Quadrature Methods for Elliptic PDEs with Random Diffusion

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FOREWORD

The thesis on hand is concerned with elliptic PDEs with random diffusion coefficients. It is partly based on the two preprints

- H. Harbrecht, M. Peters and M. Siebenmorgen. Multilevel accelerated quadrature for PDEs with log-normal distributed random coefficient. *Preprint 2013-18, Mathematisches Institut, Universität Basel, 2013.*
- H. Harbrecht, M. Peters and M. Siebenmorgen. Tractability of the quasi-Monte Carlo quadrature with Halton points for elliptic PDEs with random diffusion. *Preprint 2013-28, Mathematisches Institut, Universität Basel, 2013.*

which originated from my time at the University of Basel.

I want to dedicate this dissertation to everyone who supported me during my time as a PhD-student. First of all, I want to thank Prof. Helmut Harbrecht who has not only been a great supervisor but also became a good friend of mine. Throughout my years as his student, he always had time for discussions and provided me with new ideas whenever it was needed. I admire his broad mathematical as well as general knowledge and I am grateful that he has shared it with me. I also want to express my gratitude to Prof. Raul Tempone for being the second assessor of my thesis and for the opportunity to participate at the SRI UQ Annual Meeting 2015 at the King Abdullah University of Science and Technology.

A special thanks goes to Michael Peters. We both started our studies in mathematics at the University of Bonn in the winter term of 2004 and became roommates during our time as PhD-students. Sometimes we spent endless days discussing about mathematical or non-mathematical problems which broadened our horizon. I would describe our relation more like twins than just good friends. In addition, we developed together the code which is used for the numerical experiments in this thesis. Moreover, he performed the computations for the numerical examples in this thesis. I would also like to take this opportunity to thank the further members of the research group of Professor Harbrecht, Manuela Utzinger, Monica Bugeanu, Jürgen Dölz and the librarian of the Mathematical Institute in Basel Deborah Scheiblechner. They all supported me during my studies, but more importantly, their friendship made my time in Basel unforgettable.

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CONTENTS

Chapter I

INTRODUCTION

In this thesis, we consider elliptic boundary value problems with random diffusion coefficients. Such equations arise in many engineering applications, for example, in the modelling of subsurface flows in porous media, such as rocks, see e.g. [Del79, dM86, Kit97]. These models are of particular importance for geologists since, amongst other things, they can be used to simulate the pollution caused by the long term disposal of radioactive waste in an underground repository, see [CGSS00]. Here, it is convenient to use Darcy's law, cf. [Dar56], to describe the subsurface flow. The key ingredient in this system of equations is the hydraulic conductivity a. This parameter measures the transmissivity of a fluid through an aquifer. It depends on the permeability of the heterogenous media and on the dynamic viscosity of the fluid. Darcy's law states that the flow velocity v, more precisely the discharge per unit which is called Darcy's flux, is proportional to the gradient of the hydraulic head u times the hydraulic conductivity parameter a, i.e.

$$\begin{aligned} v + a\nabla u &= F \qquad \text{in } D, \\ \operatorname{div} v &= 0 \qquad \text{in } D. \end{aligned}$$

Herein $D \subset \mathbb{R}^d$ denotes a spatial domain and the vector field F describes the sources and sinks in the domain D. The second equation is the mass conservation law, cf. [CGSS00]. In addition, this system of equations has to be equipped with appropriate boundary conditions. For simplicity, we employ homogenous Dirichlet boundary conditions of the hydraulic head. Applying the divergence operator and setting $f = -\operatorname{div} F$ leads to the elliptic boundary value problem:

$$\begin{aligned} -\operatorname{div}(a\nabla u) &= f \qquad \text{in } D, \\ u &= 0 \qquad \text{on } \partial D. \end{aligned} \tag{0.1}$$

Elliptic boundary value problems are well understood and can be solved with high accuracy if the input data are known exactly. Unfortunately, the hydraulic conductivity is not given exactly in most cases and has to be determined from measurements. Since the media is usually heterogenous and measurements are only available at a discrete number of points, the hydraulic conductivity is endowed with uncertainty which limits the accuracy of the model. Thus, a common approach in geology is to model the uncertainty in the hydraulic conductivity as a random field a over a propability space $(\Omega, \mathcal{F}, \mathbb{P})$, see [Del79, dMDG$^+$05], where Ω denotes a set, $\mathcal{F}$ a σ-algebra on Ω and $\mathbb{P}$ a probability measure on $\mathcal{F}$. Naturally, this uncertainty propagates through the model and, therefore, even the solution

u is a random field. This yields that (0.1) becomes an elliptic boundary value problem with a random diffusion coefficient:

$$\begin{aligned} -\operatorname{div}\big(a(\mathbf{x},\omega)\nabla u(\mathbf{x},\omega)\big) &= f(\mathbf{x}) && \text{in } D\times\Omega,\\ u(\mathbf{x},\omega) &= 0 && \text{on } \partial D\times\Omega. \end{aligned} \tag{0.2}$$

This equation has been the topic in many publications in recent years. To mention only a few of them, see e.g. [BNT07, BTZ04, CGSS00, FST05, GS91, KX02a, MK05, SG11]. The reason for this is that it is quite challenging to compute an accurate approximation of the solution or even to derive statistical quantities of the solution, like the expectation, the variance, and higher order moments.

The random diffusion coefficient $a(\mathbf{x},\omega)$ itself or, in the context of geological models, the logarithm of the diffusion coefficient $\log\big(a(\mathbf{x},\omega)\big)$ is characterized by its mean field, its covariance function, and the knowledge of the probability space $\mathbb{P}$, where these quantities have to be estimated from the measurement data. Thus, the first computational challenge is to find a suitable representation of the diffusion coefficient. A widely used approach is the Karhunen-Loève expansion, see [HPS15, Loè77, ST06], which separates the spatial variable and the stochastic variable. In general, the number of random variables and spatial functions used in this representation is infinite. For the computation, this series needs to be truncated appropriately. Numerically, this can be done efficiently by a pivoted Cholesky decomposition, cf. [BJ05, HPS12, HPS15].

The random variables which appear in the Karhunen-Loève expansion are assumed to be independent and their distributions are given by density functions. It is convenient to replace them by coordinates, the so called random parameters, which are defined on the image of the corresponding random variables. How many random parameters are required in such a series expansion to get an adequate approximation of the random field depends on the smoothness of the covariance kernel and on the desired accuracy. This number is, especially for rough covariance kernels, large. This yields that (0.2) can be rewritten as a high-dimensional parameter dependent elliptic boundary value problem. Although the solution u depends quite smoothly on the random parameters, the computational effort to solve such a parametric equation by e.g. a polynomial chaos expansion, cf. [FST05, GS91, KX02b, MK05], or stochastic collocation methods, cf. [Bie11, BNT07, BNTT12, NTW08a, NTW08b], may become unfeasible due to the curse of dimensionality. Many terms which are used in such a series representation of the solution often play a negligible role. Therefore, adaptive methods have been developed to detect the most important coefficients, cf. [BNTT11, BNTT12, BNTT14, CDS11, CCS15].

In this thesis, we are interested in statistical quantities of the solution and not in the random solution itself. In particular, we consider the computation of the moments. These quantities appear as integrals over the high-dimensional parameter domain. In most cases, they cannot be determined analytically and, thus, one has to apply a quadrature method to solve the integration problem. Hence, the thesis will be mainly concerned with the investigation of the convergence of different quadrature methods.

The common approach to deal with high-dimensional integrals is the *Monte Carlo quadrature*, see e.g. [HH64], which will serve as a benchmark method throughout this thesis. There are two main reasons for the popularity of this method. On the one hand, the method is easy to implement and perfectly suited for parallel implementation. Having a random number generator at hand which produces random vectors with respect to the underlying distribution of the random parameter, so called *sample points*, the Monte Carlo estimator simply averages the evaluations of the integrand at these sample points. On the other hand, the convergence of the Monte Carlo method is dimension independent under low regularity requirements on the integrand. The disadvantages of Monte Carlo methods are that the convergence rate is of low algebraic order and that deterministic error bounds are not available since the Monte Carlo estimator itself is a random variable. Nevertheless, the estimator converges with probability 1 by the law of large numbers and further convergence properties are provided in the literature, see e.g. [Caf98, STZ01]. For example, the convergence in distribution of the Monte Carlo estimator results from the central limit theorem. Since the Monte Carlo quadrature serves throughout this thesis more as a benchmark method, we restrict ourselves in the sequel to error bounds of the *root mean square error* (RMSE).

As mentioned before, the Monte Carlo estimator works under low regularity requirements on the integrand. Conversely, this implies that the smoothness, which is provided by the integrands under consideration, is not exploited by the Monte Carlo estimator. Hence, we are more interested in quadrature rules which take this smoothness into account in order to achieve better convergence rates. Instead of choosing the sample points randomly, one can also construct a deterministic sequence of sample points and end up with the *quasi-Monte Carlo method*, see [Nie92]. The construction of such point sequences is typically performed for the integration of functions defined on the unit cube $[0,1]^m$. In order to define quasi-Monte Carlo quadrature rules on more general domains of integration, one has to map these points appropriately. The quality of the point sequence is given by its *discrepancy* which measures, roughly speaking, the difference of the point sequence and the uniform distribution. Under certain regularity requirements on the integrand, the discrepancy serves as an error estimate for the quasi-Monte Carlo quadrature. There exist point sequences such that the quasi-Monte Carlo quadrature provides a higher convergence rate in comparison to the Monte Carlo quadrature, but, in general, the convergence rate deteriorates when m gets large.

The third class of considered quadrature methods are *Gaussian type quadrature formulae* which are closely related to stochastic collocation methods with collocation points at Gaussian abscissae. The one-dimensional Gaussian quadrature points and weights are constructed in such a way that the degree of polynomial exactness is maximized. This means that the integrals of polynomials up to a certain degree are determined exactly by the quadrature method. Therefore, a univariate Gaussian quadrature provides the best possible convergence rate for smooth integrands. Unfortunately, the complexity of a tensor product Gaussian quadrature increases exponentially with the dimensionality m. If the integrand has some additional regularity, one can sparsify the tensor product Gaussian quadrature without a significant loss of accuracy. This yields the sparse Gaussian quadrature, cf. [BG04, GG98, Zen91]. This approach significantly reduces the complexity. Nevertheless, the computational cost of the classical sparse grid quadrature still grows exponentially in m.

The question that arises is how it can be explained that certain quasi-Monte Carlo methods or certain sparse-grid quadrature methods work well for some kind of integration problems even if m is large, maybe $m = 100$. The answer to that question cannot be given generally since it depends on the particular choice of the quadrature method and on the particular integration problem. We will examine in depth whether the considered quadrature methods converge (nearly) independent of m for the approximation of the moments of u. The key ingredient therefore is that the integrands, i.e. the solution u to (0.2) and its powers, have a certain anisotropic behaviour which means that the different dimensions are not equally important to the integrands. Then, the idea is to exploit this anisotropic behaviour in the construction of the quadrature methods. Therefore, of course, the anisotropic behaviour of the integrand has to be quantified which will be done with the regularity analysis of the integrands.

In recent years, a lot of work has been done to investigate whether quasi-Monte Carlo quadrature methods can be constructed with dimension- or nearly dimension-independent convergence rates for integrands belonging to a certain weighted space, see [NW10, SW97]. In this context, near dimension-independency refers to independency up to a polynomial factor. If that is the case, the integration problem is said to be *tractable* or *polynomial tractable* with respect to the weighted function space, cf. [Woź94]. But, having an existence result of a dimension-independent convergent quasi-Monte Carlo quadrature in a certain function space at hand does not necessarily imply that the construction of this quadrature is available as well. Hence, a further challenge is the construction of such a point sequence, see e.g. [DKLG+14, NC06, SKJ02]. Alternatively, one can show that some known quasi-Monte Carlo quadrature methods provide a dimension-independent convergence rate in a certain weighted setting, see e.g. [HPS13b, HW02, Wan02]. In this thesis, we will concentrate on the latter approach.

For Gaussian quadrature methods, the anisotropic behaviour of the integrand can also be exploited. This can be done by choosing the number of quadrature points in each particular dimension according to the importance of this dimension for the integrand. This yields the anisotropic Gaussian quadrature method. Of course, an anisotropic sparse Gaussian quadrature can be constructed as well. We will analyze the convergence of the anisotropic Gaussian quadrature and the anisotropic sparse Gaussian quadrature.

The attempts described above are only concerned with the approximation of the solution in the random parameters. Nevertheless, the solution has additionally to be discretized in the spatial variable since each evaluation of a quadrature point or each determination of a coefficient in the polynomial expansion of the solution corresponds to a deterministic elliptic PDE. In general, the level of spatial refinement has to be chosen in such a way that the spatial and the stochastic discretization error are equilibrated. This means that each deterministic elliptic PDE has to be solved on a fine spatial discretization level. Since only a single level of spatial refinement is used for the computation, this corresponds to a single level method. A recently popular approach to keep the computational cost low is to apply multilevel techniques, like the multilevel Monte Carlo method, cf. [BSZ11, CST13, Gil08, Hei00, Hei01], which has been extended to multi-index Monte Carlo methods, see [HNT16]. The idea of multilevel methods is to combine several spatial and stochastic levels of refinement in such a way that the coarser spatial refinement levels are combined with finer stochastic refinement levels and vice versa. In this thesis, we extend the concepts of the multilevel Monte Carlo method to arbitrary quadrature rules, yielding the related multilevel quadrature methods, which has already been mentioned in [HPS16, HPS13a]. Since stochastic collocation methods and Gaussian quadrature methods are closely related, the concepts of multilevel quadrature extend to multilevel collocation methods, cf. [TJWG15, VW14].

The remainder of this thesis is structured as follows. In Chapter II, we introduce and provide some properties of function spaces which are important for the presented analysis. In addition to Lebesgue and Sobolev spaces, the Lebesgue-Bochner spaces are considered which are the canonical function spaces when dealing with random fields. Chapter III is dedicated to the mathematical formulation of the problem at hand. We describe the representation of the random diffusion coefficient by its Karhunen-Loève expansion and analyze the truncation error in the solution arising from the truncation of this expansion. Moreover, we parametrize the stochastic diffusion problem by introducing coordinates for the random variables in the Karhunen-Loève expansion. Since the Karhunen-Loève expansion is computed from its mean field and its covariance kernel, we introduce the covariance kernels of the *Matérn* class. These covariance kernels are commonly used for the description of stochastic fields and we emphasize some properties of these kernels which are relevant for further investigations. The main part of Chapter III is devoted to estimates on the regularity of the solution u to (0.1) and the regularity of its powers u^p. Especially, the investigation of the regularity of u^p is provided here. These estimates are particularly crucial for the error analysis of the approximation of the moments by the quasi-Monte Carlo quadrature and the Gaussian quadrature.

The Monte Carlo and the quasi-Monte Carlo quadrature are analyzed in Chapter IV. Since the convergence analysis of the Monte Carlo quadrature is known, we describe the method briefly and the focus of this chapter is on the quasi-Monte Carlo method. More precisely, we concentrate on the quasi-Monte Carlo quadrature with Halton points, a classical point sequence which is easy to construct even for a high dimensionality m. It is known that the quasi-Monte Carlo quadrature based on the Halton sequence converges dimension-independently for functions defined over the hypercube $[0,1]^m$ which exhibit a certain anisotropic behaviour. This can easily be generalized to functions defined over the tensor product of arbitrary finite intervals. Hence, if the densities of the random variables in the Karhunen-Loève expansion have bounded support, like e.g. uniformly distributed random variables on $[-1/2, 1/2]$, we only need to analyze whether our integrands provide the required anisotropic behaviour. The situation is more challenging for Gaussian random variables. Here, the support of the density functions is $\mathbb{R}$ and the solution u may be unbounded when the modulus of at least one random parameter tends to infinity. Therefore, the main result of this chapter will be that the approximation of the moments of the solution u of (0.1) with a lognormally distributed diffusion coefficient by the quasi-Monte Carlo method based on the first N Halton points converges with a rate $\mathcal{O}(mN^{-1+\delta})$ for an arbitrary $\delta > 0$. This implies that the convergence rate is independent of the dimensionality m up to a linear factor. Of course, this result is only available under suitable regularity and anisotropy conditions of the solution u.

In Chapter V, we discuss the use of Gaussian quadrature rules for the approximation of the moments of u. We base our findings on one-dimensional best polynomial interpolation error results from [BNT07, Bie09]. These results are very similar for the uniformly elliptic and the lognormal situation which allows us to perform the convergence analysis for both cases simultaneously. As expected, in case of the anisotropic tensor product Gaussian quadrature, the decay requirements on the sequence $\{\gamma_k\}$ in order to get dimension-independent convergence rates turn out to be very strong. Hence, we also investigate the impact of the dimensionality m on the convergence rate when dimension-independent convergence cannot be shown. With a new estimate on the number of quadrature points in an anisotropic sparse grid, we are able to significantly improve the convergence results for the anisotropic sparse Gaussian quadrature in comparison to the anisotropic tensor product Gaussian quadrature.

Chapter VI is concerned with the multilevel acceleration of the quadrature methods. Most importantly, we provide the regularity results which are necessary for the convergence of the multilevel quadrature. Since we want to combine spatial and stochastic approximation errors, mixed regularity results in the spatial and stochastic variables are required. These regularity results are employed to establish error estimates for the multilevel quadrature approximation of the moments of u. We end this chapter with a complexity analysis of the considered single level and multilevel quadrature methods.

In order to avoid the repeated use of generic but unspecified constants, we will use the following notation. By $C \lesssim D$ we mean that C can be bounded by a multiple of D, independent of parameters which C and D may depend on. Obviously, $C \gtrsim D$ is defined as $D \lesssim C$, and $C \eqsim D$ as $C \lesssim D$ and $C \gtrsim D$.

Chapter II

PRELIMINARIES

In this chapter, we introduce the function spaces which are necessary to establish regularity results of the solution of an elliptic boundary value problem with stochastic diffusion coefficient. We start with a short review on Lebesgue and Sobolev spaces. Afterwards, we consider the Lebesgue-Bochner spaces, which are the canonical spaces for the treatment of random fields. For further details, we refer to [AE08, AF03, Alt07, LC85]. Throughout this thesis, we denote by $\mathbb{N} = \{0, 1, 2, \ldots\}$ the set of natural numbers including 0 and write $\mathbb{N} \setminus \{0\}$ whenever 0 should be excluded.

1. Lebesgue spaces

(1.1) **Definition.** Let $D \subset \mathbb{R}^d$ denote a bounded domain. The *Lebesgue space* $L^p(D)$ for $p \in [1, \infty)$ consists of the equivalence classes of measurable functions $v : D \to \mathbb{R}$ for which the p-th power is absolutely Lebesgue integrable, i.e.

$$\|v\|_{L^p(D)} := \left(\int_D |v(\mathbf{x})|^p \,\mathrm{d}\mathbf{x} \right)^{\frac{1}{p}} < \infty.$$

Two functions belong to the same equivalence class if they differ from each other at most on a Lebesgue null set. Moreover, the space $L^\infty(D)$ contains the equivalence classes of essentially bounded functions with respect to the norm

$$\|v\|_{L^\infty(D)} := \operatorname*{ess\,sup}_{\mathbf{x} \in D} |v(\mathbf{x})|.$$

The Lebesgue spaces $L^p(D)$ are Banach spaces for all $p \in [1, \infty]$ and separable for $p < \infty$, see e.g. [AF03]. Furthermore, the space $L^2(D)$ is a Hilbert space endowed with the scalar product

$$(1.2) \qquad (v, w)_{L^2(D)} = \int_D v(\mathbf{x}) w(\mathbf{x}) \,\mathrm{d}\mathbf{x}.$$

This scalar product is very important for the treatment of Lebesgue spaces, since it implies a simple characterization of the dual spaces by the *Riesz representation theorem*.

(1.3) **Theorem.** Let $1 < p < \infty$ and let p' denote the *dual exponent*, i.e. $\frac{1}{p} + \frac{1}{p'} = 1$. Then, for each $L \in \big(L^p(D)\big)'$, there exists a function $v \in L^{p'}(D)$ such that

$$L(w) = \int_D v(\mathbf{x}) w(\mathbf{x}) \,\mathrm{d}\mathbf{x}$$

for all $w \in L^p(D)$. Moreover, it holds $\|v\|_{L^{p'}(D)} = \|L\|_{(L^p(D))'}$. Thus, $(L^p(D))'$ is isometrically isomorphic to $L^{p'}(D)$.

Proof. A proof of this theorem is given in [AF03]. □

Notice that the dual space of $L^1(D)$ is isometrically isomorphic to $L^\infty(D)$, but the reverse implication does not hold in general.

Several times we will make use of the generalized Hölder inequality.

(1.4) **Lemma.** For $n \in \mathbb{N}$, let $p_i \in [1, \infty]$ for $i = 1, \dots, n$ be given with $\sum_{i=1}^n \frac{1}{p_i} = 1$. Then, it follows for functions $v_i \in L^{p_i}(D)$, $i = 1, \dots, n$, that $\prod_{i=1}^n v_i \in L^1(D)$. Moreover, these functions satisfy the *generalized Hölder inequality*

(1.5) $$\left\| \prod_{i=1}^n v_i \right\|_{L^1(D)} \le \prod_{i=1}^n \|v_i\|_{L^{p_i}(D)}.$$

Proof. For a proof, see [Alt07]. □

For $n = 2$ and dual exponents $p, p' \in [1, \infty]$, the generalized Hölder inequality (1.5) reduces for functions $v \in L^p(D)$ and $w \in L^{p'}(D)$ to

$$(v, w)_{L^2(D)} \le \|v\|_{L^p(D)} \|w\|_{L^{p'}(D)}.$$

Hence, the scalar product (1.2) extends to a duality product on $L^p(D) \times L^{p'}(D)$.

2. Sobolev spaces

It is well known that Sobolev spaces come into play when weak solutions of elliptic partial differential equations are considered. These spaces are defined as follows:

(2.1) **Definition.** We define the Sobolev space $W^{k,p}(D)$ for $k \in \mathbb{N}$ as the closure of $C^\infty(D)$ with respect to the norm

(2.2) $$\|v\|_{W^{k,p}(D)} = \begin{cases} \left(\sum_{|\boldsymbol{\alpha}| \le k} \|\partial^{\boldsymbol{\alpha}} v\|_{L^p(D)}^p \right)^{1/p} & \text{for } 1 \le p < \infty, \\ \max_{|\boldsymbol{\alpha}| \le k} \|\partial^{\boldsymbol{\alpha}} v\|_{L^\infty(D)} & \text{for } p = \infty. \end{cases}$$

Here, $\boldsymbol{\alpha} \in \mathbb{N}^d$ denotes the multi-index $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_d)$ with the usual definition $|\boldsymbol{\alpha}| = \sum_{i=1}^d \alpha_i$. Moreover, we denote by

(2.3) $$\partial^{\boldsymbol{\alpha}} v(\mathbf{x}) := \frac{\partial^{\alpha_1}}{\partial x_1} \frac{\partial^{\alpha_2}}{\partial x_2} \cdots \frac{\partial^{\alpha_d}}{\partial x_d} v(\mathbf{x})$$

the weak derivative of order $\boldsymbol{\alpha}$ of v. Analogously, the spaces $W_0^{k,p}(D)$ are given as the closure of $C_0^\infty(D)$ with respect to the norm $\|\cdot\|_{W^{k,p}(D)}$. Additionally, we define for $s \in \mathbb{R}$ the Sobolev space $W^{s,p}(D)$ as the functions $v \in W^{\lfloor s \rfloor,p}(D)$ such that

$$\|v\|_{W^{s,p}(D)} := \left(\|v\|_{W^{\lfloor s \rfloor,p}(D)}^p + |v|_{W^{s,p}(D)}^p \right)^{\frac{1}{p}} < \infty.$$

Herein, we denote by $|v|_{W^{s,p}(D)}$ the *Sobolev-Slobodeckiĭ semi-norm*

$$|v|^p_{W^{s,p}(D)} := \sum_{|\boldsymbol{\alpha}|=\lfloor s\rfloor} \int_D \int_D \frac{|\partial^{\boldsymbol{\alpha}} v(\mathbf{x}) - \partial^{\boldsymbol{\alpha}} v(\mathbf{y})|^p}{|\mathbf{x}-\mathbf{y}|^{d+(s-\lfloor s\rfloor)p}} \,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}.$$

As in the case of the Lebesgue spaces, the Sobolev spaces are Banach spaces which are separable for $p < \infty$. In particular, the Sobolev spaces $W^{s,2}(D)$, denoted by $H^s(D)$, are Hilbert spaces with respect to the scalar product

$$(u,v)_{H^s(D)} := \sum_{|\boldsymbol{\alpha}|\leq s} (\partial^{\boldsymbol{\alpha}} u, \partial^{\boldsymbol{\alpha}} v)_{L^2(D)}, \qquad \text{if } s \in \mathbb{N},$$

and with respect to the scalar product

$$\begin{aligned}(u,v)_{H^s(D)} :=& (u,v)_{H^{\lfloor s\rfloor}(D)} \\ &+ \sum_{|\boldsymbol{\alpha}|=\lfloor s\rfloor} \int_D \int_D \frac{\big(\partial^{\boldsymbol{\alpha}} u(\mathbf{x}) - \partial^{\boldsymbol{\alpha}} u(\mathbf{y})\big)\big(\partial^{\boldsymbol{\alpha}} v(\mathbf{x}) - \partial^{\boldsymbol{\alpha}} v(\mathbf{y})\big)}{|\mathbf{x}-\mathbf{y}|^{d+2(s-\lfloor s\rfloor)}} \,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y},\end{aligned}$$

if $s \in \mathbb{R}_+ \setminus \mathbb{N}$, see e.g. [Ste03]. An important property of the Sobolev space $W^{s,p}(D)$ is the compact embedding into the space $C^{q,\lambda}(D)$ for certain values of s, p, q and λ. The Banach space $C^{q,\lambda}(D)$ with $q \in \mathbb{N}$ and $\lambda \in (0,1)$ consists of q times continuously differentiable functions, whose derivatives of order q are additionally Hölder-continuous with Hölder-exponent λ. This space is equipped with the norm

$$\|v\|_{C^{q,\lambda}(D)} := \|v\|_{C^q(D)} + \sup_{|\boldsymbol{\alpha}|=q} \sup_{\mathbf{x}\neq\mathbf{y}\in D} \frac{\big|\partial^{\boldsymbol{\alpha}} v(\mathbf{x}) - \partial^{\boldsymbol{\alpha}} v(\mathbf{y})\big|}{|\mathbf{x}-\mathbf{y}|^\lambda}.$$

The embedding theorem of Sobolev provides the relation between s, p, q, λ and the dimensionality d of D to ensure this embedding property.

(2.4) **Theorem.** Let $D \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary and let $s > d/p$. Then, we have the following compact embedding of the Sobolev space $W^{s,p}(D)$:

$$W^{s,p}(D) \hookrightarrow \begin{cases} C^{\lfloor s-d/p\rfloor, s-d/p-\lfloor s-d/p\rfloor}(D), & \text{if } s-d/p \notin \mathbb{N}, \\ C^{s-d/p-1,\lambda}(D), & \text{if } s-d/p \in \mathbb{N} \text{ for arbitrary } \lambda \in (0,1). \end{cases}$$

Proof. For a proof of this result, see [DD12]. □

A semi-norm $|\cdot|_{W^{k,p}(D)}$ on $W^{k,p}(D)$ is defined for all $k \in \mathbb{N}$ if only the L^p-norms of the weak derivatives of order k are taken into account in (2.2), i.e.

$$(2.5) \qquad |v|_{W^{k,p}(D)} = \left(\sum_{|\boldsymbol{\alpha}|=k} \|\partial^{\boldsymbol{\alpha}} v\|^p_{L^p(D)} \right)^{1/p}.$$

By Sobolev's norm equivalence theorem, cf. [Ada75], this semi-norm (2.5) defines an equivalent norm on $W^{k,p}_0(D)$ on bounded domains D. More precisely, there is a constant $c \geq 1$ such that

$$(2.6) \qquad |v|_{W^{k,p}(D)} \leq \|v\|_{W^{k,p}(D)} \leq c|v|_{W^{k,p}(D)}$$

holds for all $v \in W^{k,p}_0(D)$. This result can be proven by the repeated application of the *Poincaré inequality*.

(2.7) **Lemma.** Let the domain D be bounded. Then, there exists a Poincaré constant $c_P > 0$ such that for all $v \in W_0^{1,p}(D)$ it holds the Poincaré inequality

(2.8) $$\|v\|_{L^p(D)} \le c_P |v|_{W^{1,p}(D)}.$$

Proof. A proof of this inequality is provided in [AF03]. □

(2.9) **Remark.** We denote the Sobolev space $W_0^{s,2}(D)$ by $H_0^s(D)$. From now on, the space $H_0^1(D)$ is considered to be equipped with the norm

$$\|\cdot\|_{H_0^1(D)} := |\cdot|_{H^1(D)} = \|\nabla \cdot\|_{[L^2(D)]^d},$$

which is an equivalent norm to (2.2) by (2.6). Likewise, we use corresponding norms for the Sobolev spaces $W_0^{1,p}(D)$, i.e.

$$\|\cdot\|_{W_0^{1,p}(D)} := |\cdot|_{W^{1,p}(D)} = \|\nabla \cdot\|_{[L^p(D)]^d}.$$

The space $[L^p(D)]^d$ is defined as the space of equivalence classes of $\mathbb{R}^d$-valued functions $v = [v_1, \ldots, v_d]^\intercal$ which are bounded with respect to the norm

$$\|v\|_{[L^p(D)]^d}^p := \int_D \|v\|_p^p \,\mathrm{d}\mathbf{x} \quad \text{with} \quad \|v\|_p^p := \sum_{k=1}^d |v_k|^p.$$

For Sobolev spaces $W_0^{1,p_i}(D)$, we obtain an analogue to the generalized Hölder inequality (1.5).

(2.10) **Lemma.** For $n \in \mathbb{N}$, let $p_i \in [1, \infty]$ for $i = 1, \ldots, n$ be given with $\sum_{i=1}^n \frac{1}{p_i} = 1$. Then, it follows for functions $v_i \in W_0^{1,p_i}(D)$, $i = 1, \ldots, n$, that $\prod_{i=1}^n v_i \in W_0^{1,1}(D)$. In addition, these funtions fulfill the inequality

(2.11) $$\left\| \prod_{i=1}^n v_i \right\|_{W_0^{1,1}(D)} \le n c_P^{n-1} \prod_{i=1}^n \|v_i\|_{W_0^{1,p_i}(D)}.$$

Proof. With the product rule for derivatives, it holds that

(2.12) $$\left\| \prod_{i=1}^n v_i \right\|_{W_0^{1,1}(D)} \le \sum_{i=1}^n \left\| \nabla v_i \prod_{j \ne i} v_j \right\|_{[L^1(D)]^d}.$$

We apply the generalized Hölder inequality (1.5) and the Poincaré inequality (2.8) to obtain

$$\left\| \nabla v_i \prod_{j \ne i} v_j \right\|_{[L^1(D)]^d} \le \|\nabla v_i\|_{[L^{p_i}(D)]^d} \prod_{j \ne i} \|v_j\|_{L^{p_j}(D)} \le c_P^{n-1} \prod_{j=1}^n \|v_j\|_{W_0^{1,p_j}(D)}.$$

Summation over these terms in (2.12) yields (2.11). □

We end this section with a brief note on interpolation spaces and inequalities between Sobolev spaces. For further details, we refer to [BS08] and the reference therein.

(2.13) **Definition.** Let X_0, X_1 denote two Banach spaces with $X_1 \subset X_0$. For $t > 0$, we define a measure for the approximability of X_0 with elements of X_1 by

$$K(t,v) := \inf_{w \in X1} (\|v - w\|_{X_0} + t\|w\|_{X1}).$$

Then, the interpolation space $[X_0, X_1]_{\theta,p} := \{v \in X_0 : \|v\|_{[X_0,X_1]_{\theta,p}} < \infty\}$ is given as the set of elements of X_0 which are bounded with respect to the norm

$$\|v\|_{[X_0,X_1]_{\theta,p}} := \left(\int_0^\infty t^{-\theta p} K(t,v)^p \frac{\mathrm{d}t}{t} \right)^{1/p}.$$

The interpolation space $[X_0, X_1]_{\theta,p}$ fulfills the following lemma, cf. [BS08].

(2.14) **Lemma.** Let X_0, X_1 and Y_0, Y_1 are two pairs of Banach spaces according to Definition (2.13) and $T \colon X_i \to Y_i$ a linear operator. Then T maps $[X_0, X_1]_{\theta,p} \to [Y_0, Y_1]_{\theta,p}$ and satisfies the inequality

$$\|T\|_{[X_0,X_1]_{\theta,p} \to [Y_0,Y_1]_{\theta,p}} \leq \|T\|_{X_0 \to Y_0}^{1-\theta} \|T\|_{X_1 \to Y_1}^{\theta}$$

It is obvious that it holds $X_1 \subset [X_0, X_1]_{\theta,p} \subset X_0$ for arbitrary Banach spaces $X_1 \subset X_0$. For Sobolev spaces, however, it is possible to characterize the interpolation spaces more precisely. We will only need the characterization for the Sobolev spaces $H^s(D)$. For more general resuts for Sobolev or Lebesgue spaces, we refer to [BL76].

(2.15) **Lemma.** The $[H^r(D), H^s(D)]_{\theta,2}$-norm is equivalent to the $H^{(1-\theta)r+\theta s}(D)$-norm for arbitrary $r, s \in \mathbb{R}$ and $0 < \theta < 1$ provided that D is a Lipschitz domain. This yields that

$$[H^r(D), H^s(D)]_{\theta,2} = H^{(1-\theta)r+\theta s}(D).$$

We will make use of the following norm-inequality

$$\text{(2.16)} \qquad \|v\|_{H^s(D)} \lesssim \|v\|_{L^2(D)}^{1-s/r} \|v\|_{H^r(D)}^{s/r} \quad \text{for all } v \in H^r(D), \quad 0 \leq s \leq r$$

which can be inferred from Lemma (2.14) and Lemma (2.15) combined with a duality argument.

3. Lebesgue-Bochner spaces

Let (S, Σ, μ) be a measure space with σ-algebra Σ and measure μ. Moreover, we denote by X a Banach space over $\mathbb{R}$, equipped with its Borel σ-algebra $\mathcal{B}$ and its norm $\|\cdot\|_X$. In the sequel, the Banach space X will represent a space of real valued functions which are defined on the domain $D \subset \mathbb{R}^d$ like a Lebesgue space or a Sobolev space.

Notice that a random field $v : D \times \Omega \to \mathbb{R}$ can be seen as a mapping $v : \Omega \to X$ which assigns an element of a suitable Banach space X to each element $\omega \in \Omega$. Moreover, recall that a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is particularly a measure space. Thus, the *Lebesgue-Bochner spaces*, which transfer the concept of the classical Lebesgue spaces to strongly measureable and Banach space valued functions, defined over the measure space

(S, Σ, μ), are the canonical spaces when dealing with random fields. At first defined in [Boc33], these spaces are well studied nowadays. In many textbooks, the concepts of Lebesgue spaces are introduced for the general situation of functions which map from a measure space into a Banach space, see e.g. [Alt07]. We will follow the construction of [Alt07] and start with the definition of strongly measurable functions $v : S \to X$.

(3.1) **Definition.** A μ-measurable map is a function $v : S \to X$ such that for any Borel set $B \in \mathcal{B}$ it follows $v^{-1}(B) \in \Sigma$. Moreover, a μ-measurable map v is strongly μ-measurable if there exists a μ-null set N such that $v(S \setminus N)$ is separable.

With this definition at hand, we are able to define the Lebesgue-Bochner space $L^p_\mu(S; X)$.

(3.2) **Definition.** The Lebesgue-Bochner space $L^p_\mu(S; X)$ is defined for $1 \leq p < \infty$ as the set of equivalence classes of strongly μ-measurable functions $v : S \to X$ with finite norm

$$\|v\|_{L^p_\mu(S;X)} := \left(\int_S \|v(s)\|_X^p \, \mathrm{d}\mu(s) \right)^{1/p} < \infty.$$

For $p = \infty$, the space $L^\infty_\mu(S; X)$ contains all equivalence classes of measurable functions which are essentially bounded, i.e.

$$\|v\|_{L^\infty_\mu(S;X)} := \operatorname*{ess\,sup}_{s \in S} \|v(s)\|_X := \inf_{N \subset S : \mu(N) = 0} \sup_{s \in S \setminus N} \|v(s)\|_X.$$

Two functions $v, w : S \to X$ are in the same equivalence class if v coincides with w μ-almost everywhere.

The *Bochner integral* is constructed in a similar way as the Lebesgue integral, see [AE08, Alt07]. To that end, we denote for an element $S_i \in \Sigma$ the indicator function of S_i by

$$\mathbb{1}_{S_i} \colon S \to \{0, 1\}, \quad \mathbb{1}_{S_i}(s) = \begin{cases} 1, & \text{if } s \in S_i, \\ 0, & \text{else.} \end{cases}$$

For simple functions $v(s) = \sum_{i=1}^n \mathbb{1}_{S_i}(s) x_i$, where $S_i \in \Sigma$ and $x_i \in X$, the Bochner integral is defined as

$$\int_S v(s) \, \mathrm{d}\mu(s) = \sum_{i=1}^n \mu(S_i) x_i.$$

A strongly μ-measurable function $v : S \to X$ is *Bochner integrable* if there exists a sequence $\{v_j\}_j$ of simple functions such that

$$\lim_{j \to \infty} \int_S \|v - v_j\|_X \, \mathrm{d}\mu(s) = 0.$$

The Bochner integral is then defined as

$$\int_S v \, \mathrm{d}\mu(s) = \lim_{j \to \infty} \int_S v_j \, \mathrm{d}\mu(s). \tag{3.3}$$

A simple characterization of Bochner integrable functions is provided by the *Bochner criterion* for integrability. This criterion states that a strongly μ-measurable function $v : S \to X$ is Bochner integrable if and only if

$$\int_S \|v\|_X \,\mathrm{d}\mu(s) < \infty.$$

There are several useful properties for the Lebesgue-Bochner spaces and the Bochner integral. We collect a few of them in the following lemma.

(3.4) **Lemma.** (a) The Lebesgue-Bochner spaces $L^p_\mu(S;X)$ are Banach spaces for $1 \le p \le \infty$.

(b) The Bochner integral

$$I : L^1_\mu(S;X) \to X, \qquad v \mapsto Iv := \int_S v(s)\,\mathrm{d}\mu(s) \tag{3.5}$$

is a linear map and well defined. Moreover, this map is continuous with continuity constant 1, i.e.

$$\left\| \int_S v(s)\,\mathrm{d}\mu(s) \right\|_X \le \int_S \|v(s)\|_X \,\mathrm{d}\mu(s). \tag{3.6}$$

(c) Let $v : S \to X$ be a μ-measurable function and let $w \in L^1_\mu(S;\mathbb{R})$ with $w \ge 0$. If it holds for μ-almost every $s \in S$ that $\|v(s)\|^p_X \le w(s)$ for a $p \in [1,\infty)$, then it follows that $v \in L^p_\mu(S;X)$.

Proof. For the proof of the lemma, we refer to [Alt07]. □

Notice that we will use different notations for the Bochner integral throughout the thesis when S is further specified. If S is a one-dimensional subset of $\mathbb{R}$ and μ is given by a continuous density with respect to the Lebesgue-measure, we will use I as in (3.5), the tensorization of those one-dimensional Bochner integral operators is denoted by $\mathbf{I}$, and if (S,σ,μ) is a probability space, we will employ $\mathbb{E}$.

As in the case of the Lebesgue spaces, the case $p = 2$ implies a special situation. Then, the Lebesgue-Bochner space $L^2_\mu(S;X)$ is a Hilbert space provided that X is a Hilbert space. The scalar product is defined by

$$(v,w)_{L^2_\mu(S;X)} := \int_S (v(s),w(s))_X \,\mathrm{d}\mu(s).$$

If X and S are additionally separable, we know that $L^2_\mu(S;X)$ is isometrically isomorphic to the tensor product space $L^2_\mu(S) \otimes X$, see [LC85], i.e.

$$L^2_\mu(S;X) \simeq L^2_\mu(S) \otimes X.$$

Chapter III

PROBLEM FORMULATION

1. Problem formulation

In the following, let $D \subset \mathbb{R}^d$ for $d \in \mathbb{N}$ be a domain with Lipschitz continuous boundary and let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with σ-field $\mathcal{F} \subset 2^\Omega$ and probability measure $\mathbb{P}$. The completeness of the probability space implies that, for all $A \subset B$ and $B \in \mathcal{F}$ with $\mathbb{P}[B] = 0$, it follows $A \in \mathcal{F}$.

As mentioned in the introduction, we want to approximate the random solution $u(\omega) \in H_0^1(D)$ to the stochastic elliptic diffusion problem

$$(1.1) \qquad -\operatorname{div}\left(a(\omega)\nabla u(\omega)\right) = f \text{ in } D \quad \text{for almost every } \omega \in \Omega$$

with (deterministic) loading $f \in L^2(D)$. Instead of directly approximating the solution $u(\omega)$ itself, we rather intend to compute the solution's moments

$$(1.2) \qquad \mathcal{M}^p u(\mathbf{x}) := \mathbb{E}_{u^p}(\mathbf{x}) = \int_\Omega u^p(\mathbf{x}, \omega) \, \mathrm{d}\mathbb{P}(\omega).$$

Especially, the solution's expectation given by

$$(1.3) \qquad \mathbb{E}_u(\mathbf{x}) = \int_\Omega u(\mathbf{x}, \omega) \, \mathrm{d}\mathbb{P}(\omega) \in H_0^1(D)$$

and its variance given by

$$(1.4) \qquad \mathbb{V}_u(\mathbf{x}) = \mathbb{E}_{u^2}(\mathbf{x}) - \mathbb{E}_u^2(\mathbf{x}) = \int_\Omega u^2(\mathbf{x}, \omega) \, \mathrm{d}\mathbb{P}(\omega) - \mathbb{E}_u^2(\mathbf{x}) \in W_0^{1,1}(D)$$

are of interest to us. They correspond to the first and the second (centered) moment of the solution u. As we will show later on, it holds $\mathcal{M}^p u \in W_0^{1,1}(D)$ for a sufficiently smooth diffusion coefficient a and $f \in L^p(D)$. Note, that the knowledge of all moments is sufficient to determine the distribution of the random field u.

