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The quenched critical point for self-avoiding walk on random conductors

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Abstract

Following similar analysis to that in Lacoin [16], we can show that the quenched critical point for self-avoiding walk on random conductors on $\mathbb{Z}^d$ is almost surely a constant, which does not depend on the location of the reference point. We provide upper and lower bounds which are valid for all $d \geq 1$.

1 Introduction

1.1 Background

Self-avoiding walk (SAW) is a statistical-mechanical model for chain-like solvents and linear polymers. SAW was first introduced by Flory [9, 10] in order to model and investigate the behavior of polymer chains. Since then, many rigorous mathematical results on SAW have been proven, while physicists have much more conjectures that are believed to be true. Most of them are supported by numerical simulations and physical ideas that have not been fully justified mathematically.

It would be more natural to consider an inhomogeneous environment in which polymers lie. In recent years, various models of SAW in a quenched random environment have attracted attention of chemists, physicists and mathematicians [4, 5, 13, 18]. One of them is SAW on a randomly diluted lattice, introduced by Chakrabarti and Kertész [4]. Le Doussal and Machta [8] investigated it by applying a renormalization method on a hierarchical lattice and came up to some conjectures. Lacoin [15] answered affirmatively to

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one of them by showing that, on an infinite supercritical percolation cluster in 2 dimensions, the quenched critical point (defined by divergence of the quenched susceptibility) is strictly smaller than the annealed one (defined by divergence of the average susceptibility).

In this paper, we investigate SAW in a different type of random environment, which is topologically regular, but random in energy landscape. The goal is to achieve better understanding of how the introduction of randomness changes the properties of the critical point.

1.2 The model and the main theorem

Let $\Omega(x, y)$ be the set of (nearest-neighbor) self-avoiding paths on $\mathbb{Z}^d$ from x to y, and let $\Omega(x) = \bigcup_{y\in\mathbb{Z}^d} \Omega(x, y)$. Denoting the length of ω by $|\omega|$ (i.e., $|\omega| = n$ for $\omega = (\omega_0, \dots, \omega_n)$) and the energy cost of a bond between consecutive monomers by $h \in \mathbb{R}$, we define the susceptibility as

$$\chi_h = \sum_{\omega\in\Omega(x)} e^{-h|\omega|}, \tag{1.1}$$

which is independent of the location of the reference point $x \in \mathbb{Z}^d$. Two other key observables are the number of n-step SAWs and the two-point function:

$$c(n) = \sum_{\omega\in\Omega(x)} \mathbb{1}_{\{|\omega|=n\}}, \qquad G_h(x) = \sum_{\omega\in\Omega(o,x)} e^{-h|\omega|}, \tag{1.2}$$

where o is the origin of $\mathbb{Z}^d$ and $\mathbb{1}_{\{\cdots\}}$ is the indicator function. Obviously,

$$\chi_h = \sum_{n=0}^{\infty} e^{-hn} c(n) = \sum_{x\in\mathbb{Z}^d} G_h(x). \tag{1.3}$$

Due to subadditivity of $\log c(n)$, we can readily show that $\chi_h < \infty$ if and only if $h > \log \mu$, where μ is the connective constant for SAW [17]:

$$\mu = \lim_{n\to\infty} c(n)^{1/n} = \inf_n c(n)^{1/n}. \tag{1.4}$$

Therefore, $h_0 \equiv \log \mu$ is the critical point of the susceptibility. Many rigorous results on the behavior of these observables around $h = h_0$ have been proven, especially in high dimensions $d > 4$, with the help of the lace expansion [3, 17]. However, there still remain many challenging open problems in two and three dimensions. See [21] and the references therein.

Next, we introduce randomness to the environment. Let $\mathbb{B}^d$ denote the set of nearest-neighbor bonds in $\mathbb{Z}^d$, and let $\boldsymbol{X} = \{X_b\}_{b\in\mathbb{B}^d}$ be a collection of integrable random variables whose law $\mathbb{P}$ is translation-invariant and ergodic. From a physical point of view, X_b can be regarded as the magnitude of resistance of a conductor attached to $b \in \mathbb{B}^d$, and therefore it is more natural to assume $X_b \geq 0$. However, the results in this paper are all valid without this assumption. Given the environment $\boldsymbol{X}$ and the strength of randomness $\beta \geq 0$, we define the quenched susceptibility at $x \in \mathbb{Z}^d$ as

$$\hat{\chi}_{h,\beta,\boldsymbol{X}}(x) = \sum_{\omega \in \Omega(x)} e^{-\sum_{j=1}^{|\omega|}(h+\beta X_{b_j})}, \tag{1.5}$$

where

$$b_j \equiv b_j(\omega) = (\omega_{j-1}, \omega_j). \tag{1.6}$$

Because of the inhomogeneity of $\boldsymbol{X}$, the quenched susceptibility is no longer translation invariant and does depend on the location of the reference point x. Similarly to the homogeneous case, we also define

$$\hat{c}_{\beta,\boldsymbol{X}}(x;n) = \sum_{\omega \in \Omega(x)} e^{-\beta \sum_{j=1}^{|\omega|} X_{b_j}} \mathbb{1}_{\{|\omega|=n\}}, \tag{1.7}$$

$$\hat{G}_{h,\beta,\boldsymbol{X}}(x,y) = \sum_{\omega \in \Omega(x,y)} e^{-\sum_{j=1}^{|\omega|}(h+\beta X_{b_j})}. \tag{1.8}$$

