A characterization of a class of hyperplanes of $DW(2n-1,\mathbb{F})$

Bart De Bruyn

Ghent University, Department of Mathematics, Krijgslaan 281 (S22), B-9000 Gent, Belgium,
E-mail: bdb@cage.ugent.be

Abstract

A hyperplane of the symplectic dual polar space $DW(2n-1,\mathbb{F})$, $n \geq 2$, is said to be of subspace-type if it consists of all maximal singular subspaces of $W(2n-1,\mathbb{F})$ meeting a given $(n-1)$-dimensional subspace of $\mathrm{PG}(2n-1,\mathbb{F})$. We show that a hyperplane of $DW(2n-1,\mathbb{F})$ is of subspace-type if and only if every hex F of $DW(2n-1,\mathbb{F})$ intersects it in either F, a singular hyperplane of F or the extension of a full subgrid of a quad. In the case $\mathbb{F}$ is a perfect field of characteristic 2, a stronger result can be proved, namely a hyperplane H of $DW(2n-1,\mathbb{F})$ is of subspace-type or arises from the spin-embedding of $DW(2n-1,\mathbb{F}) \cong DQ(2n,\mathbb{F})$ if and only if every hex F intersects it in either F, a singular hyperplane of F, a hexagonal hyperplane of F or the extension of a full subgrid of a quad.

Keywords: symplectic dual polar space, hyperplane (of subspace-type), spin-embedding
MSC2000: 51A50

1 Introduction

Hyperplanes have been investigated for several classes of point-line geometries. In particular, many constructions and classification results have been obtained, and the close relationship between hyperplanes and projective embeddings has been studied. This relationship originates from the fact that with every full embedding e of a point-line geometry $\mathcal{S}$ there are associated hyperplanes, the so-called hyperplanes of $\mathcal{S}$ arising from e. The connection between hyperplanes and projective embeddings has played a crucial role in Tits' classification of polar spaces [21]. The question which hyperplanes of a given point-line geometry arise from a full projective embedding has been widely investigated (see e.g. Cohen and Shult [4, Theorem 5.12] for the case of polar spaces). Sometimes hyperplanes tell you whether a point-line geometry can admit a full projective embedding (see e.g. Ronan [16, Corollary 2, p. 183]) or whether a given full projective embeddings is absolutely universal (see e.g. Shult [18]).

This note is concerned with characterizing certain hyperplanes of dual polar spaces. Hyperplanes of dual polar spaces are usually characterized in terms of their possible intersections with convex subspaces. The initial characterization results used the possible intersections with quads[1] as basis for the characterizations. In this regard, it is worth mentioning the work of Shult & Thas [20], Pasini & Shpectorov [12], Cooperstein & Pasini [5], Cardinali, De Bruyn & Pasini [3] and De Bruyn [7] on locally singular, locally subquadrangular and locally ovoidal hyperplanes. Pralle [14] investigated hyperplanes in dual polar spaces of rank 3 that do not admit subquadrangular quads and those without singular quads (for arbitrary ranks) were studied in [13]. In joint work with the author [10], he also investigated hyperplanes of symplectic dual polar spaces of rank 3 without ovoidal quads. This classification was later extended by the author to arbitrary ranks [6].

There are also a number of characterizations in terms of the possible intersections with hexes. In this regard, it is worth mentioning the result of Cardinali, De Bruyn and Pasini [3, Lemma 3.4] who showed that the singular hyperplanes of thick dual polar spaces are precisely the hyperplanes intersecting each hex F in either F or a singular hyperplane of $\widetilde{F}$. Pralle and Shpectorov [15] studied hyperplanes in thick dual polar spaces of rank 3 intersecting each hex in the extension of an ovoid of a quad. In [8], the author extended this study to hyperplanes in dual polar spaces of arbitrary rank that intersect each hex F in either F, a singular hyperplane of $\widetilde{F}$ or the extension of an ovoid of a quad. It was shown there that these hyperplanes are precisely the possible trivial extensions of the SDPS-hyperplanes, a class of hyperplanes introduced in [11].

In this note, we consider a problem that is similar to the one studied in [8]. We take a look at hyperplanes of dual polar spaces that intersect each hex F in either F, a singular hyperplane of $\widetilde{F}$ or the extension of a subquadrangle of a quad. As we will see, it is possible to classify all these hyperplanes in the case of symplectic dual polar spaces. They are precisely the hyperplanes of subspace-type, a class of hyperplanes under investigation in [9]. In the case where the field $\mathbb{F}$ is perfect of characteristic 2, it is even possible to classify all hyperplanes if one allows an additional possibility for the intersection with hexes. The following two results are the main results of this note.

Theorem 1.1 *The following are equivalent for a hyperplane H of $DW(2n-1,\mathbb{F})$, $n \geq 3$:*

(1) *H is a hyperplane of subspace-type;*

(2) *for every hex F of $DW(2n-1,\mathbb{F})$, $F \cap H$ is either F, a singular hyperplane of $\widetilde{F}$ or the extension of a full subgrid of a quad of $\widetilde{F}$.*

Theorem 1.2 *Let $n \geq 3$ and $\mathbb{F}$ a perfect field of characteristic 2. Then the following are equivalent for a hyperplane H of $DW(2n-1,\mathbb{F})$:*

(1) *H is either a hyperplane of subspace-type or arises from the spin-embedding of $DW(2n-1,\mathbb{F}) \cong DQ(2n,\mathbb{F})$;*

[1] Many notions used in this introductory section will be explicitly defined in Section 2.

