathbb{L}}^3 - \underline{\mathbb{L}} + 2)$.

We can now proceed to prove the equality $\Big[\Gamma\Big]_{\mathbb{F}_1} = \Big[\mathcal{F}(\Gamma)\Big]_{\mathbb{F}_1}$ for the loose tree Γ. We know that $\Gamma \setminus \{e\}$ has two disjoint connected components (Γ_1 and Γ_2), and each of the end points of the edge e belongs to one of these two components. Suppose that $v_1 \in \Gamma_1$ and $v_2 \in \Gamma_2$. For computing the class $\Big[\mathcal{F}(\Gamma) \setminus \mathcal{F}(\bar{e})\Big]_{\mathbb{F}_1}$ we need to calculate the classes of the constructible sets defined by $\Gamma_1 \setminus \{v_1\}$ and $\Gamma_2 \setminus \{v_2\}$, but one needs once more to take into account, as in the above example, that they define constructible sets embedded in $\mathcal{F}(\Gamma)$ in order to keep the right degree of the vertices v_1 and v_2.

Using the relations in the Grothendieck ring we deduce that the following equations are satisfied

$$\Big[\mathcal{F}(\Gamma) \setminus \mathcal{F}(\bar{e})\Big]_{\mathbb{F}_1} = \Big[\mathcal{F}(\Gamma_1 \cup \{e\} \cup \{v_2\}) \setminus \mathcal{F}(\bar{e})\Big]_{\mathbb{F}_1} + \Big[\mathcal{F}(\Gamma_2 \cup \{e\} \cup \{v_1\}) \setminus \mathcal{F}(\bar{e})\Big]_{\mathbb{F}_1}, \quad (3.14)$$

$$\Big[\Gamma_i \cup \{e\} \cup \{v_j\}\Big]_{\mathbb{F}_1} = \Big[\mathcal{F}(\Gamma_i \cup \{e\} \cup \{v_j\}) \setminus \mathcal{F}(\bar{e})\Big]_{\mathbb{F}_1} + \Big[\bar{e}\Big]_{\mathbb{F}_1}, \quad (3.15)$$

with $i, j = 1, 2$ and $i \neq j$, since $\bar{e}$ defines a closed subscheme of the Deitmar constructible set correponding to $\Gamma_i \cup \{e\} \cup \{v_j\}$. Writing all three equations (3.13), (3.14) and (3.15) together, one obtains that

$$\Big[\Gamma\Big]_{\mathbb{F}_1} = \Big[\bar{e}\Big]_{\mathbb{F}_1} + (\Big[\Gamma_1 \cup \{e\} \cup \{v_2\}\Big]_{\mathbb{F}_1} - \Big[\bar{e}\Big]_{\mathbb{F}_1}) + (\Big[\Gamma_2 \cup \{e\} \cup \{v_1\}\Big]_{\mathbb{F}_1} - \Big[\bar{e}\Big]_{\mathbb{F}_1}). \quad (3.16)$$

By induction on the number of vertices and edges, we know that the function $[\cdot]_{\mathbb{F}_1}$ is well defined for both $\Big[\Gamma_1 \cup \{e\} \cup \{v_2\}\Big]_{\mathbb{F}_1}$ and $\Big[\Gamma_2 \cup \{e\} \cup \{v_1\}\Big]_{\mathbb{F}_1}$, i.e., we can write

$$\begin{aligned} \Big[\Gamma_1 \cup \{e\} \cup \{v_2\}\Big]_{\mathbb{F}_1} &= \sum_{r=1}^{k_1} n_{1_r}\mathbb{L}^{d_{1r}} - I_1 \cdot \mathbb{L} + I_1 + E_1, \\ \Big[\Gamma_2 \cup \{e\} \cup \{v_1\}\Big]_{\mathbb{F}_1} &= \sum_{j=1}^{k_2} n_{2_j}\mathbb{L}^{d_{2j}} - I_2 \cdot \mathbb{L} + I_2 + E_2. \end{aligned} \quad (3.17)$$

Finally we observe that

i) $I = I_1 + I_2 + 1$.

ii) $E = E_1 + E_2 - 2$.

iii) $[\bar{e}\Big]_{\mathbb{F}_1} = \mathbb{L} + 1$. (Base case.)

iv) The degree of the inner vertices and the number of vertices for each degree remain the same as in Γ.

Finally, introducing these formulas in (3.16), we obtain

$$\Big[\Gamma\Big]_{\mathbb{F}_1} = -(\mathbb{L}+1) + (\sum_{r=1}^{k_1} n_{1_r}\mathbb{L}^{d_{1r}} + \sum_{j=1}^{k_2} n_{2_j}\mathbb{L}^{d_{2j}}) - (I_1 + I_2)\mathbb{L} + (I_1 + E_1 + I_2 + E_2), \quad (3.18)$$

and reordering the degrees of the vertices, we have the desired equality for Γ:

$$\begin{aligned}\left[\Gamma\right]_{\mathbb{F}_1} &= \sum_{i=1}^{k} n_i \mathbb{L}^{d_i} - (I_1 + I_2 + 1)\mathbb{L} + (I_1 + I_1 + 1 + E_1 + E_2 - 2) \\ &= \sum_{i=1}^{k} n_i \mathbb{L}^{d_i} - I \cdot \mathbb{L} + I + E. \end{aligned} \tag{3.19}$$

It is proved that our polynomial counts the number of points of $\mathcal{F}(\Gamma)$, where Γ is a finite loose tree. ∎

3.4 Lifting the class of trees in $K_0(\mathtt{Sch}_{\mathbb{F}_1})$

In section 2.4, we explained (following Deitmar's construction) how one can extend a scheme over $\mathbb{F}_1$ to a scheme over $\mathbb{Z}$ by lifting affine schemes $\mathtt{Spec}(A)$ to $\mathtt{Spec}(A) \otimes_{\mathbb{F}_1} \mathbb{Z}$, the gluing being defined by the scheme on the $\mathbb{F}_1$-level, and that the same can be done for the constructible sets $\mathcal{F}(\Gamma)$. The same base extension is also defined for any field k. Thanks to the naturality of the base change functor, we will prove that this lifting is compatible as well on the level of the Grothendieck ring of schemes of finite type "up to zeta-equivalence."

We define Ω as a map from the subring $\mathbb{Z}[\mathbb{L}]$ of the Grothendieck ring $K_0(\mathtt{Sch}_{\mathbb{F}_1})$ of Deitmar schemes of finite type to the Grothendieck ring $K_0(\mathtt{Sch}_k)$ of schemes of finite type over any field k generated by the map sending the class $\mathbb{L}$ to the class $\mathbb{L}$, i.e.

$$\begin{aligned} \Omega : \quad \mathbb{Z}[\mathbb{L}] &\longrightarrow K_0(\mathtt{Sch}_k) \\ \sum_{j=1}^{m} a_j \mathbb{L}^j &\longmapsto \sum_{j=1}^{m} a_j \mathbb{L}^j. \end{aligned}$$

As we did for the class of $\mathcal{F}(\Gamma)$ in the Grothendieck ring of schemes of finite type over $\mathbb{F}_1$, we will denote, from now on, by $[\Gamma]_k$ the class of its lifting $\mathcal{F}(\Gamma) \otimes_{\mathbb{F}_1} k$ in the Grothendieck ring of schemes of finite type over k.

Theorem 3.4.1. *Let Γ be a finite loose tree, and let $\mathcal{F}(\Gamma)$ be zeta-equivalent to an object γ of $\mathtt{CS}_{\mathbb{F}_1}$ whose class is contained in $\mathbb{Z}[\mathbb{L}]$. Then $\Omega([\gamma]_{\mathbb{F}_1}) = [\widetilde{\gamma}]_k$, where $\widetilde{\gamma}$ is an object of $\mathtt{CS}_k$ which is zeta-equivalent to $\mathcal{F}(\Gamma) \otimes_{\mathbb{F}_1} k$.*

We will succintly write $\Omega(\left[\Gamma\right]_{\mathbb{F}_1}) \equiv \left[\Gamma\right]_k \mod ZE$ for the property expressed in the statement of Theorem 3.4.1.

Proof. We will prove this theorem by induction, in the same way as in section 3.3. We will start with the basic cases.

Projective line. Let $\Gamma_{\mathbb{P}^1}$ be the graph associated to the projective line $\mathbb{P}^1_{\mathbb{F}_1}$. Then,

$$\Omega(\left[\Gamma_{\mathbb{P}^1}\right]_{\mathbb{F}_1}) = \Omega(\mathbb{L} + 1) = \mathbb{L} + 1 = [\mathbb{P}^1_k] = \left[\Gamma_{\mathbb{P}^1}\right]_k. \tag{3.20}$$

Affine spaces. Let $\Gamma_{\mathbb{A}^n}$ be the graph corresponding to the affine scheme $\mathbb{A}^n_{\mathbb{F}_1}$. Then,

$$\Omega(\left[\Gamma_{\mathbb{A}^n}\right]_{\mathbb{F}_1}) = \Omega(\underline{\mathbb{L}}^n) = \mathbb{L}^n = [\mathbb{A}^n_k] = \left[\Gamma_{\mathbb{A}^n}\right]_k. \tag{3.21}$$

Loose stars S^k_n. The Deitmar constructible sets associated to S^k_n can be written as:

$$\mathcal{F}(S^k_n) = \mathbb{A}^n_{\mathbb{F}_1} \cup \coprod_{i=1}^{k} \{v_i\}. \tag{3.22}$$

Since such a constructible set is a disjoint union of an n-dimensional affine space over $\mathbb{F}_1$ and a disjoint union of closed points (which is closed by the property "FUCP"), it follows that its base extension to k is also a disjoint union of an n-dimensional k-affine space and a union of closed points. Hence,

$$\Omega(\left[S^k_n\right]_{\mathbb{F}_1}) = \Omega(\underline{\mathbb{L}}^n + k) = \mathbb{L}^n + k = [\mathbb{A}^n_k]_k + k = \left[S^k_n\right]_k. \tag{3.23}$$

General loose trees. Let Γ be a loose tree. The way we proved the formula above for loose trees, gives us a decomposition of $\left[\Gamma\right]_{\mathbb{F}_1}$, according to the relations in the Grothendieck ring over $\mathbb{F}_1$, as the sum $\left[\bar{e}\right]_{\mathbb{F}_1} + \left[\mathcal{F}(\Gamma) \setminus \mathcal{F}(\bar{e})\right]_{\mathbb{F}_1}$, where $\bar{e}$ satisfies the condition (D) (cf. 3.3).

We know that $\bar{e}$ defines a projective line over $\mathbb{F}_1$, so its extension to k corresponds with the projective line $\mathbb{P}^1_k$, which is also a closed subscheme of the k-constructible set $\mathcal{F}(\Gamma) \otimes_{\mathbb{F}_1} k$. This gives us the following relation in the Grothendieck ring of schemes over k:

$$\left[\Gamma\right]_k = \left[\mathcal{F}(\Gamma) \setminus \mathcal{F}(\bar{e})\right]_k + \left[\bar{e}\right]_k. \tag{3.24}$$

Hence, using induction on the number of vertices and edges of the loose tree and the decomposition of (3.16), we obtain that:

$$\begin{aligned} \Omega(\left[\Gamma_1 \cup \{e\} \cup \{v_2\}\right]_{\mathbb{F}_1}) &= \left[\Gamma_1 \cup \{e\} \cup \{v_2\}\right]_k, \\ \Omega(\left[\Gamma_2 \cup \{e\} \cup \{v_1\}\right]_{\mathbb{F}_1}) &= \left[\Gamma_2 \cup \{e\} \cup \{v_1\}\right]_k, \end{aligned} \tag{3.25}$$

so, we can conclude that $\Omega(\left[\Gamma\right]_{\mathbb{F}_1}) = \left[\Gamma\right]_k$.

For general loose graphs, lifting will be further commented upon in section 3.6. Note that for each of the cases we handled up till now, we have

$$\Omega(\left[\Gamma\right]_{\mathbb{F}_1}) = \left[\Gamma\right]_k = [\mathcal{F}(\Gamma) \otimes_{\mathbb{F}_1} k]_k. \tag{3.26}$$

In the next section we introduce a process to calculate Grothendieck polynomials over fields $k \neq \mathbb{F}_1$, and we look at some consequences.

3.5 Surgery

In this section we derive a procedure that we call *surgery* in order to inductively *calculate* the Grothendieck polynomial of a k-constructible set coming from a general loose graph. In each step of the procedure, we will "resolve" an edge, so as to eventually end up with a tree in much higher dimension. So one will have to keep track of how the Grothendieck polynomials change in each step. In the entire section, k will be a field different from $\mathbb{F}_1$. Since we will determine Grothendieck polynomials, k usually will be finite (although one notes that in the context of the Grothendieck ring, the polynomials also are meaningful for infinite k).

For the sake of convenience in writing, we will say the *Grothendieck polynomial of* Γ to refer to the Grothendieck polynomial of the k-constructible set $\mathcal{F}_k(\Gamma)$.

3.5.1 Resolution of edges

Let $\Gamma = (V, E, \mathbf{I})$ be a loose graph, and let $e \in E$ be incident with two distinct vertices v_1, v_2. The *resolution* of Γ *along* e, denoted Γ_e, is the loose graph which is obtained from Γ by deleting e and adding two new loose edges (each with one vertex) e_1 and e_2, where v_i is incident with e_i, $i = 1, 2$.

One observes that

$$\dim(\mathbf{P}(\Gamma_e)) = \dim(\mathbf{P}(\Gamma)) + 2, \tag{3.27}$$

because $\overline{\Gamma_e}$ has 2 more vertices than $\overline{\Gamma}$.

3.5.2 The loose graphs $\Gamma(u, v; m)$

We define $\Gamma(u, v; m)$ (figure 3.5), with $m \in \mathbb{N}$ and u, v symbols, to be the loose graph with adjacent vertices u, v; precisely m common neighbors of u and v and no further incidences.

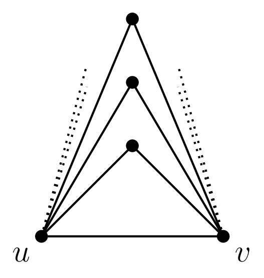


Figure 3.5: The loose graph $\Gamma(u, v; m)$.

The corresponding k-constructible set consists of two affine $(m + 1)$-spaces $\mathbb{A}_u$ and $\mathbb{A}_v$ and m additional closed points in their spaces at infinity, of which the union covers all the points of the projective $(m+1)$-space $\mathbf{P}(\Gamma(u,v;m))$ up to all points of the intersection γ of their spaces at infinity (which is a projective $(m-1)$-space), except m points in γ in general position. So the Grothendieck polynomial is

$$\sum_{i=0}^{m+1}\mathbb{L}^i - ((\sum_{i=0}^{m-1}\mathbb{L}^i) - m) = \mathbb{L}^{m+1} + \mathbb{L}^m + m. \tag{3.28}$$

Example 3.5.1. If $m = 1$, $\Gamma(u,v;1)$ is the complete graph K_3 (figure 3.6) and, according to the above formula, its Grothendieck polynomial is $\mathbb{L}^2 + \mathbb{L} + 1$, which coincides with the number of points of a projective plane over a finite field k if $\mathbb{L}$ is substituted by $|k|$. In the case where $m = 2$, one gets for $\Gamma(u,v;2)$ a complete graph K_4 without an edge (figure 3.6). Its Grothendieck polynomial will then be $\mathbb{L}^3 + \mathbb{L}^2 + 2$, which is exactly the number of points of a projective 3-space without a multiplicative group when $\mathbb{L}$ is substituted by $|k|$.

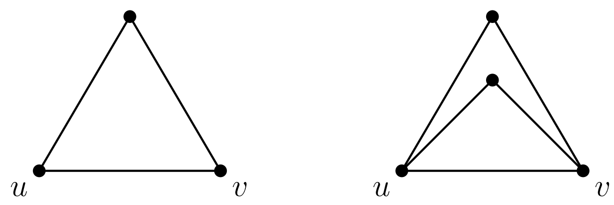


Figure 3.6: The loose graphs $\Gamma(u,v;1)$ and $\Gamma(u,v;2)$.

Adding some loose edges to u and v

Now consider a graph Γ which consists of a $\Gamma(u,v;m)$, r further edges incident with u, and s further edges on v ($r,s \in \mathbb{N}$). We suppose that these edges are loose, but as we have seen before, if they would contain some more vertices (say $c \leq r+s$ vertices), then we just add c to the Grothendieck polynomial below. For further reference, denote such a loose graph by $\Gamma((u,r),(v,s);m)$.

Obviously, on the level of constructible sets, the only intersections occur in the projective space $\mathbf{P}(\Gamma(u,v;m)) \subseteq \mathbf{P}(\Gamma((u,r),(v,s);m))$, so the Grothendieck polynomial is

$$(\mathbb{L}^{r+m+1} - \mathbb{L}^{m+1}) + (\mathbb{L}^{s+m+1} - \mathbb{L}^{m+1}) + \mathbb{L}^{m+1} + \mathbb{L}^m + m =$$

$$\mathbb{L}^{r+m+1} + \mathbb{L}^{s+m+1} - \mathbb{L}^{m+1} + \mathbb{L}^m + m. \tag{3.29}$$

Remark 3.5.2. Put $r = 0 = s$ and $m = 1$; then we obtain again the projective $\mathbb{F}_1$-plane (with Grothendieck polynomial $\mathbb{L}^2 + \mathbb{L} + 1$). In general, put $r = s = 0$; then we obtain (3.28).

Adding further graph structure on the common neighbors

Define $\Gamma(u, v; G(m))$, with u, v symbols, to be the loose graph with adjacent vertices u, v; m common neighbors of u and v, and with the graph G defined on the common neighbors. (If the graph G has no edges, we are back in formula (3.28)). For generality's sake, let r further edges be incident with u, and s further edges with v ($r, s \in \mathbb{N}$). As before, we suppose that these edges are loose.

Then in the same way as above one calculates the Grothendieck polynomial to be

$$\mathbb{L}^{r+m+1} + \mathbb{L}^{s+m+1} - \mathbb{L}^{m+1} + \mathbb{L}^{m} + \mathbb{P}(G), \tag{3.30}$$

where $\mathbb{P}(G)$ is the Grothendieck polynomial of G. Notice that $\mathbb{P}(G)$ can be simply added to the polynomial because the constructible subset associated to G is inside the projective $(m-1)$-subspace defined by the common neighbors of u and v, and this projective space is a closed constructible subset of the projective space $\mathbf{P}(\Gamma(u, v; G(m)))$. The relations in the Grothendieck ring of schemes prove the claim.

3.5.3 The loose graphs $\Gamma(u, v; m)_{uv}$

We define $\Gamma(u, v; m)_{uv}$ to be the loose graph resulting from resolving $\Gamma(u, v; m)$ along the edge uv (figure 3.7). The corresponding k-constructible sets consist of two disjoint affine $(m+1)$-spaces $\mathbb{A}_u$ and $\mathbb{A}_v$ (of which the hyperplanes at infinity intersect in the projective $(m-1)$-space generated by $v_1, \ldots, v_m$) and m additional mutually disjoint affine planes α_i, $i = 1, \ldots, m$, in the projective $(m+3)$-space $\mathbf{P}(\Gamma(u, v; m))$ such that for each j, $\alpha_j \cap \mathbb{A}_u \cong \alpha_j \cap \mathbb{A}_v$ is a projective line minus two points.

The Grothendieck polynomial is

$$2\mathbb{L}^{m+1} + m\mathbb{L}^2 - 2m(\mathbb{L} - 1). \tag{3.31}$$

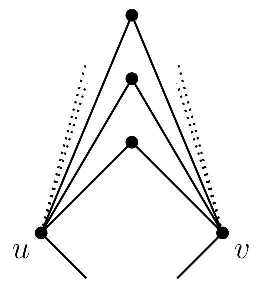


Figure 3.7: Resolution of $\Gamma(u, v; m)$ along the edge uv.

Adding some loose edges to u and v

As before, consider a more general loose graph Γ_{uv}, but which has r loose edges on u, and s loose edges on v, the case $r = 1 = s$ giving $\Gamma(u, v; m)_{uv}$. For further reference, denote such a loose graph by $\Gamma((u, r), (v, s); m)_{uv}$. Then the Grothendieck polynomial is

$$\mathbb{L}^{r+m} + \mathbb{L}^{s+m} + m\mathbb{L}^2 - 2m(\mathbb{L} - 1). \tag{3.32}$$

Remark 3.5.3. Put $r = 0 = s$ and $m = 1$; then we obtain the example 2.3.1 studied in section 2.3 with Grothendieck polynomial $\mathbb{L}^2 + 2$.

Adding further graph structure on the common neighbors

Now consider a graph Γ_{uv} defined as in the previous subsection, but where some graph G is defined on the m common neighbors of u and v. Denote this loose graph by $\Gamma((u, r), (v, s); G)_{uv}$, and remark that when G has no edges, we are back in the previous subsection.

We claim that the Grothendieck polynomial is

$$\mathbb{L}^{r+m} + \mathbb{L}^{s+m} + \mathbb{P}(G)\mathbb{L}^2 - 2\mathbb{P}(G)(\mathbb{L} - 1) = \mathbb{L}^{r+m} + \mathbb{L}^{s+m} + \mathbb{P}(G)(\mathbb{L} - 1)^2 + \mathbb{P}(G). \tag{3.33}$$

We will prove that this is indeed the right polynomial by using a construction of a "graph cone."

3.5.4 General cones

Let K_m be the complete graph on m vertices, and let G_1 and G_2 be subgraphs such that $\mathbf{P}(G_1) \cap \mathbf{P}(G_2) = \emptyset$ (noting that $\mathbf{P}(G_1) \cup \mathbf{P}(G_2) \subset \mathbf{P}(K_m)$). Define the *cone* with *base* G_2 and *vertex* G_1, denoted $C(G_2, G_1)$, as the subgraph which contains G_1 and G_2, and all the edges of K_m which connect a vertex of G_2 with a vertex of G_1. Note that $C(G_2, G_1) = C(G_1, G_2)$ and that, when G_1 is only a vertex, the cone $C(G_2, G_1)$ is indeed a cone as we know it from classical geometry.

Theorem 3.5.4. *Let the number of vertices of G_1 be m_1 and the number of vertices of G_2 be m_2. We have that the Grothendieck polynomial (in $\mathbb{Z}[\mathbb{L}]$), where k is any finite field, is given by*

$$\mathbb{P}(C(G_2, G_1)) = \mathbb{P}(G_1)\mathbb{L}^{m_2} + \mathbb{P}(G_2)\mathbb{L}^{m_1} - \mathbb{P}(G_1)\mathbb{P}(G_2)(\mathbb{L} - 1). \tag{3.34}$$

Proof. Let $k = \mathbb{F}_q$ be a finite field with q elements, and consider $\mathcal{F}(C(G_2, G_1)) \otimes_{\mathbb{F}_1} k$. Then for each point ν of $\mathcal{F}(G_1)$, respectively $\mathcal{F}(G_2)$, the cone $\mathcal{F}(C(G_2, G_1)) \otimes_{\mathbb{F}_1} k$ contains an affine m_2-space defined by ν and the vertices of G_2, respectively an affine m_1-space defined by ν and the vertices of G_1. Two by two, these affine m_2-spaces, respectively affine m_1-spaces, are disjoint and the union of $\mathcal{F}(G_1)$, $\mathcal{F}(G_2)$ and the affine m_1- and m_2-spaces thus obtained is the set of k-points of $\mathcal{F}(C(G_2, G_1)) \otimes_{\mathbb{F}_1} k$. Now take two points ν_1 of $\mathcal{F}(G_1)$ and ν_2 of $\mathcal{F}(G_2)$, then the intersection of the m_2-space associated to ν_1 and the m_1-space associated to ν_2 is a projective line minus the two points ν_1 and ν_2.

Besides, due to the fact that the m_2-affine spaces added by the cone construction on the points of $\mathcal{F}(G_1)$ are disjoint two by two, all intersections of the affine spaces associated to vertices of G_1 inside $\mathcal{F}(C(G_2, G_1))$ happen on the level of $\mathcal{F}(G_1)$. The same facts occur with the m_1-affine spaces and $\mathcal{F}(G_2)$. Hence, one can count the rational points of the set $\mathcal{F}(C(G_2, G_1)) \otimes_{\mathbb{F}_1} k$ by multiplying the polynomial $\mathbb{P}(G_1)$ (respectively $\mathbb{P}(G_2)$), which carries the information of the dimensions and intersections of the affine spaces on the level of $\mathcal{F}(G_1)$ (respectively $\mathcal{F}(G_2)$), by a factor $\mathbb{L}^{m_2}$ (respectively $\mathbb{L}^{m_1}$). The order of $\mathcal{F}(C(G_2, G_1)) \otimes_{\mathbb{F}_1} k$ is then

$$\mathbb{P}(G_1)(q)q^{m_2} + \mathbb{P}(G_2)(q)q^{m_1} - \mathbb{P}(G_1)(q)\mathbb{P}(G_2)(q)(q-1), \tag{3.35}$$

where the last term stands for the points which are double counted due to the intersections of the m_1- and m_2-affine spaces. ■

Putting G_1 equal to a graph consisting of two vertices, one obtains the formula of subsection 3.5.3 after having added r and s loose edges to these vertices. In fact, one easily calculates the Grothendieck polynomial of a general cone with base and vertex loose graphs.

Theorem 3.5.5. *Let G_1 and G_2 be loose graphs (disjoint, as above). Let the number of vertices of G_1 be m_1 and the number of vertices of G_2 be m_2. Let $\widetilde{G_1}$ and $\widetilde{G_2}$ be their respective reduced graphs (the ones without loose edges). Denote the degree of a vertex v in G_i by $\deg_{G_i}(v)$, $i = 1, 2$. Then we have that the Grothendieck polynomial of the cone $C(G_2, G_1)$ (in $\mathbb{Z}[\mathbb{L}]$), where k is any finite field, is given by*

$$\mathbb{P}(C(G_2, G_1)) = \mathbb{P}(\widetilde{G_1})\mathbb{L}^{m_2} + \mathbb{P}(\widetilde{G_2})\mathbb{L}^{m_1} - \mathbb{P}(\widetilde{G_1})\mathbb{P}(\widetilde{G_2})(\mathbb{L}-1)$$

$$+\mathbb{L}^{m_2}\sum_{v\in G_1}(\mathbb{L}^{\deg_{G_1}(v)} - \mathbb{L}^{\deg_{\widetilde{G_1}}(v)}) + \mathbb{L}^{m_1}\sum_{v\in G_2}(\mathbb{L}^{\deg_{G_2}(v)} - \mathbb{L}^{\deg_{\widetilde{G_2}}(v)}). \tag{3.36}$$

Proof. By Theorem 3.5.4, we know the Grothendieck polynomial of $C(\widetilde{G_2}, \widetilde{G_1})$ for any finite field k. Now for each vertex $u \in G_i$, add a term $\mathbb{L}^{\deg_{G_i}(u)+m_j} - \mathbb{L}^{\deg_{\widetilde{G_i}}(u)+m_j}$, where $\{i, j\} = \{1, 2\}$. ■

As a direct consequence, we obtain a simple formula when the vertex of the cone is a graph consisting of one single vertex.

Corollary 3.5.6. *Let G_1 be the graph consisting of one vertex v and let G_2 be a loose graph with m_2 vertices. Then*

$$\mathbb{P}(C(G_2, G_1)) = \mathbb{L}^{m_2} + \mathbb{P}(G_2).$$

Loose graph cones versus "classical cones" — Caution!

Let G_1 and G_2 be loose graphs, and let k be a field. As above, we see G_1 and G_2 as being embedded in some $\mathbb{F}_1$-projective space, and they generate subspaces which are disjoint. For A and B two disjoint point sets in a projective space $\mathbb{P}_k$ over k, by $A \times B$ we denote the set of points which are on lines containing a point of A and a point of B.

One might be tempted to think that the following identity holds:

$$\mathcal{F}(C(G_1, G_2)) \otimes_{\mathbb{F}_1} k \cong (\mathcal{F}(G_1) \otimes_{\mathbb{F}_1} k) \times (\mathcal{F}(G_2) \otimes_{\mathbb{F}_1} k), \tag{3.37}$$

that is, that the following diagram commutes:

$$\begin{tikzcd}
G_1, G_2 \arrow[r, "C"] \arrow[d, "(\otimes_{\mathbb{F}_1} k)\circ(\mathcal{F})"] & C(G_1, G_2) \arrow[d, "(\otimes_{\mathbb{F}_1} k)\circ(\mathcal{F})"] \\
\mathcal{F}(G_1) \otimes_{\mathbb{F}_1} k,\ \mathcal{F}(G_2) \otimes_{\mathbb{F}_1} k \arrow[r, "\times"] & (\mathcal{F}(G_1) \otimes_{\mathbb{F}_1} k) \times (\mathcal{F}(G_2) \otimes_{\mathbb{F}_1} k)
\end{tikzcd}$$

In general, this is not the case. We will give two simple examples in which (taken that k is finite), respectively, $|\mathcal{F}(C(G_1, G_2)) \otimes_{\mathbb{F}_1} k| > |(\mathcal{F}(G_1) \otimes_{\mathbb{F}_1} k) \times (\mathcal{F}(G_2) \otimes_{\mathbb{F}_1} k)|$ and $|\mathcal{F}(C(G_1, G_2)) \otimes_{\mathbb{F}_1} k| < |(\mathcal{F}(G_1) \otimes_{\mathbb{F}_1} k) \times (\mathcal{F}(G_2) \otimes_{\mathbb{F}_1} k)|$. So there isn't even a fixed direction in which inclusion would work for general examples.

Example 3.5.7. Let G_2 the graph of a projective line over $\mathbb{F}_1$, and let G_1 be the loose graph of a projective $\mathbb{F}_1$-plane without a multiplicative group (i.e., an affine $\mathbb{F}_1$-plane with two extra points at infinity). Then the Grothendieck polynomial of $C(G_2, G_1)$ (see figure 3.8) is $\mathbb{L}^4 + \mathbb{L}^3 + \mathbb{L}^2 + 2$, while the Grothendieck polynomial of $(\mathcal{F}(G_1) \otimes_{\mathbb{F}_1} k) \times (\mathcal{F}(G_2) \otimes_{\mathbb{F}_1} k)$ is

$$(\mathbb{L} + 1)(\mathbb{L}^2 + 2)(\mathbb{L} - 1) + (\mathbb{L} + 1) + (\mathbb{L}^2 + 2) = \mathbb{L}^4 + 2\mathbb{L}^2 + \mathbb{L} + 1. \tag{3.38}$$

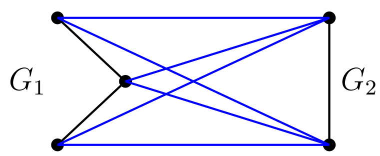


Figure 3.8: Cone constructed from a projective plane without a multiplicative group and a projective line.

Note that when the two points at infinity of G_1 wouldn't be there, both constructions *would* yield the same number of points. (When those two points are then added in G_1, the two closed points in G_2 suddenly see 4-dimensional spaces instead of planes.)

Example 3.5.8. Let both G_1 and G_2 be the loose graph of an affine plane over $\mathbb{F}_1$. Then the Grothendieck polynomial of $C(G_2, G_1)$ (figure 3.9) is $2\mathbb{L}^3 - \mathbb{L} + 1$, while the Grothendieck polynomial of $(\mathcal{F}(G_1) \otimes_{\mathbb{F}_1} k) \times (\mathcal{F}(G_2) \otimes_{\mathbb{F}_1} k)$ is

$$\mathbb{L}^2 \cdot \mathbb{L}^2 \cdot (\mathbb{L} - 1) + 2\mathbb{L}^2 = \mathbb{L}^5 - \mathbb{L}^4 + 2\mathbb{L}^2. \tag{3.39}$$

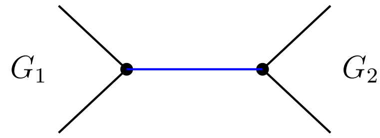


Figure 3.9: Cone constructed from two affine planes.

3.5.5 Affection Principle

Having studied a number of local situations, we now determine what happens when one resolves an edge in a general finite loose graph. For that purpose, we consider a finite loose graph Γ, and let $\mathbb{P}(\Gamma)$ be its Grothendieck polynomial. We choose an edge uv which is not loose, and we compare $\mathbb{P}(\Gamma)$ and $\mathbb{P}(\Gamma_{uv})$.

We have seen in the previous subsections that for each finite field $\mathbb{F}_q$, the number of $\mathbb{F}_q$-rational points of $\mathcal{F}(\Gamma) \otimes_{\mathbb{F}_1} \mathbb{F}_q$ is given by substituting the value q for the indeterminate in $\mathbb{P}(\Gamma)$. As locally each closed point of $\mathcal{F}(\Gamma) \otimes_{\mathbb{F}_1} \mathbb{F}_q =: \mathcal{X}_q$ corresponding to a vertex yields an affine space (of which the dimension is the degree of the vertex), its number of points can be expressed through the Inclusion-Exclusion principle. Call the vertices of Γ $v_1, \ldots, v_r$, and let for each v_i, $\mathbb{A}_i$ be the local affine space at v_i of dimension $\deg(v_i)$. Then one can calculate the number of points (over any $\mathbb{F}_q$) through the expression

$$\sum_{i=1}^{r} (-1)^{i+1} \Big(\sum_{1 \leq j_1 < \cdots < j_i \leq r} |\mathbb{A}_{j_1} \cap \cdots \cap \mathbb{A}_{j_i}| \Big). \tag{3.40}$$

So to start with, we have to control the intersections of type $\mathbb{A}_x \cap \mathbb{A}_y$ — in other words, the intersections $\overline{\mathbb{A}_x} \cap \overline{\mathbb{A}_y}$ (since the former intersections are controlled by the behavior at infinity). Here, $\overline{\mathbb{A}}$ denotes the projective completion of $\mathbb{A}$.