These quantities are reduced to χ_h, $c(n)$ and $G_h(y-x)$, respectively, when $\beta = 0$. Moreover,

$$\hat{\chi}_{h,\beta,\boldsymbol{X}}(x) = \sum_{n=0}^{\infty} e^{-hn} \hat{c}_{\beta,\boldsymbol{X}}(x;n) = \sum_{y \in \mathbb{Z}^d} \hat{G}_{h,\beta,\boldsymbol{X}}(x,y). \tag{1.9}$$

Since $\hat{\chi}_{h,\beta,\boldsymbol{X}}(x)$ is monotonic in h, we can define the quenched version of the critical point as

$$\hat{h}^{\mathsf{q}}_{\beta,\boldsymbol{X}}(x) = \inf\{h \in \mathbb{R} : \hat{\chi}_{h,\beta,\boldsymbol{X}}(x) < \infty\}. \tag{1.10}$$

Our goal is to understand how the randomness of the environment $\boldsymbol{X}$ affects the behavior of these quenched observables around the critical point. There are numerous examples in which the introduction of randomness alters the behavior of relevant observables. Classical examples are Sinai's one-dimensional random walk in a random medium [20] and Smith and Wilkinson's branching processes in random environments [22]. More recent examples are the random pinning models [7, 12] and the directed polymer models [6].

As a first step to understand the properties of the random variable $\hat{h}^{\mathsf{q}}_{\beta,\boldsymbol{X}}(x)$, we consider the mean-field approximation (or often called the annealing), i.e., to take the average of $\hat{\chi}_{h,\beta,\boldsymbol{X}}(x)$ over the environment $\boldsymbol{X}$. Let

$$h^{\mathsf{a}}_{\beta} = \{h \in \mathbb{R} : \mathbb{E}[\hat{\chi}_{h,\beta,\boldsymbol{X}}(x)] < \infty\}, \tag{1.11}$$

where $\mathbb{E}$ is the expectation for $\mathbb{P}$. Since $\mathbb{P}$ is translation-invariant, the annealed critical point h_β^{a} does not depend on the location of the reference point $x \in \mathbb{Z}^d$. We note that $\hat{h}_{\beta,\boldsymbol{X}}^{\mathsf{q}}(x) \leq h_\beta^{\mathsf{a}}$ by definition. In particular, if $\boldsymbol{X}$ is i.i.d. and the Laplace transform

$$\lambda_\beta = \mathbb{E}[e^{-\beta X_b}] \tag{1.12}$$

exists, then we can directly compute $\mathbb{E}[\hat{c}_{\beta,\boldsymbol{X}}(x;n)]$ as

$$\mathbb{E}[\hat{c}_{\beta,\boldsymbol{X}}(x;n)] = \sum_{\omega\in\Omega(x):|\omega|=n} \prod_{j=1}^{n} \mathbb{E}[e^{-\beta X_{b_j}}] = \lambda_\beta^n \, c(n), \tag{1.13}$$

and the annealed susceptibility $\mathbb{E}[\hat{\chi}_{h,\beta,\boldsymbol{X}}(x)]$ as

$$\begin{aligned} \mathbb{E}[\hat{\chi}_{h,\beta,\boldsymbol{X}}(x)] = \sum_{n=0}^{\infty} e^{-hn}\mathbb{E}[\hat{c}_{\beta,\boldsymbol{X}}(x;n)] &= \sum_{n=0}^{\infty} e^{-(h-\log\lambda_\beta)n} c(n) \\ &= \chi_{h-\log\lambda_\beta}. \end{aligned} \tag{1.14}$$

Therefore,

$$h_\beta^{\mathsf{a}} = h_0 + \log\lambda_\beta. \tag{1.15}$$

By Jensen's inequality, we immediately see

$$h_\beta^{\mathsf{a}} \geq h_0 - \beta\mathbb{E}[X_b], \tag{1.16}$$

where the gap is $O(\beta^2)$ as $\beta \to 0$.

The following theorem is the main result of this paper.

Theorem 1.1. *Let $d \geq 1$ and $\beta \geq 0$. The quenched critical point $\hat{h}_{\beta,\boldsymbol{X}}^{\mathsf{q}}(x)$ is almost surely an x-independent constant. Moreover, by abbreviating $\hat{h}_{\beta,\boldsymbol{X}}^{\mathsf{q}}(x)$ as $\hat{h}_{\beta,\boldsymbol{X}}^{\mathsf{q}}$, we have*

$$h_0 - \beta\mathbb{E}[X_b] \leq \hat{h}_{\beta,\boldsymbol{X}}^{\mathsf{q}} \leq h_\beta^{\mathsf{a}}, \quad \textit{almost surely.} \tag{1.17}$$

For $d = 1$, in particular, the lower bound is an equality.

Before proving this theorem in the next section, we give two remarks.

Remark 1.2 (On the first inequality in (1.17))**.** For $d = 1$, $\hat{h}_{\beta,\boldsymbol{X}}^{\mathsf{q}} = -\beta\mathbb{E}[X_b]$ (recall that $h_0 = 0$ for $d = 1$) is due to the fact that $c(n)$ is always two: either to the left or to the right of the reference point. Let $h = -\beta\mathbb{E}[X_b] + \delta$ and $\Delta_j = X_{(x+j-1,x+j)} - \mathbb{E}[X_b]$. Then, we have

$$\hat{\chi}_{h,\beta,\boldsymbol{X}}(x) = 1 + \sum_{n=1}^{\infty} e^{-\delta n}\Big(e^{-\beta\sum_{j=1}^{n}\Delta_j} + e^{-\beta\sum_{j=0}^{n-1}\Delta_{-j}}\Big). \tag{1.18}$$

By applying the individual ergodic theorem to those two sequences $\{\Delta_j\}_{j=1}^{\infty}$ and $\{\Delta_{-j}\}_{j=0}^{\infty}$, we can conclude that the above series almost surely converges if and only if $\delta > 0$.