We investigate two different types of diffusion coefficients. On the one hand, we consider a uniform elliptic diffusion coefficient. With the knowledge of the diffusion coefficient's mean field $\mathbb{E}_a(\mathbf{x})$ and its covariance kernel $k_a(\mathbf{x}, \mathbf{x}')$ at hand, a representation of this diffusion coefficient can be computed by the Karhunen-Loève expansion, cf. [Loè77], which is analyzed in Section 2. This expansion has the form

$$(1.5) \qquad a(\mathbf{x}, \omega) = \mathbb{E}_a(\mathbf{x}) + \sum_{k=1}^{\infty} \sqrt{\lambda_k} \varphi_k(\mathbf{x}) \psi_k(\omega).$$

We have to assume certain distribution properties on the occurring random variables $\psi_k(\omega)$. To ensure ellipticity and boundedness of the associated bilinear form in the variational formulation, the variation in the diffusion coefficient has to be bounded. We employ uniformly distributed random variables on $[-1/2, 1/2]$, i.e. $\psi_k \sim \text{UNI}(-1/2, 1/2)$. Of course, other distributions of the random variables can be treated in the same way as long as they can be described by a density function with bounded support. The uniform ellipticity and the boundedness condition imply that there exist constants $\underline{a}, \overline{a} > 0$ such that

$$\mathbb{P}\left(\underline{a} \leq \operatorname*{ess\,inf}_{x\in D} a(\mathbf{x},\omega) \leq \operatorname*{ess\,sup}_{x\in D} a(\mathbf{x},\omega) \leq \overline{a}\right) = 1. \tag{1.6}$$

We refer to this case as the *uniformly elliptic* case.

On the other hand, we consider a lognormally distributed diffusion coefficient a, where the logarithm of a is given by a centered Gaussian field. Here, the covariance kernel $k_b(\mathbf{x}, \mathbf{x}')$ of the logarithm of the diffusion coefficient $b(\mathbf{x}, \omega) = \log\big(a(\mathbf{x}, \omega)\big)$ is assumed to be known. The Karhunen-Loève expansion of b yields the representation

$$a(\mathbf{x},\omega) = \exp\big(b(\mathbf{x},\omega)\big) \quad \text{with} \quad b(\mathbf{x},\omega) = \sum_{k=1}^{\infty} \sqrt{\lambda_k}\varphi_k(\mathbf{x})\psi_k(\omega). \tag{1.7}$$

We know from Chapter I that equations involving such diffusion coefficients are of great importance for geologists and arise in the treatment of subsurface flow models. Gaussian random variables are not bounded from above or below which implies that for all $c \in \mathbb{R}$ it holds $\mathbb{P}(\psi < c) > 0$ and $\mathbb{P}(\psi > c) > 0$ if $\psi \sim \mathcal{N}(\mu, \sigma)$. Hence, it follows that a lognormal diffusion coefficient is not uniformly bounded from above or away from zero. Thus, the treatment of lognormal diffusion coefficients, which we call the *lognormal case* is more complicated in comparison to the uniformly elliptic case. Nevertheless, the lognormal case is in a certain way more flexible than the uniform elliptic case since it yields no restriction on the variation in the diffusion coefficient.

2. Karhunen-Loève expansion of random fields

In this section, we describe the computation of the Karhunen-Loève expansion in (1.5) and (1.7). This is a common representation of random fields since it separates the spatial dependency and the stochastic dependency of the random field. The expansion can be regarded as the continuous analogue of the singular value decomposition for matrices, see e.g. [HPS15]. Therefore, we assume the knowledge of the mean field and the covariance kernel of the stochastic field $a(\mathbf{x}, \omega) \in L^2_{\mathbb{P}}(\Omega; L^2(D))$ in the uniformly elliptic case. These statistics are given by the Bochner integrals

$$\mathbb{E}_a(\mathbf{x}) = \int_\Omega a(\mathbf{x},\omega)\,\mathrm{d}\mathbb{P}(\omega)$$

and

$$k_a(\mathbf{x},\mathbf{x}') = \int_\Omega \big(a(\mathbf{x},\omega) - \mathbb{E}_a(\mathbf{x})\big)\big(a(\mathbf{x}',\omega) - \mathbb{E}_a(\mathbf{x}')\big)\,\mathrm{d}\mathbb{P}(\omega).$$

It is easily obtained from $a \in L^2_{\mathbb{P}}(\Omega; L^2(D))$ that the covariance kernel k_a is in $L^2(D \times D)$. Hence, the associated covariance operator is a mapping $\mathcal{C} : L^2(D) \to L^2(D)$ defined by

(2.1) $$(\mathcal{C}u)(\mathbf{x}) = \int_D k_a(\mathbf{x}, \mathbf{x}')u(\mathbf{x}')\,\mathrm{d}\mathbf{x}'.$$

This operator is a symmetric and positive semi-definite *Hilbert-Schmidt operator*. Thus, due to the compactness of Hilbert-Schmidt operators, the eigenvalues $\{\lambda_k\}_k$ of $\mathcal{C}$ generate a decreasing sequence $\lambda_1 \geq \lambda_2 \geq \ldots \geq 0$ which tends to zero. Let us denote by (λ_k, φ_k) the eigenpairs of $\mathcal{C}$. Then, the Karhunen-Loève expansion of a is given by

(2.2) $$a(\mathbf{x}, \omega) = \mathbb{E}_a(\mathbf{x}) + \sum_{k=1}^{\infty} \sqrt{\lambda_k}\varphi_k(\mathbf{x})\psi_k(\omega)$$

with random variables $\psi_k(\omega)$ which are determined by

$$\psi_k(\omega) = \frac{1}{\sqrt{\lambda_k}} \int_D \big(a(\mathbf{x}, \omega) - \mathbb{E}_a(\mathbf{x})\big)\varphi_k(\mathbf{x})\,\mathrm{d}\mathbf{x}.$$

It can be easily verified that these random variables are uncorrelated, normalized and centered, i.e. it holds for all $k, \ell \in \mathbb{N}$ that

$$\mathbb{E}[\psi_k] = 0 \quad \text{and} \quad \int_\Omega \psi_k(\omega)\psi_\ell(\omega)\,\mathrm{d}\mathbb{P}(\omega) = \delta_{k,\ell}.$$

Since the diffusion coefficient itself is not explicitly known, we are generally not able to determine the distribution of the random variables in (2.2) and have to estimate them from measurements, for example with a *maximum likelihood estimator*, cf. [ST06]. The following assumptions on the Karhunen-Loève expansion in the uniformly elliptic case are widely used:

(2.3) **Assumption.** The family $\{\psi_k\}_k$ consists of independent random variables with image $\Gamma_k := \mathrm{Im}(\psi_k)$. Furthermore, the distribution of ψ_k is described by a density function $\rho_k : \Gamma_k \to \mathbb{R}_+$ which is continuous with respect to the Lebesgue measure.

In the lognormal case, we define the Karhunen-Loève expansion of the mean-free Gaussian random field $b(\mathbf{x}, \omega)$ analogously to (2.2). The advantage is that the assumption (2.3) is fulfilled automatically. Since we know that $b(\mathbf{x}, \omega)$ is Gaussian, we can deduce that the random variables ψ_k in the Karhunen-Loève expansion are also Gaussian. Moreover, due to the fact that they are normalized and centered, it follows immediately that each random variable ψ_k is standard normally distributed, i.e. $\psi_k \sim \mathcal{N}(0, 1)$. This implies that $\Gamma_k = \mathbb{R}$ and $\rho_k = \exp(-y_k^2/2)/\sqrt{2\pi}$. In addition, we know that a family of uncorrelated Gaussian random variables is also independent.

For the further analysis, we assume that the sequence

(2.4) $$\gamma_k := \sqrt{\lambda_k}\|\varphi_k\|_{L^\infty(D)}$$

satisfies $\{\gamma_k\}_k \in \ell^1(\mathbb{N})$. In Chapter VI, we will also require that the products of the first derivatives of the eigenfunctions with the singular values

(2.5) $$\tilde{\gamma}_k := \gamma_k + \sqrt{\lambda_k}\|\nabla\varphi_k\|_{L^\infty(D)} = \sqrt{\lambda_k}\big(\|\varphi_k\|_{L^\infty(D)} + \|\nabla\varphi_k\|_{L^\infty(D)}\big)$$

build a summable sequence, i.e. $\{\tilde{\gamma}_k\}_k \in \ell^1(\mathbb{N})$.

(2.6) **Remark.** The decay of the sequences in (2.4) and (2.5) will be very important throughout this thesis. As we will see, it quantifies the anisotropic dependency of the solution u on the different stochastic dimensions.

In practice, the Karhunen-Loève expansion needs to be truncated appropriately after m terms. Hence, we define the truncated diffusion coefficient $a_m\colon D \times \Omega \to \mathbb{R}$ via

$$(2.7) \qquad a_m(\mathbf{x},\omega) := \mathbb{E}_a(\mathbf{x}) + \sum_{k=1}^{m} \sqrt{\lambda_k}\varphi_k(\mathbf{x})\psi_k(\omega)$$

in the uniformly elliptic case. For the lognormal case, we obtain the truncated diffusion coefficient by

$$(2.8) \qquad b_m(\mathbf{x},\omega) := \sum_{k=1}^{m} \sqrt{\lambda_k}\varphi_k(\mathbf{x})\psi_k(\omega) \quad \text{and} \quad a_m(\mathbf{x},\omega) := \exp\big(b_m(\mathbf{x},\omega)\big).$$

Replacing $a(\omega)$ in (1.1) by $a_m(\omega)$ yields the truncated problem to determine the solution $u_m(\omega) \in H_0^1(D)$ of the stochastic elliptic diffusion problem

$$(2.9) \qquad -\operatorname{div}\big(a_m(\omega)\nabla u_m(\omega)\big) = f \text{ in } D \quad \text{for almost every } \omega \in \Omega.$$

The error induced by the truncation of the Karhunen-Loève expansion has to be considered. We distinguish between strong and weak truncation error estimates. The *strong truncation error* $\|u - u_m\|_{L^q_{\mathbb{P}}(\Omega;C^{1,\beta}(D))}$ for $q \geq 1$ and some $\beta > 0$ as well as the *weak truncation error* $\|\mathbb{E}(g(u_m) - g(u))\|_{C^{1,\beta}(D)}$ for sufficiently smooth functions $g\colon \mathbb{R} \to \mathbb{R}$ are analyzed for the lognormal case in [CD13, Cha12]. Especially, the weak truncation error estimate is of interest for us since it provides a bound in case of the moment computation. For the uniformly elliptic case, a truncation error estimate, pointwise for $\mathbb{P}$-almost all $\omega \in \Omega$ and measured in the $\|\cdot\|_{H^1(D)}$-norm, is given in [KSS15]. Moreover, the weak truncation error $|\mathbb{E}(G(u) - G(u_m))|$ for linear functionals $G \in \big(H_0^1(D)\big)'$ is provided there. Unfortunately, this weak truncation error is not applicable for the moment computation, but the weak truncation error estimate in [CD13] is straightforwardly transferable to the uniformly elliptic case. Thus, we will formulate this estimate for the uniformly elliptic case as well. In the lognormal case, we have the following result from [CD13].

(2.10) **Theorem ([CD13]).** Assume that the eigenfunctions in the Karhunen-Loève expansion of the Gaussian random field $b(\mathbf{x},\omega)$ are Hölder-continuous with exponent $1/2$ and that the series

$$\sum_{k=1}^{\infty} \lambda_k \|\varphi_k\|^2_{C^{0,1/2}(D)}$$

is convergent. Furthermore, let $f \in L^p(D)$ for $p > d$ and define

$$R_m^\alpha := \sum_{k=m+1}^{\infty} \lambda_k \|\varphi_k\|^2_{C^{0,\alpha}(D)}.$$

Then, the solution $u_m(\omega)$ to (2.9) with the lognormal diffusion coefficient (2.8) satisfies for all $q \geq 1$ the estimate

(2.11) $$\|u - u_m\|_{L^q_{\mathbb{P}}(\Omega;C^{1,\beta}(D))} \lesssim \sqrt{R_m^\alpha}$$

for any $\alpha, \beta \in \mathbb{R}$ with $0 < \beta < \min\{\frac{1}{2}, 1 - \frac{d}{p}\}$ and $0 < \beta < \alpha < 1/2$. The constant hidden in this estimate depends on f, p, q, α, β and on the diffusion coefficient a.

This is the strong error estimate. For the weak error estimate, the bound can be improved. In particular, one can prove that the weak error decays with twice the order of the strong error in the number of terms in the Karhunen-Loève expansion.

(2.12) **Theorem ([CD13]).** Let the assumptions of Theorem (2.10) be fulfilled and let $g \in C^6(\mathbb{R})$ such that g and its derivatives grow at most polynomially as $|x| \to \infty$. Then, it holds for all $m \in \mathbb{N}$ the weak error bound

(2.13) $$\|\mathbb{E}(g(u_m) - g(u))\|_{C^{1,\alpha}(D)} \lesssim R_m^\alpha.$$

This estimate is valid for any $\alpha > 0$ with $0 < \alpha < \min\{1/2, 1 - \frac{d}{p}\}$. The constant hidden in (2.13) is dependent on α, f, g, p and on the diffusion coefficient a.

Since the diffusion coefficient is bounded for $\mathbb{P}$-almost all $\omega \in \Omega$ in the uniformly elliptic case, it is possible to show the following truncation error bound:

(2.14) **Theorem ([KSS15]).** The solution $u_m(\omega)$ of (2.9) with a uniformly elliptic diffusion coefficient (2.7) satisfies for every $m \in \mathbb{N}$ the estimate

$$\|u(\omega) - u_m(\omega)\|_{H^1(D)} \lesssim \|f\|_{L^2(D)} \sum_{k=m+1}^{\infty} \gamma_k \quad \text{for } \mathbb{P}\text{-almost all } \omega \in \Omega.$$

The constant in this inequality depends on the ellipticity constant $\underline{a}$ and on the Poincaré constant.

The analogue of Theorem (2.12) in the uniformly elliptic case reads:

(2.15) **Theorem.** Let the assumptions of Theorem (2.10) on the eigenfunctions of the Karhunen-Loève expansion (2.2) and on the loading f be fulfilled. Additionally, let $g \in C^6(\mathbb{R})$ be such that g and its derivatives grow at most polynomially as $|x| \to \infty$. Then, we obtain for the solution $u_m(\omega)$ of (2.9) with a uniformly elliptic diffusion coefficient (2.7) the weak truncation error

$$\|\mathbb{E}(g(u_m) - g(u))\|_{C^{1,\alpha}(D)} \lesssim R_m^\alpha$$

for any $\alpha \in \mathbb{R}$ with $0 < \alpha < \min\{\frac{1}{2}, 1 - \frac{d}{p}\}$. The constant depends on β, f, g, p and the ellipticity constant $\underline{a}$.

Proof. In the proof of Theorem (2.12), see [CD13], the error function $u - u_m$ is represented by a Taylor expansion in the random variables. Therefore, estimates on the partial derivatives of u with respect to the particular random variables are required. From Section 5, we know that the estimates on the derivatives are similar in the lognormal and in the uniformly elliptic case. With these estimates at hand, each argument in [CD13] is also applicable in the uniformly elliptic case. □

(2.16) **Remark.** (a) The above estimates imply that we need a number of terms $m(\varepsilon)$ in the Karhunen-Loève expansion to achieve the accuracy ε. Moreover, this $m(\varepsilon)$ is affected by the goal of computation, i.e. whether one is interested in the solution u_m itself or if one would like to determine statistics of the solution u to (1.1) like the expectation, the variance or higher order moments. In the first case, $m(\varepsilon)$ is given by the strong error estimates and, in the second case, by the weak error estimates. Nevertheless, it turns out to be difficult to calculate the quantity R_m^α exactly or even to provide an algorithm which approximates R_m^α. We use for the computation of the Karhunen-Loève expansion the pivoted Cholesky decomposition, see [HPS12, HPS15]. Then, the truncation error of the random field a_m can rigorously be controlled in terms of the trace of the Schur complement which accounts for $\sum_{k>m} \lambda_k$, cf. [HPS15]. This usually provides a good approximation on R_m^α since the quantity is dominated by the decay of $\{\lambda_k\}_k$. But R_m^α might be slightly underestimated since the $C_0^\alpha(D)$-norms of the eigenfunctions φ_k for $k > m$ are not available and, hence, not taken into account. Thus, to rule out the truncation error, we calculate in our numerical experiments $m(\varepsilon)$ in the more conservative way such that $\sum_{k>m} \lambda_k < \varepsilon^2$ instead of $\sum_{k>m} \lambda_k < \varepsilon$.

(b) Regardless of the employed error bound, the number of terms $m(\varepsilon)$ increases as the accuracy ε tends to zero. Thus, it is crucial to find numerical methods for the approximation of the solution u_m of the truncated problem (2.9) or of its statistics which converge (nearly) independently of the dimensionality $m(\varepsilon)$.

3. Parametrization of the problem

It is convenient to consider the image of the random variables $\{\psi_k\}_k$ instead of working on the rather abstract probability space itself. Therefore, we consider the pushforward measure $\mathbb{P}_{\boldsymbol{\psi}} := \mathbb{P} \circ \boldsymbol{\psi}$ with respect to the measurable mapping

$$\boldsymbol{\psi} \colon \Omega \to \boldsymbol{\Gamma} := \bigtimes_{k=1}^{m} \Gamma_k, \quad \omega \mapsto \boldsymbol{\psi}(\omega) := (\psi_1(\omega), \ldots, \psi_m(\omega)).$$

We know from Assumption (2.3) that the family $\{\psi_k\}_k$ is independent and that the distribution of ψ_k is given by the density ρ_k. Hence, the pushforward measure $\mathbb{P}_{\boldsymbol{\psi}}$ is described by the joint density function $\boldsymbol{\rho}$ with respect to the m-dimensional Lebesgue measure

$$\boldsymbol{\rho}(\mathbf{y}) := \prod_{k=1}^{m} \rho_k(y_k). \tag{3.1}$$

With the representation (3.1) at hand, we can reformulate the stochastic problem (1.1) as a parametric deterministic problem. To that end, we substitute the random variables ψ_k by the coordinates $y_k \in \Gamma_k$. Then, we define the parameterized and truncated uniformly elliptic diffusion coefficient via

$$a_m(\mathbf{x}, \mathbf{y}) := \sum_{k=1}^{m} \sqrt{\lambda_k} \varphi_k(\mathbf{x}) y_k \tag{3.2}$$

for all $\mathbf{x} \in D$ and $\mathbf{y} = (y_1, y_2, \dots, y_m) \in \mathbf{\Gamma}$. Thus, we arrive at the following parametric and truncated boundary value problem in the uniformly elliptic case:

$$\text{(3.3)} \qquad \begin{aligned} &\text{find for all } \mathbf{y} \in \mathbf{\Gamma} \text{ a function } u_m(\mathbf{y}) \in H_0^1(D) \text{ such that} \\ &\qquad -\operatorname{div}\left(a_m(\mathbf{x},\mathbf{y})\nabla u_m(\mathbf{x},\mathbf{y})\right) = f(\mathbf{x}) \text{ in } D. \end{aligned}$$

Likewise, the parametrized and truncated Gaussian field which describes the logarithm of the diffusion coefficient in the lognormal case is given by

$$\text{(3.4)} \qquad b_m(\mathbf{x},\mathbf{y}) := \left(\sum_{k=1}^{m} \sqrt{\lambda_k}\varphi_k(\mathbf{x}) y_k \right) \quad \text{and} \quad a_m(\mathbf{x},\mathbf{y}) := \exp\left(b_m(\mathbf{x},\mathbf{y})\right)$$

for $\mathbf{x} \in D$ and $\mathbf{y} = (y_1, y_2, \dots, y_m) \in \mathbf{\Gamma} = \mathbb{R}^m$. The corresponding parametric and truncated boundary value problem in the lognormal case reads:

$$\text{(3.5)} \qquad \begin{aligned} &\text{find for all } \mathbf{y} \in \mathbb{R}^m \text{ a function } u_m(\mathbf{y}) \in H_0^1(D) \text{ such that} \\ &\qquad -\operatorname{div}\Big(\exp\left(b_m(\mathbf{x},\mathbf{y})\right)\nabla u_m(\mathbf{x},\mathbf{y})\Big) = f(\mathbf{x}) \text{ in } D. \end{aligned}$$

Furthermore, we obtain in the uniformly elliptic as well as the lognormal case the variational formulation of the problem:

$$\text{(3.6)} \qquad \begin{aligned} &\text{find for all } \mathbf{y} \in \mathbf{\Gamma} \text{ a function } u_m(\mathbf{y}) \in H_0^1(D) \text{ such that} \\ &\qquad \mathcal{B}_{m,\mathbf{y}}(u_m(\mathbf{y}), v) = L(v) \text{ for all } v \in H_0^1(D). \end{aligned}$$

The bilinear form $\mathcal{B}_{m,\mathbf{y}} \colon H_0^1(D) \times H_0^1(D) \to \mathbb{R}$ and the linear form $L \colon H_0^1(D) \to \mathbb{R}$ are defined in the usual way as

$$\text{(3.7)} \qquad \mathcal{B}_{m,\mathbf{y}}(u,v) := \int_D a_m(\mathbf{x},\mathbf{y})\nabla u(\mathbf{x})\nabla v(\mathbf{x})\,\mathrm{d}\mathbf{x}, \qquad L(v) := \int_D f(\mathbf{x})v(\mathbf{x})\,\mathrm{d}\mathbf{x}.$$

Of course, the uniform ellipticity and boundedness condition (1.6) is still fulfilled for the parametrized and truncated problem in the uniformly elliptic case. From the Lax-Milgram lemma, we know that the variational problem (3.6) admits for each $\mathbf{y} \in \mathbf{\Gamma}$ a unique solution $u_m(\mathbf{y}) \in H_0^1(D)$. This solution additionally satisfies the stability bound

$$\text{(3.8)} \qquad \|u_m(\mathbf{y})\|_{H_0^1(D)} \le \frac{c_P}{\underline{a}} \|f\|_{L^2(D)}$$

where c_P is the Poincaré constant. From this, it follows immediately that the solution u_m is contained in the Bochner space $L^p_{\boldsymbol{\rho}}(\mathbf{\Gamma}; H_0^1(D))$ for all $p \ge 1$.

The situation is a bit more involved in the lognormal case. Here, the stochastic diffusion coefficient $a_m(\mathbf{x},\mathbf{y})$ is neither uniformly bounded away from zero nor uniformly bounded from above for all $\mathbf{y} \in \mathbf{\Gamma} = \mathbb{R}^m$. Consequently, it is impossible to show unique solvability in the classical way as for elliptic boundary value problems. Especially, the Lax-Milgram lemma does not directly apply to the problem (3.5). Nevertheless, there exist functions $\underline{a}_m(\mathbf{y}), \overline{a}_m(\mathbf{y}) \colon \mathbf{\Gamma} \to \mathbb{R}$ such that

$$\text{(3.9)} \qquad 0 < \underline{a}_m(\mathbf{y}) \le \operatorname*{ess\,inf}_{\mathbf{x}\in D} a_m(\mathbf{x},\mathbf{y}) \le \operatorname*{ess\,sup}_{\mathbf{x}\in D} a_m(\mathbf{x},\mathbf{y}) \le \overline{a}_m(\mathbf{y}) < \infty.$$

These functions can obviously be chosen as

$$\underline{a}_m(\mathbf{y}) = \exp\bigg(-\sum_{k=1}^{m}\gamma_k|y_k|\bigg) \quad \text{and} \quad \overline{a}_m(\mathbf{y}) = \exp\bigg(\sum_{k=1}^{m}\gamma_k|y_k|\bigg). \tag{3.10}$$

Hence, for every fixed $\mathbf{y} \in \mathbb{R}^m$, it follows from (3.9) and (3.10) that the problem (3.5) is elliptic and admits a unique solution $u_m(\mathbf{y}) \in H_0^1(D)$ which satisfies

$$\|u_m(\mathbf{y})\|_{H^1(D)} \leq \frac{c_P}{\underline{a}_m(\mathbf{y})}\|f\|_{L^2(D)}. \tag{3.11}$$

The lower and upper bounds $\underline{a}_m(\mathbf{y})$ and $\overline{a}_m(\mathbf{y})$ in (3.10) are not defined in the usual way and in general the bounds are not sharp. The canonical choice of these functions would be

$$\underline{a}_m(\mathbf{y}) = \operatorname*{ess\,inf}_{\mathbf{x}\in D} a_m(\mathbf{x},\mathbf{y}) \quad \text{and} \quad \overline{a}_m(\mathbf{y}) = \operatorname*{ess\,sup}_{\mathbf{x}\in D} a_m(\mathbf{x},\mathbf{y}). \tag{3.12}$$

Nevertheless, these sharper bounds yield no additional benefit for our goals and the definition in (3.10) simplifies the regularity analysis in Section VI.3. Notice that in our regime $1/\underline{a}_m(\mathbf{y})$ coincides with $\overline{a}_m(\mathbf{y})$. Thus, we could state all regularity estimates either depending on $\underline{a}_m(\mathbf{y})$ or on $\overline{a}_m(\mathbf{y})$. However, for notational convenience, we present the regularity results in Section 5 and VI.3 in terms of the quotient

$$\kappa_m(\mathbf{y}) := \overline{a}_m(\mathbf{y})/\underline{a}_m(\mathbf{y}). \tag{3.13}$$

It can easily be shown that $1/\underline{a}_m(\mathbf{y})$ and accordingly $\overline{a}_m(\mathbf{y})$ are integrable with respect to the Gaussian measure. More precisely, according to e.g. [Cha12], it holds:

(3.14) **Lemma ([Cha12]).** The lower bound $\underline{a}_m(\mathbf{y})$ and the upper bound $\overline{a}_m(\mathbf{y})$ satisfy for any $p \geq 1$ that $1/\underline{a}_m(\mathbf{y}), \overline{a}_m(\mathbf{y}) \in L^p_{\boldsymbol{\rho}}(\mathbb{R}^m)$.

From this lemma, we can derive that the solution $u_m(\mathbf{y})$ of (3.5) is unique and belongs to $L^p_{\boldsymbol{\rho}}(\mathbb{R}^m; H_0^1(D))$ for any $p > 0$.

The parametrization of the problems (3.3) and (3.5), respectively, influences the computations of the moments (1.2). Instead of integrating with respect to the original measure $\mathbb{P}$, we now integrate with respect to the pushforward measure. Hence, the domain of integration becomes the parameter domain $\boldsymbol{\Gamma}$. This is possible due to the integral transform

$$\mathcal{M}^p u_m = \int_\Omega u_m^p(\omega)\,\mathrm{d}\mathbb{P}(\omega) = \int_{\boldsymbol{\Gamma}} u_m^p(\mathbf{y})\boldsymbol{\rho}(\mathbf{y})\,\mathrm{d}\mathbf{y}. \tag{3.15}$$

4. Matérn kernels

The knowledge of the covariance kernel $k(\mathbf{x},\mathbf{x}')$ is crucial for the computation of the Karhunen-Loève expansion. Moreover, the smoothness of the kernel determines the decay of the sequences $\{\gamma_k\}_k$, $\{\tilde{\gamma}_k\}_k$, cf. [GH14, ST06], and also provides an estimate on the error which arises from the truncation of the Karhunen-Loève expansion, see Section 2.

We consider kernels from the *Matérn class*, cf. [Mat86]. They are often used as covariance kernels for the definition of stochastic fields. In accordance with [RW05], they are defined as follows:

(4.1) **Definition.** Let $r := \|\mathbf{x} - \mathbf{y}\|_2$ and $\ell, \sigma \in (0, \infty)$. Then, the *Matérn covariance kernel function* of order $\nu > 0$ is given by

$$k_\nu(r) := \sigma^2 \frac{2^{1-\nu}}{\Gamma(\nu)} \left(\frac{\sqrt{2\nu} r}{\ell} \right)^\nu K_\nu \left(\frac{\sqrt{2\nu} r}{\ell} \right). \tag{4.2}$$

Here, Γ denotes the gamma function and K_ν denotes the modified Bessel function of the second kind of order ν, cf. [AS64]. The related covariance operator $\mathcal{C}_\nu$ in (2.1) is called the *Matérn covariance operator.*

The expression (4.2) simplifies, cf. [RW05], for $\nu = p + 1/2$ with $p \in \mathbb{N}$ to

$$k_{p+1/2}(r) = \sigma^2 \exp\left(\frac{-\sqrt{2\nu} r}{\ell} \right) \frac{p!}{(2p)!} \sum_{i=0}^{p} \frac{(p+i)!}{i!(p-i)!} \left(\frac{\sqrt{8\nu} r}{\ell} \right)^{p-i}. \tag{4.3}$$

The Matérn kernel for $\nu = 1/2$ coincides with the exponential kernel $k_{1/2}(r) = \sigma^2 e^{-r/\ell}$ and for $\nu \to \infty$ one obtains the Gaussian kernel $k_\infty(r) = \sigma^2 e^{-r^2/2\ell^2}$. A visualization of these kernels for different values of ν is given in Figure III.1. As it can be observed

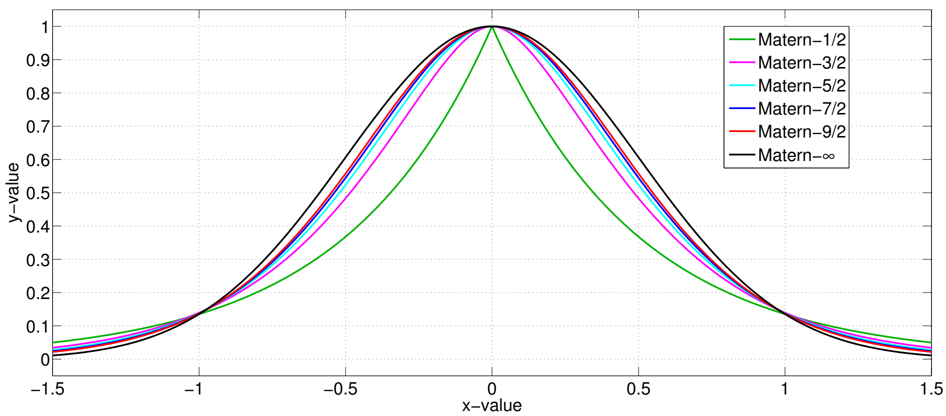


Figure III.1: Matérn kernels ($d = 1, \ell = 1, \sigma = 1$) for different values of the smoothness parameter ν.

in Figure III.1, the parameter ν is a smoothness parameter, i.e. the smoothness of the kernel k_ν increases with the modulus of ν. The following proposition on the decay of the eigenvalues of the Matérn covariance operator reflects this behaviour, see [GKN+14].

(4.4) **Proposition ([GKN+14]).** There is a constant $c > 0$ such that the eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots$ of the Matérn covariance operator decay like

$$\lambda_j \leq c j^{-(1+\frac{2\nu}{d})} \quad \text{as } j \to \infty. \tag{4.5}$$

We demonstrate now that the Assumptions (2.4) and (2.5) on the sequences $\{\gamma_j\}_j$ and $\{\tilde{\gamma}_j\}_j$ as well as the conditions of Theorem (2.10) can be verified for the Matérn covariance operator under certain conditions on the smoothness parameter ν. Therefore, we follow the arguments in e.g. [GH14, GKN+14, ST06]. It holds that the Matérn covariance operator $\mathcal{C}_\nu$, cf. Definition (4.1), is the inverse of a pseudo-differential operator of order $2\nu + d$, see [DHS14]. Thus, if the domain D is sufficiently smooth, the operator $\mathcal{C}_\nu$ is continuous from $L^2(D)$ to $H^r(D)$ for all $r < 2\nu + d$. Since the eigenfunctions φ_j are normalized in $L^2(D)$, it follows that

$$\|\varphi_j\|_{H^r(D)} = \frac{1}{\lambda_j}\|\mathcal{C}_\nu\varphi_j\|_{H^r(D)} \lesssim \frac{1}{\lambda_j}\|\varphi_j\|_{L^2(D)} = \frac{1}{\lambda_j}.$$

This implies together with the interpolation inequality (II.2.16) that

$$\|\varphi_j\|_{H^s(D)} \lesssim \lambda_j^{-\frac{s}{r}} \quad \text{for all } 0 \le s \le r.$$

From Theorem (II.2.4), we further conclude that

$$\begin{aligned} \|\varphi_j\|_{L^\infty(D)} &\lesssim \lambda_j^{-\frac{d}{2r}} && \text{for all } d/2 < r < d + 2\nu, \\ \|\varphi_j\|_{C^{0,\beta}(D)} &\lesssim \lambda_j^{-\frac{2\beta+d}{2r}} && \text{for all } d/2 + \beta < r < d + 2\nu, \\ \|\nabla\varphi_j\|_{L^\infty(D)} &\lesssim \lambda_j^{-\frac{d+2}{2r}} && \text{for all } d/2 + 1 < r < d + 2\nu. \end{aligned}$$

Thus, we obtain that

$$\|\varphi_j\|_{L^\infty(D)} \lesssim \lambda_j^{-\frac{d}{4\nu+2d}-\varepsilon} \lesssim j^{\frac{1}{2}+(1+\frac{2\nu}{d})\varepsilon} \quad \text{for all } \varepsilon > 0,$$

and

$$\|\nabla\varphi_j\|_{L^\infty(D)} \lesssim \lambda_j^{-\frac{d+2}{4\nu+2d}-\varepsilon} \lesssim j^{\frac{1}{2}+\frac{1}{d}+(1+\frac{2\nu}{d})\varepsilon} \quad \text{for all } \varepsilon > 0.$$

This implies together with (4.5) that it holds

$$\sqrt{\lambda_j}\|\varphi_j\|_{L^\infty(D)} \lesssim j^{-\frac{1}{2}-\frac{\nu}{d}+\frac{1}{2}+\varepsilon} = j^{-\frac{\nu}{d}+\varepsilon} \quad \text{for all } \varepsilon > 0 \tag{4.6}$$

and

$$\sqrt{\lambda_j}\|\nabla\varphi_j\|_{L^\infty(D)} \lesssim j^{-\frac{1}{2}-\frac{\nu}{d}+\frac{1}{2}+\frac{1}{d}+\varepsilon} = j^{-\frac{\nu-1}{d}+\varepsilon} \quad \text{for all } \varepsilon > 0. \tag{4.7}$$

To establish the Assumptions (2.4) and (2.5), the exponents in (4.6) and (4.7) have to be smaller than -1. This yields the following condition on the smoothness parameter ν for the sequence $\{\gamma_j\}_j$:

$$\frac{\nu}{d} > 1 \quad \Longleftrightarrow \quad \nu > d.$$

For the sequence $\{\tilde{\gamma}_j\}_j$, the condition reads

$$\frac{\nu - 1}{d} > 1 \quad \Longleftrightarrow \quad \nu > d + 1.$$

Thus, the Matérn kernels fulfill the Assumptions (2.4) and (2.5) provided that the parameter ν is chosen large enough. Moreover, we know that the sequence $\{\gamma_j\}_j$, and also the sequence $\{\tilde{\gamma}_j\}_j$, decays for sufficiently large ν with an algebraic rate $s_1 > 1$ and $s_2 > 1$, respectively, i.e.

(4.8) $$\gamma_j \lesssim j^{-s_1} \quad \text{and} \quad \tilde{\gamma}_j \lesssim j^{-s_2}$$

which we assume in the following.

To fulfill the conditions of Theorem (2.10), we determine the minimal parameter ν such that $\sum_{j=1}^{\infty} \lambda_j \|\varphi_j\|^2_{C^{0,1/2}(D)} < \infty$. Analogously to (4.6) and (4.7), it holds for all $\varepsilon > 0$ that

$$\lambda_j \|\varphi_j\|^2_{C^{0,1/2}(D)} \lesssim j^{-1-\frac{2\nu}{d}+\frac{(d+1)(1+2\nu/d)}{d+2\nu}+\varepsilon} = j^{-\frac{2\nu-1}{d}+\varepsilon}.$$

This implies the condition

$$\frac{2\nu-1}{d} > 1 \quad \Longleftrightarrow \quad \nu > \frac{d+1}{2}.$$

(4.9) **Remark.** Although, the above analysis shows that we loose a part of the decay of the eigenvalues when considering the sequences γ_k and $\tilde{\gamma}_k$ instead of λ_k, this behaviour is not reflected by numerical tests, see e.g. [GKN+14]. There, the decay behaviour of the sequences γ_j and $\tilde{\gamma}_j$ is similar to the decay behaviour of the eigenvalues, at least for $d = 1$ and $\nu = 3/4$ or $\nu = 3/2$. In order to corroborate this, we illustrate in Figure III.2 that the decay of the γ_k is essentially the same as the decay of the $\sqrt{\lambda_k}$ for further values of ν, namely for $\nu = 5/2$, $\nu = 7/2$ and $\nu = 9/2$.

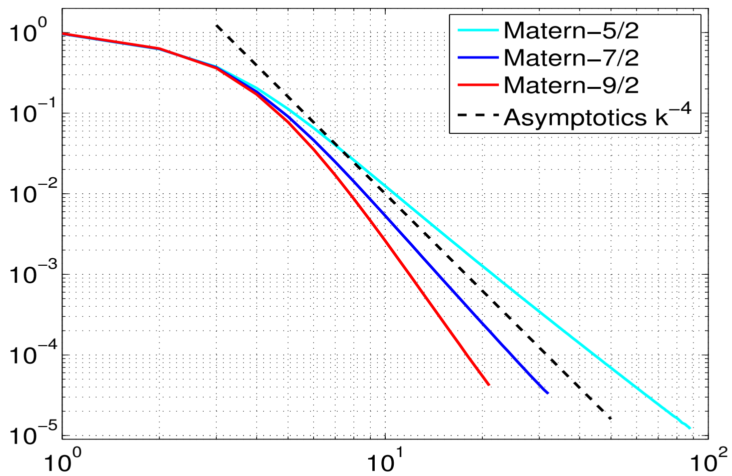


Figure III.2: Decay of the γ_k for the Matérn kernels for $\nu = 5/2$ and $\nu = 7/2$ ($d = 1,\ \ell = 1/2,\ \sigma = 1$).

5. Regularity results in the lognormal case

In this and in the next section, we provide the regularity results which are necessary to ensure convergence of our quadrature methods presented in Chapter IV and V. The main results here are the bounds on the derivatives of the powers of the solution u_m^p for some $p > 1$ which builds the basis for the error estimation of the moment approximation. For the solution itself, these results are already available, see e.g. [BNT07, Cha12, HS14, SG11]. Once more, the lognormal case is more involved and we will present the results for this case in detail and formulate the corresponding results for the uniformly elliptic case as corollaries in Section 6.

We would like to point out that the constants in these estimates are either generic or Poincaré constants which appear due to the repeated use of norm equivalences. In general, they additionally depend on p when considering the p-th power of u_m. Nevertheless, they are independent of the dimensionality m of the parameter space and of the order of differentiation. Therefore, the constants have to be treated very carefully, especially since many of the regularity results are proven by induction.

We start with the pointwise bound on the derivatives of u_m. Therefore, we shall fix some notation for multi-indices and the related derivatives. For a multi-index $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_m) \in \mathbb{N}^m$ and a vector $\boldsymbol{\delta} \in \mathbb{R}^m$, we define $\boldsymbol{\delta}^{\boldsymbol{\alpha}} := \prod_{k=1}^m \delta_k^{\alpha_k}$. Furthermore, the binomial coefficient for two multi-indices $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{N}^m$ is given by

$$\binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} = \frac{\boldsymbol{\alpha}!}{\boldsymbol{\beta}!(\boldsymbol{\alpha}-\boldsymbol{\beta})!} = \frac{\alpha_1!\alpha_2!\cdots\alpha_m!}{\beta_1!\cdots\beta_m!(\alpha_1-\beta_1)!\cdots(\alpha_m-\beta_m)!}.$$

In addition, we need a relation on multi-indices. For two multi-indices $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{N}^m$, we write $\boldsymbol{\beta} \leq \boldsymbol{\alpha}$ if $\beta_k \leq \alpha_k$ for $k = 1, \ldots, m$ and $\boldsymbol{\beta} < \boldsymbol{\alpha}$ if $\boldsymbol{\beta} \leq \boldsymbol{\alpha}$ and $\boldsymbol{\beta} \neq \boldsymbol{\alpha}$.

It turns out that the solution u_m is smooth with respect to the parametric variable $\mathbf{y}$. Hence, for $q \in \mathbb{N}$, a Banach space X and a domain $\boldsymbol{\Gamma} \subset \mathbb{R}^m$, we introduce the space $C^q(\boldsymbol{\Gamma}; X)$ of q-times continuously differentiable X-valued functions whose derivatives of order $\boldsymbol{\alpha}$ with $|\boldsymbol{\alpha}| \leq q$ are continuously extendable to $\overline{\boldsymbol{\Gamma}}$. This space is, for all $q < \infty$, a Banach space with respect to the norm

$$\|v\|_{C^q(\boldsymbol{\Gamma};X)} := \sum_{|\boldsymbol{\alpha}|\leq q} \sup_{\mathbf{y}\in\boldsymbol{\Gamma}} \left\|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}} v(\mathbf{y})\right\|_X. \tag{5.1}$$

5.1 Pointwise estimates

The differentiability of u_m follows from the differentiability of the diffusion coefficient a_m, cf. [BNTT12], more precisely from the estimate

$$\left\|\frac{\partial_{\mathbf{y}}^{\boldsymbol{\alpha}} a_m(\mathbf{y})}{a_m(\mathbf{y})}\right\|_{L^\infty(D)} \leq \boldsymbol{\gamma}^{\boldsymbol{\alpha}}, \tag{5.2}$$

where we set $\boldsymbol{\gamma} := [\gamma_1, \gamma_2, \ldots, \gamma_m]^\intercal$. In [BNTT12], a uniformly elliptic diffusion coefficient is considered which fulfills (5.2). In this case, it is possible to establish estimates uniformly in $\mathbf{y}$. In the lognormal case the pointwise regularity estimates usually depend on the parameter $\mathbf{y}$. Hence, we provide the following lemma from [HS14], which is adjusted for our purposes. For the sake of completeness and since the proof technique is applied several times in the subsequent analysis, we review the crucial parts of the proof.