Calling $\mathrm{d}(\cdot, \cdot)$ the distance function in Γ defined on $V \times V$, V being the vertex set (so that, for example, $\mathrm{d}(s, t)$, with s and t distinct vertices, is the number of edges in a shortest path from s to t), and taking into account the intersection of affine spaces explained in subsection 2.3.3, we will show we only need to consider what happens in the vertex set

$$\overline{\mathbf{B}}(u, 1) \cup \overline{\mathbf{B}}(v, 1) \tag{3.41}$$

when resolving uv, where $\overline{\mathbf{B}}(c, k) := \{v \in V \mid \mathrm{d}(c, v) \leq k\}$. Below, we will also use the notation $v^\perp = \{v\}^\perp$ for $\{w \in V \mid \mathrm{d}(v, w) = 1\}$, and then $\{v, v'\}^\perp := v^\perp \cap v'^\perp$. In particular, we have that $\overline{\mathbf{B}}(c, 1) = \{c\} \cup c^\perp$.

The two next lemmas are immediate.

Lemma 3.5.9. *Let Γ be a finite connected loose graph, and let u and v be distinct vertices. Let k be any field, and consider the k-constructible set $\mathcal{F}(\Gamma) \otimes_{\mathbb{F}_1} k$. Then if* $\mathrm{d}(u, v) \geq 2$, *we have that*

$$\mathbb{A}_u \cap \mathbb{A}_v = \emptyset. \tag{3.42}$$

(If $\mathrm{d}(u, v) \geq 3$, $\overline{\mathbb{A}_u} \cap \overline{\mathbb{A}_v} = \emptyset$.*)*

Lemma 3.5.10. *Let Γ be a finite connected loose graph, and let A be a set of distinct vertices. Let k be any field, and consider the k-constructible set $\mathcal{F}(\Gamma) \otimes_{\mathbb{F}_1} k$. If $\cap_{a \in A} \mathbb{A}_a = \emptyset$, then this intersection remains empty after resolving an arbitrary edge.*

Theorem 3.5.11 (Affection Principle)**.** *Let Γ be a finite connected loose graph, let xy be an edge on the vertices x and y, and let S be a subset of the vertex set. Let k be any finite field, and consider $\mathcal{F}(\Gamma) \otimes_{\mathbb{F}_1} k$. Then $\cap_{s \in S} \mathbb{A}_s$ changes when one resolves the edge xy only if $|S \cap \Big(\overline{\mathbf{B}}(x, 1) \cup \overline{\mathbf{B}}(y, 1)\Big)| \geq 2$.*

Proof. We first handle the case when $|S| = 2$, so we put $S = \{u, v\}$. As the loose edges on u and v clearly play no role in any change which could occur on $\mathbb{A}_u \cap \mathbb{A}_v$ when resolving xy, we will work w.l.o.g. in the reduced graph $\widetilde{\Gamma}$, which is Γ without loose edges; we will keep using the same notation for x, y, u, v.

Clearly if $xy \notin \mathbf{P}(\overline{\mathbf{B}}(u, 1) \cup \overline{\mathbf{B}}(v, 1)) =: \mathbf{P}_{u,v}$, then $\mathbb{A}_u \cap \mathbb{A}_v$ will not change through resolving xy, as $\overline{\mathbb{A}_u} \cup \overline{\mathbb{A}_v} \subseteq \mathbf{P}_{u,v}$.

Now suppose that (e.g.) $u \notin \overline{\mathbf{B}}(x, 1) \cup \overline{\mathbf{B}}(y, 1)$; then obviously $v \in \{x, y\} \cup \{x, y\}^{\perp}$ (as in each of those cases $xy \in \overline{\mathbb{A}_v}$, and otherwise not). For, if $xy \in \mathbf{P}_{u,v}$, then $\{x, y\} \subseteq (u^{\perp} \cup \{u\}) \cup (v^{\perp} \cup \{v\})$; if $u \notin \overline{\mathbf{B}}(x, 1) \cup \overline{\mathbf{B}}(y, 1)$ ($y \not\sim u \not\sim x$ and $y \neq u \neq x$), then $x, y \in v^{\perp} \cup \{v\}$, so $v \in \{x, y\} \cup \{x, y\}^{\perp}$.

Suppose first that $v = x$. If $\mathrm{d}(x, u) \geq 3$, then $\overline{\mathbb{A}_u} \cap \overline{\mathbb{A}_v} = \emptyset$ by Lemma 3.5.9, and after resolution of xy this stays the empty set. If $\mathrm{d}(x, u) = 2$, then $\overline{\mathbb{A}_u} \cap \overline{\mathbb{A}_v}$ remains unchanged when resolving xy. For, first note that $u \not\sim y$ as $u \notin \overline{\mathbf{B}}(x, 1) \cup \overline{\mathbf{B}}(y, 1)$. Consider $\mathbf{P}_{u,v} = \mathbf{P}_{u,x} = \mathbf{P}(\overline{\mathbf{B}}(u, 1) \cup \overline{\mathbf{B}}(x, 1))$ in Γ; since $\langle \overline{\mathbb{A}_u}, \overline{\mathbb{A}_x} \rangle = \mathbf{P}_{u,x}$ and $\dim(\mathbf{P}_{u,x})$ does not change after resolving xy (in Γ_{xy} we also have $\langle \overline{\mathbb{A}_u}, \overline{\mathbb{A}_x} \rangle = \mathbf{P}_{u,x}$ and the dimensions of $\overline{\mathbb{A}_u}$, $\overline{\mathbb{A}_x}$ remain the same), we have that $\overline{\mathbb{A}_u} \cap \overline{\mathbb{A}_x}$ does not change either ($\dim(\overline{\mathbb{A}_u} \cap \overline{\mathbb{A}_x})$ is in both cases determined by $|u^{\perp} \cap x^{\perp}|$).

The case $v = y$ is of course similar.

Now let $v \in x^{\perp} \cap y^{\perp}$. By considering the cases $\mathrm{d}(u, v) = 1$, $\mathrm{d}(u, v) = 2$ and $\mathrm{d}(u, v) \geq 3$, one again easily concludes that $\overline{\mathbb{A}_u} \cap \overline{\mathbb{A}_v}$ remains unchanged when resolving xy.

We conclude that *if* $\overline{\mathbb{A}_u} \cap \overline{\mathbb{A}_v}$ changes after resolving xy, then $\{u, v\} \subseteq \overline{\mathbf{B}}(x, 1) \cup \overline{\mathbf{B}}(y, 1)$. The general statement now immediately follows. ∎

In the next corollary, $\Gamma_{|\mathbf{P}_{u,v}}$ (e.g.) is the loose graph induced by Γ on the vertex set $\overline{\mathbf{B}}(u, 1) \cup \overline{\mathbf{B}}(v, 1)$.

Corollary 3.5.12 (Geometrical Affection Principle)**.** *Let Γ be a finite connected loose graph, let xy be an edge on the vertices x and y, and let k be any finite field. The difference in the number of k-points of $\mathcal{F}(\Gamma) \otimes_{\mathbb{F}_1} k$ and $\mathcal{F}(\Gamma_{xy}) \otimes_{\mathbb{F}_1} k$ is*

$$\Big|\mathcal{F}(\Gamma_{|\mathbf{P}_{x,y}}) \otimes_{\mathbb{F}_1} k\Big|_k - \Big|\mathcal{F}(\Gamma_{xy|\mathbf{P}_{x,y}}) \otimes_{\mathbb{F}_1} k\Big|_k. \tag{3.43}$$

In this expression, Γ may be chosen to be reduced (but after resolving xy, one is of course not allowed to reduce Γ_{xy}).

Proof. Suppose that u and v are distinct vertices of Γ, and suppose that $\mathbb{A}_u \cup \mathbb{A}_v$ changes through resolution of xy. By Theorem 3.5.11, we have that

$$\{u, v\} \subseteq \overline{\mathbf{B}}(x, 1) \cup \overline{\mathbf{B}}(y, 1). \tag{3.44}$$

By inspection of the possibilities, one reasons that $\{u, v\} \cap \{x, y\} \neq \emptyset$ (see section 3.7 for more details). This implies that $\Big(\mathbb{A}_u \cup \mathbb{A}_v\Big) \setminus \Big(\mathcal{F}(\Gamma_{|\mathbf{P}_{x,y}}) \otimes_{\mathbb{F}_1} k\Big)$ does not change through resolution of xy. Letting u, v vary over $V \times V$, where V is the vertex set of Γ, and considering $\Big|\cup_{v\in V} \mathbb{A}_v\Big|_k = \Big|\mathcal{F}(\Gamma) \otimes_{\mathbb{F}_1} k\Big|_k$ before and after resolution, the statement follows. ■

In terms of Grothendieck polynomials, we have the following corollary.

Corollary 3.5.13 (Polynomial Affection Principle)**.** *Let Γ be a finite connected loose graph, let xy be an edge on the vertices x and y, and let k be any finite field. Then we have*

$$\mathbb{P}(\Gamma) - \mathbb{P}(\Gamma_{xy}) = \mathbb{P}(\Gamma_{|\mathbf{P}_{x,y}}) - \mathbb{P}(\Gamma_{xy|\mathbf{P}_{x,y}}). \tag{3.45}$$

3.5.6 Polynomial Affection Principle: calculation

Let Γ be a finite connected loose graph, and let e be an edge with vertices x and y. Applying the Polynomial Affection Principle, we want to calculate

$$\mathbb{P}(\Gamma) - \mathbb{P}(\Gamma_{xy}) = \mathbb{P}(\Gamma_{|\mathbf{P}_{x,y}}) - \mathbb{P}(\Gamma_{xy|\mathbf{P}_{x,y}}), \tag{3.46}$$

in terms of certain data inside $\overline{\mathbf{B}}(x, 1) \cup \overline{\mathbf{B}}(y, 1)$.

As before, we are allowed to assume that Γ is reduced (but not after resolution of e). Also, by the Affection Principle, we only need to calculate the difference $\mathbb{P}(\Gamma_{|\mathbf{P}_{x,y}}) - \mathbb{P}(\Gamma_{xy|\mathbf{P}_{x,y}})$, so that we may replace Γ by $\Gamma_{|\mathbf{P}_{x,y}}$. We keep using the notation "Γ" for the sake of convenience.

Before resolution

Define Δ to be the graph (not the loose graph!) which is induced by Γ on the vertices of $(\overline{\mathbf{B}}(x,1) \cup \overline{\mathbf{B}}(y,1)) \setminus \{x,y\}$, and let G be the graph (not the loose graph!) which is induced by Γ on the vertices of $(\overline{\mathbf{B}}(x,1) \cap \overline{\mathbf{B}}(y,1)) \setminus \{x,y\}$. (For a representation of the ball $\overline{\mathbf{B}}(x,1) \cup \overline{\mathbf{B}}(y,1)$, see figure 3.10.) It is important to note that x and y are not contained in G. Let G^L be the *loose* graph which contains as vertex set the vertices of G, and as edge set the edges of Δ which contain a vertex of G and a vertex of Δ. (Note that G is a subgraph of G^L.) Let G^L_x be the loose graph which has as vertex set the vertices of G, and as edge set the edges of Δ which contain a vertex of G and a vertex of $\overline{\mathbf{B}}(x,1)$ $(\setminus\{y\})$. Define G^L_y similarly, and note that G is a subgraph of both G^L_x and G^L_y.

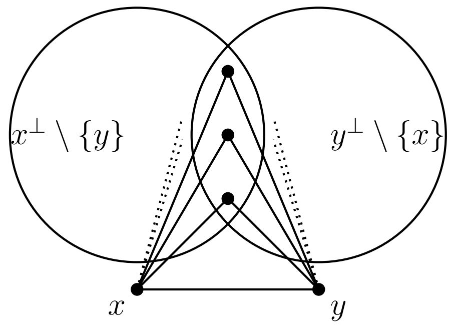


Figure 3.10: The ball $\overline{\mathbf{B}}(x,1) \cup \overline{\mathbf{B}}(y,1)$.

By the Inclusion-Exclusion principle, we know that $\mathbb{P}(\Gamma)$ is given by the expression

$$\mathbb{P}(\Gamma) = \mathbb{P}(\mathbb{A}_x) + \mathbb{P}(\mathbb{A}_y) + \mathbb{P}(C(G^L, xy)) - \mathbb{P}(\mathbb{A}_x \cap C(G^L, xy))$$

$$-\mathbb{P}(\mathbb{A}_y \cap C(G^L, xy)) - \mathbb{P}(\mathbb{A}_x \cap \mathbb{A}_y) + \mathbb{P}(\mathbb{A}_x \cap \mathbb{A}_y \cap C(G^L, xy)). \tag{3.47}$$

Here xy is seen as the graph on the vertices x and y (i.e., a projective line).

As $\mathbb{A}_x \cap \mathbb{A}_y \subseteq C(G^L, xy)$, we have that

$$\mathbb{A}_x \cap \mathbb{A}_y \cap C(G^L, xy) = \mathbb{A}_x \cap \mathbb{A}_y, \tag{3.48}$$

so that the last two terms in Equation (3.47) cancel each other. As we will see, it will be easier to calculate the following equivalent expression:

$$\mathbb{P}(\Gamma) = \mathbb{P}(\mathbb{A}_x) + \mathbb{P}(\mathbb{A}_y) + \mathbb{P}(\Delta) + (\mathbb{P}(C(G^L, xy)) - \mathbb{P}(G^L))$$

$$-\mathbb{P}((\mathbb{A}_x \cap C(G^L, xy)) \setminus G^L) - \mathbb{P}((\mathbb{A}_y \cap C(G^L, xy)) \setminus G^L). \tag{3.49}$$

Note that in this last equation, since G^L is a loose subgraph of Δ, after adding $\mathbb{P}(\Delta)$ to the polynomials we need to subtract some terms depending on $\mathbb{P}(G^L)$ to avoid double counting of points. Also, we have that

$$\begin{cases} \mathbb{P}(\mathbb{A}_x \cap C(G^L, xy)) \setminus G^L) = \mathbb{P}(C(G^L_x, xy)) - \mathbb{P}(C(G^L_x, y)) \\ \mathbb{P}(\mathbb{A}_y \cap C(G^L, xy)) \setminus G^L) = \mathbb{P}(C(G^L_y, xy)) - \mathbb{P}(C(G^L_y, x)). \end{cases} \tag{3.50}$$

Remark 3.5.14. Notice that for all computations we replaced Γ by $\Gamma_{|\mathbf{P}_{x,y}}$ but we did not consider edges of the form ux or vy, where $u \notin y^\perp$ and $v \notin x^\perp$. The reason *why* these edges are not necessary to be considered relies on the fact that the spaces $\mathbb{A}_u$, $\mathbb{A}_v$, $\overline{\mathbb{A}_u}$ and $\overline{\mathbb{A}_v}$ do not change through the resolution of the edge xy. However, we can not omit vertices that are also adjacent to some vertex of G because the spaces $\mathbb{A}_w$ and $\overline{\mathbb{A}_w}$, for $w \in G$, are indeed affected by the surgery process. For a more detailed proof of this property, we refer to Lemma 3.7.6.

All other edge types inside $\overline{\mathbf{B}}(x,1) \cup \overline{\mathbf{B}}(y,1)$ are considered either in Δ or in $C(G^L, xy)$.

After resolution

After having resolved the edge e, we make a similar calculation (which is in fact a bit easier because the intersection of $\mathbb{A}_x$ and $\mathbb{A}_y$ is also easier). We keep using the same notation. In figure 3.11, we can see how the ball $\overline{\mathbf{B}}(x,1) \cup \overline{\mathbf{B}}(y,1)$ looks like after resolving the edge xy.

Instead of considering the loose graph cone $C(G^L, xy)$ in the Inclusion-Exclusion principle, we only have to consider the loose graph $G^L_{(2)}$, which is G^L with two loose edges added per vertex (one for x and one for y), in order to reach all the points of $\mathcal{F}(\Gamma) \otimes_{\mathbb{F}_1} (k)$. So our starting point is

$$\begin{aligned} \mathbb{P}(\Gamma) = \mathbb{P}(\mathbb{A}_x) + \mathbb{P}(\mathbb{A}_y) + \mathbb{P}(\Delta) + (\mathbb{P}(G^L_{(2)}) - \mathbb{P}(G^L)) \\ - \mathbb{P}((\mathbb{A}_x \cap G^L_{(2)}) \setminus G^L) - \mathbb{P}((\mathbb{A}_y \cap G^L_{(2)}) \setminus G^L), \end{aligned} \tag{3.51}$$

remarking that we immediately started with the more simple equation, and that as in the previous subsection the terms involving $\mathbb{A}_x \cap \mathbb{A}_y$ cancel out.

We immediately obtain a simple expression in the following theorem.

Theorem 3.5.15 (Reduction to components). *Let $\{\mathcal{C}^j | j \in J\}$ be the set of connected components of G^L. Furthermore, for each $j \in J$, let $\mathcal{C}^j \cap G^L_x$ be denoted by $\mathcal{C}^j_x$ (and use a similar notation for y). Then*

$$\begin{aligned} \mathbb{P}(\Gamma) = \mathbb{P}(\mathbb{A}_x) + \mathbb{P}(\mathbb{A}_y) + \mathbb{P}(\Delta) + \sum_{j \in J} (\mathbb{L}^2 - 1)\mathbb{P}(\mathcal{C}^j) \\ - \sum_{j \in J} (\mathbb{L} - 1)\mathbb{P}(\mathcal{C}^j_x) - \sum_{j \in J} (\mathbb{L} - 1)\mathbb{P}(\mathcal{C}^j_y). \end{aligned} \tag{3.52}$$

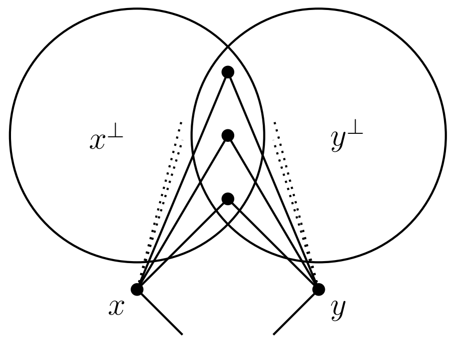


Figure 3.11: After resolution of the edge xy.

It is important to note that for each expression $\mathbb{P}(\mathcal{C}^\ell)$, $\mathbb{P}(\mathcal{C}_x^\ell)$ and $\mathbb{P}(\mathcal{C}_y^\ell)$, one has to take the embedding in respectively G^L, G_x^L and G_y^L into account.

Before starting the proof, it is useful to notice that the connected components of G^L correspond to the connected components of G. Also, note that any $\mathcal{C}_x^j$ and any $\mathcal{C}_y^j$ is connected.

Proof. First note that if G^L is connected, then it follows that

$$\mathbb{P}(\Gamma) = \mathbb{P}(\mathbb{A}_x) + \mathbb{P}(\mathbb{A}_y) + \mathbb{P}(\Delta) + (\mathbb{L}^2 - 1)\mathbb{P}(G^L)$$

$$-(\mathbb{L}-1)\mathbb{P}(G_x^L) - (\mathbb{L}-1)\mathbb{P}(G_y^L). \tag{3.53}$$

Now let there be more than one connected component, and consider an arbitrary component $\mathcal{C}^k$ ($k \in J$). Then $C(\{x,y\}, \mathcal{C}^k) \subseteq \Gamma$, and the Grothendieck polynomial of $C(\{x,y\}, \mathcal{C}^k) \setminus (\mathbb{A}_x \cup \mathbb{A}_y \cup \Delta)$ is $(\mathbb{L}^2-1)\mathbb{P}(\mathcal{C}^k) - (\mathbb{L}-1)\mathbb{P}(\mathcal{C}_x^k) - (\mathbb{L}-1)\mathbb{P}(\mathcal{C}_y^k)$. It is easy to see that Γ is covered by

$$\mathbb{A}_x,\ \mathbb{A}_y,\ \Delta,\ C(\{x,y\}, \mathcal{C}^j)_{|j \in J}, \tag{3.54}$$

so if we prove that for any field k and $i \neq j \in J$ the following holds:

$$\begin{aligned}(\mathcal{F}(C(\{x,y\}, \mathcal{C}^i)) \otimes_{\mathbb{F}_1} k) \cap (\mathcal{F}(C(\{x,y\}, \mathcal{C}^j)) \otimes_{\mathbb{F}_1} k)& \\ \subseteq\ \mathbb{A}_{k,x} \cup \mathbb{A}_{k,y} \cup (\mathcal{F}(\Delta) \otimes_{\mathbb{F}_1} k),&\end{aligned} \tag{3.55}$$

then we are done (since each of $\mathbb{A}_x,\ \mathbb{A}_y,\ \Delta,\ C(\{x,y\}, \mathcal{C}^j)_{|j\in J}$ is contained in $\mathcal{F}(\Gamma) \otimes_{\mathbb{F}_1} k$). Here, and below, as usual we work in the projective space $\langle \mathcal{F}(\Gamma) \otimes_{\mathbb{F}_1} k \rangle$. Let us remark that by $\mathbb{A}_{k,x}$ we mean the affine space over the field k associated to the vertex x of Γ.

So suppose $\mathcal{C}$ and $\widetilde{\mathcal{C}}$ are different connected components of G^L, such that there is a point z in $(\mathcal{F}(C(\{x,y\}, \mathcal{C})) \otimes_{\mathbb{F}_1} k) \cap (\mathcal{F}(C(\{x,y\}, \widetilde{\mathcal{C}})) \otimes_{\mathbb{F}_1} k)$ and which is not contained in $\mathbb{A}_{k,x} \cup \mathbb{A}_{k,y} \cup (\mathcal{F}(\Delta) \otimes_{\mathbb{F}_1} k)$. By the structure of $\xi := \mathcal{F}(C(\{x,y\}, \mathcal{C})) \otimes_{\mathbb{F}_1} k$ and

$\widetilde{\xi} := \mathcal{F}(C(\{x,y\},\widetilde{\mathcal{C}}) \otimes_{\mathbb{F}_1} k$, it follows that there are affine spaces $\mathcal{A} \subseteq \xi \cap (\mathcal{F}(\Delta) \otimes_{\mathbb{F}_1} k)$ and $\widetilde{\mathcal{A}} \subseteq \widetilde{\xi} \cap (\mathcal{F}(\Delta) \otimes_{\mathbb{F}_1} k)$, corresponding to, respectively, a vertex of $\mathcal{C}$ and $\widetilde{\mathcal{C}}$, such that $z \in \langle x, y, \mathcal{A}\rangle \cap \langle x, y, \widetilde{\mathcal{A}}\rangle$. For, $\mathcal{F}(\mathcal{C}) \otimes_{\mathbb{F}_1} k$, respectively $\mathcal{F}(\widetilde{\mathcal{C}}) \otimes_{\mathbb{F}_1} k$, is covered by affine spaces $\{\mathbb{A}_\mu\}_{|\mu}$, respectively $\{\mathbb{A}_\nu\}_{|\nu}$, corresponding to vertices of $\mathcal{C}$, respectively $\widetilde{\mathcal{C}}$, so $\mathcal{F}(C(\{x,y\},\mathcal{C})) \otimes_{\mathbb{F}_1} k$, respectively $\mathcal{F}(C(\{x,y\},\widetilde{\mathcal{C}})) \otimes_{\mathbb{F}_1} k$, is covered by affine spaces $\{\mathbb{A}_{\mu,x,y}\}_{|\mu}$, respectively $\{\mathbb{A}_{\nu,x,y}\}_{|\nu}$, together with $\mathbb{A}_x$ and $\mathbb{A}_y$. Here $\mathbb{A}_{\ell,x,y}$, corresponding to the vertex l, denotes the affine space which contains $\mathbb{A}_\ell$ and the two extra directions defined by the edges lx and ly.

It follows that $z \in \mathcal{A}_{x,y} \cap \widetilde{\mathcal{A}}_{x,y}$ (where we use the same notation as above), and this is the desired contradiction, as these spaces are obviously disjoint. ■

Remark 3.5.16 (On components)**.** The subdivision in connected components as in Theorem 3.5.15 is necessary. Consider for example the graph $\Gamma = (V, E, \mathbf{I})$ with $V = \{x, y, u, v, w\}$ and $E = \{xu, xv, yu, yv, yw, wu, wv\}$. Then G^L consists of the vertices u, v with loose edges uw, vw (embedded in a projective plane), and hence has two connected components (see figure 3.12). If G^L would be considered as being an affine plane with two additional points u, v but without the vertex w (that is, if $\mathbb{P}(G^L) = \mathbb{L}^2 + 1$), the formula

$$\mathbb{P}(\Gamma) = \mathbb{P}(\mathbb{A}_x) + \mathbb{P}(\mathbb{A}_y) + \mathbb{P}(\Delta) + (\mathbb{L}^2 - 1)\mathbb{P}(G^L)$$

$$-(\mathbb{L} - 1)\mathbb{P}(G^L_x) - (\mathbb{L} - 1)\mathbb{P}(G^L_y) \tag{3.56}$$

would yield too many points. In other words: G^L can not be considered (on the polynomial level) as being "Δ without w."

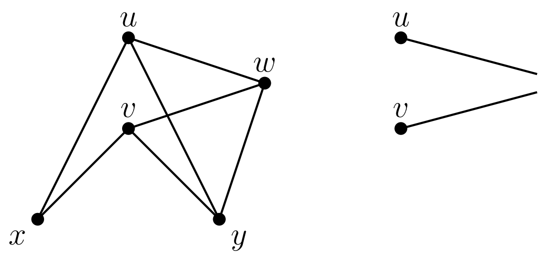


Figure 3.12: The loose graphs Γ and G^L.

The difference

The difference between the polynomials before and after resolution is now given by

$$\mathbb{P}(\Gamma) - \mathbb{P}(\Gamma_{xy}) = (\mathbb{P}(C(G^L, xy)) - \mathbb{P}(G^L)) - \mathbb{P}((\mathbb{A}_x \cap C(G^L, xy)) \setminus G^L)$$
$$-\mathbb{P}((\mathbb{A}_y \cap C(G^L, xy)) \setminus G^L) - \sum_{j\in J}(\mathbb{L}^2 - 1)\mathbb{P}(\mathcal{C}^j)$$
$$+\sum_{j\in J}(\mathbb{L} - 1)\mathbb{P}(\mathcal{C}^j_x) + \sum_{j\in J}(\mathbb{L} - 1)\mathbb{P}(\mathcal{C}^j_y). \tag{3.57}$$

3.5.7 Steps of surgery

In section 3.3 we defined a function to compute the Grothendieck polynomial of any loose tree and in the previous subsections we showed that for any loose graphs Γ and Γ_{xy}, their difference of Grothendieck polynomials can be locally computed in a neighborhood of x and y. In fact, this difference also depends on certain loose graphs with a lower number of vertices or edges. All these features motivated the idea to develop an inductive process, called *surgery*, that allows us to compute the Grothendieck polynomial of any loose graph Γ.

Before describing the process we need to prove the next result. Recall that a *spanning tree* of a graph Γ is a tree which is a subgraph of Γ and has the same set of vertices as Γ. It is well known that each connected graph has at least one connected spanning tree.

Theorem 3.5.17. *Each connected loose graph Γ has a connected loose spanning tree which contains all the loose edges.*

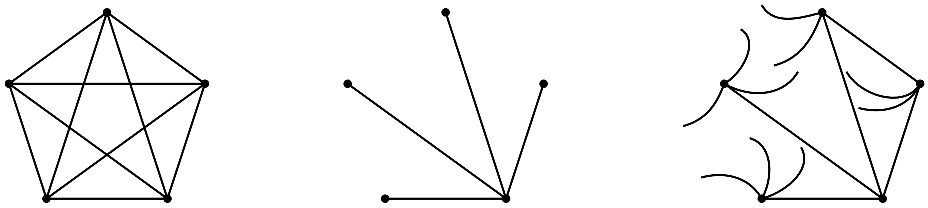
Proof. Let S be the set of loose edges of Γ. As Γ is connected, the reduced graph $\widetilde{\Gamma}$ (obtained from Γ by deleting all elements of S) is a connected graph. So it has a connected spanning tree T'. Now add S to T' to obtain a connected loose spanning tree T of Γ which contains all the loose edges. ■

Let Γ be any finite connected loose graph, let S be its set of loose edges and $\widetilde{\Gamma}$ be its reduced graph. Let T' be a spanning tree of $\widetilde{\Gamma}$, and note that if C is a cycle in Γ, it remains a cycle in $\widetilde{\Gamma}$. Define $\widetilde{S}$ to be the set of edges in $\widetilde{\Gamma} \setminus T'$; it is by definition the set of *fundamental edges* of $\widetilde{\Gamma}$ with respect to the spanning tree T'. (If one takes any edge $e \in \widetilde{S}$, then adding e to T' defines a unique cycle called "fundamental cycle.")

Now resolve each edge in $\widetilde{S}$ in the loose graph Γ (in precisely $|\widetilde{S}|$ steps) as explained above, to obtain a loose tree T, which contains T' up to a number of additional loose edges. Below, we can see an example of Γ, T' and T (example 3.5.19). We apply the map from definition 3.3.1 so as to obtain a counting polynomial for T. Take an edge e with vertices x and y that was resolved and consider the loose graph T^e in which all other edges of $\widetilde{S}$ except e are resolved, i.e., T^e is the next-to-last step in the procedure of obtaining T. Thanks to Corollary 3.5.13, we can compute the counting polynomial for T^e by restricting to $\mathbf{P}_{x,y}$ and, by repeating this process exactly $|\widetilde{S}|$ times, we inductively obtain the counting polynomial for $\mathcal{F}_k(\Gamma)$.

Observation 3.5.18. The procedure of surgery is one of the main targets of this PhD and more important than only finding the formulae for counting the number of rational points of constructible sets coming from loose graphs. Indeed, it is fundamental to see that for computing the number of rational points of a constructible set coming from a loose graph, one can "loosen" the intersections of its affine spaces, while lowering the number of cycles of the corresponding loose graph, go to a higher dimensional projective space to make computations easier and understand locally those intersections needed to eventually get the right number of points.

Example 3.5.19. A graph Γ, one of its loose spanning trees T' and the loose tree T



To see a full computation of the surgery process for this example, see Appendix A.

Theorem 3.5.20. *Let Γ be a loose graph and let $\widetilde{\Gamma}$, T' and T be defined as above. Then the Grothendieck polynomial of $\mathcal{F}_k(T)$ is independent of the choice of the spanning tree T' of $\widetilde{\Gamma}$.*

Proof. To prove this theorem it is enough to remark that resolving a fundamental edge e of $\widetilde{\Gamma}$ is equivalent to replacing the edge e in Γ by two different loose edges, one in each end point of e. Hence, the degrees of the vertices of Γ are invariant after resolving all fundamental edges of $\widetilde{\Gamma}$. This implies that for any spanning tree T' of $\widetilde{\Gamma}$, the set of vertices of T (and their respective degrees) is the same as in Γ. The result follows from the fact that the Grothendieck polynomial of a loose tree only depends on the vertices and its spectrum of degrees. ∎

We will prove now that the surgery process to compute the Grothendieck polynomial is "well defined" for any loose graph. Essentially, the reason why this holds is that we have not used any *particular field* property throughout.

Theorem 3.5.21. *Let Γ be a loose graph and $\mathcal{F}_k(\Gamma)$ be its associated k-constructible set, with k a finite field. The Grothendieck polynomial $P_\Gamma(\mathbb{L})$ (also denoted by $\mathbb{P}(\Gamma)$) obtained by surgery is independent of the choice of loose spanning tree, and the chosen order of edge resolution.*

Proof. Suppose that we carry out two different surgery processes in order to obtain a Grothendieck polynomial of Γ. A difference between two processes could occur either by choosing different loose spanning trees of Γ or by computing the edge resolution in a different order, starting from the same loose spanning tree.

Let $P_\Gamma(\mathbb{L})$ and $P'_\Gamma(\mathbb{L})$ be the two polynomials in $\mathbb{Z}[\mathbb{L}]$ obtained for Γ. Since both polynomials, for each finite field $k = \mathbb{F}_q$, count the number of k-rational points of $\mathcal{F}_k(\Gamma)$, it turns out that

$$P_\Gamma(q) = P'_\Gamma(q), \tag{3.58}$$

for any prime power q. This proves the equality of the two polynomials. ■

We finish this section by stating the theorem proved by developing the surgery process.