(2) *for every hex F of $DW(2n-1,\mathbb{F})$, $F \cap H$ is either F, a singular hyperplane of $\widetilde{F}$, a hexagonal hyperplane of $\widetilde{F}$ or the extension of a full subgrid of a quad of $\widetilde{F}$.*

It is somewhat unfortunate that the proofs of Theorems 1.1 and 1.2 are for a large extend already contained in [6]. This note could therefore also be seen as an addendum to [6]. The main purpose of [6] was to extend the classification result of [10] to arbitrary ranks. By focussing on this particular goal, we overlooked[2] then that the proof can be modified to a proof of our main results. We fear that this fact might remain unnoticed by a future reader, as this modification still requires some work. Indeed, certain arguments in [6] are not relevant for the current treatment and other arguments do not work when the underlying field is infinite (due to the use of counting arguments) or of order 2. These problems will be by-passed here by means of alternative arguments and a change of the order of the intermediate lemmas that will moreover lead to a simplification. For convenience to the reader, we still mention the whole chain of lemmas leading to the proofs of Theorems 1.1 and 1.2. The proofs of those lemmas that are basically contained in [6] will be omitted and instead an explicit reference to [6] will be given.

2 Preliminaries

With every polar space Π of rank $n \geq 2$ (in the sense of Tits [21, Chapter 7]) there is associated a dual polar space Δ of rank n. This is the point-line geometry whose points and lines are the maximal and next-to-maximal singular subspaces of Π, with incidence being reverse containment. Distances $\mathrm{d}(\cdot,\cdot)$ between points of Δ will always be measured in the collinearity graph which has diameter n.

There exists a bijective correspondence between the nonempty convex subspaces of Δ and the singular subspaces of Π: if α is a singular subspace of Π, then the set F_α consisting of all maximal singular subspaces containing α is a convex subspace of Δ. The convex subspaces of diameter 2, 3 and $n-1$ are called the *quads*, *hexes* and *maxes*, respectively. If F is a convex subspace of diameter $\delta \geq 2$ of Δ, then the point-line geometry $\widetilde{F}$ induced on F is a dual polar space of rank δ. In particular, if Q is a quad, then $\widetilde{Q}$ is a dual polar space of rank 2, i.e. a generalized quadrangle. Two points x and y of Δ at distance δ from each other are contained in a unique convex subspace $\langle x, y\rangle$ of diameter δ.

If F is a convex subspace of Δ and x is a point, then there exists a (necessarily unique) point $\pi_F(x) \in F$ such that $\mathrm{d}(x,y) = \mathrm{d}(x,\pi_F(x)) + \mathrm{d}(\pi_F(x),y)$ for every point $y \in F$. In particular, for every point-line pair (x, L), there is a unique point on L nearest to x. The convex subspaces through a given point x, ordered by inclusion, define a projective space $Res(x)$ of dimension $n-1$.

Suppose H is a hyperplane of Δ, i.e. a proper subspace meeting each line. If x is a point of H, then $\Lambda_H(x)$ denotes the set of lines through x contained in H. We will often

[2] The classification result obtained in [6], namely those of hyperplanes of $DW(2n-1,\mathbb{F})$ without ovoidal quads, is only valid when $\mathbb{F} = \mathbb{F}_q$ for a certain prime power $q \neq 2$. For infinite fields or the smallest field $\mathbb{F}_2$, other examples of such hyperplanes exist and a complete classification was and is still missing. As such, there was no need in [6] to let intermediate results work for any field.

regard $\Lambda_H(x)$ as a set of points of $Res(x)$. If F is a convex subspace of Δ, then either $F \subseteq H$ (in which case F is called *deep* with respect to H) or $F \cap H$ is a hyperplane of $\widetilde{F}$. As there are three types of hyperplanes in generalized quadrangles (ovoids, subquadrangles and perps of points), we see that for every quad Q of Δ, one of the following cases occurs:

(1) Q is deep, i.e. contained in H;

(2) $Q \cap H = x^{\perp} \cap Q$ for a certain point $x \in Q$;

(3) $Q \cap H$ is a full subquadrangle of $\widetilde{Q}$;

(4) $Q \cap H$ is an ovoid of $\widetilde{Q}$, i.e. a set of points meeting each line of $\widetilde{Q}$ in a singleton.

The quad Q will be called *deep*, *singular*, *subquadrangular* or *ovoidal* with respect to H depending on whether case (1), (2), (3) or (4) occurs. A hyperplane is called *locally singular*, *locally subquadrangular* or *locally ovoidal* if every non-deep quad is singular, subquadrangular or ovoidal with respect to H.

If x is a point of Δ, then the points of Δ at distance at most $n-1$ from x is a hyperplane H_x of Δ, the *singular hyperplane with deepest point* x. Suppose F is a convex subspace of diameter δ of Δ and H_F is a hyperplane of $\widetilde{F}$. The maximal distance from a point of Δ to F is equal to $n-\delta$. Denote by H the set consisting of those points of Δ at distance at most $n-\delta-1$ from F together with those points x of Δ at distance $n-\delta$ from F for which $\pi_F(x) \in H_F$. By [11, Proposition 1], H is a hyperplane of Δ, the so-called *extension* of H_F. This extension is called *trivial* if $\delta = n$ (in which case, $H = H_F$).

