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Modular symmetry in magnetized/intersecting D-brane models

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We study the modular symmetry in four-dimensional low-energy effective field theory, which is derived from type IIB magnetized D-brane models and type IIA intersecting D-brane models. We analyze modular symmetric behaviors of perturbative terms and non-perturbative terms induced by D-brane instanton effects. Anomalies are also investigated and such an analysis on anomalies suggests corrections in effective field theory.

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1. Introduction

T-duality in string theory relates a theory with the compact space size R to another theory with the size $1/R$. Thus, T-duality is a quite non-trivial symmetry in string theory. Indeed, one type of superstring theory is related to a different type of superstring theory by T-duality. (For a review, see Ref. [1].)

T-duality also has a remnant in four-dimensional (4D) low-energy effective field theory derived from superstring theory. In particular, 4D low-energy effective field theory of heterotic string theory with a certain compactification is invariant under the modular transformation of the moduli τ,

$$\tau \to \frac{a\tau + b}{c\tau + d}, \tag{1}$$

with $ad - bc = 1$ and $a, b, c, d \in \mathbf{Z}$, at least at the perturbative level. This is a symmetry inside a 4D effective field theory, but not between two theories. We refer to this symmetry inside one effective field theory as the modular symmetry in order to distinguish it from the T-duality between two theories.

The modular symmetry plays an important role in studies on 4D low-energy effective field theory of heterotic string theory. For example, moduli stabilization and supersymmetry breaking were studied with the assumption that non-perturbative effects are also modular invariant [2,3]. Moreover, anomalies of this symmetry were analyzed [4,5]. The anomaly structure in heterotic string theory has a definite structure.[1] Their phenomenological applications were also studied (see, e.g., Refs. [7,8]). In addition, the modular invariant potential of the modulus was studied for cosmic inflation [9]. Thus, the modular symmetry in 4D low-energy effective field theory is important from several viewpoints: theoretical, particle physics, and cosmology.

[1] See also Ref. [6].

In this paper, we study the modular symmetry in 4D low-energy effective field theory derived from type II superstring theory. In particular, we consider the 4D low-energy effective field theory derived from type IIB magnetized D-brane models and type IIA intersecting D-brane models. Their 4D low-energy effective field theories have been studied before (for a review, see Refs. [10,11]). We study the modular symmetry at perturbative level in their low-energy effective field theories. The T-duality of Yukawa couplings between magnetized D-brane models and intersecting D-brane models was studied in Ref. [12]. This is very useful for our purpose. We extend such analysis to show modular transformation of 4D low-energy effective field theory including three-point and higher-order couplings. Also, their anomalies are examined and the anomaly structure could provide non-trivial information like those in heterotic string theory. Furthermore, we discuss non-perturbative effects.

The paper is organized as follows. In Sect. 2, we study the modular symmetry of Yukawa couplings and higher-order couplings at the perturbative level in the 4D low-energy effective field theory derived from type IIB magnetized D-brane models. In Sect. 3, we study supergravity theory derived from type IIA intersecting D-brane models. In particular, we investigate the anomaly structure of the modular symmetry. In Sect. 4, we study the modular symmetry of non-perturbative terms induced by D-brane instanton effects. Section 5 provides the conclusions.

2. Modular symmetry

Here, we study the modular symmetry in the 4D low-energy effective field theory derived from type IIB magnetized D-brane models.

2.1. *Magnetized D-brane models*

We start with magnetized D9-brane models in type IIB theory. We compactify six-dimensional (6D) space to the 6D torus, e.g. three 2-tori. The metric of the rth 2-torus for $r = 1, 2, 3$ is written by

$$g = R_r^2 \begin{pmatrix} 1 & \mathrm{Re}\,\tau_r \\ \mathrm{Re}\,\tau_r & |\tau_r|^2 \end{pmatrix} \tag{2}$$

on the real basis (x_r, y_r), where τ_r denotes the complex structure modulus. We denote the volume of the rth 2-torus by $\mathcal{A}_r = R_r^2 \mathrm{Im}\,\tau_r$. We use the complex coordinate $z_r = x_r + \tau_r y_r$.

2.1.1. *Yukawa couplings*

Here, we review the analysis of Yukawa couplings in Ref. [12]. Our setup includes several stacks of D9-branes with magnetic fluxes. We assume that our setup preserves 4D $N = 1$ supersymmetry. Among several D-branes, we consider two stacks of N_a and N_b D9-branes, which correspond to the $U(N_a) \times U(N_b)$ gauge symmetry. We put magnetic fluxes, $F_r^a (= F^a_{z_r \bar{z}_r})$ and $F_r^b (= F^b_{z_r \bar{z}_r})$ on these D-branes along $U(1)_a$ and $U(1)_b$ directions of $U(N_a) = U(1)_a \times SU(N_a)$ and $U(N_b) = U(1)_b \times SU(N_b)$. The magnetic fluxes must be quantized as $F_r^a = \frac{\pi i}{\mathrm{Im}\,\tau_r} m_a^r$ in the complex basis. For simplicity, we do not include Wilson lines here [12].