Theorem 3.5.22. *Let Γ be a loose graph and let $\mathcal{F}_{\mathbb{F}_q}(\Gamma)$ be its $\mathbb{F}_q$-constructible set, for $\mathbb{F}_q$ any finite field. Then $\mathcal{F}_{\mathbb{F}_q}(\Gamma)$ is polynomial-count. Besides, $\mathcal{F}_{\mathbb{F}_q}(\Gamma)$ is zeta-equivalent to an object of $\mathtt{CS}_{\mathbb{F}_q}$ of which the Grothendieck class is a $\mathbb{Z}$-linear combination of classes of affine spaces $[\mathbb{A}^i]_{\mathbb{F}_q}$.*

Proof. To prove this statement we only need to prove that for each constructible set $\mathcal{F}_{\mathbb{F}_q}(\Gamma)$ there exists a polynomial $P_\Gamma(T) = \sum_{i\geq 1} a_i T^i \in \mathbb{Z}[T]$ such that for every finite extension $\mathbb{F}_{q^n}/\mathbb{F}_q$, we have

$$\Big|\mathcal{F}_{\mathbb{F}_q}(\Gamma)\Big|_{q^n} = P_\Gamma(q^n).$$

This is exactly the polynomial $\mathbb{P}(\Gamma)$ obtained in the surgery process, which can be expressed as a polynomial in $\mathbb{Z}[T]$. Hence, the theorem is proved. ■

Remark 3.5.23. If q is big enough, one can construct a scheme $\gamma \in \mathtt{Sch}_{\mathbb{F}_q}$ such that $[\gamma]_{\mathbb{F}_q}$ coincides with the class of the constructible set $\mathcal{F}_{\mathbb{F}_q}(\Gamma)$.

In Appendix B, we describe a full algorithm implemented in Magma to compute the Grothendieck polynomial $P_\Gamma(T)$ for any loose graph Γ.

3.6 Lifting $K_0(\mathtt{Sch}_{\mathbb{F}_1})$, II

In surgery, for all finite fields $k \neq \mathbb{F}_1$ we have counted k-rational points, and showed that for each loose graph Γ there is a unique polynomial in $\mathbb{Z}[\mathbb{L}]$ which yields this number for each finite field k, independent of the choices made in surgery. For $k = \mathbb{F}_1$, counting points works differently at first sight, as the number of vertices of the loose graph simply equals the number of closed points of the associated $\mathbb{F}_1$-constructible set (so that several polynomials could be worthy candidates). Still, on the other hand, the Grothendieck polynomial obtained from a given loose graph is independent of the field k, so this fact strongly suggests that the same polynomial should correspond to the situation evaluated over $\mathbb{F}_1$. And indeed, since any statement we made, and property we use, relies only on the intersection properties of affine spaces — which carry over without change to $k = \mathbb{F}_1$ — the polynomial $\mathbb{P}(\Gamma) = \mathbb{P}_k(\Gamma)$ with k any finite field also yields a natural definition for "the" Grothendieck polynomial of Γ over $\mathbb{F}_1$, and it also counts the number of closed points over $\mathbb{F}_1$.

3.7 Class of $\mathcal{F}_k(\Gamma)$ in $K_0(\texttt{Sch}_k)$

We have proved that every constructible set arising from a loose graph is polynomial-count and zeta-equivalent to a $\mathbb{Z}$-linear combination of classes of affine spaces. It seems natural now to ask whether the class of such type of constructible sets in the Grothendieck ring $K_0(\texttt{Sch}_k)$ belongs to the polynomial ring $\mathbb{Z}[\mathbb{L}]$. In other words, whether for a loose graph Γ, $[\mathcal{F}_k(\Gamma)]$ is a virtual mixed Tate motive.

The answer to this question is affirmative and the proof of this fact is the main purpose of this section.

Theorem 3.7.1. *Let Γ be any loose graph, and let $k \neq \mathbb{F}_1$ be any finite field. Then the class $[\mathcal{F}_k(\Gamma)] \in K_0(\texttt{Sch}_k)$ is a virtual mixed Tate motive.*

We proceed with the proof by induction on the number N which is the sum of the number of edges and the number of vertices of a loose graph. For the case of loose trees, the function defined in section 3.3 already gives the class of a tree as a polynomial in $\mathbb{Z}[\mathbb{L}]$. So, we will assume that Γ is not a loose tree.

Note that we may also suppose w.l.o.g. that Γ is connected (and note that resolving an edge on a connected loose graph not necessarily yields again a connected loose graph). Note also that resolution of edges on trees is not defined.

We state some lemmas that will allow us to prove the main Theorem 3.7.1. The following is easy. It rests on the following observation: if for each vertex $v \in \Gamma$, $\underline{\mathbb{A}}_v$ is the affine subspace of $\mathbb{A}_v$ determined by the directions which are not loose edges, then

$$\mathcal{F}_k(\Gamma) = \mathcal{F}_k(\widetilde{\Gamma}) \coprod \Big(\bigcup_{v \in \Gamma} (\mathbb{A}_v \setminus \underline{\mathbb{A}}_v) \Big), \tag{3.59}$$

where each $\mathbb{A}_v \setminus \underline{\mathbb{A}}_v = \mathbb{A}_v \cap (\mathbb{P} \setminus \underline{\mathbb{A}}_v)$ is constructible and $\mathbb{P}$ is the ambient space of $\mathcal{F}_k(\Gamma)$.

Lemma 3.7.2. *In $K_0(\texttt{Sch}_k)$, we have that $[\mathcal{F}_k(\Gamma)] \in \mathbb{Z}[\mathbb{L}]$ if and only if $[\mathcal{F}_k(\widetilde{\Gamma})] \in \mathbb{Z}[\mathbb{L}]$, with $\widetilde{\Gamma}$ the reduced graph of Γ.*

By Lemma 3.7.2, we may thus suppose that Γ is a graph. Now suppose $e = xy$ is an edge, with x and y its incident vertices. Resolve the edge xy to obtain Γ_{xy} (this is a loose graph).

Remark 3.7.3. Notice that intersecting with a projective space commutes with the functor $\mathcal{F}_k(\cdot)$. We will prove this remark in the following lemma.

Lemma 3.7.4. *Let us denote by $\mathbf{P} = K_V$ the complete graph defined on a subset V of vertices of Γ and let us call $\mathbb{P}_k$ the k-projective space defined by $\mathbf{P}$. Then $\mathcal{F}_k(\Gamma) \cap \mathbb{P}_k = \mathcal{F}_k(\Gamma \cap \mathbf{P})$.*

Proof. Let S_w be the loose star of a vertex w of Γ, that is, the loose subgraph of Γ formed by the vertex w and all its incident edges.

It is easy to check that $\mathcal{F}_k(\Gamma \cap \mathbf{P})$ is a constructible set of $\mathcal{F}_k(\Gamma) \cap \mathbb{P}_k$ since $\Gamma \cap \mathbf{P}$ is a subgraph of both Γ and $\mathbf{P}$. Consider now a point $x \in \mathcal{F}_k(\Gamma) \cap \mathbb{P}_k$. Then, from the definition of $\mathcal{F}_k$ (see definition 2.4.2), x belongs to $\mathtt{Spec}(\mathbb{A}_v) \cap \mathbb{P}_k$, for a vertex $v \in \Gamma$. The latter constructible set is defined by the part of the loose star $S_v \subseteq \Gamma$ inside $\mathbf{P}$, i.e., by $S_v \cap \mathbf{P}$. This concludes the proof since $S_v \cap \mathbf{P}$ is a subgraph of $\Gamma \cap \mathbf{P}$ and so $x \in \mathcal{F}_k(S_v \cap \mathbf{P}) \subseteq \mathcal{F}_k(\Gamma \cap \mathbf{P})$. ■

The following lemma, in the spirit of Corollary 3.5.13, shows that we can restrict ourselves to local considerations.

Lemma 3.7.5. *In $K_0(\mathtt{Sch}_k)$, we have that $[\mathcal{F}_k(\Gamma)] - [\mathcal{F}_k(\Gamma_{xy})] \in \mathbb{Z}[\mathbb{L}]$ if and only if $[\mathcal{F}_k(\Gamma \cap \mathbf{P}_{x,y})] - [\mathcal{F}_k(\Gamma_{xy} \cap \mathbf{P}_{x,y})] \in \mathbb{Z}[\mathbb{L}]$.*

Proof. In order to prove the statement, we will compute the difference of classes $[\mathcal{F}_k(\Gamma)] - [\mathcal{F}_k(\Gamma_{xy})]$. Thanks to Remark 3.7.3 and the relative topology on $\mathcal{F}_k(\Gamma)$ and $\mathcal{F}_k(\Gamma_{xy})$, we can deduce that both $\mathcal{F}_k(\Gamma \cap \mathbf{P}_{x,y})$ and $\mathcal{F}_k(\Gamma_{xy} \cap \mathbf{P}_{x,y})$ are closed in $\mathcal{F}_k(\Gamma)$ and $\mathcal{F}_k(\Gamma_{xy})$, respectively. Then, by the relations in the appropriate Grothendieck ring of schemes, we have that:

$$\begin{cases} [\mathcal{F}_k(\Gamma)] = [\mathcal{F}_k(\Gamma \cap \mathbf{P}_{x,y})] + [\mathcal{F}_k(\Gamma) \setminus \mathcal{F}_k(\Gamma \cap \mathbf{P}_{x,y})], \\ [\mathcal{F}_k(\Gamma_{xy})] = [\mathcal{F}_k(\Gamma_{xy} \cap \mathbf{P}_{x,y})] + [\mathcal{F}_k(\Gamma_{xy}) \setminus \mathcal{F}(\Gamma_{xy} \cap \mathbf{P}_{x,y})]. \end{cases} \tag{3.60}$$

We will prove that the last terms on the right-hand side of the equations are the same. Let $\Gamma' = \Gamma \cap \mathbf{P}_{x,y}$ and $\Gamma'_{xy} = \Gamma_{xy} \cap \mathbf{P}_{x,y}$. Note that thanks to the Affection Principle (see Theorem 3.5.11, and Lemma 3.5.9), in order to compare the classes of $\mathcal{F}_k(\Gamma) \setminus \mathcal{F}_k(\Gamma')$ and $\mathcal{F}_k(\Gamma_{xy}) \setminus \mathcal{F}_k(\Gamma'_{xy})$ in $K_0(\mathtt{Sch}_k)$, we (only) need to take into account the local affine spaces in $\mathcal{F}_k(\Gamma)$ ($\mathcal{F}_k(\Gamma_{xy})$) associated to vertices of Γ (Γ_{xy}) which are at distance at most one from the loose graph $\Gamma \setminus \Gamma'$ ($\Gamma_{xy} \setminus \Gamma'_{xy}$) (since vertices at distance strictly more than one give rise to affine spaces that remain unchanged through resolution of xy).

From the definition of Γ', one deduces that both vertices x and y are at least at distance two from any vertex of $\Gamma \setminus \Gamma'$, which implies that

$$\begin{cases} (\mathcal{F}_k(\Gamma) \setminus \mathcal{F}_k(\Gamma')) \cap \mathbb{A}_x = \emptyset, \\ (\mathcal{F}_k(\Gamma) \setminus \mathcal{F}_k(\Gamma')) \cap \mathbb{A}_y = \emptyset. \end{cases} \tag{3.61}$$

As resolving the edge xy only changes locally the affine spaces $\mathbb{A}_x$ and $\mathbb{A}_y$ in $\mathcal{F}(\Gamma)$ (more precisely in $\mathcal{F}(\Gamma')$), and as the distance between x (or y) and $\Gamma \setminus \Gamma'$ is preserved through resolution, this process does not affect the local affine spaces in $\mathcal{F}_k(\Gamma) \setminus \mathcal{F}_k(\Gamma')$, nor the intersection of any two of them. Notice that in the case of vertices $v \in \Gamma'$ at distance one from $\Gamma \setminus \Gamma'$, possible changes of the affine space $\mathbb{A}_v$ in $\mathcal{F}(\Gamma)$ by resolution of xy do not affect $\mathcal{F}_k(\Gamma) \setminus \mathcal{F}_k(\Gamma')$; changes only occur in the completion $\overline{\mathbb{A}_v} \cap \mathcal{F}(\Gamma')$.

It is now easy to observe that there is a natural isomorphism between $\mathcal{F}_k(\Gamma)\setminus\mathcal{F}_k(\Gamma')$ and $\mathcal{F}_k(\Gamma_{xy}) \setminus \mathcal{F}_k(\Gamma'_{xy})$ induced by the graph morphism

$$\gamma : \Gamma'' \to \Gamma''_{xy},$$

where Γ'' (respectively Γ''_{xy}) is the subgraph of Γ (respectively Γ_{xy}) defined on the vertex set $V(\Gamma \setminus \Gamma') \cup \{v \in \Gamma' \mid \mathrm{d}(v, \Gamma \setminus \Gamma') = 1\}$ (respectively $V(\Gamma_{xy} \setminus \Gamma'_{xy}) \cup \{v \in \Gamma'_{xy} \mid \mathrm{d}(v, \Gamma_{xy}\setminus\Gamma'_{xy}) = 1\}$), and where γ acts as the identity on vertices. This implies that both classes in $K_0(\mathtt{Sch}_k)$ are equal. For, note that $\mathcal{F}_k(\Gamma) \setminus \mathcal{F}_k(\Gamma')$ and $\mathcal{F}_k(\Gamma_{xy}) \setminus \mathcal{F}_k(\Gamma'_{xy})$ are constructible sets and write $\mathcal{F}_k(\Gamma) \setminus \mathcal{F}_k(\Gamma') = \coprod_{i\in F} W_i$, where F is a finite set and each W_i is a locally closed subscheme of the projective space $\mathbf{P}(\Gamma)$. Now γ induces a decomposition of $\mathcal{F}_k(\Gamma_{xy}) \setminus \mathcal{F}_k(\Gamma'_{xy})$ in disjoint pieces $\widetilde{W_i}$, $i \in F$, where $\widetilde{W_i} \cong W_i$.

We can then conclude that

$$[\mathcal{F}_k(\Gamma)] - [\mathcal{F}_k(\Gamma_{xy})] = [\mathcal{F}_k(\Gamma \cap \mathbf{P}_{x,y})] - [\mathcal{F}_k(\Gamma_{xy} \cap \mathbf{P}_{x,y})].$$

∎

By Lemma 3.7.5, we may suppose that $\Gamma = \Gamma \cap \mathbf{P}_{x,y}$.

The next lemma refines the Affection Principle (Theorem 3.5.11).

Lemma 3.7.6. *Let Γ be a graph, xy an edge with vertices x and y and Γ_{xy} the graph after resolving the edge xy. Let u, v be two vertices of Γ and consider $\mathbb{A}_u$ and $\mathbb{A}_v$, the local affine spaces at u and v in $\mathcal{F}_k(\Gamma)$. The intersection $\mathbb{A}_u \cap \mathbb{A}_v$ and the union $\mathbb{A}_u \cup \mathbb{A}_v$ change after resolution only if $u, v \in \{x, y\} \cup (x^\perp \cap y^\perp)$.*

Proof. First let us note that if $\mathbb{A}_u \cap \mathbb{A}_v = \emptyset$, then $\mathbb{A}_u \cup \mathbb{A}_v$ is stable under resolution. Consider now a vertex $u \in x^\perp \setminus ((x^\perp \cap y^\perp) \cup \{y\})$. Then, it is clear that neither $\mathbb{A}_u$ nor $\overline{\mathbb{A}_u} \cap \mathcal{F}_k(\Gamma)$ changes after resolving the edge xy. The latter is not affected by the resolution since the edge xy is not in the graph $\Gamma \cap \overline{\mathbf{B}}(u, 1)$. The same holds for vertices $u \in y^\perp \setminus ((x^\perp \cap y^\perp) \cup \{x\})$. To simplify notation we will write from now on only $\overline{\mathbb{A}_u}$ instead of $\overline{\mathbb{A}_u} \cap \mathcal{F}_k(\Gamma)$ and we consider it embedded in the ambient space of $\mathcal{F}_k(\Gamma)$.

Now suppose that $u \in x^\perp \cap y^\perp$; then the graph defined by xy is a subgraph of the "part at infinity" of the graph completion of S_u, the star associated to u. This implies that locally at u the changes that occur by resolving xy are contained in $\overline{\mathbb{A}_u} \setminus \mathbb{A}_u$, so the local affine space at u also remains invariant under resolution of xy.

Observe that $\mathbb{A}_u \cap \mathbb{A}_v$ and $\mathbb{A}_u \cup \mathbb{A}_v$ are controlled by $\mathbb{A}_u$, $\mathbb{A}_v$, $\overline{\mathbb{A}_u}$, $\overline{\mathbb{A}_v}$ and $\overline{\mathbb{A}_u} \cap \overline{\mathbb{A}_v}$. So, if $u, v \neq x, y$ and $u, v \notin x^\perp \cap y^\perp$, then indeed $\mathbb{A}_u \cap \mathbb{A}_v$ and $\mathbb{A}_u \cup \mathbb{A}_v$ are stable under resolution. In the case where one of $u, v \in x^\perp \cap y^\perp$ and $u, v \neq x, y$, changes under resolution will be controlled by $\overline{\mathbb{A}_u} \setminus \mathbb{A}_u$, $\overline{\mathbb{A}_v} \setminus \mathbb{A}_v$. This implies that changes in $\overline{\mathbb{A}_u} \cap \overline{\mathbb{A}_v}$ are contained in $(\overline{\mathbb{A}_u} \setminus \mathbb{A}_u) \cap (\overline{\mathbb{A}_v} \setminus \mathbb{A}_v) = (\overline{\mathbb{A}_u} \cap \overline{\mathbb{A}_v}) \setminus (\mathbb{A}_u \cup \mathbb{A}_v)$ so, $\mathbb{A}_u \cap \mathbb{A}_v$ and $\mathbb{A}_u \cup \mathbb{A}_v$ are also stable after resolving xy.

Suppose now that $v = x$ and $u \in x^\perp \setminus ((x^\perp \cap y^\perp) \cup \{y\})$. Then $\mathbb{A}_u$ and $\overline{\mathbb{A}_u}$ are stable after resolution. From the graph theoretical point of view, it is easy to see that

$\overline{S_u} \cap \overline{S_x}$ (as a subgraph of Γ) remains invariant after resolving the edge xy (considering the same intersection inside Γ_{xy}). Since $\mathbb{A}_u \cap \mathbb{A}_x \subset \overline{\mathbb{A}_u} \cap \overline{\mathbb{A}_x}$, we deduce that indeed $\mathbb{A}_u \cap \mathbb{A}_x$ and $\mathbb{A}_u \cup \mathbb{A}_x$ are also stable under resolution. The same reasoning holds when $v = y$ and $u \in y^\perp \setminus ((x^\perp \cap y^\perp) \cup \{x\})$. This concludes the proof. ■

Remark 3.7.7. Notice that the reason *why* the previous spaces are not affected by the resolution of the edge xy comes as a direct consequence of the fact that the (loose) subgraphs of Γ defining those spaces do not contain the edge xy.

Remark 3.7.8. Note that the equality of the last terms in the right-hand sides of (3.60) in Lemma 3.7.5 can also be obtained by applying Lemma 3.7.6.

3.7.1 Main Theorem for cones

We first handle a useful specific case of graphs: the cone $C(G_2, G_1)$ in the sense of subsection 3.5.4, where the vertex G_1 is the graph which consists of a single vertex.

Lemma 3.7.9. *Suppose that either $y^\perp = (x^\perp \cap y^\perp) \cup \{x\}$, or $x^\perp = (x^\perp \cap y^\perp) \cup \{y\}$. Then $[\mathcal{F}_k(\Gamma)] \in \mathbb{Z}[\mathbb{L}]$.*

Proof. Suppose w.l.o.g. that $y^\perp = (x^\perp \cap y^\perp) \cup \{x\}$; then for all vertices v in Γ, we have that either $v = x$ or $v \sim x$. It follows immediately that

$$[\mathcal{F}_k(\Gamma)] = [\mathbb{A}_x] + [\mathcal{F}_k(x^\perp \cap \Gamma)]. \tag{3.62}$$

By induction applied on the second term in the right-hand side, the lemma follows. ■

To finish the proof of Theorem 3.7.1, we consider from now on that Γ is a general loose graph. We will divide the proof in two cases

3.7.2 Γ has no external edges

We assume that there are no edges uv with $u \in x^\perp \setminus ((x^\perp \cap y^\perp) \cup \{y\})$ and $v \in y^\perp \setminus ((x^\perp \cap y^\perp) \cup \{x\})$ (call such edges "external") — the case where such edges exist will be handled separately below.

We also suppose that $y^\perp \neq (x^\perp \cap y^\perp) \cup \{x\}$ and $x^\perp \neq (x^\perp \cap y^\perp) \cup \{y\}$, since otherwise the statement is already true by the previous subsection.

Let $u \neq y$ be any vertex in $x^\perp \setminus (x^\perp \cap y^\perp)$; let $e := ux$. Let Γ^e be the graph Γ without the edge e (while not deleting u and x); similarly, we define Γ^e_{xy}. As Γ^e is a proper subgraph of Γ, induction implies that $[\mathcal{F}_k(\Gamma^e)] \in \mathbb{Z}[\mathbb{L}]$. Also, by Lemma 3.7.2, $[\mathcal{F}_k(\Gamma_{xy})] \in \mathbb{Z}[\mathbb{L}]$ if and only if $[\mathcal{F}_k(\widetilde{\Gamma_{xy}})] \in \mathbb{Z}[\mathbb{L}]$, and by induction, the latter expression is true since $\widetilde{\Gamma_{xy}}$ is a subgraph of Γ. In the same way, $[\mathcal{F}_k(\Gamma^e_{xy})] \in \mathbb{Z}[\mathbb{L}]$. Now consider $[\mathcal{F}_k(\Gamma)] - [\mathcal{F}_k(\Gamma^e)]$. Then obviously $[\mathcal{F}_k(\Gamma)] - [\mathcal{F}_k(\Gamma^e)] \in \mathbb{Z}[\mathbb{L}]$ if and only if

$[\mathcal{F}_k(\Gamma)] - [\mathcal{F}_k(\Gamma_{xu})] \in \mathbb{Z}[\mathbb{L}]$; by Lemma 3.7.5, this holds if and only if $[\mathcal{F}_k(\Gamma \cap \mathbf{P}_{x,u})] - [\mathcal{F}_k(\Gamma_{xu} \cap \mathbf{P}_{x,u})] \in \mathbb{Z}[\mathbb{L}]$. Now by our assumption, we have

$$\begin{cases} \Gamma \cap \mathbf{P}_{x,u} & \neq \ \Gamma; \\ \Gamma_{xu} \cap \mathbf{P}_{x,u} & \neq \ \Gamma_{xu}, \end{cases} \tag{3.63}$$

so that induction yields that $[\mathcal{F}_k(\Gamma)] - [\mathcal{F}_k(\Gamma^e)] \in \mathbb{Z}[\mathbb{L}]$. Since $[\mathcal{F}_k(\Gamma^e)] \in \mathbb{Z}[\mathbb{L}]$, it follows that $[\mathcal{F}_k(\Gamma)] \in \mathbb{Z}[\mathbb{L}]$. ■

3.7.3 Γ has external edges

Now suppose Γ *has* external edges. Suppose Γ' is the subgraph of Γ which one obtains by deleting one chosen external edge $e = uv$. By induction we know that $[\mathcal{F}_k(\Gamma')]$ is in $\mathbb{Z}[\mathbb{L}]$. Then

$$\mathcal{F}_k(\Gamma) = \mathcal{F}_k(\Gamma') \coprod \Big((\mathbb{A}_u \cup \mathbb{A}_v) \setminus (\cup_{s\in\Gamma'} \mathbb{A}'_s) \Big), \tag{3.64}$$

where $\mathbb{A}_w$ is the local affine space at w in $\mathcal{F}_k(\Gamma)$, and $\mathbb{A}'_t$ is the local affine space at t in $\mathcal{F}_k(\Gamma')$ (note that $\cup_{s\in\Gamma'}\mathbb{A}'_s = \mathcal{F}_k(\Gamma')$). Note that $(\mathbb{A}_u \cup \mathbb{A}_v) \setminus (\cup_{s\in\Gamma'}\mathbb{A}'_s)$ is constructible, as $(\mathbb{A}_u \cup \mathbb{A}_v) \setminus (\cup_{s\in\Gamma'}\mathbb{A}'_s) = (\mathbb{A}_u \cup \mathbb{A}_v) \bigcap (\mathbb{P} \setminus (\cup_{s\in\Gamma'}\mathbb{A}'_s))$, where $\mathbb{A}_u \cup \mathbb{A}_v$ and $\cup_{s\in\Gamma'}\mathbb{A}'_s$ are constructible. Then

$$[\mathcal{F}_k(\Gamma)] = [\mathcal{F}_k(\Gamma')] + \underbrace{\Big[(\mathbb{A}_u \cup \mathbb{A}_v) \setminus (\cup_{s\in\Gamma'}\mathbb{A}'_s) \Big]}_{\text{(A)}}. \tag{3.65}$$

Doing the same for Γ_{xy}, we obtain that

$$[\mathcal{F}_k(\Gamma_{xy})] = [\mathcal{F}_k(\Gamma'_{xy})] + \underbrace{\Big[(\mathbb{A}_u \cup \mathbb{A}_v) \setminus (\cup_{s\in\Gamma'_{xy}}\mathbb{A}'_s) \Big]}_{\text{(B)}}, \tag{3.66}$$

where all the local affine spaces are now considered in Γ_{xy} or Γ'_{xy}.

By Lemma 3.7.6, we have that (A) = (B). For, $(\mathbb{A}_u \cup \mathbb{A}_v) \cap \mathbb{A}'_x = \mathbb{A}_u \cap \mathbb{A}'_x$ and $(\mathbb{A}_u \cup \mathbb{A}_v) \cap \mathbb{A}'_y = \mathbb{A}_v \cap \mathbb{A}'_y$ do not change when resolving xy, and if $w \in x^\perp \cap y^\perp$, then $(\mathbb{A}_u \cup \mathbb{A}_v) \cap \mathbb{A}'_w$ also does not change through resolution. All the other cases are covered by Lemma 3.7.6.

After applying induction, we now get that $[\mathcal{F}_k(\Gamma)] - [\mathcal{F}_k(\Gamma')] \in \mathbb{Z}[\mathbb{L}]$. ■

3.7.4 End of the proof of Theorem 3.7.1

We are able to finally prove the main theorem of the section. Starting from a connected loose graph Γ, choose any edge xy that is not contained in a loose spanning tree T', and resolve xy. We have shown that

$$[\mathcal{F}_k(\Gamma)] - [\mathcal{F}_k(\Gamma_{xy})] \ \in \ \mathbb{Z}[\mathbb{L}]. \tag{3.67}$$

Now there are two ways to proceed.

(A) Carry out surgery on the constructible set $\mathcal{F}_k(\Gamma)$. Take the loose spanning tree T' of Γ and construct a loose tree T containing T', obtained after resolving in Γ all edges of $\Gamma \setminus T'$ so as to eventually wind up with a constructible set $\mathcal{F}_k(T)$, cf. section 3.5.7. We have seen that $[\mathcal{F}_k(T)] \in \mathbb{Z}[\mathbb{L}]$ in section 3.3. Since by (3.67) each difference between Grothendieck classes of consecutive steps is an element of $\mathbb{Z}[\mathbb{L}]$, we conclude that the same is true for the initial class $[\mathcal{F}_k(\Gamma)]$ as well.

(B) Use the induction hypothesis to conclude that $[\mathcal{F}_k(\Gamma_{xy})] \in \mathbb{Z}[\mathbb{L}]$, so that $\mathcal{F}_k(\Gamma) \in \mathbb{Z}[\mathbb{L}]$.

This conclude the proof of Theorem 3.7.1. ■

3.8 Mixed Tate motives in the Grothendieck ring

One very fundamental aspect of the philosophy of $\mathbb{F}_1$-geometry is that the number of $\mathbb{F}_1$-rational points of an $\mathbb{F}_1$-scheme Y of finite type should equal the Euler characteristic

$$\chi(Y_{\mathbb{C}}) \quad := \quad \sum_{i=0}^{2\dim Y} (-1)^i b_i \tag{3.68}$$

where $b_i := \dim(\mathrm{H}^i(Y_{\mathbb{C}}, \mathbb{C}))$ are the Betti numbers of the complex scheme $Y_{\mathbb{C}} := Y \otimes_{\mathbb{F}_1} \mathbb{C}$. This idea is deduced from the following thought.

Let X be a smooth projective scheme such that there is a polynomial $N(Z) \in \mathbb{Z}[Z]$ that counts $\mathbb{F}_q$-rational points, i.e. $\Big|X\Big|_{q^n} = N(q)$ for every prime power q. As a consequence of the comparison theorem for singular and ℓ-adic cohomology and Deligne's proof of the Weil conjectures ([12, 13]), we know that the counting polynomial is of the form

$$N(Z) \quad = \quad \sum_{i=0}^{n} b_{2i}\, Z^i \tag{3.69}$$

and that $b_j = 0$ if j is odd (cf. [26]). Thus $\chi = \sum_{i=0}^{n} b_{2i}$ is the Euler characteristic of $X_{\mathbb{C}}$ in this case; it equals $N(1)$, which has the interpretation as the number of $\mathbb{F}_1$-rational points of an $\mathbb{F}_1$-model $X_{\mathbb{F}_1}$ of X.

Conjecturally, after the Tate conjectures, smooth projective schemes that are equipped with a counting polynomial as above, are precisely those that come with a mixed Tate motive. As our construction defines a functor

$$\mathcal{F}_{\mathbb{Z}} : \ \mathtt{LGraph} \ \longrightarrow \ \mathtt{CS}_{\mathbb{Z}} \tag{3.70}$$

from the category of loose graphs to the category of constructible sets over $\mathbb{Z}$ which all come with a counting polynomial, a natural question is whether one can derive the

Betti numbers from loose graphs Γ for which $\mathcal{F}_{\mathbb{Z}}$ is smooth and projective. As we will see below (in Theorem 3.8.1), the answer is "yes," but it is also trivial, since such Γs always give rise to projective spaces.

In the next theorem, we will use the following property:

INJ If Γ and $\widetilde{\Gamma}$ are nonisomorphic loose graphs, then $\mathcal{F}(\Gamma) \ncong \mathcal{F}(\widetilde{\Gamma})$, and for any field k, $\mathcal{F}(\Gamma) \otimes_{\mathbb{F}_1} k \ncong \mathcal{F}(\widetilde{\Gamma}) \otimes_{\mathbb{F}_1} k$.

Theorem 3.8.1. *Let Γ be a connected loose graph, such that $\chi = \mathcal{F}(\Gamma) \otimes_{\mathbb{F}_1} \mathbb{Z}$ defines a (connected) projective $\mathbb{Z}$-variety. Then Γ is a complete graph, and χ is a projective space.*

Proof. Let k be any field, and define $\chi_k := \mathcal{F}(\Gamma) \otimes_{\mathbb{F}_1} k$; then χ_k is a projective variety which is embedded in a projective space $\mathbf{P}$ over k. For any subvariety V of $\mathbf{P}$, define $\overline{V}$ to be its projective closure. Then from an inclusion of varieties $U \subseteq V \subseteq \mathbf{P}$, we have $\overline{U} \subseteq \overline{V} \subseteq \mathbf{P}$. By our construction, we know that χ_k can be covered by a set of affine spaces $\{\mathbb{A}_v | v \in V(\Gamma)\}$ over k, all embedded in $\mathbf{P}$, where v runs through the vertices of Γ. As χ_k is a projective variety (and so equal to its projective closure), all points at infinity of these spaces are also contained in χ_k, that is, the projective spaces they generate are also subvarieties of χ_k. So Γ is a graph.

We proceed with an induction argument on the number ℓ of vertices of Γ. (Obviously, the case $v = 1$ is trivial.) Let the number of vertices of Γ be $\ell > 1$. Then there exists a vertex, say x, such that the graph $\Gamma \setminus \{x\}$ (which is the graph — not the loose graph — induced on the vertex set $V(\Gamma) \setminus \{x\}$) is connected. Then for any field k, $\mathbf{P}_x := \langle \mathcal{F}(\Gamma \setminus \{x\}) \otimes_{\mathbb{F}_1} k \rangle$ is a proper sub-projective space of $\mathbf{P}$ (by CV), and as

$$\mathcal{F}(\Gamma \setminus \{x\}) \otimes_{\mathbb{F}_1} k = \mathbf{P}_x \cap (\mathcal{F}(\Gamma) \otimes_{\mathbb{F}_1} k), \tag{3.71}$$

$\mathcal{F}(\Gamma \setminus \{x\}) \otimes_{\mathbb{F}_1} k$ is a projective variety. By induction, it is a projective space. So $\mathcal{F}(\Gamma) \otimes_{\mathbb{F}_1} k$ is a union of two projective spaces, $\mathbf{P}_1 = \mathcal{F}(\Gamma \setminus \{x\}) \otimes_{\mathbb{F}_1} k$ and $\mathbf{P}_2 = \overline{\mathbb{A}_x}$. This means that Γ is a (non-disjoint) union of two complete graphs, say K_1 and K_2 (here, we implicitly use INJ). Now let y be a vertex in $K_1 \cap K_2$. Then y is adjacent to all other vertices of Γ, so by L-D, $\mathcal{F}(\Gamma)$ contains an affine $\mathbb{F}_1$-space of dimension $\ell - 1$. As $\mathcal{F}(\Gamma) \otimes_{\mathbb{F}_1} \mathbb{Z}$ defines a projective $\mathbb{Z}$-variety, it must be a projective space of dimension $\ell - 1$ (as the projective closure of $\mathbb{A}_y$ is a projective space $\overline{\mathbb{A}_y}$). ■

Note that in the previous theorem, we did not need to ask smoothness.

4 A New Zeta Function for (Loose) Graphs

Zeta functions at their origin are designed to be counting functions. The zeta function of a number field, for instance, counts integral ideals of a given domain and Selberg zeta functions count the number of geodesics on a surface. Throughout this chapter we will introduce and compare two zeta functions on graphs; the well-known *Ihara zeta function* and a new zeta function that we will define in the category of loose graphs. The final section includes a possible connection between the new zeta function and the chromatic polynomial associated to a given graph.