For $d \geq 2$, however, since $c(n)$ grows exponentially, it is hard to control the speed of convergence along those walks at the same time. Because of this entropic effect, we strongly believe that the first inequality in (1.17) is a strict inequality. So far, we have been able to prove it to be true only for SAW on i.i.d. random conductors in a homogeneous tree of degree $\ell \geq 3$, if β is sufficiently small. In fact, we can show the equality $\hat{h}^{\mathsf{q}}_{\beta,\boldsymbol{X}} = h^{\mathsf{a}}_{\beta}$ as follows. First, we set $\delta = h^{\mathsf{a}}_{\beta} - h > 0$ and use $\mu = \ell - 1$ to obtain the rewrite

$$\hat{\chi}_{h,\beta,\boldsymbol{X}}(x) = \sum_{\omega \in \Omega(x)} e^{\delta|\omega|} \prod_{j=1}^{|\omega|} e^{-h^{\mathsf{a}}_{\beta} - \beta X_{b_j}} = 1 + \frac{\ell}{\ell - 1} \sum_{n=1}^{\infty} e^{\delta n} Z_{\beta,\boldsymbol{X}}(x;n), \quad (1.19)$$

where

$$Z_{\beta,\boldsymbol{X}}(x;n) = \sum_{\omega \in \Omega(x):|\omega|=n} \frac{1}{\ell(\ell-1)^{n-1}} \prod_{j=1}^{n} \frac{e^{-\beta X_{b_j}}}{\lambda_\beta} \quad (1.20)$$

is a positive martingale, and thus the limit $Z_{\beta,\boldsymbol{X}}(x;\infty) \equiv \lim_{n\to\infty} Z_{\beta,\boldsymbol{X}}(x;n)$ exists almost surely. Adapting the statement in [14, p.886] to our setting, we have the following dichotomy:

$$f(\beta) \equiv h^{\mathsf{a}}_{\beta} - (h^{\mathsf{a}}_{\beta})'\beta \begin{cases} > 0 & \Rightarrow \quad \mathbb{P}\big(Z_{\beta,\boldsymbol{X}}(x;\infty) > 0\big) = 1, \\ \leq 0 & \Rightarrow \quad \mathbb{P}\big(Z_{\beta,\boldsymbol{X}}(x;\infty) = 0\big) = 1. \end{cases} \quad (1.21)$$

Since $f(0) = \log(\ell - 1) > 0$ and $f'(\beta) = -(h^{\mathsf{a}}_{\beta})''\beta \leq 0$, we have $f(\beta) > 0$ for sufficiently small β, hence $\hat{\chi}_{h,\beta,\boldsymbol{X}}(x) = \infty$ almost surely.

Now we are back on $\mathbb{Z}^d$. If β is large and $\mathbb{E}[X_b] > 0$, then the gap between the lower and upper bounds in (1.17) is large, and the inequality (1.17) is no longer effective. In the following specific case, however, we may find a better bound. Suppose that $\mathbb{P}(X_b = 0)$ is bigger than the critical point for oriented percolation on $\mathbb{Z}^d_+$. Then, there is almost surely an $\boldsymbol{X}$-free infinite oriented-percolation cluster $\mathcal{C}_x$ at some $x \in \mathbb{Z}^d_+$, in which the number of n-step directed paths from x grows exponentially in n [11, Theorem 3.1(2)]. The susceptibility $\hat{\chi}_{h,\beta,\boldsymbol{X}}(x)$ can be bounded below by restricting the sum over those directed paths in $\mathcal{C}_x$, implying existence of a β-independent positive lower bound on $\hat{h}^{\mathsf{q}}_{\beta,\boldsymbol{X}}$.

Remark 1.3 (On the second inequality in (1.17))**.** Although it is trivial by definition, the second inequality in (1.17) can be proven in the following tedious way. First, by the Markov inequality, we have

$$\mathbb{P}\Big(\hat{c}_{\beta,\boldsymbol{X}}(x;n) \geq n^2 \mathbb{E}[\hat{c}_{\beta,\boldsymbol{X}}(x;n)]\Big) \leq \frac{1}{n^2}. \quad (1.22)$$

Then, by the Borel-Cantelli lemma, we can conclude that the opposite inequality $\hat{c}_{\beta,\boldsymbol{X}}(x;n) \leq n^2\mathbb{E}[\hat{c}_{\beta,\boldsymbol{X}}(x;n)]$ holds for all but finitely many n, implying almost sure convergence of $\hat{\chi}_{h,\beta,\boldsymbol{X}}(x)$ for $h > h^{\mathsf{a}}_\beta$.

We may improve it to a strict inequality in two dimensions by adapting the idea of Lacoin [16]. In his setting (i.e., SAW on an infinite supercritical percolation cluster in $\mathbb{Z}^2$), it is proven that there are $b, \theta \in (0,1)$ such that

$$\mathbb{E}[\hat{c}_{\beta,\boldsymbol{X}}(x;n)^\theta] \leq \big(b^n\mathbb{E}[\hat{c}_{\beta,\boldsymbol{X}}(x;n)]\big)^\theta. \tag{1.23}$$

Then, by the Markov inequality, we have

$$\mathbb{P}\Big(\hat{c}_{\beta,\boldsymbol{X}}(x;n) \geq n^{2/\theta}b^n\mathbb{E}[\hat{c}_{\beta,\boldsymbol{X}}(x;n)]\Big) \leq \frac{1}{n^2}. \tag{1.24}$$

By the Borel-Cantelli lemma again, we may conclude $\hat{h}^{\mathsf{q}}_\beta \leq h^{\mathsf{a}}_\beta - \log\frac{1}{b}$.