If ζ is a symplectic polarity of $\mathrm{PG}(2n-1,\mathbb{F})$, then the subspaces of $\mathrm{PG}(2n-1,\mathbb{F})$ that are totally isotropic with respect to ζ define a symplectic polar space $W(2n-1,\mathbb{F})$. The corresponding dual polar space will be denoted by $DW(2n-1,\mathbb{F})$. The dual polar space $DW(3,\mathbb{F})$ is isomorphic to the generalized quadrangle $Q(4,\mathbb{F})$. The only subquadrangles of this generalized quadrangle that are also hyperplanes are the full subgrids. If π is an $(n-1)$-dimensional subspace of $\mathrm{PG}(2n-1,\mathbb{F})$, then the set H_π of all maximal singular subspaces of $W(2n-1,\mathbb{F})$ meeting π is a hyperplane of $DW(2n-1,\mathbb{F})$, see De Bruyn [9]. We call any such hyperplane a *hyperplane of subspace-type*.

If $\mathbb{F}$ is perfect field of characteristic 2, then $(D)W(2n-1,\mathbb{F})$ is isomorphic to the orthogonal (dual) polar space $(D)Q(2n,\mathbb{F})$ arising from a nonsingular quadric $Q(2n,\mathbb{F})$ of Witt index n in $\mathrm{PG}(2n,\mathbb{F})$. The dual polar space $DQ(2n,\mathbb{F})$ has a full projective embedding e_{sp} in $\mathrm{PG}(2^n-1,\mathbb{F})$ which is called the *spin-embedding* of $DQ(2n,\mathbb{F})$ (see Buekenhout and Cameron [2, §7]). If $\mathcal{P}$ denotes the point set of $DQ(2n,\mathbb{F})$ and π is a hyperplane of $\mathrm{PG}(2^n-1,\mathbb{F})$, then $e_{sp}^{-1}(e_{sp}(\mathcal{P}) \cap \pi)$ is a hyperplane of $DQ(2n,\mathbb{F})$. By results of Shult & Thas [20] and De Bruyn [7, Theorem 1.3], we know that the locally singular hyperplanes of $DQ(2n,\mathbb{F})$ are precisely the hyperplanes of $DQ(2n,\mathbb{F})$ arising from e_{sp}. By Shult [17] (finite case) and Pralle [13] (general case), the dual polar space $DQ(6,\mathbb{F})$ has two types of locally singular hyperplanes, the singular hyperplanes and the hexagonal hyperplanes. If H is a hexagonal hyperplane of $DQ(6,\mathbb{F})$, then every quad is singular with respect to H, and $\Lambda_H(x)$ is a line of $Res(x) \cong \mathrm{PG}(2,\mathbb{F})$ for every point $x \in H$.

3 Three types of points in certain hyperplanes

In this section, we suppose that H is a hyperplane of the symplectic dual polar space $DW(2n-1,\mathbb{F})$, $n \geq 3$, such that for every hex F of $DW(2n-1,\mathbb{F})$, the intersection $H \cap F$ is either F, a singular hyperplane of $\widetilde{F}$, the extension of a full subgrid of a quad of $\widetilde{F}$, or a hexagonal hyperplane of $\widetilde{F}$. Observe that the latter type of hyperplane can only occur when $\mathbb{F}$ is a perfect field of characteristic 2.

Lemma 3.1 *There are no quads that are ovoidal with respect to H.*

Proof. Let Q be an arbitrary quad of $DW(2n-1,\mathbb{F})$ and let F denote a hex through Q. If $H \cap F = F$, then Q is deep with respect is H. If $H \cap F$ is a singular hyperplane of $\widetilde{F}$, then Q is deep or singular with respect to H. If $H \cap F$ is the extension of a full subgrid of a quad of $\widetilde{F}$, then Q is deep, singular or subquadrangular with respect to H. If $H \cap F$ is a hexagonal hyperplane of $\widetilde{F}$, then Q is singular with respect to H. ■

A proof of the following lemma is also contained in [6, Lemma 4.1], but that proof makes use of counting arguments, and is therefore not valid in the infinite case.

Lemma 3.2 *Let $x \in H$. Then $\Lambda_H(x)$ is one of the following sets of points of $Res(x)$:*
(I) a hyperplane;
(II) the union of two distinct hyperplanes;
(III) the whole space.

Proof. Put $\Lambda(x) := \Lambda_H(x)$. For a subspace α of $Res(x)$ of dimension at least 1, we show the following by induction on $\dim(\alpha)$:

(∗) the intersection $\alpha \cap \Lambda(x)$ is equal to either α, a hyperplane of α or the union of two distinct hyperplanes of α.

The lemma then follows by applying Property (∗) to the subspace $\alpha = Res(x)$.

Suppose $\dim(\alpha) = 1$. Let Q denote the quad through x corresponding to α. By Lemma 3.1, Q is deep, singular or subquadrangular with respect to H. If Q is deep, then $\alpha \cap \Lambda(x) = \alpha$. If Q is singular, then $\alpha \cap \Lambda(x)$ is either α or a hyperplane of α (i.e. a singleton). If Q is subquadrangular, then $\alpha \cap \Lambda(x)$ is the union of two hyperplanes of α (i.e. a pair).

Suppose $\dim(\alpha) = 2$. Let F denote the hex through x corresponding to α. We verify Property (∗) for each of the four possible intersections of F with H. If $F \subseteq H$, then $\alpha \cap \Lambda(x)$ is equal to α. If $F \cap H$ is a singular hyperplane of F, then $\alpha \cap \Lambda(x)$ is either α or a hyperplane of α. If $F \cap H$ is the extension of a full subgrid of a quad of F, then $\alpha \cap \Lambda(x)$ is either α, a singular hyperplane of α or the union of two distinct hyperplanes of α. If $F \cap H$ is a hexagonal hyperplane of F, then $\alpha \cap \Lambda(x)$ is a hyperplane of α.