4.1 Ihara zeta function

In 1966 Ihara [22] first introduced a zeta function for discrete subgroups of the "two by two" projective linear group over p-adic fields, analogously to Selberg's zeta function. Years later, Serre suggested that this zeta function could have a graph-theoretical interpretation. We will describe in this section how Ihara's zeta function for graphs is defined.

Let $\Gamma = (V, E, \mathbf{I})$ be a finite connected undirected graph with no vertices of degree 1 ("end points"). We define the *rank* r_Γ of Γ to be $|E| - |V| + 1$; it is the number of edges one has to delete from Γ to obtain a spanning tree. Alternatively, one can also define r_Γ as the rank of the fundamental group of Γ. Suppose that $r_\Gamma \geq 1$ — that implies that Γ is not a tree.

Let the edge set E be $E = \{e_1, \ldots, e_n\}$, and define a new oriented edge set of size $2|E|$ as follows (where below the edges of E are arbitrarily oriented):

$$e_1, \ldots, e_n; \quad e_{n+1} = e_1^{-1}, \ldots, e_{2n} = e_n^{-1}. \tag{4.1}$$

Let $D = a_1 a_2 \cdots a_r$ be a directed closed path in Γ (all the a_i are edges and they are directed in the same direction). We consider the *equivalence class* $[D]$ of a path D

to be the set

$$[D] = \{a_1a_2 \cdots a_r,\ a_2a_3 \cdots a_ra_1, \ldots,\ a_ra_1 \cdots a_{r-1}\}. \tag{4.2}$$

We define the *length* $\nu(D)$ of D to be the number r. We say a path D is *backtrackless* if $a_{i+1} \neq a_i^{-1}$ for all $i \in \{1, \ldots, r-1\}$, and it is *tailless* if $a_r \neq a_1^{-1}$. The path D is *primitive* if $D \neq F^m$ for any positive integer $m \geq 2$ and any directed path F. Remember that the mth-power F^m of a path F is the graph with the same set of vertices of F obtained by adding edges between each pair of vertices of F at distance at most m.

A *prime path* for Γ is an equivalence class $[P]$ of closed backtrackless tailless primitive (directed) paths in Γ.

Definition 4.1.1. The *Ihara zeta function* of Γ is now defined as follows:

$$\zeta(u, \Gamma) := \prod_{[P] \ \text{prime}} (1 - u^{\nu(P)})^{-1}. \tag{4.3}$$

where $u \in \mathbb{C}$ with $|u|$ sufficiently small.

Theorem 4.1.2 (Bass [1])**.** *The Ihara zeta function $\zeta(u, \Gamma)$ is a rational function satisfying:*

$$\zeta(u, \Gamma)^{-1} = (1 - u^2)^{r_\Gamma - 1} \det(\mathrm{id} - A_\Gamma u + Q_\Gamma u^2), \tag{4.4}$$

where id *is an identity matrix, A_Γ the adjacency matrix of Γ, and Q_Γ the diagonal matrix whose j-th diagonal entry is ($-1+$ degree of j-th vertex). (One has to number the vertices in order to obtain* id, A_Γ *and Q_Γ; all are $(|V| \times |V|)$-matrices.)*

Remark 4.1.3. Remember that the adjacency matrix A_Γ of a graph Γ is a $(|V| \times |V|)$-matrix where a_{ij} is 1 if there is an edge from the vertex v_i to v_j and 0 otherwise.

Remark 4.1.4. $\zeta(u, \Gamma)^{-1}$ is a polynomial of degree $2|E|$.

Example 4.1.5. If Γ is a tree, there are no closed paths so we have an empty product in the definition of the Ihara zeta function. Hence, $\zeta(u, \Gamma) = 1$ for any tree.

Now let $|E| = m$, and define a $(2m \times 2m)$-matrix $\mathcal{E}$ (the "edge adjacency matrix") by letting the ij-th entry be 1 if the terminal vertex of e_i is the initial vertex of e_j, provided that $e_j \neq e_i^{-1}$. Otherwise, the entry is 0. Then Hashimoto [19] proved that the Ihara zeta function of Γ can also be calculated as

$$\zeta(u, \Gamma)^{-1} = \det(\mathrm{id} - u\mathcal{E}). \tag{4.5}$$

In other words, the roots of $\zeta(u, \Gamma)^{-1}$ (with multiplicities) are the eigenvalues (with multiplicities) of $\mathcal{E}$. So two graphs have the same Ihara zeta function *if and only* *if they are isospectral with respect to the edge adjacency matrix.*

The following properties/values can be read from the Ihara zeta function of a (finite connected undirected) graph (with no vertices of degree 1, and rank ≥ 1):

- whether it is bipartite or not;
- its number of vertices and edges;
- whether it is regular, and if so, its regularity degree and spectrum.

For more details on how to read these properties from the zeta function, one can go to [6].

4.2 Schemes defined over $\mathbb{F}_1$ à la Kurokawa

Consider a scheme X of finite type over $\mathbb{Z}$. Recall from subsection 3.2.1 that a point $x \in X$ is closed if and only if its residue field $k(x)$ is finite. We define the *arithmetic zeta function* $\zeta_X(s)$ of the scheme X as

$$\zeta_X(s) = \prod_x \frac{1}{1 - N(x)^{-s}}.$$

where the product is taken over all closed points x of the scheme X and $N(x)$ denotes the cardinality of the finite field $k(x)$.

In [26], Kurokawa says that a $\mathbb{Z}$-scheme X is of $\mathbb{F}_1$-*type* if its arithmetic zeta function $\zeta_X(s)$ can be expressed via the Riemann zeta function $\zeta(s)$ in the form

$$\zeta_X(s) = \prod_{k=0}^{n} \zeta(s-k)^{a_k} \tag{4.6}$$

with the a_ks in $\mathbb{Z}$. A very interesting result in [26] reads as follows:

Theorem 4.2.1. *Let X be a $\mathbb{Z}$-scheme. The following are equivalent.*

(i)
$$\zeta_X(s) = \prod_{k=0}^{n} \zeta(s-k)^{a_k} \tag{4.7}$$

with the a_ks in $\mathbb{Z}$.

(ii) *For all primes p we have*

$$\zeta_{X|\mathbb{F}_p}(s) = \prod_{k=0}^{n} (1 - p^{k-s})^{-a_k} \tag{4.8}$$

with the a_ks in $\mathbb{Z}$.

(iii) *There exists a polynomial* $P_X(Y) = \sum_{i=0}^{n} a_k Y^k$ *such that*

$$\#X(\mathbb{F}_{p^m}) = P_X(p^m) \tag{4.9}$$

for all finite fields $\mathbb{F}_{p^m}$.

Kurokawa defines the $\mathbb{F}_1$-*zeta function* of a $\mathbb{Z}$-scheme X of $\mathbb{F}_1$-type as

$$\zeta_{X|\mathbb{F}_1}(s) := \prod_{k=0}^{n} (s-k)^{-a_k} \tag{4.10}$$

with the a_ks as above, and the *Euler characteristic* is by definition

$$\#X(\mathbb{F}_1) := \sum_{k=0}^{n} a_k. \tag{4.11}$$

The connection between $\mathbb{F}_1$-zeta functions and arithmetic zeta functions is explained in the following theorem, taken from [26].

Theorem 4.2.2. *Let* X *be a* $\mathbb{Z}$*-scheme which is defined over* $\mathbb{F}_1$*. Then*

$$\zeta_{X|\mathbb{F}_1}(s) = \lim_{p \longrightarrow 1} \zeta_{X|\mathbb{F}_p}(s)(p-1)^{\#X(\mathbb{F}_1)}. \tag{4.12}$$

Here, p *is seen as a complex variable (so that the left-hand term is the leading coefficient of the Laurent expansion of* $\zeta_{X|\mathbb{F}_1}(s)$ *around* $p = 1$*).*

For affine and projective spaces, we obtain the following zeta functions (over $\mathbb{Z}$, $\mathbb{F}_p$ and $\mathbb{F}_1$, with $n \in \mathbb{N}^\times$):

$$\begin{aligned} \zeta_{\mathbb{A}^n|\mathbb{Z}}(s) &= \zeta(s-n); \\ \zeta_{\mathbb{A}^n|\mathbb{F}_p}(s) &= \frac{1}{1-p^{n-s}}; \\ \zeta_{\mathbb{A}^n|\mathbb{F}_1}(s) &= \frac{1}{s-n}, \end{aligned} \tag{4.13}$$

and

$$\begin{aligned} \zeta_{\mathbf{P}^n|\mathbb{Z}}(s) &= \zeta(s)\zeta(s-1)\cdots\zeta(s-n); \\ \zeta_{\mathbf{P}^n|\mathbb{F}_p}(s) &= \frac{1}{(1-p^{-s})(1-p^{1-s})\cdots(1-p^{n-s})}; \\ \zeta_{\mathbf{P}^n|\mathbb{F}_1}(s) &= \frac{1}{s(s-1)\cdots(s-n)}. \end{aligned} \tag{4.14}$$

4.3 The new zeta function

In this section, we are ready to introduce a new zeta function for each loose graph.

Definition 4.3.1. We say that a constructible set X over $\mathbb{Z}$ is *defined over* $\mathbb{F}_1$ in Kurokawa's sense if it satisfies the property (iii) of Theorem 4.2.1.

Theorem 4.3.2. *For any loose graph* Γ*, the* $\mathbb{Z}$*-constructible set* $\chi := \mathcal{F}(\Gamma) \otimes_{\mathbb{F}_1} \mathbb{Z}$ *is defined over* $\mathbb{F}_1$ *in Kurokawa's sense.*

Proof. Let Γ be an arbitrary finite connected loose graph. As we have seen, there exists a polynomial $P_\Gamma(X) = \sum_{i=0}^{m} a_m X^m \in \mathbb{Z}[X]$ such that for each finite field $k = \mathbb{F}_q$, the number of $\mathbb{F}_q$-rational points is given by

$$N_\chi(\mathbb{F}_q) := \left|\mathcal{F}(\Gamma) \otimes_{\mathbb{F}_1} \mathbb{F}_q\right|_q = P_\Gamma(q). \tag{4.15}$$

This is precisely what we needed to prove. ■

Definition 4.3.3 (Zeta function for (loose) graphs)**.** Let Γ be a loose graph, and let $\chi := \mathcal{F}(\Gamma) \otimes_{\mathbb{F}_1} \mathbb{Z}$. Let $P_\Gamma(X) = \sum_{i=0}^{m} a_i X^i \in \mathbb{Z}[X]$ be as above. We define the $\mathbb{F}_1$-*zeta function* of Γ as:

$$\zeta_\Gamma^{\mathbb{F}_1}(t) := \prod_{k=0}^{m} (t-k)^{-a_k}. \tag{4.16}$$

In the next section, we will compare $\zeta^{\mathbb{F}_1}(\cdot)$ with the Ihara zeta function for some fundamental examples of graphs. Note that for trees, the Ihara zeta function is trivial while $\zeta^{\mathbb{F}_1}(\cdot)$ contains much information.

The new zeta function for loose trees

Let Γ be a loose tree. We use the notation as before:

- D is the set of degrees $\{d_1, \ldots, d_m\}$ of $V(\Gamma)$ such that $1 < d_1 < d_2 < \ldots < d_m$;
- n_i is the number of vertices of Γ with degree d_i, $1 \le i \le m$;
- $I = \sum_{i=1}^{m} n_i - 1$;
- E is the number of vertices of Γ with degree 1.

Then, we proved that

$$\left[\Gamma\right]_{\mathbb{F}_1} = \sum_{i=1}^{m} n_i \mathbb{L}^{d_i} - I \cdot \mathbb{L} + I + E. \tag{4.17}$$

The zeta function is thus given by

$$\zeta_\Gamma^{\mathbb{F}_1}(t) \quad = \quad \frac{(t-1)^I}{t^{E+I}} \cdot \prod_{k=1}^{m} (t-d_k)^{-n_k}. \tag{4.18}$$

4.3.1 Future steps

In future research we will try to see how much information about a loose graph Γ (or its corresponding constructible set $\mathcal{F}(\Gamma) \otimes_{\mathbb{F}_1} k$) can be obtained through $\zeta_{\{.\}}^{\mathbb{F}_1}(t)$, following [6] for example. We believe that the information obtained from both the new and the Ihara zeta functions could be very different.

4.4 Comparison with the Ihara zeta function: some examples

In this section we will consider some examples of different graphs and compare the expressions of the corresponding Ihara zeta function and the zeta function we have previously defined.

For the Ihara zeta function we will use the Bass-Hashimoto formula (cf. section 4.1) to compute its inverse with the help of Magma and for the new zeta function, the definition can be found in section 4.3. In both cases we will compute the inverse so that it is easy to compare expressions. All our computations were made in Magma.

Complete graph K_4

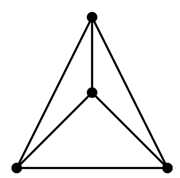


Figure 4.1: The complete graph K_4.

A computation for the graph K_4 (figure 4.1) using the Bass-Hashimoto formula gives the following inverse of the Ihara zeta function:

$$\zeta(u, K_4)^{-1} = 16u^{12} - 24u^{10} - 16u^9 - 3u^8 + 24u^7 + 16u^6 - 6u^4 - 8u^3 + 1, \qquad (4.19)$$

with $u \in \mathbb{C}$. For our zeta function we first need to compute the Grothendieck polynomial, which in this case is

$$\mathbb{P}(K_4) = \mathbb{L}^3 + \mathbb{L}^2 + \mathbb{L} + 1,$$

and hence the $\mathbb{F}_1$-zeta function is given by

$$\zeta_{K_4}^{\mathbb{F}_1\,-1}(t) = t(t-1)(t-2)(t-3) = t^4 - 6t^3 + 11t^2 - 6t.$$

The two functions appear to be very different and this is also the case for other examples.

Complete graph K_4 without an edge

Let us call K_4^* the graph $K_4 \setminus e$ (figure 4.2), where e is any edge of K_4. In this case, we obtain the following comparison

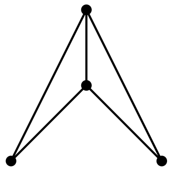


Figure 4.2: Loose graph K_4^*.

$$
\begin{aligned}
\zeta(u, K_4^*)^{-1} &= -4u^{10} + u^8 + 4u^7 + 4u^6 - 2u^4 - 4u^3 + 1, \\
\mathbb{P}(K_4^*) &= \mathbb{L}^3 + \mathbb{L}^2 + 2, \\
{\zeta_{K_4^*}^{\mathbb{F}_1}}^{-1}(t) &= t^2(t-2)(t-3) = t^4 - 5t^3 + 6t^2.
\end{aligned}
$$

Complete graph K_5

A detailed computation of the Grothendieck polynomial for K_5 (figure 4.3) is done in Appendix A.

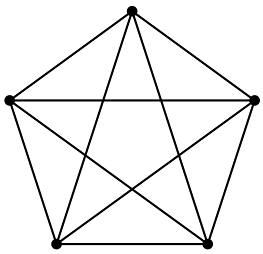


Figure 4.3: The complete graph K_5.

In this case, the two different zeta functions are the following:

$$
\begin{aligned}
\zeta(u, K_5)^{-1} &= -243u^{20} + 1080u^{18} + 180u^{17} - 1710u^{16} - 776u^{15} \\
&\quad +870u^{14} + 1200u^{13} + 505u^{12} - 660u^{11} - 708u^{10} \\
&\quad -140u^9 + 165u^8 + 240u^7 + 70u^6 - 24u^5 - 30u^4 \\
&\quad -20u^3 + 1, \\
\mathbb{P}(K_5) &= \mathbb{L}^4 + \mathbb{L}^3 + \mathbb{L}^2 + \mathbb{L} + 1, \\
{\zeta_{K_5}^{\mathbb{F}_1}}^{-1}(t) &= t(t-1)(t-2)(t-3)(t-4).
\end{aligned}
$$

Johnson graph $J(4,2)$

For the case of the Johnson graph with parameters $(4,2)$ (figure 4.4) the zeta functions are given by:

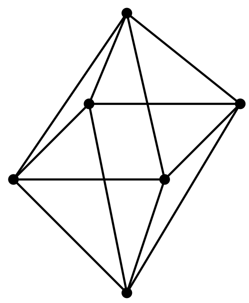


Figure 4.4: The Johnson Graph $J(4,2)$.

$$\begin{aligned}
\zeta(u, J(4,2))^{-1} &= 729u^{24} - 3888u^{22} - 432u^{21} + 7938u^{20} + 2160u^{19} \\
&\quad -6912u^{18} - 4032u^{17} + 639u^{16} + 3008u^{15} + 2976u^{14} \\
&\quad +96u^{13} - 1412u^{12} - 1248u^{11} - 384u^{10} + 320u^{9} \\
&\quad +327u^{8} + 192u^{7} + 16u^{6} - 48u^{5} - 30u^{4} - 16u^{3} + 1, \\
\mathbb{P}(J(4,2)) &= 6\mathbb{L}^4 - 12\mathbb{L}^3 + 20\mathbb{L}^2 - 16\mathbb{L} + 8, \\
{\zeta^{\mathbb{F}_1}_{J(4,2)}}^{-1}(t) &= \frac{t^8(t-2)^{20}(t-4)^6}{(t-1)^{16}(t-3)^{12}}.
\end{aligned}$$

Remark 4.4.1. Note that the Johnson graph $J(4,2)$ can be seen as a combinatorial representation of the $\mathbb{F}_1$-analogon of the Grassmannian $\mathtt{Gr}(4,2)$.

Let V be a n-dimensional vector space over a field k. The *Grassmannian* $\mathtt{Gr}(n,r)$ is defined to be the set of all r-dimensional linear subspaces of V. If we consider the analogous version for $\mathbb{F}_1$, where n-dimensional vector spaces are sets of n elements, we can define the Grassmannian over $\mathbb{F}_1$, denoted by $\mathtt{Gr}_{\mathbb{F}_1}(n,r)$, to be the set of all subsets of r elements of a set of n elements.

With this definition, it is easy to see that the order of $\mathtt{Gr}_{\mathbb{F}_1}(n,r)$ is just the binomial number $\binom{n}{r}$. If we consider the example $\mathtt{Gr}_{\mathbb{F}_1}(4,2)$, we obtain six 2-dimensional subspaces of a 4-dimensional vector space. Now construct a graph where each 2-dimensional subspace corresponds to a vertex and two vertices are adjacent if the corresponding 2-spaces share a 1-dimensional subspace. We then obtain the Johnson graph $(4,2)$.

Hexahedron

Our last example is the hexahedron (figure 4.5), which we call D. Then,

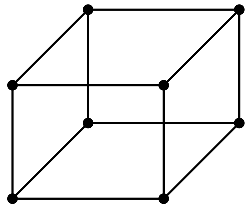


Figure 4.5: Hexahedron.

$$
\begin{aligned}
\zeta(u,D)^{-1} &= 256u^{24} - 768u^{22} + 480u^{20} + 400u^{18} - 183u^{16} - 384u^{14} \\
&\quad +68u^{12} + 144u^{10} + 30u^{8} - 32u^{6} - 12u^{4} + 1, \\
\mathbb{P}(D) &= 8\mathbb{L}^3 - 12\mathbb{L} + 12, \\
\zeta_D^{\mathbb{F}_1\,-1}(t) &= \frac{t^{12}(t-3)^8}{(t-1)^{12}}.
\end{aligned}
$$

4.5 The chromatic polynomial

G. D. Birkhoff introduced the chromatic polynomial in 1912 as an attempt to prove the four color theorem [3]. He noticed that the number of ways a certain map can be painted with at most k colors exhibits polynomial dependence on k. Nowadays, the extended definition of the chromatic polynomial for arbitrary graphs given by H. Whitney in 1932 is the most used expression [49, 50].

We will give a brief introduction to this subject. All the results and definitions are taken from [21].

Definition 4.5.1. Let G be a graph and let $V(G)$ be its set of vertices. A *k-coloring* of G is a function $\sigma : V(G) \to \{1, 2, \ldots, k\}$ which satisfies $\sigma(i) \neq \sigma(j)$ for any edge $e = ij$. Note that it is not compulsory to use all the colors. The graph is said to be *k-colorable* if such a function exists. The *chromatic number* $\chi(G)$ is the minimal k for which the graph is k-colorable and we say that G is *k-chromatic* if $\chi(G) = k$.

Remark 4.5.2. Notice that a *k-coloring* means the assignment of colors to the vertices of a graph in such a way that no two adjacent vertices share the same color.

Consider now the number of different k-colorings of a given graph G as a function of k, and denote it by $\mathtt{chr}(G,k)$. One can prove [21] that $\mathtt{chr}(G,k)$ is a polynomial of k and has degree $n = |V(G)|$, the number of vertices of G.

Definition 4.5.3. For a given G, we call $\mathtt{chr}(G,k)$ the *chromatic polynomial* of G.

Examples 4.5.4. i) The chromatic polynomial of the graph $\overline{K_n}$ on n vertices without edges is $\mathtt{chr}(\overline{K_n},k) = k^n$.

ii) The chromatic polynomial of the complete graph K_n is $\texttt{chr}(K_n, k) = k(k-1)\cdots(k-n+1)$.

iii) Any tree T_n on n vertices has $\texttt{chr}(T_n, k) = k(k-1)^{n-1}$ as a chromatic polynomial.

For computing the chromatic polynomial associated to a given graph G, one uses a recursive process called *deletion-contraction*. The idea of this process is to choose a pair of non-adjacent vertices (i, j) and classify colorings of the graph G depending on whether i and j share a color or not. In the first case, if i and j have the same color, one can consider the graph G/ij obtained by merging the two vertices into one and, in the second case, one substitutes the graph G by the graph $G + ij$ in which an edge incident with both i and j is added. The chromatic polynomial should then satisfy the following recursive formula:

$$\texttt{chr}(G, k) = \texttt{chr}(G + ij, k) + \texttt{chr}(G/ij, k).$$

Nevertheless, although some coefficients of the chromatic polynomial are easy to compute (for instance, for every graph G, the leading coefficient of $\texttt{chr}(G, k)$ is always 1 and the constant term is always 0), the problem of computing the coefficients of the chromatic polynomial is $\#P$-hard.

4.5.1 Connection with the new zeta function

Let K_n be the complete graph on n vertices and let $\mathcal{F}_k(K_n)$ be its associated $(n-1)$-dimensional projective space. The counting polynomial of K_n is

$$P_{K_n}(X) = \sum_{i=0}^{n-1} X^i$$

and hence, its $\mathbb{F}_1$-zeta function is given by the formula

$$\zeta_{K_n}^{\mathbb{F}_1}{}^{-1}(t) = t(t-1)\cdots(t-n+1),$$

which equals the chromatic polynomial $\texttt{chr}(K_n, t)$ (cf. example 4.5.4). At the moment we do not know whether there is an immediate connection between these two concepts associated to a graph. In future research, we will study this very interesting fact that could lead to another bridge between $\mathbb{F}_1$-theory and combinatorics. We thank Dimitri Leemans for bringing these ideas to us during the private defense of this thesis.

5 Automorphism Groups

In the previous two chapters we studied the functor $\mathcal{F}$ in the context of Algebraic Geometry and the theory of motives. In this last chapter of the PhD we focus more on a connection with Projective Geometry. We will study some relations between the automorphism group of a loose graph Γ and several automorphism groups of the constructible set $\mathcal{F}_k(\Gamma)$ associated to it, with k a finite field. Regarding the constructible set $\mathcal{F}_k(\Gamma)$, we will consider three different automorphism groups (projective, combinatorial and topological) that will be defined in section 5.2. In the end we will consider some future steps that are needed to extend the results showed in the course of the chapter.

For this chapter we assume some basic level of familiarity in Projective Geometry and Group Theory, although some notions will be recalled when necessary.

5.1 Automorphism group of loose graphs

We recall the definition of the automorphism group of a loose graph Γ in a nutshell. For more information and examples on morphisms of loose graph, see subsection 2.2.1.

Definition 5.1.1. Let $\Gamma = (V, E, \mathbf{I})$ be a loose graph. We say that a map $f : \Gamma \to \Gamma$ is an *automorphism* of Γ if:

i) f is bijective in the set of vertices and in the set of edges.

ii) $f(e_\emptyset) = e_\emptyset$.

iii) A vertex v is incident with an edge e in Γ if and only if $f(v)$ is incident with $f(e)$ in Γ.

5.2 Automorphism groups of constructible sets

Let Γ be a loose graph, $\mathcal{F}(\Gamma)$ its associated Deitmar constructible set and $\mathcal{X}_k := \mathcal{F}_k(\Gamma)$ its k-constructible set. Below we define the three different automorphism groups of $\mathcal{X}_k$ that we will use for the purpose of this chapter.

5.2.1 Projective automorphism group

Definition 5.2.1. We define the *projective automorphism group* of $\mathcal{X}_k$, denoted by $\mathrm{Aut}^{\mathrm{proj}}(\mathcal{X}_k)$, as the group of automorphisms of the ambient projective space of $\mathcal{X}_k$ stabilizing $\mathcal{X}_k$ setwise, modulo the group of such automorphisms acting trivially on $\mathcal{X}_k$.

Remark 5.2.2. Recall that the projective general linear automorphism group of a projective space $\mathbb{P}^n_k$ is the *projective linear group* $\mathbf{PGL}_{n+1}(k)$, i.e., the group of invertible $(n+1)\times(n+1)$-matrices modulo the group of scalar multiples of the identity. An element of the *general automorphism group* $\mathbf{P\Gamma L}_{n+1}(k)$ is an element of $\mathbf{PGL}_{n+1}(k)$ "twisted" by an automorphism of the field k. In fact, the so-called "projective semilinear group" $\mathbf{P\Gamma L}_{n+1}(k)$ can be described as

$$\mathbf{P\Gamma L}_{n+1}(k) = \mathbf{PGL}_{n+1}(k) \rtimes \mathrm{Aut}(k).$$

When the automorphism group of the field k is trivial, then $\mathbf{P\Gamma L}_{n+1}(k) = \mathbf{PGL}_{n+1}(k)$.

5.2.2 Combinatorial automorphism group

Before defining the *combinatorial automorphism group* of $\mathcal{X}_k = \mathcal{F}(\Gamma) \otimes_{\mathbb{F}_1} k$, with k a field, we need to give a combinatorial structure to the constructible set $\mathcal{X}_k$. We define $\mathcal{X}_k$ as an *incidence geometry of rank 2*, i.e., $\mathcal{X}_k := (\mathcal{P}, \mathcal{L}, \mathbf{I})$ where $\mathcal{P}$ is a set of points, $\mathcal{L}$ is a set of lines and $\mathbf{I}$ is a relation of incidence on the disjoint union $\mathcal{P} \cup \mathcal{L}$. We consider the set of points to be the set of k-rational points of $\mathcal{X}_k$ and the set of lines to be consisting of both projective lines (over k) and *complete affine lines*. A *complete affine* line l of $\mathcal{X}_k$ is a line whose projective completion $\bar{l}$ intersects $\mathcal{X}_k$ in the whole projective line $\bar{l}$ minus one point.

Definition 5.2.3. Let Γ be a loose graph and let $\mathcal{X}_k$ the k-constructible set associated to Γ by $\mathcal{F}_k$ considered as an incidence geometry as before. A *combinatorial automorphism* g of $\mathcal{X}_k$ is a bijective map on the set of points and on the set of lines preserving incidence, i.e., a point p is on the line L of $\mathcal{X}_k$ if and only if the image of p is on the image of L in $\mathcal{X}_k$. We will denote by $\mathrm{Aut}^{\mathrm{comb}}(\mathcal{X}_k)$ the group of combinatorial automorphisms of $\mathcal{X}_k$.

The next two results show that combinatorial automorphisms automatically preserve the linear subspace structure of the constructible sets. The results are direct, but we provide proofs anyhow.

Lemma 5.2.4. *Let $\mathcal{X}_k$ be a constructible set coming from a loose tree and let g be a combinatorial automorphism of $\mathcal{X}_k$. If $\mathbb{A}$ is a d-dimensional affine space contained in $\mathcal{X}_k$, then $\mathbb{A}^g$ is also an affine space of dimension d, contained in $\mathcal{X}_k$ and isomorphic to $\mathbb{A}$.*

Proof. To prove that $\mathbb{A}^g$ is an affine space isomorphic to $\mathbb{A}$, it is sufficient to recall the axiomatic definition of an affine space in terms of an incidence geometry of rank 2 in which one has a set of points $\mathcal{P}$, a set of lines $\mathcal{L}$ and an equivalence relation "$\|$" of parallelism defined on the set of lines. By using the axioms one observes that $\mathbb{A}^g$ is an axiomatic affine space with the same dimension of $\mathbb{A}$. It is then obvious that g is an isomorphism between (axiomatic) affine spaces (and so $\mathbb{A}^g$ is also defined over k). The axioms are the following:

- Each pair P, Q of distinct points is contained in a unique line l.
- For each point P and each line l, there is a unique line l' such that $P \in l'$ and $l \| l'$.
- *Trapezoid axiom.* Let PQ and RS be distinct parallel lines and let T be a point of $PR \setminus \{P, R\}$. Then, there must be a point incident with PQ and TS.
- *Parallelogram axiom.* If no line has more than two points, and if P, Q and R are three distinct points, then the line through R parallel to PQ must have a point in common with the line through P parallel to QR.
- *Thickness.* Each line contains at least two points.
- *Space axiom.* There exists two disjoint lines l and l' such that $l \nparallel l'$. Notice that this axiom is only required if the dimension of the affine space (see below) is greater than 2.

Every axiom is satisfied in $\mathbb{A}^g$ since the automorphism g preserves the incidence relations and g is injective on points and lines of $\mathbb{A}$. Hence, $\mathbb{A}^g$ is an affine space. It remains to prove that it is indeed of dimension d.

Recall that the *geometric dimension* of the affine space $\mathbb{A}$ is given recursively by the largest number (d in this case) for which there exists a strictly ascending chain of subspaces of the form:

$$\emptyset \subset X_0 \subset X_1 \subset \cdots \subset X_d = \mathbb{A}. \tag{5.1}$$

A subspace X of $\mathbb{A}$ is a subset such that any line containing two points of X is a subset of X (where lines are seen as point sets) and this line then is a line of X. A subspace X_i in such a chain is said to have geometric dimension i. Since g is an automorphism, applying g to this chain we will obtain a new chain of the form

$$\emptyset \subset X_0^g \subset X_1^g \subset \cdots \subset X_d^g = \mathbb{A}^g,$$

where all subspaces X_i^g are of dimension greater or equal to i, since g is injective. Let us now suppose that $\dim(A^g) = j > d$; then there will exist a chain of the form

$$\emptyset \subset Y_0 \subset Y_1 \subset \cdots \subset Y_j = \mathbb{A}^g.$$

Applying g^{-1} to this new chain, we will obtain a chain for $\mathbb{A}$ longer than (5.1) since g^{-1} is injective as well. But this is not possible since (5.1) is a chain of maximal length. ■

Lemma 5.2.5. *With the same conditions of the previous observation, if $\mathbb{P}$ is a d-dimensional projective space contained in $\mathcal{X}_k$, then $\mathbb{P}^g$ is also a projective space of dimension d, contained in $\mathcal{X}_k$ and isomorphic to $\mathbb{P}$.*

Proof. As before we just have to recall the axiomatic definition of a projective space in terms of an incidence geometry of rank 2. The axioms, described already in chapter 1, are the following:

- Two different points are exactly incident with one line.
- *Thickness.* Each line has at least three points.
- *Veblen's axiom.* If a, b, c and d are different points and the lines ab and cd meet, then so do the lines ac and bd.

For the same reason as in the affine case, every axiom is satisfied in $\mathbb{P}^g$, so $\mathbb{P}^g$ is a projective space. The fact that $\dim(\mathbb{P}^g) = d$ is proven in the same way as for the affine case. ■

After these two lemmas, one realizes that there is another "natural" way of giving a structure of incidence geometry to $\mathcal{X}_k$, in which we not only consider points and lines but also all affine and projective subspaces. Let $r := \max\{\deg(v) \mid v \in V(\Gamma)$ and v defines an affine space $\mathbb{A}_v$ such that $\overline{\mathbb{A}_v}$ is not contained in $\mathcal{X}_k\}$ and $s := \max\{n-1 \mid K_n \subseteq \Gamma\}$. We will consider $\mathcal{X}_k$ as an incidence geometry of "*double rank* (r, s)." We define $\mathcal{X}_k$ to be the $(r+s+2)$-tuple $(K, A_1, \ldots, A_r, P_1, \ldots, P_s, \mathbf{I})$, where A_i is the set of i-dimensional affine subspaces of $\mathcal{X}_k$ whose completion is not contained in $\mathcal{X}_k$; P_k is the set of k-dimensional projective subspaces of $\mathcal{X}_k$, $K = A_0 = P_0$, and $\mathbf{I}$ is the natural incidence relation between these spaces. Note that the sets A_i and P_j are non empty for all i, j.

Remark 5.2.6. If $\mathcal{X}_k$ is, e.g., a projective space of dimension d, then the double rank is $(0, d)$. If Γ is a tree, then the double rank is $(r, 1)$ or $(0, 0)$ (if Γ is a vertex).

With this definition of $\mathcal{X}_k$, the two aforementioned lemmas lead to the following result.