(5.3) **Lemma.** The derivatives $\partial_{\mathbf{y}}^{\boldsymbol{\alpha}} u_m$ of the solution u_m to (3.5) satisfy for every multi-index $\boldsymbol{\alpha} \in \mathbb{N}^m$ and every $\mathbf{y} \in \mathbb{R}^m$ the estimate

$$\|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}} u_m(\mathbf{y})\|_{H_0^1(D)} \leq |\boldsymbol{\alpha}|! \left(\frac{\boldsymbol{\gamma}}{\log 2}\right)^{\boldsymbol{\alpha}} \sqrt{\kappa_m(\mathbf{y})} \|u_m(\mathbf{y})\|_{H_0^1(D)}. \tag{5.4}$$

Proof. We differentiate the weak formulation (3.6) with respect to the stochastic parameter $\mathbf{y}$. For $|\boldsymbol{\alpha}| > 0$, this leads to

$$\int_D \partial_{\mathbf{y}}^{\boldsymbol{\alpha}} \big(a_m(\mathbf{x},\mathbf{y}) \nabla u_m(\mathbf{x},\mathbf{y})\big) \nabla v(\mathbf{x}) \,\mathrm{d}\mathbf{x} = 0$$

since the right-hand side is independent of $\mathbf{y}$. With the help of the Leibniz rule

$$\partial_{\mathbf{y}}^{\boldsymbol{\alpha}} \big(g(\mathbf{y}) h(\mathbf{y})\big) = \sum_{\boldsymbol{\beta} \leq \boldsymbol{\alpha}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} \partial_{\mathbf{y}}^{\boldsymbol{\beta}} g(\mathbf{y}) \partial_{\mathbf{y}}^{\boldsymbol{\alpha}-\boldsymbol{\beta}} h(\mathbf{y}),$$

we obtain that

$$\begin{aligned} &\int_D a_m(\mathbf{x},\mathbf{y}) \nabla \partial_{\mathbf{y}}^{\boldsymbol{\alpha}} u_m(\mathbf{x},\mathbf{y}) \nabla v(\mathbf{x}) \,\mathrm{d}\mathbf{x} \\ &\qquad = - \sum_{\mathbf{0} \neq \boldsymbol{\beta} \leq \boldsymbol{\alpha}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} \int_D \partial_{\mathbf{y}}^{\boldsymbol{\beta}} a_m(\mathbf{x},\mathbf{y}) \nabla \partial_{\mathbf{y}}^{\boldsymbol{\alpha}-\boldsymbol{\beta}} u_m(\mathbf{x},\mathbf{y}) \nabla v(\mathbf{x}) \,\mathrm{d}\mathbf{x}. \end{aligned} \tag{5.5}$$

We now fix the parametric variable $\mathbf{y}$ and choose the test function $v = \partial_{\mathbf{y}}^{\boldsymbol{\alpha}} u_m(\mathbf{y})$. This, in combination with (5.2) and the Hölder inequality, yields

$$\begin{aligned} &\int_D a_m(\mathbf{x},\mathbf{y}) \nabla \partial_{\mathbf{y}}^{\boldsymbol{\alpha}} u_m(\mathbf{x},\mathbf{y}) \nabla \partial_{\mathbf{y}}^{\boldsymbol{\alpha}} u_m(\mathbf{x},\mathbf{y}) \,\mathrm{d}\mathbf{x} \\ &\quad = - \sum_{\mathbf{0} \neq \boldsymbol{\beta} \leq \boldsymbol{\alpha}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} \int_D \partial_{\mathbf{y}}^{\boldsymbol{\beta}} a_m(\mathbf{x},\mathbf{y}) \nabla \partial_{\mathbf{y}}^{\boldsymbol{\alpha}-\boldsymbol{\beta}} u_m(\mathbf{x},\mathbf{y}) \nabla \partial_{\mathbf{y}}^{\boldsymbol{\alpha}} u_m(\mathbf{x},\mathbf{y}) \,\mathrm{d}\mathbf{x} \\ &\quad \leq \sum_{\mathbf{0} \neq \boldsymbol{\beta} \leq \boldsymbol{\alpha}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} \left\| \frac{\partial_{\mathbf{y}}^{\boldsymbol{\beta}} a_m(\mathbf{y})}{a_m(\mathbf{y})} \right\|_{L^\infty(D)} \left(\int_D a_m(\mathbf{x},\mathbf{y}) \nabla \partial_{\mathbf{y}}^{\boldsymbol{\alpha}} u_m(\mathbf{x},\mathbf{y}) \nabla \partial_{\mathbf{y}}^{\boldsymbol{\alpha}} u_m(\mathbf{x},\mathbf{y}) \,\mathrm{d}\mathbf{x} \right)^{1/2} \\ &\qquad\qquad \cdot \left(\int_D a_m(\mathbf{x},\mathbf{y}) \nabla \partial_{\mathbf{y}}^{\boldsymbol{\alpha}-\boldsymbol{\beta}} u_m(\mathbf{x},\mathbf{y}) \nabla \partial_{\mathbf{y}}^{\boldsymbol{\alpha}-\boldsymbol{\beta}} u_m(\mathbf{x},\mathbf{y}) \,\mathrm{d}\mathbf{x} \right)^{1/2}. \end{aligned}$$

Inserting (5.2) and dividing both sides by $\left(\int_D a_m(\mathbf{x},\mathbf{y}) \nabla \partial_{\mathbf{y}}^{\boldsymbol{\alpha}} u_m(\mathbf{x},\mathbf{y}) \nabla \partial_{\mathbf{y}}^{\boldsymbol{\alpha}} u_m(\mathbf{x},\mathbf{y}) \,\mathrm{d}\mathbf{x} \right)^{1/2}$ leads to

$$\begin{aligned} &\left(\int_D a_m(\mathbf{x},\mathbf{y}) \nabla \partial_{\mathbf{y}}^{\boldsymbol{\alpha}} u_m(\mathbf{x},\mathbf{y}) \nabla \partial_{\mathbf{y}}^{\boldsymbol{\alpha}} u_m(\mathbf{x},\mathbf{y}) \,\mathrm{d}\mathbf{x} \right)^{1/2} \\ &\qquad \leq \sum_{\mathbf{0} \neq \boldsymbol{\beta} \leq \boldsymbol{\alpha}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} \boldsymbol{\gamma}^{\boldsymbol{\beta}} \left(\int_D a_m(\mathbf{x},\mathbf{y}) \nabla \partial_{\mathbf{y}}^{\boldsymbol{\alpha}-\boldsymbol{\beta}} u_m(\mathbf{x},\mathbf{y}) \nabla \partial_{\mathbf{y}}^{\boldsymbol{\alpha}-\boldsymbol{\beta}} u_m(\mathbf{x},\mathbf{y}) \,\mathrm{d}\mathbf{x} \right)^{1/2}. \end{aligned} \tag{5.6}$$

The rest of the proof is performed by induction. We would like to show that

$$
\begin{aligned}
&\bigg(\int_D a_m(\mathbf{x},\mathbf{y})\nabla\partial_{\mathbf{y}}^{\boldsymbol{\alpha}} u_m(\mathbf{x},\mathbf{y})\nabla\partial_{\mathbf{y}}^{\boldsymbol{\alpha}} u_m(\mathbf{x},\mathbf{y})\,\mathrm{d}\mathbf{x}\bigg)^{1/2}\\
&\qquad\leq \gamma^{\boldsymbol{\alpha}} B_{|\boldsymbol{\alpha}|}\bigg(\int_D a_m(\mathbf{x},\mathbf{y})\nabla u_m(\mathbf{x},\mathbf{y})\nabla u_m(\mathbf{x},\mathbf{y})\,\mathrm{d}\mathbf{x}\bigg)^{1/2},
\end{aligned}
\tag{5.7}
$$

where B_k denotes the k-th *ordered Bell number* which is defined by the recurrence relation

$$
B_0 := 1, \qquad B_n := \sum_{k=0}^{n-1}\binom{n}{k} B_k,
\tag{5.8}
$$

see [Bel34]. The inequality (5.7) is obviously fulfilled for $|\boldsymbol{\alpha}| = 0$. For the induction step, we assume that (5.7) holds for all $|\boldsymbol{\alpha}| < n$. Then, we obtain for $|\boldsymbol{\alpha}| = n$ by inserting (5.7) into (5.6) that

$$
\begin{aligned}
&\bigg(\int_D a_m(\mathbf{x},\mathbf{y})\nabla\partial_{\mathbf{y}}^{\boldsymbol{\alpha}} u_m(\mathbf{x},\mathbf{y})\nabla\partial_{\mathbf{y}}^{\boldsymbol{\alpha}} u_m(\mathbf{x},\mathbf{y})\,\mathrm{d}\mathbf{x}\bigg)^{1/2}\\
&\qquad\leq \sum_{\mathbf{0}\neq\boldsymbol{\beta}\leq\boldsymbol{\alpha}}\binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}}\boldsymbol{\gamma}^{\boldsymbol{\alpha}} B_{|\boldsymbol{\alpha}-\boldsymbol{\beta}|}\bigg(\int_D a_m(\mathbf{x},\mathbf{y})\nabla u_m(\mathbf{x},\mathbf{y})\nabla u_m(\mathbf{x},\mathbf{y})\,\mathrm{d}\mathbf{x}\bigg)^{1/2}.
\end{aligned}
$$

Hence, the induction step follows due to

$$
\sum_{\mathbf{0}\neq\boldsymbol{\beta}\leq\boldsymbol{\alpha}}\binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} B_{|\boldsymbol{\alpha}-\boldsymbol{\beta}|} = \sum_{k=1}^{n}\sum_{|\boldsymbol{\beta}|=k}\binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} B_{n-k} = \sum_{k=1}^{n}\binom{n}{k} B_{n-k} = \sum_{k=0}^{n-1}\binom{n}{k} B_k = B_{|\boldsymbol{\alpha}|}.
\tag{5.9}
$$

Finally, we arrive at the assertion by

$$
\begin{aligned}
\|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}} u_m(\mathbf{y})\|_{H_0^1(D)} &\leq \sqrt{\frac{1}{\underline{a}_m(\mathbf{y})}}\bigg(\int_D a_m(\mathbf{x},\mathbf{y})\nabla\partial_{\mathbf{y}}^{\boldsymbol{\alpha}} u_m(\mathbf{x},\mathbf{y})\nabla\partial_{\mathbf{y}}^{\boldsymbol{\alpha}} u_m(\mathbf{x},\mathbf{y})\,\mathrm{d}\mathbf{x}\bigg)^{1/2}\\
&\leq \boldsymbol{\gamma}^{\boldsymbol{\alpha}} B_{|\boldsymbol{\alpha}|}\sqrt{\frac{1}{\underline{a}_m(\mathbf{y})}}\bigg(\int_D a_m(\mathbf{x},\mathbf{y})\nabla u_m(\mathbf{x},\mathbf{y})\nabla u_m(\mathbf{x},\mathbf{y})\,\mathrm{d}\mathbf{x}\bigg)^{1/2}\\
&\leq \boldsymbol{\gamma}^{\boldsymbol{\alpha}} B_{|\boldsymbol{\alpha}|}\sqrt{\kappa_m(\mathbf{y})}\|u_m(\mathbf{y})\|_{H_0^1(D)}
\end{aligned}
$$

and the estimate on the ordered Bell numbers $B_n \leq n!(\log 2)^{-n}$, cf. [BNTT12]. □

With this bound on the derivatives of the solution u_m to (3.5) at hand, we can perform the convergence analysis for the quadrature of the mean of u_m. To use the same error estimates for the higher order moments, we have to provide similar bounds on the derivatives of the powers u_m^p of u_m. This is different for the second moment and the higher order moments, since for the second moment the preceding estimates on the derivatives of u_m can directly be used to prove bounds on the derivatives of u_m^2. For the higher order moments, we have to derive pointwise bounds on the $W_0^{1,p}(D)$-norm of the solution's derivatives with respect to the random parameter $\mathbf{y}$. Therefore, we have to assume that the right-hand side f belongs to $L^p(D)$ instead of $L^2(D)$. This estimate is the $L^p(D)$-extension of (5.4), but the proofs are different since we cannot exploit the symmetry in the bilinear form anymore. Hence, the factor $\kappa_m(\mathbf{y})$ is involved in the estimate on the $W_0^{1,p}(D)$-norm instead of the factor $\sqrt{\kappa_m(\mathbf{y})}$ in the bound on the $H_0^1(D)$-norm (5.4). Of course, this leads to stronger estimates for the derivatives of u_m^2 in terms of $\kappa_m(\mathbf{y})$ in comparison to the bounds on the derivatives of u_m^p for $p > 2$. We therefore distinguish the two cases $p = 2$ and $p > 2$ and start with the pointwise bound of the derivatives of u_m^2.

(5.10) **Lemma.** The derivatives of u_m^2, where u_m is the solution to (3.5), satisfy

$$\|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}} u_m^2(\mathbf{y})\|_{W_0^{1,1}(D)} \lesssim (|\boldsymbol{\alpha}|+1)!\left(\frac{\boldsymbol{\gamma}}{\log 2}\right)^{\boldsymbol{\alpha}} \kappa_m(\mathbf{y})\|u_m(\mathbf{y})\|_{H_0^1(D)}^2 \tag{5.11}$$

for all multi-indices $\boldsymbol{\alpha} \in \mathbb{N}^m$.

Proof. With the help of the Leibniz rule, we deduce, similar to (5.5), that

$$\|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}} u_m^2(\mathbf{y})\|_{W_0^{1,1}(D)} \leq \sum_{\boldsymbol{\beta}\leq\boldsymbol{\alpha}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} \|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}-\boldsymbol{\beta}} u_m(\mathbf{y}) \partial_{\mathbf{y}}^{\boldsymbol{\beta}} u_m(\mathbf{y})\|_{W_0^{1,1}(D)}. \tag{5.12}$$

Each summand in (5.12) can be estimated due to Lemma (II.2.10) as follows

$$\|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}-\boldsymbol{\beta}} u_m(\mathbf{y}) \partial_{\mathbf{y}}^{\boldsymbol{\beta}} u_m(\mathbf{y})\|_{W_0^{1,1}(D)} \lesssim \|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}-\boldsymbol{\beta}} u_m(\mathbf{y})\|_{H_0^1(D)} \|\partial_{\mathbf{y}}^{\boldsymbol{\beta}} u_m(\mathbf{y})\|_{H_0^1(D)}.$$

The application of Lemma (5.3) yields that

$$\|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}-\boldsymbol{\beta}} u_m(\mathbf{y}) \partial_{\mathbf{y}}^{\boldsymbol{\beta}} u_m(\mathbf{y})\|_{W_0^{1,1}(D)} \lesssim |\boldsymbol{\beta}|!\,|(\boldsymbol{\alpha}-\boldsymbol{\beta})|!\left(\frac{\boldsymbol{\gamma}}{\log 2}\right)^{\boldsymbol{\alpha}} \kappa_m(\mathbf{y})\|u_m(\mathbf{y})\|_{H_0^1(D)}^2.$$

By inserting this inequality into (5.12), we conclude that

$$\begin{aligned}\|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}} u_m^2(\mathbf{y})\|_{W_0^{1,1}(D)} &\lesssim \sum_{\boldsymbol{\beta}\leq\boldsymbol{\alpha}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} |\boldsymbol{\beta}|!\,|(\boldsymbol{\alpha}-\boldsymbol{\beta})|!\left(\frac{\boldsymbol{\gamma}}{\log 2}\right)^{\boldsymbol{\alpha}} \kappa_m(\mathbf{y})\|u_m(\mathbf{y})\|_{H_0^1(D)}^2 \\ &= \left(\frac{\boldsymbol{\gamma}}{\log 2}\right)^{\boldsymbol{\alpha}} \kappa_m(\mathbf{y})\|u_m(\mathbf{y})\|_{H_0^1(D)}^2 \sum_{k=0}^{|\boldsymbol{\alpha}|} (|\boldsymbol{\alpha}|-k)!\,k! \sum_{|\boldsymbol{\beta}|=k} \binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}}.\end{aligned}$$

In view of

$$\sum_{k=0}^{|\boldsymbol{\alpha}|} (|\boldsymbol{\alpha}|-k)!\,k! \sum_{|\boldsymbol{\beta}|=k} \binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} = \sum_{k=0}^{|\boldsymbol{\alpha}|} (|\boldsymbol{\alpha}|-k)!\,k! \binom{|\boldsymbol{\alpha}|}{k} = \sum_{k=0}^{|\boldsymbol{\alpha}|} |\boldsymbol{\alpha}|! = (|\boldsymbol{\alpha}|+1)!,$$

we finally arrive at the assertion (5.11). □

For the derivatives of the higher powers of u_m, we need, as mentioned above, regularity results for the derivatives of u_m in the $W_0^{1,p}(D)$-norm. After that, we shall apply the Faà di Bruno formula, which is the multivariate analogue to the chain rule, to achieve estimates on the derivatives $\partial_{\mathbf{y}}^{\boldsymbol{\alpha}} u_m^p$.

(5.13) **Lemma.** Let $p > 2$ and let the right-hand side f be contained in $L^p(D)$. Then, the solution u_m to (3.5) is contained in $W_0^{1,p}(D)$ and satisfies the regularity estimate

$$\|u_m(\mathbf{y})\|_{W_0^{1,p}(D)} \lesssim \frac{1}{\underline{a}_m(\mathbf{y})}\|f\|_{L^p(D)}. \tag{5.14}$$

The derivatives of u_m with respect to the parametric variable $\mathbf{y}$ can be estimated by

$$\|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}} u_m(\mathbf{y})\|_{W_0^{1,p}(D)} \leq |\boldsymbol{\alpha}|!\left(\frac{C(p,D)\boldsymbol{\gamma}}{\log 2}\right)^{\boldsymbol{\alpha}} \kappa_m(\mathbf{y})\|u_m(\mathbf{y})\|_{W_0^{1,p}(D)} \tag{5.15}$$

with a constant $C(p,D) \geq 1$. Additionally, the derivatives of the powers u_m^p with respect to the parametric variable $\mathbf{y}$ fulfill the bound

$$\|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}} u_m^p(\mathbf{y})\|_{W_0^{1,1}(D)} \lesssim |\boldsymbol{\alpha}|!\left(\frac{C(p,D)p\boldsymbol{\gamma}}{\log 2}\right)^{\boldsymbol{\alpha}} \kappa_m(\mathbf{y})^p\|u_m(\mathbf{y})\|_{W_0^{1,p}(D)}^p. \tag{5.16}$$

Proof. At first, we notice that the bilinear form

$$(u,v)_{H_0^1(D)} := (\nabla v, \nabla u)_{[L^2(D)]^d} \tag{5.17}$$

defines a scalar product on the Hilbert space $H_0^1(D)$. Let $1 < p, p' < \infty$ be dual exponents, i.e. $1/p + 1/p' = 1$. Then, the scalar product (5.17) extends continuously to a duality product on $W_0^{1,p}(D) \times W_0^{1,p'}(D)$. It is proven in [Sim72] that, for each function $u \in W_0^{1,p}(D)$, the estimate

$$\begin{aligned}
\|u\|_{W_0^{1,p}(D)} = \|\nabla u\|_{[L^p(D)]^d} &= \sup_{0\neq v\in[L^{p'}(D)]^d} \frac{(\nabla u, v)_{L^2(D)}}{\|v\|_{[L^{p'}(D)]^d}} \\
&\leq C(p,D) \sup_{0\neq v\in W_0^{1,p'}(D)} \frac{(u,v)_{H_0^1(D)}}{\|v\|_{W_0^{1,p'}(D)}}
\end{aligned}$$

is valid with a constant $C(p,D) \geq 1$. This follows from the fact that $W_0^{1,p'}(D)$ is densely embedded into $[L^{p'}(D)]^d$ by the mapping $v \mapsto \nabla v$, cf. [Sim72]. From this, we conclude that it holds for each fixed $\mathbf{y} \in \mathbb{R}^m$

$$\begin{aligned}
\|u_m(\mathbf{y})\|_{W_0^{1,p}(D)} &\lesssim \sup_{0\neq v\in W_0^{1,p'}(D)} \frac{(u_m(\mathbf{y}), v)_{H_0^1(D)}}{\|v\|_{W_0^{1,p'}(D)}} \\
&\leq \frac{1}{\underline{a}_m(\mathbf{y})} \sup_{0\neq v\in W_0^{1,p'}(D)} \frac{\mathcal{B}_{m,\mathbf{y}}(u_m(\mathbf{y}), v)}{\|v\|_{W_0^{1,p'}(D)}},
\end{aligned}$$

with the continuous extension of the bilinear form from (3.7) onto $W_0^{1,p}(D) \times W_0^{1,p'}(D)$, that is

$$\mathcal{B}_{m,\mathbf{y}}(u_m(\mathbf{y}), v) = \int_D a(\mathbf{x},\mathbf{y})\nabla u_m(\mathbf{x},\mathbf{y})\nabla v(\mathbf{x})\,\mathrm{d}\mathbf{x}.$$

Regard that $H_0^1(D) \subset W_0^{1,p'}(D)$ since $p' < 2$. In view of $f \in L^p(D)$, it is easy to verify by a density argument that equation (3.6) is still valid for $v \in W_0^{1,p'}(D)$. Therefore, we have

$$\sup_{0 \neq v \in W_0^{1,p'}(D)} \frac{\mathcal{B}_{m,\mathbf{y}}(u_m(\mathbf{y}), v)}{\|v\|_{W_0^{1,p'}(D)}} = \sup_{0 \neq v \in W_0^{1,p'}(D)} \frac{(f, v)_{L^2(D)}}{\|v\|_{W_0^{1,p'}(D)}} \lesssim \|f\|_{L^p(D)},$$

which follows from the Hölder inequality and the estimate $\|v\|_{L^{p'}(D)} \lesssim \|v\|_{W_0^{1,p'}(D)}$. This establishes the inequality (5.14).

The second assertion follows similarly to the case $p = 2$ which is considered in Lemma (5.3). We show the parts of the proof which are different and refer to the proof of Lemma (5.3) for the identical parts. We prove by induction that it holds

$$\|a_m(\mathbf{y})\nabla\partial_{\mathbf{y}}^{\boldsymbol{\alpha}} u_m(\mathbf{y})\|_{[L^p(D)]^d} \leq |\boldsymbol{\alpha}|!\left(\frac{C(p,D)\boldsymbol{\gamma}}{\log 2}\right)^{\boldsymbol{\alpha}} \|a_m(\mathbf{y})\nabla u_m(\mathbf{y})\|_{[L^p(D)]^d}. \tag{5.18}$$

The case $|\boldsymbol{\alpha}| = 0$ is trivial. Let now (5.18) be satisfied for all $|\boldsymbol{\alpha}| < n$. Then, we have for $|\boldsymbol{\alpha}| = n$ that

$$\|a_m(\mathbf{y})\nabla\partial_{\mathbf{y}}^{\boldsymbol{\alpha}} u_m(\mathbf{y})\|_{[L^p(D)]^d} \leq C(p,D) \sup_{0 \neq v \in W_0^{1,p'}(D)} \frac{\mathcal{B}_{m,\mathbf{y}}(\partial_{\mathbf{y}}^{\boldsymbol{\alpha}} u_m(\mathbf{y}), v)}{\|v\|_{W_0^{1,p'}(D)}}. \tag{5.19}$$

Now, differentiation of the bilinear form (3.7) with respect to $\mathbf{y}$ yields

$$\begin{aligned}
&\partial_{\mathbf{y}}^{\boldsymbol{\alpha}} \mathcal{B}_{m,\mathbf{y}}(u_m(\mathbf{y}), v) \\
&\quad = \mathcal{B}_{m,\mathbf{y}}(\partial_{\mathbf{y}}^{\boldsymbol{\alpha}} u_m(\mathbf{y}), v) + \sum_{\mathbf{0} \neq \boldsymbol{\beta} \leq \boldsymbol{\alpha}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} \int_D \partial_{\mathbf{y}}^{\boldsymbol{\beta}} a_m(\mathbf{x},\mathbf{y}) \nabla \partial_{\mathbf{y}}^{\boldsymbol{\alpha}-\boldsymbol{\beta}} u_m(\mathbf{x},\mathbf{y}) \nabla v(\mathbf{x}) \,\mathrm{d}\mathbf{x}.
\end{aligned}$$

From the differentiation of the variational formulation (3.6), we know that the left-hand side vanishes. Therefore, we obtain

$$\begin{aligned}
&\mathcal{B}_{m,\mathbf{y}}(\partial_{\mathbf{y}}^{\boldsymbol{\alpha}} u_m(\mathbf{y}), v) \\
&\quad \leq \sum_{\mathbf{0} \neq \boldsymbol{\beta} \leq \boldsymbol{\alpha}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} \left\| \frac{\partial_{\mathbf{y}}^{\boldsymbol{\beta}} a_m(\mathbf{y})}{a_m(\mathbf{y})} \right\|_{L^\infty(D)} \int_D |a_m(\mathbf{x},\mathbf{y}) \nabla \partial_{\mathbf{y}}^{\boldsymbol{\alpha}-\boldsymbol{\beta}} u_m(\mathbf{x},\mathbf{y}) \nabla v(\mathbf{x})| \,\mathrm{d}\mathbf{x} \\
&\quad \leq \sum_{\mathbf{0} \neq \boldsymbol{\beta} \leq \boldsymbol{\alpha}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} \boldsymbol{\gamma}^{\boldsymbol{\beta}} \|a_m(\mathbf{y}) \nabla \partial_{\mathbf{y}}^{\boldsymbol{\alpha}-\boldsymbol{\beta}} u_m(\mathbf{y})\|_{[L^p(D)]^d} \|v\|_{W_0^{1,p'}(D)}.
\end{aligned}$$

Inserting this into (5.19) leads to

$$\begin{aligned}
&\|a_m(\mathbf{y})\nabla\partial_{\mathbf{y}}^{\boldsymbol{\alpha}} u_m(\mathbf{y})\|_{[L^p(D)]^d} \\
&\qquad\qquad \leq \sum_{\mathbf{0} \neq \boldsymbol{\beta} \leq \boldsymbol{\alpha}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} (C(p,D)\boldsymbol{\gamma})^{\boldsymbol{\beta}} \|a_m(\mathbf{y}) \nabla \partial_{\mathbf{y}}^{\boldsymbol{\alpha}-\boldsymbol{\beta}} u_m(\mathbf{y})\|_{[L^p(D)]^d}.
\end{aligned}$$

The inequality (5.18) follows then by inserting the induction hypothesis and the same combinatorial estimates as in the proof of Lemma (5.3).

Finally, to establish estimate (5.16), we apply Faà di Bruno's formula, cf. [CS96]. For $n := |\boldsymbol{\alpha}|$, this formula provides that

$$\partial_{\mathbf{y}}^{\boldsymbol{\alpha}} u_m^p(\mathbf{y}) = \sum_{r=1}^{n} p(p-1)\cdots(p-r+1) u_m^{p-r}(\mathbf{y}) \sum_{P(\boldsymbol{\alpha},r)} \boldsymbol{\alpha}! \prod_{j=1}^{n} \frac{\left(\partial_{\mathbf{y}}^{\boldsymbol{\beta}_j} u_m(\mathbf{y})\right)^{k_j}}{k_j! \boldsymbol{\beta}_j!}. \tag{5.20}$$

The set $P(\boldsymbol{\alpha}, r)$ contains restricted integer partitions of the multi-index $\boldsymbol{\alpha}$ into r non-vanishing multi-indices. It is defined by

$$\begin{aligned} P(\boldsymbol{\alpha}, r) := \Bigg\{ &\big((k_1, \boldsymbol{\beta}_1), \ldots, (k_n, \boldsymbol{\beta}_n)\big) \in (\mathbb{N} \times \mathbb{N}^m)^n : \sum_{i=1}^{n} k_i \boldsymbol{\beta}_i = \boldsymbol{\alpha}, \sum_{i=1}^{n} k_i = r, \\ &\text{and } \exists\, 1 \le s \le n \big|\ k_i = 0 \text{ and } \boldsymbol{\beta}_i = \mathbf{0} \text{ for all } 1 \le i \le n-s, \\ &k_i > 0 \text{ for all } n-s+1 \le i \le n \text{ and } \mathbf{0} \prec \boldsymbol{\beta}_{n-s+1} \prec \cdots \prec \boldsymbol{\beta}_n \Bigg\}. \end{aligned}$$

For multi-indices $\boldsymbol{\beta}, \boldsymbol{\beta}' \in \mathbb{N}^m$, the relation $\boldsymbol{\beta} \prec \boldsymbol{\beta}'$ means either $|\boldsymbol{\beta}| < |\boldsymbol{\beta}'|$ or, if $|\boldsymbol{\beta}| = |\boldsymbol{\beta}'|$, it denotes the lexicographical order which means that $\beta_1 = \beta_1', \ldots, \beta_k = \beta_k'$ and $\beta_{k+1} < \beta_{k+1}'$ for some $0 \le k < m$. Equation (5.20) together with (5.15) and Lemma (II.2.10) yield that

$$\begin{aligned} &\|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}} u_m^p(\mathbf{y})\|_{W_0^{1,1}(D)} \\ &\le \sum_{r=1}^{n} p(p-1)\cdots(p-r+1) \sum_{P(\boldsymbol{\alpha},r)} \frac{\boldsymbol{\alpha}!}{\prod_{j=1}^{n} k_j! \boldsymbol{\beta}_j!} \left\| u_m^{p-r}(\mathbf{y}) \prod_{j=1}^{n} \left(\partial_{\mathbf{y}}^{\boldsymbol{\beta}^j} u_m(\mathbf{y})\right)^{k_j} \right\|_{W_0^{1,1}(D)} \\ &\lesssim \left(\frac{C(p,D)\boldsymbol{\gamma}}{\log 2}\right)^{\boldsymbol{\alpha}} \kappa_m(\mathbf{y})^p \|u_m(\mathbf{y})\|_{W_0^{1,p}(D)}^p \\ &\qquad \cdot \sum_{r=1}^{n} p(p-1)\cdots(p-r+1) \sum_{P(\boldsymbol{\alpha},r)} \frac{\boldsymbol{\alpha}!}{\prod_{j=1}^{n} k_j! \boldsymbol{\beta}_j!} \prod_{j=1}^{n} \left(|\boldsymbol{\beta}_j|!\right)^{k_j}. \end{aligned}$$

From [CS96], we know that

$$\sum_{P(\boldsymbol{\alpha},r)} \frac{\boldsymbol{\alpha}!}{\prod_{j=1}^{n} k_j! \boldsymbol{\beta}_j!} = S_{n,r},$$

where $S_{n,r}$ denote the Stirling numbers of the second kind, cf. [AS64]. Moreover, since $\prod_{j=1}^{n} \left(|\boldsymbol{\beta}_j|!\right)^{k_j} \le |\boldsymbol{\alpha}|!$, we can further estimate that

$$\begin{aligned} &\sum_{r=1}^{n} p(p-1)\cdots(p-r+1) \sum_{P(\boldsymbol{\alpha},r)} \frac{\boldsymbol{\alpha}!}{\prod_{j=1}^{n} k_j! \boldsymbol{\beta}_j!} \prod_{j=1}^{n} \left(|\boldsymbol{\beta}_j|!\right)^{k_j} \\ &\qquad \le |\boldsymbol{\alpha}|! \sum_{r=1}^{n} p(p-1)\cdots(p-r+1) S_{n,r} = |\boldsymbol{\alpha}|! p^n. \end{aligned}$$

The last inequality follows from the theory of generating functions for the Stirling numbers of the second kind, see e.g. [AS64]. This finally completes the proof. □

The constant $C(p,D)$, which arises from the inequality

$$\sup_{0\neq v\in[L^{p'}(D)]^d} \frac{(\nabla u, v)_{L^2(D)}}{\|v\|_{[L^{p'}(D)]^d}} \leq C(p,D) \sup_{0\neq v\in W_0^{1,p'}(D)} \frac{(u,v)_{H_0^1(D)}}{\|v\|_{W_0^{1,p'}(D)}}, \tag{5.21}$$

will be used several times in the rest of this thesis. Therefore, whenever $C(p,D)$ occurs in the sequel, it is associated with the constant in (5.21).

5.2 Analytic extension

The Gauss-Legendre quadrature on a finite interval as well as the Gauss-Hermite quadrature on the real line converge exponentially if the integrand can be extended analytically into a region of the complex plane. Therefore, we establish in this subsection that the solution u_m and its powers can be extended analytically with respect to the parametric variable $\mathbf{y}$ into a subset of the m-dimensional space $\mathbb{C}^m$.

Since the diffusion coefficient $a_m(\mathbf{x},\mathbf{y})$ is not uniformly elliptic with respect to $\mathbf{y}$ in the lognormal case, we cannot expect the solution u_m of (3.5) to be uniformly bounded in $\mathbf{y}$. Thus, u_m may not be contained in the Banach space $C^q(\mathbb{R}^m; H_0^1(D))$ for $q \in \mathbb{N}$. Nonetheless, we can multiply u_m by an auxiliary weight and end up with a bounded product in the sense of a weighted space which is defined as follows, cf. [BNT07].

(5.22) **Definition.** Let X denote some Banach space, for example $X = H_0^1(D)$ or $X = W_0^{1,1}(D)$. For X and a weight $w : \mathbb{R}^m \to \mathbb{R}_+$, we define the weighted space

$$C_w^0(\mathbb{R}^m; X) := \left\{ v\colon \mathbb{R}^m \to X : v \text{ is continuous and } \sup_{\mathbf{y}\in\mathbb{R}^m} w(\mathbf{y})\|v(\mathbf{y})\|_X < \infty \right\}$$

equipped with the norm

$$\|v\|_{C_w^0(\mathbb{R}^m;X)} := \sup_{\mathbf{y}\in\mathbb{R}^m} w(\mathbf{y})\|v(\mathbf{y})\|_X.$$

For our problem setting, the weight w should be chosen in such a way that the solution u_m is contained in $C_w^0(\mathbb{R}^m; H_0^1(D))$. Additionally, the weight should be integrable with respect to the Gaussian density and the value of this integral should be independent of the dimensionality m.

(5.23) **Definition.** For $\boldsymbol{\gamma} := (\gamma_1, \ldots, \gamma_m)$ from (2.4), let $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_m) \in \mathbb{R}^m$ be such that $\boldsymbol{\beta} \geqslant \boldsymbol{\gamma}$. Then, we define the auxiliary weight

$$\boldsymbol{\sigma}(\mathbf{y}) := \prod_{k=1}^m \sigma_k(y_k) \quad \text{with} \quad \sigma_k(y_k) := \exp(-\beta_k|y_k|).$$

If we choose $w = \boldsymbol{\sigma}$ in Definition (5.22), then the space $C_{\boldsymbol{\sigma}}^0(\mathbb{R}^m; X)$ satisfies that $C_{\boldsymbol{\sigma}}^0(\mathbb{R}^m; X) \subset L_{\boldsymbol{\rho}}^p(\mathbb{R}^m; X)$ for all $p \in \mathbb{N}$. This originates from the fact that

$$\|v\|_{L_{\boldsymbol{\rho}}^p(\mathbb{R}^m;X)} \leq \left(\int_{\mathbb{R}^m} (\boldsymbol{\sigma}(\mathbf{y}))^{-p} \boldsymbol{\rho}(\mathbf{y})\,\mathrm{d}\mathbf{y} \right)^{\frac{1}{p}} \|v\|_{C_{\boldsymbol{\sigma}}^0(\mathbb{R}^m;X)} < \infty$$

for all $v \in C^0_{\boldsymbol{\sigma}}(\mathbb{R}^m; X)$ because of $p\beta_k y_k = o(y_k^2)$ for $y_k \to \infty$ and the integrability of the normal distribution's tails. In particular, we will use several times the continuity of the Bochner integral operator

$$\mathbf{I} : C^0_{\boldsymbol{\sigma}}(\mathbb{R}^m; X) \to X, \quad v \mapsto \mathbf{I}v := \int_{\mathbb{R}^m} v(\mathbf{y}) \, \mathrm{d}\mathbf{y},$$

which satisfies

$$\|\mathbf{I}v\|_X \le C(\boldsymbol{\sigma}) \|v\|_{C^0_{\boldsymbol{\sigma}}(\mathbb{R}^m; X)} \quad \text{with} \quad C(\boldsymbol{\sigma}) = \int_{\mathbb{R}^m} \big(\boldsymbol{\sigma}(\mathbf{y})\big)^{-1} \boldsymbol{\rho}(\mathbf{y}) \, \mathrm{d}\mathbf{y}. \tag{5.24}$$

The constant $C(\boldsymbol{\sigma})$ depends on the choice of $\boldsymbol{\beta}$ and may grow exponentially in m since the domain of integration is $\mathbb{R}^m$. This is however not necessarily the case here. If we integrate the particular weight

$$\boldsymbol{\sigma}_s(\mathbf{y}) := \prod_{k=1}^m \sigma_{s,k}(y_k) := \prod_{k=1}^m \exp(-s\gamma_k |y_k|), \quad s \ge 1 \tag{5.25}$$

with respect to the Gaussian measure, we obtain, cf. [Git10], that

$$C(\boldsymbol{\sigma}_s) = \left(\int_{\mathbb{R}^m} \boldsymbol{\sigma}_s(\mathbf{y})^{-1} \rho(\mathbf{y}) \, \mathrm{d}\mathbf{y} \right) \le \exp\left(s^2 \sum_{k=1}^m \gamma_k^2 + s\sqrt{\frac{2}{\pi}} \sum_{k=1}^m \gamma_k \right). \tag{5.26}$$

This expression depends only on s and the decay of the sequence $\{\gamma_k\}_k$. It is bounded independently of the dimensionality m due to (2.4). In Proposition (5.27), we will show that u_m belongs to $C^0_{\boldsymbol{\sigma}_s}(\mathbb{R}^m; H^1_0(D))$. Hence, the weight $\boldsymbol{\sigma}_s$ fulfills the desired properties and we assume from now on that the weight is given by (5.25).

(5.27) **Proposition.** The solution u_m of (3.5) is contained in $C^0_{\boldsymbol{\sigma}_s}(\mathbb{R}^m; H^1_0(D))$ for all $s \ge 1$. In particular, it holds that

$$\|u_m\|_{C^0_{\boldsymbol{\sigma}_s}(\mathbb{R}^m; H^1_0(D))} \lesssim \|f\|_{L^2(D)}.$$

Moreover, the square u_m^2 satisfies $u_m^2 \in C^0_{\boldsymbol{\sigma}_s}(\mathbb{R}^m; W^{1,1}_0(D))$ for all $s \ge 2$ with

$$\|u_m^2\|_{C^0_{\boldsymbol{\sigma}_s}(\mathbb{R}^m; W^{1,1}_0(D))} \lesssim \|f\|^2_{L^2(D)}.$$

Proof. In view of inequality (3.11) and since $1/\underline{a}_m(\mathbf{y}) = \exp\big(\sum_{k=1}^m \gamma_k |y_k|\big)$, we derive, for all $s \ge 1$, the first estimate from

$$\boldsymbol{\sigma}_s(\mathbf{y}) \|u_m(\mathbf{y})\|_{H^1_0(D)} \lesssim \exp\left(\sum_{k=1}^m (\gamma_k - s\gamma_k)|y_k| \right) \|f\|_{L^2(D)} \le \|f\|_{L^2(D)}.$$

For all $s \ge 2$, the second estimate follows from

$$\begin{aligned} \boldsymbol{\sigma}_s(\mathbf{y}) \|u_m^2(\mathbf{y})\|_{W^{1,1}_0(D)} &\le \boldsymbol{\sigma}_s(\mathbf{y}) \|2u_m(\mathbf{y}) \nabla u_m(\mathbf{y})\|_{L^1(D)} \\ &\lesssim 2\boldsymbol{\sigma}_s(\mathbf{y}) \|u_m(\mathbf{y})\|^2_{H^1_0(D)} \lesssim \|f\|^2_{L^2(D)}. \end{aligned}$$

□

Due to the results of Subsection 5.1, we can bound all derivatives with respect to $\mathbf{y}$ of the solution u_m and its square u_m^2 in $C^0_{\boldsymbol{\sigma}_s}(\mathbb{R}^m; X)$. Moreover, if the right-hand side f belongs to $L^p(D)$, we obtain similar results for arbitrary powers u_m^p of u_m.

