Polarized non-abelian representations of slim near-polar spaces

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Abstract

In [15], Shult introduced a class of parapolar spaces, the so-called *near-polar spaces*. We introduce here the notion of a polarized non-abelian representation of a slim near-polar space, that is, a near-polar space in which every line is incident with precisely three points. For such a polarized non-abelian representation, we study the structure of the corresponding representation group, enabling us to generalize several of the results obtained in [14] for non-abelian representations of slim dense near hexagons. We show that with every polarized non-abelian representation of a slim near-polar space, there is an associated polarized projective embedding.

Keywords. Near-polar space, (universal, polarized) non-abelian representation, (universal) projective embedding, (minimal) polarized embedding, extraspecial 2-group, combinatorial group theory
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1 Introduction

Projective embeddings of point-line geometries have been widely studied. A projective embedding is a map from the point set of a point-line geometry $\mathcal{S}$ to the point set of a projective space $\mathrm{PG}(V)$ mapping lines of $\mathcal{S}$ to full lines of $\mathrm{PG}(V)$. In case $\mathcal{S}$ has three points per line, the underlying field of V is $\mathbb{F}_2$. For such a geometry, a projective embedding can alternatively be viewed as a map $p \mapsto \bar{v}_p$ from the point set of $\mathcal{S}$ to the nontrivial elements of the additive group of V such that if $\{p_1, p_2, p_3\}$ is a line of $\mathcal{S}$, then $\bar{v}_{p_3} = \bar{v}_{p_1} + \bar{v}_{p_2}$. This alternative point of view allows to generalize the notion of projective embeddings to so-called representations, where points of the slim geometry are no longer mapped to points of a projective space or to nonzero vectors of a vector space, but to involutions of a group R, the so-called representation group. If R is a non-abelian group, then the representation itself is also called *non-abelian*.

Non-abelian representations have been studied for a variety of geometries, including polar spaces and dense near polygons. In this paper, we study non-abelian representations for a class of parapolar spaces that includes both the polar spaces and the dense near

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polygons. This class of parapolar spaces was introduced by Shult in [15] and called near-polar spaces in [2].

In this paper, we restrict to those near-polar spaces that are slim and to a particular family of non-abelian representations, the so-called polarized ones. For polarized non-abelian representations of slim near-polar spaces, we derive quite some information about the representation groups. We show that these representation groups are closely related to extraspecial 2-groups, and obtain information about the centers of these groups. We also show that with every polarized non-abelian representation of a slim near-polar space, there is an associated polarized projective embedding (by taking a suitable quotient).

2 Preliminaries

2.1 Partial linear spaces and their projective embeddings

Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a point-line geometry with nonempty point set $\mathcal{P}$, line set $\mathcal{L}$ and incidence relation $\mathrm{I} \subseteq \mathcal{P} \times \mathcal{L}$.

We call $\mathcal{S}$ a *partial linear space* if every two distinct points of $\mathcal{S}$ are incident with at most one line. We call $\mathcal{S}$ *slim* if every line of $\mathcal{S}$ is incident with precisely three points. In the sequel, all considered point-line geometries will be partial linear spaces. We will often identify a line with the set of points incident with it. The incidence relation then corresponds to "containment".

A *subspace* of $\mathcal{S}$ is a set X of points with the property that if a line L has at least two of its points in X then all the points of L are in X. A *hyperplane* of $\mathcal{S}$ is a subspace, distinct from $\mathcal{P}$, meeting each line of $\mathcal{S}$.

The distance $\mathrm{d}(x_1, x_2)$ between two points x_1 and x_2 of $\mathcal{S}$ will be measured in the collinearity graph of $\mathcal{S}$. A path of minimal length between two points of $\mathcal{S}$ is called a *geodesic*. A subspace X of $\mathcal{S}$ is called *convex* if every point on a geodesic between two points of X is also contained in X. If x_1 and x_2 are two points of $\mathcal{S}$, then the intersection of all convex subspaces containing $\{x_1, x_2\}$ is denoted by $\langle x_1, x_2 \rangle$. (This is well-defined since $\mathcal{P}$ is a convex subspace.) The set $\langle x_1, x_2 \rangle$ itself is a convex subspace and hence it is the smallest convex subspace of $\mathcal{S}$ containing $\{x_1, x_2\}$. The subspace $\langle x_1, x_2 \rangle$ is called the *convex closure* of x_1 and x_2.

A *full projective embedding* of $\mathcal{S}$ is a map e from $\mathcal{P}$ to the point set of a projective space Σ satisfying: (i) $\langle e(\mathcal{P}) \rangle_\Sigma = \Sigma$; and (ii) $e(L) := \{e(x) \mid x \in L\}$ is a full line of Σ for every line L of $\mathcal{S}$. If e is moreover injective, then the full projective embedding e is called *faithful*. A full projective embedding e from $\mathcal{S}$ into a projective space Σ will shortly be denoted by $e : \mathcal{S} \to \Sigma$.

If N is the maximum dimension of a projective space into which $\mathcal{S}$ is fully embeddable, then the number $N+1$ is called the *embedding rank* of $\mathcal{S}$ and is denoted by $er(\mathcal{S})$. The number $er(\mathcal{S})$ is only defined when $\mathcal{S}$ is fully embeddable.

Two full projective embeddings $e_1 : \mathcal{S} \to \Sigma_1$ and $e_2 : \mathcal{S} \to \Sigma_2$ of $\mathcal{S}$ are called *isomorphic* (denoted by $e_1 \cong e_2$) if there exists an isomorphism θ from Σ_1 to Σ_2 such that $e_2 = \theta \circ e_1$.

Let $e : \mathcal{S} \to \Sigma$ be a full projective embedding of $\mathcal{S}$ and suppose α is a subspace of Σ satisfying the following two properties:

(Q1) $e(p) \notin \alpha$ for every point p of $\mathcal{S}$;

(Q2) $\langle \alpha, e(p_1) \rangle \neq \langle \alpha, e(p_2) \rangle$ for any two distinct points p_1 and p_2 of $\mathcal{S}$.

We denote by Σ/α the quotient projective space whose points are those subspaces of Σ that contain α as a hyperplane. Since α satisfies properties (Q1) and (Q2), it is easily verified that the map which associates with each point x of $\mathcal{S}$ the point $\langle \alpha, e(x) \rangle$ of Σ/α defines a full projective embedding of $\mathcal{S}$ into Σ/α. We call this embedding a *quotient* of e and denote it by e/α.

If $\mathcal{S}$ is a fully embeddable slim partial linear space, then by Ronan [12], $\mathcal{S}$ admits up to isomorphism a unique full projective embedding $\widetilde{e} : \mathcal{S} \to \widetilde{\Sigma}$ such that every full projective embedding e of $\mathcal{S}$ is isomorphic to a quotient of $\widetilde{e}$. The full projective embedding $\widetilde{e}$ is called the *universal embedding* of $\mathcal{S}$. We have $er(\mathcal{S}) = \dim(\widetilde{\Sigma}) + 1$. If $\mathcal{S}$ admits a faithful full projective embedding, then the universal embedding $\widetilde{e}$ of $\mathcal{S}$ is also faithful.

2.2 Near polygons

A partial linear space $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is called a *near polygon* if for every point p and every line L, there exists a unique point on L nearest to p. If $d \in \mathbb{N}$ is the maximal distance between two points of $\mathcal{S}$ (= the *diameter* of $\mathcal{S}$), then the near polygon is also called a *near* $2d$*-gon*. A near 0-gon is a point, a near 2-gon is a line. Near quadrangles are usually called *generalized quadrangles*. A near polygon is called *dense* if every line is incident with at least three points and if every two points at distance 2 have at least two common neighbors.

2.3 Polar and dual polar spaces

A partial linear space $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is called a *polar space* if for every point p and every line L, either one or all points of L are collinear with p. The *radical* of a polar space is the set of all points x which are collinear with any other point. A polar space is called *nondegenerate* if its radical is empty. A subspace of a polar space is said to be *singular* if any two of its points are collinear. The *rank* r of a nondegenerate polar space is the maximal length r of a chain $S_0 \subset S_1 \subset \cdots \subset S_r$ of singular subspaces where $S_0 = \emptyset$ and $S_i \neq S_{i+1}$ for all $i \in \{0, \ldots, r-1\}$. A nondegenerate polar space of rank 2 is just a nondegenerate generalized quadrangle. The *rank* of a singular subspace S of a nondegenerate polar space is the maximal length k of a chain $S_0 \subset S_1 \subset \cdots \subset S_k$ of singular subspaces such that $S_0 = \emptyset$, $S_k = S$ and $S_i \neq S_{i+1}$ for all $i \in \{0, \ldots, k-1\}$. Singular subspaces of rank r are also called *maximal singular subspaces*, those of rank $r-1$ are called *next-to-maximal singular subspaces*. A nondegenerate polar space is called *thick* if every line is incident with at least three points and if every next-to-maximal singular subspace is contained in at least three maximal singular subspaces.

With every (thick) polar space $\mathcal{S}$ of rank $r \geq 1$, there is associated a partial linear space Δ, which is called a *(thick) dual polar space of rank* r. The points of Δ are the maximal singular subspaces of $\mathcal{S}$, the lines of Δ are the next-to-maximal singular subspaces of $\mathcal{S}$, and incidence is reverse containment. Every thick dual polar space of rank r is a dense near $2r$-gon.

2.4 Near-polar spaces

In [15], Shult introduced a class of point-line geometries. These point-line geometries were called *near-polar spaces* in [2]. Near-polar spaces of diameter n are inductively defined as follows.

A near-polar space of diameter 0 is just a point and a near-polar space of diameter 1 is a line having at least three points. A near-polar space of diameter $n \geq 2$ is a point-line geometry $\mathcal{S}$ satisfying the following five axioms:

(E1) $\mathcal{S}$ is connected and its diameter is equal to n;

(E2) Every line of $\mathcal{S}$ is incident with at least three points;

(E3) Every geodesic $x_0, x_1, \ldots, x_k$ in $\mathcal{S}$ can be completed to a geodesic $x_0, x_1, \ldots, x_k, x_{k+1}, \ldots, x_n$ of length n;

(E4) For every point x of $\mathcal{S}$, the set H_x of points of $\mathcal{S}$ at distance at most $n-1$ from x is a hyperplane of $\mathcal{S}$;

(E5) If x_1 and x_2 are two points of $\mathcal{S}$ with $k := \mathrm{d}(x_1, x_2) < n$, then the subgeometry of $\mathcal{S}$ induced on the convex closure $\langle x_1, x_2 \rangle$ is a near-polar space of diameter k.

The hyperplane H_x mentioned in Axiom (E4) is called the *singular hyperplane of* $\mathcal{S}$ *with deepest point* x.

The near-polar spaces of diameter 2 are precisely the nondegenerate polar spaces in which each line is incident with at least three points. Every near-polar space of diameter $n \geq 2$ is a strong parapolar space in the sense of Cohen and Cooperstein [4]. The convex closures of the pairs of points at distance 2 from each other are also called *symplecta*.