Corollary 5.2.7. *Let Γ be a loose graph, $\mathcal{X}_k$ its corresponding constructible set over k and g a combinatorial automorphism of $\mathcal{X}_k$. If we define the numbers r and s as above, then g is also an automorphism of $\mathcal{X}_k$ as an incidence geometry of double rank (r, s).*

Proof. The proof of this corollary follows immediately after Lemma 5.2.4 and Lemma 5.2.5 and the fact that g (and g^{-1}) preserves incidence relations when $\mathcal{X}_k$ is considered as an incidence geometry of rank 2. ■

5.2.3 Topological automorphism group

Definition 5.2.8. We define a *topological automorphism* g of the constructible set $\mathcal{X}_k$ as a homeomorphism of its underlying topological space, i.e, a bijective continuous map with a continuous inverse map. In a natural way, we obtain the *topological automorphism group* of $\mathcal{X}_k$, denoted by $\mathrm{Aut}^{\mathrm{top}}(\mathcal{X}_k)$.

Relation with the combinatorial automorphism group

Let k be a finite field and $\mathcal{X}_k$ be the affine scheme $\mathtt{Spec}(k[X])$. The closed set topology of $\mathtt{Spec}(k[X])$ consists of (0), all closed points (all prime ideals are maximal since they correspond to monic irreducible polynomials in $k[X]$) and all finite sets of such points that contain (0). So $\mathrm{Aut}^{\mathrm{top}}(\mathtt{Spec}(k[X]))$ is isomorphic to the symmetric group on the set of closed points. On the other hand, $\mathrm{Aut}^{\mathrm{comb}}(\mathtt{Spec}(k[X]))$ is isomorphic to the symmetric group on the k-rational points, so as soon as k is not algebraically closed, the groups are not the same. Now let $\mathcal{X}_k$ be $\mathtt{Spec}(k[X_1, \ldots, X_m])$ with $m \geq 2$, and let U be an affine subline. Then $\mathrm{Aut}^{\mathrm{top}}(\mathcal{X}_k)$ induces $\mathrm{Aut}^{\mathrm{top}}(U)$ on the topology of U, which, as we have seen, is isomorphic to the symmetric group on the closed points of U. The combinatorial automorphism group of $\mathcal{X}_k$ induces the affine group $\mathbf{A\Gamma L}_1(k)$ on U (acting on the k-rational points). So in general these two groups are not isomorphic.

The next proposition deals with the other direction.

Proposition 5.2.9. *The combinatorial group of $\mathcal{X}_k$ is a subgroup of the topological automorphism group of $\mathcal{X}_k$.*

Proof. Let us first take a combinatorial automorphism f of $\mathcal{X}_k$. It is possible to reduce our proof w.l.o.g. to the case of an affine space defined by one of the loose stars corresponding to a vertex of Γ, since an automorphism of the constructible set $\mathcal{X}_k$ can be constructed as "pasting" local morphisms of affine spaces.

Let $\mathtt{Spec}(A_v)$ be the affine space of $\mathcal{X}_k$ corresponding to the vertex $v \in \Gamma$. If f is a combinatorial automorphism of $\mathcal{X}_k$, by Lemma 5.2.4 we know that f induces also a combinatorial isomorphism f_v from $\mathtt{Spec}(A_v) \otimes_{\mathbb{F}_1} k$ to $\mathtt{Spec}(A_{f(v)}) \otimes_{\mathbb{F}_1} k$. The affine spaces $\mathtt{Spec}(A_v) \otimes_{\mathbb{F}_1} k$ and $\mathtt{Spec}(A_{f(v)}) \otimes_{\mathbb{F}_1} k$ are isomorphic to $\mathtt{Spec}(k[X_1, \ldots, X_n])$ and $\mathtt{Spec}(k[Y_1, \ldots, Y_n])$, with $n = \deg(v) = \deg(f(v))$, respectively.

Let $X_i = 0$, $i = \{1, \ldots, n\}$ be the coordinate hyperplanes inside the affine space $\mathtt{Spec}(A_v) \otimes_{\mathbb{F}_1} k$ and consider the action of f_v induced on them. As a consequence, we obtain an isomorphism on the coordinate rings $k[X_1, \ldots, X_n]$ and $k[Y_1, \ldots, Y_n]$ that gives, by functoriality, an isomorphism of affine schemes between $\mathtt{Spec}(A_v) \otimes_{\mathbb{F}_1} k$ and $\mathtt{Spec}(A_{f(v)}) \otimes_{\mathbb{F}_1} k$. For each vertex v of the graph Γ we hence obtain an induced topological isomorphism between the local affine k-schemes corresponding to v and $f(v)$. By considering the union of these isomorphisms we finally obtain the topological automorphism of $\mathcal{X}_k$. ∎

5.3 Trees and constructible sets

We are now ready to study how automorphisms of loose graphs are related to those automorphisms of constructible sets described in the previous section. We will start analyzing a basic example, which we call "toy example," but we first recall some notions on actions of groups that are needed for the complete understanding of the section.

5.3.1 Group action

Definition 5.3.1. Let G be a group and X be a set. We say that *the group G acts on X* if there exists a function

$$\phi : G \times X \longrightarrow X$$

$$(g, x) \longmapsto g.x$$

that satisfies the following two axioms:

- ⋆ $e.x = x$ for all $x \in X$, where e is the identity element of the group G.
- ⋆ $(gh).x = g.(h.x)$ for all $g, h \in G$ and all $x \in X$.

Sometimes we will write $g(x)$ or x^g for $g.x$. So, $h(g(x)) = (g^x)^h = h.(g.x)$.

Definition 5.3.2 (Types of actions). Let G be a group acting on a non-empty set X. We say that the action of G is

- *Transitive* if for all $x, y \in X$ there exists $g \in G$ such that $g.x = y$;
- *Free* if $g.x = x$ for some $x \in X$ implies that g is the identity element of G;
- *Faithful* if for each element $g \neq e \in G$ there exists an element $x \in X$ such that $g.x \neq x$;
- *Sharply transitive* if it is transitive and free, i.e., if for each pair $x, y \in X$ there exists a unique $g \in G$ such that $g.x = y$;
- *n-transitive* if X has at least n elements and for all pairwise distinct $x_1, \ldots, x_n$ and pairwise distinct $y_1, \ldots, y_n$ there is an element $g \in G$ such that $g.x_k = y_k$ for all $1 \leq k \leq n$.

Definition 5.3.3. Let G be a group acting on a set X. The *stabilizer* of a point $x \in X$ is the subgroup of G given by

$$G_x = \{g \in G \mid g.x = x\} \leq G.$$

Let X' be a subset of X. The *pointwise stabilizer* of X' in G, denoted by $G_{[X']}$, is the set of elements $g \in G$ such that $g.x = x$ for all $x \in X'$. Similarly, the *setwise stabilizer* of X', denoted by $G_{X'}$, is the set of elements $g \in G$ such that $g.x \in X'$ for all $x \in X'$.

We define the *orbit* of an element $x \in X$ to be the subset of X given by

$$x^G = \{g.x \mid g \in G\} \subseteq X$$

and the *kernel of the action* to be the subgroup of G defined by the set

$$K = \{g \in G \mid g.a = a \ \forall a \in A\}.$$

Definition 5.3.4. Let G be a group and let A and B be two subgroups of G. We say that G is an *internal central product* of A and B if the following conditions are satisfied:

- $\star$ $G = \langle A, B \rangle$, i.e., G is generated by A and B.
- $\star$ $[A, B] = \{\mathrm{id}\}$, i.e., every element of A commutes with every element of B.

In this case, we say that both A and B are *central factors* of G. We will denote by $A * B$ the internal central product of A and B

5.3.2 Toy example

Let Γ be the connected loose graph on two vertices (x and y) of regular degree 2 (figure 5.1) and denote the loose edge on x by L_x and the loose edge on y by L_y. In this section we show that

$$\mathrm{Aut}(\mathcal{F}(\Gamma) \otimes_{\mathbb{F}_1} k) \cong \mathrm{Aut}^{\mathsf{proj}}(\mathcal{F}(\Gamma) \otimes_{\mathbb{F}_1} k) \tag{5.2}$$

for any field k. Here (and throughout the rest of the chapter), $\mathrm{Aut}(\cdot)$ denotes the *combinatorial* automorphism group.

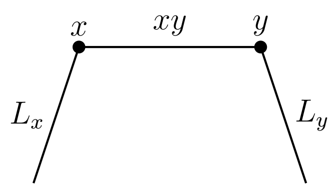


Figure 5.1: Toy example.

For the rest of this section, fix a field k. Also, let $\mathbb{A}_x$ and $\mathbb{A}_y$ be the affine planes corresponding (respectively) to the vertices x and y through $\mathcal{F}_k$. We keep the notation $\mathcal{X}_k$ of the previous section for $\mathcal{F}(\Gamma) \otimes_{\mathbb{F}_1} k$ (and in particular $\mathcal{X}_{\mathbb{F}_1} = \mathcal{X}$).

For now, we want to see $\mathcal{X}_k$ coming together with its embedding

$$\mathcal{X}_k \hookrightarrow \mathbf{PG}(3,k). \tag{5.3}$$

It makes sense to projectively complete $\mathbb{A}_x$ and $\mathbb{A}_y$ — that is, to add the respective lines at infinity X and Y (see figure 5.2) since any element of $\mathrm{Aut}^{\mathrm{proj}}(\mathcal{X}_k)$ also fixes the "projective completion" $\overline{\mathcal{X}_k}$. By projective completion of $\mathcal{X}_k$ we mean the union of all projective completions of the affine spaces associated to vertices.

Consider now a configuration of the form

$$\begin{gathered}\rho := (\mathcal{P} = \{x, y\}, \mathcal{L} = \{X, xy, Y\}, \mathbf{I}) = \\ \{(x,Y),(Y,x),(y,X),(X,y),(x,xy),(xy,x),(y,xy),(xy,y)\}\end{gathered} \tag{5.4}$$

defined as an incidence geometry. We will call such configuration a *root* of $\mathbf{PG}(3,k)$ and denote it by (Y, x, xy, y, X). As X is incident with y (as a point of $\mathcal{X}_k$) and Y with x (as a point of $\mathcal{X}_k$), we can deduce that an element $\alpha \in \mathbf{P\Gamma L}_4(k)$ is an element of $\mathrm{Aut}^{\mathrm{proj}}(\mathcal{X}_k)$ if and only if α stabilizes the root ρ. Hence, we can translate the problem of describing the group $\mathrm{Aut}^{\mathrm{proj}}(\mathcal{X}_k)$ in describing the elements of $\mathbf{P\Gamma L}_4(k)$ stabilizing the root ρ setwise.

For what is to follow it is very important to notice that X and Y are *projective* lines and not *affine* lines. Now let $\mathbf{PG}(n,k)$ be an n-dimensional projective space and Ω be a hyperplane of $\mathbf{PG}(n,k)$; we denote by $T(\Omega)$ the *group of translations of* $\mathbf{PG}(n,k)$ *with axis* Ω. A *translation* of $\mathbf{PG}(n,k)$ is an automorphism of $\mathbf{PG}(n,k)$ with *axis* Ω and *center* $p \in \Omega$, meaning that it fixes Ω pointwise and all hyperplanes through p setwise. It is known that $T(\Omega)$ is a subgroup of $\mathbf{PGL}_{n+1}(k)$ which acts sharply transitively on the points of $\mathbf{PG}(n,k) \setminus \Omega$.

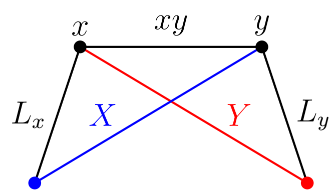


Figure 5.2: Projective completion of $\mathbb{A}_x$ and $\mathbb{A}_y$.

Proposition 5.3.5. $\mathbf{P\Gamma L}_4(k)$ *acts transitively on the roots of* $\mathbf{PG}(3,k)$.

Proof. Obviously $\mathbf{P\Gamma L}_4(k)$ acts transitively on the ordered triples (u, uv, v), with $u \neq v$ points of $\mathbf{PG}(3,k)$ (as it acts transitively on the lines, and a line stabilizer induces the natural action of $\mathbf{P\Gamma L}_2(k)$, which is 3-transitive). Fix such a triple (x, xy, y). Let Y, Y' be different lines on x, both different from xy. Consider a plane ν containing xy but not Y nor Y'. Then there is an element in $T(\nu)$ that maps Y' to Y, so from now on, we also fix Y. Now let X, X' be lines on y different from xy, and not meeting Y. Define the plane $\rho := \langle Y, xy\rangle$, and note that it does not contain X nor X'. Then $T(\rho)$ contains an element which maps X' to X. The claim follows. ■

Note that roots are *ordered*. By the proof of the previous proposition, we immediately have the following.

Corollary 5.3.6. $\mathbf{PGL}_4(k)$ *acts transitively on the roots of* $\mathbf{PG}(3, k)$.

Proof. One can replace $\mathbf{P\Gamma L}_4(k)$ by $\mathbf{PGL}_4(k)$ in the proof of Proposition 5.3.5. Furthermore, all translations are elements in $\mathbf{PGL}_4(k)$. ■

The following is immediate.

Proposition 5.3.7. *The kernel of the action of* $\mathbf{P\Gamma L}_4(k)_{\mathcal{X}_k}$ *on* $\mathcal{X}_k$ *is trivial.*

Proposition 5.3.8. *Let* $\mathbb{P}_x$ *be the projective* k*-plane generated by* x, xy *and* Y*. Let* $A := \mathrm{Aut}(\mathbb{P}_x)_{(Y,x,xy,y)}$ *be the elementwise stabilizer of* $\{Y, x, y, xy\}$ *in* $\mathrm{Aut}(\mathbb{P}_x)$*, where the latter is the combinatorial automorphism group of* $\mathbb{P}_x$ *(so isomorphic to* $\mathbf{P\Gamma L}_3(k)$*). (For later purposes, we similarly define* $\mathbb{P}_y$*.) Then each element of* A *extends to an element of* $\mathbf{P\Gamma L}_4(k)_{\mathcal{X}_k}$ *(in a not necessarily unique fashion).*

Proof. Let $\alpha \in A$ be arbitrary; then α extends to elements of $\mathbf{P\Gamma L}_4(k)$, for instance to $\widetilde{\alpha}$. Note that $\widetilde{\alpha}$ fixes y. Suppose that $X^{\widetilde{\alpha}} =: X'$. Now let β be an element in $T(\mathbb{P}_x)$ which maps X' back to X; then $\beta \circ \widetilde{\alpha}$ fixes the root (Y, x, xy, y, X) and induces α on $\mathbb{P}_x$. ■

Remark 5.3.9. It is important to note that $\mathrm{Aut}(\mathbb{P}_x)$ coincides with the automorphism group of $\mathbb{P}_x$ induced by the automorphisms of $\mathbf{P\Gamma L}_4(k)$.

The "number" of ways to extend an element α is easy to determine. For, if γ and γ' are two such elements, then $\gamma^{-1} \circ \gamma'$ fixes $\mathbb{P}_x$ pointwise, while fixing X. This group faithfully induces $\mathbf{PGL}_2(k)_y$ on the projective line X (we are in the projective group, since $\mathbb{P}_x$ is pointwise fixed). Its order is $|k|(|k| - 1)$ (the group acts sharply 2-transitively on $X \setminus \{y\}$ and is isomorphic to $k \rtimes k^\times$).

We now have all the ingredients for writing down $\mathrm{Aut}^{\mathrm{proj}}(\mathcal{X}_k)$. First of all, it is clear that $\mathrm{Aut}(\Gamma) \cong \langle\varphi\rangle$, with $\varphi \neq \mathrm{id}$ an involution. By Proposition 5.3.5, there is an element in $\mathbf{P\Gamma L}_4(k)$ which stabilizes the root ρ, and which has the same action as φ. And obviously, the subgroup of $\mathrm{Aut}^{\mathrm{proj}}(\mathcal{X}_k)$ which fixes both x and y is a normal subgroup of $\mathrm{Aut}^{\mathrm{proj}}(\mathcal{X}_k)$. We will denote by $\mathrm{Aut}^{\mathrm{proj}}(\mathcal{X}_k)_{(x,y)}$ such a subgroup.

Theorem 5.3.10. *Let* $\mathbf{P\Gamma L}_2(k)$ *be the automorphism group of the projective line* $\mathbf{PG}(1, k)$*, and let* u, v *be distinct points of the latter. Let* $C := \mathbf{P\Gamma L}_2(k)_{(u,v)} \cong k^\times \rtimes \mathrm{Aut}(k)$*, and* $D := \mathbf{PGL}_2(k)_u \cong k \rtimes k^\times$*. Then*

$$\mathrm{Aut}^{\mathrm{proj}}(\mathcal{F}(\Gamma) \times_{\mathbb{F}_1} k) \cong (D \rtimes (D \rtimes C)) \rtimes \langle\varphi\rangle. \tag{5.5}$$

In the latter expression,

- $(D \rtimes (D \rtimes C))$ *is the elementwise stabilizer of* $\{x, y\}$ *in* $\mathrm{Aut}^{\mathrm{proj}}(\mathcal{X}_k)$*;*
- *the "first D" is* $T(\mathbb{P}_x) \cap \mathrm{Aut}^{\mathrm{proj}}(\mathcal{X}_k)$*;*
- $E := D \rtimes C$ *is the pointwise stabilizer of* X *in* $\mathrm{Aut}^{\mathrm{proj}}(\mathcal{X}_k)$*;*
- D *("in E") is the pointwise stabilizer of* xy *in* E*, and* C *("in E") is the action induced by* E *on* xy*.*

For later purposes, we need an approach which allows a possibility to extend to more general cases. Let $\alpha \in \mathrm{Aut}^{\mathrm{proj}}(\mathcal{X}_k)_{(x,y)}$; then α induces an element α_x of $\mathrm{Aut}(\mathbb{P}_x)$ which fixes (Y, x, xy, y), and also an element α_y of $\mathrm{Aut}(\mathbb{P}_y)$ which fixes (x, xy, y, X), and both elements have the same action on the projective line xy. And vice versa, we have that $\mathrm{Aut}^{\mathrm{proj}}(\mathcal{X}_k)_{(x,y)}$ is completely determined by the data

$$\left\{(\alpha_x, \alpha_y) \mid \alpha_x \in \mathrm{Aut}(\mathbb{P}_x)_{(Y,x,xy,y)}, \alpha_y \in \mathrm{Aut}(\mathbb{P}_y)_{(x,xy,y,X)}, \alpha_x\Big|_{xy} \equiv \alpha_y\Big|_{xy}\right\}. \tag{5.6}$$

Before using this observation, we prove the next theorem.

Theorem 5.3.11. *Let* Γ *be the connected loose graph on two vertices (*x *and* y*) of regular degree* 2*. Then*

$$\mathrm{Aut}(\mathcal{F}(\Gamma) \otimes_{\mathbb{F}_1} k) \cong \mathrm{Aut}^{\mathrm{proj}}(\mathcal{F}(\Gamma) \otimes_{\mathbb{F}_1} k) \tag{5.7}$$

for any field k*.*

Proof. Recall that it is obvious by definition that $\mathrm{Aut}^{\mathrm{proj}}(\mathcal{X}_k) \leq \mathrm{Aut}(\mathcal{X}_k)$. Let γ be an element of $\mathrm{Aut}(\mathcal{X}_k) \setminus \mathrm{Aut}^{\mathrm{proj}}(\mathcal{X}_k)$; then there also exists an element γ' in $\mathrm{Aut}(\mathcal{X}_k) \setminus \mathrm{Aut}^{\mathrm{proj}}(\mathcal{X}_k)$ which fixes both x and y (that is, which fixes the root (Y, x, xy, y, X) elementwise). For, it is obvious that there is an $\epsilon \in \mathrm{Aut}^{\mathrm{proj}}(\mathcal{X}_k)$ which switches x and y (and Y and X) (by using Proposition 5.3.5). If γ already fixes x and y, there is nothing to prove. If not, $\epsilon \circ \gamma$ fixes x, y, and is not in $\mathrm{Aut}^{\mathrm{proj}}(\mathcal{X}_k)$. Now γ' induces an element γ'_x in $\mathrm{Aut}(\mathbb{P}_x)_{(Y,x,xy,y)}$ and an element γ'_y in $\mathrm{Aut}(\mathbb{P}_y)_{(x,xy,y,X)}$ which agree on xy. We have seen that there exists an element γ^* in $\mathrm{Aut}^{\mathrm{proj}}(\mathcal{X}_k)$ which also yields the data (γ'_x, γ'_y); composing γ' with γ^{*-1}, we obtain the identity of $\mathrm{Aut}(\mathcal{X}_k)$. The isomorphism follows. ■

If L is a line of $\mathbf{PG}(3, k)$, recall that $\mathbf{P\Gamma L}_4(k)_{[L]}$ denotes the pointwise stabilizer of L in $\mathbf{P\Gamma L}_4(k)$. (Note that it is a subgroup of $\mathbf{PGL}_4(k)$.) More generally, if S is a set of points in $\mathbf{PG}(3, k)$, $\mathbf{P\Gamma L}_4(k)_{[S]}$ denotes its pointwise stabilizer (and this is not necessarily a subgroup of $\mathbf{PGL}_4(k)$).

Lemma 5.3.12. *Define* $A := \mathrm{Aut}^{\mathrm{proj}}(\mathcal{X}_k) \cap \mathbf{P\Gamma L}_4(k)_{[Y]}$*, and* $B := \mathrm{Aut}^{\mathrm{proj}}(\mathcal{X}_k) \cap \mathbf{P\Gamma L}_4(k)_{[X]}$*. Then* $A \cong \mathrm{Aut}(\mathbb{P}_y)_{(x,xy,y,X)} \cap \mathbf{PGL}_3(k)$ *(where the latter expression means the projective general elements in* $\mathrm{Aut}(\mathbb{P}_y)_{(x,xy,y,X)}$*), and* $B \cong \mathrm{Aut}(\mathbb{P}_x)_{(Y,x,xy,y)} \cap \mathbf{PGL}_3(k)$*.*

Proof. We prove the assertion for A. Let α be any element in $\mathrm{Aut}(\mathbb{P}_y)_{(x,xy,y,X)} \cap \mathbf{PGL}_3(k)$; we have seen that α extends to some element $\widetilde{\alpha}$ of $\mathrm{Aut}(\mathcal{X}_k)$, and that any such element induces a projective general linear element on Y. So there is a unique element in $\mathbf{P\Gamma L}_4(k)_{[\mathbb{P}_y]}$ with the same action on Y. Composing with the inverse of $\widetilde{\alpha}$, we obtain an element of A which induces α on $\mathbb{P}_y$. The required isomorphism easily follows. ■

Theorem 5.3.13. *Let $\mathbf{PGL}(\mathcal{X}_k)_{(x,y)}$ be defined as*

$$\mathrm{Aut}^{\mathrm{proj}}(\mathcal{X}_k)_{(x,y)} \cap \mathbf{PGL}_4(k). \tag{5.8}$$

Then $\mathbf{PGL}(\mathcal{X}_k)_{(x,y)}$ is isomorphic to the internal central product of A and B.

Proof. It is obvious that $\langle A, B\rangle = \mathbf{PGL}(\mathcal{X}_k)_{(x,y)}$, so we only have to show that $[A, B] = \{\mathrm{id}\}$. Now if $a \in A$ and $b \in B$, we have that $[a, b] = a^{-1}b^{-1}ab$ fixes Y and X pointwise. On the other hand, both a and b induce elements in $\mathrm{Aut}(xy)_{(x,y)} \cap \mathbf{PGL}_2(k) \cong k^\times$, and this is an abelian group. So $[a, b]$ acts as the identity on xy. It now easily follows that $[a, b]$ acts trivially on $\mathcal{X}_k$ so, by proposition 5.3.7, $[a, b] = \{\mathrm{id}\}$. ■

In general, we have the next conclusion.

Theorem 5.3.14. *We have that*

$$\mathrm{Aut}^{\mathrm{proj}}(\mathcal{X}_k) \;\cong\; \mathrm{Aut}(\mathcal{X}_k) \;\cong\; ((A * B) \rtimes \mathrm{Aut}(k)) \rtimes \langle\varphi\rangle. \tag{5.9}$$

Proof. Follows from Theorem 5.3.10, and the identities

$$\mathbf{PGL}(\mathcal{X}_k)_{(x,y)} \;\trianglelefteq\; \mathrm{Aut}^{\mathrm{proj}}(\mathcal{X}_k)_{(x,y)} \;\trianglelefteq\; \mathrm{Aut}^{\mathrm{proj}}(\mathcal{X}_k). \tag{5.10}$$

■

5.3.3 Loose trees

If Γ is a connected loose tree, and k a field, one of the first things to hope is that:

- $\mathrm{Aut}(\mathcal{X}_k)$ *acts* on the set of affine spaces defined by the vertices Γ;
- this action is induced by $\mathrm{Aut}(\Gamma)$.

These properties are not true in general — look for instance at a projective plane (coming from a triangle): for no field $k \neq \mathbb{F}_1$ one has that $\mathrm{Aut}(\mathcal{X}_k)$ induces an action on the three subplanes corresponding to the vertices.

If the toy example would generalize naturally, one candidate for $\mathrm{Aut}^{\mathrm{proj}}(\mathcal{X}_k)$ would be

$$(U \rtimes \mathrm{Aut}(k)) \rtimes \mathrm{Aut}(\Gamma)^*, \tag{5.11}$$

where $U \rtimes \mathrm{Aut}(k)$ is the part that fixes all vertices of Γ (once pulled to k), and U is the projective general linear part of the latter. After the toy example, U should be isomorphic to a central product of the appropriate groups. Also, $\mathrm{Aut}(\Gamma)^*$ is the automorphism group of the reduced graph $\widetilde{\Gamma}$ of Γ.

The first thing in order to obtain a similar result to the case of general loose graphs is to generalize the little theory of roots in order. We will do that using what we call *fundaments*.

5.3.4 Fundaments

Consider a $\mathbf{PG}(a+b-1,k)=\pi$ over the field k, with $a,b\geq 2$. A *fundament* of *type* (a,b) of π is a triple (α,xy,β), where α is an $(a-1)$-dimensional projective subspace of π, β a $(b-1)$-dimensional projective subspace, and xy a projective line for which $\alpha\cap xy=\{x\}$ and $\beta\cap xy=\{y\}$, and such that

$$\langle\alpha,xy\rangle\cap\langle\beta,xy\rangle=xy. \tag{5.12}$$

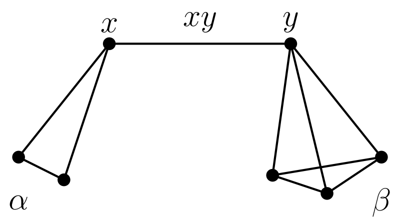


Figure 5.3: A fundament of type (3,4).

You can see an example of a fundament on figure 5.3. Note that $\langle\alpha,\beta\rangle=\pi$, and that a fundament of $\mathbf{PG}(3,k)$ is a root. We define now a fundament *with ends* to be a 5-tuple (α,A,xy,β,B) where (α,xy,β) is a fundament (of type (a,b)), A a projective subspace of α which does not contain x, and B is a projective subspace of β not containing y. Such a fundament has *type* $(a,b;c,d)$ if A and B respectively have dimension c and d. A root of $\mathbf{PG}(3,k)$ is then a fundament with ends of type $(2,2;0,0)$.

The proof of the next proposition is different from that of Proposition 5.3.5 (but it also works for the latter).

Proposition 5.3.15. $\mathbf{P\Gamma L}_{a+b}(k)$ *acts transitively on the fundaments with ends of* $\mathbf{PG}(a+b-1,k)=\pi$ *of type* $(a,b;c,d)$*. In particular,* $\mathbf{P\Gamma L}_{a+b}(k)$ *acts transitively on the fundaments of* $\mathbf{PG}(a+b-1,k)$ *of type* (a,b)*.*

Proof. Let (α,A,xy,β,B) and $(\alpha',A',x'y',\beta',B')$ be two fundaments with ends, both of type $(a,b;c,d)$, both in π. Let $(x,x_1,\ldots,x_{a-1},y,\,y_1,\ldots,y_{b-1})$ be an ordered base of π such that

- $(x, x_1, \ldots, x_{a-1})$ is an ordered base of α and $(y, y_1, \ldots, y_{b-1})$ an ordered base of β;
- $(x_{a-c-1}, \ldots, x_{a-1})$ is an ordered base of A and $(y_{b-d-1}, \ldots, y_{b-1})$ is an ordered base of B.

Define in a similar way an ordered base $(x', x'_1, \ldots, x'_{a-1}, y', y'_1, \ldots, y'_{b-1})$ with respect to $(\alpha', A', x'y', \beta', B')$. Then $\mathbf{PGL}_{a+b}(k)$ contains an element sending the first ordered base to the second, as it acts transitively on the ordered bases of $\mathbf{PG}(a+b-1, k)$. ∎

Corollary 5.3.16. $\mathbf{PGL}_{a+b}(k)$ *acts transitively on the fundaments of* $\mathbf{PG}(a+b-1, k)$ *of type* (a, b).

Proof. The proof is the same as for Proposition 5.3.15. ∎

Let Γ be the connected loose graph on two inner vertices x and y, respectively of degree a and b. Suppose $c \leq a-1$ edges on x different from xy have an end point, and that $d \leq b-1$ edges on y different than xy have end points (see figure 5.4). We will show that

$$\mathrm{Aut}(\mathcal{F}(\Gamma) \otimes_{\mathbb{F}_1} k) \cong \mathrm{Aut}^{\mathrm{proj}}(\mathcal{F}(\Gamma) \otimes_{\mathbb{F}_1} k) \tag{5.13}$$

for any field k, where again $\mathrm{Aut}(\cdot)$ denotes the combinatorial group.

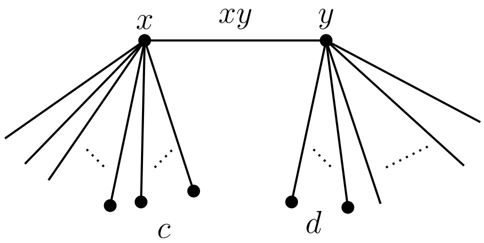


Figure 5.4: The graph Γ.

Let $\mathbb{A}_x$ and $\mathbb{A}_y$ be the affine spaces corresponding respectively to the vertices x and y through the functor $\mathcal{F}$. Write $\mathcal{X}$ for $\mathcal{F}(\Gamma)$, and $\mathcal{X}_k$ for $\mathcal{F}(\Gamma) \otimes_{\mathbb{F}_1} k$. As before, we want to see $\mathcal{X}_k$ coming together with its embedding

$$\mathcal{X}_k \hookrightarrow \mathbf{PG}(a+b-1, k). \tag{5.14}$$

We projectively complete $\mathbb{A}_x$ and $\mathbb{A}_y$; the space at infinity of $\mathbb{A}_x$ is $\langle \alpha_x, y \rangle$, with α_x the projective space defined by the points at infinity of the edges on x different from xy, and the space at infinity of $\mathbb{A}_y$ is $\langle \beta_y, x \rangle$, with β_y the projective space defined by the points at infinity of the edges on y different from xy. Also, let A be the projective subspace of α_x defined by the c end points different from y and adjacent to x, and let B be the projective subspace of β_x defined by the d end points different from x and adjacent to y.

Any element of $\mathrm{Aut}^{\mathrm{proj}}(\mathcal{X}_k)$ also fixes the projective completion $\overline{\mathcal{X}_k}$.

Proposition 5.3.17. *We have that α is in $\mathrm{Aut}^{\mathrm{proj}}(\mathcal{X}_k)$ (with $\alpha \in \mathbf{P\Gamma L}_{a+b}(k)$) if and only if α stabilizes the incidence geometry of the fundament $(\alpha_x, A, xy, \beta_y, B)$.*

The following is immediate.

Proposition 5.3.18. *The kernel of the action of $\mathbf{P\Gamma L}_{a+b}(k)_{\mathcal{X}_k}$ on $\mathcal{X}_k$ is trivial.*

Proof. Let $\gamma \in \mathbf{P\Gamma L}_{a+b}(k)_{\mathcal{X}_k}$ fix all the (k-rational) points of $\mathcal{X}_k$. Then γ fixes $\Pi_x := \overline{\mathbb{A}_x}$ and $\Pi_y := \overline{\mathbb{A}_y}$ pointwise. Consider any point z in $\mathbf{PG}(a+b-1,k)$ outside $\mathcal{X}_k$. Then $\langle \Pi_x, z\rangle$ is an $(a+1)$-dimensional projective space which meets the b-space Π_y in a plane ρ containing xy, and not contained in Π_x. So $\langle \Pi_x, z\rangle = \langle \Pi_x, \rho\rangle$. Hence

$$\langle \Pi_x, z\rangle^\gamma = \langle \Pi_x, \rho\rangle^\gamma = \langle \Pi_x^\gamma, \rho^\gamma\rangle = \langle \Pi_x, \rho\rangle, \tag{5.15}$$

and the latter is pointwise fixed by γ, since Π_x and ρ are. (If γ fixes $\mathcal{X}_k$ pointwise, it also fixes each local affine space pointwise, so also their completions.) So $z^\gamma = z$, and γ is the identity. ■

For further purposes, let $\pi_x = \langle \alpha_x, y\rangle$ be the space at infinity of $\mathbb{A}_x$, and $\pi_y = \langle \beta_y, x\rangle$ be the space at infinity of $\mathbb{A}_y$.

The next couple of results carry over from roots to fundaments in a straightforward way.