(5.28) **Proposition.** For all weights $\boldsymbol{\sigma}_s$ with $s \geq 2$, the partial derivatives of the solution u_m to (3.5) satisfy that

$$(5.29) \qquad \|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}} u_m\|_{C^0_{\boldsymbol{\sigma}_s}(\mathbb{R}^m; H^1_0(D))} \lesssim |\boldsymbol{\alpha}|! \left(\frac{\boldsymbol{\gamma}}{\log 2}\right)^{\boldsymbol{\alpha}} \|f\|_{L^2(D)}.$$

Especially, it holds that $\partial_{\mathbf{y}}^{\boldsymbol{\alpha}} u \in C^0_{\boldsymbol{\sigma}_s}(\mathbb{R}^m; H^1_0(D))$. The partial derivatives of u_m^2 fulfill $\partial_{\mathbf{y}}^{\boldsymbol{\alpha}} u^2 \in C^0_{\boldsymbol{\sigma}_s}(\mathbb{R}^m; W^{1,1}_0(D))$ for all $\boldsymbol{\sigma}_s$ with $s \geq 4$ and the bound

$$(5.30) \qquad \|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}} u_m^2\|_{C^0_{\boldsymbol{\sigma}_s}(\mathbb{R}^m; W^{1,1}_0(D))} \lesssim (|\boldsymbol{\alpha}|+1)! \left(\frac{\boldsymbol{\gamma}}{\log 2}\right)^{\boldsymbol{\alpha}} \|f\|^2_{L^2(D)} \leq |\boldsymbol{\alpha}|! \left(\frac{2\boldsymbol{\gamma}}{\log 2}\right)^{\boldsymbol{\alpha}} \|f\|^2_{L^2(D)}.$$

Proof. From Lemma (5.3), we obtain that

$$\begin{aligned}
\|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}} u_m\|_{C^0_{\boldsymbol{\sigma}_s}(\mathbb{R}^m; H^1_0(D))} &= \sup_{\mathbf{y} \in \mathbb{R}^m} \boldsymbol{\sigma}_s(\mathbf{y}) \|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}} u_m(\mathbf{y})\|_{H^1_0(D)} \\
&\leq |\boldsymbol{\alpha}|! \left(\frac{\boldsymbol{\gamma}}{\log 2}\right)^{\boldsymbol{\alpha}} \sup_{\mathbf{y} \in \mathbb{R}^m} \sqrt{\kappa_m(\mathbf{y})} \boldsymbol{\sigma}_s(\mathbf{y}) \|u_m(\mathbf{y})\|_{H^1_0(D)}.
\end{aligned}$$

In view of (3.10), we know that $\sqrt{\kappa_m(\mathbf{y})} = \exp\left(\sum_{k=1}^m \gamma_k |y_k|\right)$. This implies that

$$\begin{aligned}
\|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}} u_m\|_{C^0_{\boldsymbol{\sigma}_s}(\mathbb{R}^m; H^1_0(D))} &\leq |\boldsymbol{\alpha}|! \left(\frac{\boldsymbol{\gamma}}{\log 2}\right)^{\boldsymbol{\alpha}} \sup_{\mathbf{y} \in \mathbb{R}^m} \exp\left(\sum_{k=1}^m (1-s)\gamma_k |y_k|\right) \|u_m(\mathbf{y})\|_{H^1_0(D)} \\
&\lesssim |\boldsymbol{\alpha}|! \left(\frac{\boldsymbol{\gamma}}{\log 2}\right)^{\boldsymbol{\alpha}} \|f\|_{L^2(D)} \sup_{\mathbf{y} \in \mathbb{R}^m} \exp\left(\sum_{k=1}^m (2-s)\gamma_k |y_k|\right).
\end{aligned}$$

Then, the inequality (5.29) follows from $s \geq 2$. The bound (5.30) is obtained in the same way by using Lemma (5.10) instead of Lemma (5.3). □

Analogously, we obtain that the higher powers u_m^p of u_m fulfill the following proposition.

(5.31) **Proposition.** Under the assumptions of Lemma (5.13), the derivatives of u_m^p with respect to $\mathbf{y}$ belong to $C^0_{\boldsymbol{\sigma}_s}(\mathbb{R}^m; W^{1,1}_0(D))$ for all $\boldsymbol{\sigma}_s$ with $s \geq 3p$ and it holds that

$$(5.32) \qquad \|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}} u_m^p\|_{C^0_{\boldsymbol{\sigma}_s}(\mathbb{R}^m; W^{1,1}_0(D))} \lesssim |\boldsymbol{\alpha}|! \left(\frac{C(p,D)p\boldsymbol{\gamma}}{\log 2}\right)^{\boldsymbol{\alpha}} \|f\|^p_{L^p(D)}.$$

We finish this section with the analytic extension of functions $v \in C^0_{\boldsymbol{\sigma}_s}(\mathbb{R}^m; X)$ whose derivatives admit certain decay properties. We distinguish between the analytic extension in a particular stochastic direction and the extension in all directions at the same time. For the first result, we follow the notation in [BNT07] and introduce

$$(5.33) \qquad \begin{gathered}
\boldsymbol{\sigma}^\star_{s,k}(\mathbf{y}^\star_k) := \prod_{\substack{i=1 \\ i \neq k}}^m \sigma_{s,i}(y_i), \quad \boldsymbol{\rho}^\star_k(\mathbf{y}^\star_k) := \prod_{\substack{i=1 \\ i \neq k}}^m \rho(y_i) \\
\text{and} \quad \mathbf{y}^\star_k := (y_1, \ldots, y_{k-1}, y_{k+1}, \ldots, y_m) \in \mathbb{R}^{m-1}.
\end{gathered}$$

Then, we have the following statement which is adopted from Lemma (3.2) in [BNT07]. We shortly review the most important steps of the proof.

(5.34) **Theorem.** For $v \in C^0_{\boldsymbol{\sigma}_s}(\mathbb{R}^m; X)$, let the derivatives of v with respect to $\mathbf{y}$ fulfill the bound

$$\|\partial^j_{y_k} v\|_{C^0_{\boldsymbol{\sigma}_s}(\mathbb{R}^m;X)} \lesssim j!\mu_k^j \tag{5.35}$$

with some constant $\mu_k \in (0, \infty)$. Then, for $\tau_k \in (0, 1/\mu_k)$, the function

$$v \colon \mathbb{R} \to C^0_{\boldsymbol{\sigma}^\star_{s,k}}(\mathbb{R}^{m-1}; X), \quad y_k \mapsto v(\mathbf{x}, y_k, \mathbf{y}^\star_k)$$

admits an analytic extension $v(\mathbf{x}, z, \mathbf{y}^\star_k)$ for $z \in \Sigma(\mathbb{R}, \tau_k) := \{z \in \mathbb{C} : \operatorname{dist}(z, \mathbb{R}) \le \tau_k\}$. Moreover, the function v is bounded in the norm

$$\|v\|_{C^0_{\sigma_{s,k}}\left(\Sigma(\mathbb{R},\tau_k);C^0_{\boldsymbol{\sigma}^\star_{s,k}}(\mathbb{R}^{m-1};X)\right)} := \sup_{z\in\Sigma(\mathbb{R},\tau_k)} \sigma_{s,k}(\operatorname{Re}(z))\|v(z)\|_{C^0_{\boldsymbol{\sigma}^\star_{s,k}}(\mathbb{R}^{m-1};X)}. \tag{5.36}$$

Proof. For every $y_k \in \mathbb{R}$, we consider the Taylor expansion around y_k in $z \in \mathbb{C}$

$$v(\mathbf{x}, z, \mathbf{y}^\star_k) = \sum_{j=0}^{\infty} \frac{(z - y_k)^j}{j!} \partial^j_{y_k} v(\mathbf{x}, y_k, \mathbf{y}^\star_k).$$

Thus, given an arbitrary $y_k \in \mathbb{R}$, we can estimate that

$$\begin{aligned}\sigma_{s,k}(y_k)\|v(z)\|_{C^0_{\boldsymbol{\sigma}^\star_{s,k}}(\mathbb{R}^{m-1};X)} &\le \sum_{j=0}^{\infty} \frac{|z - y_k|^j}{j!} \sigma_{s,k}(y_k)\|\partial^j_{y_k} v(y_k)\|_{C^0_{\boldsymbol{\sigma}^\star_{s,k}}(\mathbb{R}^{m-1};X)} \\ &\le \sum_{j=0}^{\infty} \frac{|z - y_k|^j}{j!} \|\partial^j_{y_k} v\|_{C^0_{\boldsymbol{\sigma}_s}(\mathbb{R}^m;X)} \lesssim \sum_{j=0}^{\infty} \left(|z - y_k|\mu_k\right)^j.\end{aligned}$$

The last expression converges for all $|z - y_k| \le \tau_k < 1/\mu_k$. Since we can cover $\Sigma(\mathbb{R}, \tau_k)$ by the union of balls $|z - y_k| \le \tau_k$, the function v can be extended analytically to the whole region $\Sigma(\mathbb{R}, \tau_k)$ and is bounded with respect to the norm (6.7). □

This theorem establishes that the solution u_m to (3.5) as well as its powers u_m^p are analytically extendable in each stochastic dimension in a strip around the real line. This is sufficient to perform the error analysis of a tensor product Gauss-Hermite quadrature. Since we will also consider a sparse grid Gaussian quadrature, we have to provide the analytic extension of the integrand into $\boldsymbol{\Sigma}(\mathbb{R}^m, \boldsymbol{\tau})$, where $\boldsymbol{\Sigma}(\mathbb{R}^m, \boldsymbol{\tau})$ is an m-fold Cartesian product of the form

$$\boldsymbol{\Sigma}(\mathbb{R}^m, \boldsymbol{\tau}) := \bigtimes_{k=1}^{m} \Sigma(\mathbb{R}, \tau_k) \tag{5.37}$$

with $\Sigma(\mathbb{R}, \tau_k)$ defined as in Theorem (5.34). This is proven in the following theorem which is a modified version of Theorem (2.1) in [BNTT12].

(5.38) **Theorem.** Let $v \in C^0_{\boldsymbol{\sigma}_s}(\mathbb{R}^m; X)$ satisfy the estimate

(5.39) $$\|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}} v(\mathbf{y})\|_X \lesssim h(\mathbf{y})|\boldsymbol{\alpha}|!\boldsymbol{\mu}^{\boldsymbol{\alpha}}$$

for some vector $\boldsymbol{\mu} \in \mathbb{R}^m$ and a continuous function $h : \mathbb{R}^m \to \mathbb{R}$ with $h(\mathbf{y}) \lesssim \boldsymbol{\sigma}_s(\mathbf{y})$. Then, the function v is analytically extendable to the domain $\boldsymbol{\Sigma}(\mathbb{R}^m, \boldsymbol{\tau})$ where

(5.40) $$0 < \tau_k < \frac{1}{C(\delta)k^{1+\delta}\mu_k} \qquad \text{and} \qquad C(\delta) := \sum_{k=1}^{\infty} k^{-1-\delta}.$$

Moreover, v is bounded in the norm

$$\|v\|_{C^0_{\boldsymbol{\sigma}_s}(\boldsymbol{\Sigma}(\mathbb{R}^m,\boldsymbol{\tau});X)} := \sup_{\mathbf{z}\in\boldsymbol{\Sigma}(\mathbb{R}^m,\boldsymbol{\tau})} \boldsymbol{\sigma}_s(\operatorname{Re}(\mathbf{z}))\|v(\mathbf{z})\|_X,$$

where $\operatorname{Re}(\mathbf{z})_k := \operatorname{Re}(z_k)$.

Proof. We assign to each vector $\mathbf{z} \in \boldsymbol{\Sigma}(\mathbb{R}^m, \boldsymbol{\tau})$ the vector $\mathbf{y} = \operatorname{Re}(\mathbf{z}) \in \mathbb{R}^m$. Then, the Taylor expansion of u in $\mathbf{y}$ is given by

$$v(\mathbf{z}) = \sum_{\boldsymbol{\alpha}\in\mathbb{N}^m} \frac{1}{\boldsymbol{\alpha}!}\partial_{\mathbf{y}}^{\boldsymbol{\alpha}} v(\mathbf{y})(\mathbf{z}-\mathbf{y})^{\boldsymbol{\alpha}}.$$

If we define the coordinatewise modulus of $\mathbf{z} \in \mathbb{C}^m$ by $\operatorname{abs}(\mathbf{z})$, it holds that

$$\|v(\mathbf{z})\|_X \lesssim \sum_{\boldsymbol{\alpha}\in\mathbb{N}^m} h(\mathbf{y})\frac{|\boldsymbol{\alpha}|!}{\boldsymbol{\alpha}!}\boldsymbol{\mu}^{\boldsymbol{\alpha}} \operatorname{abs}(\mathbf{z}-\mathbf{y})^{\boldsymbol{\alpha}} = h(\mathbf{y})\sum_{k=0}^{\infty}\sum_{|\boldsymbol{\alpha}|=k} \frac{k!}{\boldsymbol{\alpha}!}\boldsymbol{\mu}^{\boldsymbol{\alpha}} \operatorname{abs}(\mathbf{z}-\mathbf{y})^{\boldsymbol{\alpha}}.$$

By the generalized Newton binomial formula, see [BNTT12],

$$\sum_{|\boldsymbol{\alpha}|=k} \frac{k!}{\boldsymbol{\alpha}!}\boldsymbol{\mu}^{\boldsymbol{\alpha}} \operatorname{abs}(\mathbf{z}-\mathbf{y})^{\boldsymbol{\alpha}} = \left(\sum_{n=1}^{m} \mu_n \operatorname{abs}(\mathbf{z}-\mathbf{y})_n\right)^k$$

it follows, since $\operatorname{abs}(\mathbf{z}-\mathbf{y})_n \le \tau_n$, that

$$\|v(\mathbf{z})\|_X \lesssim h(\mathbf{y})\sum_{k=0}^{\infty}\left(\sum_{n=1}^{m} \mu_n \operatorname{abs}(\mathbf{z}-\mathbf{y})_n\right)^k \le h(\mathbf{y})\sum_{k=0}^{\infty}\left(\sum_{n=1}^{m} \mu_n\tau_n\right)^k.$$

Due to (6.9), we obtain that $\sum_{n=1}^{\infty} \mu_n\tau_n < 1$ which yields

$$\|v(\mathbf{z})\|_X \lesssim h(\mathbf{y}).$$

Furthermore, we deduce that

$$\|v\|_{C^0_{\boldsymbol{\sigma}_s}(\boldsymbol{\Sigma}(\mathbb{R}^m,\boldsymbol{\tau});X)} = \sup_{\mathbf{z}\in\boldsymbol{\Sigma}(\mathbb{R}^m,\boldsymbol{\tau})} \boldsymbol{\sigma}_s(\operatorname{Re}(\mathbf{z}))\|v(\mathbf{z})\|_X \lesssim \boldsymbol{\sigma}_s(\mathbf{y})h(\mathbf{y}) < \infty$$

which completes the proof. □

6. Regularity results in the uniformly elliptic case

In this section, we state the corresponding results to those from Section 5 for the uniformly elliptic case. Since the proofs of these results are either known or can immediately be derived from the proofs of the results in the lognormal case, there are omitted.

We present the associated results to the pointwise regularity estimates Lemma (5.3), Lemma (5.10) and Lemma (5.13) for the uniformly elliptic case. According to [CDS10], the derivatives of the solution u_m to (3.3) fulfill the following lemma.

(6.1) **Lemma ([CDS10]).** For all $\boldsymbol{\alpha} \in \mathbb{N}^m$ the derivatives of the solution u_m to (3.3) satisfy the estimate

(6.2) $$\|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}} u_m(\mathbf{y})\|_{H_0^1(D)} \leq |\boldsymbol{\alpha}|! \left(\frac{\boldsymbol{\gamma}}{\underline{a}}\right)^{\boldsymbol{\alpha}} \|u_m(\mathbf{y})\|_{H_0^1(D)}.$$

With the same techniques as in the lognormal case, we arrive at estimates for the derivatives of the powers u_m^p of the solution u_m to (3.3). Again, in the case $p = 2$ we can use the bound (6.2) directly, whereas in the case $p > 2$ the L^p-extension of (6.2) has to be applied. This yields a better constant in the case $p = 2$ in terms of the ratio of the continuity constant $\overline{a}$ and the ellipticity constant $\underline{a}$. We state these results in the following corollary.

(6.3) **Corollary.** Let u_m be the solution to (3.3). Then, it holds for all $\boldsymbol{\alpha} \in \mathbb{N}^m$ that

(6.4) $$\|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}} u_m^2(\mathbf{y})\|_{W_0^{1,1}(D)} \lesssim (|\boldsymbol{\alpha}| + 1)! \left(\frac{\boldsymbol{\gamma}}{\underline{a}}\right)^{\boldsymbol{\alpha}} \|u(\mathbf{y})\|_{H_0^1(D)}^2.$$

Moreover, if $f \in L^p(D)$, we obtain the analogous bounds to Lemma (5.13):

$$\|u_m(\mathbf{y})\|_{W_0^{1,p}(D)} \lesssim \frac{1}{\underline{a}} \|f\|_{L^p(D)},$$
$$\|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}} u_m(\mathbf{y})\|_{W_0^{1,p}(D)} \lesssim |\boldsymbol{\alpha}|! \left(\frac{C(p,D)\boldsymbol{\gamma}}{\underline{a}}\right)^{\boldsymbol{\alpha}} \frac{\overline{a}}{\underline{a}} \|u_m(\mathbf{y})\|_{W_0^{1,p}(D)},$$
$$\|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}} u_m^p(\mathbf{y})\|_{W_0^{1,1}(D)} \lesssim |\boldsymbol{\alpha}|! \left(\frac{C(p,D)p\boldsymbol{\gamma}}{\underline{a}}\right)^{\boldsymbol{\alpha}} \left(\frac{\overline{a}}{\underline{a}}\right)^p \|u_m(\mathbf{y})\|_{W_0^{1,p}(D)}^p.$$

With the aid of an example, we illustrates that the estimates on the derivatives of the solution u_m in Lemma (6.1) are sharp.

(6.5) **Remark.** The bound (6.2) on the derivative of u_m can be motivated by the following example from [BNTT12, CDS10]: Consider equation (3.3) with the diffusion coefficient $a_m(y) = 1 + \sum_{i=1}^m b_i y_i$, where $b_i > 0$, $y_i \sim \text{UNI}(-1/2, 1/2)$ and $\sum_{i=1}^m b_i < 2$. Moreover, let $f \in L^2(D)$ and denote by u_m the solution to

$$-\operatorname{div}(a_m(y)\nabla u_m(\mathbf{y})) = f \text{ in } D, \quad u_m = 0 \text{ on } \partial D.$$

Then, the derivatives of u_m with respect to $\mathbf{y}$ can be determined analytically as

$$\partial_{\mathbf{y}}^{\boldsymbol{\alpha}} u_m(\mathbf{x}, \mathbf{y}) = g(\mathbf{x}) \frac{|\boldsymbol{\alpha}|! \mathbf{b}^{\boldsymbol{\alpha}}}{a_m(\mathbf{y})^{|\boldsymbol{\alpha}|+1}},$$

where g is the solution to

$$-\Delta g = f \text{ in } D, \quad g = 0 \text{ on } \partial D.$$

In the uniformly elliptic case, the pointwise regularity estimates in Lemma (6.1) and Corollary (6.3) directly imply that the solution u_m is contained in $C^k(\mathbf{\Gamma}; H_0^1(D))$ and that its powers belong to $C^k(\mathbf{\Gamma}; W_0^{1,1}(D))$. Thus, we obtain, analogously to Theorem (5.34) and Theorem (5.38), the following two results on the analytic extendability.

(6.6) **Corollary.** For $v \in C^0(\mathbf{\Gamma}; X)$, let the derivatives of v with respect to $\mathbf{y}$ fulfill the bound

$$\|\partial_{y_k}^j v\|_{C^0(\mathbf{\Gamma};X)} \lesssim j! \mu_k^j$$

with some constant $\mu_k \in (0, \infty)$. Then, for $\tau_k \in (0, 1/\mu_k)$, the function

$$v \colon \Gamma_k \to C^0(\mathbf{\Gamma}; X), \quad y_k \mapsto v(\mathbf{x}, y_k, \mathbf{y}_k^\star)$$

admits an analytic extension $v(\mathbf{x}, z, \mathbf{y}_k^\star)$ for $z \in \Sigma(\Gamma_k, \tau_k) := \{z \in \mathbb{C} : \operatorname{dist}(z, \Gamma_k) \leq \tau_k\}$. In addition, the function v is bounded with respect to the norm

$$\|v\|_{C^0(\Sigma(\Gamma_k,\tau_k); C^0(\mathbf{\Gamma}_k^\star;X))} := \sup_{z \in \Sigma(\Gamma_k,\tau_k)} \|v(z)\|_{C^0(\mathbf{\Gamma}_k^\star;X)} \tag{6.7}$$

where $\mathbf{\Gamma}_k^\star := \bigtimes_{i=1, i\neq k} \Gamma_i$.

(6.8) **Corollary.** The function $v \in C^0(\mathbf{\Gamma}; X)$ is analytically extendable into the domain $\mathbf{\Sigma}(\mathbf{\Gamma}, \boldsymbol{\tau}) := \bigtimes_{k=1}^m \Sigma(\Gamma_k, \tau_k)$ where

$$\tau_k < \frac{1}{C(\delta) k^{1+\delta} \mu_k} \qquad \text{and} \qquad C(\delta) = \sum_{k=1}^\infty k^{-1-\delta}, \tag{6.9}$$

if the derivatives of v satisfy for all multi-indices $\boldsymbol{\alpha} \in \mathbb{N}^m$ the bound $\|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}} v(\mathbf{y})\|_X \lesssim |\boldsymbol{\alpha}|! \mu^{\boldsymbol{\alpha}}$. Furthermore, v is bounded in the norm

$$\|v\|_{C^0(\mathbf{\Sigma}(\mathbf{\Gamma},\boldsymbol{\tau});X)} := \sup_{\mathbf{z} \in \mathbf{\Sigma}(\mathbf{\Gamma},\boldsymbol{\tau})} \|v(\mathbf{z})\|_X \lesssim \frac{1}{1-\gamma}$$

where γ is given by $\gamma = \sum_{k=1}^m \tau_k \mu_k < 1$.

Chapter IV

(QUASI-) MONTE CARLO QUADRATURE

The computation of the moments (III.3.15) corresponds to the evaluation of the m-dimensional integration operator $\mathbf{I}\colon L^1_{\boldsymbol{\rho}}(\boldsymbol{\Gamma}; X) \to X$

$$(0.1)\qquad \mathbf{I}v(\mathbf{x}) = \int_{\boldsymbol{\Gamma}} v(\mathbf{x}, \mathbf{y})\boldsymbol{\rho}(\mathbf{y})\,\mathrm{d}\mathbf{y}$$

for $v(\mathbf{x}, \mathbf{y}) = u_m^p(\mathbf{x}, \mathbf{y})$. For the approximation, we will use quadrature rules $\mathbf{Q}_N^{(m)}$ given by

$$(0.2)\qquad \left(\mathbf{Q}_N^{(m)} v\right)(\mathbf{x}) = \sum_{i=1}^{N} w_i v(\mathbf{x}, \boldsymbol{\xi}^i).$$

Herein, N denotes the number of *samples* and $\boldsymbol{\xi}^i \in \mathbb{R}^m$ is a *sample point* which is combined with the quadrature weight w_i.

In this chapter, we discuss Monte Carlo and quasi-Monte Carlo quadrature rules. These quadrature rules are classically of the form

$$(0.3)\qquad \left(\mathbf{Q}_{(\mathrm{Q})\mathrm{MC},N}^{(m)} v\right)(\mathbf{x}) = \frac{1}{N}\sum_{i=1}^{N} v(\mathbf{x}, \boldsymbol{\xi}^i).$$

Hence, such quadrature rules always use the same weight $w_i = 1/N$ for all sample points $\boldsymbol{\xi}^i$.

(0.4) **Remark.** The evaluation of the integrand in each sample point in (0.2) corresponds to the solution of an elliptic boundary value problem which is defined on the spatial domain D. In general, such boundary value problems cannot be solved analytically and so we need to solve them numerically. Of course, this introduces a discretization error in the spatial variable which is discussed in Chapter VI. In addition, the cost complexity of the quadrature rules from Chapters IV and V is firstly analyzed in terms of the number of quadrature points. This corresponds to the assumption that each sample can be evaluated in constant time, i.e. it requires $\mathcal{O}(1)$ operations. Of course, this assumption does not hold in general. Hence, we will also analyze the computational cost for solving the associated deterministic boundary value problem later on in Chapter VI.

In our applications, the dimensionality m of the domain of integration in (0.1) corresponds to the number of random parameters which are required for an accurate representation of the diffusion coefficient in (III.3.3) or (III.3.5). Hence, the dimensionality increases with the desired accuracy ε and tends, in general, to infinity as $\varepsilon \to 0$. Therefore, it is important to construct quadrature methods which converge as far as possible independently of m.

Monte Carlo quadrature is the method of choice for integration problems where the dimensionality of the integration domain is large, at least if the integrand depends almost equally on each dimension. We refer to this situation as the *unweighted case*. This is due to the fact that the Monte Carlo method yields a convergence rate, in terms of the number of quadrature points, which is independent of the dimensionality. For deterministic quadrature methods, which means that the sample points are chosen deterministically, the convergence rate usually deteriorates when the dimensionality is large, at least in the unweighted case. In this case, the cost(ε), which is the number of quadrature points to get an error of $\mathcal{O}(\varepsilon)$, may grow exponentially in the dimension m. The situation changes when we take into account that the integrands under consideration depend anisotropically on the different parameter dimensions. In particular, the higher dimensions are of less importance to the integrands. The dependencies are reflected by the regularity results of the solution and its powers in Section III.5 and Section III.6. This is the *weighted case* which, however, offers possibilities to get rid of the exponential dependency on the dimensionality of deterministic quadrature rules.

(0.5) **Remark.** In the literature, certain integration problems are investigated in the context of *tractability*, see e.g.[SW97, Woź94]. Instead of the integration of a single function, the standard setting there is to consider the integration in certain Banach spaces $\mathcal{Y} \subset L^1_{\boldsymbol{\rho}}(\boldsymbol{\Gamma})$ of real valued functions. The cost(ε) corresponds to the number of quadrature points which are required to get a worst case error $\mathcal{O}(\varepsilon)$ with respect to all normalized functions in $\mathcal{Y}$. Roughly speaking, the associated integration problem is then called *(polynomially) tractable* if there exists a quadrature method such that the cost(ε) can be bounded by a polynomial in m and ε. Otherwise, the problem is called intractable. There have been established a lot of positive results, i.e. proofs of tractability, as well as negative results, i.e. proofs of intractability, in the past 15 years. For an insightful overview of this topic see [NW08, NW10, NW12]. Nevertheless, even if an integration problem is proven to be tractable, this implies only the existence and not necessarily the constructability of the respective quadrature rule, where the latter one is known as a constructive proof of tractability. Thus, it has also become a challenge to find the quadrature rules which corroborate the proven tractability results.

We will not focus on the construction of new quadrature methods in this chapter. The main challenge for us is to prove that the quasi-Monte Carlo method based on Halton points converges, under certain decay conditions on the sequence $\{\gamma_k\}_k$, nearly dimension-independent for the approximation of the moments in the lognormal case. For the uniformly elliptic case, this can straightforwardly be derived with the application of findings from [KSS12a, Wan02]. But, as we will see, this result cannot immediately be transferred to the lognormal case.

1. Monte Carlo quadrature

In the case of the Monte Carlo quadrature, the sample points in (0.3) are chosen randomly. Hence, we need a (pseudo-) random number generator which produces m-dimensional random vectors, independent and identically distributed according to the underlying density function $\boldsymbol{\rho}$. There are two main advantages of the Monte Carlo quadrature: The method does not suffer from the curse of dimensionality and the convergence result requires only weak regularity conditions on the integrand. The drawback of this method is that it convergences only with the low algebraic rate $\mathcal{O}(N^{-1/2})$ in the root mean square sense, see e.g. [HH64]. More precisely, one has that

$$(1.1)\qquad \Big(\mathbb{E}\|(\mathbf{I}-\mathbf{Q}_N)v\|_X^2\Big)^{\frac{1}{2}} \lesssim N^{-\frac{1}{2}}\|v\|_{L^2_{\boldsymbol{\rho}}(\boldsymbol{\Gamma};X)},$$

see e.g. [BSZ11]. For the sake of simplicity, we restrict ourselves to this estimate, although, as mentioned in Chapter I, convergence in distribution could be considered as well. From (1.1), we derive the following proposition.

(1.2) **Proposition.** The Monte Carlo quadrature for the approximation of the moments of the solution u_m in the uniformly elliptic (III.3.3) as well as in the lognormal case (III.3.5) converges in the root mean square sense dimension-independently with an algebraic rate of 1/2. It holds

$$(1.3)\qquad \Big(\mathbb{E}\|(\mathbf{I}-\mathbf{Q}_N)u_m\|_{H_0^1(D)}^2\Big)^{\frac{1}{2}} \lesssim N^{-\frac{1}{2}}\|f\|_{L^2(D)}$$

for the approximation of the mean and

$$(1.4)\qquad \Big(\mathbb{E}\|(\mathbf{I}-\mathbf{Q}_N)u_m^p\|_{W_0^{1,1}(D)}^2\Big)^{\frac{1}{2}} \lesssim N^{-\frac{1}{2}}\|f\|_{L^p(D)}^p$$

for the approximation of the higher order moments with constants which depend on p but not on m.

Proof. The regularity results of the previous chapter provide that the solution u_m belongs to $L^2_{\boldsymbol{\rho}}(\boldsymbol{\Gamma};H_0^1(D))$ and that its powers u_m^p belong to $L^2_{\boldsymbol{\rho}}(\boldsymbol{\Gamma};W_0^{1,1}(D))$. This follows in the uniformly elliptic case immediately from the stability estimate (III.3.8) for u_m and from the combination of Corollary (III.6.3) with

$$(1.5)\qquad \|u_m^p(\mathbf{y})\|_{W_0^{1,1}(D)} \lesssim \|u_m(\mathbf{y})\|_{W_0^{1,p}(D)}^p$$

for u_m^p.

In the lognormal case, we establish the assertion by analogous results. On the one hand, for u_m, we exploit the stability estimate (III.3.11) and the integrability of $1/\underline{a}_m(\mathbf{y})$ with respect to the Gaussian density, see (III.5.26). On the other hand, for u_m^p, we use the above estimate (1.5), and apply afterwards the L^p-stability estimate (III.5.14) and the integrability of $1/\underline{a}_m(\mathbf{y})^p$ with respect to the Gaussian density.

This implies that it holds for all $p\in\mathbb{N}$ in the uniformly elliptic as well as in the lognormal case that

$$(1.6)\qquad \|u_m\|_{L^2_{\boldsymbol{\rho}}(\boldsymbol{\Gamma};H_0^1(D))} \lesssim \|f\|_{L^2(D)} \quad\text{and}\quad \|u_m^p\|_{L^2_{\boldsymbol{\rho}}(\boldsymbol{\Gamma};W_0^{1,1}(D))} \lesssim \|f\|_{L^p(D)}^p$$

which establishes the assertion. □

2. Quasi-Monte Carlo quadrature

From now on, we assume that the parameter domain is given by $\mathbf{\Gamma} = [-1/2, 1/2]^m$ in the uniformly elliptic case and that the distribution of the parameter $\mathbf{y}$ is given by the uniform density $\boldsymbol{\rho}(\mathbf{y}) \equiv 1$. Hence, the random parameters y_k are independent for all $k = 1, \ldots, m$ and uniformly distributed on $[-1/2, 1/2]$. This simplifies the analysis of quasi-Monte Carlo quadrature methods which are usually constructed over the unit cube $[0, 1]^m$. The construction of quadrature rules over $[-1/2, 1/2]^m$ is then simply obtained by shifting each quadrature point by the vector $-\mathbf{1/2} := [-1/2, \ldots, -1/2]^\intercal \in \mathbb{R}^m$. Likewise, in this section, we will make use of the vectors $\mathbf{0}, \mathbf{1/2}$ and $\mathbf{1}$ which are defined analogously to the vector $-\mathbf{1/2}$.

2.1 General remarks

The error estimation of the quasi-Monte Carlo quadrature is usually performed for functions $f : [0, 1]^m \to \mathbb{R}$ of *bounded variation in the sense of Hardy and Krause*, i.e.

$$V_{HK}(f) := \sum_{\|\boldsymbol{\alpha}\|_\infty = 1} V^{(|\boldsymbol{\alpha}|)}(f(\mathbf{y}_{\boldsymbol{\alpha}}, \mathbf{1})) < \infty,$$

where $V^{(m)}(f)$ is the variation of f on $[0, 1]^m$ in the sense of Vitali, see e.g. [Nie92]. For a given vector $\mathbf{y} \in \mathbb{R}^m$, we denote by $\mathbf{y}_{\boldsymbol{\alpha}} \in \mathbb{R}^{|\boldsymbol{\alpha}|}$ the compressed vector which contains those components y_k of $\mathbf{y}$ where $\alpha_k = 1$. Additionally, for $\mathbf{z} \in \mathbb{R}^m$, we write $(\mathbf{y}_{\boldsymbol{\alpha}}, \mathbf{z}) \in \mathbb{R}^m$ for the vector whose k-th component is given either by y_k if $\alpha_k = 1$ or by z_k if $\alpha_k = 0$. For $\mathbf{z} = \mathbf{1}$, the vector $(\mathbf{y}_{\boldsymbol{\alpha}}, \mathbf{z})$ is contained in the $|\boldsymbol{\alpha}|$-dimensional face $\{\mathbf{y} \in [0, 1]^m : y_j = 1 \text{ for } \alpha_j = 0\}$, see [Nie92]. Thus, $f(\mathbf{y}_{\boldsymbol{\alpha}}, \mathbf{1})$ corresponds to the restriction of f to this $|\boldsymbol{\alpha}|$-dimensional face. Note that the variation in the sense of Vitali has a simple expression if the function f has continuous partial derivatives. Then, it holds that

$$V^{(m)}(f) = \int_{[0,1]^m} \left| \frac{\partial^m f}{\partial y_1 \cdots \partial y_m}(\mathbf{y}) \right| \mathrm{d}\mathbf{y}.$$

Hence, the variation in the sense of Hardy and Krause can be written as

$$V_{HK}(f) = \sum_{\|\boldsymbol{\alpha}\|_\infty = 1} \int_{[0,1]^{|\boldsymbol{\alpha}|}} |\partial_{\mathbf{y}}^{\boldsymbol{\alpha}} f(\mathbf{y}_{\boldsymbol{\alpha}}, \mathbf{1})| \, \mathrm{d}\mathbf{y}_{\boldsymbol{\alpha}}.$$

The error of a quasi-Monte Carlo method over the unit cube $[0, 1]^m$ is estimated for functions of bounded variation by means of the *star discrepancy* $\mathcal{D}_\infty^\star(\Xi_N)$ of the set $\Xi_N = \{\boldsymbol{\xi}^1, \ldots, \boldsymbol{\xi}^N\} \subset [0, 1]^m$ of sample points. It is defined by

$$\mathcal{D}_\infty^\star(\Xi_N) := \sup_{\mathbf{t} \in [0,1]^m} |\mathrm{discr}_{\Xi_N}(\mathbf{t})|.$$

Herein, the *local discrepancy function* $\mathrm{discr}_{\Xi_N} : [0, 1]^m \to \mathbb{R}$ is given by

$$\mathrm{discr}_{\Xi_N}(\mathbf{t}) := \mathrm{Vol}\left([\mathbf{0}, \mathbf{t})\right) - \frac{1}{N} \sum_{i=1}^{N} \mathbb{1}_{[\mathbf{0},\mathbf{t})}(\boldsymbol{\xi}^i),$$

where $\operatorname{Vol}([\mathbf{0},\mathbf{t}))$ denotes the Lebesgue measure of the cuboid $[\mathbf{0},\mathbf{t})$. More precisely, the quadrature error can be estimated for functions f of bounded variation in the sense of Hardy and Krause by

$$|(\mathbf{I}-\mathbf{Q}_{\Xi_N})f| \leq \mathcal{D}^\star_\infty(\Xi_N)V_{HK}(f). \tag{2.1}$$

This estimate is known as the *Koksma-Hlawka* inequality, cf. [Nie92]. The discrepancy measures the deviation of the sequence Ξ_N from the uniform distribution on $[0,1]^m$. In case of certain, so-called *low discrepancy* point sequences, e.g. the Sobol sequence, the Niederreiter sequence or the Halton sequence, the star discrepancy can typically be estimated to be of the order $\mathcal{O}(N^{-1}(\log N)^m)$, see e.g. [Ata04, Nie92].

Estimate (2.1) does not reflect any anisotropic behaviour of the integrand. Nevertheless, one can derive a weighted version of the Koksma-Hlawka inequality, cf. [KSS12b]. To that end, we start with the *Zaremba-Hlawka* identity, see [Hla61, Zar68], which provides an explicit representation of the quadrature error

$$(\mathbf{I}-\mathbf{Q}_{\Xi_N})f = \sum_{\|\boldsymbol{\alpha}\|_\infty=1}(-1)^{|\boldsymbol{\alpha}|}\int_{[0,1]^{|\boldsymbol{\alpha}|}}\partial^{\boldsymbol{\alpha}}_{\mathbf{y}}f(\mathbf{y}_{\boldsymbol{\alpha}},\mathbf{1})\operatorname{discr}_{\Xi_N}(\mathbf{y}_{\boldsymbol{\alpha}},\mathbf{1})\,\mathrm{d}\mathbf{y}_{\boldsymbol{\alpha}}. \tag{2.2}$$

Following [KSS12b], we insert weights $\omega_{\boldsymbol{\alpha}}\in\mathbb{R}$ independent of $\mathbf{x}$ and $\mathbf{y}$ in (2.2) which yields

$$(\mathbf{I}-\mathbf{Q}_{\Xi_N})f = \sum_{\|\boldsymbol{\alpha}\|_\infty=1}(-1)^{|\boldsymbol{\alpha}|}\int_{[0,1]^{|\boldsymbol{\alpha}|}}\partial^{\boldsymbol{\alpha}}_{\mathbf{y}}f(\mathbf{y}_{\boldsymbol{\alpha}},\mathbf{1})\omega_{\boldsymbol{\alpha}}^{-1/2}\omega_{\boldsymbol{\alpha}}^{1/2}\operatorname{discr}_{\Xi_N}(\mathbf{y}_{\boldsymbol{\alpha}},\mathbf{1})\,\mathrm{d}\mathbf{y}_{\boldsymbol{\alpha}}.$$

The application of Hölder's inequality for the integral as well as for the sum in the above equation yields the desired weighted and generalized Koksma-Hlawka inequality

$$|(\mathbf{I}-\mathbf{Q}_{\Xi_N})v| \leq \mathcal{D}^{r,s}_{\mathbf{w}}(\Xi_N)\|f\|_{\mathcal{W}^{r',s'}_{\mathbf{w}}} \tag{2.3}$$

with dual exponents r,r' and s,s', respectively. The weighted discrepancy $\mathcal{D}^{r,s}_{\mathbf{w}}(\Xi_N)$ is defined by

$$\mathcal{D}^{r,s}_{\mathbf{w}}(\Xi_N) := \left(\sum_{\|\boldsymbol{\alpha}\|_\infty=1}\|\omega_{\boldsymbol{\alpha}}^{1/2}\operatorname{discr}_{\Xi_N}(\mathbf{y}_{\boldsymbol{\alpha}},\mathbf{1})\|^s_{L^r([0,1]^m)}\right)^{\frac{1}{s}} \tag{2.4}$$

and the norm $\|\cdot\|_{\mathcal{W}^{r',s'}_{\mathbf{w}}}$ by

$$\|f\|_{\mathcal{W}^{r',s'}_{\mathbf{w}}} := \left(\sum_{\|\boldsymbol{\alpha}\|_\infty=1}\|\omega_{\boldsymbol{\alpha}}^{-1/2}\partial^{\boldsymbol{\alpha}}_{\mathbf{y}}f(\mathbf{y}_{\boldsymbol{\alpha}},\mathbf{1})\|^{s'}_{L^{r'}([0,1]^m)}\right)^{\frac{1}{s'}}. \tag{2.5}$$

The modifications for the cases $r,s\in\{1,\infty\}$ are defined as usual. The norm (2.5) defines a Banach space $\mathcal{W}^{r',s'}_{\mathbf{w}}$. The weighted Koksma-Hlawka inequality implies that the worst case quadrature error for functions in the Banach space $\mathcal{W}^{r',s'}_{\mathbf{w}}$ of a quasi-Monte Carlo quadrature with respect to the point set Ξ_N is bounded by the weighted discrepancy $\mathcal{D}^{r,s}_{\mathbf{w}}(\Xi_N)$.