The open strings between these magnetized branes have massless modes. There appear I_{ab}^r zero-modes on the rth 2-torus, where $I_{ab}^r = m_a^r - m_b^r$, and the total number of massless modes is given by their product, $I_{ab} = \prod_{r=1}^3 I_{ab}^r$. Their zero-mode profiles on the rth 2-torus are written by [12]

$$\psi^{j,N}(\tau_r, z_r) = \mathcal{N}_r \cdot e^{i\pi N z_r \mathrm{Im}\, z_r/\mathrm{Im}\,\tau_r} \cdot \vartheta \begin{bmatrix} \frac{j}{N} \\ 0 \end{bmatrix} (N z_r, N\tau_r) \tag{3}$$

for $N = I^r_{ab} > 0$, where j denotes the zero-mode index for $j = 1, \ldots, N$ (mod N), and $\mathcal{N}_r$ is the normalization factor given by

$$\mathcal{N}_r = \left(\frac{2 \mathrm{Im}\, \tau_r |N|}{\mathcal{A}_r^2} \right)^{1/4}. \tag{4}$$

The ϑ-function is defined as

$$\vartheta \begin{bmatrix} a \\ b \end{bmatrix} (\nu, \tau) = \sum_{l \in \mathbf{Z}} e^{\pi i (a+l)^2 \tau} e^{2\pi i (a+l)(\nu+b)}. \tag{5}$$

These zero-modes are also written by another basis,

$$\chi^{j,N}(\tau_r, z_r) = \frac{\mathcal{N}_r}{\sqrt{N}} \cdot e^{i\pi N z_r \mathrm{Im}\, z_r / \mathrm{Im}\, \tau_r} \cdot \vartheta \begin{bmatrix} 0 \\ \frac{j}{N} \end{bmatrix} (z_r, \tau_r/N), \quad j = 1, \ldots, N. \tag{6}$$

These bases are related as

$$\chi^{j,N} = \frac{1}{\sqrt{N}} \sum_k e^{2\pi i \frac{jk}{N}} \psi^{k,N}. \tag{7}$$

Note that the zero-mode profiles of bosonic and fermionic modes are the same in supersymmetric models. For $N = I^r_{ab} < 0$, the zero-mode profiles are obtained by $\psi^{j,N}(\tau_r, z_r)^*$.

In addition to the above two stacks of D-branes, we consider another stack of N_c D9-branes. Then, there appear three types of massless modes, a–b, b–c, and c–a modes. Their Yukawa couplings among canonically normalized fields can be obtained by overlap integral of wavefunctions,

$$y_{ijk} = C_{abc}\, e^{\phi_{10}/2} \prod_{r=1}^{3} \int dz_r d\bar{z}_r\ \psi^{i, I^r_{ab}}(z_r) \cdot \psi^{j, I^r_{ca}}(z_r) \cdot \left(\psi^{k, I^r_{cb}}(z_r) \right)^*, \tag{8}$$

where C_{abc} is the moduli-independent coefficient and ϕ_{10} denotes the ten-dimensional dilaton. Here, we set $I^r_{ab} + I^r_{ca} = -I^r_{bc} = I^r_{cb}$, because of gauge invariance. To be exact, we should replace the zero-mode indexes i, j, k by i^r, j^r, k^r. However, we denote them as i, j, k to simplify the equations. Hereafter, we use a similar simplification. In this computation, the following relation of zero-mode profiles,

$$\begin{aligned} \psi^{i, I^r_{ab}} \cdot \psi^{j, I^r_{ca}} &= \mathcal{A}_r^{-1/2} (2 \mathrm{Im}\, \tau_r)^{1/4} \left| \frac{I^r_{ab} I^r_{ca}}{I^r_{bc}} \right|^{1/4} \\ &\quad \cdot \sum_m \psi^{i+j+I^r_{ab} m, I^r_{cb}}(z) \cdot \vartheta \begin{bmatrix} \frac{I^r_{ca} i - I^r_{ab} j + I^r_{ab} I^r_{ca} m}{-I^r_{ab} I^r_{bc} I^r_{ca}} \\ 0 \end{bmatrix} \left(0, \tau_r \left| I^r_{ab} I^r_{bc} I^r_{ca} \right| \right), \end{aligned} \tag{9}$$

is very useful. Then, the Yukawa coupling is written by [12]

$$y_{ijk} = C_{abc}\, e^{\phi_{10}/2} \prod_{r=1}^{3} \left(\frac{2 \mathrm{Im}\, \tau_r}{\mathcal{A}_r{}^2} \right)^{1/4} \left| \frac{I^r_1 I^r_2}{I^r_1 + I^r_2} \right|^{1/4} \cdot \vartheta \begin{bmatrix} \delta^r_{ijk} \\ 0 \end{bmatrix} \left(0, \tau_r \left| I^r_{ab} I^r_{bc} I^r_{ca} \right| \right), \tag{10}$$

where

$$\delta^r_{ijk} = \frac{i}{I^r_{ab}} + \frac{j}{I^r_{ca}} + \frac{k}{I^r_{bc}}.$$

Similarly, the Yukawa couplings can be written in the basis χ,

$$y_{lmn} = C_{abc}\, e^{\phi_{10}/2} \prod_{r=1}^{3} \left(\frac{2\mathrm{Im}\,\tau_r}{\mathcal{A}_r{}^2} \right)^{1/4} \left| \frac{I_1^r I_2^r}{I_1^r + I_2^r} \right|^{1/4} \cdot |I_{ab} I_{bc} I_{ca}|^{-1/2}$$

$$\cdot\, \vartheta \begin{bmatrix} 0 \\ \delta^r_{ijk} \end{bmatrix} \left(0, \tau_r / \left| I^r_{ab} I^r_{bc} I^r_{ca} \right| \right). \tag{11}$$