Proposition 5.3.19. *Let $E := \mathrm{Aut}(\Pi_x)_{(\pi_y,x,xy,y)}$ be the elementwise stabilizer of $\{\pi_y, x, xy, y\}$ in $\mathrm{Aut}(\Pi_x)$. (Here, $\mathrm{Aut}(\Pi_x)$ is the combinatorial automorphism group of Π_x, isomorphic to $\mathbf{P\Gamma L}_{a+1}(k)$, and it is induced by $\mathbf{P\Gamma L}_{a+b}(k)$.) Then each element of E extends to an element of $\mathbf{P\Gamma L}_{a+b}(k)_{\mathcal{X}_k}$ (in a not necessarily unique fashion).*

Theorem 5.3.20. *Let Γ be the loose graph defined in the beginning of this section (figure 5.4). Then*

$$\mathrm{Aut}(\mathcal{F}(\Gamma) \times_{\mathbb{F}_1} k) \cong \mathrm{Aut}^{\mathrm{proj}}(\mathcal{F}(\Gamma) \times_{\mathbb{F}_1} k) \tag{5.16}$$

for any field k.

If S is a set of points in $\mathbf{PG}(a+b-1,k)$, $\mathbf{P\Gamma L}_{a+b}(k)_{[S]}$ denotes its pointwise stabilizer.

Lemma 5.3.21. *Define $F := \mathrm{Aut}^{\mathrm{proj}}(\mathcal{X}_k) \cap \mathbf{P\Gamma L}_{a+b}(k)_{[\pi_y]}$, and $G := \mathrm{Aut}^{\mathrm{proj}}(\mathcal{X}_k) \cap \mathbf{P\Gamma L}_{a+b}(k)_{[\pi_x]}$. Then $F \cong \mathrm{Aut}(\Pi_y)_{(\pi_x,x,xy,y)} \cap \mathbf{PGL}_{a+b}(k)$, and $G \cong \mathrm{Aut}(\Pi_x)_{(x,xy,y,\pi_y)} \cap \mathbf{PGL}_{a+b}(k)$.*

The following theorem is proved in exactly the same way as in the case of roots.

Theorem 5.3.22. *Let $\mathbf{PGL}(\mathcal{X}_k)_{(x,y)}$ be defined as*

$$\mathrm{Aut}^{\mathrm{proj}}(\mathcal{X}_k)_{(x,y)} \cap \mathbf{PGL}_{a+b}(k). \tag{5.17}$$

Then $\mathbf{PGL}(\mathcal{X}_k)_{(x,y)}$ is isomorphic to the internal central product of F and G.

The general version is the following.

Theorem 5.3.23 (Trees on two inner vertices)**.** *We have that*

$$\mathrm{Aut}^{\mathrm{proj}}(\mathcal{X}_k) \;\cong\; \mathrm{Aut}(\mathcal{X}_k) \;\cong\; ((F * G) \rtimes \mathrm{Aut}(k)) \rtimes \langle \varphi \rangle. \tag{5.18}$$

In the latter expression, φ is trivial, unless the type of the fundament has the form $(a, a; c, c)$, in which case φ is an involution in the automorphism group of Γ which switches x and y.

Proof. If the type is $(a, a; c, c)$, then obviously there is an involution as in the statement of the theorem. And any element in $\mathrm{Aut}(\Gamma)$ fixes both x and y if $a \neq b$ or $c \neq d$. ∎

5.3.5 General loose trees

Let $T = (V, E, \mathbf{I})$ be a finite loose tree, and assume its number of vertices is at least 3. Let $\overline{T}$ be the minimal graph of T and define the *boundary* of T, denoted $\partial(T)$, as the set of vertices of degree 1 in $\overline{T}$. Let x be a vertex of T which is at distance 1 from $\partial(T)$ (i.e., is adjacent with at least one vertex of $\partial(T)$). As $|V| \geq 3$, x is an inner vertex of degree at least 2.

Define k and $\mathcal{X}_k$ as before. Let $\mathbf{PG}(m-1, k)$ be the ambient projective space of $\mathcal{X}_k$.

Proposition 5.3.24. *The kernel of the action of $\mathbf{P\Gamma L}_m(k)_{\mathcal{X}_k}$ on $\mathcal{X}_k$ is trivial.*

Proof. Let $\gamma \in \mathbf{P\Gamma L}_m(k)_{\mathcal{X}_k}$ fix all the k-rational points of $\mathcal{X}_k$. If T is an affine $\mathbb{F}_1$-space (with some end points), then there is nothing to prove. So suppose T is not.

Define $\Pi_x := \overline{\mathbb{A}_x}$ as before, and let $y \sim x \neq y$ be not in $\partial(T)$ (such a point exists). Let Π_y be the projective completion of the image under $\mathcal{F}_k$ of the loose graph T_y induced on the vertex set $V_y := \{v \in V \mid \mathrm{d}(v, x) \geq 2\} \cup \{v \in V \mid \mathrm{d}(v, x) = 1,\ \deg(v) > 1\}$ (by "induced," we mean, besides inheriting the induced *loose graph* structure, that if e is a loose edge in T which is incident with a vertex of V_y, then e is in T_y).

Now repeat the argument of Proposition 5.3.18, using induction on the loose tree T_y. ∎

In the next couple of results, we keep using the notation introduced in the beginning of this subsection. Also, with I the set of inner vertices of $\overline{T}$, and $w \in I$, let $S(w)$ be the subgroup of $\mathrm{Aut}^{\mathrm{proj}}(\mathcal{X}_k)$ which fixes the k-rational points of $\mathcal{X}_k$ inside all affine subspaces $\widetilde{\mathbb{A}_v}$ (over k) which are generated (over $\mathbb{F}_1$) by a vertex v different from w and all directions on v which are not incident with w. In figure 5.5, we can see in a more clear way which points are fixed by $S(w)$. In fact, if the distance of v to w is at least 2, the local space at v is fixed pointwise, and if the distance is 1, $\widetilde{\mathbb{A}_v}$ is an affine space of dimension one less than the dimension of $\mathbb{A}_v$. (In particular, the points in $I \cap \mathbf{B}(w, 1)$ are fixed.)

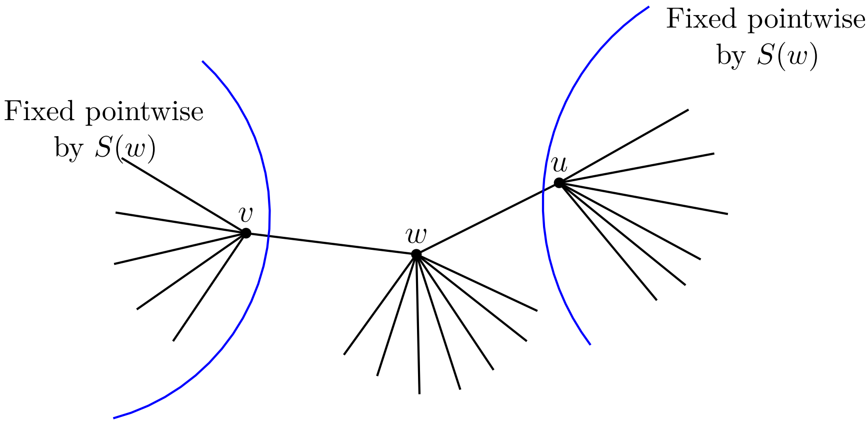


Figure 5.5: Part of $\mathcal{X}_k$ fixed pointwise by the subgroup $S(w)$.

In the next theorem, one recalls that $\mathcal{X}_k$ comes with an embedding

$$T \overset{\iota}{\hookrightarrow} \mathcal{X}_k \hookrightarrow \mathbf{PG}(m-1,k), \tag{5.19}$$

so that it makes sense to consider stabilizers of substructures of T in, e.g., $\mathbf{PGL}(\mathcal{X}_k)$. By the first embedding of the above formula we mean that $\iota(T)$ is isomorphic to T as a point-line incidence structure inside $\mathcal{X}_k$.

Theorem 5.3.25. *Let* $\mathbf{PGL}(\mathcal{X}_k)_{[I]}$ *be defined as*

$$\mathrm{Aut}^{\mathrm{proj}}(\mathcal{X}_k)_{[I]} \ \cap \ \mathbf{PGL}_m(k). \tag{5.20}$$

Then $\mathbf{PGL}(\mathcal{X}_k)_{[I]}$ *is isomorphic to*

$$\prod_{w \in I}^{\mathrm{centr}} S(w). \tag{5.21}$$

Proof. Let $x \in I$ be at distance 1 from $\partial(T)$. Also, let $y \sim x \neq y$, $y \notin \partial(T)$ and $y \in I$. Let T_y be the loose graph induced on the vertex set $V_y := \{v \in V \mid \mathrm{d}(v,x) \geq 2\} \cup \{v \in V \mid \mathrm{d}(v,x) = 1,\ \deg(v) > 1\}$. Let $H(y)$ be the subgroup of $\mathrm{Aut}^{\mathrm{proj}}(\mathcal{X}_k)$ which fixes pointwise the affine subspace of $\mathbf{PG}(m-1,k)$ that is generated by all edges on x in T except xy, and which fixes each element of I. It is important to observe that $S(y) \leq H(y)$ for the induction argument later on. Then in the same way as in the proof of Theorem 5.3.13, one shows that

$$\mathbf{PGL}(\mathcal{X}_k)_{[V]} = S(x) * H(y). \tag{5.22}$$

Now perform induction on T_y to conclude that

$$\mathbf{PGL}(\mathcal{X}_k)_{[I]} = S(x) * \Big(S(y) * \Big(\cdots\Big)\Big). \tag{5.23}$$

Note that $\mathbf{PGL}(\mathcal{F}_k(T_y))_{[V_y]} = H(y)$, and that loose trees need not be connected anymore. Also, for each $u, v \in I$, it follows that

$$[S(u), S(v)] \;=\; \{\mathrm{id}\}. \tag{5.24}$$

■

Determination of $S(w)$

We start by remarking that although in general $S(w)$ fixes a lot of points, it is not necessarily a subgroup of $\mathbf{PGL}_m(k)$ (see for instance Lemma 5.3.27 below). What we *do* know — by its mere definition — is that it is a subgroup of $\mathbf{P\Gamma L}_m(k)$.

We will distinguish two cases in order to determine $S(w)$.

† w is the only inner point

Then all the edges are incident with w. Call E' the set of such edges with an end point, and L the set of loose edges. Put $|E'| = e'$ and $|L| = \ell$. Then obviously

$$S(w) \cong \Big(\mathbf{P\Gamma L}_{e'+\ell+1}(k)_{L,E'}\Big)_{[\{w\}]}. \tag{5.25}$$

By the first remark of this subsection, it is not contained in the projective linear subgroup.

‡ w is not the only inner point

Then there is some inner vertex $v \sim w$ different from w which is itself incident to some edge $W \neq wv$. Now over k, the projective line which is the completion of the affine line determined by the incident vertex-edge pair (v, W), is fixed pointwise by $S(w)$, so $S(w)$ must be a subgroup of $\mathbf{PGL}_m(k)$.

Let E' be the set of edges incident with w which have an end point, let L be the set of loose edges incident with w, and let I be the set of edges incident with w which are incident with another inner point. Put $|E'| = e'$, $|L| = \ell$ and $|I| = i$. Let δ be an element of $S(w)$; then it induces an element of $\mathbf{PGL}(\overline{\mathbb{A}_w})$ (the latter meaning the projective linear group of the local projective space at w). If δ' is another such element which induces the same action, it is obvious that $\delta\delta'^{-1}$ is the identity on the entire ambient space $\mathbf{PG}(m-1, k)$. So $S(w)$ faithfully is a subgroup of $\Big(\mathbf{PGL}_{e'+\ell+i+1}(k)_{L,E'}\Big)_{[I\cup\{w\}]}$.

Note that the projective space generated (over k) by the points at distance at least 2 from w in $\overline{\Gamma}$, is fixed pointwise by $S(w)$. So in particular π_I, the projective space generated by the inner vertices adjacent to w, is also fixed pointwise. It now follows easily that

$$S(w) \cong \Big(\mathbf{PGL}_{e'+\ell+i+1}(k)_{L,E'}\Big)_{[\pi_I\cup\{w\}]}. \tag{5.26}$$

Caution: central and direct products

On the graph theoretical level (that is, on the combinatorial $\mathbb{F}_1$-level), the groups which occur in Theorem 5.3.25 are much easier to describe, replacing the central product by a direct product. The central product is needed as soon as $k^\times$ is not trivial.

Inner Tree Theorem

The following theorem is a crucial ingredient in the proof of our main theorem for trees.

Theorem 5.3.26 (Inner Tree Theorem). *Let T be a loose tree, and let k be any field. As usual, put $\mathcal{X}_k = \mathcal{F}(T) \otimes_{\mathbb{F}_1} k$, and consider the embedding*

$$\iota : T \hookrightarrow \mathcal{X}_k. \tag{5.27}$$

as in formula 5.19. Let $\mathrm{Aut}(\mathcal{X}_k)$ *be any of the automorphism groups which are considered in this chapter (i.e., combinatorial, induced by projective space or topological). Let I be the set of inner vertices of T, and let $T(I)$ be the subtree of T induced on I. Then if $|I| \geq 2$, we have that* $\mathrm{Aut}(\mathcal{X}_k)$ *stabilizes $\iota(T(I))$. Moreover,* $\mathrm{Aut}(\iota(T(I)))$ *is induced by* $\mathrm{Aut}(\mathcal{X}_k)$.

Proof. Each edge of $\iota(T(I))$ defines a projective line over k which is a full line of the ambient space of $\mathcal{X}_k$. Let $\iota(T(I))_k$ be this set of projective lines. Now define $\underline{\mathcal{X}_k}$ as the *projective part* of $\mathcal{X}_k$ — by definition, it is the union of all projective k-lines which are completely contained in $\mathcal{X}_k$. As each local affine space at a vertex of T is an affine space with some possible end points at infinity, one observes that $\underline{\mathcal{X}_k}$ consists precisely of the projective k-lines which are defined by the edges with two different vertices of T. That is, $\underline{\mathcal{X}_k}$ consists of $\iota(T(I))_k$ together with additional projective lines defined by edges which contain both an inner vertex and an end point of T. As $|I| \geq 2$, the first part of the theorem easily follows.

That $\mathrm{Aut}(\iota(T(I)))$ is induced follows by functoriality (and the discussion in subsection 5.3.6). ■

Note that if $|I| = 1$, T defines an affine space with some end points, so the theorem is not true, unless its dimension is 0. If $|I| = 0$, then either T is the empty tree, or T is an edge with one or two vertices.

The general group

Before proceeding, we need another lemma. We use the notation of the previous subsection.

Lemma 5.3.27 (Field automorphisms). *Let* $\mathbf{PG}(m-1,k)$ *be the ambient space of $\mathcal{X}_k$. We have that*

$$\mathbf{P\Gamma L}_m(k)_{\mathcal{X}_k} \Big/ \mathbf{PGL}_m(k)_{\mathcal{X}_k} \cong \mathrm{Aut}(k). \tag{5.28}$$

Proof. Let Δ be the base of $\mathbf{PG}(m-1,k)$ corresponding to the vertices of $\overline{T}$. Then it is well known that

$$\mathbf{P\Gamma L}_m(k)_{[\Delta]}\Big/\mathbf{PGL}_m(k)_{[\Delta]} \cong \mathrm{Aut}(k). \tag{5.29}$$

(In fact, working with homogeneous coordinates with respect to Δ, $\mathbf{P\Gamma L}_m(k)_{[\Delta]}$ contains all elements of the form $\mathbf{x} \mapsto \mathrm{id}_m\mathbf{x}^\tau$, with $\mathbf{x}$ a column vector representing points in homogeneous coordinates, id_m the identity $(m \times m)$-matrix and $\tau \in \mathrm{Aut}(k)$.) As $\mathbf{P\Gamma L}_m(k)_{[\Delta]} \leq \mathbf{P\Gamma L}_m(k)_{\mathcal{X}_k}$ and $\mathbf{PGL}_m(k)_{[\Delta]} \leq \mathbf{PGL}_m(k)_{\mathcal{X}_k}$, the lemma easily follows. ■

Theorem 5.3.28 (Projective automorphism group)**.** *Let T be a loose tree, and let k be any field. Put $\mathcal{X}_k = \mathcal{F}(T) \otimes_{\mathbb{F}_1} k$, and consider the embedding*

$$\iota : T \hookrightarrow \mathcal{X}_k. \tag{5.30}$$

Let I be the set of inner vertices of T, and let $T(I)$ be the subtree of T induced on I. We have $\mathbf{P\Gamma L}(\mathcal{X}_k) = \mathrm{Aut}^{\mathrm{proj}}(\mathcal{X}_k)$ is isomorphic to

$$\Big(\Big(\prod_{w\in I}^{\mathrm{centr}} S(w)\Big) \rtimes \mathrm{Aut}(T(I))\Big) \rtimes \mathrm{Aut}(k). \tag{5.31}$$

Proof. First note that by Proposition 5.3.24, the kernel of the action of $\mathbf{P\Gamma L}_m(k)_{\mathcal{X}_k}$ on $\mathcal{X}_k$ is trivial. Then by Lemma 5.3.27, we only have to show that

$$\mathbf{PGL}_m(k)_{\mathcal{X}_k} \ \cong \ \Big(\prod_{w\in I}^{\mathrm{centr}} S(w)\Big) \rtimes \mathrm{Aut}(T(I)). \tag{5.32}$$

By Theorem 5.3.25, we have that $\mathbf{PGL}(\mathcal{X}_k)_{[I]}$ is isomorphic to

$$\prod_{w\in I}^{\mathrm{centr}} S(w), \tag{5.33}$$

and obviously $\mathbf{PGL}(\mathcal{X}_k)_{[I]} \trianglelefteq \mathbf{PGL}_m(k)_{\mathcal{X}_k}$.

The theorem now follows from the Inner Tree Theorem. ■

5.3.6 More on the different automorphism group types

By Theorem 5.3.28, we can now determine the combinatorial group as well.

Theorem 5.3.29 (Combinatorial automorphism group)**.** *Let T be a loose tree, and let k be any field. Put $\mathcal{X}_k = \mathcal{F}(T) \otimes_{\mathbb{F}_1} k$, let I be the set of inner vertices, and suppose that $|I| \geq 2$. Let ι be as in Theorem 5.3.28. Then*

$$\mathrm{Aut}^{\mathrm{comb}}(\mathcal{X}_k) \cong \mathrm{Aut}^{\mathrm{proj}}(\mathcal{X}_k). \tag{5.34}$$

Proof. As in the proof of Theorem 5.3.11, we assume by way of contradiction that there is some $\alpha \in \mathrm{Aut}^{\mathrm{comb}}(\mathcal{X}_k) \setminus \mathrm{Aut}^{\mathrm{proj}}(\mathcal{X}_k)$. As in that theorem, by the fact that $\mathrm{Aut}^{\mathrm{proj}}(\mathcal{X}_k)$ induces $\mathrm{Aut}(\iota(T(I)))$ by the Inner Tree Theorem, we may assume that α fixes all vertices of $\iota(T(I))$. Now α induces projective automorphisms in each $\overline{\mathbb{A}_x}$ with x an inner vertex, which are compatible on edges of $\iota(T(I))$. By Theorem 5.3.28, we can end in the same way as in the proof of Theorem 5.3.11. ■

We have shown in Proposition 5.2.9 that for each $\mathcal{X}_k$, the combinatorial automorphism group is a subgroup of the topological automorphism group. Also it is clear that any projectively induced automorphism is combinatorial, but the other direction is in general *not* true. Let Γ be, e.g., an edge with two different vertices, so that for all k, $\mathcal{X}_k$ is a projective k-line. Then each permutation of the k-points yields a combinatorial automorphism, but not all of these come from projective automorphisms for all k. So

$$\begin{cases} \mathrm{Aut}^{\mathrm{top}}(\mathcal{X}_k) \geq \mathrm{Aut}^{\mathrm{comb}}(\mathcal{X}_k) \\ \mathrm{Aut}^{\mathrm{comb}}(\mathcal{X}_k), \mathrm{Aut}^{\mathrm{top}}(\mathcal{X}_k) \geq \mathrm{Aut}^{\mathrm{proj}}(\mathcal{X}_k). \end{cases} \tag{5.35}$$

5.4 Convexity

Let T be a loose tree, and for any field k, consider $\mathcal{X}_k := \mathcal{F}(T) \otimes_{\mathbb{F}_1} k$. In this section we will prove a useful convexity property for the spaces $\mathcal{X}_k$.

The following lemma is trivial, but also useful.

Lemma 5.4.1. *Let G be any subgraph of $\overline{T}$, not necessarily connected. Then the dimension of the projective space generated over $\mathbb{F}_1$ by G equals the number of vertices of G minus* 1.

Theorem 5.4.2 (Convexity)**.** *Let k and $\mathcal{X}_k$ be as in the beginning of this section. Let $\mathbb{A}_u$ and $\mathbb{A}_v$ be local affine spaces over k with $u, v \neq u$ vertices of T. If $x \in \mathbb{A}_u$, but not contained in any of the lines determined by the local loose star of u, and $y \in \mathbb{A}_v$ is not contained in any of the lines determined by the local loose star of v, then the projective k-line xy only meets $\mathcal{X}_k$ in x and y.*

Proof. Suppose by way of contradiction that $z \in (\mathcal{X}_k \cap xy) \setminus \{x, y\}$. Obviously $z \notin \mathbb{A}_u \cup \mathbb{A}_v$, so z is in some other local affine k-space $\mathbb{A}_w$, with w a vertex of T. There are (essentially) five possible configurations to be considered:

(1) $u \sim v$ and $u \not\sim w \not\sim v$;

(2) $u \sim v$ and $u \not\sim w \sim v$;

(3) $u \not\sim v$ and $u \not\sim w \not\sim v$;

(4) $u \not\sim v$ and $u \not\sim w \sim v$;

(5) $u \not\sim v$ and $u \sim w \sim v$.

Note that u, v, w cannot form a triangle. In each of the cases, consider the projective space generated by $\mathbb{A}_u, \mathbb{A}_v$ and $\mathbb{A}_w$, calculate its dimension, and apply Lemma 5.4.1 to find a contradiction. ■

5.5 The edge-relation dichotomy

The fact that the calculations for loose trees T are so successful rests largely on the fact that there are no cycles; that property leads to the fact that we can apply the Inner Tree Theorem, and this makes it possible to determine the various automorphism groups of $\mathcal{F}(T) \otimes k$, k any field.

The examples which are the farthest from satisfying the Inner Tree Theorem are affine and projective spaces. In case of affine spaces $\mathbb{A}_k^n$, the automorphism group (assumed combinatorial) acts transitively on the k-points, so obviously the Inner Tree Theorem, formulated for loose graphs (see subsection 5.6.1), cannot hold. In fact, we have the following observation the trivial proof of which we leave to the reader.

Theorem 5.5.1. *Let Γ be the loose graph of an affine or projective $\mathbb{F}_1$-space. Then for any field k and any of the considered automorphism groups* $\mathrm{Aut}(\cdot)$*, we have that* $\mathrm{Aut}(\mathcal{F}(\Gamma) \otimes k)$ *acts transitively on the set of subgeometries isomorphic to* Γ*. (Here, as before a subgeometry consists of k-points and affine or projective k-lines.)*

5.5.1 Examples close to trees

Consider the following loose graph Γ_1 (see figure 5.6), which, for each field k, defines in the ambient projective 3-space $\mathbf{PG}(3, k)$, four affine planes each with two extra points at infinity and cyclically denoted by α_i $(i = 1, 2, 3, 4)$, in which "adjacent planes" meet in a projective line, and "opposite planes" meet precisely in the end points. Denote the constructible set by $\mathcal{X}_k$.

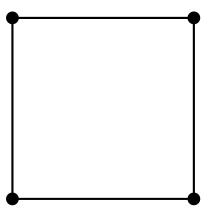


Figure 5.6: The loose graph Γ_1

Obviously we have

$$\mathrm{Aut}^{\mathrm{proj}}(\mathcal{X}_k) \;\cong\; \mathbf{P\Gamma L}_4(k)_{\Gamma_1}, \tag{5.36}$$

where Γ_1 comes with the embedding

$$\iota : \Gamma_1 \hookrightarrow \mathcal{X}_k. \tag{5.37}$$

The complement Γ_1^c of Γ_1 in its ambient projective $\mathbb{F}_1$-space is also fixed by $\mathrm{Aut}^{\mathrm{proj}}(\mathcal{X}_k)$, as that complement just defines two disjoint multiplicative groups. Notice however that

$$\Big(\mathcal{F}(\Gamma_1) \otimes_{\mathbb{F}_1} k\Big) \coprod \Big(\mathcal{F}(\Gamma_1^c) \otimes_{\mathbb{F}_1} k\Big) \neq \mathbf{PG}(3,k)! \tag{5.38}$$

The example Γ_1 easily generalizes to the class of polygonal graphs $\Gamma(m)$ with $m+1$ vertices, $m \geq 0$, $m \neq 1,2$; for $m = 0,1$ we get spaces $\mathtt{Proj}(k[X])$ and $\mathtt{Proj}(k[X,Y])$ which satisfy the Inner Tree Property; for $m = 2$ we get a projective k-plane, and for $m \geq 3$, we obtain a constructible set consisting of $m+1$ affine k-planes each with two extra points at infinity, which intersect two by two according to their graph intersection (in a projective k-line, a point or no intersection). All of them except $\Gamma(1)$ and $\Gamma(2)$ have the property that

$$\mathrm{Aut}^{\mathrm{proj}}(\mathcal{X}_k) \;\cong\; \mathbf{P\Gamma L}_{n+1}(k)_{\Gamma(m)}. \tag{5.39}$$

The graph complements are also fixed by $\mathrm{Aut}^{\mathrm{proj}}(\mathcal{X}_k)$.

5.5.2 Missing piece

Let Γ be a loose graph, k any field, $\mathbb{P}_k := \mathbf{PG}(m-1,k)$ the ambient space over k, and Γ^c the complement in $\mathbb{P}_{\mathbb{F}_1}$ of Γ. We have a decomposition

$$\Big(\mathcal{F}(\Gamma) \otimes_{\mathbb{F}_1} k\Big) \coprod \Big(\mathcal{F}(\Gamma^c) \otimes_{\mathbb{F}_1} k\Big) \coprod \mathcal{Y}_k(\Gamma) \;=\; \mathbf{PG}(m-1,k), \tag{5.40}$$

for some constructible set $\mathcal{Y}_k(\Gamma)$. The constructible set $\mathcal{Y}_k$ measures a difference in behavior of $\mathcal{F}(\Gamma) \otimes_{\mathbb{F}_1} k$ with respect to fields k and $k = \mathbb{F}_1$, since, for instance, for $k = \mathbb{F}_1$ we have that $\mathcal{F}(\Gamma) \coprod \mathcal{F}(\Gamma^c)$ partitions the Deitmar line set of $\mathbf{PG}(m-1,\mathbb{F}_1)$. (Note however that one has to be careful with decompositions in terms of loose graphs: e.g., an affine $\mathbb{F}_1$-plane minus a multiplicative group $\mathbb{G}_m$ is *not* an affine line! — one might want to think in terms of the Grothendieck ring of $\mathbb{F}_1$-schemes $K_0(\mathtt{Sch}_{\mathbb{F}_1})$ to see this more clearly.)

It might be interesting to study the maps

$$\mathcal{Y}_k : \Gamma \longmapsto \mathcal{Y}_k(\Gamma). \tag{5.41}$$

5.5.3 Examples close to the ambient space

Now consider the following example Γ_2 (see figure 5.7), which, for each field k, defines a projective 3-space $\mathbf{PG}(3, k)$ without one multiplicative group $\mathbb{G}_m$ (corresponding to the missing diagonal edge). (Denote the constructible set by $\mathcal{X}_k$.)

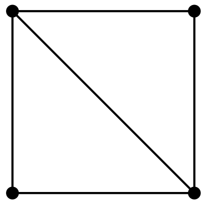


Figure 5.7: The loose graph Γ_2

Let x and y be the two k-points of $\mathbf{PG}(3, k)$ in the projective line defined by $\mathbb{G}_m$ which are not contained in $\mathbb{G}_m$. Then obviously

$$\mathrm{Aut}^{\mathrm{proj}}(\mathcal{X}_k) \ \cong\ \mathbf{P\Gamma L}_4(k)_{(x,y)}, \tag{5.42}$$

so $\mathrm{Aut}^{\mathrm{proj}}\mathcal{X}_k$ does not fix the graph defined by

$$\iota : \Gamma_2 \hookrightarrow \mathcal{X}_k. \tag{5.43}$$

What it *does* fix, is the complement of Γ_2 in the projective $\mathbb{F}_1$-space defined by Γ_1 (considered in the same embedding).

5.6 Future steps

5.6.1 Constructible sets satisfying the Inner Graph Property

One essential ingredient in the proof of our main theorem for trees, is the *inner tree property*, which we define as follows for general loose graphs.

> Let Γ be a loose graph, and let k be any field. Put $\mathcal{X}_k = \mathcal{F}(\Gamma) \otimes_{\mathbb{F}_1} k$, and consider the embedding
>
> $$\iota : \Gamma \hookrightarrow \mathcal{X}_k. \tag{5.44}$$
>
> Let $\mathrm{Aut}(\mathcal{X}_k)$ be one of the automorphism groups considered in this chapter — combinatorial, induced by projective space or topological. Let I be the set of inner vertices of T, and let $\Gamma(I)$ be the subgraph of Γ induced on I. Suppose $|I| \geq 2$. Then we say that Γ satisfies the *inner graph property* if $\mathrm{Aut}(\mathcal{X}_k)$ stabilizes $\iota(\Gamma(I))$.

Question 5.6.1. *Characterize (the) loose graphs that do/do not have the inner graph property.*

Let `InnGraph` be the category of loose graphs which have the inner graph property. Following the same lines of the proof of Theorem 5.3.28, one can determine the map

$$\mathrm{Aut} : \mathtt{InnGraph} \longrightarrow \mathtt{Group} : \ \Gamma \longmapsto \mathrm{Aut}(\Gamma). \tag{5.45}$$

5.6.2 Heisenberg principle

Let `LGraph` be the category of loose graphs, `LTree` the category of loose trees, and `CGraph` the category of complete graphs. We end the chapter with the following questions.

Question 5.6.2. *Does there exist a distance function*

$$\delta : \mathtt{LGraph} \times \mathtt{LGraph} \longrightarrow (S, \leq), \tag{5.46}$$

with $(S, \leq)$ a (totally) ordered set, such that the following properties hold?

- *The distance between a loose tree and its completion in* `CGraph` *is maximal.*
- *If* $\min\{\delta(\Gamma, T) \mid T \in \mathtt{LTree}, T \leq \Gamma\} \ll$, *then* Γ *satisfies the inner graph property.*
- *If* $\delta(\Gamma, \overline{\Gamma}) \ll$, *with* $\overline{\Gamma}$ *the completion of* Γ *in* `CGraph`, *then* Γ *does not satisfy the inner graph property.*

We strongly suspect that δ should be expressed in terms of cycles.

Question 5.6.3. *Let δ be as in the previous question. Let Γ be in* `LGraph`, *and suppose that*

$$\min\{\delta(\Gamma, T) \mid T \in \mathtt{LTree}, T \leq \Gamma\} \cdot \delta(\Gamma, \overline{\Gamma}) \tag{5.47}$$

is "quadratic," when can one decide that Γ satisfies the inner graph property?

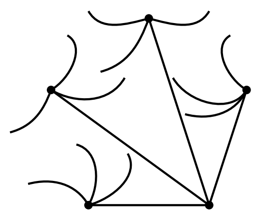
Appendices

A Computation of the Grothendieck Polynomial of K_5

In this appendix we will give a detailed computation of the Grothendieck polynomial for the complete graph on 5 vertices, K_5. As we already know, the associated $\mathbb{F}_1$-constructible set for K_5 is the projective space $\mathbb{P}^4_{\mathbb{F}_1}$, and so its Grothendieck polynomial must be

$$[K_5] = \mathbb{L}^4 + \mathbb{L}^3 + \mathbb{L}^2 + \mathbb{L} + 1. \tag{A.1}$$

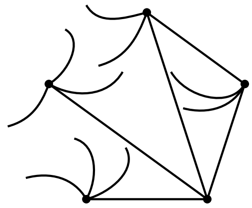
First of all, we will choose one loose spanning tree for the graph K_5, i.e, a loose tree obtained after resolving all fundamental edges of a spanning tree of K_5. Let us remark that we can choose any loose spanning tree since its Grothendieck polynomial is an invariant for a given graph. In our case, let us call Γ the following loose spanning tree of K_5:



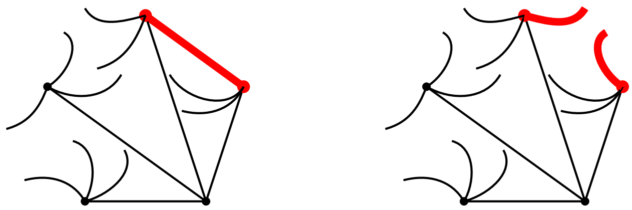
After definition 3.3.1, the Grothendieck polynomial of Γ can be easily computed:

$$[\Gamma] = 5\mathbb{L}^4 - 4\mathbb{L} + 4. \tag{A.2}$$

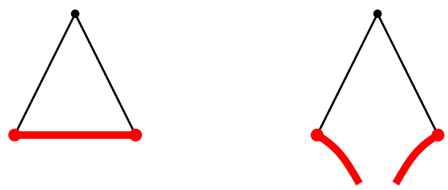
The process to compute the Grothendieck polynomial of the original graph K_5 consists of "unresolving" in each step one of the fundamental edges of the graph and, using the Affection Principle, keeping track of the list of resulting alternations to the Grothendieck polynomial. By the term "unresolving an edge of a graph Γ," one means choosing one graph Γ_1 in a way that Γ is obtained from Γ_1 after resolving a fundamental edge. In our case, we take Γ_1 to be



Let us compare both loose graphs to see more clearly which are the vertices that must be taken into account according to the Affection Principle.