Next, we consider functions $v \in C^{1,\text{mix}}([0,1]^m; W_0^{1,q}(D))$ for $q \in \mathbb{N} \setminus \{0\}$. The Banach space $C^{1,\text{mix}}([0,1]^m; W_0^{1,q}(D))$ consists of all continuously differentiable functions $v : [0,1]^m \to W_0^{1,q}(D)$ whose derivatives $\partial_{\mathbf{y}}^{\boldsymbol{\alpha}} v$ exist and are bounded for all $\|\boldsymbol{\alpha}\|_\infty \leq 1$. We equip this space with the norm

$$\|v\|_{C^{1,\text{mix}}([0,1]^m;W_0^{1,q}(D))} := \sup_{\|\boldsymbol{\alpha}\|_\infty \leq 1} \sup_{\mathbf{y}\in[0,1]^m} \|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}} v(\mathbf{y})\|_{W_0^{1,q}(D)} < \infty.$$

It holds that the function $v(\mathbf{x},\cdot)$ is a continuously differentiable function from $[0,1]^m \to \mathbb{R}$ for almost all $\mathbf{x} \in D$. Thus, we can apply the Zaremba-Hlawka identity (2.2) for almost all $\mathbf{x} \in D$, which leads to

$$\begin{aligned}
&\|(\mathbf{I} - \mathbf{Q}_{\Xi_N})v\|_{W_0^{1,q}(D)} \\
&= \left(\sum_{i=1}^d \int_D \left| \frac{\partial}{\partial x_i} \Big[(\mathbf{I} - \mathbf{Q}_{\Xi_N}) v(\mathbf{x},\cdot) \Big] \right|^q \mathrm{d}\mathbf{x} \right)^{1/q} \\
&= \left(\sum_{i=1}^d \int_D \left| \frac{\partial}{\partial x_i} \left[\sum_{\|\boldsymbol{\alpha}\|_\infty = 1} (-1)^{|\boldsymbol{\alpha}|} \int_{[0,1]^{|\boldsymbol{\alpha}|}} \partial_{\mathbf{y}}^{\boldsymbol{\alpha}} v(\mathbf{x}, \mathbf{y}_{\boldsymbol{\alpha}}, \mathbf{1}) \operatorname{discr}_{\Xi_N}(\mathbf{y}_{\boldsymbol{\alpha}}, \mathbf{1}) \,\mathrm{d}\mathbf{y}_{\boldsymbol{\alpha}} \right] \right|^q \mathrm{d}\mathbf{x} \right)^{1/q} \qquad (2.6) \\
&= \left\| \sum_{\|\boldsymbol{\alpha}\|_\infty = 1} (-1)^{|\boldsymbol{\alpha}|} \int_{[0,1]^{|\boldsymbol{\alpha}|}} \partial_{\mathbf{y}}^{\boldsymbol{\alpha}} v(\cdot, \mathbf{y}_{\boldsymbol{\alpha}}, \mathbf{1}) \operatorname{discr}_{\Xi_N}(\mathbf{y}_{\boldsymbol{\alpha}}, \mathbf{1}) \,\mathrm{d}\mathbf{y}_{\boldsymbol{\alpha}} \right\|_{W_0^{1,q}(D)}.
\end{aligned}$$

Then, we obtain from the Bochner inequality (II.3.6) that

$$\begin{aligned}
\|(\mathbf{I} - \mathbf{Q}_{\Xi_N})v\|_{W_0^{1,q}(D)} &\leq \sum_{\|\boldsymbol{\alpha}\|_\infty = 1} \int_{[0,1]^{|\boldsymbol{\alpha}|}} \|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}} v(\cdot, \mathbf{y}_{\boldsymbol{\alpha}}, \mathbf{1}) \operatorname{discr}_{\Xi_N}(\mathbf{y}_{\boldsymbol{\alpha}}, \mathbf{1})\|_{W^{1,q}(D)} \,\mathrm{d}\mathbf{y}_{\boldsymbol{\alpha}} \\
&\leq \left(\sum_{\|\boldsymbol{\alpha}\|_\infty = 1} \int_{[0,1]^{|\boldsymbol{\alpha}|}} \|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}} v(\cdot, \mathbf{y}_{\boldsymbol{\alpha}}, \mathbf{1})\|_{W_0^{1,q}(D)} \,\mathrm{d}\mathbf{y}_{\boldsymbol{\alpha}} \right) \mathcal{D}_\infty^\star(\Xi_N).
\end{aligned}$$

This is the analogue to the Koksma-Hlawka inequality for the evaluation of Bochner integrals in $W_0^{1,q}(D)$. Of course, likewise to (2.4), one can obtain a weighted version of this inequality by replacing $\|f\|_{\mathcal{W}_{\mathbf{w}}^{r',s'}}$ in (2.3) by $\|v\|_{\mathcal{W}_{\mathbf{w}}^{r',s'}(W_0^{1,q}(D))}$, where

$$\|v\|_{\mathcal{W}_{\mathbf{w}}^{r',s'}(W_0^{1,q}(D))} := \left(\sum_{\|\boldsymbol{\alpha}\|_\infty = 1} \|\omega_{\boldsymbol{\alpha}}^{-1/2} \partial_{\mathbf{y}}^{\boldsymbol{\alpha}} v(\mathbf{y}_{\boldsymbol{\alpha}}, \mathbf{1})\|_{L^{r'}([0,1]^m; W_0^{1,q}(D))}^{s'} \right)^{\frac{1}{s'}}.$$

2.2 Uniformly elliptic case

For elliptic partial differential equations (III.3.3) with uniformly elliptic diffusion coefficients, the p-th moment of the solution u_m corresponds to the integral

$$\int_{[-1/2,1/2]^m} u_m^p(\mathbf{x},\mathbf{y}) \,\mathrm{d}\mathbf{y} = \int_{[0,1]^m} u_m^p(\mathbf{x}, \mathbf{y} - \mathbf{1/2}) \,\mathrm{d}\mathbf{y}. \qquad (2.7)$$

In order to obtain error estimates for quasi-Monte Carlo quadrature methods, we will show that the integrand on the right-hand side belongs to the Banach space $\mathcal{W}_{\mathbf{w}}^{r',s'}(X)$

with $X = H_0^1(D)$ if $p = 1$ and $X = W_0^{1,1}(D)$ if $p > 1$. Then, we will use the bound for the corresponding discrepancy $\mathcal{D}_{\mathbf{w}}^{r,s}(\Xi_N)$ as an estimate for the quadrature error. Throughout this subsection, we assume that the sequence of integration points is given by the *Halton sequence*, cf. [Hal60].

(2.8) **Definition.** Let $b_1, \ldots, b_m$ denote the first m prime numbers. Then, the m-dimensional *Halton sequence* is given by

$$\boldsymbol{\xi}^i = [h_{b_1}(i), \ldots, h_{b_m}(i)]^\intercal, \quad i = 0, 1, 2, \ldots,$$

where $h_{b_j}(i)$ denotes the i-th element of the *van der Corput sequence* according to b_j. That is, if $i = \cdots c_3c_2c_1$ in radix b_j, then $h_{b_j}(i) = 0.c_1c_2c_3\cdots$ in radix b_j.

For the weighted discrepancy $\mathcal{D}_{\mathbf{w}}^{r,s}(\Xi_N)$ of the Halton sequence, we know the following result from [KSS12b].

(2.9) **Theorem.** Assume that the weights $\omega_{\boldsymbol{\alpha}}$ in (2.3) are product weights. This means that they are given by a sequence $\{w_k\}_k$ such that

$$\omega_{\boldsymbol{\alpha}} := \prod_{k=1}^{m} w_k^{\alpha_k}.$$

Moreover, let these weights satisfy

$$\sum_{k=1}^{\infty} w_k^\nu k \log k < \infty \tag{2.10}$$

for some $\nu \geq 1/2$. Let $\mathbf{w} = [w_1, \ldots, w_m]$ denote the first m elements of the sequence $\{w_k\}_k$. Then, the discrepancy $\mathcal{D}_{\mathbf{w}}^{r,s}(\Xi_N)$ of the first N points of the m-dimensional Halton sequence Ξ_N is bounded for all $r \geq 1$, $s \geq 2\nu$ and $\delta > 0$ by

$$\mathcal{D}_{\mathbf{w}}^{r,s}(\Xi_N) \lesssim N^{-\frac{1}{2\nu}+\delta}. \tag{2.11}$$

Here, the hidden constant depends on ν and δ but not on m and tends to infinity when $\delta \to 0$.

Proof. The result is shown for $\nu = 1/2$ in [Wan02] and extended in [KSS12a] for general $\nu > 1/2$. □

To establish convergence results for the moment computation, it remains to investigate the conditions under which the shifted powers $u_m^p(\mathbf{y} - \mathbf{1/2})$ of u_m belong to the Banach space $\mathcal{W}_{\mathbf{w}}^{r',s'}(X)$ for a weight vector $\mathbf{w}$ that satisfies the condition (2.10) of Theorem (2.9).[1] This condition is satisfied if the weights are of product form and fulfill

$$w_k \leq C_1 k^{\frac{-2-\eta}{\nu}} \quad \text{for } \eta > 0 \text{ and an arbitrary constant } C_1 > 0. \tag{2.12}$$

[1]The weight vector $\mathbf{w} \in \mathbb{R}^m$ contains the first m elements of the sequence $\{w_k\}_k$. It is obvious that condition (2.10) can be fulfilled for all fixed m and every weight vector $\mathbf{w} \in \mathbb{R}^m$ which is extended to a sequence $\{w_k\}_k$ by $w_k = 0$ for $k > m$. But the aim is that the bound in (2.10) holds independently of m and, hence, even for $m \to \infty$. Thus, whenever we refer to the summability condition of a m-dimensional vector in the sequel, it has to be understood in the sense that the bound holds even for $m \to \infty$ with a dimension-independent constant.

(2.13) **Theorem.** Let u_m be the solution of (III.3.3) and let $f \in L^p(D)$ for some integer $p \geq 2$. Furthermore, let the sequence $\{\gamma_k\}_k$ in (III.2.4) fulfill the algebraic decay assumption (III.4.8), i.e. $\gamma_k \lesssim k^{-s_1}$ with $s_1 > 2 + (1+\eta)/2\nu$ for $\nu \geq 1/2$ and arbitrary $\eta > 0$ from (2.12). Then, for all $\ell = 1, \ldots, p$, there is a weight vector $\mathbf{w}$ which satisfies (2.12) such that the shifted power $u_m^\ell(\mathbf{y} - \mathbf{1/2})$ is contained in $\mathcal{W}_{\mathbf{w}}^{r',s'}(X)$ for all $r' \geq 1$ and $s' = 2\nu/(2\nu - 1)$ with $X = H_0^1(D)$ if $\ell = 1$ and $X = W_0^{1,1}(D)$ if $\ell = 2, \ldots, p$. Moreover, the norm $\|u_m^\ell\|_{\mathcal{W}_{\mathbf{w}}^{r',s'}(X)}$ is bounded independent of m. This implies that the quasi-Monte Carlo quadrature based on Halton points for approximating the moments up to order p satisfies the following error estimate

$$(2.14) \qquad \|(\mathbf{I} - \mathbf{Q}_{\Xi_N})u_m^\ell\|_X \lesssim N^{-\frac{1}{2\nu}+\delta}$$

for arbitrary $\delta > 0$ from Theorem (2.9) with a constant which depends on δ and ν but which is independent of m.

Proof. We use the regularity results of Section III.5 to establish the connection between ν and the anisotropic behaviour of u_m and its powers. To this end, we consider a function $v\colon [0,1]^m \to X$ which satisfies

$$(2.15) \qquad \|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}} v(\mathbf{y})\|_X \lesssim |\boldsymbol{\alpha}|!\boldsymbol{\mu}^{\boldsymbol{\alpha}}.$$

Note that the estimate (2.15) is available for $v(\mathbf{x}, \mathbf{y}) = u_m^\ell(\mathbf{x}, \mathbf{y} - \mathbf{1/2})$ with a vector $\boldsymbol{\mu}$ determined by $\mu_k = C(\ell, D)\ell\gamma_k/\underline{a}$, see Corollary (III.6.3). Hence, due to the assumption on the decay of $\{\gamma_k\}_k$, there is for every fixed ℓ a constant $C_2 > 0$ such that $\mu_k \leq C_2 k^{-s_1}$.

In accordance with Theorem (2.9), there are results for the quasi-Monte Carlo quadrature based on Halton points for product weights available. Thus, we set $\tilde{\mu}_k = k\mu_k$ and obtain for all $\boldsymbol{\alpha}$ with $\|\boldsymbol{\alpha}\|_\infty \leq 1$ that

$$\|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}} v(\mathbf{y})\|_X \lesssim \tilde{\boldsymbol{\mu}}^{\boldsymbol{\alpha}} = \prod_{k=1}^m \tilde{\mu}_k^{\alpha_k}.$$

For such a function, the norm $\|v\|_{W_{\mathbf{w}}^{r',s'}(X)}$ fulfills the estimate

$$\|v\|_{W_{\mathbf{w}}^{r',s'}(X)} = \left(\sum_{\|\boldsymbol{\alpha}\|_\infty = 1} \|\omega_{\boldsymbol{\alpha}}^{-1/2} \partial_{\mathbf{y}}^{\boldsymbol{\alpha}} v(\mathbf{y}_{\boldsymbol{\alpha}}, \mathbf{1})\|_{L^{r'}([0,1]^m;X)}^{s'} \right)^{\frac{1}{s'}}$$
$$\lesssim \left(\sum_{\|\boldsymbol{\alpha}\|_\infty = 1} (\omega_{\boldsymbol{\alpha}}^{-1/2} \tilde{\boldsymbol{\mu}}^{\boldsymbol{\alpha}})^{s'} \right)^{\frac{1}{s'}}.$$

Using the product form of the weights $\omega_{\boldsymbol{\alpha}}$, the sum in the above expression is rewritten with $s' = 2\nu/(2\nu - 1)$ by

$$(2.16) \qquad \sum_{\|\boldsymbol{\alpha}\|_\infty = 1} (\omega_{\boldsymbol{\alpha}}^{-1/2} \tilde{\boldsymbol{\mu}}^{\boldsymbol{\alpha}})^{\frac{2\nu}{2\nu-1}} = \sum_{\|\boldsymbol{\alpha}\|_\infty = 1} \prod_{k=1}^m \left((w_k^{-1/2} \tilde{\mu}_k)^{\frac{2\nu}{2\nu-1}} \right)^{\alpha_k}.$$

Hence, we have to ensure that the right-hand side of (2.16) is bounded independently of $m \in \mathbb{N}$. A necessary condition to achieve this goal is that the sequence $\Big\{(w_k^{-1/2}\tilde{\mu}_k)^{\frac{2\nu}{2\nu-1}}\Big\}_k$ is summable. A sufficient condition is that the ℓ^1-norm of this sequence is smaller than 1.

Choosing the weights w_k in such a way that (2.12) is satisfied with equality and employing the decay properties of μ_k leads to

$$(2.17)\qquad \sum_{k=1}^{\infty}(w_k^{-1/2}\tilde{\mu}_k)^{\frac{2\nu}{2\nu-1}} = \left(\frac{C_2^2}{C_1}\right)^{\frac{\nu}{2\nu-1}} \sum_{k=1}^{\infty} k^{\frac{-2(s_1-1)\nu}{2\nu-1}} k^{\frac{2+\eta}{2\nu-1}}.$$

This is summable if

$$2\nu(s_1-1) - 2 - \eta > 2\nu - 1 \quad\Longleftrightarrow\quad s_1 > 2 + \frac{1+\eta}{2\nu} \quad \text{or} \quad \nu > \frac{1+\eta}{2(s_1-2)}.$$

Since $C_1 > 0$ is an arbitrary constant in (2.12), we can choose C_1 such that (2.17) is smaller than 1. Thus, we have established that $v(\mathbf{x},\mathbf{y}) = u_m^{\ell}(\mathbf{x},\mathbf{y}-\mathbf{1/2})$ belongs to $\mathcal{W}_{\mathbf{w}}^{r',s'}(X)$ for a weight vector $\mathbf{w}$ which satisfies (2.12) and that the norm $\|v\|_{\mathcal{W}_{\mathbf{w}}^{r',s'}(X)}$ can be bounded independently of m.

The error of the quasi-Monte Carlo method based on the Halton sequence to approximate (2.7) can, therefore, be estimated by the associated discrepancy $\mathcal{D}_{\mathbf{w}}^{r,s}(\Xi_N)$ which is bounded, according to Theorem (2.9), by $\mathcal{D}_{\mathbf{w}}^{r,s}(\Xi_N) \lesssim N^{-\frac{1}{2\nu}+\delta}$. □

2.3 Lognormal case: QMC quadrature with auxiliary density

In the lognormal case, the moments are given by the Bochner integral, see (III.3.15),

$$(2.18)\qquad (\mathcal{M}^p u_m)(\mathbf{x}) = \int_{\mathbb{R}^m} u_m^p(\mathbf{x},\mathbf{y}) \exp\left(-\frac{\|\mathbf{y}\|_2^2}{2}\right) \mathrm{d}\mathbf{y} = (\mathbf{I}u_m^p)(\mathbf{x}).$$

In order to obtain a quasi-Monte Carlo method for the domain of integration $\mathbb{R}^m$, one can map the sample points from $[0,1]^m$ to $\mathbb{R}^m$ by the inverse normal distribution function. Numerically, this can be done very efficiently by employing a rational interpolant of the inverse distribution function, cf. [Mor95].

At first, we will consider the integration of functions $f\colon \mathbf{\Gamma} \to \mathbb{R}$ which are defined on a general product domain $\mathbf{\Gamma} = \times_{k=1}^m \Gamma_k \subset \mathbb{R}^m$ equipped with general density functions $\boldsymbol{\rho}\colon \mathbf{\Gamma} \to \mathbb{R}$ of product form. Following the lines of [HSW04], we present a strategy for estimating the error of a quasi-Monte Carlo method in this situation. We will apply the results for the lognormal case. Nevertheless, due to the quite general representation, we observe as a by-product that the restriction $\mathbf{\Gamma} = [-1/2,1/2]^m$ and $\boldsymbol{\rho}(\mathbf{y}) \equiv 1$ from the uniformly elliptic case can be weakened.

With the density function $\boldsymbol{\rho} = \prod_{k=1}^m \rho_k$ and the domain $\mathbf{\Gamma} = \times_{k=1}^m \Gamma_k$ on hand, we define the associated distribution function $\mathbf{W} = [W_1,\ldots,W_m]\colon \mathbf{\Gamma} \to [0,1]^m$ by

$$W_k\colon \Gamma_k \to [0,1], \qquad y \mapsto \int_{\Gamma_k} \rho_k(y')\mathbb{1}_{y'\leq y}\,\mathrm{d}y'.$$

Furthermore, we denote the restriction of $\mathbf{W}$ to $\mathbf{\Gamma}_{\boldsymbol{\alpha}} := \{\mathbf{y}_{\boldsymbol{\alpha}} : \mathbf{y} \in \mathbf{\Gamma}\}$ by $\mathbf{W}_{\boldsymbol{\alpha}}$.

It has been shown in [HSW04] that the error of a quasi-Monte Carlo quadrature given by the quadrature points $\Xi_N = \{\boldsymbol{\xi}^1, \ldots, \boldsymbol{\xi}^N\} \subset \boldsymbol{\Gamma}$ for the integral

$$\int_{\boldsymbol{\Gamma}} f(\mathbf{y})\boldsymbol{\rho}(\mathbf{y})\,\mathrm{d}\mathbf{y}$$

can again be represented by a generalized version of the Zaremba-Hlawka identity. Therefore, we denote by $\{\hat{\boldsymbol{\xi}}^1, \ldots, \hat{\boldsymbol{\xi}}^N\} = \hat{\Xi}_N := \mathbf{W}(\Xi_n) \subset [0,1]^m$ the set of quadrature points which are mapped to the unit cube. Moreover, we define the *local same-quadrant discrepancy function* $\operatorname{discr}_{\hat{\Xi}_N}(\mathbf{z};\mathbf{d})$, anchored at $\mathbf{d} \in [0,1]^m$, by

$$\operatorname{discr}_{\hat{\Xi}_N}(\mathbf{z};\mathbf{d}) := (-1)^{|\boldsymbol{\alpha}-\boldsymbol{\delta}_{\mathbf{d}\leq\mathbf{z}}|}\bigg(\operatorname{Vol}\left(B(\mathbf{z};\mathbf{d})\right) - \frac{1}{N}\sum_{i=1}^{N} \mathbb{1}_{B(\mathbf{z};\mathbf{d})}(\hat{\boldsymbol{\xi}}^i)\bigg), \qquad \mathbf{z} \in [0,1]^m.$$

Herein, $B(\mathbf{z};\mathbf{d})$ denotes the box with vertices $\mathbf{z}$ and $\boldsymbol{\delta}_{\mathbf{d}\leq\mathbf{z}}$, where the latter vector is given by

$$(\boldsymbol{\delta}_{\mathbf{d}\leq\mathbf{z}})_k := \begin{cases} 1, & \text{if } d_k \leq z_k, \\ 0, & \text{if } d_k > z_k. \end{cases}$$

Then, for an arbitrary *anchor point* $\mathbf{c} \in \boldsymbol{\Gamma}$, the generalized version of the Zaremba-Hlawka identity reads

$$(2.19) \qquad (\mathbf{I} - \mathbf{Q}_{\Xi_N})f = \sum_{\|\boldsymbol{\alpha}\|_\infty = 1} \int_{\boldsymbol{\Gamma}_{\boldsymbol{\alpha}}} \partial_{\mathbf{y}}^{\boldsymbol{\alpha}} f(\mathbf{y}_{\boldsymbol{\alpha}}, \mathbf{c}) \operatorname{discr}_{(\hat{\Xi}_N)_{\boldsymbol{\alpha}}}(\mathbf{W}_{\boldsymbol{\alpha}}(\mathbf{y}_{\boldsymbol{\alpha}}); \mathbf{W}_{\boldsymbol{\alpha}}(\mathbf{c}_{\boldsymbol{\alpha}}))\,\mathrm{d}\mathbf{y}_{\boldsymbol{\alpha}}.$$

The identity (2.19) implies the error estimate, see [HSW04],

$$(2.20) \qquad |(\mathbf{I} - \mathbf{Q}_{\Xi_N})f| \leq \|f\|_{W^{1,1}_{\mathrm{mix},\mathbf{c}}(\boldsymbol{\Gamma})} \mathcal{D}_{\infty,\mathbf{W}(\mathbf{c})}(\hat{\Xi}_N),$$

where

$$\|f\|_{W^{1,1}_{\mathrm{mix},\mathbf{c}}(\boldsymbol{\Gamma})} := \sum_{\|\boldsymbol{\alpha}\|_\infty = 1} \int_{\boldsymbol{\Gamma}_{\boldsymbol{\alpha}}} |\partial_{\mathbf{y}}^{\boldsymbol{\alpha}} f(\mathbf{y}_{\boldsymbol{\alpha}}, \mathbf{c})|\,\mathrm{d}\mathbf{y}_{\boldsymbol{\alpha}}.$$

Moreover, $\mathcal{D}_{\infty,\mathbf{W}(\mathbf{c})}(\hat{\Xi}_N)$ is the L^∞*-same-quadrant discrepancy*, anchored at $\mathbf{W}(\mathbf{c})$, which is defined, cf. [HSW04], by

$$\mathcal{D}_{\infty,\mathbf{W}(\mathbf{c})}(\hat{\Xi}_N) := \sup_{|\boldsymbol{\alpha}|\leq 1} \sup_{\mathbf{y}_{\boldsymbol{\alpha}} \in [0,1]^{|\boldsymbol{\alpha}|}} \left|\operatorname{discr}_{\Xi_N}(\mathbf{y}_{\boldsymbol{\alpha}}; \mathbf{c})\right|.$$

Note that this discrepancy coincides with $\mathcal{D}^\star_\infty(\hat{\Xi}_N)$ if $\mathbf{W}(\mathbf{c}) = \mathbf{1}$ and describes the L^∞-*centered discrepancy* for $\mathbf{W}(\mathbf{c}) = \mathbf{1/2}$, see [Hic98] and Definition (2.40). The estimate (2.20) is obtained from (2.19) by the Hölder inequality in the same way as (2.3) is derived from (2.2) with $r = s = \infty$. Of course, the Hölder inequality for integrals and sums can be applied as in (2.3) with arbitrary dual exponents r, r' and s, s', leading to different discrepancies and norms in (2.20).

In complete analogy to (2.6), we can apply the generalized Zaremba-Hlawka identity (2.19) to functions $v \in C^{1,\mathrm{mix}}(\boldsymbol{\Gamma}; W_0^{1,q}(D))$ pointwise for almost all $\mathbf{x} \in D$. This leads to the estimate

$$(2.21) \qquad \|(\mathbf{I} - \mathbf{Q}_{\Xi_N})v\|_X \leq \|v\|_{W^{1,1}_{\mathrm{mix},\mathbf{c}}(\boldsymbol{\Gamma},X)} \mathcal{D}_{\infty,\mathbf{W}(\mathbf{c})}(\hat{\Xi}_N).$$

It is also possible to introduce weights in (2.19) to get an error estimate in terms of a weighted same-quadrant discrepancy if the integrand provides some anisotropic behaviour.

As mentioned before, we accomplished the identity (2.21) for general product domains and general density functions of product type to illustrate that we can easily handle the uniformly elliptic case with more general random parameters y_k which are distributed on a bounded interval Γ_k with respect to the density ρ_k.

Now, we turn our attention back to the lognormal case, i.e. the domain $\mathbf{\Gamma} = \mathbb{R}^m$ and the density function $\boldsymbol{\rho}(\mathbf{y}) = (2\pi)^{-m/2} \exp\big(-\|\mathbf{y}\|_2^2/2\big)$, and we will apply (2.21) with respect to the anchor point $\mathbf{c} = \mathbf{0}$. In particular, the univariate normal distribution function is given by

$$\Phi\colon \mathbb{R} \to (0,1) \quad \text{with} \quad \Phi(y) := \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y'^2}{2}\right) \mathrm{d}y'$$

and its inverse by

$$\Phi^{-1}\colon (0,1) \to \mathbb{R}.$$

The multivariate distribution function $\mathbf{\Phi}\colon \mathbb{R}^m \to [0,1]^m$ is simply defined by the coordinatewise application of the one-dimensional distribution functions, which means that $\mathbf{\Phi}(\mathbf{y}) := [\Phi(y_1), \ldots, \Phi(y_m)]^\intercal$.

In order to apply (2.21), the norm $\|v\|_{W^{1,1}_{\text{mix},\mathbf{c}}(\mathbb{R}^m;X)}$ needs to be bounded. Unfortunately, this condition is generally very restrictive and not fulfilled in our applications since the norm corresponds to the unweighted integration of $\|\partial^{\boldsymbol{\alpha}}_{\mathbf{y}} v(\mathbf{y})\|_X$ over $\mathbb{R}^m$ for multi-indices $\boldsymbol{\alpha}$ with $\|\boldsymbol{\alpha}\|_\infty \leq 1$. To overcome this obstruction, we follow the lines of [HSW04] and rewrite the integral $\mathbf{I}v(\mathbf{x})$ according to

$$\mathbf{I}v(\mathbf{x}) = \int_{\mathbb{R}^m} v(\mathbf{x},\mathbf{y})\boldsymbol{\rho}(\mathbf{y})\,\mathrm{d}\mathbf{y} = \overline{\rho} \int_{\mathbb{R}^m} v(\mathbf{x},\mathbf{y})\sqrt{\boldsymbol{\rho}(\mathbf{y})}\,\frac{\sqrt{\boldsymbol{\rho}(\mathbf{y})}}{\overline{\rho}}\,\mathrm{d}\mathbf{y}$$

with the scaling factor $\overline{\rho}$ being defined by $\overline{\rho} := \int_{\mathbb{R}^m} \sqrt{\boldsymbol{\rho}(\mathbf{y})}\,\mathrm{d}\mathbf{y}$. We now employ a quasi-Monte Carlo method with respect to the auxiliary density function $\sqrt{\boldsymbol{\rho}(\mathbf{y})}/\overline{\rho}$ and obtain, with respect to (2.21), the error estimate

$$\|(\mathbf{I} - \mathbf{Q}_{\Xi_N})v\|_X \lesssim \mathcal{D}_{\infty,\mathbf{1/2}}\big(\hat{\mathbf{\Phi}}(\Xi_N)\big)\|v\sqrt{\boldsymbol{\rho}}\|_{W^{1,1}_{\text{mix},\mathbf{0}}(\mathbb{R}^m;X)}.$$

Herein, we denote by $\hat{\mathbf{\Phi}}$ the distribution function according to the modified Gaussian density $\sqrt{\boldsymbol{\rho}(\mathbf{y})}/\overline{\rho}$. Moreover, the norm involved here is bounded in case of the moment computation as it is proven in the next theorem.

(2.22) **Theorem.** For the solution u_m to (III.3.5), the following bound is valid

$$\|u_m\sqrt{\boldsymbol{\rho}}\|_{W^{1,1}_{\text{mix},\mathbf{0}}(\mathbb{R}^m;H^1_0(D))} \lesssim \left(\sum_{\|\mathbf{q}\|_\infty \leq 1} \frac{1}{2^{|\mathbf{q}|}} \sum_{\boldsymbol{\alpha}\leq\mathbf{q}} \left(\frac{2\gamma}{\log 2}\right)^{\boldsymbol{\alpha}} |\boldsymbol{\alpha}|!\right)\|f\|_{L^2(D)} < \infty.$$

Furthermore, if $f \in L^p(D)$ for $p \geq 2$, it holds for the p-th power u_m^p of u_m that

$$\|u_m^p\sqrt{\boldsymbol{\rho}}\|_{W^{1,1}_{\text{mix},\mathbf{0}}(\mathbb{R}^m;W^{1,1}_0(D))} \lesssim \left(\sum_{\|\mathbf{q}\|_\infty \leq 1} \frac{1}{2^{|\mathbf{q}|}} \sum_{\boldsymbol{\alpha}\leq\mathbf{q}} \left(\frac{2C(p,D)p\gamma}{\log 2}\right)^{\boldsymbol{\alpha}} |\boldsymbol{\alpha}|!\right)\|f\|^p_{L^p(D)} < \infty.$$

Proof. Each summand in the expression

$$\|v\sqrt{\boldsymbol{\rho}}\|_{W^{1,1}_{\mathrm{mix},\mathbf{0}}(\mathbb{R}^m;X)} = \sum_{\|\boldsymbol{\alpha}\|_\infty=1} \int_{\mathbb{R}^{|\boldsymbol{\alpha}|}} \Big\|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}}\Big(v(\mathbf{y}_{\boldsymbol{\alpha}},\mathbf{0})\sqrt{\boldsymbol{\rho}(\mathbf{y}_{\boldsymbol{\alpha}},\mathbf{0})}\Big)\Big\|_X \,\mathrm{d}\mathbf{y}_{\boldsymbol{\alpha}}$$

can be estimated by

$$\begin{aligned}
&\int_{\mathbb{R}^{|\boldsymbol{\alpha}|}} \Big\|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}}\Big(v(\mathbf{y}_{\boldsymbol{\alpha}},\mathbf{0})\sqrt{\boldsymbol{\rho}(\mathbf{y}_{\boldsymbol{\alpha}},\mathbf{0})}\Big)\Big\|_X \,\mathrm{d}\mathbf{y}_{\boldsymbol{\alpha}}\\
&\qquad= \sum_{\boldsymbol{\beta}\le\boldsymbol{\alpha}} \frac{\boldsymbol{\alpha}!}{\boldsymbol{\beta}!(\boldsymbol{\alpha}-\boldsymbol{\beta})!} \int_{\mathbb{R}^{|\boldsymbol{\alpha}|}} \Big\|\partial_{\mathbf{y}}^{\boldsymbol{\beta}} v(\mathbf{y}_{\boldsymbol{\alpha}},\mathbf{0})\partial_{\mathbf{y}}^{\boldsymbol{\alpha}-\boldsymbol{\beta}}\sqrt{\boldsymbol{\rho}(\mathbf{y}_{\boldsymbol{\alpha}},\mathbf{0})}\Big\|_X \,\mathrm{d}\mathbf{y}_{\boldsymbol{\alpha}}.
\end{aligned}$$

Due to the product structure of the auxiliary density and since we consider only mixed first order derivatives, we find that

$$\partial_{\mathbf{y}}^{\boldsymbol{\alpha}-\boldsymbol{\beta}}\sqrt{\boldsymbol{\rho}(\mathbf{y}_{\boldsymbol{\alpha}},\mathbf{0})} = \frac{(-1)^{|\boldsymbol{\alpha}-\boldsymbol{\beta}|}}{2^{|\boldsymbol{\alpha}-\boldsymbol{\beta}|}}\sqrt{\boldsymbol{\rho}(\mathbf{y}_{\boldsymbol{\alpha}},\mathbf{0})}(\mathbf{y}_{\boldsymbol{\alpha}},\mathbf{0})^{\boldsymbol{\alpha}-\boldsymbol{\beta}}, \quad \text{and} \quad \boldsymbol{\alpha}! = \boldsymbol{\beta}! = (\boldsymbol{\alpha}-\boldsymbol{\beta})! = 1.$$

Hence, we arrive at

$$\begin{aligned}
&\int_{\mathbb{R}^{|\boldsymbol{\alpha}|}} \Big\|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}}\Big(v(\mathbf{y}_{\boldsymbol{\alpha}},\mathbf{0})\sqrt{\boldsymbol{\rho}(\mathbf{y}_{\boldsymbol{\alpha}},\mathbf{0})}\Big)\Big\|_X \,\mathrm{d}\mathbf{y}_{\boldsymbol{\alpha}}\\
&\qquad= \sum_{\boldsymbol{\beta}\le\boldsymbol{\alpha}} \frac{1}{2^{|\boldsymbol{\alpha}-\boldsymbol{\beta}|}} \int_{\mathbb{R}^{|\boldsymbol{\alpha}|}} \Big\|\partial_{\mathbf{y}}^{\boldsymbol{\beta}} v(\mathbf{y}_{\boldsymbol{\alpha}},\mathbf{0})\Big\|_X (\mathbf{y}_{\boldsymbol{\alpha}},\mathbf{0})^{\boldsymbol{\alpha}-\boldsymbol{\beta}}\sqrt{\boldsymbol{\rho}(\mathbf{y}_{\boldsymbol{\alpha}},\mathbf{0})} \,\mathrm{d}\mathbf{y}_{\boldsymbol{\alpha}}.
\end{aligned}$$

For all functions v whose mixed first order derivatives grow at most exponentially in $\|\mathbf{y}\|$, the norm $\|v\sqrt{\boldsymbol{\rho}}\|_{W^{1,1}_{\mathrm{mix},\mathbf{0}}(\mathbb{R}^m;X)}$ is bounded since $\sqrt{\boldsymbol{\rho}(\mathbf{y}_{\boldsymbol{\alpha}},\mathbf{0})} = \prod_{k=1}^m \big(\exp(-y_k^2/4)\big)^{\alpha_k}$ decays double exponentially in $\|\mathbf{y}_{\boldsymbol{\alpha}}\|$. Thus, the integrals on the right-hand side of this equation are all finite.

For the solution u_m, we obtain the first assertion with σ_s from the regularity result (III.5.29) as follows

$$\begin{aligned}
&\int_{\mathbb{R}^{|\boldsymbol{\alpha}|}} \Big\|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}}\Big(u_m(\mathbf{y}_{\boldsymbol{\alpha}},\mathbf{0})\sqrt{\boldsymbol{\rho}(\mathbf{y}_{\boldsymbol{\alpha}},\mathbf{0})}\Big)\Big\|_{H_0^1(D)} \,\mathrm{d}\mathbf{y}_{\boldsymbol{\alpha}}\\
&\quad\lesssim \sum_{\boldsymbol{\beta}\le\boldsymbol{\alpha}} \frac{|\boldsymbol{\beta}|!}{2^{|\boldsymbol{\alpha}-\boldsymbol{\beta}|}}\left(\frac{\boldsymbol{\gamma}}{\log 2}\right)^{\boldsymbol{\beta}} \|f\|_{L^2(D)} \int_{\mathbb{R}^{|\boldsymbol{\alpha}|}} (\mathbf{y}_{\boldsymbol{\alpha}},\mathbf{0})^{\boldsymbol{\alpha}-\boldsymbol{\beta}}\boldsymbol{\sigma_s}^{-1}(\mathbf{y}_{\boldsymbol{\alpha}},\mathbf{0})\sqrt{\boldsymbol{\rho}(\mathbf{y}_{\boldsymbol{\alpha}},\mathbf{0})}\,\mathrm{d}\mathbf{y}_{\boldsymbol{\alpha}}\\
&\quad\lesssim \sum_{\boldsymbol{\beta}\le\boldsymbol{\alpha}} \frac{|\boldsymbol{\beta}|!}{2^{|\boldsymbol{\alpha}-\boldsymbol{\beta}|}}\left(\frac{\boldsymbol{\gamma}}{\log 2}\right)^{\boldsymbol{\beta}} \|f\|_{L^2(D)}.
\end{aligned}$$

Note that the last step holds since $\int_{\mathbb{R}^{|\boldsymbol{\alpha}|}} (\mathbf{y}_{\boldsymbol{\alpha}},\mathbf{0})^{\boldsymbol{\alpha}-\boldsymbol{\beta}}\boldsymbol{\sigma_s}^{-\ell}(\mathbf{y}_{\boldsymbol{\alpha}},\mathbf{0})\sqrt{\boldsymbol{\rho}(\mathbf{y}_{\boldsymbol{\alpha}},\mathbf{0})}\,\mathrm{d}\mathbf{y}_{\boldsymbol{\alpha}} < \infty$ for all $\ell\in\mathbb{N}$.

For the p-th power u_m^p of the solution, the assertion follows analogously using (III.5.30) for $p=2$ and (III.5.32) for $p>2$. The constant $C(p,D)$ from (III.5.32) is equal to 1 for $p = 2$. By setting $v = u_m^p$, we thus arrive at

$$\begin{aligned}
&\int_{\mathbb{R}^{|\boldsymbol{\alpha}|}} \left\| \partial_{\mathbf{y}}^{\boldsymbol{\alpha}} \left(u_m^p(\mathbf{y}_{\boldsymbol{\alpha}}, \mathbf{0}) \sqrt{\boldsymbol{\rho}(\mathbf{y}_{\boldsymbol{\alpha}}, \mathbf{0})} \right) \right\|_{W_0^{1,1}(D)} \mathrm{d}\mathbf{y}_{\boldsymbol{\alpha}} \\
&\lesssim \sum_{\boldsymbol{\beta} \leq \boldsymbol{\alpha}} \frac{|\boldsymbol{\beta}|!}{2^{|\boldsymbol{\alpha}-\boldsymbol{\beta}|}} \left(\frac{C(p,D)p\boldsymbol{\gamma}}{\log 2} \right)^{\boldsymbol{\beta}} \|f\|_{L^p(D)}^p \int_{\mathbb{R}^{|\boldsymbol{\alpha}|}} (\mathbf{y}_{\boldsymbol{\alpha}}, \mathbf{0})^{\boldsymbol{\alpha}-\boldsymbol{\beta}} \boldsymbol{\sigma}_{\boldsymbol{s}}{}^{-p}(\mathbf{y}_{\boldsymbol{\alpha}}, \mathbf{0}) \sqrt{\boldsymbol{\rho}(\mathbf{y}_{\boldsymbol{\alpha}}, \mathbf{0})} \, \mathrm{d}\mathbf{y}_{\boldsymbol{\alpha}} \\
&\lesssim \sum_{\boldsymbol{\beta} \leq \boldsymbol{\alpha}} \frac{|\boldsymbol{\beta}|!}{2^{|\boldsymbol{\alpha}-\boldsymbol{\beta}|}} \left(\frac{C(p,D)p\boldsymbol{\gamma}}{\log 2} \right)^{\boldsymbol{\beta}} \|f\|_{L^p(D)}^p.
\end{aligned}$$

This implies the second assertion. □

(2.23) **Remark.** The presented approach in this subsection is well suited for lognormal distributed diffusion coefficients which depend on a small number of random parameters. Nevertheless, there arise the following problems for a large dimensionality m.

(a) Unfortunately, the introduction of the auxiliary density function does not preserve the structure of the anisotropy in the derivatives or at least we are not able to prove this with the analysis presented above. Thus, a weighted version of the Koksma-Hlawka inequality like in the uniformly elliptic case is not applicable and so we cannot make use of Theorem (2.9). Hence, the quadrature error cannot be bounded by the weighted discrepancy of the point sequence $\hat{\boldsymbol{\Phi}}(\Xi_N)$ and, thus, the convergence rate generally deteriorates for classical low discrepancy sequences when m gets large.

(b) It is proven in [HSW04] that in $[0,1]^m$ there exist point sets of cardinality N for all $N \in \mathbb{N}$ such that the (unweighted) L^∞-same-quadrant discrepancy is bounded by $\mathcal{O}(\sqrt{m/N})$ and, hence, that the integration problem is polynomially tractable, see Remark (0.5). Nevertheless, this proof is non-constructive. Moreover, there are constructions of N sample points whose star discrepancy is nearly as good as the non-constructive proven one, cf. [NW08], but the computation of these points has exponential runtime in m. To find point sets with polynomially bounded (in m and N) L^∞-same-quadrant discrepancy which are in addition constructable in polynomial time in N and m is still an open problem, cf. [NW08].

2.4 Lognormal case: QMC quadrature based on Halton points

Next, we propose another approach which is available for the quasi-Monte Carlo method based on the Halton sequence. Hence, we assume as in Subsection 2.2 that the set of quadrature points Ξ_N is given by the Halton sequence (2.8). Instead of introducing an auxiliary density function, we simply transform the integral onto the unit cube $[0,1]^m$. The integrand may then tend to infinity around a region of the boundary. But, according to [Owe06], the first N points of the Halton sequence provide the interesting feature that they avoid the region around the boundary of the unit cube. Thus, this method takes only into account the behaviour of the integrand inside a certain set $K_N \subset [0,1]^m$, which contains these first N points, but introduces a truncation error. We will show, under certain anisotropy conditions, that this truncation error can be estimated nearly independently of the dimension m and that the anisotropic behaviour of the integrand is preserved inside K_N up to a certain extent. This will then prove that the quasi-Monte Carlo quadrature based on Halton points converges nearly dimension-independent for the computations of the moments of the solution u_m to (III.3.5).

In the sequel, we will focus our analysis on the quadrature of the expectation of the solution u_m to (III.3.5). Analogous results can be established in the same way for the computations of the moments, see Corollary (2.47), when applying the corresponding regularity results for the powers of u_m.

We transform the Bochner integral (2.18) onto the unit cube $[0,1]^m$. It is well known that it holds

$$\int_{\mathbb{R}} f(y)\rho(y)\,\mathrm{d}y = \int_{\mathbb{R}} f(y)\frac{1}{\sqrt{2\pi}}\exp\Big(-\frac{y^2}{2}\Big)\,\mathrm{d}y = \int_0^1 f\big(\Phi^{-1}(z)\big)\,\mathrm{d}z$$

for a function $f \in L^1_\rho(\mathbb{R})$ due to the substitution $z = \Phi(y)$. With the definition of $\mathbf{\Phi}$ at hand, we can extend the above integral transform to the multivariate case, i.e. $f \in L^1_{\boldsymbol{\rho}}(\mathbb{R}^m)$ and

$$\int_{\mathbb{R}^m} f(\mathbf{y})\rho(\mathbf{y})\,\mathrm{d}\mathbf{y} = \int_{(0,1)^m} f\big(\mathbf{\Phi}^{-1}(\mathbf{z})\big)\,\mathrm{d}\mathbf{z}.$$

Although we have $f\circ\mathbf{\Phi}^{-1} \in L^1\big((0,1)^m\big)$, the integrand might be unbounded in a neighbourhood of the hypercube's boundary in our application since the diffusion coefficient may tend to zero. This implies that the variation in the sense of Hardy and Krause might be unbounded, too. As a consequence, the Koksma-Hlawka inequality (2.1) or a weighted version of this estimate is not applicable. Due to the definition of the Halton sequence, cf. (2.8), the first N points $\Xi_N = \boldsymbol{\xi}^1,\dots,\boldsymbol{\xi}^N$ of this sequence are included in the cuboid

$$K_N := \bigtimes_{k=1}^m [(b_kN)^{-1}, 1-(b_kN)^{-1}].$$

Let now $\hat{u}_m(\mathbf{x},\mathbf{z}) := u_m(\mathbf{x},\mathbf{\Phi}^{-1}(\mathbf{z}))$. For $\mathbf{z} \in (0,1)^m \setminus K_N$ and almost every $\mathbf{x} \in D$, we replace $\hat{u}_m$ by its *low-variation extension* $\hat{u}_{m,\mathrm{ext}}$, cf. [Owe06],

(2.24) $$\hat{u}_{m,\mathrm{ext}}(\mathbf{x},\mathbf{z}) := \hat{u}_m(\mathbf{x},\mathbf{c}) + \sum_{\|\boldsymbol{\alpha}\|_\infty=1}\int_{[\mathbf{c}_{\boldsymbol{\alpha}},\mathbf{z}_{\boldsymbol{\alpha}}]} \mathbb{1}_{(\mathbf{y}_{\boldsymbol{\alpha}},\mathbf{c})\in K_N}\partial^{\boldsymbol{\alpha}}_{\mathbf{y}}\hat{u}_m(\mathbf{x},\mathbf{y}_{\boldsymbol{\alpha}},\mathbf{c})\,\mathrm{d}\mathbf{y}_{\boldsymbol{\alpha}}.$$

For a given anchor point $\mathbf{c} \in K_N$, the extension coincides by definition (2.24) with the function $\hat{u}_m$ on K_N, i.e. $\hat{u}_{\mathrm{m,ext}}(\mathbf{x},\mathbf{z}) = \hat{u}_m(\mathbf{x},\mathbf{z})$ for all $\mathbf{z} \in K_N$ and almost all $\mathbf{x} \in D$. We are now ready to prove the main result of this chapter.