Every thick dual polar space and more generally every dense near polygon is a near-polar space. The class of near-polar spaces also includes some half-spin geometries, some Grassmann spaces and some exceptional geometries, see Shult [15, Section 6].

We will now discuss full projective embeddings of near-polar spaces. Most of what we say here is based on De Bruyn [5].

Suppose $e : \mathcal{S} \to \Sigma$ is a full projective embedding of a near-polar space $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$. By Shult [15, Lemma 6.1(ii)], every singular hyperplane H_x, $x \in \mathcal{P}$, of $\mathcal{S}$ is a maximal (proper) subspace. This implies that $\Pi_x := \langle e(H_x) \rangle_\Sigma$ is either Σ or a hyperplane of Σ. The embedding e is called *polarized* if Π_x is a hyperplane of Σ for every point x of $\mathcal{S}$. If e is polarized, then the subspace $\mathcal{N}_e := \bigcap_{x \in \mathcal{P}} \Pi_x$ is called the *nucleus* of e. By De Bruyn [5, Proposition 3.4], the nucleus $\mathcal{N}_e$ satisfies the conditions (Q1) and (Q2) of Section 2.1 and the embedding $\bar{e} := e/\mathcal{N}_e$ is polarized.

Suppose now that $\mathcal{S}$ is a slim near-polar space. Then $\mathcal{S}$ admits a faithful full polarized embedding, see Brouwer & Shpectorov [3] or De Bruyn [5, Proposition 3.11(i)]. So, $\mathcal{S}$ also has a universal embedding $\widetilde{e} : \mathcal{S} \to \widetilde{\Sigma}$. This universal embedding necessarily is polarized and faithful. The embedding $\widetilde{e}/\mathcal{N}_{\widetilde{e}}$ is called *the minimal full polarized embedding* of $\mathcal{S}$. For every full polarized embedding e of $\mathcal{S}$, the embedding $\bar{e} = e/\mathcal{N}_e$ is isomorphic to $\widetilde{e}/\mathcal{N}_{\widetilde{e}}$. Every full embedding of $\mathcal{S}$ is isomorphic to $\widetilde{e}/\alpha$ for some subspace α of $\widetilde{\Sigma}$ satisfying Properties (Q1) and (Q2). If α_1 and α_2 are two subspaces of $\widetilde{\Sigma}$ satisfying (Q1) and (Q2), then $e/\alpha_1 \cong e/\alpha_2$ if and only if $\alpha_1 = \alpha_2$.

Suppose again that $\mathcal{S}$ is a slim near-polar space and that $e : \mathcal{S} \to \Sigma$ is a full polarized embedding of $\mathcal{S}$. This means that for every point x of $\mathcal{S}$, the subspace $\langle e(H_x)\rangle_\Sigma$ is a hyperplane Π_x of Σ. By De Bruyn [5, Propositions 3.5 and 3.11(ii)], the map $x \mapsto \Pi_x$ defines a polarized full embedding e^* of $\mathcal{S}$ into a subspace of the dual Σ^* of Σ. The embedding e^* is called the *dual embedding* of e. The nucleus of e^* is empty. So, the dual embedding e^* is isomorphic to the minimal full polarized embedding of $\mathcal{S}$.

2.5 Extraspecial 2-groups

In the sequel, we will adopt the following conventions when dealing with groups. For elements a, b of a group G, we write $[a, b] = a^{-1}b^{-1}ab$ and $a^b = b^{-1}ab$. For elements x, y, z of G, we have $[xy, z] = [x, z]^y[y, z]$ and $[x, yz] = [x, z][x, y]^z$. We denote by C_n the cyclic group of order n.

A finite 2-group G is called *extraspecial* if its Frattini subgroup $\Phi(G)$, commutator subgroup $G' = [G, G]$ and center $Z(G)$ coincide and have order 2. We refer to [7, Section 20, pp.78–79] or [8, Chapter 5, Section 5] for the properties of finite extraspecial 2-groups which we will mention now.

An extraspecial 2-group is of order 2^{1+2n} for some integer $n \geq 1$. Let D_8 and Q_8, respectively, denote the dihedral and the quaternion groups of order 8. A non-abelian 2-group of order 8 is extraspecial and is isomorphic to either D_8 or Q_8. If G is an extraspecial 2-group of order 2^{1+2n}, $n \geq 1$, then the exponent of G is 4 and G is either a central product of n copies of D_8, or a central product of $n-1$ copies of D_8 and one copy of Q_8. If the former (respectively, latter) case occurs, then the extraspecial 2-group is denoted by 2^{1+2n}_+ (respectively, 2^{1+2n}_-).

Suppose G is an extraspecial 2-group of order 2^{2n+1}, $n \geq 1$, and set $G' = \{1, \lambda\}$. Then $V = G/G'$ is an elementary abelian 2-group and hence can be regarded as a $2n$-dimensional vector space over $\mathbb{F}_2$. For all $x, y \in G$, we define

$$f(xG', yG') = \begin{cases} 0 \in \mathbb{F}_2 & \text{if } [x, y] = 1, \\ 1 \in \mathbb{F}_2 & \text{if } [x, y] = \lambda. \end{cases}$$

Then f is a nondegenerate alternating bilinear form on V. For all $x \in G$, $x^2 \in G' = \{1, \lambda\}$ as G/G' is elementary abelian. We define

$$q(xG') = \begin{cases} 0 \in \mathbb{F}_2 & \text{if } x^2 = 1, \\ 1 \in \mathbb{F}_2 & \text{if } x^2 = \lambda. \end{cases}$$

Then q is a nondegenerate quadratic form on V. The bilinear form associated with q is precisely f, that is,

$$q(xG'yG') = q(xG') + q(yG') + f(xG', yG')$$

for all $x, y \in G$. The nondegenerate quadratic form q defines a nonsingular quadric of $\mathrm{PG}(V)$, which is of hyperbolic type if $G = 2_+^{1+2n}$ or of elliptic type if $G = 2_-^{1+2n}$.

2.6 Representations of slim partial linear spaces

Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a slim partial linear space. A *representation* [10, p.525] of $\mathcal{S}$ is a pair (R, ψ), where R is a group and ψ is a mapping from $\mathcal{P}$ to the set of involutions in R, satisfying:

(i) R is generated by the image of ψ;

(ii) $\psi(x)\psi(y) = \psi(z)$ for every line $\{x, y, z\}$ of $\mathcal{S}$.

If $\{x, y, z\}$ is a line of $\mathcal{S}$, then condition (ii) implies that $\psi(x), \psi(y), \psi(z)$ are mutually distinct and $[\psi(x), \psi(y)] = [\psi(x), \psi(z)] = [\psi(y), \psi(z)] = 1$. The group R is called a *representation group* of $\mathcal{S}$. The representation (R, ψ) of $\mathcal{S}$ is called *faithful* if ψ is injective. Depending on whether R is abelian or not, the representation (R, ψ) itself will be called *abelian* or *non-abelian*. For an abelian representation, the representation group is an elementary abelian 2-group and hence can be considered as a vector space over the field $\mathbb{F}_2$ with two elements. In this case, the representation thus corresponds to a full projective embedding of $\mathcal{S}$.

We refer to [9] and [13, Sections 1 and 2] for representations of partial linear spaces with $p + 1$ points per line, where p is a prime.

Suppose $\mathcal{S}_1$ and $\mathcal{S}_2$ are two slim partial linear spaces. Let (R_i, ψ_i), $i \in \{1, 2\}$, be a representation of $\mathcal{S}_i$. The representations (R_1, ψ_1) and (R_2, ψ_2) are called *equivalent* if there exists an isomorphism θ_1 from $\mathcal{S}_1$ to $\mathcal{S}_2$ and a group isomorphism θ_2 from R_1 to R_2 such that $\psi_2 \circ \theta_1(x) = \theta_2 \circ \psi_1(x)$ for every point x of $\mathcal{S}_1$. If $\mathcal{S}_1 = \mathcal{S}_2$, then (R_1, ψ_1) and (R_2, ψ_2) are called *isomorphic* if there exists a group isomorphism θ from R_1 to R_2 such that $\psi_2(x) = \theta \circ \psi_1(x)$ for every point x of $\mathcal{S}_1$.

Suppose (R, ψ) is a representation of a slim partial linear space $\mathcal{S}$. Let N be a normal subgroup of R such that $\psi(x) \notin N$ for every point x of $\mathcal{S}$. For every point x of $\mathcal{S}$, let $\psi_N(x)$ denote the element $\psi(x)N$ of the quotient group R/N. Then $(R/N, \psi_N)$ is a representation of $\mathcal{S}$ which is called a *quotient* of (R, ψ). If (R_1, ψ_1) and (R_2, ψ_2) are two representations of $\mathcal{S}$, then (R_2, ψ_2) is isomorphic to a quotient of (R_1, ψ_1) if and only if there exists a group epimorphism θ from R_1 to R_2 such that $\psi_2(x) = \theta \circ \psi_1(x)$. If this is the case, then (R_2, ψ_2) is isomorphic to $(R_1/N, (\psi_1)_N)$, where $N = ker(\theta)$.

2.7 Polarized and universal representations of slim near-polar spaces

Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a slim near-polar space of diameter $n \geq 2$.

- A representation (R, ψ) of $\mathcal{S}$ is called *quasi-polarized* if $[\psi(x), \psi(y)] = 1$ for every two points x and y of $\mathcal{S}$ at distance at most $n - 1$ from each other.

- An abelian representation (R, ψ) of $\mathcal{S}$ is called *polarized* if the corresponding full projective embedding (in the sense of Section 2.6) is polarized.

- A non-abelian representation (R, ψ) of $\mathcal{S}$ is called *polarized* if $[\psi(x), \psi(y)] = 1$ for every two points x and y of $\mathcal{S}$ at distance at most $n-1$ from each other, that is, if the representation is quasi-polarized.

We will later show that with every polarized non-abelian representation of $\mathcal{S}$, there is an associated full polarized embedding of $\mathcal{S}$ (which is obtained by taking a suitable quotient).

(1) Let $\widetilde{R}_u$ be the group defined by the generators r_x, $x \in \mathcal{P}$, and the following relations:

- $r_x^2 = 1$, where $x \in \mathcal{P}$;
- $r_x r_y r_z = 1$, where $x, y, z \in \mathcal{P}$ such that $\{x, y, z\} \in \mathcal{L}$.

For every point x of $\mathcal{S}$, we define $\widetilde{\psi}_u(x) := r_x \in \widetilde{R}_u$.

(2) Let $\widetilde{R}_p$ be the group defined by the generators r_x, $x \in \mathcal{P}$, and the following relations:

- $r_x^2 = 1$, where $x \in \mathcal{P}$;
- $[r_x, r_y] = 1$, where $x, y \in \mathcal{P}$ such that $\mathrm{d}(x, y) < n$;
- $r_x r_y r_z = 1$, where $x, y, z \in \mathcal{P}$ such that $\{x, y, z\} \in \mathcal{L}$.

For every point x of $\mathcal{S}$, we define $\widetilde{\psi}_p(x) := r_x \in \widetilde{R}_p$.