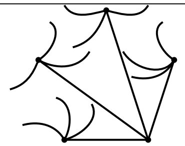
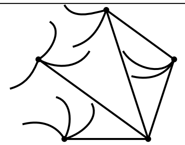
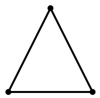
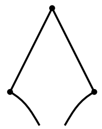
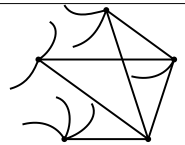
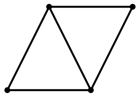
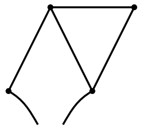
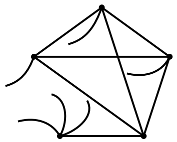
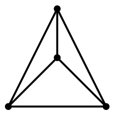
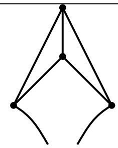
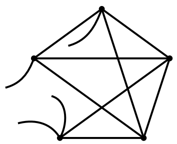
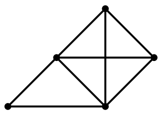
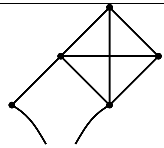
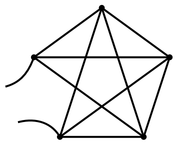
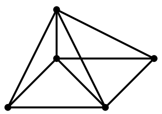
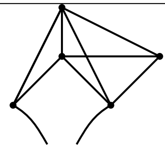
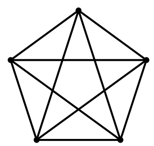
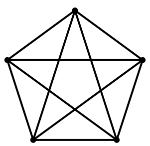
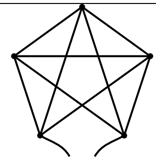
The edge that has been resolved to go from Γ_1 to Γ is the red one. Using the Affection Principle to compute the polynomial of Γ_1, we only need to know the difference between the polynomials of the following two loose graphs:



The polynomials of these graphs are easy to compute since the graph on the left-hand side is the one corresponding with $\mathbb{P}^2_{\mathbb{F}_1}$ and the graph on the right is a tree. So the Grothendieck polynomials are $\mathbb{L}^2+\mathbb{L}+1$ and $3\mathbb{L}^2-2\mathbb{L}+2$, respectively. Calling Δ_1 the difference between these two polynomials, we obtain the Grothendieck polynomial for the graph Γ_1 as follows

$$\begin{cases} \Delta_1 = 2\mathbb{L}^2 - 3\mathbb{L} + 1, \\ [\Gamma_1] = [\Gamma] - \Delta_1 = 5\mathbb{L}^4 - 4\mathbb{L} + 4 - (2\mathbb{L}^2 - 3\mathbb{L} + 1) = 5\mathbb{L}^4 - 2\mathbb{L}^2 - \mathbb{L} + 3. \end{cases} \tag{A.3}$$

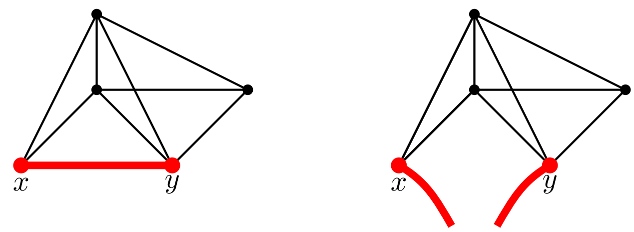
It is important to remark that, even though in this first case it was easy to compute Δ_1 due to the well-known Grothendieck polynomials of the two loose graphs from above, the general formulas for Grothendieck polynomials given in subsection 3.5.6 would (of course) give the same expressions. In the following table, the reader can find all the necessary differences and steps which must be computed in order to get the Grothendieck polynomial for the scheme associated to K_5.

	GRAPH	BEFORE RESOLVING	AFTER RESOLVING	Δ_i	GROTH. POLYNOMIAL
Γ					$[\Gamma] = 5\mathbb{L}^4 - 4\mathbb{L} + 4$
Γ_1				$\Delta_1 = 2\mathbb{L}^2 - 3\mathbb{L} + 1$	$[\Gamma_1] = 5\mathbb{L}^4 - 2\mathbb{L}^2 - \mathbb{L} + 3$
Γ_2				$\Delta_2 = \mathbb{L}^3 - \mathbb{L}^2$	$[\Gamma_2] = 5\mathbb{L}^4 - \mathbb{L}^3 - \mathbb{L}^2 - \mathbb{L} + 3$
Γ_3				$\Delta_3 = 2\mathbb{L}^3 - 2\mathbb{L}^2 - \mathbb{L} + 1$	$[\Gamma_3] = 5\mathbb{L}^4 - 3\mathbb{L}^3 + \mathbb{L}^2 + 2$
Γ_4				$\Delta_4 = \mathbb{L}^4 - 2\mathbb{L}^3 + 2\mathbb{L}^2 - \mathbb{L}$	$[\Gamma_4] = 4\mathbb{L}^4 - \mathbb{L}^3 - \mathbb{L}^2 + \mathbb{L} + 2$
Γ_5				$\Delta_5 = \mathbb{L}^4 - 2\mathbb{L}^2 + \mathbb{L}$	$[\Gamma_5] = 3\mathbb{L}^4 - \mathbb{L}^3 + \mathbb{L}^2 + 2$
K_5				$\Delta_6 = 2\mathbb{L}^4 - 2\mathbb{L}^3 - \mathbb{L} + 1$	$[K_5] = \mathbb{L}^4 + \mathbb{L}^3 + \mathbb{L}^2 + \mathbb{L} + 1$

As already mentioned before, when we apply the formulas for the loose graphs before and after resolving, we have to take into account that the loose graphs we use are *embedded* in the original graph for which we are trying to obtain the Grothendieck polynomial. We will make a detailed study of step 5 (difference between Γ_4 and Γ_5) to clarify this remark.

Computation of $[\Gamma_5]$ starting from the Grothendieck polynomial of Γ_4

The polynomial associated to Γ_4 is $4\mathbb{L}^4 - \mathbb{L}^3 - \mathbb{L}^2 + \mathbb{L} + 2$. Then according to the Affection Principle, in order to obtain $[\Gamma_5]$, we only have to compute the difference between the polynomials for the following two graphs:



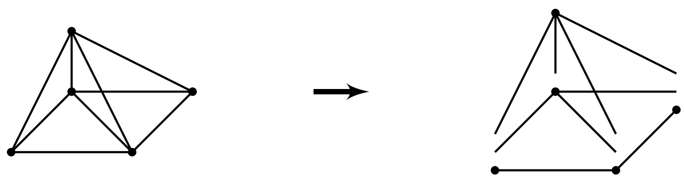
Before resolution

To compute the Grothendieck polynomial of the loose graph before resolution we could use the formula given in subsection 3.5.6 for this, but, we will use the cone construction instead because it allows us to obtain the polynomial without much computation. Thanks to the decomposition in the Grothendieck ring of varieties and using Corollary 3.5.6 for a cone $C(G_1, G_2)$, where G_1 is a vertex v of the loose graph having *maximal* degree — the degree of v is equal to the number of vertices minus one — and G_2 is the loose subgraph induced by all the vertices except v, we obtain the following decomposition:

$$[\Gamma] = \mathbb{P}(\mathbb{A}_v) + [\Gamma \setminus v], \tag{A.4}$$

where $\Gamma \setminus v$ means the "subgraph of Γ resulting after deleting v and its incident edges."

In our case, applying two times the cone construction, we can decompose the loose graph in three parts



and translated in terms of Grothendieck polynomials, one obtains

$$[\Gamma] = \mathbb{L}^4 + \mathbb{L}^3 + \mathbb{L}^2 + 2. \tag{A.5}$$

After resolution

According to subsection 3.5.6, the formula for the Grothendieck polynomial in this case is

$$\begin{aligned} \mathbb{P}(\Gamma) = \mathbb{P}(\mathbb{A}_x) + \mathbb{P}(\mathbb{A}_y) + \mathbb{P}(\Delta) + (\mathbb{L}^2 - 1)\mathbb{P}(G^L) \\ -(\mathbb{L} - 1)\mathbb{P}(G^L_x) - (\mathbb{L} - 1)\mathbb{P}(G^L_y). \end{aligned} \tag{A.6}$$

Therefore, we only have to find what the different parts of the formula mean in terms of loose graphs and number of rational points. Let us start with $\mathbb{A}_x$ and $\mathbb{A}_y$, which are isomorphic to $\mathbb{A}^3_{\mathbb{F}_1}$ and $\mathbb{A}^4_{\mathbb{F}_1}$, respectively. Also, observe that Δ is the subgraph generated by all the vertices except x and y, i.e., a complete subgraph on 3 vertices. That is, Δ defines a projective plane over $\mathbb{F}_1$.

Now the CV condition in the definition of the functor $\mathcal{F}$ (see definition 2.3.1) comes into play and, as G^L is embedded in Δ, we deduce that it defines a projective plane minus one point; the same holds for G^L_y. Besides, G^L_x has a projective line as associated Deitmar scheme since the only neighbors of x are the common ones. Once we know all different and necessary parts of the formula (A.6), we obtain the Grothendieck polynomial of Γ as follows:

$$\begin{aligned} \mathbb{P}(\mathbb{A}_x) &= \mathbb{L}^3, & \mathbb{P}(\mathbb{A}_y) &= \mathbb{L}^4, \\ \mathbb{P}(\Delta) &= \mathbb{L}^2 + \mathbb{L} + 1, & \mathbb{P}(G^L) &= \mathbb{L}^2 + \mathbb{L}, \\ \mathbb{P}(G^L_x) &= \mathbb{L} + 1, & \mathbb{P}(G^L_y) &= \mathbb{L}^2 + \mathbb{L}; \end{aligned}$$

$$\begin{aligned} \mathbb{P}(\Gamma) &= \mathbb{L}^3 + \mathbb{L}^4 + (\mathbb{L}^2 + \mathbb{L} + 1) + (\mathbb{L}^2 - 1)(\mathbb{L}^2 + \mathbb{L}) \\ &\quad -(\mathbb{L} - 1)(\mathbb{L}^2 + \mathbb{L}) - (\mathbb{L} - 1)(\mathbb{L} + 1) \\ &= 2\mathbb{L}^4 + \mathbb{L}^3 - \mathbb{L}^2 + \mathbb{L} + 2. \end{aligned} \tag{A.7}$$

Hence, using the difference between the formulas obtained before and after resolution, we get that

$$\begin{cases} \Delta_5 &= (2\mathbb{L}^4 + \mathbb{L}^3 - \mathbb{L}^2 + \mathbb{L} + 2) - (\mathbb{L}^4 + \mathbb{L}^3 + \mathbb{L}^2 + 2) \\ &= \mathbb{L}^4 - 2\mathbb{L}^2 + \mathbb{L}, \\ & \\ \mathbb{P}(\Gamma_5) &= \mathbb{P}(\Gamma_4) - \Delta_5 \\ &= (4\mathbb{L}^4 - \mathbb{L}^3 - \mathbb{L}^2 + \mathbb{L} + 2) - (\mathbb{L}^4 - 2\mathbb{L}^2 + \mathbb{L}) \\ &= 3\mathbb{L}^4 - \mathbb{L}^3 + \mathbb{L}^2 + 2. \end{cases}$$

B Computations

In this last appendix we will briefly explain the code in Magma that we use to obtain the Grothendieck polynomial for any loose graph. We first have to remark that the main difficulty we found in writing this code is that computations are only allowed in the category of graphs. Therefore, all computations of the Grothendieck polynomial of a loose graph Γ will be done in what we called the *minimal graph* of Γ (and denoted by $\overline{\Gamma}$), i.e. the minimal graph in which Γ is embedded. However, computations do not get much more complicated since Γ and $\overline{\Gamma}$ differ just by a finite number of vertices and, thanks to the FUCP property (cf. example 2.3.2), this only implies a difference by a constant in the level of Grothendieck polynomials. That is why, for some programs, we also need to keep track of the differences between the number of vertices.

Let us recall that to calculate the Grothendieck polynomial of a loose graph, we use the procedure called "surgery" consisting of unresolving one fundamental edge in each step and, keeping track of the differences in the graphs, ending up in a loose tree where the Grothendieck polynomial is well defined and known. Taking all this into account, we first create a program called *PolTree* that given a tree computes its Grothendieck polynomial using Definition 3.3.1. However, since we need to calculate the zeta function for any loose graph and *PolTree* is only defined for trees, we use a program called *LooseSPTree* to construct a list with all different loose graphs of the surgery procedure having as its last term a loose spanning tree of the original graph.

The program *LooseSPTree* receives a graph Γ as data and chooses a spanning tree of Γ with the command *SpanningTree* (already implemented in Magma). Then, we compare both Γ and its spanning tree and we add a new graph to the output list constructed from Γ by replacing one of the fundamental edges by two different new edges, one on each vertex incident with the fundamental edge. The process continues comparing this new graph with the chosen spanning tree of Γ and the program will finish when all the fundamental edges are resolved, i.e., when the graph constructed is a tree.

To wrap up we use the program called *PolSPTree* that takes a graph Γ as input and computes the Grothendieck polynomial of the tree constructed from Γ by the program *LooseSPTree*. You can see here the code:

```
function PolLSPTree(G)
      B:=LooseSPTree(G);
      n:=#(B);
      A:=PolTree(B[n]);
      return A;
end function;
```

Thanks to Theorem 3.5.20, the polynomial obtained in *PolSPTree* is independent of the list given by *LooseSPTree*.

Now that we have programmed everything we need for the computation of a polynomial of a tree, we start describing how to calculate the difference in each step of the surgery. For this, let us recall the general formula for a difference after resolving the edge xy:

$$\begin{aligned}\Delta_{xy} = (\mathbb{L}^2)[G^L] - (\mathbb{L}-1)[G^L_x] - (\mathbb{L}-1)[G^L_y] - [C(G^L, xy)] \\ +[C(G^L_x, xy)] - [C(G^L_x, y)] + [C(G^L_y, xy)] - [C(G^L_y, x)].\end{aligned} \tag{B.1}$$

Thanks to the list obtained in *LooseSPTree* we will be able to get the new loose graphs needed to calculate the differences. For, we have created several programs called *GL, GLx, GLy, GL2, GL2x, GL3x, GL2y* and *GL3y* that compute, respectively, the loose graphs G^L, G^L_x, G^L_y, $C(G^L, xy)$, $C(G^L_x, xy)$, $C(G^L_x, y)$, $C(G^L_y, xy)$ and $C(G^L_y, x)$. The algorithm of all 8 programs is the same. Each of them receives two different graphs as data (the first one is the graph which arises after having resolved one edge of the second one) and, checking the neighbors of each vertex in both graphs, it identifies the resolved edge and constructs a list with the one of the aforementioned loose subgraphs. There is an important fact to be remarked here regarding connectedness of the graphs. In the previous formula, the loose graphs G^L, G^L_x and G^L_y might not be connected and, in fact, the formula from above considers them as being decomposed in connected components. That is why their corresponding programs give a list where different connected components are considered as different graphs. For the other six programs this is not necessary since adding any vertex x or y makes the loose graphs become connected. Besides, the impossibility of working with loose graphs in Magma forces us to create new programs, *mL, mLx* and *mLy,* which will keep track, respectively, of the number of vertices of the minimal graphs $\overline{G^L}$, $\overline{G^L_x}$ and $\overline{G^L_x}$ that are not vertices of their corresponding loose graphs.

Before being able to write the final program that will compute the polynomial for any graph, we need to keep track of all different polynomials of type G^L, G^L_x, etc. in all the different steps of the surgery process. For this, we will combine both the program *LooseSPTree* and one program of the previous ones (*GL, GLx, GLy, GL2, GL2x, GL3x, GL2y* or *GL3y*). We only describe the program keeping track of all the G^L's but there is one (exactly with the same algorithm) for each of the 8 programs listed before. We denote by *ListGL* a program that takes a list of graphs, applies the program *GL* to any two consecutive graphs from the list and creates a new list where the elements are all the graphs obtained by the program *GL*.

Now we can describe the last two programs calculating the Grothendieck polynomial for an arbitrary loose graph. The first one, *PolynGraph1,* receives a graph (or the minimal graph of a loose graph) and follows the following recursive algorithm. Initially we set the polynomial to be zero and we settle the two different situations in which the current program should stop: the case in which the graph is empty, giving 0 as answer; and the case where the graph is a tree, giving *PolTree(G)* as the answer. This is set in the following way

```
Q:=0;
if IsNull(G) then
   return Q;
   else
      if IsTree(G) then
         return Q + PolTree(G);
         else
            --------------
      end if;
end if;
```

The next step is finding out whether the graph has a vertex of maximal degree, in which case we will use the cone construction with one vertex of maximal degree as the vertex of the cone (cf. Corollary 3.5.6). This is necessary to avoid the program to go in an infinite computation. For instance, if in a graph Γ, after having resolved an edge, all the other vertices are common neighbors of the two vertices from the resolved edge, then the programs *GL2, GL2x* and *GL2y* will give as result the same graph Γ and recursion will be impossible to use. So, with the code

```
V:=Vertices(G);
         {D:=SetToIndexedSet(Alldeg(G, #(V) -1));
         if IsEmpty(D) then
         --------------
         else
            G1:= G - D[1];
            G2:= Components(G1);
            Q:=Q + x^(Degree(D[1]));
            for i in [1..#(G2)] do
                  Q:= Q + PolynGraph1(sub< G1 | G2[i]>);
            end for;
         end if;
```

we add, in case there is a vertex v of maximal degree, the corresponding term $\mathbb{L}^{\deg(v)}$ to the polynomial Q and use recursion on the connected components of the remaining graph. Once all these cases are checked we only have to take the graph, construct the list of all intermediate steps of the surgery, compute the polynomial for the loose spanning tree (last graph of the list) and use recursion on all different graphs obtained by the programs of the type *ListGL* with their corresponding coefficients from the formula (B.1). Let us remark that resolving edges in Magma not only means adding two edges, but also adding two more vertices, as we compute everything in the category of graphs. So when we calculate the polynomial of a spanning tree of a graph, we keep track of the number of extra vertices added in the resolving procedure. This number of extra vertices must be subtracted from the polynomial of the chosen spanning tree and, in that way, it is expressed in the algorithm with the following code:

```
L:=LooseSPTree(G);
tL:=#(L);
Q:=Q + PolLSPTree(L[tL]) - 2*(tL - 1);
```

Finally, there is only one step left to end the computation. For that purpose, we make the last program called *PolynGraph* that receives two arguments, a graph Γ and a number m (the number of loose edges). The program gives as output the polynomial *PolynGraph1*$(\Gamma) - m$.

```
function PolynGraph(G,m)
   Q:= PolynGraph1(G) - m;
   return Q;
end function;
```

Complete code in Magma

```
// Zeta Polynomial for loose trees

function PolTree(G)
      V:=Vertices(G);
      Q:=0;
      if #(V) eq 0 then return Q;
      else
         for i in [1..#(V)] do
            if Degree(V[i]) eq 1 then
               Q:=Q + 1;
            else
               Q:=Q + x^(Degree(V[i])) - x + 1;
            end if;
         end for;
```

```
      Q:=Q + x - 1;
      return Q;
      end if;
end function;

//Given a loose graph G, create a spanning tree breaking a cycle in each
   step

function LooseSPTree(G)
      L:=[* G *];
      if IsTree(G) then
         return L;
      else

         G1:=SpanningTree(G);
         V1:=Vertices(G1);
         V:=Vertices(G);
         G2:=G;
         t:=#(V);
         t1:=#(V1);
         for i in [1 .. t] do
         A:=Neighbors(V[i]);
         A1:=Neighbors(V1[i]);
            for c in [i+1 .. t1] do
            if V[c] in A then
               if V1[c] notin A1 then
                  R:=G2 + 2;
                  t2:=#(Vertices(R));
                  R:=AddEdge(R, Vertices(R)[i], Vertices(R)[t2-1]);
                  R:= R - { {Vertices(R)[i], Vertices(R)[c]} };
                  G2:=AddEdge(R, Vertices(R)[c], Vertices(R)[t2]);
                  L:=Append(L, G2);
               end if;
            end if;
            end for;
         end for;
         return L;
      end if;
end function;

//Given a graph, compute its Loose Spanning Tree

function PolLSPTree(G)
      B:=LooseSPTree(G);
      n:=#(B);
      A:=PolTree(B[n]);
```

```
      return A;
end function;

//Computing the graph G^L

function GL(A,B)
   V1:=Vertices(A);
   V2:=Vertices(B);
   t1:=#(V1);
   t2:=#(V2);
   L:=[* *];
   for i in [1 .. t2] do
      N2:=Neighbors(V2[i]);
      for c in [i+1 .. t2] do
         if V2[c] adj V2[i] then
            if V1[c] notadj V1[i] then
               H1:= Exclude(Neighbors(V2[c]), V2[i]);
               H2:= Exclude(Neighbors(V2[i]), V2[c]);
               H:= H1 meet H2;
               H3:= H1 sdiff H2;
               if IsEmpty(H) then
                  L:=Append(L, NullGraph());
               else
                  G3:=sub<B | H>;
                  T:=Components(G3);
                  lT:=#(T);
                  for i in [1..lT] do
                        G4:=sub<G3 | T[i]>;
                        V4:=Vertices(G4);
                        lV4:=#(V4);
                        if #(H3) ge 1 then
                           G5:=G4;
                           VG5:=Vertices(G5);
                           for j in [1..#(H3)] do
                              V5:={@ v : v in VG5 | (Vertices(B) ! v) adj
                                  SetToIndexedSet(H3)[j] @};
                              if IsEmpty(V5) then
                              else
                                 G5:= G5 + 1;
                                 lG5:=#(Vertices(G5));
                                 if IsSubgraph(G5,G4) then
                                    V6:={@ (VertexSet(G5) ! V5[s]) : s in
                                        [1..#(V5)] @};
                                    G5:= G5 + {{V6[s], Vertices(G5)[lG5]} : s
                                        in [1..#(V6)]};
                                 end if;
```

```
                        end if;
                     end for;
                     L:=Append(L,G5);
                  else
                     L:=[* sub<G3 | T[i]> : i in [1..lT] *];
                  end if;
              end for;

            end if;
            return L;
         end if;
      end if;
   end for;
 end for;
end function;

//Computing the graph C(G^L, xy)

function GL2(A,B)
  V1:=Vertices(A);
  V2:=Vertices(B);
  t1:=#(V1);
  t2:=#(V2);
  for i in [1 .. t2] do
     N2:=Neighbors(V2[i]);
     for c in [i+1 .. t2] do
        if V2[c] adj V2[i] then
           if V1[c] notadj V1[i] then
              H1:= Exclude(Neighbors(V2[c]), V2[i]);
              H2:= Exclude(Neighbors(V2[i]), V2[c]);
              H:= H1 meet H2;
              H5:=Include(Include(H, V2[c]), V2[i]);
              H3:= H1 sdiff H2;
              G2:=sub<B | H5>;
              G3:=sub<B | H>;
              if IsEmpty(H) then
                 L:=[* G2 *];
              else
                 T:=Components(G3);
                 lT:=#(T);
                 G5:=G2;
                 for i in [1..lT] do
                       G4:=sub<G3 | T[i]>;
                       V4:=Vertices(G4);
                       lV4:=#(V4);
                       if #(H3) ge 1 then
```

```
                  for j in [1..#(H3)] do
                    if IsSubgraph (G2,G4) then
                    V5:={@ (VertexSet(B) ! (VertexSet(G3) ! v)) : v
                        in V4 | (VertexSet(B) ! (VertexSet(G3) ! v))
                        adj SetToIndexedSet(H3)[j] @};
                      if not IsEmpty(V5) then
                        G5:=G5 + 1;
                        lG5:=#(Vertices(G5));
                        if IsSubgraph (G5,G2) then
                          V6:={@ (VertexSet(G5) ! (VertexSet(G2) !
                              V5[s])) : s in [1..#(V5)] @};
                          G5:= G5 + {{V6[s], Vertices(G5)[lG5]} :
                              s in [1..#(V6)]};
                        end if;
                      end if;
                    end if;
                  end for;
                end if;
            end for;
            L:=[* G5 *];
          end if;
          return L;
        end if;
      end if;
    end for;
  end for;
end function;

//Computing the graph G^L_x

function GLx(A,B)
  V1:=Vertices(A);
  V2:=Vertices(B);
  t1:=#(V1);
  t2:=#(V2);
  L:=[* *];
  for i in [1 .. t2] do
    N2:=Neighbors(V2[i]);
    for c in [i+1 .. t2] do
      if V2[c] adj V2[i] then
        if V1[c] notadj V1[i] then
          H1:= Exclude(Neighbors(V2[c]), V2[i]);
          H2:= Exclude(Neighbors(V2[i]), V2[c]);
          H:= H1 meet H2;
          H3:= H1 diff H2;
          if IsEmpty(H) then
```

```
            L:=Append(L, NullGraph());
         else
            G3:=sub<B | H>;
            T:=Components(G3);
            lT:=#(T);
            for i in [1..lT] do
                  G4:=sub<G3 | T[i]>;
                  V4:=Vertices(G4);
                  lV4:=#(V4);
                  if #(H3) ge 1 then
                     G5:=G4;
                     VG5:=Vertices(G5);
                     for j in [1..#(H3)] do
                        V5:={@ v : v in VG5 | (Vertices(B) ! v) adj
                            SetToIndexedSet(H3)[j] @};
                        if IsEmpty(V5) then
                        else
                           G5:= G5 + 1;
                           lG5:=#(Vertices(G5));
                           if IsSubgraph(G5,G4) then
                              V6:={@ (VertexSet(G5) ! V5[s]) : s in
                                   [1..#(V5)] @};
                              G5:= G5 + {{V6[s], Vertices(G5)[lG5]} : s
                                   in [1..#(V6)]};
                           end if;
                        end if;
                     end for;
                     L:=Append(L,G5);
                  else
                     L:=[* sub<G3 | T[i]> : i in [1..lT] *];
                  end if;
            end for;

         end if;
         return L;
      end if;
   end if;
  end for;
 end for;
end function;

//Computing the graph C(G^L_x, xy)

function GL2x(A,B)
 V1:=Vertices(A);
 V2:=Vertices(B);
```

```
t1:=#(V1);
t2:=#(V2);
for i in [1 .. t2] do
   N2:=Neighbors(V2[i]);
   for c in [i+1 .. t2] do
      if V2[c] adj V2[i] then
         if V1[c] notadj V1[i] then
            H1:= Exclude(Neighbors(V2[c]), V2[i]);
            H2:= Exclude(Neighbors(V2[i]), V2[c]);
            H:= H1 meet H2;
            H5:=Include(Include(H, V2[c]), V2[i]);
            H3:= H1 diff H2;
            G2:=sub<B | H5>;
            G3:=sub<B | H>;
            if IsEmpty(H) then
               L:=[* G2 *];
            else
               T:=Components(G3);
               lT:=#(T);
               G5:=G2;
               for i in [1..lT] do
                     G4:=sub<G3 | T[i]>;
                     V4:=Vertices(G4);
                     lV4:=#(V4);
                     if #(H3) ge 1 then
                        for j in [1..#(H3)] do
                           if IsSubgraph (G2,G4) then
                           V5:={@ (VertexSet(B) ! (VertexSet(G3) ! v)) : v
                               in V4 | (VertexSet(B) ! (VertexSet(G3) ! v))
                               adj SetToIndexedSet(H3)[j] @};
                              if not IsEmpty(V5) then
                                 G5:=G5 + 1;
                                 lG5:=#(Vertices(G5));
                                 if IsSubgraph (G5,G2) then
                                    V6:={@ (VertexSet(G5) ! (VertexSet(G2) !
                                        V5[s])) : s in [1..#(V5)] @};
                                    G5:= G5 + {{V6[s], Vertices(G5)[lG5]} :
                                        s in [1..#(V6)]};
                                 end if;
                              end if;
                           end if;
                        end for;
                     end if;
               end for;
               L:=[* G5 *];
            end if;
```

```
            return L;
         end if;
      end if;
   end for;
end for;
end function;

//Computing the graph C(G^L_x, y)

function GL3x(A,B)
   V1:=Vertices(A);
   V2:=Vertices(B);
   t1:=#(V1);
   t2:=#(V2);
   for i in [1 .. t2] do
      N2:=Neighbors(V2[i]);
      for c in [i+1 .. t2] do
         if V2[c] adj V2[i] then
            if V1[c] notadj V1[i] then
               H1:= Exclude(Neighbors(V2[c]), V2[i]);
               H2:= Exclude(Neighbors(V2[i]), V2[c]);
               H:= H1 meet H2;
               H5:=Include(H, V2[i]);
               H3:= H1 diff H2;
               G2:=sub<B | H5>;
               G3:=sub<B | H>;
               if IsEmpty(H) then
                  L:=[* G2 *];
               else
                  T:=Components(G3);
                  lT:=#(T);
                  G5:=G2;
                  for i in [1..lT] do
                        G4:=sub<G3 | T[i]>;
                        V4:=Vertices(G4);
                        lV4:=#(V4);
                        if #(H3) ge 1 then
                           for j in [1..#(H3)] do
                              if IsSubgraph (G2,G4) then
                              V5:={@ (VertexSet(B) ! (VertexSet(G3) ! v)) : v
                                  in V4 | (VertexSet(B) ! (VertexSet(G3) ! v))
                                  adj SetToIndexedSet(H3)[j] @};
                                 if not IsEmpty(V5) then
                                    G5:=G5 + 1;
                                    lG5:=#(Vertices(G5));
                                    if IsSubgraph (G5,G2) then
```

```
                                    V6:={@ (VertexSet(G5) ! (VertexSet(G2) !
                                       V5[s])) : s in [1..#(V5)] @};
                                    G5:= G5 + {{V6[s], Vertices(G5)[lG5]} :
                                       s in [1..#(V6)]};
                                 end if;
                              end if;
                           end if;
                        end for;
                     end if;
                 end for;
                 L:=[* G5 *];
              end if;
              return L;
           end if;
        end if;
     end for;
  end for;
end function;

//Computing the graph G^L_y

function GLy(A,B)
   V1:=Vertices(A);
   V2:=Vertices(B);
   t1:=#(V1);
   t2:=#(V2);
   L:=[* *];
   for i in [1 .. t2] do
      N2:=Neighbors(V2[i]);
      for c in [i+1 .. t2] do
         if V2[c] adj V2[i] then
            if V1[c] notadj V1[i] then
               H1:= Exclude(Neighbors(V2[c]), V2[i]);
               H2:= Exclude(Neighbors(V2[i]), V2[c]);
               H:= H1 meet H2;
               H4:= H2 diff H1;
               if IsEmpty(H) then
                  L:=Append(L, NullGraph());
               else
                  G3:=sub<B | H>;
                  T:=Components(G3);
                  lT:=#(T);
                  for i in [1..lT] do
                        G4:=sub<G3 | T[i]>;
                        V4:=Vertices(G4);
                        lV4:=#(V4);
```

```
                    if #(H4) ge 1 then
                       G5:=G4;
                       VG5:=Vertices(G5);
                       for j in [1..#(H4)] do
                          V5:={@ v : v in VG5 | (Vertices(B) ! v) adj
                              SetToIndexedSet(H4)[j] @};
                          if IsEmpty(V5) then
                          else
                             G5:= G5 + 1;
                             lG5:=#(Vertices(G5));
                             if IsSubgraph(G5,G4) then
                                V6:={@ (VertexSet(G5) ! V5[s]) : s in
                                    [1..#(V5)] @};
                                G5:= G5 + {{V6[s], Vertices(G5)[lG5]} : s
                                    in [1..#(V6)]};
                             end if;
                          end if;
                       end for;
                       L:=Append(L,G5);
                    else
                       L:=[* sub<G3 | T[i]> : i in [1..lT] *];
                    end if;
                 end for;

              end if;
              return L;
           end if;
        end if;
     end for;
  end for;
end function;

//Computing the graph C(G^L_y, xy)

function GL2y(A,B)
  V1:=Vertices(A);
  V2:=Vertices(B);
  t1:=#(V1);
  t2:=#(V2);
  for i in [1 .. t2] do
     N2:=Neighbors(V2[i]);
     for c in [i+1 .. t2] do
        if V2[c] adj V2[i] then
           if V1[c] notadj V1[i] then
              H1:= Exclude(Neighbors(V2[c]), V2[i]);
              H2:= Exclude(Neighbors(V2[i]), V2[c]);
```

```
          H:= H1 meet H2;
          H5:=Include(Include(H, V2[c]), V2[i]);
          H3:= H2 diff H1;
          G2:=sub<B | H5>;
          G3:=sub<B | H>;
          if IsEmpty(H) then
             L:=[* G2 *];
          else
             T:=Components(G3);
             lT:=#(T);
             G5:=G2;
             for i in [1..lT] do
                   G4:=sub<G3 | T[i]>;
                   V4:=Vertices(G4);
                   lV4:=#(V4);
                   if #(H3) ge 1 then
                      for j in [1..#(H3)] do
                         if IsSubgraph (G2,G4) then
                         V5:={@ (VertexSet(B) ! (VertexSet(G3) ! v)) : v
                             in V4 | (VertexSet(B) ! (VertexSet(G3) ! v))
                             adj SetToIndexedSet(H3)[j] @};
                            if not IsEmpty(V5) then
                               G5:=G5 + 1;
                               lG5:=#(Vertices(G5));
                               if IsSubgraph (G5,G2) then
                                  V6:={@ (VertexSet(G5) ! (VertexSet(G2) !
                                       V5[s])) : s in [1..#(V5)] @};
                                  G5:= G5 + {{V6[s], Vertices(G5)[lG5]} :
                                       s in [1..#(V6)]};
                               end if;
                            end if;
                         end if;
                      end for;
                   end if;
             end for;
             L:=[* G5 *];
          end if;
          return L;
        end if;
      end if;
    end for;
  end for;
end function;

//Computing the graph C(G^L_y, x)
```

```
function GL3y(A,B)
   V1:=Vertices(A);
   V2:=Vertices(B);
   t1:=#(V1);
   t2:=#(V2);
   for i in [1 .. t2] do
      N2:=Neighbors(V2[i]);
      for c in [i+1 .. t2] do
         if V2[c] adj V2[i] then
            if V1[c] notadj V1[i] then
               H1:= Exclude(Neighbors(V2[c]), V2[i]);
               H2:= Exclude(Neighbors(V2[i]), V2[c]);
               H:= H1 meet H2;
               H5:=Include(H, V2[c]);
               H3:= H2 diff H1;
               G2:=sub<B | H5>;
               G3:=sub<B | H>;
               if IsEmpty(H) then
                  L:=[* G2 *];
               else
                  T:=Components(G3);
                  lT:=#(T);
                  G5:=G2;
                  for i in [1..lT] do
                        G4:=sub<G3 | T[i]>;
                        V4:=Vertices(G4);
                        lV4:=#(V4);
                        if #(H3) ge 1 then
                           for j in [1..#(H3)] do
                              if IsSubgraph (G2,G4) then
                              V5:={@ (VertexSet(B) ! (VertexSet(G3) ! v)) : v
                                  in V4 | (VertexSet(B) ! (VertexSet(G3) ! v))
                                  adj SetToIndexedSet(H3)[j] @};
                                 if not IsEmpty(V5) then
                                    G5:=G5 + 1;
                                    lG5:=#(Vertices(G5));
                                    if IsSubgraph (G5,G2) then
                                       V6:={@ (VertexSet(G5) ! (VertexSet(G2) !
                                           V5[s])) : s in [1..#(V5)] @};
                                       G5:= G5 + {{V6[s], Vertices(G5)[lG5]} :
                                           s in [1..#(V6)]};
                                    end if;
                                 end if;
                              end if;
                           end for;
                        end if;
```

```
            end for;
            L:=[* G5 *];
          end if;
          return L;
        end if;
      end if;
    end for;
  end for;
end function;

//Create all list with all different graph from the whole surgery process

function ListGL (L)
  tL:=#(L);
  L1:=[* *];
  for i in [1..(tL-1)] do
    L1:=L1 cat GL(L[tL - i +1], L[tL - i]);
  end for;
  return L1;
end function;

function ListGLx (L)
  tL:=#(L);
  L1:=[* *];
  for i in [1..(tL-1)] do
    L1:=L1 cat GLx(L[tL - i +1], L[tL - i]);
  end for;
  return L1;
end function;

function ListGLy (L)
  tL:=#(L);
  L1:=[* *];
  for i in [1..(tL-1)] do
    L1:=L1 cat GLy(L[tL - i +1], L[tL - i]);
  end for;
  return L1;
end function;

function ListGL2 (L)
  tL:=#(L);
  L1:=[* *];
  for i in [1..(tL-1)] do
    L1:=L1 cat GL2(L[tL - i +1], L[tL - i]);
  end for;
  return L1;
```

```
end function;

function ListGL2x (L)
   tL:=#(L);
   L1:=[* *];
   for i in [1..(tL-1)] do
      L1:=L1 cat GL2x(L[tL - i +1], L[tL - i]);
   end for;
   return L1;
end function;

function ListGL2y (L)
   tL:=#(L);
   L1:=[* *];
   for i in [1..(tL-1)] do
      L1:=L1 cat GL2y(L[tL - i +1], L[tL - i]);
   end for;
   return L1;
end function;

function ListGL3x (L)
   tL:=#(L);
   L1:=[* *];
   for i in [1..(tL-1)] do
      L1:=L1 cat GL3x(L[tL - i +1], L[tL - i]);
   end for;
   return L1;
end function;

function ListGL3y (L)
   tL:=#(L);
   L1:=[* *];
   for i in [1..(tL-1)] do
      L1:=L1 cat GL3y(L[tL - i +1], L[tL - i]);
   end for;
   return L1;
end function;

//Keeping track of all vertices added in the surgery process

function mL (A,B)
   V1:=Vertices(A);
   V2:=Vertices(B);
   t1:=#(V1);
   t2:=#(V2);
   t:=0;
```

```
  for i in [1 .. t2] do
    N2:=Neighbors(V2[i]);
    for c in [i+1 .. t2] do
      if V2[c] adj V2[i] then
        if V1[c] notadj V1[i] then
          H1:= Exclude(Neighbors(V2[c]), V2[i]);
          H2:= Exclude(Neighbors(V2[i]), V2[c]);
          H:= H1 meet H2;
          L:=GL(A,B);
          for i in [1..#(L)] do
            t:= t + #(Vertices(L[i]));
          end for;
          t:= t - #(H);
        end if;
      end if;
    end for;
  end for;
  return t;
end function;

function ListmL(L)
  tL:=#(L);
  L2:=[mL(L[tL - i +1], L[tL - i]) : i in [1..(tL-1)]];
  return L2;
end function;

function mLx (A,B)
  V1:=Vertices(A);
  V2:=Vertices(B);
  t1:=#(V1);
  t2:=#(V2);
  t:=0;
  for i in [1 .. t2] do
    N2:=Neighbors(V2[i]);
    for c in [i+1 .. t2] do
      if V2[c] adj V2[i] then
        if V1[c] notadj V1[i] then
          H1:= Exclude(Neighbors(V2[c]), V2[i]);
          H2:= Exclude(Neighbors(V2[i]), V2[c]);
          H:= H1 meet H2;
          L:=GLx(A,B);
          for i in [1..#(L)] do
            t:= t + #(Vertices(L[i]));
          end for;
          t:= t - #(H);
        end if;
```

```
        end if;
      end for;
   end for;
   return t;
end function;

function ListmLx(L)
   tL:=#(L);
   L2:=[mLx(L[tL - i +1], L[tL - i]) : i in [1..(tL-1)]];
   return L2;
end function;

function mLy (A,B)
   V1:=Vertices(A);
   V2:=Vertices(B);
   t1:=#(V1);
   t2:=#(V2);
   t:=0;
   for i in [1 .. t2] do
      N2:=Neighbors(V2[i]);
      for c in [i+1 .. t2] do
         if V2[c] adj V2[i] then
            if V1[c] notadj V1[i] then
               H1:= Exclude(Neighbors(V2[c]), V2[i]);
               H2:= Exclude(Neighbors(V2[i]), V2[c]);
               H:= H1 meet H2;
               L:=GLy(A,B);
                  for i in [1..#(L)] do
               t:= t + #(Vertices(L[i]));
               end for;
               t:= t - #(H);
            end if;
         end if;
      end for;
   end for;
   return t;
end function;

function ListmLy(L)
   tL:=#(L);
   L2:=[mLy(L[tL - i +1], L[tL - i]) : i in [1..(tL-1)]];
   return L2;
end function;

//Compute the Grothendieck polynomial for a graph
```

```
function PolynGraph1(G)
  Q:=0;
  if IsNull(G) then
    return Q;
  else

    if IsTree(G) then
      return Q + PolTree(G);
    else
      V:=Vertices(G);
      D:=SetToIndexedSet(Alldeg(G, #(V) -1));
      if IsEmpty(D) then
        L:=LooseSPTree(G);
        tL:=#(L);
        Q:=Q + PolLSPTree(L[tL]) - 2*(tL - 1);
        L1:=ListGL(L);
        L2:=ListGLx(L);
        L3:=ListGLy(L);
        L4:=ListGL2(L);
        L5:=ListGL2x(L);
        L6:=ListGL2y(L);
        L7:=ListGL3x(L);
        L8:=ListGL3y(L);
        L9:=ListmL(L);
        L10:=ListmLx(L);
        L11:=ListmLy(L);
        for i in [1..#(L1)] do
          Q:= Q - x^2*PolynGraph1(L1[i]);
        end for;
        for i in [1..#(L2)] do
          Q:= Q + (x-1)*PolynGraph1(L2[i]);
        end for;
        for i in [1..#(L3)] do
          Q:= Q + (x-1)*PolynGraph1(L3[i]);
        end for;
        for i in [1..#(L9)] do
          Q:= Q + (x^2-1)*L9[i];
        end for;
        for i in [1..#(L10)] do
          Q:= Q - (x-1)*L10[i];
        end for;
        for i in [1..#(L11)] do
          Q:= Q - (x-1)*L11[i];
        end for;
        for i in [1..#(L4)] do
          Q:= Q + PolynGraph1(L4[i]);
```

```
          end for;
          for i in [1..#(L5)] do
             Q:= Q  - PolynGraph1(L5[i]);
          end for;
          for i in [1..#(L6)] do
             Q:= Q  - PolynGraph1(L6[i]);
          end for;
          for i in [1..#(L7)] do
             Q:= Q  + PolynGraph1(L7[i]);
          end for;
          for i in [1..#(L8)] do
             Q:= Q  + PolynGraph1(L8[i]);
          end for;
       else
          G1:= G - D[1];
          G2:= Components(G1);
          Q:=Q + x^(Degree(D[1]));
          for i in [1..#(G2)] do
                Q:= Q + PolynGraph1(sub<G1 | G2[i]>);
          end for;
       end if;
       return Q;
    end if;
  end if;
end function;

//Compute the Grothendieck polynomial for a loose graph

function PolynGraph(G,m)
  Q:= PolynGraph1(G) - m;
  return Q;
end function;
```