(2.25) **Theorem.** Let the sequence $\{\gamma_k\}_k$ satisfy the decay property $\gamma_k \lesssim k^{-4-2\eta}$ for arbitrary $\eta > 0$. Then, the convergence of the quasi-Monte Carlo quadrature using Halton points for approximating the expectation of the solution u_m to (III.3.5) depends at most linearly on the dimensionality m and is algebraic in the number of quadrature points. More precisely, there exists for each $\delta > 0$ a sequence $\{\delta_k\}_k \in \ell^1(\mathbb{N})$ with $\delta_k \simeq k^{-1-\eta}$ and a $\tilde{\delta} > 0$ with $\tilde{\delta} + \sum_{k=1}^\infty \delta_k < \delta$ such that the error of the quasi-Monte Carlo quadrature with N Halton points satisfies

(2.26) $$\begin{aligned}\|(\mathbf{I}-\mathbf{Q}_{\Xi_N})\hat{u}_m\|_{H^1_0(D)} &\lesssim \|f\|_{L^2(D)}\big(mN^{-1+\|\boldsymbol{\delta}\|_\infty} + N^{-1+\tilde{\delta}+|\boldsymbol{\delta}|}\big)\\ &\leq \|f\|_{L^2(D)}(m+1)N^{-1+\delta}.\end{aligned}$$

Herein, the vector $\boldsymbol{\delta} \in \mathbb{R}^m$ is given by $\boldsymbol{\delta} = [\delta_1, \ldots, \delta_m]^\intercal$. The constant hidden in the above inequality depends on the sequence $\{\delta_k\}_k$, on $\tilde{\delta}$ and on δ, but is independent of m.

The proof of this theorem is performed by splitting the error of integration into three parts. Namely, with respect to the extension $\hat{u}_{\mathrm{m,ext}}$, we write

$$\begin{aligned} &\|(\mathbf{I} - \mathbf{Q}_{\Xi_N})\hat{u}_m\|_{H_0^1(D)} \\ &\quad \leqslant \|\mathbf{I}(\hat{u}_m - \hat{u}_{m,\mathrm{ext}})\|_{H_0^1(D)} + \|\mathbf{Q}_{\Xi_N}(\hat{u}_m - \hat{u}_{m,\mathrm{ext}})\|_{H_0^1(D)} + \|(\mathbf{I} - \mathbf{Q}_{\Xi_N})\hat{u}_{m,\mathrm{ext}}\|_{H_0^1(D)}. \end{aligned} \tag{2.27}$$

Due to $\hat{u}_m\big|_{K_N} = \hat{u}_{m,\mathrm{ext}}\big|_{K_N}$, the second term on the right-hand side of (2.27) vanishes. The first term on the right-hand side of (2.27), which corresponds to the truncation error of the quasi-Monte Carlo quadrature based on the Halton sequence, will be estimated by Lemma (2.28). Finally, the third term on the right-hand side of (2.27), which reflects the integration error inside K_N, will be estimated in Lemma (2.41).

(2.28) **Lemma.** Let the conditions of Theorem (2.25) be satisfied and let $\hat{u}_{m,\mathrm{ext}}$ be defined according to (2.24). Then, it holds

$$\|\mathbf{I}(\hat{u}_m - \hat{u}_{m,\mathrm{ext}})\|_{H_0^1(D)} \lesssim \|f\|_{L^2(D)} N^{-1+\|\boldsymbol{\delta}\|_\infty} m. \tag{2.29}$$

Proof. We organize the proof in four steps.

($i.$) On the one hand, from [Fan13], we know that

$$\Phi^{-1}(z) < \sqrt{-\log\left(2\pi(1-z)^2(1-\log(2\pi(1-z)^2))\right)} \quad \text{for all } z \in [0.9, 1].$$

Furthermore, we have from [PR96] that

$$\begin{aligned} \Phi^{-1}(z) \leq{} & \sqrt{-2\log(1-z)} \\ & - \frac{2.30753 + 0.27061\sqrt{-2\log(1-z)}}{1 + 0.99229\sqrt{-2\log(1-z)} - 0.08962\log(1-z)} + 0.003 \end{aligned}$$

for all $z \in [0.5, 1]$. These inequalities imply that

$$\Phi^{-1}(z) \leq \sqrt{-2\log(1-z)} \quad \text{for all } z \in [0.5, 1].$$

Due to the symmetry of the distribution, this shows that

$$|\Phi^{-1}(z)| \leq \sqrt{-2\log(\min\{z, 1-z\})} \quad \text{for all } z \in [0, 1].$$

The derivative of the distribution function Φ is the Gaussian density function. Hence, the derivative of its inverse can easily be determined. We derive that

$$\frac{\mathrm{d}}{\mathrm{d}z}\Phi^{-1}(z) = \sqrt{2\pi}\exp\left(\frac{\Phi^{-1}(z)^2}{2}\right) \leq \sqrt{2\pi}\min\{z, 1-z\}^{-1},$$

which implies the estimate

$$\left|\prod_{k=1}^m \left(\frac{\mathrm{d}}{\mathrm{d}z_k}\Phi^{-1}(z_k)\right)^{\alpha_k}\right| \leq \prod_{k=1}^m \left(\sqrt{2\pi}\min\{z_k, 1-z_k\}^{-1}\right)^{\alpha_k}$$

for all non-negative integers α_k.

(*ii.*) On the other hand, one verifies

$$\exp\left(\gamma_k|\Phi^{-1}(z)|\right) \le C(\delta_k,\gamma_k)\min\{z,1-z\}^{-\delta_k}$$

for all $\delta_k > 0$ with the constant

$$C(\delta_k,\gamma_k) = \begin{cases} \exp\left(\frac{\gamma_k^2}{2\delta_k}\right), & \text{if } \delta_k \le \frac{\gamma_k}{\sqrt{2\log 2}},\\ \frac{\exp(\sqrt{2\log 2}\gamma_k)}{\exp(\delta_k \log 2)}, & \text{else.}\end{cases}$$

Hence, we find by the definition of $\overline{a}_m$ and $\underline{a}_m$ that

$$\begin{aligned}\sqrt{\frac{\overline{a}_m(\mathbf{\Phi}^{-1}(\mathbf{z}))}{\underline{a}_m(\mathbf{\Phi}^{-1}(\mathbf{z}))^3}} &= \exp\left(\sum_{k=1}^m 2\gamma_k|\Phi^{-1}(z_k)|\right)\\ &\le \prod_{k=1}^m \Big(C(\delta_k,2\gamma_k)\min\{z_k,1-z_k\}^{-\delta_k}\Big).\end{aligned}$$

Consequently, with Lemma (III.5.3) and the stability estimate (III.3.11), we deduce for any multi-index $\boldsymbol{\alpha}$ that

$$\begin{aligned}&\|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}} u_m(\mathbf{\Phi}^{-1}(\mathbf{z}))\|_{H_0^1(D)}\\ &\leqslant |\boldsymbol{\alpha}|!\Big(\frac{\boldsymbol{\gamma}}{\log 2}\Big)^{\boldsymbol{\alpha}}\sqrt{\frac{\overline{a}_m(\mathbf{\Phi}^{-1}(\mathbf{z}))}{\underline{a}_m(\mathbf{\Phi}^{-1}(\mathbf{z}))}}\|u_m(\mathbf{\Phi}^{-1}(\mathbf{z}))\|_{H_0^1(D)}\\ &\lesssim |\boldsymbol{\alpha}|!\Big(\frac{\boldsymbol{\gamma}}{\log 2}\Big)^{\boldsymbol{\alpha}}\sqrt{\frac{\overline{a}_m(\mathbf{\Phi}^{-1}(\mathbf{z}))}{\underline{a}_m(\mathbf{\Phi}^{-1}(\mathbf{z}))^3}}\|f\|_{L^2(D)}\\ &\le \|f\|_{L^2(D)}|\boldsymbol{\alpha}|!\Big(\frac{\boldsymbol{\gamma}}{\log 2}\Big)^{\boldsymbol{\alpha}}\prod_{k=1}^m\Big(C(\delta_k,2\gamma_k)\min\{z_k,1-z_k\}^{-\delta_k}\Big).\end{aligned}$$

(*iii.*) For an arbitrary multi-index $\boldsymbol{\alpha}$, it holds for all $\mathbf{z}\in(0,1)^m$ that

$$\begin{aligned}\|\partial_{\mathbf{z}}^{\boldsymbol{\alpha}}\hat{u}_m(\mathbf{z})\|_{H_0^1(D)} &= \|\partial_{\mathbf{z}}^{\boldsymbol{\alpha}} u_m(\mathbf{\Phi}^{-1}(\mathbf{z}))\|_{H_0^1(D)}\\ &= \left\|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}} u_m(\mathbf{\Phi}^{-1}(\mathbf{z}))\prod_{k=1}^m\left(\frac{\mathrm{d}}{\mathrm{d}z_k}\Phi^{-1}(z_k)\right)^{\alpha_k}\right\|_{H_0^1(D)}\\ &= \left|\prod_{k=1}^m\left(\frac{\mathrm{d}}{\mathrm{d}z_k}\Phi^{-1}(z_k)\right)^{\alpha_k}\right|\|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}} u_m(\mathbf{\Phi}^{-1}(\mathbf{z}))\|_{H_0^1(D)}.\end{aligned} \tag{2.30}$$

From now on, we choose the anchor point $\mathbf{c} = \mathbf{1/2}$ and define

$$\tilde{C} := \frac{\sqrt{2\pi}\max_{k\in\mathbb{N}} C(\delta_k,2\gamma_k)}{\log 2}. \tag{2.31}$$

Note that $\tilde{C} < \infty$ since there is a $k_0 \in \mathbb{N}$ such that $C(\delta_k, 2\gamma_k) \leq 1$ for all $k \geq k_0$ under the decay assumptions on the sequences $\{\delta_k\}_k$ and $\{\gamma_k\}_k$. Due to $\Phi^{-1}(1/2) = 0$, we easily get from item (*ii.*) that

$$\begin{aligned}&\|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}} u_m(\boldsymbol{\Phi}^{-1}(\mathbf{z}_{\boldsymbol{\alpha}}, \mathbf{c}))\|_{H_0^1(D)} \\ &\quad\lesssim \|f\|_{L^2(D)} |\boldsymbol{\alpha}|! \Big(\frac{\boldsymbol{\gamma}}{\log 2}\Big)^{\boldsymbol{\alpha}} \prod_{k=1}^{m} \Big(C(\delta_k, 2\gamma_k) \min\{z_k, 1 - z_k\}^{-\delta_k}\Big)^{\alpha_k}\end{aligned} \tag{2.32}$$

holds for all $\boldsymbol{\alpha}$ with $\|\boldsymbol{\alpha}\|_\infty = 1$. Thus, by combining (2.30) with item (*i.*) and inequality (2.32), we arrive at the estimate

$$\|\partial_{\mathbf{z}}^{\boldsymbol{\alpha}} \hat{u}_m(\mathbf{z}_{\boldsymbol{\alpha}}, \mathbf{c})\|_{H_0^1(D)} \lesssim |\boldsymbol{\alpha}|! \|f\|_{L^2(D)} \prod_{k=1}^{m} \Big(\gamma_k \tilde{C} \min\{z_k, 1 - z_k\}^{-1-\delta_k}\Big)^{\alpha_k}. \tag{2.33}$$

From (2.24), we infer the identity

$$\hat{u}_m(\mathbf{x}, \mathbf{z}) - \hat{u}_{m,\text{ext}}(\mathbf{x}, \mathbf{z}) = \sum_{\|\boldsymbol{\alpha}\|_\infty = 1} \int_{[\mathbf{c}_{\boldsymbol{\alpha}}, \mathbf{z}_{\boldsymbol{\alpha}}]} \mathbb{1}_{(\mathbf{y}_{\boldsymbol{\alpha}}, \mathbf{c}) \notin K_N} \partial_{\mathbf{y}}^{\boldsymbol{\alpha}} \hat{u}_m(\mathbf{x}, \mathbf{y}_{\boldsymbol{\alpha}}, \mathbf{c}) \, \mathrm{d}\mathbf{y}_{\boldsymbol{\alpha}}.$$

This, together with the estimate (2.33) on the derivates of $\hat{u}_m$ yields for every $\mathbf{z} \notin K_N$, cf. [Owe06], that

$$\begin{aligned}&\|\hat{u}_m(\mathbf{z}) - \hat{u}_{m,\text{ext}}(\mathbf{z})\|_{H_0^1(D)} \\ &\quad\leq \sum_{\|\boldsymbol{\alpha}\|_\infty = 1} \int_{[\mathbf{c}_{\boldsymbol{\alpha}}, \mathbf{z}_{\boldsymbol{\alpha}}]} \mathbb{1}_{(\mathbf{y}_{\boldsymbol{\alpha}}, \mathbf{c}) \notin K_N} \|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}} \hat{u}_m(\mathbf{y}_{\boldsymbol{\alpha}}, \mathbf{c})\|_{H_0^1(D)} \, \mathrm{d}\mathbf{y}_{\boldsymbol{\alpha}} \\ &\quad\lesssim \|f\|_{L^2(D)} \sum_{\|\boldsymbol{\alpha}\|_\infty = 1} |\boldsymbol{\alpha}|! \prod_{k=1}^{m} (\gamma_k \tilde{C})^{\alpha_k} \\ &\qquad\qquad \cdot \int_{[\mathbf{c}_{\boldsymbol{\alpha}}, \mathbf{z}_{\boldsymbol{\alpha}}]} \mathbb{1}_{(\mathbf{y}_{\boldsymbol{\alpha}}, \mathbf{c}) \notin K_N} \prod_{k=1}^{m} \Big(\min\{y_k, 1 - y_k\}^{-1-\delta_k}\Big)^{\alpha_k} \mathrm{d}\mathbf{y}_{\boldsymbol{\alpha}} \\ &\quad\leq \|f\|_{L^2(D)} \sum_{\|\boldsymbol{\alpha}\|_\infty = 1} |\boldsymbol{\alpha}|! \prod_{k=1}^{m} \Big(\gamma_k \tilde{C} \int_{\min\{z_k, 1-z_k\}}^{1/2} y_k^{-1-\delta_k} \, \mathrm{d}y_k\Big)^{\alpha_k}.\end{aligned}$$

Herein, the integral can simply be bounded via its lower limit according to

$$\begin{aligned}&\|\hat{u}_m(\mathbf{z}) - \hat{u}_{m,\text{ext}}(\mathbf{z})\|_{H_0^1(D)} \\ &\quad\lesssim \|f\|_{L^2(D)} \sum_{\|\boldsymbol{\alpha}\|_\infty = 1} |\boldsymbol{\alpha}|! \prod_{k=1}^{m} \Big(\gamma_k \tilde{C} \min\{z_k, 1 - z_k\}^{-\delta_k}\Big)^{\alpha_k} \\ &\quad\leq \|f\|_{L^2(D)} \sum_{\|\boldsymbol{\alpha}\|_\infty = 1} \prod_{k=1}^{m} \Big(k \gamma_k \tilde{C} \min\{z_k, 1 - z_k\}^{-\delta_k}\Big)^{\alpha_k} \\ &\quad= \|f\|_{L^2(D)} \bigg(\prod_{k=1}^{m} \Big(1 + \frac{\min\{z_k, 1 - z_k\}^{-\delta_k} k \gamma_k \tilde{C}}{\delta_k}\Big) - 1\bigg) \\ &\quad\leq \|f\|_{L^2(D)} \prod_{k=1}^{m} \Big(1 + \frac{k \gamma_k \tilde{C}}{\delta_k}\Big) \min\{z_k, 1 - z_k\}^{-\delta_k}.\end{aligned} \tag{2.34}$$

Next, due to Bochner's inequality (II.3.6) and due to the fact that $\hat{u}_m$ coincides with $\hat{u}_{m,\text{ext}}$ in K_N, it follows that

$$\begin{aligned}\|\mathbf{I}(\hat{u}_m - \hat{u}_{m,\text{ext}})\|_{H_0^1(D)} &\leqslant \int_{(0,1)^m} \|\hat{u}_m(\mathbf{z}) - \hat{u}_{\text{ext}}(\mathbf{z})\|_{H_0^1(D)} \,\mathrm{d}\mathbf{z}\\ &= \int_{(0,1)^m\setminus K_N} \|\hat{u}_m(\mathbf{z}) - \hat{u}_{m,\text{ext}}(\mathbf{z})\|_{H_0^1(D)} \,\mathrm{d}\mathbf{z}.\end{aligned}$$

In view of the estimate (2.34), we conclude that

$$\begin{aligned}&\|\mathbf{I}(\hat{u}_m - \hat{u}_{m,\text{ext}})\|_{H_0^1(D)}\\ &\lesssim \|f\|_{L^2(D)} \int_{(0,1)^m\setminus K_N} \prod_{k=1}^m \min\{z_k, 1-z_k\}^{-\delta_k} \,\mathrm{d}\mathbf{z} \prod_{k=1}^m \left(1 + \frac{k\gamma_k\tilde{C}}{\delta_k}\right)\\ &\leq \|f\|_{L^2(D)} 2^m \sum_{j=1}^m \int_0^{(b_jN)^{-1}} z_j^{-\delta_j} \,\mathrm{d}z_j \prod_{\substack{i=1\\ i\neq j}}^m \int_0^{1/2} z_i^{-\delta_i} \,\mathrm{d}z_i \prod_{k=1}^m \left(1 + \frac{k\gamma_k\tilde{C}}{\delta_k}\right)\\ &\leq \|f\|_{L^2(D)} 2^m \sum_{j=1}^m (b_jN)^{\delta_j - 1} 2^{-m+1} 2^{|\boldsymbol{\delta}|} \prod_{k=1}^m \left[\left(1 + \frac{k\gamma_k\tilde{C}}{\delta_k}\right)\left(\frac{1}{1-\delta_k}\right)\right]\\ &\lesssim \|f\|_{L^2(D)} N^{\|\boldsymbol{\delta}\|_\infty - 1} m \prod_{k=1}^m \left[\left(1 + \frac{k\gamma_k\tilde{C}}{\delta_k}\right)\left(\frac{1}{1-\delta_k}\right) 2^{\delta_k}\right].\end{aligned}$$

(*iv.*) It remains to prove that the appearing constants are bounded independently of the dimensionality m. Therefore, it is sufficient to show that

$$\prod_{k=1}^\infty \left(1 + \frac{k\gamma_k\tilde{C}}{\delta_k}\right)\left(\frac{1}{1-\delta_k}\right) 2^{\delta_k} < \infty. \tag{2.35}$$

Since we may choose $\delta_k > 0$ arbitrarily, we can assume that the sequence $\{\delta_k\}$ satisfies the conditions of Theorem (2.25). Then, it holds

$$\prod_{k=1}^\infty 2^{\delta_k} = 2^{\sum_{k=1}^\infty \delta_k} \leq 2^\delta \quad \text{and} \quad \prod_{k=1}^\infty \frac{1}{1-\delta_k} = \exp\left(-\sum_{k=1}^\infty \log(1-\delta_k)\right). \tag{2.36}$$

We make use of the fact that the Taylor expansion of the logarithm $\log(x)$ at $x = 1$ is given by

$$\log(1-h) = -\sum_{k=1}^\infty \frac{h^k}{k} = -h - \mathcal{O}(h^2), \quad h > 0.$$

By inserting this into the equation on the right of (2.36), we obtain

$$\prod_{k=1}^\infty \frac{1}{1-\delta_k} \leq \exp\left(\sum_{k=1}^\infty (\delta_k + \mathcal{O}(\delta_k^2))\right) \lesssim \exp(\delta + c\delta^2) \tag{2.37}$$

for some $c > 0$. Since the sequence $\{\gamma_k\}_k$ decays asymptotically faster than $k^{-4-2\eta}$, it follows that

$$(2.38)\qquad \prod_{k=1}^{\infty}\left(1+\frac{\tilde{C}k\gamma_k}{\delta_k}\right) \lesssim \prod_{k=1}^{\infty}\left(1+\tilde{c}k^{-2-\eta}\right) < \infty$$

for some $\tilde{c} > 0$. This establishes estimate (2.35) and, thus, finally the assertion (2.29). □

(2.39) **Remark.** Notice that the last estimate of item (*iii.*) is quite rough. In fact, if we sum up $\sum_{j=1}^{m} b_j^{\delta_j-1}$, we end up with a factor $\log(m)$ or even $\log(\log(m))$, cf. (V.3.46), than a factor m since $b_k \eqsim k\log k$. Moreover, for this lemma, the weaker decay condition $\{\gamma_k\}_k \lesssim k^{-3-2\eta}$ is sufficient. This can easily be seen from equation (2.38) and the definition of the constant $\tilde{C}$, cf. (2.31). These are the only parts in the proof of Lemma (2.28) where the decay properties of $\{\gamma_k\}_k$ enter. Especially, they remain valid under the weaker assumption $\{\gamma_k\}_k \lesssim k^{-3-2\eta}$.

Lastly, we bound the third term in (2.27). Therefore, we apply the estimate (2.20) on $[0,1]^m$ with the L^∞-centered discrepancy. We will bound this L^∞-centered discrepancy roughly by the *extreme discrepancy*. The L^∞-centered discrepancy can be represented in a more explicit form than the L^∞-same-quadrant discrepancy and is defined as follows, see [HW02].

(2.40) **Definition.** The *local centered discrepancy* function is defined for a given set of N sample points $\Xi_N \subset [0,1]^m$ as $\mathrm{discr}_c(\Xi_N)\colon [0,1]^m \to \mathbb{R}$,

$$\mathrm{discr}_c(\mathbf{z},\Xi_N) := \prod_{k=1}^{m}\Big(-z_k + \mathbb{1}_{\{z_k>1/2\}}\Big) - \frac{1}{N}\sum_{i=1}^{N}\prod_{k=1}^{m}\Big(\mathbb{1}_{\{z_k>1/2\}} - \mathbb{1}_{\{z_k>\xi_k^i\}}\Big).$$

Then, the L^∞*-centered discrepancy* is given by

$$\mathcal{D}_c(\Xi_N) := \sup_{\mathbf{z}\in[0,1]^m}\big|\mathrm{discr}_c(\mathbf{z},\Xi_N)\big|.$$

Furthermore, we introduce the *extreme discrepancy* by

$$\mathcal{D}_{\mathrm{extr}}(\Xi_N) := \sup_{\mathbf{x},\mathbf{y}\in[0,1]^m}\left|\mathrm{Vol}\left([\mathbf{x},\mathbf{y})\right) - \frac{1}{N}\sum_{i=1}^{N}\mathbb{1}_{[\mathbf{x},\mathbf{y})}(\boldsymbol{\xi}^i)\right|.$$

It follows directly from the definition that the L^∞-centered discrepancy can be bounded by the extreme discrepancy. This fact can be used to estimate the third term on the right-hand side of (2.27).

(2.41) **Lemma.** Let the conditions of Theorem (2.25) be satisfied and let $\hat{u}_{\mathrm{m,ext}}$ be defined by (2.24). Then, it holds

$$(2.42)\qquad \big\|(\mathbf{I}-\mathbf{Q}_{\Xi_N})\hat{u}_{m,\mathrm{ext}}\big\|_{H_0^1(D)} \lesssim \|f\|_{L^2(D)} N^{-1+\tilde{\delta}+|\boldsymbol{\delta}|}.$$

Proof. Application of the identity (2.19) with $\mathbf{\Gamma} = [0,1]^m$ and the anchor point $\mathbf{c} = \mathbf{1/2}$ leads for almost all $\mathbf{x} \in D$ to the representation of the quadrature error

$$(\mathbf{I} - \mathbf{Q}_{\Xi_N})\hat{u}_{m,\mathrm{ext}}(\mathbf{x}) = \sum_{\|\boldsymbol{\alpha}\|=1} \int_{[0,1]^{|\boldsymbol{\alpha}|}} \partial_{\mathbf{y}}^{\boldsymbol{\alpha}} \hat{u}_{m,\mathrm{ext}}(\mathbf{x}, \mathbf{y}_{\boldsymbol{\alpha}}, \mathbf{c}) \mathrm{discr}_{(\Xi_N)_{\boldsymbol{\alpha}}}(\mathbf{y}_{\boldsymbol{\alpha}}; \mathbf{c}_{\boldsymbol{\alpha}}) \,\mathrm{d}\mathbf{y}_{\boldsymbol{\alpha}}.$$

From

$$\mathrm{discr}_{(\Xi_N)_{\boldsymbol{\alpha}}}(\mathbf{y}_{\boldsymbol{\alpha}}; \mathbf{c}_{\boldsymbol{\alpha}}) = (-1)^{|\boldsymbol{\alpha}| - |\boldsymbol{\delta}_{\mathbf{1/2} \le \mathbf{y}_{\boldsymbol{\alpha}}}|} \mathrm{discr}_c(\mathbf{y}_{\boldsymbol{\alpha}}, (\Xi_N)_{\boldsymbol{\alpha}}),$$

we obtain by Bochner's inequality (II.3.6) the error estimate

$$\begin{aligned} &\big\|(\mathbf{I} - \mathbf{Q}_{\Xi_N})\hat{u}_{m,\mathrm{ext}}\big\|_{H_0^1(D)} \\ &\quad\le \sum_{\|\boldsymbol{\alpha}\|_\infty=1} \int_{[0,1]^{|\boldsymbol{\alpha}|}} \big\|\partial_{\mathbf{z}}^{\boldsymbol{\alpha}} \hat{u}_{m,\mathrm{ext}}(\mathbf{z}_{\boldsymbol{\alpha}}, \mathbf{c})\big\|_{H_0^1(D)} \,\mathrm{d}\mathbf{z}_{\boldsymbol{\alpha}} \sup_{\mathbf{z}_{\boldsymbol{\alpha}} \in [0,1]^{|\boldsymbol{\alpha}|}} \mathrm{discr}_c(\mathbf{z}_{\boldsymbol{\alpha}}, (\Xi_N)_{\boldsymbol{\alpha}}). \end{aligned} \tag{2.43}$$

The next step is to introduce weights $w_k \in (0,\infty)$ for $k = 1,\ldots,m$ and define the associated product weights with respect to the multi-index $\boldsymbol{\alpha}$ by $\omega_{\boldsymbol{\alpha}} := \prod_{k=1}^m w_k^{\alpha_k}$. Later on, we will specify these weights by exploiting the decay properties of the occurring derivatives of the integrand. By inserting the weights into (2.43), we deduce that

$$\begin{aligned} &\big\|(\mathbf{I} - \mathbf{Q}_{\Xi_N})\hat{u}_{m,\mathrm{ext}}\big\|_{H_0^1(D)} \\ &\quad\le \sum_{\|\boldsymbol{\alpha}\|_\infty=1} \Big\{\omega_{\boldsymbol{\alpha}}^{-1/2} \int_{[0,1]^{|\boldsymbol{\alpha}|}} \big\|\partial_{\mathbf{z}}^{\boldsymbol{\alpha}} \hat{u}_{m,\mathrm{ext}}(\mathbf{z}_{\boldsymbol{\alpha}}, \mathbf{c})\big\|_{H_0^1(D)} \,\mathrm{d}\mathbf{z}_{\boldsymbol{\alpha}}\Big\}\Big\{\omega_{\boldsymbol{\alpha}}^{1/2} \mathcal{D}_c\big((\Xi_N)_{\boldsymbol{\alpha}}\big)\Big\} \\ &\quad\le \Big\{\sup_{\|\boldsymbol{\alpha}\|_\infty=1} \omega_{\boldsymbol{\alpha}}^{-1/2} \int_{[0,1]^{|\boldsymbol{\alpha}|}} \big\|\partial_{\mathbf{z}}^{\boldsymbol{\alpha}} \hat{u}_{m,\mathrm{ext}}(\mathbf{z}_{\boldsymbol{\alpha}}, \mathbf{c})\big\|_{H_0^1(D)} \,\mathrm{d}\mathbf{z}_{\boldsymbol{\alpha}}\Big\}\Big\{\sum_{\|\boldsymbol{\alpha}\|_\infty=1} \omega_{\boldsymbol{\alpha}}^{1/2} \mathcal{D}_c\big((\Xi_N)_{\boldsymbol{\alpha}}\big)\Big\}. \end{aligned} \tag{2.44}$$

This corresponds in the terminology of the beginning of section 2 to the weighted and generalized centered Koksma-Hlawka inequality with the choices $r = \infty$ and $s = 1$, see (2.3). Due to the definition of $\hat{u}_{m,\mathrm{ext}}$, cf. (2.24), the derivative $\partial_{\mathbf{z}}^{\boldsymbol{\alpha}} \hat{u}_{m,\mathrm{ext}}(\mathbf{z}_{\boldsymbol{\alpha}}, \mathbf{c})$ vanishes in $[0,1]^{|\boldsymbol{\alpha}|} \setminus (K_N)_{\boldsymbol{\alpha}}$ and coincides with the derivative of $\hat{u}_m$ in $(K_N)_{\boldsymbol{\alpha}}$. Therefore, with $\tilde{C}$ defined as in (2.31), we can estimate

$$\begin{aligned} &\sup_{\|\boldsymbol{\alpha}\|_\infty=1} \omega_{\boldsymbol{\alpha}}^{-1/2} \int_{[0,1]^{|\boldsymbol{\alpha}|}} \big\|\partial_{\mathbf{z}}^{\boldsymbol{\alpha}} \hat{u}_{m,\mathrm{ext}}(\mathbf{z}_{\boldsymbol{\alpha}}, \mathbf{c})\big\|_{H_0^1(D)} \,\mathrm{d}\mathbf{z}_{\boldsymbol{\alpha}} \\ &\quad\lesssim \|f\|_{L^2(D)} \sup_{\|\boldsymbol{\alpha}\|_\infty=1} \omega_{\boldsymbol{\alpha}}^{-1/2} |\boldsymbol{\alpha}|! \int_{(K_N)_{\boldsymbol{\alpha}}} \prod_{\substack{k=1 \\ \alpha_k=1}}^m \Big(\gamma_k \tilde{C} \min\{z_k, 1-z_k\}^{-1-\delta_k}\Big) \,\mathrm{d}\mathbf{z}_{\boldsymbol{\alpha}} \\ &\quad\le \|f\|_{L^2(D)} \sup_{\|\boldsymbol{\alpha}\|_\infty=1} \omega_{\boldsymbol{\alpha}}^{-1/2} 2^{|\boldsymbol{\alpha}|} \prod_{\substack{k=1 \\ \alpha_k=1}}^m \bigg(k\gamma_k \tilde{C} \int_{(b_k N)^{-1}}^{1/2} z_k^{(-1-\delta_k)} \,\mathrm{d}z_k\bigg) \\ &\quad\le \|f\|_{L^2(D)} \sup_{\|\boldsymbol{\alpha}\|_\infty=1} \omega_{\boldsymbol{\alpha}}^{-1/2} \prod_{k=1}^m \bigg(\frac{2k\gamma_k \tilde{C}}{\delta_k}\bigg)^{\alpha_k} (b_k N)^{\alpha_k \delta_k}. \end{aligned}$$

The specific choice of the weights

$$w_k = \frac{8\pi C(\delta_k, 2\gamma_k)^2 k^2 \gamma_k^2}{\delta_k^2 \log^2 2}, \quad k = 1, \ldots, m, \tag{2.45}$$

yields that

$$\omega_{\boldsymbol{\alpha}}^{1/2} = \prod_{k=1}^{m} \left(\frac{2k\gamma_k \tilde{C}}{\delta_k}\right)^{\alpha_k}.$$

Therefore, we obtain that

$$\sup_{\|\boldsymbol{\alpha}\|_\infty=1} \omega_{\boldsymbol{\alpha}}^{-1/2} \prod_{k=1}^{m} \left(\frac{2k\gamma_k \tilde{C}}{\delta_k}\right)^{\alpha_k} (b_k N)^{\alpha_k \delta_k} \leq N^{|\boldsymbol{\delta}|} \prod_{k=1}^{m} b_k^{\delta_k}.$$

Now, the prime number theorem, see e.g. [SMC95], implies that $b_k < 2k\log(k+2)$. Hence, we deduce that

$$\prod_{k=1}^{\infty} b_k^{\delta_k} = \exp\left(\sum_{k=1}^{\infty} \delta_k \log b_k\right) \lesssim \exp\left(\sum_{k=1}^{\infty} k^{-1-\eta} \log\big(2k\log(k+2)\big)\right) < \infty.$$

From this, we finally conclude the estimate

$$\sup_{\|\boldsymbol{\alpha}\|_\infty=1} \omega_{\boldsymbol{\alpha}}^{-1/2} \int_{[0,1]^{|\boldsymbol{\alpha}|}} \|\partial_{\mathbf{z}}^{\boldsymbol{\alpha}} \hat{u}_{m,\text{ext}}(\mathbf{z}_{\boldsymbol{\alpha}}, \mathbf{c})\|_{H_0^1(D)} \, \mathrm{d}\mathbf{z}_{\boldsymbol{\alpha}} \lesssim N^{|\boldsymbol{\delta}|} \|f\|_{L^2(D)}.$$

This bounds the first term on the right-hand side of (2.44).

In order to bound the weighted sum of the L^∞-centered discrepancies, i.e. the second term on the right-hand side of (2.44), we use the following result from [Nie92]:

$$\mathcal{D}_{\text{extr}}(\Xi_N) \leq 2^m \mathcal{D}^\star(\Xi_N).$$

Thus, it follows that

$$\sum_{\|\boldsymbol{\alpha}\|_\infty=1} \omega_{\boldsymbol{\alpha}}^{1/2} \mathcal{D}_c\big((\Xi_N)_{\boldsymbol{\alpha}}\big) \leq \sum_{\|\boldsymbol{\alpha}\|_\infty=1} \omega_{\boldsymbol{\alpha}}^{1/2} \mathcal{D}_{\text{extr}}\big((\Xi_N)_{\boldsymbol{\alpha}}\big) \leq \sum_{\|\boldsymbol{\alpha}\|_\infty=1} \omega_{\boldsymbol{\alpha}}^{1/2} 2^{|\boldsymbol{\alpha}|} \mathcal{D}^\star\big((\Xi_N)_{\boldsymbol{\alpha}}\big).$$

Under the decay property

$$\sum_{k=1}^{\infty} \tilde{w}_k^{1/2} k \log k < \infty$$

of the weights $\tilde{w}_k := 4w_k$, it is shown in [Wan02], that

$$\sum_{\|\boldsymbol{\alpha}\|_\infty=1} \tilde{w}_{\boldsymbol{\alpha}}^{1/2} \mathcal{D}_\infty^\star\big((\Xi_N)_{\boldsymbol{\alpha}}\big) \lesssim N^{-1+\tilde{\delta}} \tag{2.46}$$

holds for all $\tilde{\delta} > 0$ with a constant which depends on $\tilde{\delta}$ but not on the dimensionality m. This condition is satisfied if the weights fulfill $\tilde{w}_k^{1/2} \lesssim k^{-2-\eta}$. Hence, we get the following condition on the decay of γ_k:

$$\frac{4k\gamma_k\tilde{C}}{\delta_k} \lesssim k^{-2-\eta} \quad \Longrightarrow \quad \gamma_k \lesssim \frac{\delta_k}{4\tilde{C}} k^{-3-\eta} \simeq k^{-4-2\eta}.$$

□

With the preceding two lemmata at hand, we can establish the estimate (2.27). This completes the proof of Theorem (2.25). Note that the estimation of the L^∞-centered discrepancy in Lemma (2.41) by the extreme discrepancy is not sharp and might be improvable. This would lead to a better constant in the determination of the weights w_k. Nevertheless, this does not affect the summability condition of the weights and, therefore, does not influence the requirements on the decay of $\{\gamma_k\}_k$. Moreover, in the last proof, we used the result of [Wan02] about the estimation of the discrepancy, which can be extended by the result in [KSS12a] as already pointed out in the uniformly elliptic case. This would make the proof of Lemma (2.41) more technical and is therefore omitted. It would imply that the integration error considered in Lemma (2.41) can still be bounded independently of m with a lower convergence rate provided that the sequence $\{\gamma_k\}_k$ decays only faster than $\{k^{-3-\eta}\}_k$ for some $\eta > 0$. We would like to point out that the decay condition, needed in the proof of Lemma (2.28) to bound the truncation error with constants nearly independent of m, is $\{k^{-3-\eta}\}_k$. Nevertheless, we would not claim that this is a sharp condition. In particular, we observe in our numerical results dimension-independent convergence rates even for a weaker decay of the sequence $\{\gamma_k\}_k$.

In Theorem (2.25), we have shown approximation results of the quasi-Monte Carlo quadrature based on Halton points for the mean of the solution u_m to (III.3.5). Note that, due to the regularity estimates proven in Section III.5, the result can be extended in complete analogy for the computation of the p-th moment $\mathcal{M}^p u_m$ provided that $f \in L^p(D)$. This is due to the similar behaviour of the estimates on the solution's derivatives and the derivatives of its powers.

(2.47) **Corollary.** Let $f \in L^p(D)$ for $p \geqslant 2$. Under the conditions of Theorem (2.25), the quasi-Monte Carlo quadrature using the first N Halton points for approximating the p-th moment of the solution u_m to (III.3.5) satisfies the error estimate

$$\|(\mathbf{I} - \mathbf{Q}_{\Xi_N})\hat{u}_m^p\|_{W_0^{1,1}(D)} \lesssim \|f\|_{L^p(D)}^p m N^{-1+\delta} \tag{2.48}$$

with a constant depending on p and δ, but not on the dimensionality m.

Proof. In the proof of Theorem (2.25), we exploited the bounds on the derivative of the integrand u_m. The p-th power of u_m satisfies the following estimate on the derivatives

$$\|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}} u_m^p(\mathbf{y})\|_{W_0^{1,1}(D)} \lesssim |\boldsymbol{\alpha}|!\left(\frac{C(p,D)p\boldsymbol{\gamma}}{\log 2}\right)^{\boldsymbol{\alpha}}\left(\frac{\overline{a}_m(\mathbf{y})}{\underline{a}_m(\mathbf{y})}\right)^p \|u_m(\mathbf{y})\|_{W_0^{1,p}(D)}^p.$$

This estimate is very similar to the bound on the derivatives of u_m. To apply all the steps of the proof of Theorem (2.25), we observe that the modified sequence $\{C(p,D)p\gamma_k\}_k$ has, for each fixed p, up to a constant the same decay behaviour as $\{\gamma_k\}_k$. Moreover, with (III.5.14), we can estimate that

$$\left(\frac{\overline{a}_m(\mathbf{y})}{\underline{a}_m(\mathbf{y})}\right)^p \|u_m(\mathbf{y})\|_{W_0^{1,p}(D)}^p \leq \exp\left(\sum_{k=1}^m 3p\gamma_k|y_k|\right)\|f\|_{L^p}^p.$$

This leads in item (*ii.*) of the proof of Lemma (2.28) to the constant $C(\delta_k, 3p\gamma_k)$. Since $3p\gamma_k$ also has up to a constant the same behaviour as $2\gamma_k$, all the steps of the proofs of the Lemmata (2.28) and (2.41) remain valid which verifies (2.48). Nevertheless, we would like to mention that the constants in (2.48) may grow exponentially in p. □

3. Numerical results

For the numerical validation of the theoretical findings, we consider the computation of the first four moments of the solution u_m of the one-dimensional diffusion problem

$$-\partial_x(a_m(x,\mathbf{y})\partial_x u_m(x,\mathbf{y})) = 1 \text{ in } D = (0,1) \tag{3.1}$$

with homogenous Dirichlet boundary conditions, i.e. $u_m(0,\mathbf{y}) = u_m(1,\mathbf{y}) = 0$. The diffusion coefficient $a_m(x,\mathbf{y})$ is determined by a truncated Karhunen-Loève expansion with respect to a covariance kernel of the Matérn class. More precisely, we investigate the two different kernels $k_{7/2}$ and $k_{5/2}$ from (III.4.3) where the correlation length ℓ is set to $\ell = 1/2$ and the variance $\sigma^2 = 1/4$. The decay of the related sequences $\{\gamma_k\}_k$ is depicted in Figure III.2. The computation of the truncated Karhunen-Loève expansion is performed by a pivoted Cholesky decomposition of the associated covariance operator $\mathcal{C}$, see (III.2.1). Although, a truncation with respect to the weak error estimates in Section III.2 would be sufficient, we truncate the Karhunen-Loève expansion in a more conservative way. As already mentioned in Remark (III.2.16), we choose m in such a way that the trace error in the covariance operator $\mathcal{C}$ is smaller than ε^2 in order to rule out the truncation error. Herein, the accuracy ε reflects the precision of the spatial discretization.