Suppose $\dim(\alpha) = 3$. By the induction hypothesis, property (∗) holds for any line or plane of α. If every line of α intersects $\Lambda(x)$ in the whole line or a singleton, then $\alpha \cap \Lambda(x)$ is either α or a hyperplane of α. So, we may suppose that there exists a line L in α that intersects $\Lambda(x)$ in two points x_1 and x_2. Every plane of α through L intersects

$\Lambda(x)$ in the union of a line through x_1 and a line through x_2. Now, let $\beta_1, \beta_2, \beta_3$ be three distinct planes of α through L. For every $i \in \{1,2,3\}$, let L_i, respectively M_i, denote the unique line through x_1, respectively x_2, contained in $\beta_i \cap \Lambda(x)$. Put $\gamma_1 := \langle L_1, L_2 \rangle$, $\gamma_2 := \langle M_1, M_2 \rangle$, $\{u_1\} = \gamma_1 \cap M_3$ and $\{v_1\} = \gamma_2 \cap L_3$. Since $L_1 \cup L_2 \cup \{u_1\} \subseteq \Lambda(x)$ and $u_1 \notin L_1 \cup L_2$, we have $\gamma_1 \subseteq \Lambda(x)$ by the induction hypothesis applied to the plane γ_1. In a similar way, one shows that $\gamma_2 \subseteq \Lambda(x)$. Now, every plane of α through L intersects $\gamma_1 \cup \gamma_2$ in the union of a line through x_1 and a line through x_2. This forces $\Lambda(x) \cap \alpha$ to be equal to $\gamma_1 \cup \gamma_2$.

Suppose that $\dim(\alpha) \geq 4$ and that property $(*)$ holds for any subspace of dimension less than $\dim(\alpha)$. If every line of α intersects $\Lambda(x)$ in the whole line or a singleton, then $\alpha \cap \Lambda(x)$ is either α or a hyperplane of α. So, we may suppose that there exists a line L in α that intersects $\Lambda(x)$ in two points x_1 and x_2. For every plane $\beta \subseteq \alpha$ through L, let $k(\beta)$ denote the unique point of β such that $\beta \cap \Lambda(x)$ is the union of the two lines $k(\beta)x_1$ and $k(\beta)x_2$. The set K consisting of all these $k(\beta)$'s completely determines $\alpha \cap \Lambda(x)$. It suffices to show that K is a subspace of dimension $\dim(\alpha) - 2$ disjoint from x_1x_2, as this would imply that $\alpha \cap \Lambda(x) = \langle K, x_1 \rangle \cup \langle K, x_2 \rangle$.

Let β_1 and β_2 be two distinct planes of α through L. By the induction hypothesis, the three-space $\langle \beta_1, \beta_2 \rangle$ intersects $\Lambda(x)$ in the union of two planes δ_1 and δ_2. The line $\delta_1 \cap \delta_2$ coincides with the line through $k(\beta_1)$ and $k(\beta_2)$, and every point of $\delta_1 \cap \delta_2$ is of the form $k(\beta)$ for some plane β of $\langle \beta_1, \beta_2 \rangle$ through L. This proves that K is a subspace. Since L is disjoint from K, $\dim(K) \leq \dim(\alpha) - 2$. Since every plane of α through L meets K, $\dim(K) = \dim(\alpha) - 2$. ∎

A point $x \in H$ is said to be of type $X \in \{I, II, III\}$ if Case (X) of Lemma 3.2 occurs. A point $x \in H$ is of type II if and only if there is a line of $Res(x)$ intersecting $\Lambda_H(x)$ in precisely two points. This implies the following.

Lemma 3.3 *A point $x \in H$ has type II if and only if there exists a subquadrangular quad containing x.*

4 Some properties of hyperplanes of subspace-type

In this section, we suppose again that $DW(2n-1, \mathbb{F})$, $n \geq 2$, is the symplectic dual polar space arising from a symplectic polarity ζ of $\mathrm{PG}(2n-1, \mathbb{F})$. Consider a hyperplane H_π of subspace-type associated with an $(n-1)$-dimensional subspace π of $\mathrm{PG}(2n-1, \mathbb{F})$.

Lemma 4.1 *If $n \geq 3$ and F is a hex of $DW(2n-1, \mathbb{F})$, then F intersects H_π in either F, a singular hyperplane of $\widetilde{F}$ or the extension of a full subgrid of a quad of $\widetilde{F}$.*

Proof. By Proposition 2.12 of [9], we know that every hyperplane of subspace-type of $DW(5, \mathbb{F})$ is either a singular hyperplane or the extension of a full subgrid of a quad. By Proposition 2.9 of [9], we know that if F is a convex subspace of diameter $\delta \geq 2$ of $DW(2n-1, \mathbb{F})$, then either $F \subseteq H_\pi$ or $F \cap H_\pi$ is a hyperplane of subspace-type of

$\widetilde{F} \cong DW(2\delta-1,\mathbb{F})$. In particular, every hex F will intersect H_π in either F, a singular hyperplane of $\widetilde{F}$ or the extension of a full subgrid of $\widetilde{F}$. ∎

By Lemmas 3.2 and 4.1, we thus know that there are three possible types of points in H_π (types I, II and III). By [6, Lemma 3.3] and [9, Proposition 2.5] we know the following.