C Nederlandse Samenvatting

Deze thesis presenteert nieuwe resultaten die gevonden zijn in de theorie van schema's en constructieve verzamelingen over het veld met één element, $\mathbb{F}_1$. Essentieel gaat het om een studie van bepaalde functoren $\mathcal{F}_k$, met k een eindig veld of $\mathbb{F}_1$, die van de categorie van "losse grafen" — die grafen veralgemenen — gaan naar de categorie van Deitmar constructieve verzamelingen met extra structuur. Die functoren worden gebruikt om Grothendieck polynomen van losse grafen Γ te vinden, en die polynomen tellen het aantal rationale punten van de geassocieerde constructieve verzamelingen $\mathcal{F}_k(\Gamma)$. Eigenlijk is het zelfs zo dat de virtuele motieven van de constructieve verzamelingen $\mathcal{F}_k(\Gamma)$ in de Grothendieck ring van k-schema's van eindig type mixed Tate zijn. We gebruiken de functoren ook om een nieuwe zeta functie in te voeren voor de categorie van (losse) grafen. We bepalen tevens automorfismegroepen van deze objecten.

C.1 Deitmar schema's en constructieve verzamelingen

In Hoofdstuk 1 geven we een motivatie voor $\mathbb{F}_1$-theorie, en geven we tevens een kleine introductie tot Lineaire Algebra en Algebraïsche Meetkunde over het veld met één element. Een van de hoofdrolspelers in dit proefwerk is de notie van "Deitmar schema's", en die gaan we nu even definiëren.

Beschouw een "$\mathbb{F}_1$-ring" A; dit is een multiplicatieve commutatieve monoïde met een extra opslorpend element 0. Onderstel dat $\texttt{Spec}(A)$ de verzameling is van alle priemidealen van A, met daarop een Zariski topologie. De topologische ruimte $\texttt{Spec}(A)$ met een structuurschoof van $\mathbb{F}_1$-ringen noemen we een *affien Deitmar schema*. We definiëren een *monoïdale ruimte* als zijnde een paar $(X, \mathcal{O}_X)$ waarbij X een topologische ruimte is, en $\mathcal{O}_X$ een schoof van $\mathbb{F}_1$-ringen gedefinieerd over X. Een *Deitmar schema* is nu een monoïdale ruimte waarvoor elk punt $x \in X$ een open omgeving $U \subseteq X$ heeft zodat $(U, \mathcal{O}_X|_U)$ isomorf is met een affien Deitmar schema.

Voor meer details over priemidealen van een monoïde, Deitmar schema's en structuurschoven over $\mathbb{F}_1$-ringen, refereren we aan [8].

C.2 Losse grafen

In Hoofdstuk 2 introduceren we een nieuwe categorie van combinatorische objecten die we *losse grafen* noemen.

Definitie C.2.1. Een losse graaf is een punt-rechte meetkunde $\Gamma = (V, E, \mathbf{I})$, met V een verzameling *toppen*, E een verzameling *bogen* en $\mathbf{I}$ een symmetrische relatie op $V \cup E$, disjunct van $V \times V$ en $E \times E$, die uitdrukt wanneer bogen en toppen incident zijn, en waar *elke boog incident is met ten hoogste twee verschillende toppen*. We noemen bogen met minder dan twee toppen *losse bogen* en de bogen met twee verschillende toppen *echte bogen*.

Merk op dat losse grafen de notie van grafen veralgemenen, want een boog kan nu één of zelfs geen top(pen) hebben. Elke graaf is dus een losse graaf. We onderstellen dat elke losse graaf komt met een *ledige boog*, genoteerd $e_\emptyset$, die per definitie helemaal nooit incident is met een top.

In deze thesis beschouwen we meestal samenhangende enkelvoudige losse grafen zonder richting.

Definitie C.2.2. Onderstel dat $\Gamma_1 = (V_1, E_1, \mathbf{I}_1)$ en $\Gamma_2 = (V_2, E_2, \mathbf{I}_2)$ losse grafen zijn. Dan is een afbeelding $f : \Gamma_1 \to \Gamma_2$ een *morfisme van losse grafen* van Γ_1 naar Γ_2 indien:

i) f toppen naar toppen stuurt en bogen naar bogen, i.e., $f|_{V_1} : V_1 \to V_2$ en $f|_{E_1} : E_1 \to E_2$.

ii) $f(e_{1,\emptyset}) = e_{2,\emptyset}$.

iii) Indien $e \in E$ een echte boog is, dan is $f(e) = e_{2,\emptyset}$ als en slechts als de twee toppen incident met e in Γ_1 hetzelfde beeld hebben onder f.

iv) Indien een top $v \in V_1$ incident is met een boog $e \in E_1$, dan zijn $f(v)$ en $f(e)$ ook incident indien $f(e) \neq e_{2,\emptyset}$.

Een morfisme $f : \Gamma_1 \to \Gamma_1$ van losse grafen is een *automorfisme* indien

a) f bijectief is op de verzamelingen van toppen en bogen.

b) Een top v incident is met een boog e als en slechts als $f(v)$ incident is met $f(e)$.

We definiëren de *categorie van losse grafen*, genoteerd `LGraph`, als de categorie met objecten de losse grafen en morfismen morfismen van losse grafen. Een van de hoofdstellingen die ons toelaat om de functor $\mathcal{F}$ te definiëren, die zelf essentieel is voor het huidige werk, is de inbeddingsstelling (cf. [45] voor meer details).

Stelling C.2.3 (Inbeddingsstelling)**.** *Onderstel dat Γ een losse graaf is en dat $\overline{\Gamma}$ de minimale graaf is, i.e., de graaf die we bekomen door aan elke losse boog toppen toe te voegen om zo echte bogen te bekomen. Dan kan Γ gezien worden als een losse deelgraaf van de combinatorische projectieve ruimte $\mathbf{P}_c(\Gamma)$, i.e, de projectieve $\mathbb{F}_1$-ruimte gedefinieerd door de complete graaf op de toppenverzameling van $\overline{\Gamma}$.*

We definiëren de functor $\mathcal{F}$ die gaat van de categorie `LGraph` van losse grafen naar de categorie $\mathtt{CS}_{\mathbb{F}_1}$ van Deitmar constructieve verzamelingen als een functor die aan de volgende eigenschappen voldoet:

CV Indien $\Gamma \subset \widetilde{\Gamma}$ een strikte inclusie is van losse grafen, dan is $\mathcal{F}(\Gamma)$ een strikt constructieve deelverzameling van $\mathcal{F}(\widetilde{\Gamma})$.

L-D Indien x een top is van graad $m \in \mathbb{N}^\times$ in Γ, dan is er een affiene ruimte van dimensie m die x bevat, en volledig bevat is in $\mathcal{F}(\Gamma)$.

FIN De constructieve verzameling $\mathcal{F}(\Gamma)$ is de unie van de affiene ruimten uit het vorige punt.

CO Indien K_m een complete deelgraaf is op m toppen in Γ, dan is $\mathcal{F}(K_m)$ een gesloten projectieve deelruimte van dimensie $m-1$ in $\mathcal{F}(\Gamma)$.

MG Een boog zonder toppen moet corresponderen met de multiplicatieve groep.

Er is nu een eenvoudige manier om $\mathcal{F}$ te definiëren:

(F1) Voor elke losse ster S_n (dit is de losse graaf die bestaat uit een enkele top samen met n bogen incident met deze top), is $\mathcal{F}(S_n)$ een affiene $\mathbb{F}_1$-ruimte van dimensie n.

(F2) Onderstel dat Γ een samenhangende losse graaf is, en laat $\overline{\Gamma}$ de minimale graaf zijn. Onderstel dat $\overline{\Gamma}$ precies $m+1$ toppen heeft. Onderstel dat $\mathbf{P}(\overline{\Gamma})$ de projectieve $\mathbb{F}_1$-ruimte is van dimensie m gedefinieerd door deze toppen; dan is $\mathcal{F}(\Gamma)$ de unie in $\mathbf{P}(\overline{\Gamma})$ van de affiene $\mathbb{F}_1$-ruimten gedefinieerd door de sterren gehecht aan de toppen, zonder de gesloten punten die corresponderen met de toppen die toegevoegd werden om $\overline{\Gamma}$ te bekomen.

(F3) Indien we homogene coördinaten kiezen (in $\mathbf{P}(\overline{\Gamma})$) zodat elke top als coördinaten een vector in $\{0,1\}^{m+1}$ met precies één niet-nul term heeft, kan $\mathcal{F}(\Gamma)$ expliciet analytisch beschreven worden. Het niet-samenvallende geval volgt makkelijk uit het samenvallende geval.

Voor een eindig veld k (of $\mathbb{Z}$) definiëren we de functor $\mathcal{F}_k$ als de functor $\mathcal{F}_k(\cdot) = \mathcal{F}(\cdot) \otimes k$ die gaat van de categorie van losse grafen naar de categorie van k-constructieve verzamelingen. Merk op dat dit gedefinieerd is via de basis-extensie functor van de categorie van $\mathbb{F}_1$-ringen naar de categorie van k-ringen.

Het hoofdresultaat van Hoofdstuk 2 is het bewijs dat $\mathcal{F}_k$ weldegelijk een functor is voor k een willekeurig eindig veld, $\mathbb{F}_1$ of $\mathbb{Z}$.

C.3 Telveelterm

In Hoofdstuk 3 beginnen we door de *Grothendieck ring van schema's van eindig type over* $\mathbb{F}_1$ te definiëren, genoteerd $K_0(\mathtt{Sch}_{\mathbb{F}_1})$, op een gelijkaardige manier als de Grothendieck ring $K_0(\mathtt{Sch}_k)$ gedefinieerd is voor schema's van eindig type over een veld k, en we benadrukken de connectie met de theorie van motieven. Inderdaad kan men ook de Grothendieck ring $K_0(\mathcal{M}(k))$ van motieven over een veld k (verschillend van $\mathbb{F}_1$) beschouwen, waarbij de klasse van het motief van de affiene rechte $\mathbb{A}^1_k$ het *Lefschetz* motief genoemd wordt, genoteerd $\mathbb{L}$, en een gelijkaardige rol speelt als de klasse van de affiene rechte $\mathbb{A}^1_k$ in $K_0(\mathtt{Sch}_k)$. Om deze connectie tussen schema's en motieven te benadrukken, noteren we de klasse van de affiene rechte $\mathbb{A}^1_k$ in $K_0(\mathtt{Sch}_k)$ door $\mathbb{L}$, en $\underline{\mathbb{L}}$ is de notatie voor de klasse van $\mathbb{A}^1_{\mathbb{F}_1}$ in $K_0(\mathtt{Sch}_{\mathbb{F}_1})$.

Definitie C.3.1. Onderstel dat X een constructieve verzameling is over een veld k. We zeggen dat X *veelterm-telbaar* is indien er een (noodzakelijk unieke) veelterm $P_X(T) = \sum_{i=0}^m a_iT^i \in \mathbb{Z}[T]$ bestaat zodat voor elke eindige extensie $\mathbb{F}_{q^n}/\mathbb{F}_q$, we hebben dat

$$\Big|X\Big|_{q^n} = P_X(q^n). \tag{C.1}$$

We noemen deze veelterm de *Grothendieck veelterm* van X.

Definitie C.3.2. We zeggen dat X *een gemengd Tate motief heeft* indien zijn klasse in de Grothendieck ring van schema's van eindig type over k behoort tot de deelring $\mathbb{Z}[\mathbb{L}]$.

Refererend aan een van de Tate vermoedens, dat als gevolg het volgende zegt:

> *"Onderstel dat X een variëteit is; dan is X veelterm-telbaar voor alle behalve een endig aantal priemgetallen als en slechts als X een gemengd Tate motief heeft,"*

bewijzen we in Hoofdstuk 3 de volgende resultaten over de constructieve verzamelingen $\mathcal{F}_k(\Gamma)$.

Stelling C.3.3. *Onderstel dat Γ een losse graaf is, en stel dat $\mathcal{F}_{\mathbb{F}_q}(\Gamma)$ zijn $\mathbb{F}_q$-constructieve verzameling is, met $\mathbb{F}_q$ een willekeurig eindig veld. Dan is $\mathcal{F}_{\mathbb{F}_q}(\Gamma)$ veelterm-telbaar. Daarnaast is $\mathcal{F}_{\mathbb{F}_q}(\Gamma)$ zeta-equivalent met een $\mathbb{F}_q$-constructieve verzameling met Grothendieck klasse een $\mathbb{Z}$-lineaire combinatie van klassen van affiene ruimten $[\mathbb{A}^i]_{\mathbb{F}_q}$, i.e., de Grothendieck klasse is een element van de ring $\mathbb{Z}[\mathbb{L}]$.*

En in verband met het eerder vermelde gevolg van een van de Tate vermoedens, bewijzen we ook het volgende resultaat in Hoofdstuk 3.

Stelling C.3.4. *Onderstel dat Γ een losse graaf is, en laat $k \neq \mathbb{F}_1$ een veld zijn. Dan is de klasse $[\mathcal{F}_k(\Gamma)] \in K_0(\mathtt{Sch}_k)$ een virtueel gemengd Tate motief.*

C.4 Een nieuwe zeta functie voor (losse) grafen

In Hoofdstuk 4 definiëren we, geïnspireerd door Kurokawa [26], een nieuwe zeta functie voor losse grafen. In [26] stelt Kurokawa dat een $\mathbb{Z}$-schema X *van* $\mathbb{F}_1$*-type* is indien haar aritmetische zeta functie $\zeta_X(s)$ uitgedrukt kan worden via de Riemann zeta functie $\zeta(s)$ op de volgende manier

$$\zeta_X(s) := \prod_{k=0}^{n} \zeta(s-k)^{a_k}, \tag{C.2}$$

waarbij de a_is elementen zijn van $\mathbb{Z}$, en hij toont aan dat deze definitie equivalent is aan de voorwaarde dat er een veelterm $P_X(Y) = \sum_{i=0}^{n} a_k Y$ is zodat $\#X(\mathbb{F}_{p^m}) = P_X(p^m)$ voor alle eindige velden $\mathbb{F}_{p^m}$. We verkrijgen dan het volgende resultaat.

Definitie C.4.1. We zeggen dat een constructieve verzameling X *gedefinieerd is over* $\mathbb{F}_1$ in Kurokawa's aanpak indien X een veelterm $P_x(Y)$ heeft zodat $\#X(\mathbb{F}_{p^m}) = P_X(p^m)$.

Stelling C.4.2. *Voor elke losse graaf* Γ *is het* $\mathbb{Z}$*-constructieve verzameling* $\mathcal{X}_{\mathbb{Z}} := \mathcal{F}(\Gamma) \otimes_{\mathbb{F}_1} \mathbb{Z}$ *gedefinieerd over* $\mathbb{F}_1$ *in Kurokawa's aanpak.*

Definitie C.4.3 (Zeta functie voor (losse) grafen)**.** Laat Γ een losse graaf zijn, en laat $\mathcal{X}_{\mathbb{Z}} := \mathcal{F}(\Gamma) \otimes_{\mathbb{F}_1} \mathbb{Z}$. Onderstel dat $P_{\mathcal{X}}(X) = \sum_{i=0}^{m} a_i X^i \in \mathbb{Z}[X]$ de zeta veelterm is uit Hoofdstuk 3. We definiëren de $\mathbb{F}_1$-*zeta functie* van Γ als:

$$\zeta_{\Gamma}^{\mathbb{F}_1}(t) := \prod_{k=0}^{m} (t-k)^{-a_k}. \tag{C.3}$$

Voorbeeld C.4.4. Onderstel dat Γ een boom is, en laat D de verzameling van graden $\{d_1, \ldots, d_m\}$ zijn van de toppen van Γ zodat $1 < d_1 < d_2 < \ldots < d_m$; de zeta functie wordt gegeven door

$$\zeta_{\Gamma}^{\mathbb{F}_1}(t) \quad = \quad \frac{(t-1)^I}{t^{E+I}} \cdot \prod_{k=1}^{m} (t-d_k)^{-n_k}, \tag{C.4}$$

waar n_i het aantal toppen is van Γ met graad d_i, $1 \leq i \leq m$, E het aantal toppen van Γ met graad 1 en $I = \sum_{i=1}^{m} n_i - 1$.

C.5 Automorfismegroepen van $\mathcal{F}(\Gamma)$

Onderstel dat Γ een losse graaf is, dat $\mathcal{F}(\Gamma)$ zoals eerder is, en stel $\mathcal{X}_k = \mathcal{F}(\Gamma) \otimes_{\mathbb{F}_1} k$ de extensie tot het eindig veld k. In Hoofdstuk 5 introduceren we de volgende automorfismegroepen van $\mathcal{X}_k$.

Definitie C.5.1. We definiëren de *projectieve automorfismegroep* van $\mathcal{X}_k$, genoteerd $\mathrm{Aut}^{\mathrm{proj}}(\mathcal{X}_k)$, als de groep van automorfismen van de omhullende projectieve ruimte van $\mathcal{X}_k$ die $\mathcal{X}_k$ stabiliseert, modulo de groep van zulke automorfismen die triviaal werken op $\mathcal{X}_k$.

Definitie C.5.2. We beschouwen $\mathcal{X}_k$ nu als een *incidentiemeetkunde van rang* 2, waar de puntenverzameling $\mathcal{P}$ de verzameling van k-rationale punten is van $\mathcal{X}_k$ en de rechtenverzameling $\mathcal{L}$ bestaat uit projectieve rechten (over k) en *volledige affiene rechten*. Een *volledige affiene rechte* l van $\mathcal{X}_k$ is een rechte waarvan de projectieve completering $\bar{l}$ de constructieve verzameling $\mathcal{X}_k$ snijdt in de projectieve rechte $\bar{l}$ min een punt. We definiëren de *combinatorische automorfismegroep* van $\mathcal{X}_k$, genoteerd $\mathrm{Aut}^{\mathrm{comb}}(\mathcal{X}_k)$, als de groep van bijectieve afbeeldingen $\mathcal{P} \cup \mathcal{L} \to \mathcal{P} \cup \mathcal{L}$ die $\mathcal{P}$ en $\mathcal{L}$ behouden, en die incidentie behouden in beide richtingen.

Definitie C.5.3. We definiëren de *topologische automorfismegroep* van $\mathcal{X}_k$, genoteerd als $\mathrm{Aut}^{\mathrm{top}}(\mathcal{X}_k)$, als de groep van homeomorfismen van de onderliggende topologische ruimte.

Een van de eerste eigenschappen die we bekomen is de volgende.

Propositie C.5.4. *De combinatorische groep van $\mathcal{X}_k$ is een deelgroep van de topologische automorfismegroep van $\mathcal{X}_k$.*

C.6 Automorfismen van algemene losse bomen

Onderstel dat $T = (V, E, \mathbf{I})$ een eindige losse boom is, en onderstel dat het aantal toppen tenminste 3 is. Definieer $\mathcal{X}_k$ als eerder, waarbij k een eindig veld is. Een van de hoofddoelen van Hoofdstuk 5 is de bepaling van de projectieve automorfismegroep van $\mathcal{X}_k$. Vooraleer we overgaan op het beschrijven van de hoofdresultaten, hebben we extra notatie nodig.

Onderstel dat $\overline{T}$ de minimale graaf is van T, i.e., de boom die ontstaat door alle mogelijke eindpunten toe te voegen aan T, en laat $\mathbf{PG}(m-1,k)$ de omhullende projectieve ruimte zijn van $\mathcal{X}_k$. Door de inbedding stelling, kan T gezien worden als een deelmeetkunde van een projectieve $\mathbb{F}_1$-ruimte.

Onderstel dat I de verzameling is van inwendige toppen van T; voor elke $w \in I$ definiëren we $S(w)$ als de deelgroep van $\mathrm{Aut}^{\mathrm{proj}}(\mathcal{X}_k)$ die de k-rationale punten van $\mathcal{X}_k$ fixeert binnen elke affiene deelruimte $\widetilde{\mathbb{A}_v}$ (over k) die voortgebracht is (over $\mathbb{F}_1$) door een top v verschillend van w en alle richtingen door v die niet incident zijn met w. Laat nu S een verzameling punten zijn in $\mathbf{PG}(m-1,k)$; dan is $\mathbf{P\Gamma L}_m(k)_{[S]}$ de puntsgewijze stabilisator. De hoofdresultaten van Hoofdstuk 5 zijn de volgende.

Stelling C.6.1. *Laat $\mathbf{PGL}(\mathcal{X}_k)_{[I]}$ gedefinieerd zijn als*

$$\mathrm{Aut}^{\mathrm{proj}}(\mathcal{X}_k)_{[I]} \ \cap \ \mathbf{PGL}_m(k). \tag{C.5}$$

Dan is $\mathbf{PGL}(\mathcal{X}_k)_{[I]}$ isomorf met het centrale product

$$\prod_{w \in I}^{\mathrm{centr}} S(w). \tag{C.6}$$

Stelling C.6.2 (Inwendige Boom Stelling)**.** *Onderstel dat T een losse boom is, en k een veld. Stel $\mathcal{X}_k = \mathcal{F}(T) \otimes_{\mathbb{F}_1} k$, en beschouw de inbedding*

$$\iota : T \hookrightarrow \mathcal{X}_k, \tag{C.7}$$

waar $\iota(T)$ isomorf met T is als punt-rechte deelmeetkunde van de punt-rechte meetkunde van $\mathcal{X}_k$.

Onderstel dat $\mathrm{Aut}(\mathcal{X}_k)$ *gelijk welke van de automorfismegroepen is die we in dit hoofdstuk beschouwen (i.e., combinatorisch, geïnduceerd via de projectieve ruimte of topologisch). Onderstel dat I de verzameling van inwendige toppen is van T, and laat $T(I)$ de deelboom zijn van T die geïnduceerd wordt op I. Indien $|I| \geq 2$ hebben we dat* $\mathrm{Aut}(\mathcal{X}_k)$*, $\iota(T(I))$ vast houdt. Meer nog,* $\mathrm{Aut}(\iota(T(I)))$ *is geïnduceerd door* $\mathrm{Aut}(\mathcal{X}_k)$*.*

Stelling C.6.3 (Projectieve automorfismegroep)**.** *Onderstel dat T een losse boom is, en k een veld. Stel $\mathcal{X}_k = \mathcal{F}(T) \otimes_{\mathbb{F}_1} k$, en beschouw de inbedding*

$$\iota : T \hookrightarrow \mathcal{X}_k. \tag{C.8}$$

Onderstel dat I de verzameling van inwendige toppen is van T, and laat $T(I)$ de deelboom zijn van T die geïnduceerd wordt op I. We hebben dat $\mathbf{P\Gamma L}(\mathcal{X}_k) = \mathrm{Aut}^{\mathrm{proj}}(\mathcal{X}_k)$ *isomorf is met*

$$\Big(\Big(\prod_{w \in I}^{\mathrm{centr}} S(w)\Big) \rtimes \mathrm{Aut}(T(I))\Big) \rtimes k^{\times}. \tag{C.9}$$

Stelling C.6.4 (Combinatorische automorfismegroep)**.** *Onderstel dat T een losse boom is, en k een veld. Stel $\mathcal{X}_k = \mathcal{F}(T) \otimes_{\mathbb{F}_1} k$. Onderstel dat I de verzameling is van de inwendige toppen van T, en onderstel dat $|I| \geq 2$. Laat ι zijn zoals in Stelling C.6.2. Dan is*

$$\mathrm{Aut}^{\mathrm{comb}}(\mathcal{X}_k) \cong \mathrm{Aut}^{\mathrm{proj}}(\mathcal{X}_k). \tag{C.10}$$

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