Analyzing fractional moments, as in (1.23), has been a standard method to investigate disordered systems. To see how it is used in other settings, we refer to [23] for random walks in random environments, and to [1, 2] for random pinning models.

2 Proof of the main result

We prove Theorem 1.1 as follows. In Section 2.1, we prove the first half of Theorem 1.1 by showing that the quenched critical point is a degenerate random variable that does not depend on the location of the reference point. In Section 2.2, we complete the proof of Theorem 1.1 by showing the first inequality in (1.17). Recall that its reduction to an equality for $d = 1$ and the second inequality in (1.17) for all $d \geq 1$ have already been mentioned in the previous section.

2.1 Degeneration of the quenched critical point

Recall that $\boldsymbol{X} = \{X_b\}_{b\in\mathbb{B}^d}$ is a collection of integrable (thus almost surely finite) random variables whose law $\mathbb{P}$ is translation-invariant and ergodic. Following similar analysis to that in Lacoin [16], we first prove that the quenched critical point is independent of the location of the reference point.

Lemma 2.1. *The quenched critical point $\hat{h}^{\mathsf{q}}_{\beta,\boldsymbol{X}}(x)$ is almost surely a constant function of $x \in \mathbb{Z}^d$.*

Proof. We will show below that

$$\hat{\chi}_{h,\beta,\boldsymbol{X}}(u) \leq \hat{\chi}_{h,\beta,\boldsymbol{X}}(v)^2 + e^{h+\beta X_{(v,u)}}\hat{\chi}_{h,\beta,\boldsymbol{X}}(v) \tag{2.1}$$

holds for any pair of neighboring vertices $u, v \in \mathbb{Z}^d$. Since $X_{(u,v)}$ is almost surely finite, it implies $\hat{\chi}_{h,\beta,\boldsymbol{X}}(u) < \infty$ if and only if $\hat{\chi}_{h,\beta,\boldsymbol{X}}(v) < \infty$. Repeated applications of this inequality to all neighboring vertices in $\mathbb{Z}^d$, we conclude that all vertices are in the same equivalent class, i.e., either $\hat{\chi}_{h,\beta,\boldsymbol{X}}(x) < \infty$ for all $x \in \mathbb{Z}^d$ or $\hat{\chi}_{h,\beta,\boldsymbol{X}}(x) = \infty$ for all $x \in \mathbb{Z}^d$. Therefore, $\hat{h}^{\mathsf{q}}_{\beta,\boldsymbol{X}}(x)$ does not depend on $x \in \mathbb{Z}^d$, almost surely.

It remains to show (2.1). First, we split the sum into two as

$$\hat{\chi}_{h,\beta,\boldsymbol{X}}(u) = \sum_{\omega\in\Omega(u)} e^{-\sum_{j=1}^{|\omega|}(h+\beta X_{b_j})}\big(\mathbb{1}_{\{v\in\omega\}} + \mathbb{1}_{\{v\notin\omega\}}\big). \tag{2.2}$$

Due to subadditivity and reversibility, the contribution from $\mathbb{1}_{\{v\in\omega\}}$ is bounded as

$$\begin{aligned}\sum_{\omega\in\Omega(u):v\in\omega} e^{-\sum_{j=1}^{|\omega|}(h+\beta X_{b_j})} &\le \underbrace{\sum_{\omega\in\Omega(u,v)} e^{-\sum_{j=1}^{|\omega|}(h+\beta X_{b_j(\omega)})}}_{\hat{G}_{h,\beta,\boldsymbol{X}}(u,v)} \underbrace{\sum_{\eta\in\Omega(v)} e^{-\sum_{j=1}^{|\eta|}(h+\beta X_{b_j(\eta)})}}_{\hat{\chi}_{h,\beta,\boldsymbol{X}}(v)} \\ &= \hat{G}_{h,\beta,\boldsymbol{X}}(v,u)\ \hat{\chi}_{h,\beta,\boldsymbol{X}}(v) \\ &\le \hat{\chi}_{h,\beta,\boldsymbol{X}}(v)^2. \end{aligned} \tag{2.3}$$

On the other hand, by adding an extra step from v to u, the contribution from $\mathbb{1}_{\{v\notin\omega\}}$ is bounded as

$$\begin{aligned}\sum_{\omega\in\Omega(u):v\notin\omega} e^{-\sum_{j=1}^{|\omega|}(h+\beta X_{b_j})} &= e^{h+\beta X_{(v,u)}} \sum_{\omega\in\Omega(u):v\notin\omega} e^{-(h+\beta X_{(v,u)})} e^{-\sum_{j=1}^{|\omega|}(h+\beta X_{b_j(\omega)})} \\ &= e^{h+\beta X_{(v,u)}} \sum_{\bar{\omega}\in\Omega(v):\bar{\omega}_1=u} e^{-\sum_{j=1}^{|\bar{\omega}|}(h+\beta X_{b_j(\bar{\omega})})} \\ &\le e^{h+\beta X_{(v,u)}} \hat{\chi}_{h,\beta,\boldsymbol{X}}(v), \end{aligned} \tag{2.4}$$

where we have used the symmetry $X_{(u,v)} = X_{(v,u)}$ to form $\bar{\omega}$. This completes the proof. ∎

In the rest of this section, we simply denote $\hat{h}^{\mathsf{q}}_{\beta,\boldsymbol{X}}(x)$ by $\hat{h}^{\mathsf{q}}_{\beta,\boldsymbol{X}}$.