(3) As mentioned before, $\mathcal{S}$ has faithful full projective embeddings. The universal projective embedding of $\mathcal{S}$ can be constructed as follows. Let V be a vector space over $\mathbb{F}_2$ with a basis B whose elements are indexed by the points of $\mathcal{P}$, say $B = \{\bar{v}_x \,|\, x \in \mathcal{P}\}$. Let W be the subspace of V generated by all vectors $\bar{v}_{x_1} + \bar{v}_{x_2} + \bar{v}_{x_3}$ where $\{x_1, x_2, x_3\}$ is some line of $\mathcal{S}$. Let $\widetilde{V}$ be the quotient vector space V/W and for every point x of $\mathcal{S}$, let $\widetilde{v_x}$ be the vector $\bar{v}_x + W$ of $\widetilde{V}$. The map $x \mapsto \langle \widetilde{v_x} \rangle$ defines a full projective embedding $\widetilde{e}$ of $\mathcal{S}$ into $\mathrm{PG}(\widetilde{V})$ which is isomorphic to the universal embedding of $\mathcal{S}$.

Proposition 2.1. (1) *$(\widetilde{R}_u, \widetilde{\psi}_u)$ is a faithful representation of $\mathcal{S}$.*

(2) *$(\widetilde{R}_p, \widetilde{\psi}_p)$ is a faithful polarized representation of $\mathcal{S}$.*

(3) *If (R, ψ) is a representation of $\mathcal{S}$, then (R, ψ) is isomorphic to a quotient of $(\widetilde{R}_u, \widetilde{\psi}_u)$.*

(4) *If (R, ψ) is a quasi-polarized representation of $\mathcal{S}$, then (R, ψ) is isomorphic to a quotient of $(\widetilde{R}_p, \widetilde{\psi}_p)$.*

Proof. We show that $(\widetilde{R}_p, \widetilde{\psi}_p)$ is a faithful representation. Since $\widetilde{v_x} + \widetilde{v_x} = W$ for every $x \in \mathcal{P}$, $(-\widetilde{v_x}) + (-\widetilde{v_y}) + \widetilde{v_x} + \widetilde{v_y} = W$ for all $x, y \in \mathcal{P}$ and $\widetilde{v_x} + \widetilde{v_y} + \widetilde{v_z} = W$ for every line $\{x, y, z\}$ of $\mathcal{S}$, we know from von Dyck's theorem that there exists an epimorphism from $\widetilde{R}_p$ to the additive group of $\widetilde{V}$ mapping r_x to $\widetilde{v_x}$ for every point x of $\mathcal{S}$. Since $\widetilde{e}$ is a full projective embedding, $\widetilde{v_x} \neq W$ and hence $r_x \neq_{\widetilde{R}_p} 1$ for every $x \in \mathcal{P}$. The latter fact implies that $(\widetilde{R}_p, \widetilde{\psi}_p)$ is a representation. Since $\widetilde{e}$ is a faithful projective embedding, we have $\widetilde{v_x} \neq \widetilde{v_y}$ for any two distinct points $x, y \in \mathcal{P}$. This implies that also $r_x \neq_{\widetilde{R}_p} r_y$. So, $(\widetilde{R}_p, \widetilde{\psi}_p)$ is a faithful representation.

In a completely similar way, one can show that $(\widetilde{R}_u, \widetilde{\psi}_u)$ is a faithful representation.

Claims (3) and (4) are straightforward consequences of von Dyck's theorem.

By construction, the representation $(\widetilde{R}_p, \widetilde{\psi}_p)$ is quasi-polarized and hence polarized if $\widetilde{R}_p$ is non-abelian. Suppose $\widetilde{R}_p$ is abelian. Then let e_p denote the full projective embedding of $\mathcal{S}$ corresponding to $(\widetilde{R}_p, \widetilde{\psi}_p)$. Let $(\widetilde{R}, \widetilde{\psi})$ denote the abelian representation corresponding to the universal projective embedding $\widetilde{e}$ of $\mathcal{S}$. By Claim (4), $(\widetilde{R}, \widetilde{\psi})$ is isomorphic to a quotient of $(\widetilde{R}_p, \widetilde{\psi}_p)$, and hence $\widetilde{e}$ is isomorphic to a quotient of e_p. As $\widetilde{e}$ cannot be a proper quotient of some full embedding of $\mathcal{S}$, the projective embeddings $\widetilde{e}$ and e_p are isomorphic. So, e_p is polarized, or equivalently, $(\widetilde{R}_p, \widetilde{\psi}_p)$ is polarized. □

The representation $(\widetilde{R}_u, \widetilde{\psi}_u)$ is called the *universal representation* of $\mathcal{S}$. The representation $(\widetilde{R}_p, \widetilde{\psi}_p)$ is called the *universal polarized representation* of $\mathcal{S}$.

From Section 5 (see Lemma 5.3) it will follow that there exists a $\widetilde{\lambda} \in \widetilde{R}_p$ such that $[\widetilde{\psi}_p(x), \widetilde{\psi}_p(y)] = \widetilde{\lambda}$ for every two points x and y at distance n from each other. If $\widetilde{\lambda} = 1$, then the universal polarized representation is abelian and hence corresponds to the universal projective embedding of $\mathcal{S}$ (which is always polarized). If $\widetilde{\lambda} \neq 1$, then the universal polarized representation of $\mathcal{S}$ is non-abelian. Both instances can occur. Indeed, the slim dual polar space $DW(2n-1, 2)$ and the slim dense near hexagons $Q(5,2) \times \mathbb{L}_3$, $Q(5,2) \otimes Q(5,2)$ have non-abelian polarized representations [6, 11], while no finite slim nondegenerate polar space has non-abelian representations [13, Theorem 1.5(i)]. Computer computations showed that other dense near polygons (like the dual polar space $DH(5,4)$) also have non-abelian polarized representations (in extraspecial 2-groups), but the authors are still looking for computer free descriptions of these representations.

3 Main results

For a finite slim near-polar space $\mathcal{S}$, we denote the embedding rank $er(\mathcal{S})$ also by $er^+(\mathcal{S})$. The vector space dimension of the minimal full polarized embedding of $\mathcal{S}$ will be denoted by $er^-(\mathcal{S})$. We will see in Proposition 4.2 that the number $er^-(\mathcal{S})$ is even. By [14], every non-abelian representation of a slim dense near hexagon is polarized. The following theorem is the first main theorem of this paper. It generalizes some results regarding slim dense near hexagons obtained in [14]. We will prove it in Section 5.

Theorem 3.1. *Suppose $\mathcal{S}$ is a finite slim near-polar space of diameter $n \geq 2$ having some polarized non-abelian representation (R, ψ). Then $n \geq 3$ and the universal polarized representation $(\widetilde{R}_p, \widetilde{\psi}_p)$ of $\mathcal{S}$ is also non-abelian. Moreover,*

(i) ψ is faithful and $\psi(x) \notin Z(R)$ for every point x of $\mathcal{S}$.

(ii) R *is a* 2*-group of exponent* 4*,* $|R'| = 2$ *and* $R' = \Phi(R) \subseteq Z(R)$*.*

(iii) *If* $|Z(R)| = 2^{l+1}$*, then* $Z(R)$ *is isomorphic*[1] *to either* $(C_2)^{l+1}$ *or* $(C_2)^{l-1} \times C_4$*.*

(iv) R *is of order* 2^β *for some integer* β *satisfying* $1+er^-(\mathcal{S}) \leq \beta \leq 1+er^+(\mathcal{S})$*. We have* $\beta = 1+er^-(\mathcal{S})$ *if and only if* R *is an extraspecial* 2*-group. We have* $\beta = 1+er^+(\mathcal{S})$ *if and only if* (R,ψ) *is isomorphic to* $(\widetilde{R}_p, \widetilde{\psi}_p)$*.*

(v) *If* $l = er^+(\mathcal{S}) - er^-(\mathcal{S})$*, then* $Z(\widetilde{R}_p)$ *has order* 2^{l+1} *and so is isomorphic to either* $(C_2)^{l+1}$ *or* $(C_2)^{l-1} \times C_4$*.*

In Section 6, we prove the following results.

Theorem 3.2. *Suppose* $\mathcal{S}$ *is a finite slim near-polar space of diameter* $n \geq 3$ *having polarized non-abelian representations. Then the following hold:*

(i) *The polarized representations of* $\mathcal{S}$ *are precisely the representations of the form* $(\widetilde{R}_p/N, (\widetilde{\psi}_p)_N)$*, where* N *is a subgroup of* $\widetilde{R}_p$ *contained in* $Z(\widetilde{R}_p)$*.*

(ii) *If* N_1 *and* N_2 *are two subgroups of* $Z(\widetilde{R}_p)$*, then the representations* $(\widetilde{R}_p/N_1, (\widetilde{\psi}_p)_{N_1})$ *and* $(\widetilde{R}_p/N_2, (\widetilde{\psi}_p)_{N_2})$ *of* $\mathcal{S}$ *are isomorphic if and only if* $N_1 = N_2$*.*

Remark. If $l = er^+(\mathcal{S}) - er^-(\mathcal{S})$, then we will see in Section 6 that Theorems 3.1(v) and 3.2 imply that the number of nonisomorphic polarized non-abelian representations of $\mathcal{S}$ is equal to the sum[2] $\sum_{i=0}^{l+1} \begin{bmatrix} l+1 \\ i \end{bmatrix}_2 - \sum_{i=0}^{l} \begin{bmatrix} l \\ i \end{bmatrix}_2$ if $Z(\widetilde{R}_p) \cong (C_2)^{l+1}$, and equal to $\sum_{i=0}^{l} \begin{bmatrix} l \\ i \end{bmatrix}_2 - \sum_{i=0}^{l-1} \begin{bmatrix} l-1 \\ i \end{bmatrix}_2$ if $Z(\widetilde{R}_p) \cong (C_2)^{l-1} \times C_4$.

Theorem 3.3. *Suppose* $\mathcal{S}$ *is a finite slim near-polar space of diameter* $n \geq 3$ *having polarized non-abelian representations. Set* $l := er^+(\mathcal{S}) - er^-(\mathcal{S})$*. Then* $\mathcal{S}$ *has a polarized non-abelian representation* (R,ψ) *with* $|R| = 2^{1+er^-(\mathcal{S})}$ *if and only if* $Z(\widetilde{R}_p) \cong (C_2)^{l+1}$*. If this is the case then there are up to isomorphism* 2^l *such representations. Moreover, the representation groups of any two of them are isomorphic (to either* $2_+^{1+er^-(\mathcal{S})}$ *or* $2_-^{1+er^-(\mathcal{S})}$*).*

Theorem 3.4. *Suppose* $\mathcal{S}$ *is a finite slim near-polar space of diameter* $n \geq 3$ *having polarized non-abelian representations. Suppose* $Z(\widetilde{R}_p) \cong C_2^{l-1} \times C_4$*, where* $l = er^+(\mathcal{S}) - er^-(\mathcal{S}) \geq 1$*. Then* $|R| \geq 2^{2+er^-(\mathcal{S})}$ *for every polarized non-abelian representation* (R,ψ) *of* $\mathcal{S}$*. Moreover, there are up to isomorphism* 2^{l-1} *polarized non-abelian representations* (R,ψ) *with* $|R| = 2^{2+er^-(\mathcal{S})}$*. If* (R,ψ) *is such a representation, then* $Z(R) \cong C_4$*.*

[1]If $l = 0$, then $(C_2)^{-1}$ is not defined. In this case, this sentence should be understood as "$Z(R)$ is isomorphic to C_2".