We have discretized (3.1) by piecewise linear finite elements and choose piecewise constant elements for the discretization of the diffusion coefficient. The discretization level is in all computations set to 14 which results in a meshwidth $h = 2^{-14}$. This fine meshwidth prevents that the overall error is dominated by the finite element discretization error. Notice that for higher-dimensional spatial domains D, it would be unfeasible to employ such high refinement levels. Hence, the one-dimensional toy problem under consideration is perfectly suited for the investigation of the convergence of the moment computation.

Since the solution of (3.1) is not known analytically, we have to provide a reference solution. This reference solution is computed by the quasi-Monte Carlo quadrature with Halton points and $N = 10 \cdot 2^{20} \approx 10^7$ samples.

For the investigation of the convergence rates of the Monte Carlo as well as the quasi-Monte Carlo method based on Halton points, we have kept the spatial discretization level fixed and successively increased the number of quadrature points N by $N = 10\cdot 2^j$ for $j = 1, 2, \ldots, 20$. Additionally, in case of the Monte-Carlo, we compute the RMSE based on five realizations of the Monte-Carlo estimator. The error with respect to the reference solution is measured in the $H^1(D)$-norm for the approximation of the mean and in the $W^{1,1}(D)$-norm for the approximations of the higher order moments, respectively.

3.1 Results for lognormal diffusion

The Matérn kernel for $\nu = 7/2$

For the smoothness parameter $\nu = 7/2$, we have truncated the Karhunen-Loève expansion after $m = 30$ terms. As can be seen from Figure III.2, the sequence $\{\gamma_k\}_k$ decays in this case exactly with a rate of k^{-4} which is the limiting case in order to fulfill the requirements of Theorem (2.25).

The error plots of the Monte Carlo method for the approximation of the first four moments and the according plots of the quasi-Monte Carlo method are visualized in

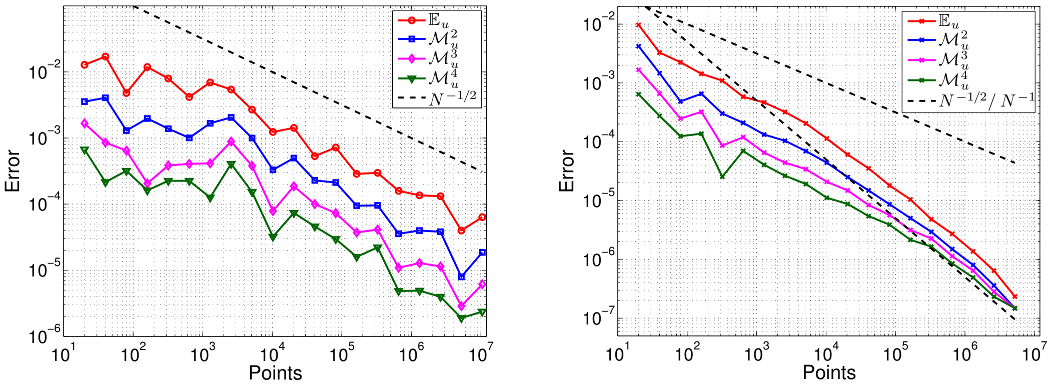


Figure IV.1: Errors for $\nu = 7/2$ of the Monte Carlo quadrature (left) and the quasi-Monte Carlo quadrature (right) in the lognormal case.

Figure IV.1. In these plots, we see that the convergence rate of the quasi-Monte Carlo method is in each case superior to the convergence rate of the Monte Carlo method. As expected, we observe algebraic convergence rates. In case of the quasi-Monte Carlo method, we obtain a convergence rate of $N^{-5/6}$ for the computation of the mean and a slightly lower rate for the second and higher order moments. This can be explained by the exponential dependence of the constant in Corollary (2.47) on p for the quasi-Monte Carlo method. Since we plot absolute errors and the considered norms of the moments decrease from the first to the fourth moment, the initial errors decrease from the first to the fourth moment as well. For the Monte-Carlo method, we observe for all moments the expected convergence rate 1/2.

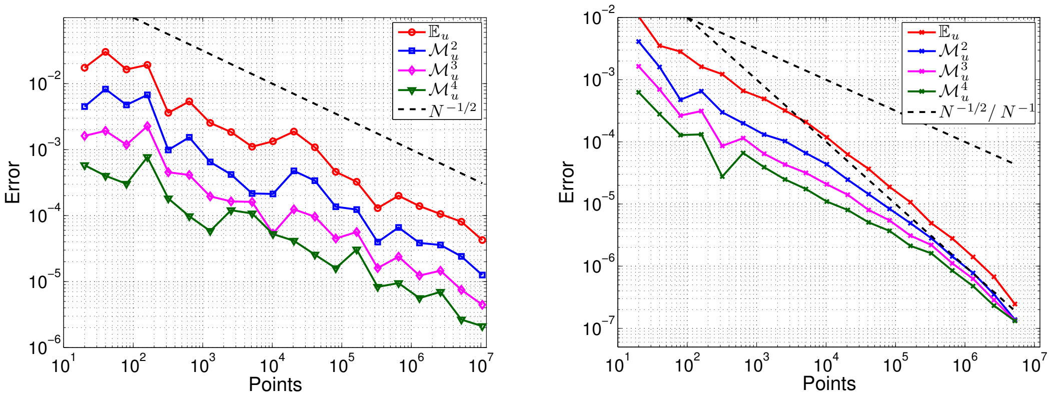


Figure IV.2: Errors for $\nu = 5/2$ of the Monte Carlo quadrature (left) and the quasi-Monte Carlo quadrature (right) in the lognormal case.

The Matérn kernel for $\nu = 5/2$

For the smoothness parameter $\nu = 5/2$, we have truncated the Karhunen-Loève expansion after $m = 64$ terms. As expected, the convergence rates of the Monte Carlo quadrature remain unchanged in comparison to the previous example. Although, the correlation kernel $k_{5/2}$ does not meet the required smoothness assumptions of Theorem (2.25) anymore, we observe the same convergence rates, as for the correlation kernel $k_{7/2}$. A visualization of the corresponding errors plots is given in Figure IV.2. From this figure, we immediately deduce that the quasi-Monte Carlo quadrature is preferable to the Monte Carlo quadrature for this example.

In our examples, we observe essentially the same convergence behavior of the Monte Carlo and the quasi-Monte Carlo quadrature based on Halton points with respect to an increasing length of the Karhunen-Loève expansion. This behaviour was expected for the Monte Carlo quadrature. For the quasi-Monte Carlo quadrature, this indicates that the linear dependency on the dimensionality m in the convergence rate in Theorem (2.25) can be removed or at least weakened, see also Remark (2.39). In addition, the results for the correlation kernel $k_{5/2}$ imply that the claimed decay conditions on $\{\gamma_k\}_k$ in Theorem (2.25) can most likely be improved.

3.2 Results for uniformly elliptic diffusion

The Matérn kernel for $\nu = 7/2$

We consider (3.1) with a diffusion coefficient which is given by a truncated Karhunen-Loève expansion

$$a_m(x, \mathbf{y}) = \mathbb{E}_a(x) + \sum_{k=1}^{m} \sqrt{\lambda_k}\varphi_k(x)y_k \quad x \in [0, 1].$$

In addition to the covariance kernel, the expectation field of a_m has to be known. We set $\mathbb{E}_a(x) \equiv 2.5$ which is sufficient to guarantee the positivity of a_m.

The convergence results for the Monte Carlo and quasi-Monte Carlo quadrature based on Halton points for the approximation of the first for moments of u_m are visualized in Figure IV.3. As in the lognormal case, the Monte Carlo quadrature yields for all moments a convergence rate $1/2$. The convergence results for the quasi-Monte Carlo quadrature are expected to be slightly better than in the lognormal case. Indeed, all the moments converge with a rate of approximately $10/11$. Hence, the convergence rate comes close to the upper bound 1. The convergence rate of a quasi Monte Carlo quadrature based on Halton points is even for $m = 1$ limited by 1 since, in this case, the discrepancy is bounded by $\mathcal{O}(N^{-1})$. This demonstrates that the dimensionality m does not significantly impair the convergence of the quasi-Monte Carlo quadrature in the numerical examples.

The Matérn kernel for $\nu = 5/2$

As in the previous example, we set the expectation field of a to $\mathbb{E}_a(x) \equiv 2.5$. On the left-hand side of Figure IV.4, the convergence rate of the Monte Carlo quadrature is shown. As in all other examples, the convergence rate is $1/2$ for the moment computation. Nevertheless, we observe that the convergence of the Monte Carlo quadrature is not monotonically

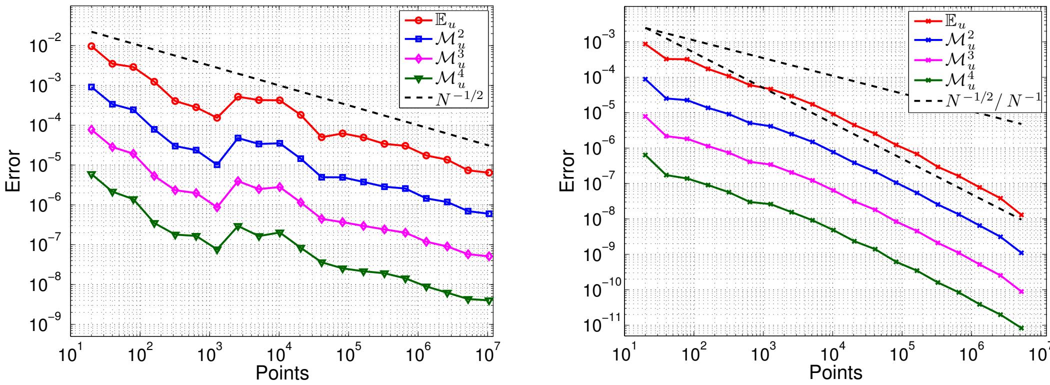


Figure IV.3: Errors for $\nu = 7/2$ of the Monte Carlo quadrature (left) and the quasi-Monte Carlo quadrature (right) in the uniformly elliptic case.

with respect to the number of quadrature points. This can be explained since the method provides error estimates in the mean square sense.

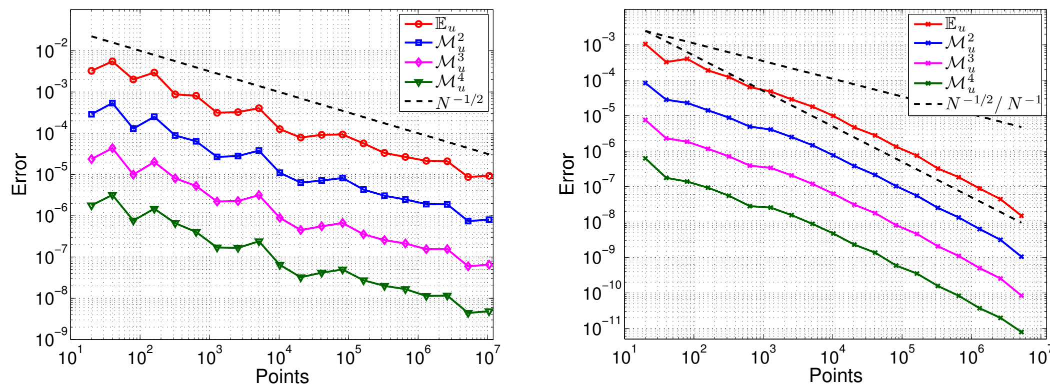


Figure IV.4: Errors for $\nu = 5/2$ of the Monte Carlo quadrature (left) and the quasi-Monte Carlo quadrature (right) in the uniformy elliptic case.

For the quasi-Monte Carlo quadrature, we observe essentially the same error plots, visualized on the right-hand side of Figure IV.4, as for the Matérn kernel with smoothness parameter $\nu = 7/2$. This demonstrates that the quasi-Monte Carlo quadrature works robust with respect to the increase of the dimensionality which is, as for the lognormal examples, $m = 30$ for $\nu = 7/2$ and $m = 64$ for $\nu = 5/2$.

We conclude that the quasi-Monte Carlo method based on Halton points outperformes the Monte Carlo quadrature in the numerical examples of this section. Especially in the lognormal case, the quasi-Monte Carlo quadrature performes even better than expected. More precisely, the method converges dimension-independent with a rate close to 1 even if the Matern kernel with $\nu = 5/2$ is used. In particular, this case is not covered by our analysis anymore.

Chapter V

GAUSSIAN QUADRATURE

In this chapter, we discuss and analyze the application of Gaussian quadrature rules for approximating the moments of the solution u_m to (III.3.3) or to (III.3.5). We will consider full tensor product quadrature methods as well as sparse Smolyak type quadrature methods. The uniformly elliptic case (III.3.3) is treated with the Gauss-Legendre quadrature, assuming that the random parameters in the Karhunen-Loève expansion are uniformly distributed. As long as the ellipticity of the diffusion coefficient is guaranteed, in principle any arbitrary distribution on a finite interval can be considered. This would lead to a different system of orthogonal polynomials which can be determined by a 3-term-recurrence relation, see e.g. [HB09]. Then, a Gaussian quadrature rule with n points is constructed based on the roots of the n-th orthogonal polynomial. Since the underlying density function for the lognormal case (III.3.5) is the Gaussian density, the orthogonal polynomials in this case are given by the Hermite polynomials and, thus, the resulting quadrature is the Gauss-Hermite quadrature.

The main focus of this chapter will be, as in the previous one, the investigation whether the convergence rate of such methods deteriorates with the dimensionality m or not. We consider anisotropic tensor product Gaussian quadrature and anisotropic sparse Gaussian quadrature methods. The convergence analysis is based on one-dimensional best polynomial interpolation error results from [BNT07, Bie09]. These error estimates are quite similar in the lognormal as well as in the uniformly elliptic case. Hence, we summarize the results into a more general one-dimensional Gaussian quadrature error estimate and treat both cases simultaneously. In order to obtain dimension-independent convergence rates for the anisotropic Gaussian quadrature, the decay requirements on the sequence $\{\gamma_k\}_k$ turn out to be quite strong, namely $\gamma_k \lesssim \exp(k^{-1-\eta})$ for $\eta > 0$. Thus, we additionally analyze the impact of the dimensionality in case of an algebraically decaying sequence $\{\gamma_k\}_k$.

For the anisotropic sparse Gaussian quadrature, we present a new bound on the number of indices in an anisotropic sparse grid in order to improve the results of the anisotropic tensor product Gaussian quadrature. We formulate this estimate as a conjecture since it is only proven for two dimensions in this chapter. In the appendix, the proof is extended up to $m = 5$ dimensions [1]. Nevertheless, we checked this bound numerically for various values of m and various anisotropic settings. With this new estimate at hand, we can at least show that the convergence rate of the anisotropic sparse Gaussian quadra-

[1] In the meantime, the conjecture is proven by Abdul-Lateef Haji-Ali

ture is dimension-independent for arbitrary exponentially decaying sequences $\{\gamma_k\}_k$. This is, for example, the case for Gaussian covariance kernels. Moreover, in case of algebraic decay properties on $\{\gamma_k\}_k$ and a moderate dimensionality m, our analysis still provides satisfactory results, since the dimensionality enters only with the factor $\log(\log(m))$ into the convergence rate. Although these estimates require the validity of the conjecture, the findings help to better understand why the convergence rate of anisotropic sparse grid quadratures behaves quite well even for moderately smooth covariance kernels. At the end of this chapter, the theoretical findings are validated by numerical examples.

1. Univariate Gaussian quadrature

Let $\rho > 0$ be a density function on $\Gamma \subset \mathbb{R}$. Then, we define an scalar product on $L^2_\rho(\Gamma)$ by

$$(v, w)_{L^2_\rho(\Gamma)} = \int_\Gamma v(y)w(y)\rho(y)\,\mathrm{d}y.$$

A sequence of orthonormal polynomials in $L^2_\rho(\Gamma)$ can be constructed by the 3-term-recurrence relation

$$\beta_{k+1}p_{k+1}(y) = (y - \alpha_{k+1})p_k(y) - \beta_k p_{k-1}(y), \quad k \geq 1 \tag{1.1}$$

with $p_{-1} \equiv 0$ and $p_0 \equiv 1$. The coefficients α_k and β_k are given by

$$\begin{aligned} \alpha_{k+1} &= (p_k(y), y p_k(y))_{L^2_\rho(\Gamma)} && \text{and} \\ \beta_{k+1} &= \|q_{k+1}\|_{L^2_\rho(\Gamma)} && \text{with} \quad q_{k+1} = (y - \alpha_{k+1})p_k(y) - \beta_k p_{k-1}(y). \end{aligned}$$

The N-point Gaussian quadrature on Γ with respect to the density ρ is then defined by the quadrature points $\mathcal{J}_N := \{\eta_1, \ldots, \eta_N\}$ and the corresponding quadrature weights $\{\omega_1, \ldots, \omega_N\}$. The quadrature points are determined as the N roots of p_N and the weights by $\omega_k = (L_k, 1)_{L^2_\rho(\Gamma)}$, where L_k denotes the k-th Lagrange polynomial with respect to the point set $\mathcal{J}_N$. Numerically, these quadrature rules can be constructed by determining the eigenpairs of the *Jacobi matrix* $\mathbf{J}_N$ which is associated with the 3-term-recurrence relation (1.1), see [HB09]. This Jacobi matrix is the $N \times N$ tridiagonal matrix

$$\mathbf{J}_N = \begin{pmatrix} \beta_1 & \alpha_1 & & 0 \\ \alpha_1 & \beta_2 & \ddots & \\ & \ddots & \ddots & \alpha_{N-1} \\ 0 & & \alpha_{N-1} & \beta_N \end{pmatrix}.$$

Denoting by $(\lambda_k, \mathbf{v}_k)_{k=1}^N$ the eigenpairs of $\mathbf{J}_N$, where the eigenvectors $\mathbf{v}_k$ are assumed to be normalized, i.e. $\|\mathbf{v}_k\|_2 = 1$, it holds that $\eta_k = \lambda_k$ and $\omega_k = v_{k,1}^2$ for $k = 1, \ldots, N$.

With the quadrature points and weights at hand, we define the univariate N point Gaussian quadrature rule

$$(Q_{G,N}v)(\mathbf{x}) = \sum_{k=1}^N \omega_k v(\mathbf{x}, \eta_k)$$

to approximate the Bochner integral

$$(Iv)(\mathbf{x}) := \int_{\Gamma} v(\mathbf{x}, y)\rho(y)\,\mathrm{d}y.$$

This quadrature rule corresponds to the quadrature operator $Q_{G,N} \colon C^0_\sigma(\Gamma; X) \to X$, where σ is given by (III.5.23) for $m = 1$ in the lognormal case and $\sigma \equiv 1$ in the uniformly elliptic case, respectively. With the construction above, we are in principle able to consider arbitrary density functions. Nevertheless, we restrict ourselves to the Gauss-Legendre quadrature on $\Gamma = [-1/2, 1/2]$ in the uniformly elliptic case and denote the associated quadrature operator with N points by $Q_{GL,N}$. Likewise, we write $Q_{GH,N}$ for the Gauss-Hermite quadrature operator with N points.

In the sequel, we adapt the analysis and the notation presented in [BNT07], where the approximation error of the stochastic collocation method is analyzed. According to [BNT07], we shall introduce the one-dimensional Gaussian auxiliary weight $\sqrt{\rho(y)} = (2\pi)^{1/4}\exp(-y^2/4)$ for the lognormal case and denote the corresponding space by $C^0_{\sqrt{\rho}}(\mathbb{R}; X)$. The norm in $C^0_{\sqrt{\rho}}(\mathbb{R}; X)$ is weaker than the norm in $C^0_\sigma(\mathbb{R}; X)$ which yields that $C^0_\sigma(\mathbb{R}; X) \subset C^0_{\sqrt{\rho}}(\mathbb{R}; X)$.

The following two lemmata bound the one-dimensional quadrature error by the polynomial best approximation error. They are slight modifications of the corresponding lemmata in [BNT07] for the polynomial interpolation.

(1.2) **Lemma.** The quadrature operator $Q_{GH,N} \colon C^0_\sigma(\mathbb{R}; X) \to X$ is continuous.

Proof. Consider $v \in C^0_\sigma(\mathbb{R}; X)$. By using the triangle inequality and exploiting the positivity of the weights w_k of the Gauss-Hermite quadrature, we deduce that

$$\begin{aligned}\|Q_{GH,N}v\|_X &= \left\| \sum_{i=1}^{N} \omega_i v(\eta_i) \right\|_X \leq \sum_{i=1}^{N} \omega_i \|v(\eta_i)\|_X = \sum_{i=1}^{N} \frac{\omega_i}{\sigma(\eta_i)} \big\|\sigma(\eta_i)v(\eta_i)\big\|_X \\ &\leq \max_{i=1,\dots,N} \big\|\sigma(\eta_i)v(\eta_i)\big\|_X \sum_{i=1}^{N} \frac{\omega_i}{\sigma(\eta_i)} \lesssim \|v\|_{C^0_\sigma(\mathbb{R};X)}.\end{aligned}$$

The last inequality follows from [Usp28], where the convergence

$$\sum_{i=1}^{N} \frac{\omega_i}{\sigma(\eta_i)} \xrightarrow{N\to\infty} \int_{\mathbb{R}} \frac{\rho(y)}{\sigma(y)}\,\mathrm{d}y < \infty$$

is shown. □

The above lemma is also valid for the space $C^0_{\sqrt{\rho}}(\mathbb{R}; X)$. Moreover, the Gauss-Legendre quadrature operator $Q_{GL,N} \colon C^0([-1/2, 1/2]; X) \to X$ is obviously continuous with continuity constant $1 = \sum_{i=1}^{N} w_i$, i.e. it holds for all $v \in C^0([-1/2, 1/2]; X)$ that

$$\|Q_{GL,N}v\|_X \leq \|v\|_{C^0([-1/2,1/2];X)}.$$

The continuity and the polynomial exactness of the quadrature operators are exploited to relate the quadrature error to the polynomial best approximation error in $C^0_{\sqrt{\rho}}(\mathbb{R}; X)$ in the lognormal case and in $C^0([-1/2; 1/2]; X)$ in the uniformly elliptic case, respectively. Therefore, let us denote by $\mathcal{P}_n(\Gamma)$ the space of polynomials of degree at most n.

(1.3) **Lemma.** For every $v \in C^0_{\sqrt{\rho}}(\mathbb{R};X)$, the quadrature error of the N-point Gauss-Hermite quadrature is bounded by

$$\|Iv - Q_{GH,N}v\|_X \lesssim \inf_{w\in\mathcal{P}_{2N-1}(\mathbb{R})\otimes X} \|v-w\|_{C^0_{\sqrt{\rho}}(\mathbb{R};X)}.$$

Accordingly, for every $v \in C^0([-1/2,1/2];X)$, it holds for the N-point Gauss-Legendre quadrature that

$$\|Iv - Q_{GL,N}v\|_X \lesssim \inf_{w\in\mathcal{P}_{2N-1}([-1/2,1/2])\otimes X} \|v-w\|_{C^0([-1/2,1/2];X)}.$$

Proof. Since the N-point Gauss-Hermite quadrature is exact for all polynomials of degree $2N-1$, it holds that $Q_{GH,N}w = Iw$ for all $w \in \mathcal{P}_{2N-1}(\mathbb{R})\otimes X$. Moreover, we have by the continuity of the integration operator I in $C^0_{\sqrt{\rho}}(\mathbb{R};X)$ that

$$\|v\|_{L^1_\rho(\mathbb{R};X)} = \int_{\mathbb{R}} \|v\|_X \rho(x)\,\mathrm{d}x \lesssim \|v\|_{C^0_{\sqrt{\rho}}(\mathbb{R};X)}.$$

Using additionally Bochner's inequality (II.3.6) and the continuity of the quadrature operator $Q_{GH,N}$, cf. Lemma (1.2), we deduce for arbitrary $w \in \mathcal{P}_{2N-1}(\mathbb{R})\otimes X$ that

$$\begin{aligned}\|Iv - Q_{GH,N}v\|_X &\le \|I(v-w)\|_X + \|Q_{GH,N}(v-w)\|_X\\ &\lesssim \|v-w\|_{L^1_\rho(\mathbb{R};X)} + \|v-w\|_{C^0_{\sqrt{\rho}}(\mathbb{R};X)}\\ &\lesssim \|v-w\|_{C^0_{\sqrt{\rho}}(\mathbb{R};X)}.\end{aligned}$$

The assertion follows in complete analogy in case of the Gauss-Legendre quadrature. □

Hence, in order to estimate the approximation error of the univariate Gauss-quadrature, we have to estimate the error of the polynomial best approximation for certain functions $v \in C^0_{\sqrt{\rho}}(\mathbb{R};X)$ in the lognormal and for certain functions $v \in C^0([-1/2,1/2];X)$ in the uniformly elliptic case, respectively. This would not lead to useful results for general functions v contained in these spaces, but the specific integrands under consideration provide additional regularity. In particular, they are analytic and, moreover, analytically extendable into a region of the complex plane according to Theorem (III.5.34) and Corollary (III.6.6). For such kind of functions, approximation results are available. In the lognormal case, we exploit the one-dimensional approximation error discussed in [Bie09]. The proof of this result is based on estimates for the Fourier-Hermite coefficients which are calculated with the help of Cauchy's integral theorem. Therefore, recall that $\sigma_s(y) = \exp(-s\gamma|y|)$ and X denote a Banach space, for example $X = H^1_0(D)$ or $X = W^{1,1}_0(D)$.

(1.4) **Lemma ([Bie09]).** [2] Suppose that $v \in C^0_{\sigma_s}(\mathbb{R};X)$ admits an analytic extension in $\Sigma(\mathbb{R},\tau)$ for some $1/\sqrt{2} < \tau < 1/\gamma$. Then, the error of the best approximation by

[2] Unfortunately, this lemma turned out to be wrong after the defense of the thesis. The proof of this lemma is based on an estimate of the Hermite coefficients of the function v by means of Cauchy's integral formula. But, there is a factor $\sqrt{n!}$ missing in each Hermite coefficient and, hence, even convergence of the remainder of the Hermite series cannot be proven with this estimate. For a fixed dimension m, the estimate (1.8) can be used instead to obtain convergence results. However, for small values of τ, the estimate (1.8) is not very accurate due to the constant $C(\tau)$ and, thus, it does not yield good enough results for a convergence analysis with respect to a growing dimensionality m and increasing regions of analycity τ_k for $k = 1,\ldots,m$. Therefore, the following analysis in this chapter is only valid for the uniformly elliptic case or, in more generality, for multivariate quadrature problems where an estimate (2.9) is available for the univariate quadrature.

polynomials of degree at most n can be bounded by

$$(1.5)\qquad \inf_{w\in\mathcal{P}_n(\mathbb{R})\otimes X} \|v-w\|_{C^0_{\sqrt{\rho}}(\mathbb{R};X)} \le C(\sigma_s,\tau)e^{-\log(\sqrt{2}\tau)n}\|v\|_{C^0_{\sigma_s}(\Sigma(\mathbb{R},\tau);X)}$$

where $\|v\|_{C^0_\sigma(\Sigma(\mathbb{R},\tau);X)} = \sup_{z\in\Sigma(\mathbb{R},\tau)} \sigma(\mathrm{Re}(z))\|v(z)\|_X$ and

$$(1.6)\qquad C(\sigma_s,\tau) = \frac{2\exp\left(\frac{(s\gamma)^2}{4}\right)\exp(s\gamma\tau)}{\sqrt{2}\tau-1}.$$

(1.7) **Remark.** The above lemma is only useful when the quantity τ which describes the region of analyticity is greater than $1/\sqrt{2}$. For smaller values of τ, there are estimates available of the form

$$(1.8)\qquad \inf_{w\in\mathcal{P}_n(\mathbb{R})\otimes X} \|v-w\|_{C^0_{\sqrt{\rho}}(\mathbb{R};X)} \le C(\tau)\sqrt{n}\exp(-\tau\sqrt{n}),$$

cf. [BNT07, Bie11]. Although, the estimate (1.8) looks quite promising, the constant $C(\tau)$ may grow considerably fast in τ, approximately like $\exp(\tau^2/2)$ and, thus, (1.8) is not useful for large values of τ.

Nevertheless, the regions of analyticity in our application grow with the dimensionality which implies that at least after a certain fixed number k_0 of dimensions the conditions of Lemma (1.4) on τ_k are fulfilled for all $k \ge k_0$. In this case, we would have to deal with (1.8) in the first k_0 dimensions which further complicates, but does not impair our analysis. Hence, we assume in the following that each τ_k for $k = 1,\dots,m$ meets the assumptions of Lemma (1.4).

The numerator in the constant in (1.6) decreases for decreasing $s\gamma$. Hence, for fixed s and $\gamma < \sqrt{2}$, the numerator in (1.6) can be bounded by a generic constant $C > 0$. Therefore, we can reduce estimate (1.5) to

$$(1.9)\qquad \inf_{w\in\mathcal{P}_n(\mathbb{R})\otimes X} \|v-w\|_{C^0_{\sqrt{\rho}}(\mathbb{R};X)} \le \frac{C}{\sqrt{2}\tau-1}e^{-\log(\sqrt{2}\tau)n}\|v\|_{C^0_{\sigma_s}(\Sigma(\mathbb{R},\tau);X)}.$$

Lemma (1.4) shows that the univariate Gauss-Hermite quadrature converges exponentially in terms of the number of quadrature points for functions which are analytically extendable into some strip around the real line. The corresponding result for the Gauss-Legendre quadrature is provided by the next lemma from [BNT07].

(1.10) **Lemma ([BNT07]).** Suppose that $v \in C^0([-1/2,1/2];X)$ admits an analytic extension in $\Sigma([-1/2,1/2],\tau)$ for some $\tau > 0$. Then, the error of the best approximation by polynomials of degree at most n can be bounded by

$$(1.11)\qquad \inf_{w\in\mathcal{P}_n([-1/2,1/2])\otimes X} \|v-w\|_{C^0([-1/2,1/2];X)} \le \frac{2}{\kappa-1}e^{-n\log\kappa}\|v\|_{C^0_\sigma(\Sigma([-1/2,1/2],\tau);X)},$$

with $\kappa = 2\tau + \sqrt{1+4\tau^2}$.

2. Anisotropic tensor product Gaussian quadrature

The integrals in our approximation are defined over the m-dimensional domain $\mathbf{\Gamma}$. Hence, we have to construct a multivariate quadrature formula. This can simply be done by tensorization of the univariate quadrature rules. Therefore, we define the tensor product Gaussian quadrature operator with respect to a multi-index $\mathbf{N} \in \mathbb{N}^m$ by

$$(2.1)\qquad (\mathbf{Q}_{G,\mathbf{N}}v)(\mathbf{x}) := \left(\bigotimes_{k=1}^{m} Q_{G,N_k}^{(k)} v\right)(\mathbf{x}) = \sum_{\boldsymbol{\alpha}\in\mathcal{J}_{\mathbf{N}}} \omega_{\boldsymbol{\alpha}} v(\mathbf{x},\boldsymbol{\eta}_{\boldsymbol{\alpha}})$$

as an approximation to the multivariate integration operator $\mathbf{I}$ defined in (IV.0.1). Herein, the index set is given by $\mathcal{J}_{\mathbf{N}} = \times_{k=1}^{m} \mathcal{J}_{N_k}$. Moreover, the quadrature points and weights are defined according to

$$(2.2)\qquad \boldsymbol{\eta}_{\boldsymbol{\alpha}} := \left(\eta_{\alpha_1}^{(1)},\ldots,\eta_{\alpha_m}^{(m)}\right) \quad \text{and} \quad \omega_{\boldsymbol{\alpha}} := \prod_{k=1}^{m} \omega_{\alpha_k}^{(k)}.$$

Note that $\mathbf{I}$ coincides, due to the product structure of the measure $\boldsymbol{\rho}\,\mathrm{d}\mathbf{x}$, with the tensor product integration operator

$$\mathbf{I} := \bigotimes_{k=1}^{m} I^{(k)}.$$

2.1 Continuity of the Gaussian quadrature operator

It is crucial to investigate the continuity of $\mathbf{Q}_{G,\mathbf{N}}$ and $\mathbf{I}$ in order to establish error estimates with constants which are independent of the dimension m of the parameter domain $\mathbf{\Gamma}$. In analogy to the univariate case, we denote the tensor product Gauss-Hermite quadrature operator by $\mathbf{Q}_{GH,\mathbf{N}}$ and the tensor product Gauss-Legendre quadrature operator by $\mathbf{Q}_{GL,\mathbf{N}}$. In the uniformly elliptic case, the continuity of $\mathbf{Q}_{GL,\mathbf{N}}$ is again fulfilled with continuity constant 1:

$$(2.3)\qquad \begin{aligned} \|\mathbf{Q}_{GL,\mathbf{N}}v\|_X &= \left\| \sum_{\boldsymbol{\alpha}\in\mathcal{J}_{\mathbf{N}}} \omega_{\boldsymbol{\alpha}} v(\mathbf{x},\boldsymbol{\eta}_{\boldsymbol{\alpha}})\right\|_X \\ &\le \|v\|_{C^0([-1/2,1/2]^m;X)} \sum_{\boldsymbol{\alpha}\in\mathcal{J}_{\mathbf{N}}} \omega_{\boldsymbol{\alpha}} = \|v\|_{C^0([-1/2,1/2]^m;X)}. \end{aligned}$$

(2.4) **Lemma.** Let the weight $\boldsymbol{\sigma}_s$ be defined as in (III.5.25). Then, the multivariate tensor product Gaussian quadrature operator $\mathbf{Q}_{GH,\mathbf{N}}$ as well as the integration operator $\mathbf{I}$ are continuous as mappings $C^0_{\boldsymbol{\sigma}_s}(\mathbb{R}^m;X) \to X$ with continuity constants which depend on s but not on m.

Proof. For the integration operator $\mathbf{I}$, the assertion is directly obtained by (III.5.26) with the continuity constant $C(\boldsymbol{\sigma}_s)$. For the quadrature operator $\mathbf{Q}_{GH,\mathbf{N}}$, we conclude in the same way as in the univariate case that

$$(2.5)\qquad \|\mathbf{Q}_{GH,\mathbf{N}}v\|_X \le \|v\|_{C^0_{\boldsymbol{\sigma}_s}(\mathbb{R}^m;X)} \sum_{\boldsymbol{\alpha}\in\mathcal{J}_{\mathbf{N}}} \omega_{\boldsymbol{\alpha}} \boldsymbol{\sigma}_s^{-1}(\boldsymbol{\eta}_{\boldsymbol{\alpha}}).$$

It remains to show that $\sum_{\boldsymbol{\alpha}\in\mathcal{J}_{\mathbf{N}}}\omega_{\boldsymbol{\alpha}}\boldsymbol{\sigma}_s^{-1}(\boldsymbol{\eta}_{\boldsymbol{\alpha}})$ is bounded by a constant independent of m. Therefore, we exploit the fact that the function $\boldsymbol{\sigma}_s(\mathbf{y}) = \prod_{k=1}^m \exp(-s\gamma_k|y_k|)$ as well as the weights $\omega_{\boldsymbol{\alpha}}$ have tensor product structure. Hence, the sum on the right-hand side of (2.5) can be rewritten into a product of univariate Gauss-Hermite quadrature formulae

$$\sum_{\boldsymbol{\alpha}\in\mathcal{J}_{\mathbf{N}}}\omega_{\boldsymbol{\alpha}}\boldsymbol{\sigma}_s^{-1}(\boldsymbol{\eta}_{\boldsymbol{\alpha}}) = \prod_{k=1}^m\sum_{\ell=1}^{N_k}\omega_\ell^{(k)}\exp\left(s\gamma_k\left|\eta_\ell^{(k)}\right|\right).$$

From [Usp28], we know that the univariate Gauss-Hermite quadrature formula converges for $\sigma(y) = \exp(\gamma|y|)$ and arbitrary $\gamma\in\mathbb{R}$. To establish dimension independence of the continuity constant of the tensor product operator, we have to use the decay properties of the sequence γ_k. Since $\{\gamma_k\}_k\in\ell^1(\mathbb{N})$, there exists an index k_0 such that $s\gamma_k<1$ for all $k>k_0$. We assume now, without loss of generality, that it holds $s\gamma_k<1$ for all $k\in\mathbb{N}$. Otherwise, the continuity constant will additionally depend on k_0, but will remain independent on m. This assumption allows us to construct an estimate on the continuity constant of the univariate Gauss-Hermite quadrature operator. In particular, we show that

$$\sum_{\ell=1}^{N_k}\omega_\ell^{(k)}\exp\left(s\gamma_k\left|\eta_\ell^{(k)}\right|\right)\le\frac{1}{1-s\gamma_k}. \tag{2.6}$$

With this estimate at hand, we can conclude with the Taylor expansion of the logarithm at 1 and the summability of the sequence $\{\gamma_k\}_k$, see (IV.2.37), that

$$\begin{aligned}\sum_{\boldsymbol{\alpha}\in\mathcal{J}_{\mathbf{N}}}\omega_{\boldsymbol{\alpha}}\boldsymbol{\sigma}_s^{-1}(\boldsymbol{\eta}_{\boldsymbol{\alpha}}) &\le \prod_{k=1}^m\frac{1}{1-s\gamma_k} = \exp\left(-\sum_{k=1}^m\log(1-s\gamma_k)\right)\\ &\le \exp\left(\sum_{k=1}^\infty\gamma_k + cs^2\sum_{k=1}^\infty\gamma_k^2\right)<\infty\end{aligned}$$

holds with a constant $c>0$. For the proof of (2.6), we follow some arguments from [Usp28]. From there, we know that the Gauss-Hermite quadrature approximates the moments

$$\mathcal{M}^p := \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} y^p\exp\left(-\frac{y^2}{2}\right)\mathrm{d}y,\qquad p\in\mathbb{N},$$

from below. Moreover, it is well known that the moments can be determined by

$$\mathcal{M}^p = \begin{cases}0, & \text{if } p \text{ is odd},\\ (p-1)!!, & \text{if } p \text{ is even},\end{cases}\qquad\text{where}\quad (p-1)!! := (p-1)\cdot(p-3)\cdots 3\cdot 1.$$

For notational convenience, we omit the dependency on k in (2.6) and set $\gamma = s\gamma_k$ with $\gamma<1$. This leads, in combination with the series expansion of the exponential, to

$$\sum_{\ell=1}^N\omega_\ell\exp(\gamma|\eta_\ell|) = \sum_{\ell=1}^N\omega_\ell\sum_{j=1}^\infty\frac{(\gamma|\eta_\ell|)^j}{j!} = \sum_{j=1}^\infty\frac{\gamma^j}{j!}\sum_{\ell=1}^N\omega_\ell|\eta_\ell|^j. \tag{2.7}$$

For even j, the convergence of the Gauss-Hermite quadrature for the moments from below implies that

$$\sum_{\ell=1}^{N} \omega_\ell |\eta_\ell|^j \leq \mathcal{M}^j = (j-1)!!.$$

With the application of the Cauchy-Schwarz inequality, we additionally obtain that it holds

$$\begin{aligned}\sum_{\ell=1}^{N} \omega_\ell |\eta_\ell|^j &= \sum_{\ell=1}^{N} \sqrt{\omega_\ell |\eta_\ell|^{j+1}} \sqrt{\omega_\ell |\eta_\ell|^{j-1}} \leq \sqrt{\sum_{\ell=1}^{N} \omega_\ell |\eta_\ell|^{j+1}} \sqrt{\sum_{\ell=1}^{N} \omega_\ell |\eta_\ell|^{j-1}} \\ &\leq \sqrt{\mathcal{M}^{j-1}\mathcal{M}^{j+1}} = \sqrt{j}\mathcal{M}^{j-1}\end{aligned}$$

for odd $j \in \mathbb{N}$. It is now easily validated that $\sum_{\ell=1}^{N} \omega_\ell |\eta_\ell|^j \leq j!$ for all $j \in \mathbb{N}$. Inserting this into (2.7) yields that

$$\sum_{\ell=1}^{N} \omega_\ell \exp(\gamma |\eta_\ell|) \leq \sum_{j=0}^{\infty} \gamma^j = \frac{1}{1-\gamma}.$$

Hence, we have established the estimate (2.6) which completes the proof. □

With the convergence results of the univariate quadrature at hand, we are able to estimate the error of the tensor product Gaussian quadrature. A crucial ingredient is the analytic extendability of the integrands under consideration which is provided in Lemma (III.5.34) for the lognormal case and in Corollary (III.6.6) for the uniformly elliptic case. The error analysis leads to an anisotropic tensor product quadrature formula since the number of quadrature points in each direction is determined by the region of analytic extension in this particular direction.

2.2 Error estimate for the anisotropic Gaussian quadrature

In this subsection, we analyze the convergence of the anisotropic tensor product Gaussian quadrature. The basis for this analysis are the one-dimensional error estimates in Lemma (1.4) and Lemma (1.10) and the continuity of the asociated multivariate integral and quadrature operators, cf. (2.3) and Lemma (2.4). To estimate the error in the lognormal and in the uniformly elliptic case simultaneously, we make the following assumption on the one-dimensional error estimates.

(2.8) **Assumption.** Let the function $v \in C^0_\sigma(\Gamma; X) \subset L^2_\rho(\Gamma; X)$ be analytically extendable into $\Sigma(\Gamma, \tau)$ where X denotes a Banach space. Moreover, let $Q_{G,N}$ denote the N-point Gaussian quadrature with respect to the density function ρ. Then, there exist functions $g, h\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that the error of the Gaussian quadrature can be bounded by

(2.9) $$\|(I - Q_{G,N})v\|_X \leq g(\tau) \exp\big(-h(\tau)(2N-1)\big) \|v\|_{C^0_\sigma(\Sigma(\Gamma,\tau);X)}.$$

(2.10) **Remark.** According to Lemma (1.3) and Lemma (1.4), the Assumption (2.8) is fulfilled for the Gauss-Hermite quadrature with $\sigma(y) = \sigma_s(y)$, $\Gamma = \mathbb{R}$, $h(\tau) = \log(\sqrt{2}\tau)$ and $g(\tau) = C/(\sqrt{2}\tau - 1)$. The constant C depends, besides on the continuity constants of I and $Q_{GH,N}$, only on a generic constant, cf. (1.9).