Lemma 2.2. *The quenched critical point $\hat{h}^{\mathsf{q}}_{\beta,\boldsymbol{X}}$ is a degenerate random variable.*

Proof. Due to Lemma 2.1, the event $\{\hat{h}^{\mathsf{q}}_{\beta,\boldsymbol{X}} = h\}$ for any $h \in \mathbb{R}$ is translation invariant. Since $\mathbb{P}$ is ergodic, we can conclude that $\mathbb{P}(\hat{h}^{\mathsf{q}}_{\beta,\boldsymbol{X}} = h)$ is either zero or one. ∎

2.2 Lower bound on the quenched critical point

In this section, we prove the first inequality in (1.17) by showing almost sure divergence of the quenched susceptibility at $h = h_0 - \beta\mathbb{E}[X_b] - \beta\delta$ for any $\beta > 0$ and $\delta > 0$.

Let $\Delta_b = X_b - \mathbb{E}[X_b]$ and define

$$\Omega(x; n) = \{\omega \in \Omega(x) : |\omega| = n\}, \tag{2.5}$$

$$\hat{\Omega}^{\mathsf{good}}_{\delta,\boldsymbol{X}}(x; n) = \Big\{\omega \in \Omega(x; n) : \big|\tfrac{1}{n}\textstyle\sum_{j=1}^{n}\Delta_{b_j(\omega)}\big| < \delta\Big\}. \tag{2.6}$$

Using this random set, we can bound $\hat{\chi}_{h,\beta,\boldsymbol{X}}(x)$ at $h = h_0 - \beta\mathbb{E}[X_b] - \beta\delta$ as

$$\hat{\chi}_{h,\beta,\boldsymbol{X}}(x) = \sum_{\omega\in\Omega(x)} \frac{1}{\mu^{|\omega|}} e^{\beta|\omega|\left(\delta - \frac{1}{|\omega|}\sum_{j=1}^{|\omega|}\Delta_{b_j}\right)} \geq \sum_{n=1}^{\infty} \frac{1}{\mu^n} |\hat{\Omega}^{\mathsf{good}}_{\delta,\boldsymbol{X}}(x; n)|. \tag{2.7}$$

If there are infinitely many n such that $|\hat{\Omega}^{\mathsf{good}}_{\delta,\boldsymbol{X}}(x; n)| \geq \frac{1}{2}c(n)$, then, by $c(n) \geq \mu^n$ (cf., (1.4)), we obtain divergence of the susceptibility. Therefore,

$$\begin{aligned}\mathbb{P}(\hat{\chi}_{h,\beta,\boldsymbol{X}} = \infty) &\geq \underbrace{\mathbb{P}\Big(\hat{\chi}_{h,\beta,\boldsymbol{X}} = \infty \Big| \limsup_{n\to\infty}\big\{|\hat{\Omega}^{\mathsf{good}}_{\delta,\boldsymbol{X}}(x; n)| \geq \tfrac{1}{2}c(n)\big\}\Big)}_{1} \\ &\quad \times \mathbb{P}\Big(\limsup_{n\to\infty}\big\{|\hat{\Omega}^{\mathsf{good}}_{\delta,\boldsymbol{X}}(x; n)| \geq \tfrac{1}{2}c(n)\big\}\Big) \\ &\geq \lim_{n\to\infty} \mathbb{P}\big(|\hat{\Omega}^{\mathsf{good}}_{\delta,\boldsymbol{X}}(x; n)| \geq \tfrac{1}{2}c(n)\big).\end{aligned} \tag{2.8}$$

To complete the proof, since $\mathbb{P}(\hat{\chi}_{h,\beta,\boldsymbol{X}}(x) = \infty)$ is either zero or one, it suffice to show that the rightmost limit is positive. Here, we use the following Paley-Zygmund (PZ) inequality [19]: for a random variable $Z \geq 0$ whose second moment is finite and for $\varepsilon \in (0, 1)$,

$$\mathbb{P}(Z \geq \varepsilon\mathbb{E}[Z]) \geq (1-\varepsilon)^2 \frac{\mathbb{E}[Z]^2}{\mathbb{E}[Z^2]}. \tag{2.9}$$

Let $Z = |\hat{\Omega}^{\mathsf{good}}_{\delta,\boldsymbol{X}}(x; n)|$. Notice that, by definition and ergodicity, we can bound $\mathbb{E}\big[|\hat{\Omega}^{\mathsf{good}}_{\delta,\boldsymbol{X}}(x; n)|\big]$ from below as

$$\mathbb{E}\big[|\hat{\Omega}^{\mathsf{good}}_{\delta,\boldsymbol{X}}(x; n)|\big] = \sum_{\omega\in\Omega(x;n)} \mathbb{P}\big(\big|\tfrac{1}{n}\textstyle\sum_{j=1}^{n}\Delta_{b_j(\omega)}\big| < \delta\big) \geq c(n)\big(1 - o(1)\big). \tag{2.10}$$

Using this and the trivial inequality $\mathbb{E}\big[|\hat{\Omega}^{\mathsf{good}}_{\delta,\boldsymbol{X}}(x; n)|^2\big] \leq c(n)^2$, we obtain[1]

$$\lim_{n\to\infty} \mathbb{P}\big(|\hat{\Omega}^{\mathsf{good}}_{\delta,\boldsymbol{X}}(x; n)| \geq \tfrac{1}{2}c(n)\big) \geq \frac{1}{4} > 0, \tag{2.12}$$

as required. ■

[1] One of two anonymous referees found the following much simpler proof of (2.12). First,

3 Another application of the PZ inequality

Application of the PZ inequality is often dubbed the second-moment method. It has been a standard tool to investigate disordered systems. We show below that the PZ inequality may also be used to investigate critical behavior for SAW on i.i.d. random conductors. From now on, we assume that $\lambda_\beta < \infty$ for all $\beta \geq 0$.