[2]The terms occurring in this sum are Gaussian binomial coefficients.

4 Some properties of near-polar spaces

Let $\mathcal{S}$ be a near-polar space of diameter $n \geq 1$. Two points x and y of $\mathcal{S}$ are called *opposite* if they are at a maximum distance from each other, that is, $\mathrm{d}(x, y) = n$. For two distinct points x, y of $\mathcal{S}$, we write $x \sim y$ if they are collinear.

Proposition 4.1. *Let $\mathcal{S}$ be a near-polar space of diameter $n \geq 1$. Let Γ be the graph whose vertices are the ordered pairs of opposite points of $\mathcal{S}$, with two distinct vertices (x_1, y_1) and (x_2, y_2) being adjacent whenever either $x_1 = x_2$ and $y_1 \sim y_2$; or $x_1 \sim x_2$ and $y_1 = y_2$. Then Γ is connected.*

Proof. Let (x_1, y_1) and (x_2, y_2) be two arbitrary vertices of Γ. We prove that (x_1, y_1) and (x_2, y_2) are contained in the same connected component of Γ.

For every point x of $\mathcal{S}$, the subgraph of the collinearity graph of $\mathcal{S}$ induced on the set of points at distance n from x is connected by Shult [15, Lemma 6.1(ii)]. So, if $x_1 = x_2$ or $y_1 = y_2$, then (x_1, y_1) and (x_2, y_2) belong to the same connected component of Γ.

Assume that $x_1 \neq x_2$ and $y_1 \neq y_2$. We prove that there exists a point y_3 at distance n from x_1 and x_2. If y_3 is such a point, then (a_1, b_1) and (a_2, b_2) belong to the same connected component of Γ for every $(a_1, b_1, a_2, b_2) \in \{(x_1, y_1, x_1, y_3), (x_1, y_3, x_2, y_3), (x_2, y_3, x_2, y_2)\}$, proving that (x_1, y_1) and (x_2, y_2) also belong to the same connected component of Γ.

The point y_3 alluded to in the previous paragraph is defined as a point of $\mathcal{S}$ at distance n from x_1 which lies as far away from x_2 as possible. Suppose $\mathrm{d}(y_3, x_2) \leq n - 1$ for such a point y_3. Then by Axiom (E3), there exists a point y_4 collinear with y_3 which lies at distance $k := \mathrm{d}(y_3, x_2) + 1$ from x_2. By Axiom (E5), a near-polar space of diameter k can be defined on the convex closure $\langle x_2, y_4 \rangle$. By applying Axiom (E4) to this near-polar space of diameter k, we see that the points of the line $y_3 y_4$ distinct from y_3 lie at distance $k = d(y_3, x_2) + 1$ from x_2. By Axioms (E2) and (E4) applied to $\mathcal{S}$, at least one of the points of $y_3 y_4 \setminus \{y_3\}$ lies at distance n from x_1. This contradicts the maximality of $\mathrm{d}(y_3, x_2)$. So, $\mathrm{d}(x_1, y_3) = \mathrm{d}(x_2, y_3) = n$ as we needed to prove. $\square$

Proposition 4.2. *Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a finite slim near-polar space of diameter $n \geq 1$, let V be a finite-dimensional vector space over $\mathbb{F}_2$ and let $e : \mathcal{S} \to \mathrm{PG}(V)$ be a full polarized embedding of $\mathcal{S}$ into $\mathrm{PG}(V)$. Then there exists a unique alternating bilinear form f on V for which the following holds:*

> *If x is a point of $\mathcal{S}$ and $\bar{v}$ is the unique vector of V for which $e(x) = \langle \bar{v} \rangle$, then $\mathrm{PG}(\bar{v}^{\perp_f})$ is a hyperplane of $\mathrm{PG}(V)$ which contains all the points $e(y)$, where $y \in \mathcal{P}$ and $d(x, y) \leq n - 1$, and none of the points $e(z)$, where $z \in \mathcal{P}$ and $d(x, z) = n$.*

If e is isomorphic to the minimal full polarized embedding of $\mathcal{S}$, then the alternating bilinear form f is nondegenerate and hence $er^-(\mathcal{S}) = \dim(V)$ is even.

Proof. For every point x of $\mathcal{S}$, let Π_x denote the unique hyperplane of $\mathrm{PG}(V)$ which contains all the points $e(y)$, where $y \in \mathcal{P}$ and $\mathrm{d}(x, y) \leq n - 1$, and none of the points $e(z)$, where $z \in \mathcal{P}$ and $\mathrm{d}(x, z) = n$.

(1) We first prove the existence of the alternating bilinear form in the case e is isomorphic to the minimal full polarized embedding of $\mathcal{S}$. Then $\bigcap_{x \in \mathcal{P}} \Pi_x = \emptyset$.

Recall that the map $x \mapsto \Pi_x$ defines a full projective embedding e^* of $\mathcal{S}$ into the dual $\mathrm{PG}(V)^*$ of $\mathrm{PG}(V)$. This embedding e^* is called the dual embedding of e and is isomorphic to the minimal full polarized embedding of $\mathcal{S}$. So, there exists an isomorphism ϕ from $\mathrm{PG}(V)$ to $\mathrm{PG}(V)^*$ mapping $e(x)$ to Π_x for every point x of $\mathcal{S}$.

We prove that ϕ is a polarity of $\mathrm{PG}(V)$, or equivalently that $\phi^2 = 1$. Since ϕ^2 defines a collineation of $\mathrm{PG}(V)$, it suffices to prove that $\phi^2(p) = p$ for every point p belonging to a generating set of $\mathrm{PG}(V)$. So, it suffices to prove that $\phi(\Pi_x) = \phi^2(e(x)) = e(x)$ for every point x of $\mathcal{P}$. If y is a point at distance at most $n-1$ from x, then $e(y) \in \Pi_x$ implies that $\phi(\Pi_x) \in \Pi_y$. Hence, $\phi(\Pi_x)$ is contained in the intersection I of all hyperplanes Π_y, where $y \in \mathcal{P}$ and $\mathrm{d}(x,y) \leq n-1$. Since e^* is polarized, the hyperplanes Π_y, where $y \in \mathcal{P}$ and $\mathrm{d}(x,y) \leq n-1$, generate a hyperplane of $\mathrm{PG}(V)^*$. So, I is a singleton. Since $e(x) \in \Pi_y$ for every $y \in \mathcal{P}$ satisfying $\mathrm{d}(x,y) \leq n-1$, we also have $e(x) \in I$. Hence, $\phi(\Pi_x) = e(x)$ as we needed to prove.

We now prove that ϕ is a symplectic polarity of $\mathrm{PG}(V)$. To that end, it suffices to prove that $p \in p^\phi$ for every point p of $\mathrm{PG}(V)$. Since $\mathrm{PG}(V) = \langle Im(e) \rangle$, it suffices to prove the following:

(a) $e(x) \in e(x)^\phi$ for every $x \in \mathcal{P}$;

(b) if $L = \{p_1, p_2, p_3\}$ is a line of $\mathrm{PG}(V)$ such that $p_1 \in p_1^\phi$ and $p_2 \in p_2^\phi$, then also $p_3 \in p_3^\phi$.

Since $e(x)^\phi = \Pi_x$ and $e(x) \in \Pi_x$, Property (a) clearly holds. If $p_2 \in p_1^\phi$, then $\{p_3\} \subseteq L \subseteq p_1^\phi \cap p_2^\phi = L^\phi \subseteq p_3^\phi$. If $p_2 \not\in p_1^\phi$, then $p_1^\phi = \langle L^\phi, p_1 \rangle$, $p_2^\phi = \langle L^\phi, p_2 \rangle$ and p_3^ϕ is the unique hyperplane through L^ϕ distinct from p_1^ϕ and p_2^ϕ, implying that $p_3^\phi = \langle L^\phi, p_3 \rangle$. So, Property (b) also holds in that case.

If f is the nondegenerate alternating bilinear form of V corresponding to the symplectic polarity ϕ of $\mathrm{PG}(V)$, then f satisfies the required conditions.

(2) Suppose e is not isomorphic to the minimal full polarized embedding of $\mathcal{S}$. Let α be the intersection of all subspaces Π_x, $x \in \mathcal{P}$, let U be the subspace of V corresponding to α and let W be a subspace of V such that $V = U \oplus W$. For every point x of $\mathcal{S}$, let $e'(x)$ denote the unique point of $\mathrm{PG}(W)$ contained in $\langle \alpha, e(x) \rangle$. Then e' is isomorphic to the minimal full polarized embedding of $\mathcal{S}$. By part (1) above, we know that there exists a nondegenerate alternating bilinear form f_W on W such that if x is a point of $\mathcal{S}$ and $\bar{w}$ is the unique vector of W for which $e'(x) = \langle \bar{w} \rangle$, then the hyperplane $\mathrm{PG}(\bar{w}^{\perp_{f_W}})$ of $\mathrm{PG}(W)$ contains all points $e'(y)$, where $y \in \mathcal{P}$ and $\mathrm{d}(x,y) \leq n-1$, and none of the points $e(z)$, where $z \in \mathcal{P}$ and $\mathrm{d}(x,z) = n$. Now, for all $\bar{u}_1, \bar{u}_2 \in U$ and all $\bar{w}_1, \bar{w}_2 \in W$, we define

$$f(\bar{u}_1 + \bar{w}_1, \bar{u}_2 + \bar{w}_2) := f_W(\bar{w}_1, \bar{w}_2).$$

Then f is an alternating bilinear form on V.

Suppose x is a point of $\mathcal{S}$. Let $\bar{v}$ be the unique vector of V for which $e(x) = \langle \bar{v} \rangle$ and let $\bar{w}$ be the unique vector of W for which $e'(x) = \langle \bar{w} \rangle$. Then $\langle \bar{w} \rangle = \langle U, \bar{v} \rangle \cap W$. We also have $\langle \bar{v}^{\perp_f} \rangle = \langle U, \bar{w}^{\perp_{f_W}} \rangle$. Since $\mathrm{PG}(\bar{w}^{\perp_{f_W}})$ contains all points $e'(y)$, where $y \in \mathcal{P}$ and $\mathrm{d}(x,y) \leq n-1$, and none of the points $e'(z)$, where $z \in \mathcal{P}$ and $\mathrm{d}(x,z) = n$, we have that $\mathrm{PG}(\bar{v}^{\perp_f})$ contains all points $e(y)$, where $y \in \mathcal{P}$ and $\mathrm{d}(x,y) \leq n-1$, and none of the points $e(z)$, where $z \in \mathcal{P}$ and $\mathrm{d}(x,z) = n$. So, the alternating bilinear form f satisfies the required conditions.