Lemma 4.2 ([6, 9])

- *The points of type I of H_π correspond to those maximal singular subspaces α of $W(2n-1,\mathbb{F})$ for which $\alpha \cap \pi = \alpha \cap \pi^\zeta$ is a point. If x is a point of type I of H_π, then there exists a unique deep max $A(x)$ through x such that $\Lambda(x)$ consists of those lines through x that are contained in $A(x)$.*
- *The points of type II of H_π correspond to those maximal singular subspaces α of $W(2n-1,\mathbb{F})$ for which $\dim(\pi \cap \alpha) = \dim(\pi^\zeta \cap \alpha) = 0$ and $\alpha \cap \pi \neq \alpha \cap \pi^\zeta$. If x is a point of type II of H_π, then there exist two distinct deep maxes $A_1(x)$ and $A_2(x)$ through x such that $\Lambda(x)$ consists of those lines through x that are contained in either $A_1(x)$ or $A_2(x)$.*
- *The points of type III of H_π correspond to those maximal singular subspaces α of $W(2n-1,\mathbb{F})$ for which $\dim(\pi \cap \alpha) = \dim(\pi^\zeta \cap \alpha) \geq 1$. If x is a point of type III, then $\Lambda(x)$ consists of all lines through x.*

Lemma 4.3 *Let x be a point of $DW(2n-1,\mathbb{F})$, $n \geq 3$, and let M be a max through x. If every max through x distinct from M is contained in H_π, then also M is contained in H_π and H_π is the singular hyperplane of $DW(2n-1,\mathbb{F})$ with deepest point x.*

Proof. Let α be the maximal singular subspace of $W(2n-1,\mathbb{F})$ corresponding to x. By Proposition 2.6 of [9], we know that a max of $DW(2n-1,\mathbb{F})$ is contained in H_π if and only if the point of $W(2n-1,\mathbb{F})$ corresponding to M belongs to $\pi \cup \pi^\zeta$. So, we see that there is at most 1 point in α which is not covered by $\pi \cup \pi^\zeta$. It follows that every point of α is covered by $\pi \cup \pi^\zeta$. Hence, also M is contained in H_π. So, the singular hyperplane H_x of $DW(2n-1,\mathbb{F})$ with deepest point x is contained in H_π. This implies that $H_x = H_\pi$ since H_x is a maximal proper subspace of $DW(2n-1,\mathbb{F})$ (Blok & Brouwer [1, Proposition 7.3], Shult [19, Lemma 6.1]). ∎

5 Proofs of Theorems 1.1 and 1.2

The following lemma and Lemma 4.1 show that certain classes of hyperplanes of $DW(2n-1,\mathbb{F})$ satisfy the intersection properties stated in Theorems 1.1 and 1.2.

Lemma 5.1 *Suppose $\mathbb{F}$ is a perfect field of characteristic 2 and H is a hyperplane of $DW(2n-1,\mathbb{F}) \cong DQ(2n,\mathbb{F})$ arising from its spin-embedding. Then for every hex F of $DW(2n-1,\mathbb{F})$, the intersection $H \cap F$ is either F, a singular hyperplane of $\widetilde{F}$ or a hexagonal hyperplane of $\widetilde{F}$.*

Proof. By De Bruyn [7, Proposition 1.2], the hyperplane H is locally singular. So, for every hex F, the intersection $H \cap F$ is either F or a locally singular hyperplane of $\widetilde{F} \cong DW(5, \mathbb{F}) \cong DQ(6, \mathbb{F})$. By results of Shult [17] (finite case) and Pralle [14] (general case), every locally singular hyperplane of $DQ(6, \mathbb{F})$ is singular or hexagonal. ■

In the sequel, we suppose that H is a hyperplane of $DW(2n-1, \mathbb{F})$, $n \geq 3$, such that for every hex F of $DW(2n-1, \mathbb{F})$, the intersection $H \cap F$ a either F, a singular hyperplane of $\widetilde{F}$, the extension of a full subgrid of a quad of $\widetilde{F}$ or a hexagonal hyperplane of $\widetilde{F}$ (the latter possibility can only occur when $\mathbb{F}$ is a perfect field of characteristic 2). By Lemma 3.2, we then know that there are three possible types of points (types I, II and III).

Lemma 5.2 *Suppose there are no points of type II in H. Then the following hold:*

- *If $\mathbb{F}$ is perfect field of characteristic 2, then H arises from the spin-embedding of $DW(2n-1, \mathbb{F}) \cong DQ(2n, \mathbb{F})$.*
- *If $\mathbb{F}$ is not a perfect field of characteristic 2, then H is a singular hyperplane.*

Proof. The fact that there are no points of type II implies by Lemma 3.3 that there are no subquadrangular quads. As there are also no ovoidal quads by Lemma 3.1, we know that the hyperplane H is locally singular. We distinguish two cases.

Suppose $\mathbb{F}$ is a perfect field of characteristic 2. Then $DW(2n-1, \mathbb{F}) \cong DQ(2n, \mathbb{F})$ and H must arise from the spin-embedding. Indeed, as already mentioned, the locally singular hyperplanes of $DQ(2n, \mathbb{F})$ are precisely the hyperplanes arising from its spin-embedding.

Suppose $\mathbb{F}$ is not a perfect field of characteristic 2. Then $DW(2n-1, \mathbb{F})$ is non-isomorphic to $DQ(2n, \mathbb{F})$. Theorem 3.5 of Cardinali, De Bruyn and Pasini [3] then implies that every locally singular hyperplane of $DW(2n-1, \mathbb{F})$ is singular. In particular, H must be singular. ■

Notice that every singular hyperplane of $DW(2n-1, \mathbb{F})$ is also a hyperplane of subspace-type. Lemmas 4.1, 5.1 and 5.2 already prove certain parts of Theorems 1.1 and 1.2. To complete the proofs of these theorems, it suffices to show that if there are points of type II in H, then H must be a hyperplane of subspace-type. The remainder of this section is devoted to the proof of that claim. The proof will be by induction on $n \geq 3$.