It would be convenient to use the 4D dilaton,

$$e^{\phi_4} = e^{\phi_{10}} \prod_{r=1}^{3} (\mathcal{A}_r)^{-1/2}, \tag{12}$$

and we define $\tilde{I}^r = I^r / \mathcal{A}_r$. Then, we can write the Yukawa coupling

$$y_{ijk} = C_{abc}\, e^{\phi_4/2} \prod_{r=1}^{3} (2\mathrm{Im}\,\tau_r)^{1/4} \left| \frac{\tilde{I}_1^r \tilde{I}_2^r}{\tilde{I}_1^r + \tilde{I}_2^r} \right|^{1/4} \cdot \vartheta \begin{bmatrix} \delta^r_{ijk} \\ 0 \end{bmatrix} \left(0, \tau_r \left| I^r_{ab} I^r_{bc} I^r_{ca} \right| \right). \tag{13}$$

2.1.2. Modular symmetry

Now, let us study the modular transformation of the complex structure moduli τ_r. Recall that we use the basis, so that the fields are normalized canonically. Thus, we just investigate the modular transformation of the Yukawa couplings. The modular transformation (1) is generated by the two generators, s and t,

$$s : \tau \to -\frac{1}{\tau}, \qquad t : \tau \to \tau + 1. \tag{14}$$

The modular function satisfies

$$f(-1/\tau) = \tau^n f(\tau), \tag{15}$$

where n is called its modular weight. It is obvious that $\mathrm{Im}\,\tau$ is invariant under t. Under s, we have

$$\mathrm{Im}\,\tau \to \frac{1}{|\tau|^2} \mathrm{Im}\,\tau. \tag{16}$$

The ϑ-function $\vartheta\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right](0, \tau)$ is the modular function with the modular weight 1/2.

The ϑ-function part in the Yukawa coupling is transformed under $s : \tau \to -1/\tau$,

$$\vartheta \begin{bmatrix} \delta_{ijk} \\ 0 \end{bmatrix} (0, \tau\, |I_{ab} I_{bc} I_{ca}|) \to \vartheta \begin{bmatrix} \delta_{ijk} \\ 0 \end{bmatrix} (0, -|I_{ab} I_{bc} I_{ca}|/\tau)\,. \tag{17}$$

Furthermore, using the Poisson resummation formula, we find

$$\vartheta \begin{bmatrix} \delta_{ijk} \\ 0 \end{bmatrix} (0, -|I_{ab} I_{bc} I_{ca}|/\tau) = (-i\tau)^{1/2}\, |I_{ab} I_{bc} I_{ca}|^{-1/2}\, \vartheta \begin{bmatrix} 0 \\ \delta_{ijk} \end{bmatrix} (0, \tau / \,|I_{ab} I_{bc} I_{ca}|)\,. \tag{18}$$

Thus, the τ-dependent part in the Yukawa coupling transforms under s as

$$(\mathrm{Im}\,\tau)^{1/4} \cdot \vartheta \begin{bmatrix} \delta_{lmn} \\ 0 \end{bmatrix} (0, \tau\, |I_{ab} I_{bc} I_{ca}|) \to (\mathrm{Im}\,\tau)^{1/4} \cdot |I_{ab} I_{bc} I_{ca}|^{-1/2} \cdot \vartheta \begin{bmatrix} 0 \\ \delta_{lmn} \end{bmatrix} (0, \tau / \,|I_{ab} I_{bc} I_{ca}|). \tag{19}$$

This is nothing but the τ-dependent part of the Yukawa coupling in the χ basis. Therefore, the Yukawa coupling terms in 4D low-energy effective field theory are invariant under modular transformation, including basis change.

The above results can be extended to the magnetic flux,

$$F_{z\bar{z}} = \frac{\pi i}{\operatorname{Im}\tau}\begin{pmatrix} \frac{m_a}{n_a}\mathbf{1}_{n_a} & & \\ & \frac{m_a}{n_a}\mathbf{1}_{n_b} & \\ & & \frac{m_a}{n_a}\mathbf{1}_{n_c} \end{pmatrix}, \tag{20}$$

by replacing I_{ab} as $I_{ab} = n_b m_a - n_a m_b$.

2.1.3. Higher-order couplings

We can study higher-order couplings in a similar way [13]. For example, the four-point coupling can be obtained by computing the integral of zero-mode profiles,

$$C_{abcd}e^{\phi_{10}}\prod_{r=1}^{3}\int dz_r d\bar{z}_r\ \psi^{i,I^r_{ab}}(z_r)\cdot\psi^{j,I^r_{bc}}(z_r)\cdot\psi^{k,I^r_{cd}}(z_r)\cdot\left(\psi^{l,I^r_{ad}}(z_r)\right)^*. \tag{21}$$