Proposition 3.1. *Suppose that*

$$B_1 \equiv \mathbb{E}\bigg[\sum_{y\in\mathbb{Z}^d} \hat{G}_{h,\beta,\boldsymbol{X}}(x,y)^2\bigg] < \infty \tag{3.1}$$

and

$$B_2 \equiv \mathbb{E}\bigg[\sum_{y,z\in\mathbb{Z}^d} \hat{G}_{h,\beta,\boldsymbol{X}}(x,z)\,\hat{G}_{h,\beta,\boldsymbol{X}}(z,y)^2\,\hat{G}_{h,\beta,\boldsymbol{X}}(y,x)\bigg] < \infty \tag{3.2}$$

hold uniformly in $h > h_\beta^{\mathsf{a}}$. Then, for any slowly-varying function $L(h) \downarrow 0$ as $h \downarrow h_\beta^{\mathsf{a}}$, we have

$$\liminf_{h\downarrow h_\beta^{\mathsf{a}}} \mathbb{P}\bigg(\hat{\chi}_{h,\beta,\boldsymbol{X}}(x) \geq \frac{L(h)}{h-h_\beta^{\mathsf{a}}}\bigg) \geq 1 - O(\beta^2). \tag{3.3}$$

Although the above result is conditional and still weak to establish a decisive conclusion, it provides evidence to support the belief that, in high dimensions, the coincidence $\hat{h}^{\mathsf{q}}_{\beta,\boldsymbol{X}} = h^{\mathsf{a}}_\beta$ occurs and the critical exponent for $\hat{\chi}_{h,\beta,\boldsymbol{X}}(x)$, if it exists, is bounded below by its mean-field value 1. For SAW in a homogeneous environment, the conditions (3.1)–(3.2) (in fact, the former implies the latter because $B_2 \leq B_1^2$, which is a result of translation invariance and the Cauchy-Schwarz inequality) are known to hold in dimensions $d > 4$, via the lace expansion [3, 17]. The lace expansion yields a convolution equation for the two-point function, which is applicable in both homogeneous and inhomogeneous settings. In the current random setting, however, because of the lack of translation invariance, we have not been able to fully control the lace-expansion coefficients. This is under investigation in an ongoing project.

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by the trivial inequality $|\hat{\Omega}^{\mathsf{good}}_{\delta,\boldsymbol{X}}(x;n)| \leq c(n)$, we obtain

$$\begin{aligned}\mathbb{E}\big[|\hat{\Omega}^{\mathsf{good}}_{\delta,\boldsymbol{X}}(x;n)|\big] &\leq \frac{1}{2}c(n)\,\mathbb{P}\big(|\hat{\Omega}^{\mathsf{good}}_{\delta,\boldsymbol{X}}(x;n)| < \tfrac{1}{2}c(n)\big) + c(n)\,\mathbb{P}\big(|\hat{\Omega}^{\mathsf{good}}_{\delta,\boldsymbol{X}}(x;n)| \geq \tfrac{1}{2}c(n)\big)\\ &= \frac{1}{2}c(n)\Big(1 + \mathbb{P}\big(|\hat{\Omega}^{\mathsf{good}}_{\delta,\boldsymbol{X}}(x;n)| \geq \tfrac{1}{2}c(n)\big)\Big).\end{aligned} \tag{2.11}$$

Combining this with (2.10), we can readily conclude $\mathbb{P}\big(|\hat{\Omega}^{\mathsf{good}}_{\delta,\boldsymbol{X}}(x;n)| \geq \frac{1}{2}c(n)\big) \geq 1 - o(1)$.

Proof of Proposition 3.1 First, by replacing Z in (2.9) by $\hat{\chi}_{h,\beta,\boldsymbol{X}}(x)$, we have

$$\mathbb{P}\Big(\hat{\chi}_{h,\beta,\boldsymbol{X}}(x) \geq \varepsilon\mathbb{E}[\hat{\chi}_{h,\beta,\boldsymbol{X}}(x)]\Big) \geq (1-\varepsilon)^2 \frac{\mathbb{E}[\hat{\chi}_{h,\beta,\boldsymbol{X}}(x)]^2}{\mathbb{E}[\hat{\chi}_{h,\beta,\boldsymbol{X}}(x)^2]}. \tag{3.4}$$

Since $\mathbb{E}[\hat{\chi}_{h,\beta,\boldsymbol{X}}(x)] = \chi_{h-\log\lambda_\beta}$ (cf., (1.14)) and $\chi_h \geq (h-h_0)^{-1}$ for all $h > h_0$ (cf., (1.4)), we have $\mathbb{E}[\hat{\chi}_{h,\beta,\boldsymbol{X}}(x)] \geq (h-h_\beta^{\mathsf{a}})^{-1}$ for all $h > h_\beta^{\mathsf{a}}$. Replacing ε in (3.4) by a slowly-varying function $L(h) \downarrow 0$ as $h \downarrow h_\beta^{\mathsf{a}}$, we can conclude (3.3) as soon as we can show

$$\frac{\mathbb{E}[\hat{\chi}_{h,\beta,\boldsymbol{X}}(x)^2] - \mathbb{E}[\hat{\chi}_{h,\beta,\boldsymbol{X}}(x)]^2}{\mathbb{E}[\hat{\chi}_{h,\beta,\boldsymbol{X}}(x)]^2} \leq O(\beta^2), \tag{3.5}$$

in the neighborhood of h_β^{a}.