Suppose first that $n = 3$. Then the dual polar space itself is a hex, and the hyperplane H is either a singular hyperplane, the extension of a full subgrid of a quad or a hexagonal hyperplane. The existence of points of type II implies that H is the extension of a full subgrid. Such a hyperplane is of subspace-type by [9, Proposition 2.12]. In the sequel, we will therefore suppose that $n \geq 4$ and that the claim is valid for symplectic dual polar spaces of smaller rank (induction hypothesis).

Let P_1, P_2 and P_3 denote the set of those points of H that have type I, II and III, respectively. For every $x \in P_2$, let $A_1(x)$ and $A_2(x)$ denote the two maxes through x such that $x^\perp \cap H = (A_1(x) \cap x^\perp) \cup (A_2(x) \cap x^\perp)$. Let $I(x)$ denote the convex subspace $A_1(x) \cap A_2(x)$ of diameter $n-2$. If x and y are two distinct collinear points of P_3, then every quad Q through xy is deep, since $Q \cap H$ contains both $x^\perp \cap Q$ and $y^\perp \cap Q$. Hence:

Lemma 5.3 *P_3 is a subspace of $DW(2n-1,\mathbb{F})$.*

A proof of the following lemma will be omitted since it is essentially contained in [6, Lemma 4.5]. Its proof relies on Lemma 4.2 and a use of the induction hypothesis.

Lemma 5.4 *The following holds for every point $x \in P_2$.*
(i) If $y \in I(x)$ with $d(y,x) \leq n-3$, then $y \in H$.
(ii) If $y' \in (A_1(x) \cup A_2(x)) \setminus I(x)$ with $d(y',x) \leq n-2$, then $y' \in H$.

The following lemma was proved in [9, Lemma 4.6] using an argument that required the field $\mathbb{F}$ to have at least three elements. We replace it with a different argument which works for any field.

Lemma 5.5 *For every point $x \in P_2$, $I(x) \subseteq H$.*

Proof. By the induction hypothesis, either $A_1(x) \subseteq H$, $A_1(x) \cap H$ is a hyperplane of subspace-type of $A_1(x)$ or $A_1(x) \cap H$ is a locally singular hyperplane of $A_1(x)$. If $A_1(x) \subseteq H$, then also $I(x) \subseteq H$.

Suppose $A_1(x) \cap H$ is a locally singular hyperplane of $A_1(x)$. Let y be a point of $I(x)$ at distance at most $n-3$ from x. Since there are no subquadrangular quads in $A_1(x)$ through y, Lemma 3.3 implies that y has type I or III with respect to the hyperplane $A_1(x) \cap H$ of $A_1(x)$. By Lemma 5.4, every line of $A_1(x)$ through y not contained in $I(x)$ is contained in H. This implies that y must have type III with respect to the hyperplane $A_1(x) \cap H$ of $A_1(x)$. Since this holds for every point y of $I(x)$ at distance at most $n-3$ from x, we have $I(x) \subseteq H$.

Suppose $A_1(x) \cap H$ is a hyperplane of subspace-type of $A_1(x)$. By Lemma 5.4, every max of $A_1(x)$ through x distinct from $I(x)$ is contained in $A_1(x) \cap H$. Hence, by Lemma 4.3 also $I(x)$ is contained in $A_1(x) \cap H$. ■

The following is an immediate consequence of Lemmas 5.4 and 5.5.

Corollary 5.6 *For every point $x \in P_2$, all points of $A_1(x) \cup A_2(x)$ at distance at most $n-2$ from x belong to H.* ■

A proof of the following lemma is contained in [6, Lemma 4.8 and 4.9] and is based on a double use of Corollary 5.6.

Lemma 5.7 *For every point $x \in P_2$, $A_1(x)$ and $A_2(x)$ are the only two deep maxes through x.*

By relying on previous lemmas, the following can be proved (see [6, Lemma 4.20]).

Lemma 5.8 *No point of type II is collinear with a point of type I.*

The proof of the following lemma is more complicated, but can be found in [6, Lemmas 4.22, 4.23 and 4.24]. Besides previous lemmas, it also relied on a use of the induction hypothesis.

Lemma 5.9 *If there exists a deep max M containing a point of P_1, then H is a hyperplane of subspace-type that extends a hyperplane of $\widetilde{M}$.*

In view of Lemma 5.9, we may now suppose that no point of P_1 is contained in a deep max. Since H is a proper subspace of $DW(2n-1,\mathbb{F})$, every deep max contains at least one point of P_2. We now define a relation R on the set $\mathcal{M}$ of all deep maxes. For $M_1, M_2 \in \mathcal{M}$, we say that $(M_1, M_2) \in R$ if and only if either $M_1 = M_2$ or $M_1 \cap M_2 \subseteq P_3$. The proof of the following lemma can be found in [6, Lemma 4.29].

Lemma 5.10 *The relation R is an equivalence relation.* ∎

For every point x of P_2 and every $i \in \{1,2\}$, let $\mathcal{C}_i(x)$ denote the equivalence class of R containing the deep max $A_i(x)$. Since $A_1(x) \cap A_2(x)$ contains $x \in P_2$, we have $\mathcal{C}_1(x) \neq \mathcal{C}_2(x)$. Similarly as in [6, Lemma 4.31], the following can be proved.