We use the relation (9), and then we obtain [13]

$$\begin{aligned} y_{ijk\bar{l}} = Ce^{\phi_{10}}\prod_{r=1}^{3}\left(\frac{2\operatorname{Im}\tau_r}{\mathcal{A}_r^2}\right)^{\frac{2}{4}}\left|\frac{I^r_{ab}I^r_{bc}}{M^r}\right|^{\frac{1}{4}}\cdot\left|\frac{M^r I^r_{cd}}{I^r_{ad}}\right|^{\frac{1}{4}} \\ \sum_{m\in\mathbf{Z}_{I^r_{ab}+I^r_{bc}}}\vartheta\begin{bmatrix}\frac{I^r_{bc}i-I^r_{ab}j+I^r_{ab}I^r_{bc}m}{I^r_{ab}I^r_{bc}M^r}\\ 0\end{bmatrix}\left(0,\tau_r I^r_{ab}I^r_{bc}M^r\right)\cdot\vartheta\begin{bmatrix}\frac{I^r_{cd}l-I^r_{ad}k+I^r_{cd}I^r_{ad}r}{I^r_{cd}I^r_{ad}M^r}\\ 0\end{bmatrix}\left(0,\tau_r I^r_{cd}I^r_{ad}M^r\right), \end{aligned} \tag{22}$$

where $M^r = I^r_{ab} + I^r_{bc}$ and $i + j + k + I^r_{ab}m + (I^r_{ab} + I^r_{bc})n = \ell + kI_{ad}r$ with a certain integer n.

Similarly, we can compute generic n-point couplings [13], whose τ dependence as well as ϕ_4 dependence appears in the form

$$e^{(n-2)\phi_4/2}\prod_{i=1}^{n-2}\prod_{r=1}^{3}(\operatorname{Im}\tau_r)^{1/4}\cdot\vartheta\begin{bmatrix}\delta^r_i\\ 0\end{bmatrix}\left(0,\tau_r\alpha^r_i\right) \tag{23}$$

for proper values of δ^r_i and α^r_i, because we use the relation (9). Note that the ϑ-function multiplied by $\operatorname{Im}\tau^{-1/4}$ is invariant under modular transformation. Thus, 4D low-energy effective field theory is invariant at the perturbative level under modular transformation of the complex structure moduli, up to a change of field basis.

Similarly, we can study the orientifold and orbifold compactifications. For example, the zero-mode profiles on the Z_2 orbifold can be written by linear combinations of zero-mode profiles on the torus [14],

$$\psi^{j,N}(z)_{\text{orbifold}} = \frac{1}{\sqrt{2}}\left(\psi^{j,N}(z) + \psi^{N-j,N}(z)\right). \tag{24}$$

Thus, the Yukawa couplings on the orbifold, as well as higher-order couplings, can be written by linear combinations of Yukawa couplings on the torus [14]. Then, the Yukawa couplings on the orbifold are also modular invariant in the same way as those on the torus. Furthermore, the modular symmetry in magnetized D5- and D7-brane models can be studied in a similar way.

3. Supergravity and anomaly

In this section, we study modular symmetry within the framework of string-derived supergravity and investigate its anomaly.

3.1. Intersecting D-brane models

In the previous section, we studied modular symmetry in 4D low-energy effective field theory of magnetized D-brane models for canonically normalized fields. Here, we study type IIA intersecting D-brane models, which are T-dual to magnetized D-brane models. In intersecting D-brane models, the Kähler metric of matter fields was computed [10,15–18]. In this section, we study the modular symmetry from the viewpoint of supergravity derived from intersecting D-brane models. In particular, we study intersecting D6-brane models, where two sets of D6-branes, e.g. $D6_a$ and $D6_b$, intersect each other at an angle $\pi\theta^r_{ab}$ on the rth 2-torus.

First, we write the supergravity fields in type IIB theory as

$$\mathrm{Re}\, S = e^{-\phi_{10}} \prod_{r=1}^{3} \mathcal{A}_r, \qquad \mathrm{Re}\, T_r = e^{-\phi_{10}} \mathcal{A}_r, \qquad U_r = i\tau_r, \tag{25}$$

where the imaginary parts of S and T_r correspond to certain axion fields. Their Kähler potential is written by

$$K = -\ln(S+\bar{S}) - \sum_{r=1}^{3} \ln(T_r + \bar{T}_r) - \sum_{r=1}^{3} \ln(U_r + \bar{U}_r). \tag{26}$$

We take the T-dual along the x_r direction on each 2-torus from magnetized D9-branes to intersecting D6-branes. Then, we replace

$$T_r \longleftrightarrow U_r. \tag{27}$$

We have seen that low-energy effective field theory of canonically normalized fields is modular symmetric for τ_r in type IIB magnetized D-brane models. Thus, the low-energy effective field theory of type IIA intersecting D-brane models must have symmetry under the modular transformation

$$T_r \to \frac{a_r T_r - ib_r}{ic_r T_r + d_r}, \qquad a_r, b_r, c_r, d_r \in \mathbf{Z}, \qquad a_r d_r - b_r c_r = 1, \tag{28}$$

both in a canonically normalized field basis and in a supergravity basis.