(3) We now prove the uniqueness of the alternating bilinear form. Suppose f_1 and f_2 are two alternating bilinear forms on V satisfying the required conditions. Then $g := f_1 - f_2$ is also an alternating bilinear form on V.

Suppose x_1 and x_2 are two points of $\mathcal{S}$ and let $\bar{v}_i$, $i \in \{1,2\}$, be the unique vector of V for which $e(x) = \langle \bar{v}_i \rangle$. If $\mathrm{d}(x_1,x_2) \leq n-1$, then $f_1(\bar{v}_1,\bar{v}_2) = 0 = f_2(\bar{v}_1,\bar{v}_2)$ and hence $g(\bar{v}_1,\bar{v}_2) = 0$. If $\mathrm{d}(x_1,x_2) = n$, then $f_1(\bar{v}_1,\bar{v}_2) = 1 = f_2(\bar{v}_1,\bar{v}_2)$ and hence $g(\bar{v}_1,\bar{v}_2) = 0$. Since $\mathrm{PG}(V) = \langle e(x) \,|\, x \in \mathcal{P} \rangle$, we get $g = 0$. Hence $f_1 = f_2$. □

5 Structure of the representation groups

Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a finite slim near-polar space of diameter $n \geq 2$ and suppose (R, ψ) is a polarized non-abelian representation of $\mathcal{S}$. In this section, we will prove all the claims mentioned in Theorem 3.1.

Lemma 5.1. *We have $n \geq 3$.*

Proof. By [13, Theorem 1.5(i)], every representation of a finite slim nondegenerate polar space is abelian. So, $\mathcal{S}$ is not a polar space and hence $n \geq 3$. □

Lemma 5.2. *The universal polarized representation $(\widetilde{R}_p, \widetilde{\psi}_p)$ is non-abelian. Moreover, $|\widetilde{R}_p| \geq 2^{1+er^+(\mathcal{S})}$.*

Proof. As (R, ψ) is a quotient of $(\widetilde{R}_p, \widetilde{\psi}_p)$, the universal polarized representation $(\widetilde{R}_p, \widetilde{\psi}_p)$ itself should also be non-abelian. Since the abelian representation corresponding to the universal projective embedding of $\mathcal{S}$ is quasi-polarized, it should be a quotient of $(\widetilde{R}_p, \widetilde{\psi}_p)$ by Proposition 2.1(4). This implies that $|\widetilde{R}_p| \geq 2^{1+er^+(\mathcal{S})}$. □

Later (Lemma 5.12) we will show that $|\widetilde{R}_p| = 2^{1+er^+(\mathcal{S})}$.

Lemma 5.3. *Let Γ be the graph as defined in Proposition 4.1. Then there exists an involution $\lambda \in R$ such that $\lambda = [\psi(x), \psi(y)]$ for every vertex (x,y) of Γ.*

Proof. We first show that $[\psi(x_1), \psi(y_1)] = [\psi(x_2), \psi(y_2)]$ for any two adjacent vertices (x_1, y_1) and (x_2, y_2) of Γ. Suppose $x_1 = x_2$ and $y_1 \sim y_2$. Let y_3 be the unique third point of the line $y_1 y_2$. Then $\mathrm{d}(x_1, y_3) = n-1$. Since $\psi(y_3)$ commutes with $\psi(x_1)$ and $\psi(y_2)$, we have $[\psi(x_1), \psi(y_1)] = [\psi(x_1), \psi(y_2)\psi(y_3)] = [\psi(x_1), \psi(y_2)]$. The case where $x_1 \sim x_2$ and $y_1 = y_2$ is treated in a similar way.

Now let x and y be two opposite points of $\mathcal{S}$ and set $\lambda = [\psi(x), \psi(y)]$. By Proposition 4.1, Γ is connected. So, by the first paragraph, λ is independent of the opposite points x and y. Also $\lambda \neq 1$ since (R, ψ) is polarized and non-abelian. Since $\lambda^{-1} = [\psi(x), \psi(y)]^{-1} = [\psi(y), \psi(x)] = \lambda$, we get $\lambda^2 = 1$. □

Corollary 5.4. *$\langle \psi(x), \psi(y) \rangle \cong D_8$ for every two opposite points x and y of $\mathcal{S}$.*

Proof. Since x and y are opposite points, $(\psi(x)\psi(y))^2 = [\psi(x), \psi(y)] = \lambda$ by Lemma 5.3 and so $\psi(x)\psi(y)$ is of order 4. Hence $\langle \psi(x), \psi(y) \rangle \cong D_8$ [1, 45.1]. $\square$

Lemma 5.5. *ψ is faithful and $\psi(x) \notin Z(R)$ for every point x of $\mathcal{S}$.*

Proof. Let x and y be two distinct points of $\mathcal{S}$ and let z be a point that is opposite to x, but not to y (such a point exists by Axiom (E3)). Then $[\psi(y), \psi(z)] = 1$ and $[\psi(x), \psi(z)] = \lambda \neq 1$ by Lemma 5.3. Hence, $\psi(x) \neq \psi(y)$.

For a given point x, choose a point w opposite to x. Then $[\psi(x), \psi(w)] = \lambda \neq 1$. So $\psi(x) \notin Z(R)$. $\square$

Lemma 5.6. *$R' = \{1, \lambda\} \subseteq Z(R)$.*

Proof. Set $T = \langle \lambda \rangle = \{1, \lambda\}$. Then $T \subseteq R'$ by Lemma 5.3. We first show that $T \subseteq Z(R)$. Since $R = \langle \psi(x) \mid x \in \mathcal{P} \rangle$, it is sufficient to prove that $[\psi(x), \lambda] = 1$ for every point x of $\mathcal{S}$. Let y be a point of $\mathcal{S}$ opposite to x. Since $\psi(x), \psi(y), \lambda = [\psi(x), \psi(y)]$ all are involutions, a direct calculation shows that $[\psi(x), \lambda] = 1$.

Being a central subgroup, T is normal in R. In the quotient group R/T, the generators $\psi(x)T$, $x \in \mathcal{P}$, commute pairwise. So R/T is abelian and hence $R' \subseteq T$. $\square$

Corollary 5.7. *For $a, b, c \in R$, $[ab, c] = [a, c][b, c]$ and $[a, bc] = [a, b][a, c]$.*

Proof. By Lemma 5.6, we have $[ab, c] = [a, c]^b[b, c] = [a, c] \cdot [b, c]$ and $[a, bc] = [a, c] \cdot [a, b]^c = [a, b] \cdot [a, c]$. $\square$

Lemma 5.8. (1) *For every $r \in R$, we have $r^2 \in \{1, \lambda\}$.*

(2) *R is a finite 2-group of exponent 4 and $R' = \Phi(R)$.*

Proof. We show that $r^2 \in \{1, \lambda\}$ for every $r \in R \backslash \{1\}$. Set $r = \psi(x_1)\psi(x_2) \cdots \psi(x_n)$, where $x_1, x_2, \ldots, x_n$ are points of $\mathcal{S}$. Since $\lambda^2 = 1$, $\psi(x_i)^2 = 1$ and $[\psi(x_i), \psi(x_j)] \in \{1, \lambda\} \subseteq Z(R)$ for all $i, j \in \{1, \ldots, n\}$, we have $r^2 \in \{1, \lambda\}$. It follows that $r^4 = 1$. Since R is non-abelian, the exponent of R cannot be 2 and hence equals 4.

Since $R = \langle \psi(x) \,|\, x \in \mathcal{P} \rangle$ and $\mathcal{S}$ is finite, the quotient group $R/R' = \langle \psi(x)R' \,|\, x \in \mathcal{P} \rangle$ is a finite elementary abelian 2-group. Since $|R'| = 2$ by Lemma 5.6, we get that R is also a finite 2-group. Then the two facts that R' is the smallest normal subgroup K of R such that R/K is abelian and that $\Phi(R)$ is the smallest normal subgroup H of R such that R/H is elementary abelian [1, 23.2, p.105] imply $R' = \Phi(R)$. $\square$

Since the quotient group R/R' is an elementary abelian 2-group, we can consider $V = R/R'$ as a vector space over $\mathbb{F}_2$. For every point x of $\mathcal{S}$, let $e(x)$ be the projective point $\langle \psi(x)R' \rangle$ of $\mathrm{PG}(V)$. Notice that, by Lemmas 5.5 and 5.6, $\psi(x)R'$ is indeed a nonzero vector of V.

Lemma 5.9. *The map e defines a faithful full projective embedding of $\mathcal{S}$ into $\mathrm{PG}(V)$.*

Proof. Since $R/R' = \langle \psi(x)R' \mid x \in \mathcal{P} \rangle$, the image of e generates $\mathrm{PG}(V)$.

We prove that $\psi(x_1)R' \neq \psi(x_2)R'$ for every two distinct points x_1 and x_2 of $\mathcal{S}$. Suppose to the contrary that $\psi(x_1)R' = \psi(x_2)R'$. Since ψ is faithful by Lemma 5.5, we have $\psi(x_1) \neq \psi(x_2)$. So, $\psi(x_1) = \psi(x_2)\lambda$. By Axiom (E3), there exists a point x_3 opposite to x_1, but not to x_2. Then $\lambda = [\psi(x_1), \psi(x_3)] = [\psi(x_2)\lambda, \psi(x_3)] = [\psi(x_2), \psi(x_3)] = 1$, a contradiction.

Let $L = \{x_1, x_2, x_3\}$ be a line of $\mathcal{S}$. We have $e(x_i) = \langle \psi(x_i)R' \rangle$, for $i \in \{1, 2, 3\}$. Since $\psi(x_1)\psi(x_2) = \psi(x_3)$, we have $\psi(x_3)R' = \psi(x_1)R'\psi(x_2)R'$. Hence $\{e(x_1), e(x_2), e(x_3)\}$ is a line of $\mathrm{PG}(V)$. $\square$

Definition. For all $a, b \in R$, we define

$$f(aR', bR') = \begin{cases} 1 & \text{if } [a,b] = \lambda, \\ 0 & \text{if } [a,b] = 1. \end{cases}$$

Since $R' = \{1, \lambda\} \subseteq Z(R)$, the map $f : V \times V \to \mathbb{F}_2$ is well-defined.

Lemma 5.10. *The map $f : V \times V \to \mathbb{F}_2$ is an alternating bilinear form of V.*

Proof. The claim that f is an alternating bilinear form follows from the following facts.

- Since $[a,a] = [1,a] = [a,1] = 1$, we have $f(aR', aR') = f(R', aR') = f(aR', R') = 0$ for all $a \in R$.
- Let $x_1, x_2, y_1 \in R$. Since $[x_1x_2, y_1] = [x_1, y_1][x_2, y_1]$, we have $f(x_1R'x_2R', y_1R') = f(x_1R', y_1R') + f(x_2R', y_1R')$.
- Let $x_1, y_1, y_2 \in R$. Since $[x_1, y_1y_2] = [x_1, y_1][x_1, y_2]$, we have $f(x_1R', y_1R'y_2R') = f(x_1R', y_1R') + f(x_1R', y_2R')$. $\square$

Lemma 5.11. *The embedding e of $\mathcal{S}$ into* $\mathrm{PG}(V)$ *is polarized.*

Proof. For every point x of $\mathcal{S}$, we define a certain subspace Π_x of $\mathrm{PG}(V)$. Let $\bar{v}$ be the unique vector of V for which $e(x) = \langle \bar{v} \rangle$. Then Π_x is the subspace of $\mathrm{PG}(V)$ corresponding[3] to the subspace $\bar{v}^{\perp_f}$ of V.