Lemma 5.11 *If x and y are two collinear points of P_2, then $\{\mathcal{C}_1(x), \mathcal{C}_2(x)\} = \{\mathcal{C}_1(y), \mathcal{C}_2(y)\}$.*

In [6, Lemma 4.30], it was proved that every two points $x \in H$ and $y \in H$ of type II are connected by a path that entirely consists of points of H that have type II. This required an advanced argument that needed a prior treatment of the case $n = 4$ in the induction process. By simplifying and rearranging the order of the lemmas, it turns out that a weaker version of that claim is already sufficient to complete the proofs of our main results, namely it suffices to prove this claim in the special case where the convex subspace $\langle x, y \rangle$ is contained in H. This is realized in the following lemma. Subsequently, the weaker lemma is used to show that R has precisely two equivalence classes (see Lemma 5.13).

Lemma 5.12 *Let x and y be two points of type II of H such that the convex subspace $\langle x, y \rangle$ is contained in H. Then there exists a path in $DW(2n-1,\mathbb{F})$ connecting x and y that entirely consists of points of H that have type II.*

Proof. We will prove this by induction on $\mathrm{d}(x, y)$, the cases $\mathrm{d}(x, y) = 0$ and $\mathrm{d}(x, y) = 1$ being obvious. Suppose therefore that $\mathrm{d}(x, y) \geq 2$.

Let L_x denote an arbitrary line through x contained in $\langle x, y \rangle$, let z denote the unique point on L_x at distance $\mathrm{d}(x, y) - 1$ from y and let L_y be a line of $\langle x, y \rangle$ through y not contained in $\langle y, z \rangle$. Then every point of L_x has distance $\mathrm{d}(x, y) - 1$ from a unique point of L_y. By Lemma 5.8, all points of $L_x \cup L_y$ belong to $P_2 \cup P_3$. Lemma 5.3 implies that each of $|L_x \cap P_3|$ and $|L_y \cap P_3|$ has size at most 1. Since $|L_x|, |L_y| \geq 3$, we then know that there exist points $x' \in L_x \setminus P_3$ and $y' \in L_y \setminus P_3$ at distance $\mathrm{d}(x, y) - 1$ from each other. Since $x', y' \in P_2$ and $\langle x', y' \rangle \subseteq \langle x, y \rangle \subseteq H$, the induction hypothesis applies: the points x' and y' are connected by a path that entirely consists of points of H that have type II. Hence, also x and y are connected by such a path. ∎

Lemma 5.13 *The equivalence relation R has precisely two classes.*

Proof. Let x be an arbitrary point of P_2 and let M be an arbitrary element of $\mathcal{M}$. We will prove that either $M \in \mathcal{C}_1(x)$ or $M \in \mathcal{C}_2(x)$.

By our assumption, no point of type I is contained in a deep max. In particular, every point of $A_1(x)$ belongs to P_2 or P_3. If $M \cap A_1(x) \subseteq P_3$, then $M \in \mathcal{C}_1(x)$ and we are done. Suppose therefore that there exists a point $y \in M \cap A_1(x) \cap P_2$. By Lemma 5.12, there exists a path in $DW(2n-1, \mathbb{F})$ connecting x and y that entirely consists of points of H that have type II. By applying Lemma 5.11 a number of times, we see that $\{\mathcal{C}_1(x), \mathcal{C}_2(x)\} = \{\mathcal{C}_1(y), \mathcal{C}_2(y)\}$. Since either $M \in \mathcal{C}_1(y)$ or $M \in \mathcal{C}_2(y)$, we have that either $M \in \mathcal{C}_1(x)$ or $M \in \mathcal{C}_2(x)$. ■

Let $\mathcal{C}_1$ and $\mathcal{C}_2$ denote the two classes of the equivalence relation R, and let π_i, $i \in \{1, 2\}$, be the set of points of $W(2n-1, \mathbb{F})$ corresponding to the maxes of $\mathcal{C}_i$.

Lemma 5.14 *Every maximal singular subspace α meeting $\pi_1 \cup \pi_2$ belongs to H.*

Proof. If x is a common point of α and $\pi_1 \cup \pi_2$, then the max M_x corresponding to x is deep as it belongs to $\mathcal{C}_1$ or $\mathcal{C}_2$. This implies that the point α of M_x is contained in H. ■

Lemma 5.15 *The sets π_1 and π_2 are two disjoint subspaces of* $\mathrm{PG}(2n-1, \mathbb{F})$.

Proof. As $\mathcal{C}_1$ and $\mathcal{C}_2$ are disjoint, also the sets π_1 and π_2 are disjoint. It remains to show that each π_i is a subspace of $\mathrm{PG}(2n-1, \mathbb{F})$. Let L be a line of $\mathrm{PG}(2n-1, \mathbb{F})$ containing two distinct points x and y of π_i. For every point z of L, let M_z denote the max of $DW(2n-1, \mathbb{F})$ consisting of all singular subspaces of $W(2n-1, \mathbb{F})$ containing z.

Suppose L is a hyperbolic line of $W(2n-1, \mathbb{F})$. Then the maxes M_z, $z \in L$, are mutually disjoint, and each of them is covered by the lines meeting M_x and M_y. As H is a subspace containing M_x and M_y, the latter implies that each M_z is contained in H. If $z \in L \setminus \{x, y\}$, then $M_z \cap M_x = \emptyset$ implies that M_x and M_z belong to the same equivalence class, i.e. to $\mathcal{C}_i$. So, each point of L belongs to π_i.