We take the T-dual of the Yukawa coupling (13) of the magnetized D9-brane models, and then we can write the Yukawa coupling of intersecting D-brane models:

$$y_{ijk} = C_{abc}\, e^{\phi_4/2} \prod_{r=1}^{3} (2\mathrm{Re}\, T_r)^{1/4} \left| \frac{\tilde{I}_1^r \tilde{I}_2^r}{\tilde{I}_1^r + \tilde{I}_2^r} \right|^{1/4} \cdot \vartheta \begin{bmatrix} \delta^r_{ijk} \\ 0 \end{bmatrix} \left(0, T_r \left| I^r_{ab} I^r_{bc} I^r_{ca} \right| \right), \tag{29}$$

where

$$e^{\phi_4} = \frac{(\mathrm{Re}\, U_1 U_2 U_3)^{1/2}}{\mathrm{Re}\, S}. \tag{30}$$

Within the framework of supergravity, physical Yukawa couplings are written by

$$y_{ijk} = (K_{ab}K_{bc}K_{ca})^{-1/2}\, e^{K/2} W_{ijk}, \tag{31}$$

where W_{ijk} denotes the holomorphic Yukawa coupling in the superpotential, i.e.,

$$W = W_{ijk}\Phi_i\Phi_j\Phi_k + \cdots, \tag{32}$$

K is the Kähler potential, and K_{ab}, K_{bc}, K_{ca} are the Kähler metrics of the a–b, b–c, c–a sectors, respectively. Then, the relation (31) requires that

$$K_{ab}K_{bc}K_{ca} \propto \prod_r (T_r + \bar{T}_r)^{-3/2}. \tag{33}$$

The Kähler metric of matter fields has been computed [10,15–18]. The Kähler metric of the a–b sector would be written as

$$K_{ab} = \prod_r (T_r + \bar{T}_r)^{\nu(\theta^r_{ab})}. \tag{34}$$

For example, in Refs. [16–18], the ansatz

$$\nu(\theta^r_{ab}) = -\frac{1}{2} \pm \frac{1}{2}\mathrm{sign}(I_{ab})\theta^r_{ab} \tag{35}$$

was discussed by comparing the holomorphic and physical gauge couplings and threshold corrections. They satisfy the above relation (33) when

$$\mathrm{sign}(I_{ab})\theta^r_{ab} + \mathrm{sign}(I_{bc})\theta^r_{bc} + \mathrm{sign}(I_{ca})\theta^r_{ca} = 0. \tag{36}$$

Similarly, the n-point couplings in magnetized D-brane models include the τ-dependent factor (23). Then, its T-dual intersecting D-brane models include $(2\mathrm{Re}\,T_r)^{n-2}/4$. This requires that the product of the Kähler metric satisfies

$$K_{a_1a_2}K_{a_2a_3}\cdots K_{a_na_1} = \prod_r (T_r + \bar{T}_r)^{-n/2}. \tag{37}$$

This relation is also satisfied by Eq. (35) when

$$\mathrm{sign}(I_{a_1a_2})\theta^r_{a_1a_2} + \mathrm{sign}(I_{a_2a_3})\theta^r_{a_2a_3} + \cdots + \mathrm{sign}(I_{a_na_1})\theta^r_{a_na_1} = 0. \tag{38}$$

We can take the T-dual of type IIA intersecting D-brane models along the y_r direction,

$$\text{type IIB model X} \underset{\text{T-dual along } x_r}{\Longleftrightarrow} \text{type IIA model} \underset{\text{T-dual along } y_r}{\Longleftrightarrow} \text{type IIB model Y}, \tag{39}$$

and then obtain type IIB magnetized D-brane models, which are different from the one discussed in the previous section. The relation between these two type IIB models was studied in Ref. [12], in particular the Yukawa couplings. Our results in the previous section can be understood as two different theories through double T-duality such as Ref. [12], but in any rate we are interested in the modular symmetry in one 4D low-energy effective field theory, as mentioned in Sect. 1.

3.2. Anomaly

In the previous section, the modular symmetry in the supergravity basis was studied. The chiral multiplet Φ_{ab} in the a–b sector has the Kähler metric (34). Thus, the chiral multiplet, Φ_{ab}, transforms as

$$\Phi_{ab} \to (ic_r T_r + d_r)^{-\nu(\theta^r_{ab})}\Phi_{ab} \tag{40}$$

under the modular transformation (28). That is, the matter field has the modular weight $\nu(\theta^r_{ab})$ under the modular transformation of the rth 2-torus.

Such a modular transformation may be anomalous. The supergravity Lagrangian includes the following couplings:

$$\left(\frac{1}{2}\mathrm{Re} f\bar{\lambda}\gamma^\mu\lambda - \frac{1}{2}K_{i\bar{j}}\bar{\psi}_j\gamma^\mu\psi_i\right)\frac{1}{2}V^{\text{Kähler}}_\mu + \left(\frac{1}{2}K_{i\bar{j}}\bar{\psi}_j\gamma^\mu\psi_l(-i\Gamma_{ikl}\partial_\mu\psi_k) + \text{h.c.}\right), \tag{41}$$

where λ denotes the gaugino, K_{ii} is the Kähler metric of Φ_i with the bosonic and fermionic components, ϕ_i and ψ_i,

$$\Gamma_{ijk} = \frac{\partial}{\partial\phi^i}\ln K_{jk}, \qquad V^{\text{Kähler}}_\mu = -i\left(\frac{\partial K}{\partial\phi_i}\partial_\mu\phi_i - \frac{\partial K}{\partial\bar{\phi}_j}\partial_\mu\bar{\phi}_j\right). \tag{42}$$

These couplings induce the anomaly of modular symmetry. Its anomaly coefficient of mixed anomaly with the $SU(N_a)$ gauge group is written by [4]

$$A^r_a = -C_2(G_a) + \sum_{\text{matter},b} T(R_a)(1 + 2\nu(\theta^r_{ab})), \tag{43}$$

where $C_2(G_a)$ is the quadratic Casimir and $T(R_a)$ is the Dynkin index of the representation R_a. For simplicity, we consider the intersecting D-brane models on the torus. In this case, we can write

$$A^r_a = -N_a + \frac{1}{2}\sum_b N_b I_{ab}(1 + 2\nu(\theta^r_{ab})). \tag{44}$$

This anomaly can be canceled in two ways [4,5]. One is moduli-dependent threshold corrections and the other is the generalized Green–Schwarz mechanism. The latter would lead to mixing of moduli, e.g. in the Kähler potential. In order to see this, we first review briefly anomalous $U(1)$ and the Green–Schwarz mechanism in the next subsection [10,11,19].