Let x_1 and x_2 be two points of $\mathcal{S}$ and let $\bar{v}_i = \psi(x_i)R'$, $i \in \{1, 2\}$. So $e(x_i) = \langle \bar{v}_i \rangle$. Then the following holds:

$$\begin{aligned} \mathrm{d}(x_1, x_2) \leq n-1 \quad &\Leftrightarrow \quad [\psi(x_1), \psi(x_2)] = 1 \\ &\Leftrightarrow \quad f(\psi(x_1)R', \psi(x_2)R') = 0 \\ &\Leftrightarrow \quad f(\bar{v}_1, \bar{v}_2) = 0 \\ &\Leftrightarrow \quad \bar{v}_2 \in \bar{v}_1^{\perp_f} \\ &\Leftrightarrow \quad e(x_2) \in \Pi_{x_1}. \end{aligned}$$

Now from the above it follows that $\Pi_x = \langle e(H_x) \rangle_{\mathrm{PG}(V)}$ is a hyperplane of $\mathrm{PG}(V)$ for every point x of $\mathcal{S}$, where H_x is the singular hyperplane of $\mathcal{S}$ with deepest point x. So e is polarized. $\square$

[3] The map $\phi_x : R \mapsto R'$ defined by $\phi_x(r) = [\psi(x), r]$ is a homomorphism (see Corollary 5.7) which is surjective. The kernel of ϕ_x is $C_R(\psi(x))$ which has index 2 in R by the first isomorphism theorem. Then $\bar{v}^{\perp_f}$ is precisely the image of $C_R(\psi(x))$ in V under the canonical homomorphism $R \to V; r \mapsto rR'$.

Definition. We call e the *full polarized embedding of $\mathcal{S}$ associated with the non-abelian representation* (R, ψ).

Lemma 5.12. (1) *R is of order 2^β for some β satisfying $1 + er^-(\mathcal{S}) \le \beta \le 1 + er^+(\mathcal{S})$.*

(2) *The following are equivalent:*

- *(R, ψ) is isomorphic to $(\widetilde{R}_p, \widetilde{\psi}_p)$;*
- *$\beta = 1 + er^+(\mathcal{S})$;*
- *e is isomorphic to the universal embedding of $\mathcal{S}$.*

(3) *The following are equivalent:*

- *$\beta = 1 + er^-(\mathcal{S})$;*
- *R is an extraspecial 2-group;*
- *e is isomorphic to the minimal full polarized embedding of $\mathcal{S}$.*

Proof. By Lemmas 5.9 and 5.11, e defines a full polarized embedding of $\mathcal{S}$ into $\mathrm{PG}(V)$. So, $er^-(\mathcal{S}) \le \dim(V) \le er^+(\mathcal{S})$. Since $|R/R'| = 2^{\beta-1}$, we have $\dim(V) = \beta - 1$ and hence $1 + er^-(\mathcal{S}) \le \beta \le 1 + er^+(\mathcal{S})$. The lower bound occurs if and only if e is isomorphic to the minimal full polarized embedding of $\mathcal{S}$. The upper bound occurs if and only if e is isomorphic to the universal embedding of $\mathcal{S}$. From Lemma 5.2, the upper bound and the fact that (R, ψ) is isomorphic to a quotient of $(\widetilde{R}_p, \widetilde{\psi}_p)$, it follows that $\beta = 1 + er^+(\mathcal{S})$ if and only if (R, ψ) is isomorphic to $(\widetilde{R}_p, \widetilde{\psi}_p)$.

Now, R is extraspecial if and only if $R' = Z(R)$, that is, if and only if the alternating bilinear form f is nondegenerate. For every point x of $\mathcal{S}$, let $\bar{v}_x$ be the unique vector of V for which $e(x) = \langle \bar{v}_x \rangle$. Then $\langle e(H_x) \rangle = \mathrm{PG}(\langle \bar{v}_x \rangle^{\perp_f})$ (see the proof of Lemma 5.11) is a hyperplane of $\mathrm{PG}(V)$ for every point x of $\mathcal{S}$. It follows that f is nondegenerate if and only if the nucleus $\mathcal{N}_e$ of e is empty, that is, if and only if e is a minimal full polarized embedding of $\mathcal{S}$. Thus R is extraspecial if and only if $er^-(\mathcal{S}) = \dim(V) = \beta - 1$. □

For every $r \in R$, we set $\theta(r) := rR' \in V$. Observe that if $r_1, r_2 \in R$, then $f(\theta(r_1), \theta(r_2)) = 0$ if $[r_1, r_2] = 1$ and $f(\theta(r_1), \theta(r_2)) = 1$ if $[r_1, r_2] = \lambda$. We denote by $\mathcal{R}_f$ the radical of the alternating bilinear form f. The subspace of $\mathrm{PG}(V)$ corresponding to $\mathcal{R}_f$ is precisely $\mathcal{N}_e$.

Lemma 5.13. *If N is a subgroup of R contained in $Z(R)$, then $\theta(N) \subseteq \mathcal{R}_f$.*

Proof. Let $g \in N$ and $h \in R$. Then $[g, h] = 1$ implies that $f(\theta(g), \theta(h)) = 0$. Since $\theta(R) = V$, it follows that $\theta(g) \in \mathcal{R}_f$. Hence, $\theta(N) \subseteq \mathcal{R}_f$. □

Lemma 5.14. *If U is a subspace of $\mathcal{R}_f$, then $\theta^{-1}(U)$ is a subgroup of R contained in $Z(R)$. If $\dim(U) = l$, then $\theta^{-1}(U)$ is an abelian subgroup isomorphic to either C_2^{l+1} or $C_2^{l-1} \times C_4$.*

Proof. Clearly, $\theta^{-1}(U)$ is a subgroup of R. If $g \in \theta^{-1}(U)$ and $h \in R$, then we have $f(\theta(g), \theta(h)) = 0$ since $\theta(g) \in U \subseteq \mathcal{R}_f$. This implies that $[g, h] = 1$. So, $\theta^{-1}(U) \subseteq Z(R)$. In particular, $\theta^{-1}(U)$ is abelian. By the classification of finite abelian groups, $\theta^{-1}(U)$ is isomorphic to the direct product of a number of cyclic groups. Since the exponent of R is equal to 4, each of these cyclic groups has order 2 or 4. Lemma 5.8(1) then implies that there is at most one cyclic group of order 4 in this direct product. If $\dim(U) = l$, then $|\theta^{-1}(U)| = 2^{l+1}$ and hence $\theta^{-1}(U)$ must be isomorphic to either $(C_2)^{l+1}$ or $(C_2)^{l-1} \times C_4$. $\square$

Corollary 5.15. (1) *We have* $\mathcal{R}_f = \theta(Z(R))$.

(2) *We have* $|Z(R)| = |R| \cdot 2^{-er^-(\mathcal{S})}$.

(3) *If* $l = er^+(\mathcal{S}) - er^-(\mathcal{S})$, *then the center* $Z(\widetilde{R}_p)$ *of* $\widetilde{R}_p$ *is isomorphic to either* C_2^{l+1} *or* $C_2^{l-1} \times C_4$.

Proof. (1) By Lemmas 5.13 and 5.14, we have $\theta(Z(R)) \subseteq \mathcal{R}_f$ and $\theta^{-1}(\mathcal{R}_f) \subseteq Z(R)$, implying that $\mathcal{R}_f = \theta(Z(R))$.

(2) Since $\lambda \in Z(R)$, we have $|Z(R)| = 2 \cdot |\theta(Z(R))| = 2 \cdot |\mathcal{R}_f| = 2 \cdot |V| \cdot 2^{-er^-(\mathcal{S})} = |R| \cdot 2^{-er^-(\mathcal{S})}$.

(3) This follows from Lemma 5.14 and Claim (2). $\square$

We will now study the quotient representations of (R, ψ). For such quotient representations, we need normal subgroups N of R such that $\psi(x) \notin N$ for every point x of $\mathcal{S}$.

Lemma 5.16. *The normal subgroups of* R *are the following:*

(1) *the subgroups of* R *containing* λ;

(2) *the subgroups of* R *not containing* λ *that are contained in* $Z(R)$.

Proof. Clearly, the subgroups in (1) and (2) above are normal in R. Suppose N is a normal subgroup of R not containing λ. For all $n \in N$ and all $r \in R$, we then have $[n, r] \in N \cap R' = N \cap \{1, \lambda\} = \{1\}$, implying that $N \subseteq Z(R)$. $\square$

Remark. If N is a (normal) subgroup of R contained in $Z(R)$, then the condition that $\psi(x) \notin N$ for every point x of $\mathcal{S}$ is automatically satisfied by Lemma 5.5.

Lemma 5.17. *Let* N *be a normal subgroup of* R *such that* $\psi(x) \notin N$ *for every point* x *of* $\mathcal{S}$. *Then the quotient representation* $(R/N, \psi_N)$ *is abelian if and only if* $\lambda \in N$.

Proof. The representation $(R/N, \psi_N)$ is abelian if and only if $[\psi(x)N, \psi(y)N] = [\psi(x), \psi(y)] \cdot N = N$ for every two points x and y of $\mathcal{S}$, that is, if and only if $\lambda \in N$. $\square$

Lemma 5.18. *Let* N *be a (necessarily normal) subgroup of* R *containing* λ *such that* $\psi(x) \notin N$ *for every point* x *of* $\mathcal{S}$. *Set* $U := \theta(N)$, *and let* α *denote the subspace of* $\mathrm{PG}(V)$ *corresponding to* U. *Then* $e(x) \notin \alpha$ *for every point* x *of* $\mathcal{S}$, *and the full projective embedding of* $\mathcal{S}$ *corresponding to the abelian representation* $(R/N, \psi_N)$ *is isomorphic to* e/α.

Proof. If $x \in \mathcal{P}$, then the facts that $\psi(x) \notin N$, $\lambda \in N$ and $R' = \{1, \lambda\}$ imply that $\psi(x)R'$ does not belong to the set $\{nR' \,|\, n \in N\}$, that is, $e(x) \notin \alpha$. So, e/α is well-defined as a projective embedding.