Suppose L is a line of $W(2n-1, \mathbb{F})$. Then there exists a convex subspace A of diameter $n-2$ in $DW(2n-1, \mathbb{F})$ which is contained in all maxes M_z, $z \in L$. We have $A = M_x \cap M_y \subseteq P_3$. If $z \in L$, then every point of M_z has distance at most 1 from a point of A, implying that M_z is contained in H. If $z \in L \setminus \{x, y\}$, then $M_z \cap M_x = M_x \cap M_y = A \subseteq P_3$, implying that $M_z \in \mathcal{C}_i$. So, also here every point of L belongs to π_i. ■

At this stage, the proofs of Theorems 1.1 and 1.2 can be completed as in [6]. The idea is first to show that $\dim(\pi_2) = n-1$. It is impossible that $\dim(\pi_2) \geq n$ as this would imply that every maximal singular subspace of $W(2n-1, \mathbb{F})$ meets π_2, and so H would coincide with the whole point set by Lemma 5.14. If $\dim(\pi_2) \leq n-2$ and $u \in \pi_1$, then it can be shown (see [6, Lemma 4.36]) that there exists a singular subspace through u disjoint from π_2. So, if $M \in \mathcal{C}_1$ denotes the max corresponding to u, then there exists a point in M that is not contained in an element of $\mathcal{C}_2$. Such a point of M cannot exist by [6, Lemma 4.34]. So, $\dim(\pi_2) = n-1$. The following lemma finishes the proofs of Theorems 1.1 and 1.2.

Lemma 5.16 *The hyperplane H is of subspace-type.*

Proof. Since $\dim(\pi_2) = n - 1$, we can consider the hyperplane H_{π_2} of subspace-type. By Lemma 5.14, $H_{\pi_2} \subseteq H$. But this implies that $H_{\pi_2} = H$ as H_{π_2} is a maximal proper subspace of $DW(2n-1, \mathbb{F})$ by Blok & Brouwer [1, Proposition 7.3] or Shult [19, Lemma 6.1]. ■

References

[1] R. J. Blok and A. E. Brouwer. The geometry far from a residue. pp. 29–38 in "Groups and geometries (Siena, 1996)", Trends Math., Birkhäuser, 1998.

[2] F. Buekenhout and P. J. Cameron. *Projective and affine geometry over division rings.* Chapter 2 of the "Handbook of Incidence Geometry" (ed. F. Buekenhout), North-Holland, 1995.

[3] I. Cardinali, B. De Bruyn and A. Pasini. Locally singular hyperplanes in thick dual polar spaces of rank 4. *J. Combin. Theory Ser. A* 113 (2006), 636–646.

[4] A. M. Cohen and E. E. Shult. Affine polar spaces. *Geom. Dedicata* 35 (1990), 43–76.

[5] B. N. Cooperstein and A. Pasini. The non-existence of ovoids in the dual polar space $DW(5, q)$. *J. Combin. Theory Ser. A* 104 (2003), 351–364.

[6] B. De Bruyn. Hyperplanes of $DW(2n-1, q)$, $q \neq 2$, without ovoidal quads. *Discrete Math.* 307 (2007), 2680–2694.

[7] B. De Bruyn. The hyperplanes of $DQ(2n, \mathbb{K})$ and $DQ^-(2n+1, q)$ which arise from their spin-embeddings. *J. Combin. Theory Ser. A* 114 (2007), 681–691.

[8] B. De Bruyn. A characterization of the SDPS-hyperplanes of dual polar spaces. *European J. Combin.* 28 (2007), 705–714.

[9] B. De Bruyn. On a class of hyperplanes of the symplectic and hermitian dual polar spaces. *Electron. J. Combin.* 16 (2009), Research Paper 1, 20 pp.

[10] B. De Bruyn and H. Pralle. The hyperplanes of $DW(5, q)$ with no ovoidal quad. *Glasg. Math. J.* 48 (2006), 75–82.

[11] B. De Bruyn and P. Vandecasteele. Valuations and hyperplanes of dual polar spaces. *J. Combin. Theory Ser. A* 112 (2005), 194–211.

[12] A. Pasini and S. Shpectorov. Uniform hyperplanes of finite dual polar spaces of rank 3. *J. Combin. Theory Ser. A* 94 (2001), 276–288.

[13] H. Pralle. A remark on non-uniform hyperplanes of finite thick dual polar spaces. *European J. Combin.* 22 (2001), 1003–1007.

[14] H. Pralle. Hyperplanes of dual polar spaces of rank 3 with no subquadrangular quad. *Adv. Geom.* 2 (2002), 107–122.

[15] H. Pralle and S. Shpectorov. The ovoidal hyperplanes of a dual polar space of rank 4. *Adv. Geom.* 7 (2007), 1–17.

[16] M. A. Ronan. Embeddings and hyperplanes of discrete geometries. *European J. Combin.* 8 (1987), 179–185.

[17] E. E. Shult. Generalized hexagons as geometric hyperplanes of near hexagons. pp. 229–239 in "Groups, Combinatorics and Geometry" (eds. M. Liebeck and J. Saxl), Cambridge Univ. Press, 1992.

[18] E. E. Shult. Embeddings and hyperplanes of Lie incidence geometries. pp. 215–232 in "Groups of Lie type and their geometries (Como, 1993)", London Math. Soc. Lecture Note Ser. 207, Cambridge Univ. Press, 1995.

[19] E. E. Shult. On Veldkamp lines. *Bull. Belg. Math. Soc. Simon Stevin* 4 (1997), 299–316.

[20] E. E. Shult and J. A. Thas. Hyperplanes of dual polar spaces and the spin module. *Arch. Math.* 59 (1992), 610–623.

[21] J. Tits. *Buildings of Spherical Type and Finite BN-pairs.* Lecture Notes in Mathematics 386, Springer, 1974.