To prove (3.5) under the assumptions (3.1)–(3.2), we introduce the notation

$$H_{\boldsymbol{X}}(\omega) = -\sum_{j=1}^{|\omega|} \big(h + \beta X_{b_j(\omega)}\big). \tag{3.6}$$

Let $\boldsymbol{Y} = \{Y_b\}_{b\in\mathbb{B}^d}$ be an independent copy of $\boldsymbol{X}$. Then, we obtain

$$\begin{aligned} &\mathbb{E}[\hat{\chi}_{h,\beta,\boldsymbol{X}}(x)^2] - \mathbb{E}[\hat{\chi}_{h,\beta,\boldsymbol{X}}(x)]^2 \\ &= \sum_{\omega,\eta\in\Omega(x)} \mathbb{E}\bigg[e^{H_{\boldsymbol{X}}(\omega)}\mathbb{E}_{\boldsymbol{Y}}\Big[e^{H_{\boldsymbol{X}}(\eta)} - e^{H_{\boldsymbol{Y}}(\eta)}\Big]\bigg]. \end{aligned} \tag{3.7}$$

By the telescopic-sum representation, we can decompose $e^{H_{\boldsymbol{X}}(\eta)} - e^{H_{\boldsymbol{Y}}(\eta)}$ as

$$e^{H_{\boldsymbol{X}}(\eta)} - e^{H_{\boldsymbol{Y}}(\eta)} = \sum_{j=1}^{|\eta|} e^{H_{\boldsymbol{X}}(\eta_{<j})} e^{-h}\Big(e^{-\beta X_{b_j(\eta)}} - e^{-\beta Y_{b_j(\eta)}}\Big) e^{H_{\boldsymbol{Y}}(\eta_{>j})}, \tag{3.8}$$

where $\eta_{<j} = (\eta_0, \ldots, \eta_{j-1})$ and $\eta_{>j} = (\eta_{j+1}, \ldots, \eta_{|\eta|})$, with the convention $H_{\boldsymbol{X}}(\varnothing) = 0$. Substituting this back into (3.7) and changing variables from $\eta_{<j}$ to η^1, from η_j to a bond b, and from $\eta_{>j}$ to η^2, we obtain

$$\begin{aligned} &\mathbb{E}[\hat{\chi}_{h,\beta,\boldsymbol{X}}(x)^2] - \mathbb{E}[\hat{\chi}_{h,\beta,\boldsymbol{X}}(x)]^2 \\ &= \sum_{\substack{\omega\in\Omega(x) \\ \eta^1\circ b\circ\eta^2\in\Omega(x)}} \mathbb{E}\bigg[e^{H_{\boldsymbol{X}}(\omega)+H_{\boldsymbol{X}}(\eta^1)}\mathbb{E}_{\boldsymbol{Y}}\Big[e^{-h}\Big(e^{-\beta X_b} - e^{-\beta Y_b}\Big)e^{H_{\boldsymbol{Y}}(\eta^2)}\Big]\bigg], \end{aligned} \tag{3.9}$$

where $\eta^1 \circ b \circ \eta^2$ is the concatenation of those three paths, whose lengths are not fixed any more (due to the sum over j). Since b is not contained in η^2, Y_b is independent of $H_{\boldsymbol{Y}}(\eta^2)$, hence

$$\begin{aligned} \mathbb{E}_{\boldsymbol{Y}}\Big[\Big(e^{-\beta X_b} - e^{-\beta Y_b}\Big)e^{H_{\boldsymbol{Y}}(\eta^2)}\Big] &= \mathbb{E}_{\boldsymbol{Y}}\Big[e^{-\beta X_b} - e^{-\beta Y_b}\Big]\ \mathbb{E}_{\boldsymbol{Y}}\Big[e^{H_{\boldsymbol{Y}}(\eta^2)}\Big] \\ &= \big(e^{-\beta X_b} - \lambda_\beta\big)\ \mathbb{E}_{\boldsymbol{Y}}\Big[e^{H_{\boldsymbol{Y}}(\eta^2)}\Big]. \end{aligned} \tag{3.10}$$

Substituting this back into (3.9) yields

$$
\begin{aligned}
&\mathbb{E}[\hat{\chi}_{h,\beta,\boldsymbol{X}}(x)^2]-\mathbb{E}[\hat{\chi}_{h,\beta,\boldsymbol{X}}(x)]^2\\
&=e^{-h}\sum_{\substack{\omega\in\Omega(x)\\ \eta^1\circ b\circ\eta^2\in\Omega(x)}}\mathbb{E}\Big[\underbrace{e^{H_{\boldsymbol{X}}(\omega)+H_{\boldsymbol{X}}(\eta^1)}\big(e^{-\beta X_b}-\lambda_\beta\big)}_{0\text{ if }b\notin\omega}\Big]\;\mathbb{E}_{\boldsymbol{Y}}\Big[e^{H_{\boldsymbol{Y}}(\eta^2)}\Big]\\
&\le e^{-2h}\big(\lambda_{2\beta}-\lambda_\beta^2\big)\;\mathbb{E}[\hat{\chi}_{h,\beta,\boldsymbol{X}}(x)]\sum_{\substack{\omega^1\circ b\circ\omega^2\in\Omega(x)\\ \eta^1\circ b\in\Omega(x)}}\mathbb{E}\Big[e^{H_{\boldsymbol{X}}(\omega^1)+H_{\boldsymbol{X}}(\omega^2)+H_{\boldsymbol{X}}(\eta^1)}\Big],
\end{aligned}
\tag{3.11}
$$

where the restricted sum over η^2 is bounded above by $\mathbb{E}[\hat{\chi}_{h,\beta,\boldsymbol{X}}(x)]$, which is translation invariant and independent of $x\in\mathbb{Z}^d$.