3.2.1. Anomalous $U(1)$

First, let us consider the D6$_b$-branes wrapping the 3-cycle $[\Pi_b]$, whose wrapping numbers are (n^r_b, m^r_b) along (x_r, y_r). We introduce the basis of 3-cycles, $[\alpha^0]$ and $[\alpha^k]$ with $k = 1, 2, 3$, such that $[\alpha^0]$ is along $(1, 0)$ for all of (x_r, y_r), while $[\alpha^k]$ is along $(1, 0)$ only for $r = k$ and $(0, 1)$ for the others. We also introduce their duals $[\beta^k]$ such that $[\alpha^i]\cdot[\beta^k] = \delta_{ik}$. These D6-branes correspond to the $U(N_b)$ gauge group, and its gauge kinetic function f_b is written by

$$f_b = q^0_b S - q^r_b U_r, \tag{45}$$

where

$$q_b^0 = [\Pi_b] \cdot [\beta^0] = n_b^1 n_b^2 n_b^3, \qquad q_b^i = [\Pi_b] \cdot [\beta^i] = n_b^i m_b^j m_b^k, \tag{46}$$

where $i \neq j \neq k \neq i$.

Now, we study the $U(1)_a - SU(N_b)^2$ mixed anomaly. Its anomaly coefficient can be written by

$$N_a I_{ab} = q_b^0 Q_a^0 + \sum_i q_b^i Q_a^i, \tag{47}$$

where

$$Q_b^0 = [\Pi_b] \cdot [\alpha^0], \qquad Q_b^i = [\Pi_b] \cdot [\alpha^i]. \tag{48}$$

This anomaly can be canceled by a shift of moduli,

$$S \to S + Q_a^0 \Lambda_a, \qquad U_r - Q_a^r \Lambda_a, \tag{49}$$

in the gauge kinetic function f_b under the $U(1)$ transformation,

$$V_a \to V_a + \Lambda_a + \bar{\Lambda}_a. \tag{50}$$

This means that the Kähler potential is not invariant, but the following Kähler potential is invariant:

$$K = -\ln(S + \bar{S} - Q_a^0 V_v) - \sum_r \ln(U_r + \bar{U}_r - Q_a^r V_a) - \sum_i \ln(T_r + \bar{T}_r). \tag{51}$$

The Green–Schwarz mechanism is the same in the toroidal, orientifold, and orbifold compactifications.

3.2.2. *Anomaly cancelation of modular symmetry*

As mentioned above, the modular anomaly can be canceled in two ways [4,5]. One is moduli-dependent threshold corrections and the other is the generalized Green–Schwarz mechanism. In general, the gauge kinetic function has one-loop threshold corrections due to massive modes as

$$f_a^{(\text{one-loop})} = f_a + \sum_i \Delta_a(T_r), \tag{52}$$

where the first term on the right-hand side corresponds to Eq. (45). The threshold corrections are computed explicitly [17,18,20,21], and their typical form is

$$\Delta_a(T_r) = \frac{\tilde{b}}{4\pi^2} \ln[\eta(iT_r)], \tag{53}$$

where $\tilde{b}$ is beta-function coefficient due to massive modes, and $\eta(iT)$ is the Dedekind eta function, which has the modular weight $1/2$. This threshold correction can partially cancel the anomaly. The other part of the anomaly can be canceled by the generalized Green–Schwarz mechanism, where we impose the transformation

$$S \to \frac{1}{8\pi^2} \sum_r \delta_{\text{GS}}^r \ln(ic_r T_r + d_r), \qquad U_i \to \frac{-1}{8\pi^2} \sum_r \delta_{\text{GS}}^{r,i} \ln(ic_r T_r + d_r), \tag{54}$$

under the modular transformation (28). That is, the generalized Green–Schwarz mechanism could cancel the anomaly proportional to

$$q_a^0 \delta_{\mathrm{GS}}^r + \sum_i q_a^i \delta_{\mathrm{GS}}^{r,i}. \tag{55}$$

By comparison with the total anomaly as well as the $U(1)$ anomaly, a plausible ansatz would be

$$\delta_{\mathrm{GS}}^i = \sum_b Q_b^0(\nu(\theta_{ab}^{(i)}) + c), \qquad \delta_{\mathrm{GS}}^{i,r} = \sum_b Q_b^r(\nu(\theta_{ab}^{(i)}) + c), \tag{56}$$

where c is constant. In this case, the coefficient $\tilde{b}$ may be obtained,

$$\tilde{b} = N_a - \frac{1}{2}\sum_b N_b I_{ab}(1-2c), \tag{57}$$

to cancel the modular anomaly. Indeed, the threshold correction,

$$\Delta_a = \frac{N_a}{4\pi^2}\ln[\eta(iT_i)], \tag{58}$$

was discussed in Refs. [17,18].