Consider the quotient vector space V/U and the associated projective space $\mathrm{PG}(V/U)$. The map which sends each point x of $\mathcal{S}$ to the point $\langle (\psi(x)R') \cdot U \rangle$ of $\mathrm{PG}(V/U)$ is then a full projective embedding isomorphic to e/α. The map

$$\phi : R/N \to V/U;\ rN \mapsto \theta(r) \cdot U \qquad (r \in R),$$

which is well-defined as $U = \theta(N)$, is an isomorphism of groups. (The injectivity of the map follows from the fact that $\theta^{-1}(U) = N$ which is a consequence of the fact that $\lambda \in N$.) The fact that $\phi \circ \psi_N(x) = \phi(\psi(x) \cdot N) = \theta(\psi(x)) \cdot U = (\psi(x)R') \cdot U$ for every point x of $\mathcal{S}$ implies that the full projective embedding of $\mathcal{S}$ corresponding to the abelian representation $(R/N, \psi_N)$ is isomorphic to e/α. □

Lemma 5.19. *Let N be a subgroup of R contained in $Z(R)$. Set $U := \theta(N) \subseteq \mathcal{R}_f$, and let $\alpha \subseteq \mathcal{N}_e$ denote the subspace of $\mathrm{PG}(V)$ corresponding to U. Then:*

(1) *If $\lambda \notin N$, then the projective embedding associated with the non-abelian representation $(R/N, \psi_N)$ is isomorphic to e/α.*

(2) *The representation $(R/N, \psi_N)$ is polarized.*

Proof. (1) Consider the normal subgroup $\overline{N} := \langle N, \lambda \rangle \subseteq Z(R)$ of R. By Lemma 5.5, this group does not contain any element $\psi(x)$ where $x \in \mathcal{P}$. We have $\theta(\overline{N}) = \theta(N) = U$. So, by Lemma 5.18, the projective embedding corresponding to the abelian representation $(R/\overline{N}, \psi_{\overline{N}})$ is isomorphic to e_α where α is the subspace of $\mathrm{PG}(V)$ corresponding to U. It is straightforward to verify that the projective embedding associated with the non-abelian representation $(R/N, \psi_N)$ is isomorphic to the projective embedding corresponding to the abelian representation $(R/\overline{N}, \psi_{\overline{N}})$. (Observe that $R/\overline{N} \cong (R/N)/(\overline{N}/N)$ and $(R/N)' = \overline{N}/N$.)

(2) If $(R/N, \psi_N)$ is non-abelian, then the fact that $[\psi(x)N, \psi(y)N] = [\psi(x), \psi(y)] \cdot N = N$ for any two non-opposite points x and y implies that $(R/N, \psi_N)$ is polarized. If $(R/N, \psi_N)$ is abelian, then the fact that $\alpha \subseteq \mathcal{N}_e$ implies that e/α is polarized and hence that $(R/N, \psi_N)$ is polarized by Lemma 5.18. □

Lemma 5.20. *Let N be a normal subgroup of R such that $\psi(x) \notin N$ for every point x of $\mathcal{S}$. Then the representation $(R/N, \psi_N)$ is polarized if and only if $N \subseteq Z(R)$.*

Proof. If $N \subseteq Z(R)$, then $(R/N, \psi_N)$ is polarized by Lemma 5.19. Conversely, suppose that $(R/N, \psi_N)$ is polarized. If $\lambda \notin N$, then $N \subseteq Z(R)$ by Lemma 5.16. So, we may suppose that $\{1, \lambda\} \subseteq N$. Then $(R/N, \psi_N)$ is an abelian representation of $\mathcal{S}$. Now, $R/N \cong (R/R')/(N/R')$, where $R' = \{1, \lambda\}$. The embedding e has $\mathrm{PG}(V)$ as target projective space, where $V = R/R'$ is regarded as an $\mathbb{F}_2$-vector space. The full projective embedding e' corresponding to $(R/N, \psi_N)$ has $\mathrm{PG}(R/N)$ as target projective space, where the elementary abelian 2-group R/N is again regarded as an $\mathbb{F}_2$-vector space. Since e' is polarized, we should have $\theta(N) = N/R' \subseteq \mathcal{R}_f$, that is, $N \subseteq Z(R)$. □

6 Classification of the polarized non-abelian representations

In this section, we shall prove all the claims mentioned in Theorems 3.2, 3.3 and 3.4. Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a finite slim near-polar space of diameter $n \geq 3$ that has polarized non-abelian representations. We set $(\widetilde{R}, \widetilde{\psi})$ equal to $(\widetilde{R}_p, \widetilde{\psi}_p)$, the universal polarized representation of $\mathcal{S}$. There then exists an element $\widetilde{\lambda} \in \widetilde{R} \setminus \{1\}$ such that $[\widetilde{\psi}(x), \widetilde{\psi}(y)] = \widetilde{\lambda}$ for every two opposite points x and y of $\mathcal{S}$. Recall also that $\widetilde{R}' = \{1, \widetilde{\lambda}\}$ and that the quotient group $\widetilde{R}/\widetilde{R}'$ is an elementary abelian 2-group which can be regarded as a vector space $\widetilde{V}$ over $\mathbb{F}_2$. Let $\widetilde{f}$ denote the alternating bilinear form on $\widetilde{V}$ associated with $(\widetilde{R}, \widetilde{\psi})$ as described in Section 5 (see Lemma 5.10). The radical of $\widetilde{f}$ is denoted by $\mathcal{R}_{\widetilde{f}}$. For every $r \in \widetilde{R}$, we put $\widetilde{\theta}(r) := r\widetilde{R}' \in \widetilde{V}$ and for every point x of $\mathcal{S}$, we put $\widetilde{e}(x)$ equal to the point $\langle \widetilde{\psi}(x)\widetilde{R}' \rangle$ of $\mathrm{PG}(\widetilde{V})$. Then $\widetilde{e}$ is isomorphic to the universal embedding of $\mathcal{S}$. By Section 5, we also know the following.

Proposition 6.1. *The polarized representations of $\mathcal{S}$ are precisely the representations of the form $(\widetilde{R}/N, \widetilde{\psi}_N)$, where N is a subgroup contained in $Z(\widetilde{R})$.*

Recall that if N is a subgroup contained in $Z(\widetilde{R})$, then N necessarily is normal and $\widetilde{\psi}(x) \notin Z(\widetilde{R})$ for every point x of $\mathcal{S}$, implying that the quotient representation $(\widetilde{R}/N, \widetilde{\psi}_N)$ is well-defined.

Proposition 6.2. *If N_1 and N_2 are two subgroups of $\widetilde{R}$ contained in $Z(\widetilde{R})$, then the quotient representations $(\widetilde{R}/N_1, \widetilde{\psi}_{N_1})$ and $(\widetilde{R}/N_2, \widetilde{\psi}_{N_2})$ of $\mathcal{S}$ are isomorphic if and only if $N_1 = N_2$.*

Proof. We prove that if the representations $(\widetilde{R}/N_1, \widetilde{\psi}_{N_1})$ and $(\widetilde{R}/N_2, \widetilde{\psi}_{N_2})$ are isomorphic, then $N_1 \subseteq N_2$. By symmetry, we then also have that $N_2 \subseteq N_1$.

Let ϕ be a group isomorphism from $\widetilde{R}/N_1$ to $\widetilde{R}/N_2$ such that $\phi(\widetilde{\psi}(x)N_1) = \widetilde{\psi}(x)N_2$ for every point x of $\mathcal{S}$.

Let $g \in N_1$. Since $\widetilde{R} = \langle \widetilde{\psi}(x) \,|\, x \in \mathcal{P} \rangle$, there exist (not necessarily distinct) points $x_1, x_2, \ldots, x_k$ such that $g = \widetilde{\psi}(x_1)\widetilde{\psi}(x_2) \cdots \widetilde{\psi}(x_k)$. Then $N_2 = \phi(N_1) = \phi(gN_1) = \phi(\widetilde{\psi}(x_1)N_1 \cdots \widetilde{\psi}(x_k)N_1) = \phi(\widetilde{\psi}(x_1)N_1) \cdots \phi(\widetilde{\psi}(x_k)N_1) = \widetilde{\psi}(x_1)N_2 \cdots \widetilde{\psi}(x_k)N_2 = gN_2$. Hence, $g \in N_2$. Since g is an arbitrary element of N_1, we have $N_1 \subseteq N_2$. $\square$

By Corollary 5.15(3), we know that $Z(\widetilde{R})$ is isomorphic to either $(C_2)^{l+1}$ or $(C_2)^{l-1} \times C_4$, where $l := er^+(\mathcal{S}) - er^-(\mathcal{S})$.

Proposition 6.3. *(i) The number of nonisomorphic polarized representations of $\mathcal{S}$ is equal to the sum $\sum_{i=0}^{l+1} \begin{bmatrix} l+1 \\ i \end{bmatrix}_2$ if $Z(\widetilde{R}) \cong (C_2)^{l+1}$, and equal to $2 \cdot \sum_{i=0}^{l} \begin{bmatrix} l \\ i \end{bmatrix}_2 - \sum_{i=0}^{l-1} \begin{bmatrix} l-1 \\ i \end{bmatrix}_2$ if $l \geq 1$ and $Z(\widetilde{R}) \cong (C_2)^{l-1} \times C_4$.*

(*ii*) *The number of nonisomorphic polarized non-abelian representations of* $\mathcal{S}$ *is equal to* $\sum_{i=0}^{l+1} \begin{bmatrix} l+1 \\ i \end{bmatrix}_2 - \sum_{i=0}^{l} \begin{bmatrix} l \\ i \end{bmatrix}_2$ *if* $Z(\widetilde{R}) \cong (C_2)^{l+1}$, *and equal to* $\sum_{i=0}^{l} \begin{bmatrix} l \\ i \end{bmatrix}_2 - \sum_{i=0}^{l-1} \begin{bmatrix} l-1 \\ i \end{bmatrix}_2$ *if* $l \geq 1$ *and* $Z(\widetilde{R}) \cong (C_2)^{l-1} \times C_4$.

Proof. By Lemma 5.17 and Propositions 6.1 and 6.2, the number of nonisomorphic polarized (non-abelian) representations of $\mathcal{S}$ is equal to the number of subgroups of $Z(\widetilde{R})$ (not containing $\widetilde{\lambda}$).

If $Z(\widetilde{R}) \cong (C_2)^{l+1}$, then $Z(\widetilde{R})$ is an elementary abelian 2-group and so the number of subgroups of $Z(\widetilde{R})$ (containing $\widetilde{\lambda}$) is equal to $\sum_{i=0}^{l+1} \begin{bmatrix} l+1 \\ i \end{bmatrix}_2 \left(\sum_{i=0}^{l} \begin{bmatrix} l \\ i \end{bmatrix}_2 \right)$.

If $Z(\widetilde{R}) \cong (C_2)^{l-1} \times C_4$, then $Z(\widetilde{R})/\langle\widetilde{\lambda}\rangle \cong (C_2)^l$ and hence the total number of subgroups of $Z(\widetilde{R})$ containing $\widetilde{\lambda}$ is equal to $\sum_{i=0}^{l} \begin{bmatrix} l \\ i \end{bmatrix}_2$. If G is a subgroup of $Z(\widetilde{R})$ not containing $\widetilde{\lambda}$, then G only has elements of order 1 and 2. The subgroup of $Z(\widetilde{R})$ consisting of all elements of order 1 and 2 is isomorphic to $(C_2)^l$ and hence the number of subgroups of $Z(\widetilde{R})$ not containing $\widetilde{\lambda}$ is is equal to $\sum_{i=0}^{l} \begin{bmatrix} l \\ i \end{bmatrix}_2 - \sum_{i=0}^{l-1} \begin{bmatrix} l-1 \\ i \end{bmatrix}_2$. □

Lemma 6.4. *The following are equivalent:*

(1) $Z(\widetilde{R})$ *is elementary abelian, that is, isomorphic to* C_2^{l+1};

(2) $\mathcal{S}$ *has a non-abelian representation* (R, ψ), *where* R *is some extraspecial group;*

(3) $\mathcal{S}$ *has a non-abelian representation* (R, ψ), *where* $|R| = 2^{1+er^-(\mathcal{S})}$.