Next, we investigate the remaining sum

$$
\sum_{\substack{\omega^1\circ b\circ\omega^2\in\Omega(x)\\ \eta^1\circ b\in\Omega(x)}}\mathbb{E}\Big[e^{H_{\boldsymbol{X}}(\omega^1)+H_{\boldsymbol{X}}(\omega^2)+H_{\boldsymbol{X}}(\eta^1)}\Big]\big(\mathbb{1}_{\{\omega^2\cap\eta^1=\varnothing\}}+\mathbb{1}_{\{\omega^2\cap\eta^1\neq\varnothing\}}\big).
\tag{3.12}
$$

Due to the independence among the variables in $\boldsymbol{X}$, the contribution from $\mathbb{1}_{\{\omega^2\cap\eta^1=\varnothing\}}$ is bounded by

$$
\begin{aligned}
&\sum_{\substack{\omega^1\circ b\circ\omega^2\in\Omega(x)\\ \eta^1\circ b\in\Omega(x)}}\mathbb{E}\Big[e^{H_{\boldsymbol{X}}(\omega^1)+H_{\boldsymbol{X}}(\eta^1)}\Big]\;\mathbb{E}\Big[e^{H_{\boldsymbol{X}}(\omega^2)}\Big]\\
&\quad\le\mathbb{E}[\hat{\chi}_{h,\beta,\boldsymbol{X}}(x)]\sum_{\substack{\omega^1\circ b\in\Omega(x)\\ \eta^1\circ b\in\Omega(x)}}\mathbb{E}\Big[e^{H_{\boldsymbol{X}}(\omega^1)+H_{\boldsymbol{X}}(\omega^2)}\Big]\\
&\quad\le\mathbb{E}[\hat{\chi}_{h,\beta,\boldsymbol{X}}(x)]\,2dB_1.
\end{aligned}
\tag{3.13}
$$

To bound the contribution from $\mathbb{1}_{\{\omega^2\cap\eta^1\neq\varnothing\}}$ in (3.12), we split ω^2 as $\omega^3\circ\omega^4$ at the last visit to η^1, so that $\omega^4\cap\eta^1=\{\omega^4_0\}$. Then, by using the independence among the variables in $\boldsymbol{X}$, we can bound the sum over ω^4 by $\mathbb{E}[\hat{\chi}_{h,\beta,\boldsymbol{X}}(x)]$. As a result, the contribution from $\mathbb{1}_{\{\omega^2\cap\eta^1\neq\varnothing\}}$ is bounded by

$$
\begin{aligned}
&\sum_{y\in\mathbb{Z}^d}\sum_{\substack{\omega^1\circ b\circ\omega^3\in\Omega(x,y)\\ \omega^4\in\Omega(y)}}\mathbb{1}_{\{\omega^1\circ b\circ\omega^3\circ\omega^4\in\Omega(x)\}}\sum_{\substack{\eta^3\in\Omega(x,y)\\ \eta^4\circ b\in\Omega(y)}}\mathbb{1}_{\{\eta^3\circ\eta^4\circ b\in\Omega(x)\}}\;\mathbb{1}_{\{\omega^4\cap(\eta^3\circ\eta^4)=\{y\}\}}\\
&\qquad\qquad\times\mathbb{E}\Big[e^{H_{\boldsymbol{X}}(\omega^1)+H_{\boldsymbol{X}}(\omega^3)+H_{\boldsymbol{X}}(\eta^3)+H_{\boldsymbol{X}}(\eta^4)}\Big]\;\mathbb{E}\Big[e^{H_{\boldsymbol{X}}(\omega^4)}\Big]\\
&\le\mathbb{E}[\hat{\chi}_{h,\beta,\boldsymbol{X}}(x)]\sum_{y\in\mathbb{Z}^d}\sum_{\substack{\omega^1\circ b\circ\omega^3\in\Omega(x,y)\\ \eta^3\in\Omega(x,y)\\ \eta^4\circ b\in\Omega(y)}}\mathbb{1}_{\{b\notin\eta^3\}}\,\mathbb{E}\Big[e^{H_{\boldsymbol{X}}(\omega^1)+H_{\boldsymbol{X}}(\omega^3)+H_{\boldsymbol{X}}(\eta^3)+H_{\boldsymbol{X}}(\eta^4)}\Big]
\end{aligned}
$$

$$= e^h \lambda_\beta^{-1} \mathbb{E}[\hat{\chi}_{h,\beta,\boldsymbol{X}}(x)] \sum_{y,z\in\mathbb{Z}^d} \sum_{\substack{\omega^1\in\Omega(x,z)\\ bo\omega^3\in\Omega(z,y)\\ \eta^3\in\Omega(x,y)\\ \eta^4\in\Omega(y,z)}} \mathbb{E}\Big[e^{H_{\boldsymbol{X}}(\omega^1)+H_{\boldsymbol{X}}(bo\omega^3)+H_{\boldsymbol{X}}(\eta^3)+H_{\boldsymbol{X}}(\eta^4)}\Big]$$

$$= e^h \lambda_\beta^{-1} \mathbb{E}[\hat{\chi}_{h,\beta,\boldsymbol{X}}(x)]\, B_2. \tag{3.14}$$

Finally, by summarizing (3.11)–(3.14), we arrive at

$$\frac{\mathbb{E}[\hat{\chi}_{h,\beta,\boldsymbol{X}}(x)^2] - \mathbb{E}[\hat{\chi}_{h,\beta,\boldsymbol{X}}(x)]^2}{\mathbb{E}[\hat{\chi}_{h,\beta,\boldsymbol{X}}(x)]^2} \le e^{-2h}(2dB_1 + e^h\lambda_\beta^{-1}B_2)(\underbrace{\lambda_{2\beta} - \lambda_\beta^2}_{O(\beta^2)}), \tag{3.15}$$

which proves (3.5). This completes the proof of Proposition 3.1 ■

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