The transformation (54) implies that the Kähler potential is not invariant under the modular transformation. The Kähler potential must be modified as

$$K = -\ln(S+\bar{S} - \sum_i \frac{\delta_{\mathrm{GS}}^i}{8\pi^2}(T_i+\bar{T}_i)) - \sum_j \ln(U_j + \bar{U}_j - \sum_i \frac{\delta_{\mathrm{GS}}^{i,j}}{8\pi^2}(T_i+\bar{T}_i)) - \sum_i \ln(T_i+\bar{T}_i). \tag{59}$$

That is, the moduli mix, and instead of S and U^i, the linear combinations

$$S+\bar{S} - \sum_i \frac{\delta_{\mathrm{GS}}^i}{8\pi^2}(T_i+\bar{T}_i), \qquad U_j+\bar{U}_j - \sum_i \frac{\delta_{\mathrm{GS}}^{i,j}}{8\pi^2}(T_i+\bar{T}_i), \tag{60}$$

must appear in 4D low-energy effective field theory. Similar linear combinations were discussed in Ref. [18], although the linear combinations in Ref. [18] include a mixture of all the moduli.[2]

Here, we return to the type IIB model studied in Sect. 2. Similar to the above, we may need to replace

$$S+\bar{S} \to S+\bar{S} - \sum_i \frac{\delta_{\mathrm{GS}}^i}{8\pi^2}(U_i+\bar{U}_i), \qquad T_j+\bar{T}_j \to T_j+\bar{T}_j - \sum_i \frac{\delta_{\mathrm{GS}}^{i,j}}{8\pi^2}(U_i+\bar{U}_i), \tag{61}$$

in 4D low-energy effective field theory. For example, the 4D dilaton factor in the Yukawa coupling would be modified as

$$e^{\phi_4} \to \frac{1}{2}\frac{\left(\prod_i T_j+\bar{T}_j - \sum_i \frac{\delta_{\mathrm{GS}}^{i,j}}{8\pi^2}(U_i+\bar{U}_i)\right)^{1/2}}{S+\bar{S} - \sum_i \frac{\delta_{\mathrm{GS}}^i}{8\pi^2}(U_i+\bar{U}_i)}. \tag{62}$$

[2] The sigma model anomaly concerning U_i is also discussed in Ref. [18].

4. D-brane instanton effects

In Sect. 2, we studied the modular symmetry of perturbative terms in the Lagrangian. In this section, we study terms due to non-perturbative effects, in particular terms induced by D-brane instanton effects. First, we study an illustrative example, and then we will discuss generic aspects.

4.1. *Example*

In this subsection, we study a Majorana mass term induced by an E5-brane in type IIB magnetized orientifold models with O9-planes compactified on a $Z_2 \times Z_2'$ torus. In these models, the non-perturbative corrections to the superpotential are written as [22,23][3]

$$\Delta W = \int d\alpha^1 \cdots d\alpha^n e^{-S_{\rm int}} e^{-S}. \tag{63}$$

In Eq. (63), α^i denotes a fermionic zero-mode of the E5-brane and S denotes the classical action of the E5-brane. $S_{\rm int}$ denotes interaction terms including fermionic zero-modes as

$$S_{\rm int} \sim y_{i_1 \ldots i_n, j_1 \ldots j_m} \alpha^{i_1} \cdots \alpha^{i_n} \Phi_{j_1} \cdots \Phi_{j_m}, \tag{64}$$

where $y_{i_1 \ldots i_n, j_1 \ldots j_m}$ is an $(n+m)$-point coupling and Φ_j is the chiral superfield of the models. Then, we can obtain a Majorana mass term if there are two fermionic zero-modes and three-point couplings like $y_{ijk}\alpha^i \beta^j \Phi_k$. The Majorana mass is generated as

$$M_s^2 \int d^2\alpha d^2\beta e^{y_{ijk}\alpha^i \beta^j \Phi_k} = M_s^2 \epsilon_{ij}\epsilon_{kl} y_{ikm} y_{jln} \Phi_m \Phi_n. \tag{65}$$

In this subsection, we concentrate on the rth two-dimensional torus with two D-branes wrapping the whole compact space for simplicity. We put the magnetic fluxes $\frac{\mathrm{Im}\,\tau}{\pi i} F_r^a = 2$ on one D-brane and $\frac{\mathrm{Im}\,\tau}{\pi i} F_r^b = -2$ on the other D-brane. For simplicity, all Wilson lines are set to zero in this subsection too. Then, there are three chiral fermions between these two branes. These modes are given by the linear combinations of the wave functions on the covering torus ψ^i,

$$\psi^i(z, \bar{z}) = \left(\frac{4 \cdot 2\mathrm{Im}\,\tau}{\mathcal{A}^2}\right)^{1/4} e^{i\pi 4z \mathrm{Im}\, z/\mathrm{Im}\,\tau} \vartheta \begin{bmatrix} i/4 \\ 0 \end{bmatrix} (4z, 4\tau), \tag{66}$$