If one of these conditions hold, then the number of nonisomorphic polarized non-abelian representations (R, ψ) *with* $|R| = 2^{1+er^-(\mathcal{S})}$ *is equal to* 2^l.

Proof. In Lemma 5.12(3), we already showed that (2) and (3) are equivalent. By Lemma 5.17 and Proposition 6.1, $\mathcal{S}$ has polarized non-abelian representations (R, ψ) where $|R| = 2^{1+er^-(\mathcal{S})}$ if and only if $Z(\widetilde{R})$ has subgroups of order 2^l not containing $\widetilde{\lambda}$. Such subgroups do not exist if $l \geq 1$ and $Z(\widetilde{R}) \cong (C_2)^{l-1} \times C_4$. If $Z(\widetilde{R}) \cong (C_2)^{l+1}$, then the number of such subgroups is equal to $\begin{bmatrix} l+1 \\ l \end{bmatrix}_2 - \begin{bmatrix} l \\ l-1 \end{bmatrix}_2 = 2^l$. □

Lemma 6.5. *If* $l \geq 1$ *and* $Z(\widetilde{R}) \cong (C_2)^{l-1} \times C_4$, *then* $|R| \geq 2^{2+er^-(\mathcal{S})}$ *for every polarized non-abelian representation* (R, ψ) *of* $\mathcal{S}$. *The number of such polarized non-abelian representations (up to isomorphism) is equal to* 2^{l-1}. *If* (R, ψ) *is a polarized non-abelian representation of* $\mathcal{S}$ *for which* $|R| = 2^{2+er^-(\mathcal{S})}$, *then* $Z(R) \cong C_4$.

Proof. By Lemmas 5.12 and 6.4, we know that $|R| \geq 2^{2+er^-(\mathcal{S})}$ for every polarized non-abelian representation (R, ψ) of $\mathcal{S}$. The number of such polarized non-abelian representations (up to isomorphism) is equal to the number of subgroups of order 2^{l-1} of $Z(\widetilde{R})$ that do not contain $\widetilde{\lambda}$, that is, equal to $\begin{bmatrix} l \\ l-1 \end{bmatrix}_2 - \begin{bmatrix} l-1 \\ l-2 \end{bmatrix}_2 = 2^{l-1}$. Suppose (R, ψ) is a polarized non-abelian representation of $\mathcal{S}$ with $|R| = 2^{2+er^-(\mathcal{S})}$. Then $Z(R)$ is isomorphic to either C_4 or $C_2 \times C_2$ by Lemma 5.14 and Corollary 5.15(2). If $Z(R) \cong C_2 \times C_2$, then $Z(R)$ contains subgroups of order 2 not containing R' and so (R, ψ) has a proper quotient which is a polarized non-abelian representation. This is impossible as the size of the representation group R is already as small as possible. □

Lemma 6.6. *If N_1 and N_2 are two subgroups of $\widetilde{R}$ contained in $Z(\widetilde{R})$ such that $\widetilde{\lambda} \notin N_1 \cup N_2$ and $\widetilde{\theta}(N_1) = \widetilde{\theta}(N_2)$, then there exists an automorphism of $\widetilde{R}$ mapping N_1 to N_2. As a consequence, the quotient groups $\widetilde{R}/N_1$ and $\widetilde{R}/N_2$ are isomorphic.*

Proof. Set $U := \widetilde{\theta}(N_1) = \widetilde{\theta}(N_2) = \langle \bar{v}_1, \bar{v}_2, \ldots, \bar{v}_k \rangle$ for some vectors $\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_k$ of $\widetilde{V}$ where $k = \dim(U)$. Put $d := \dim(\widetilde{V})$ and extend $\{\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_k\}$ to a basis $\{\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_d\}$ of $\widetilde{V}$. For every $i \in \{1, 2, \ldots, d\}$, let g_i be an arbitrary element of $\widetilde{\theta}^{-1}(\bar{v}_i)$. For all $i, j \in \{1, 2, \ldots, d\}$, put $a_{ij} := 1$ if $\widetilde{f}(\bar{v}_i, \bar{v}_j) = 1$ and $a_{ij} := 0$ otherwise. The group $\widetilde{R}$ has order 2^{d+1} and consists of all elements of the form

$$\widetilde{\lambda}^{\epsilon_0} g_1^{\epsilon_1} g_2^{\epsilon_2} \cdots g_d^{\epsilon_d},$$

where $\epsilon_0, \epsilon_1, \ldots, \epsilon_d \in \{0, 1\}$. If $i, j \in \{1, 2, \ldots, d\}$, we have $[g_i, g_j] = 1$ if $\widetilde{f}(\bar{v}_i, \bar{v}_j) = 0$ and $[g_i, g_j] = \widetilde{\lambda}$ if $\widetilde{f}(\bar{v}_i, \bar{v}_j) = 1$. So, the multiplication inside the group $\widetilde{R}$ should be as follows. If $\epsilon_0, \epsilon_1, \ldots, \epsilon_d, \epsilon'_0, \epsilon'_1, \ldots, \epsilon'_d \in \{0, 1\}$, then

$$(\widetilde{\lambda}^{\epsilon_0} g_1^{\epsilon_1} g_2^{\epsilon_2} \cdots g_d^{\epsilon_d}) \cdot (\widetilde{\lambda}^{\epsilon'_0} g_1^{\epsilon'_1} g_2^{\epsilon'_2} \cdots g_d^{\epsilon'_d}) = \widetilde{\lambda}^{\epsilon_0+\epsilon'_0+\epsilon''_0} g_1^{\epsilon_1+\epsilon'_1} g_2^{\epsilon_2+\epsilon'_2} \cdots g_d^{\epsilon_d+\epsilon'_d},$$

where $\epsilon''_0 := \sum_{i=1}^d \sum_{j=i+1}^d a_{ij} \epsilon'_i \epsilon_j$. Recall that $\widetilde{\lambda} \notin N_1 \cup N_2$. So, for every $i \in \{1, 2, \ldots, k\}$, there exists a unique element $g_i^{(1)} \in \{g_i, g_i\widetilde{\lambda}\}$ belonging to N_1 and a unique element $g_i^{(2)} \in \{g_i, g_i\widetilde{\lambda}\}$ belonging to N_2. Then $N_1 = \langle g_1^{(1)}, g_2^{(1)}, \ldots, g_k^{(1)} \rangle$ and $N_2 = \langle g_1^{(2)}, g_2^{(2)}, \ldots, g_k^{(2)} \rangle$. Now, let I denote the subset of $\{1, 2, \ldots, k\}$ consisting of all $i \in \{1, 2, \ldots, k\}$ for which $g_i^{(1)} \neq g_i^{(2)}$, or equivalently, for which $g_i^{(2)} = g_i^{(1)}\widetilde{\lambda}$. Then the permutation of $\widetilde{R}$ defined by

$$\widetilde{\lambda}^{\epsilon_0} g_1^{\epsilon_1} g_2^{\epsilon_2} \cdots g_d^{\epsilon_d} \mapsto \widetilde{\lambda}^{\epsilon_0+\epsilon'_0} g_1^{\epsilon_1} g_2^{\epsilon_2} \cdots g_d^{\epsilon_d},$$

where $\epsilon'_0 := \sum_{i \in I} \epsilon_i$, is an automorphism ϕ of R. Since $\phi(g_i^{(1)}) = g_i^{(2)}$ for every $i \in \{1, 2, \ldots, k\}$, we have $\phi(N_1) = N_2$. □

Corollary 6.7. *If (R_1, ψ_1) and (R_2, ψ_2) are two polarized non-abelian representations of $\mathcal{S}$ for which the associated full polarized embeddings are isomorphic, then also the representation groups R_1 and R_2 are isomorphic.*

Proof. Let N_1 and N_2 be the subgroups of $\widetilde{R}$ contained in $Z(\widetilde{R})$ such that $(R_1, \psi_1) \cong (\widetilde{R}/N_1, \widetilde{\psi}_{N_1})$ and $(R_2, \psi_2) \cong (\widetilde{R}/N_2, \widetilde{\psi}_{N_2})$. Then $\widetilde{\lambda} \notin N_1 \cup N_2$. Let α_1 and α_2 be the subspaces of $\mathcal{N}_{\widetilde{e}}$ corresponding to, respectively, $U_1 := \widetilde{\theta}(N_1) \subseteq \mathcal{R}_{\widetilde{f}}$ and $U_2 := \widetilde{\theta}(N_2) \subseteq \mathcal{R}_{\widetilde{f}}$. By Lemma 5.19(1), the projective embeddings e/α_1 and e/α_2 are isomorphic. This implies that $\alpha_1 = \alpha_2$. Hence, $\widetilde{\theta}(N_1) = \widetilde{\theta}(N_2)$. By Lemma 6.6, $R_1 \cong R_2$. □

Proposition 6.8. *If* (R_1, ψ_1) *and* (R_2, ψ_2) *are two polarized non-abelian representations of* $\mathcal{S}$ *such that* $|R_1| = |R_2| = 2^\beta$*, where* $\beta = 1 + er^-(\mathcal{S})$*, then* R_1 *and* R_2 *are isomorphic (to either* 2^β_+ *or* 2^β_-*).*

Proof. Let N_1 and N_2 be the unique normal subgroups of $\widetilde{R}$ contained in $Z(\widetilde{R})$ such that $\widetilde{\lambda} \notin N_1 \cup N_2$ and $(\widetilde{R}/N_1, \widetilde{\psi}_{N_1}) \cong (R_1, \psi_1)$ and $(\widetilde{R}/N_2, \widetilde{\psi}_{N_2}) \cong (R_2, \psi_2)$. Then $|N_1| = |N_2| = \frac{|\widetilde{R}|}{|R_1|} = 2^l$, where $l = er^+(\mathcal{S}) - er^-(\mathcal{S})$. Since $\widetilde{\lambda} \notin N_1 \cup N_2$ and $|Z(\widetilde{R})| = 2^{l+1}$, we have $Z(\widetilde{R}) = \langle N_1, \widetilde{\lambda}\rangle = \langle N_2, \widetilde{\lambda}\rangle$. Hence, $\mathcal{R}_{\widetilde{f}} = \theta(Z(\widetilde{R})) = \theta(\langle N_1, \widetilde{\lambda}\rangle) = \theta(N_1) = \theta(\langle N_2, \widetilde{\lambda}\rangle) = \theta(N_2)$. By Lemma 6.6, $R_1 \cong \widetilde{R}/N_1 \cong \widetilde{R}/N_2 \cong R_2$. □

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