where $i \in \{0, 1, 2, 3\}$. The three zero-modes on the orbifold are given by Eq. (24) [14]. That is, two of them, Φ_0 and Φ_2, correspond to ψ^0 and ψ^2, respectively, while Φ_1 is given by

$$\frac{1}{\sqrt{2}}(\psi^1 + \psi^3). \tag{67}$$

In addition, an E5-brane with no magnetic flux induces two zero-modes between the E-brane and the D-branes. These zero-modes are given by

$$\alpha^j(z, \bar{z}) = \left(\frac{2 \cdot 2\mathrm{Im}\,\tau}{\mathcal{A}^2}\right)^{1/4} e^{i\pi 2z \mathrm{Im}\, z/\mathrm{Im}\,\tau} \vartheta \begin{bmatrix} j/2 \\ 0 \end{bmatrix} (2z, 2\tau), \tag{68}$$

$$\beta^k(z, \bar{z}) = \left(-\frac{2 \cdot 2|\mathrm{Im}\,\bar{\tau}|}{\mathcal{A}^2}\right)^{1/4} e^{i\pi 2\bar{z} \mathrm{Im}\, \bar{z}/\mathrm{Im}\,\bar{\tau}} \vartheta \begin{bmatrix} k/2 \\ 0 \end{bmatrix} (2\bar{z}, -2\bar{\tau}). \tag{69}$$

[3] For explicit computations on intersecting D-brane orbifold models, see e.g. Ref. [24].

Then, Yukawa couplings are written by

$$y_{ijk} = \left(\frac{4|\mathrm{Im}\,\bar{\tau}|}{\mathcal{A}^2}\right)^{\frac{1}{2}} \sum_{m=0}^{3} \vartheta \begin{bmatrix} \frac{2j-2k+4m}{16} \\ 0 \end{bmatrix} (0, -16\bar{\tau}) \int_{T^2} dz d\bar{z}$$
$$\begin{cases} \left(\frac{4\cdot 2\mathrm{Im}\,\tau}{\mathcal{A}^2}\right)^{\frac{1}{4}} \vartheta \begin{bmatrix} \frac{i}{4} \\ 0 \end{bmatrix} (4z, 4\tau) \vartheta \begin{bmatrix} \frac{j+k+2m}{4} \\ 0 \end{bmatrix} (-4\bar{z}, -4\bar{\tau}) & i = 0, 2, \\ \frac{1}{\sqrt{2}} \left(\frac{4\cdot 2\mathrm{Im}\,\tau}{\mathcal{A}^2}\right)^{\frac{1}{4}} \left(\vartheta \begin{bmatrix} \frac{1}{4} \\ 0 \end{bmatrix} (4z, 4\tau) + \vartheta \begin{bmatrix} \frac{3}{4} \\ 0 \end{bmatrix} (4z, 4\tau)\right) \vartheta \begin{bmatrix} \frac{j+k+2m}{4} \\ 0 \end{bmatrix} (-4\bar{z}, -4\bar{\tau}) & i = 1. \end{cases} \tag{70}$$

Complete three-point couplings are products of three-point couplings of those on each two-dimensional torus and ten-dimensinal string coupling. The Majorana mass term is written as Eq. (65). This Majorana mass term is invariant under the modular transformation of the complex structure moduli since its dependence on complex structure moduli is determined by that of perturbative three-point couplings and it is invariant under the modular transformation. The modular symmetry is not violated by the non-perturbative effects in this case.

4.2. Generic discussion

The example in the previous subsection shows the modular symmetry of non-perturbative terms induced by D-brane instanton effects for the complex structure moduli in type IIB magnetized D-brane models. Moreover, this example suggests a generic aspect. The D-brane instantons induce the non-perturbative terms such as

$$Ce^{-\mathrm{Vol}(E5)} \left(\prod_i y^{(n_i)}(\tau)\right) \Phi_1 \cdots \Phi_m, \tag{71}$$

where C is a moduli-independent coefficient.[4] Here, Vol($E5$) denotes the volume of the D-brane instanton in the compact space, and it depends only on $\mathcal{A}^r$, but not τ. Furthermore, $y^{(n)}$ denotes the couplings among zero-modes and 4D fields Φ_i, and these are computed in the same way as the perturbative couplings shown in Sect. 2. The τ dependence appears only through these couplings $y^{(n)}$. Therefore, terms induced by D-brane instanton effects are also modular symmetric.

In this section, we have not taken into account the moduli mixing so far. However, the discussion in Sect. 3 would lead to modification such as Eq. (61).

5. Conclusion

We have studied the 4D low-energy effective field theory, which is derived from type IIB magnetized D-brane models and type IIA intersecting D-brane models. We have studied the modular symmetric behavior of perturbative terms. Also, such analysis has been extended to non-perturbative terms

[4] More precisely, the coefficient C may include a functional determinant of the Dirac operator as well as a bosonic Laplacian operator produced by the integration of massive modes [22,25]. However, these coefficients are canceled if the SUSY is not broken. Even if the SUSY is broken, eigenvalues of the Dirac operator and Laplacian operator depend only on $\mathcal{A}^r$, but they are independent of the complex structures [12]. Thus our conclusion would not be affected by this coefficient.

induced by D-brane instanton effects. We have also investigated the anomaly of the modular symmetry. Its cancelation would require moduli mixing correction terms in low-energy effective field theory. Thus, the modular symmetry is important in understanding the 4D low-energy effective field theory of superstring theory.

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