For the Gauss-Legendre quadrature, the combination of Lemma (1.3) and Lemma (1.10) yields that the Assumption (2.8) is fulfilled with $\sigma(y) \equiv 1$, $\Gamma = [-1/2, 1/2]$, $h(\tau) = \log(\kappa)$ and $g(\tau) = 4/(\kappa - 1)$ with $\kappa = 2\tau + \sqrt{1 + 4\tau^2}$.

With the one-dimensional error estimate (2.9) at hand, we can establish the following result for the error of the anisotropic tensor product Gaussian quadrature.

(2.11) **Theorem.** For each $k = 1, \ldots, m$, let $v \in C^0_{\boldsymbol{\sigma}}(\boldsymbol{\Gamma}; X) \subset L^2_\rho(\boldsymbol{\Gamma}; X)$ be analytically extendable into $\Sigma(\Gamma_k, \tau_k)$. Additionally, let the tensor product Gaussian quadrature operator $\mathbf{Q}_{G,\mathbf{N}} \colon C^0_{\boldsymbol{\sigma}}(\boldsymbol{\Gamma}; X) \to X$ as well as the integration operator $\mathbf{I} \colon C^0_{\boldsymbol{\sigma}}(\boldsymbol{\Gamma}; X) \to X$ be continuous. If we choose the number of quadrature points N_k in the k-th direction such that

(2.12) $$N_k \geqslant \frac{|\log \varepsilon|}{2h(\tau_k)} + \frac{1}{2},$$

we obtain that the error of $\mathbf{Q}_{G,\mathbf{N}} v$ for approximating the integral $\mathbf{I}v$ is bounded by

(2.13) $$\|(\mathbf{I} - \mathbf{Q}_{G,\mathbf{N}})v\|_X \lesssim \varepsilon \max_{k=1,\ldots,m} \|v\|_{C^0_{\sigma_k}\left(\Sigma(\Gamma_k,\tau_k); C^0_{\boldsymbol{\sigma}^\star_k}(\boldsymbol{\Gamma}^\star_k; X)\right)} \sum_{k=1}^m g(\tau_k),$$

where we use the notation $\boldsymbol{\Gamma}^\star_k = \bigtimes_{i=1, i\neq k} \Gamma_i$, cf. Lemma (III.6.6). Moreover, the constant in (2.13) is only dependent on the continuity constants of $\mathbf{Q}_{G,\mathbf{N}}$ and $\mathbf{I}$.

Proof. We estimate the tensor product quadrature error as usual by the sum of the one dimensional quadrature errors

(2.14) $$\begin{aligned} &\|(\mathbf{I} - \mathbf{Q}_{G,\mathbf{N}})v\|_X \\ &\quad \leq \sum_{k=1}^m \left\| \left(Q^{(1)}_{G,N_1} \otimes \ldots \otimes Q^{(k-1)}_{G,N_{k-1}} \otimes \left(I^{(k)} - Q^{(k)}_{G,N_k} \right) \otimes I^{(k+1)} \otimes \ldots \otimes I^{(m)} \right) v \right\|_X. \end{aligned}$$

With the continuity of the multivariate integral as well as the multivariate Gaussian quadrature operator, we can further deduce that

(2.15) $$\begin{aligned} &\left\| \left(Q^{(1)}_{G,N_1} \otimes \ldots \otimes Q^{(k-1)}_{G,N_{k-1}} \otimes \left(I^{(k)} - Q^{(k)}_{G,N_k} \right) \otimes I^{(k+1)} \otimes \ldots \otimes I^{(m)} \right) v \right\|_X \\ &\qquad \lesssim \sup_{\mathbf{y}^\star_k \in \boldsymbol{\Gamma}^\star_k} \boldsymbol{\sigma}^\star_k(\mathbf{y}^\star_k) \left\| \left(I^{(k)} - Q^{(k)}_{G,N_k} \right) v(\mathbf{y}^\star_k) \right\|_X. \end{aligned}$$

Next, we employ the one-dimensional error estimate (2.9) which yields that

$$\begin{aligned} &\left\| \left(I^{(k)} - Q^{(k)}_{G,N_k} \right) v(\mathbf{y}^\star_k) \right\|_X \\ &\qquad \leq g(\tau_k) \exp\Big(-h(\tau_k)(2N_k - 1) \Big) \sup_{z \in \Sigma(\Gamma_k,\tau_k)} \sigma_k(\mathrm{Re}(z)) \|v(z, \mathbf{y}^\star_k)\|_X. \end{aligned}$$

Inserting this estimate into (2.15), we derive that

$$\begin{aligned}(2.16)\qquad &\Big\|\Big(Q_{G,N_1}^{(1)}\otimes\ldots\otimes Q_{G,N_{k-1}}^{(k-1)}\otimes\Big(I^{(k)}-Q_{G,N_k}^{(k)}\Big)\otimes I^{(k+1)}\otimes\ldots\otimes I^{(m)}\Big)v\Big\|_X\\ &\qquad\lesssim g(\tau_k)\exp\Big(-h(\tau_k)(2N_k-1)\Big)\|v(z)\|_{C^0_{\sigma_k}\left(\Sigma(\Gamma_k,\tau_k);C^0_{\boldsymbol{\sigma}_k^\star}(\boldsymbol{\Gamma}_k^\star;X)\right)}.\end{aligned}$$

With the choice (2.12) of the number of quadrature points and for a desired accuracy $\varepsilon<1$, the assertion follows from

$$\exp\Big(-h(\tau_k)(2N_k-1)\Big)\le\exp\bigg(-h(\tau_k)\frac{|\log\varepsilon|}{h(\tau_k)}\bigg)=\exp(-|\log(\varepsilon)|)=\varepsilon$$

and the summation in (2.14) over the terms in (2.16). □

Theorem (2.11) implies that the error of the anisotropic tensor product Gaussian quadrature is of order ε when we can bound the associated one-dimensional Gaussian quadrature errors by (2.9). If the sequence $\{g(\tau_k)\}_k$ is additionally summable, we can also bound the constant independently of m. Therefore, we investigate whether the sequence $\{g(\tau_k)\}_k$ is summable in case of the Gauss-Hermite and the Gauss-Legendre quadrature. To this end, we have to exploit the behaviour of τ_k for the integrands under consideration. In particular, we relate the increase of τ_k to the decrease of γ_k in the lognormal case by the following remark.

(2.17) **Remark.** (a) Notice that the regularity result (III.5.29) for the solution u_m to (III.3.5), the regularity result (III.5.30) for u_m^2, and, if $f\in L^p(D)$, the regularity result (III.5.32) for u_m^p imply the conditions of Lemma (III.5.34). This, together with Lemma (2.4) verifies the conditions of Theorem (2.11) in the lognormal case.

(b) Furthermore, the regions of analytic extendability $\{\tau_k\}_k$ can be chosen in such a way that $\{\tau_k^{-1}\}_k$ is summable due to Assumption (III.4.8) on the weights $\{\gamma_k\}_k$. Indeed, we have by (III.4.8) that the sequence $\{\gamma_k\}_k$ belongs to $\ell^1(\mathbb{N})$. Now, we have that the solution u_m to (III.3.5) and the powers of u_m, respectively, fulfill the requirements (III.5.35) in Theorem (III.5.34) with $\mu_k=\gamma_k/\log 2$ and $\mu_k=pC(p,D)\gamma_k/\log 2$, respectively. This implies that μ_k has the same asymptotic decay behaviour as γ_k. Since the region of analyticity in the k-th direction has to satisfy $\tau_k<1/\mu_k$, we can, for example, bound the region of analyticity by $\tau_k=1/2\mu_k$. This ensures that τ_k^{-1} has the same asymptotic decay behaviour as μ_k and, hence, as γ_k.

(c) For each $k=1,\ldots,m$, the quantity

$$\|v(z)\|_{C^0_{\sigma_k}\left(\Sigma(\Gamma_k,\tau_k);C^0_{\boldsymbol{\sigma}_k^\star}(\boldsymbol{\Gamma}_k^\star;X)\right)}=\sup_{z\in\Sigma(\mathbb{R},\tau_k)}\sigma_{s,k}\big(\operatorname{Re}(z)\big)\|v(z)\|_{C^0_{\boldsymbol{\sigma}_{s,k}^\star}(\mathbb{R}^{m-1};X)}$$

in Theorem (2.11) is bounded by a constant times $\|f\|_{L^2(D)}$ for the computation of the mean and by a constant times $\|f\|_{L^p(D)}^p$ for the computation of the p-th moment. Nevertheless, the constant obviously depends on the choice of τ_k and tends to infinity if τ_k comes close to the boundary of the analyticity region, i.e. if $\tau_k\to 1/\mu_k$, as can be seen from the proof of Lemma (III.5.34).

From Remark (2.17), we conclude that τ_k^{-1} has, up to a constant, the same behaviour as γ_k in the lognormal case. Due to the similarity of the regularity results from Section III.6 and with the same argumentation as above, we also obtain that $\tau_k^{-1} \eqsim \gamma_k$ in the uniformly elliptic case. Hence, the summability of the sequence $\{g(\tau_k)\}_k$ follows immediately from the summability of $\{\tau_k^{-1}\}_k$ in the uniformly elliptic as well as the lognormal case. This can be seen in the lognormal case from

$$g(\tau_k) = \frac{C}{\sqrt{2}\tau_k - 1} \tag{2.18}$$

and the assumption that $\tau_k < \sqrt{2}$ for all k, cf. Remark (1.7). In the uniformly elliptic case, the summability of $\{g(\tau_k)\}_k$ follows from the summability of $\{\tau_k^{-1}\}_k$ due to

$$g(\tau_k) = \frac{4}{\kappa_k - 1} \leq \frac{2}{\tau_k}. \tag{2.19}$$

Hence, the quadrature error in Theorem (2.11) is, for the multivariate Gauss-Hermite and the multivariate Gauss-Legendre quadrature, of order ε with a constant which is independent of m.

2.3 Cost complexity of the anisotropic Gaussian quadrature

The next step is to analyze the cost, i.e. the number of quadrature points which are required to provide the error $\mathcal{O}(\varepsilon)$. It is obvious that the number of quadrature points is given by the cardinality of the set $\mathcal{J}_{\mathbf{N}}$ which is simply the product of the number of quadrature points of the univariate quadrature formulae, i.e.

$$\operatorname{cost}(\mathbf{Q}_{G,\mathbf{N}(\varepsilon)}, m) = \#(\mathcal{J}_{\mathbf{N}(\varepsilon)}) = \prod_{k=1}^{m} N_k(\varepsilon).$$

The number of quadrature points in each direction is determined by (2.12). Thus, it holds that

$$\operatorname{cost}(\mathbf{Q}_{G,\mathbf{N}(\varepsilon)}, m) = \prod_{k=1}^{m} \left\lceil \frac{|\log\varepsilon|}{2h(\tau_k)} + \frac{1}{2} \right\rceil \leq \prod_{k=1}^{m} \left(\frac{|\log\varepsilon|}{h(\tau_k)} + 1 \right). \tag{2.20}$$

The inequality follows from $\lceil \frac{x}{2} \rceil \leq x$ for $x \geq 1$. From the following lemma, we know that the cost in (2.20) can be bounded independently of m if the sequence $\{h(\tau_k)_k\}_k$ is summable.

(2.21) **Lemma.** If the sequence $\{h(\tau_k)_k\}_k$ in (2.12) is summable, then there exists for each $\delta_1, \delta_2 > 0$ a constant $C(\delta_1, \delta_2)$ independent of m and ε such that the cost in (2.20) can be bounded by

$$\operatorname{cost}(\mathbf{Q}_{G,\mathbf{N}(\varepsilon)}, m) \leq C(\delta_1, \delta_2)\varepsilon^{-\delta_1-\delta_2}. \tag{2.22}$$

Proof. From the summability of $\{h(\tau_k)^{-1}\}_k$, it follows that there exists a $j_0 \in \mathbb{N}$ for each $\delta_1 > 0$ such that

$$\sum_{k=j_0+1}^{\infty} h(\tau_k)^{-1} \leq \delta_1. \tag{2.23}$$

Hence, we can split the product in (2.20) into

$$\text{cost}(\mathbf{Q}_{G,\mathbf{N}(\varepsilon)}, m) \leq \prod_{k=1}^{j_0} \left(\frac{|\log \varepsilon|}{h(\tau_k)} + 1 \right) \prod_{k=j_0+1}^{m} \left(\frac{|\log \varepsilon|}{h(\tau_k)} + 1 \right). \tag{2.24}$$

We assume here that $m > j_0$ since we are interested in the asymptotic behaviour when $\varepsilon \to 0$ which implies $m \to \infty$. With (2.23), the second factor can simply be estimated by

$$\prod_{k=j_0+1}^{m} \left(\frac{|\log \varepsilon|}{h(\tau_k)} + 1 \right) = \exp\left(\sum_{k=j_0+1}^{m} \log\left(\frac{|\log \varepsilon|}{h(\tau_k)} + 1 \right) \right) \leq \varepsilon^{-\delta_1}.$$

The number of factors j_0 in the first product in (2.24) is fixed and depends only on the choice of δ_1 and on the decay properties of $\{h(\tau_k)^{-1}\}_k$, cf. (2.23). Hence, j_0 is particularly independent of the desired accuracy ε and, thus, also independent of m. Since j_0 is a fixed natural number, there exists for all $\delta_2 > 0$ a constant $C(\delta_1, \delta_2)$, independent of ε and m, such that

$$\prod_{k=1}^{j_0} \left(\frac{|\log \varepsilon|}{h(\tau_k)} + 1 \right) \leq C(\delta_1, \delta_2) \varepsilon^{-\delta_2}.$$

Indeed, there is for arbitrary $\delta_2 > 0$ a suitable constant $c > 0$ which depends on δ_2 and j_0 such that $\log(x)^{j_0} \leq c x^{\delta_2}$. This establishes the desired inequality (2.22). □

(2.25) **Remark.** Unfortunately, the summability of the sequence $\{h(\tau_k)^{-1}\}_k$ requires that the regions of analyticity τ_k increase like $\tau_k \gtrsim \exp(ck^{1+\eta})$ for some $\eta > 0$. This corresponds to a superexponentially decay of $\{\gamma_k\}_k$, i.e. $\gamma_k \lesssim \exp(-ck^{1+\eta})$. Such a decay rate is not obtained by any correlation kernel of the Matérn class. An example for a correlation kernel which provides such a decay behaviour for γ_k is given in [NTW08b].

Since correlation kernels of the Matérn class do not provide the necessary decay properties on the sequence $\{\gamma_k\}_k$ to obtain dimension-independent convergence rates for the moment computation with anisotropic Gaussian quadrature rules, we investigate how the convergence rate deteriorates with the dimension for an algebraic decay $\gamma_k \lesssim k^{-s_1}$, see (III.4.8). Hence, the sequence $\{\tau_k\}_k$ increases with the same rate $\tau_k \gtrsim k^{s_1}$, cf. Remark (2.17). This implies that the sequence $\{h(\tau_k)\}_k$ increases in the lognormal case like

$$h(\tau_k) = \log(\sqrt{2}\tau_k) \geq \log\left(c_1 k^{s_1}\right) \tag{2.26}$$

and in the uniform elliptic case like

$$h(\tau_k) = \log(\kappa_k) \geq \log(4\tau_k) \geq \log\left(c_2 k^{s_1}\right). \tag{2.27}$$

For simplicity of the further calculation, we assume that $c_1 = c_2 = 2$. Then, we can estimate the right-hand side in the cost estimate (2.20) by

$$\begin{aligned}\prod_{k=1}^{m} \left(\frac{|\log \varepsilon|}{h(\tau_k)} + 1 \right) &= \left(\frac{|\log \varepsilon|^2}{\log(2)} + 1 \right) \left(\frac{|\log \varepsilon|^2}{\log(2 \cdot 2^{s_1})} + 1 \right) \exp\left(\sum_{k=3}^{m} \log\left(\frac{|\log(\varepsilon)|}{\log(2k^{s_1})} + 1 \right) \right) \\ &\leq \left(\frac{|\log \varepsilon|^2}{\log(2)} + 1 \right) \left(\frac{|\log \varepsilon|^2}{\log(2 \cdot 2^{s_1})} + 1 \right) \exp\left(\sum_{k=3}^{m} \frac{|\log(\varepsilon)|}{\log(2k^{s_1})} \right).\end{aligned}$$

The first two factors can be estimated as in (2.21) by $C(\delta)\varepsilon^{\delta}$ for arbitrary $\delta > 0$. Therefore, the essential term which dominates the cost for large values of m is the third factor. We can estimate the sum in the third factor by

$$\sum_{k=3}^{m} \frac{1}{\log(2k^{s_1})} \leq \sum_{k=1}^{m} \frac{1}{\log(k^{s_1})} \leq \frac{1}{s_1} \int_2^m \log(x)^{-1} \, \mathrm{d}x = \frac{\mathrm{Li}(m)}{s_1}. \tag{2.28}$$

The evaluation of the Eulerian logarithmic integral $\mathrm{Li}(x)$ at m in (2.28) behaves like $\mathcal{O}(m/\log(m))$, see [AS64]. Both expressions, $\mathrm{Li}(m)$ and $m/\log(m)$, are used as estimates for the number of primes contained in the first m natural numbers. Thus, it is known that $\mathrm{Li}(m)\log(m)/m \to 1$ for $m \to \infty$. Moreover, it is also known that $\mathrm{Li}(m) > \log(m)/m$ and, hence, the constant in the $\mathcal{O}$-notation converges to 1 from above for $m \to \infty$.

Employing the estimate on the Eulerian logarithmic integral, we end up with the estimate of the cost

$$\mathrm{cost}(\mathbf{Q}_{GL,\mathbf{N}(\varepsilon)}, m) \lesssim \varepsilon^{-\frac{C}{s_1}\frac{m}{\log(m)}}. \tag{2.29}$$

The bound (2.29) implies that the convergence rate of the anisotropic tensor product Gaussian quadrature is algebraic and deteriorates quite quickly with the dimensionality.

(2.30) **Remark.** (a) To get an impression, how the term $m/(s_1 \log(m))$ behaves, we calculate for $s_1 = 4$ and $m = 15$, that $m/(s_1 \log(m)) \approx 1.3848$. Using the correct Eulerian logarithmic integral instead of the approximation $m/\log(m)$, we achieve $\mathrm{Li}(15)/4 \approx 1.7774$. The above setting corresponds to the consideration of the kernel of the Matérn class with $\nu = 7/2$, defined on the unit interval $[0,1]$ and truncated after 15 stochastic dimensions. For this example, we expect a convergence rate of the anisotropic Gaussian quadrature method of $\varepsilon^{-1.7774}$ which would be still superior to the convergence rate of a Monte-Carlo method. Nevertheless, any offset in the decay of the sequence $\{\gamma_k\}_k$ remarkably compromises the convergence results.

(b) In this section, we investigated the convergence and cost complexity of the anisotropic tensor product Gaussian quadrature. It turns out that the decay conditions on the sequence $\{\gamma_k\}_k$ are very strong, i.e. $\gamma_k \lesssim \exp(k^{-1-\eta})$ for arbitrary $\eta > 0$, in order to obtain dimension-independent convergence rates. We further analyzed how the convergence rate deteriorates with the dimensionality m for algebraic decaying sequences $\{\gamma_k\}_k$. As expected, for large values of m, the convergence rate becomes worse than the convergence rate of a Monte Carlo quadrature. Nevertheless, we achieve considerably improved results in comparison with an isotropic Gaussian quadrature rule. This yields that the anisotropic tensor product Gaussian quadrature performs, for moderate values of m, like $m = 10$ or $m = 15$, and sufficiently algebraic decay of $\{\gamma_k\}_k$, still comparable to a Monte Carlo quadrature.

3. Anisotropic sparse Gaussian quadrature

3.1 Definition of the sparse Gaussian quadrature

In this section, we construct anisotropic sparse Gaussian quadrature methods. Such methods are sparse Smolyak type quadratures, cf. [Smo63], and very similar to the anisotropic sparse collocation method based on Gaussian collocation points which has been introduced in [NTW08b].

We start by considering an increasing sequence of univariate Gaussian quadrature points

$$\theta_j := \{\eta_{i,j}\}_{i=1}^{N_j} \subset \mathbb{R}, \quad N_j \in \mathbb{N}, \quad j = 1, 2, \ldots, \tag{3.1}$$

where $N_1 \le N_2 \le \ldots$. The associated Gaussian quadrature weights are denoted by $\{\omega_{i,j}\}_{i=1}^{N_j}$ and the associated Gaussian quadrature operators are denoted by $Q_{G,j}$. For a multi-index $\boldsymbol{\alpha} \in \mathbb{N}^m$, we define the multidimensional tensor product quadrature operator by

$$\mathbf{Q}_{G,\boldsymbol{\alpha}} := Q_{G,\alpha_1}^{(1)} \otimes \cdots \otimes Q_{G,\alpha_m}^{(m)}.$$

Following the notation of [NTW08a], we introduce for $j \in \mathbb{N}$ the difference quadrature operator

$$\Delta_j := Q_{G,j} - Q_{G,j-1}, \quad \text{where} \quad Q_{G,-1} := 0. \tag{3.2}$$

With the telescoping sum $Q_{G,j} = \sum_{\ell=0}^{j} \Delta_\ell$, the isotropic tensor product quadrature operator $\mathbf{Q}_{G,\mathbf{j}}$, which uses in each direction N_j quadrature points, can be rewritten by

$$\mathbf{Q}_{G,\mathbf{j}} = \sum_{\|\boldsymbol{\alpha}\|_\infty \le j} \Delta_{\alpha_1}^{(1)} \otimes \ldots \otimes \Delta_{\alpha_m}^{(m)}, \quad \text{where } \mathbf{j} = (j, \ldots, j) \in \mathbb{N}^m. \tag{3.3}$$

The cost of applying the isotropic full tensor product quadrature operator (3.3) is obviously given by the number of points N_j^m contained in it. Thus, this isotropic tensor product quadrature extremely suffers from the curse of dimensionality. The classical *sparse Gaussian quadrature*, cf. [GG98, BG04], can overcome this obstruction up to a certain extent. It is based on linear combinations of tensor product quadrature formulae of relatively small size. To define the sparse Gaussian quadrature, we introduce as in [NTW08a, BNR00] for each *approximation level* q the sets of multi-indices

$$X(q,m) := \left\{ \mathbf{0} \le \boldsymbol{\alpha} \in \mathbb{N}^m \colon \sum_{n=1}^{m} \alpha_n \le q \right\}$$

and

$$Y(q,m) := \left\{ \mathbf{0} \le \boldsymbol{\alpha} \in \mathbb{N}^m \colon q - m + 1 \le \sum_{n=1}^{m} \alpha_n \le q \right\}.$$

The Smolyak quadrature operator, cf. [Smo63, GG98], is then given by

$$\mathcal{A}_G(q,m) := \sum_{\boldsymbol{\alpha} \in X(q,m)} \Delta_{G,\alpha_1}^{(1)} \otimes \ldots \otimes \Delta_{G,\alpha_m}^{(m)}. \tag{3.4}$$

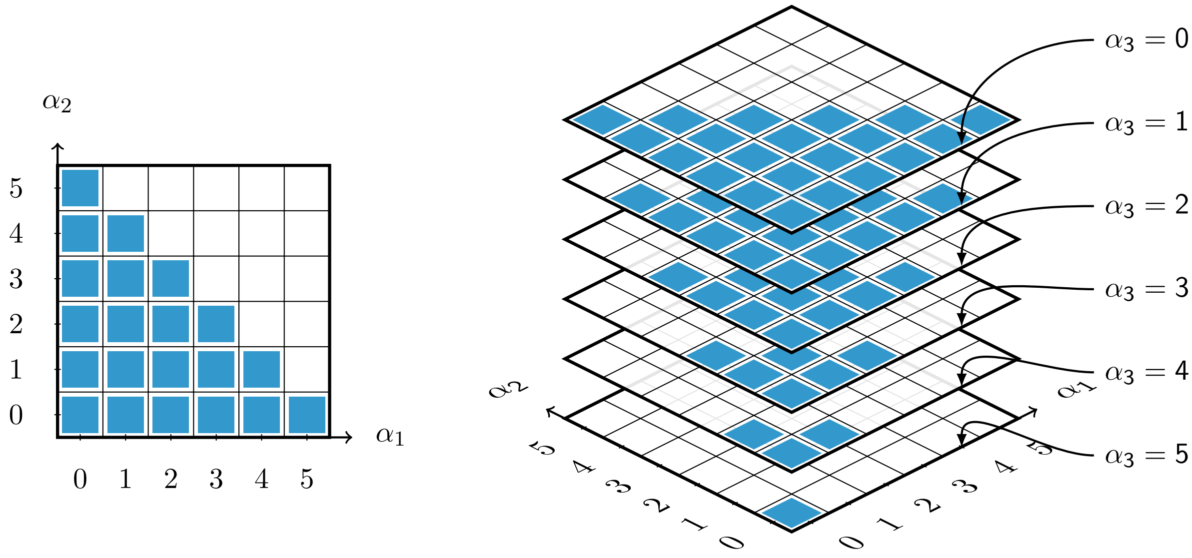


Figure V.1: The 21 indices contained in the sparse grid $X(5, 2)$ on the left and the 56 indices contained in $X(5, 3)$ on the right.

An equivalent expression is obtained by the *combination technique* [GSZ92]

$$(3.5) \qquad \mathcal{A}_G(q, m) = \sum_{\boldsymbol{\alpha} \in Y(q,m)} (-1)^{q-|\boldsymbol{\alpha}|} \binom{m-1}{q-|\boldsymbol{\alpha}|} \overbrace{\left(Q^{(1)}_{G,\alpha_1} \otimes \ldots \otimes Q^{(m)}_{G,\alpha_m}\right)}^{=\mathbf{Q}_{G,\boldsymbol{\alpha}}}.$$

A visualization of the set of indices $X(q, m)$ is given in Figure V.1.

The number of quadrature points used in the above approach is considerably reduced compared to the full tensor product quadrature. But it is not taken into account that the different stochastic dimensions are of different importance to the solution u_m. In fact, the cardinality of the set $X(q, m)$ is given by

$$\#X(q, m) = \binom{q+m}{m}$$

which still grows exponentially in the stochastic dimension m. Thus, we equip each stochastic dimension with a weight and use a weighted version of the Smolyak quadrature operator. Let $\mathbf{w} \in \mathbb{R}^m_+$ denote a weight vector for the different stochastic dimensions. We assume in the following that the weight vector is sorted in ascending order, i.e. $w_1 \leq w_2 \leq \ldots \leq w_m$. Otherwise, we would rearrange the stochastic dimensions. We modify the sparse grid sets $X(q, m)$ and $Y(q, m)$ in the following way, see also [NTW08b],

$$(3.6) \qquad X_{\mathbf{w}}(q, m) := \left\{ \mathbf{0} \leq \boldsymbol{\alpha} \in \mathbb{N}^m \colon \sum_{n=1}^{m} \alpha_n w_n \leq q \right\}$$

and

$$(3.7)\qquad Y_{\mathbf{w}}(q,m) := \left\{ \mathbf{0} \le \boldsymbol{\alpha} \in \mathbb{N}^m \colon q - |\mathbf{w}| < \sum_{n=1}^{m} \alpha_n w_n \le q \right\}.$$

With this notation at hand, the *anisotropic Smolyak quadrature operator* of level $q \in \mathbb{N}$ is defined by

$$(3.8)\qquad \mathcal{A}_{G,\mathbf{w}}(q,m) := \sum_{\boldsymbol{\alpha} \in X_{\mathbf{w}}(q,m)} \Delta_{G,\alpha_1}^{(1)} \otimes \ldots \otimes \Delta_{G,\alpha_m}^{(m)}$$

which can equivalently be expressed as, cf. [NTW08b],

$$(3.9)\qquad \mathcal{A}_{G,\mathbf{w}}(q,m) = \sum_{\boldsymbol{\alpha} \in Y_{\mathbf{w}}(q,m)} c_{\mathbf{w}}(\boldsymbol{\alpha}) \mathbf{Q}_{G,\boldsymbol{\alpha}}, \quad \text{with} \quad c_{\mathbf{w}}(\boldsymbol{\alpha}) := \sum_{\substack{\boldsymbol{\beta} \in \{0,1\}^m \\ \boldsymbol{\alpha}+\boldsymbol{\beta} \in X_{\mathbf{w}}(q,m)}} (-1)^{|\boldsymbol{\beta}|}.$$

The formula (3.9) can be regarded as the *anisotropic combination technique quadrature*. For the evaluation of this formula, we only need to determine the coefficients $c_{\mathbf{w}}(\boldsymbol{\alpha})$ and to apply tensor product quadrature formulae of relatively small size. Thus, in order to compute the approximation of the moments to the solution u_m to (III.3.3) or to (III.3.5) with the anisotropic Smolyak quadrature (3.9), it is sufficient to evaluate the integrand $v = u_m^p$ on the *anisotropic sparse grid*

$$\mathcal{J}_{\mathbf{w}}(q,m) := \bigcup_{\boldsymbol{\alpha} \in Y_{\mathbf{w}}(q,m)} \theta_{\alpha_1} \times \ldots \times \theta_{\alpha_m}.$$

Note that the Smolyak quadrature operator (3.4) coincides with the anisotropic Smolyak quadrature operator (3.8) for the special weight vector $\mathbf{w} = \mathbf{1}$.

In Figure V.2, the indices of the weighted sparse grid $X_{(1,2.5)}(5,2)$ and of the weighted sparse grid $X_{(1,2,3)}(5,3)$ are visualized. We observe that the number of indices is drastically reduced in comparison to the according isotropic sparse grids visualized in Figure V.1.

3.2 Preliminaries for the convergence analysis

In order to analyze the approximation error of the anisotropic sparse Gaussian quadrature method, we provide in this subsection some preliminary results. Firstly, we remember that the solution u_m to (III.3.3) or (III.3.5) and its powers are, in view of Corollary (III.6.8) and Theorem (III.5.38), analytically extendable in the following sense:

(3.10) **Lemma.** Let $X = H_0^1(D)$ for $p = 1$ and $X = W_0^{1,1}(D)$ for $p \ge 2$. The powers u_m^p for $p \ge 1$ of the solution u_m to (III.3.3) in the uniformly elliptic case admit an analytic extension into the region $\boldsymbol{\Sigma}(\boldsymbol{\Gamma}, \boldsymbol{\tau})$ for all $\boldsymbol{\tau}$ with

$$\tau_k < \frac{\underline{a}}{C(\delta) k^{1+\delta} C(p,D) p \gamma_k}, \quad \text{where} \quad C(\delta) = \sum_{k=1}^{\infty} k^{-1-\delta} \text{ for arbitrary } \delta > 0.$$

In accordance with the regularity estimate (III.6.2) for u_m and the estimate (III.6.4) for u_m^2, we have that $C(1,D) = C(2,D) = 1$. For $p > 2$, the constant $C(p,D)$ is given as in

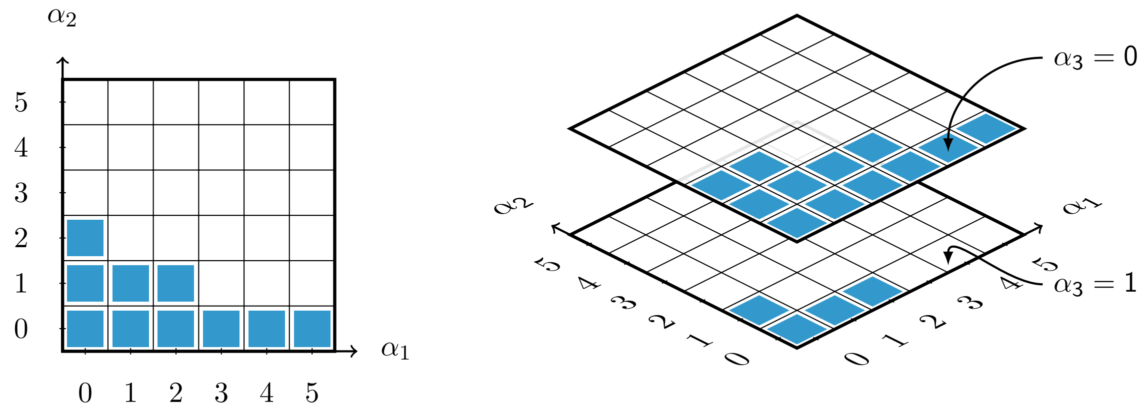


Figure V.2: The 10 indices contained in the weighted sparse grid $X_{(1,2.5)}(5,2)$ on the left and the 16 indices contained in $X_{(1,2,3)}(5,3)$ on the right.

(III.5.21) according to the regularity estimates for the higher order moments in Corollary (III.6.3). In addition, it holds that

$$\|u_m^p\|_{C^0(\boldsymbol{\Sigma}([-1/2,1/2]^m,\boldsymbol{\tau});X)} \lesssim \|f\|^p_{L^{p+\delta_{1,p}}(D)} \quad \text{with} \quad \delta_{1,p} = \begin{cases} 1, & \text{if } p = 1, \\ 0, & \text{if } p > 1. \end{cases}$$

In the lognormal case, the solution u_m to (III.3.5) and its powers are analytically extendable into $\boldsymbol{\Sigma}(\mathbb{R}^m, \boldsymbol{\tau})$ provided that

$$\tau_k < \frac{\log 2}{C(\delta)k^{1+\delta}C(p,D)p\gamma_k}.$$

Moreover, the p-th moment u_m^p is bounded in accordance with

$$\|u_m^p\|_{C^0_{\boldsymbol{\sigma}_s}(\boldsymbol{\Sigma}(\mathbb{R}^m,\boldsymbol{\tau});X)} \lesssim \|f\|^p_{L^{p+\delta_{1,p}}(D)}$$

for a weight function $\boldsymbol{\sigma}_s$ with $s \geq 2p$ if $p \in \{1,2\}$ and with $s \geq 3p$ if $p \geq 3$. The constants which are involved in the estimates depend on the choice of $\boldsymbol{\tau}$ and s, but are independent of m.

Notice that, due to the Assumption (III.4.8) which states that $\gamma_k \lesssim k^{-s_1}$ for a $s_1 > 1$, there always is an $r > 0$ such that the sequence $\{\tau_k\}_k$ which describes the region of analytic extension satisfies

$$\tau_k \gtrsim k^r. \tag{3.11}$$

The sequence $\{\tau_k\}_k$ is therefore increasing and tends to infinity.

For the further error analysis, we extend some results established in [NTW08b]. As for the Gaussian tensor product quadrature, we base our convergence results on the one-dimensional error estimate (2.9). According to the estimate (2.9), we choose in (3.1) the sequence $\{N_j\}_j$ of the number of quadrature points as

$$N_j = \left\lceil \frac{1}{2}(j+2) \right\rceil. \tag{3.12}$$

Then, we can estimate the error of the difference Gaussian quadrature operator $\Delta_{G,j} := Q_{G,j} - Q_{G,j-1}$ for all $j \geq 1$ and for all functions $v \in C_\sigma^0(\Gamma; X)$ which are analytically extendable in $\Sigma(\Gamma, \tau)$ by

$$\begin{aligned} \|\Delta_{G,j} v\|_X &\leq \|v - Q_{G,j} v\|_X + \|v - Q_{G,j-1} v\|_X \\ &\leq g(\tau)\Big(e^{-h(\tau)(j+1)} + e^{-h(\tau)j}\Big)\|v\|_{C_\sigma^0(\Sigma(\Gamma,\tau);X)} \\ &\leq g(\tau)\Big(1 + e^{-h(\tau)}\Big)e^{-h(\tau)j}\|v\|_{C_\sigma^0(\Sigma(\Gamma,\tau);X)} \\ &\leq 2g(\tau)e^{-h(\tau)j}\|v\|_{C_\sigma^0(\Sigma(\Gamma,\tau);X)}. \end{aligned} \tag{3.13}$$

For $j = 0$, the difference Gaussian quadrature operator coincides with the function evaluation at the midpoint z of Γ which implies that

$$\|\Delta_{G,0} v\|_X = \|Q_{G,0} v\|_X = \|v(z)\|_X \leq e^{-h(\tau)\cdot 0}\|v\|_{C_\sigma^0(\Sigma(\Gamma,\tau);X)}. \tag{3.14}$$

Analogously, it follows from (3.12) and (2.9) that

$$\|Iv - Q_{G,j} v\|_X \leq g(\tau)e^{-h(\tau)(j+1)}\|v\|_{C_\sigma^0(\Sigma(\Gamma;\tau);X)}. \tag{3.15}$$

Now, let us assume that the multivariate integrand can be analytically extended into the region $\boldsymbol{\Sigma}(\boldsymbol{\Gamma}, \boldsymbol{\tau})$ as validated for the integrands under consideration in Lemma (3.10). Then, it follows that the error of the tensor product of the operators $\Delta_{G,j}$ is bounded by the product of the one-dimensional errors. Indeed, we obtain for a multi-index $\boldsymbol{\alpha} \in \mathbb{N}^m$ that

$$\begin{aligned} &\Big\|\Big(\Delta_{G,\alpha_1}^{(1)} \otimes \cdots \otimes \Delta_{G,\alpha_m}^{(m)}\Big)v\Big\|_X \\ &\quad \leq \max(2g(\tau_1), 1-\alpha_1)e^{-h(\tau_1)\alpha_1} \sup_{z\in\Sigma(\Gamma_1,\tau_1)} \sigma_1(\mathrm{Re}(z))\Big\|\Big(\Delta_{G,\alpha_2}^{(2)} \otimes \cdots \otimes \Delta_{G,\alpha_m}^{(m)}\Big)v(z)\Big\|_X \\ &\quad \leq \left(\prod_{k=1}^m \max(2g(\tau_k), 1-\alpha_k)\right)e^{-\sum_{k=1}^m h(\tau_k)\alpha_k}\|v\|_{C_{\boldsymbol{\sigma}}^0(\boldsymbol{\Sigma}(\boldsymbol{\Gamma},\boldsymbol{\tau});X)}. \end{aligned} \tag{3.16}$$

The product in the above estimate is bounded by a constant which is independent of the dimensionality m if the sequence $\{g(\tau_k)\}_k$ tends to zero as k increases. In addition, we only take the maximum in this product in order to ensure that the constant is not less than 1 if $\alpha_k = 0$.

3.3 Error estimate for the anisotropic sparse Gaussian quadrature

With the above preliminaries, we are able to establish error estimates for the anisotropic sparse Gaussian quadrature. Therefore, we adapt some parts of the analysis in [NTW08b], but then conclude in a different way.

(3.17) **Lemma.** Let $v : \boldsymbol{\Gamma} \to X$ be analytically extendable into the region $\boldsymbol{\Sigma}(\boldsymbol{\Gamma}, \boldsymbol{\tau})$ and let the univariate Gaussian quadrature estimate (2.9) hold for each dimension $k = 1, \ldots, m$. Moreover, let the sequence of quadrature points be chosen as in (3.12) and let the weight vector $\mathbf{w}$ be given by $w_k = h(\tau_k)$. Then, the error of the anisotropic sparse Gaussian quadrature (3.4) is bounded by

$$\begin{aligned} &\left\|(\mathbf{I} - \mathcal{A}_{G,\mathbf{w}}(q,m))v\right\|_X \\ &\lesssim \max_{k=1,\ldots,m-1} C(k) e^{-q}\left(g(\tau_1) + \sum_{k=1}^{m-1} g(\tau_{k+1})\#X_{\mathbf{w}_{1:k}}(q,k)\right)\|v\|_{C^0_{\boldsymbol{\sigma}}(\boldsymbol{\Sigma}(\boldsymbol{\Gamma},\boldsymbol{\tau});X)} \end{aligned} \tag{3.18}$$

where we use the notation $\mathbf{w}_{1:k} = [w_1, \ldots, w_k]^\intercal$ and

$$X_{\mathbf{w}_{1:k}}(q,k) = \left\{\boldsymbol{\alpha} \in \mathbb{N}^k : \sum_{n=1}^{k} \alpha_n w_n \leq q\right\}.$$

Furthermore, the constants $C(k)$ in (3.18) are given by

$$C(k) = \prod_{n=1}^{k} \max(g(\tau_n), 1)$$

and the constant hidden in (3.18) is the continuity constant of $\mathbf{I}$.

Proof. In the same way as in [NTW08b], the error of the sparse quadrature is rewritten, with the notation $\mathbf{I} = \bigotimes_{n=1}^{m} I^{(n)}$, by

$$\mathbf{I} - \mathcal{A}_{G,\mathbf{w}}(q,m) = \sum_{k=1}^{m} R(q,k) \bigotimes_{n=k+1}^{m} I^{(n)}. \tag{3.19}$$

The quantity $R(q,k)$ is defined for $k \geq 2$ by

$$R(q,k) := \sum_{\boldsymbol{\alpha}\in X_{\mathbf{w}_{1:k-1}}(q,k-1)} \bigotimes_{n=1}^{k-1} \Delta^{(n)}_{G,\alpha_n} \otimes \left(I^{(k)} - Q_{G,\left\lfloor\left(q-\sum_{n=1}^{k-1}\alpha_n w_n\right)/w_k\right\rfloor}\right)$$

and for $k = 1$ by

$$R(q,1) := I^{(1)} - Q_{G,\lfloor q/w_1\rfloor}.$$

For $k > 2$, each summand in (3.19) can be estimated with (3.15), (3.16) and with the continuity of the integration operator by

$$\begin{aligned} &\left\|\left(R(q,k) \bigotimes_{n=k+1}^{m} I^{(n)}\right)v\right\|_X \\ &\lesssim \sum_{\boldsymbol{\alpha}\in X_{\mathbf{w}_{1:k-1}}(q,k-1)} \left(\prod_{n=1}^{k-1} \max(g(\tau_n),1)\right) e^{-\sum_{n=1}^{k-1}\alpha_n h(\tau_n)} \\ &\qquad \cdot g(\tau_k) e^{-h(\tau_k)\left(\left\lfloor\left(q-\sum_{n=1}^{k-1}\alpha_n w_n\right)/w_k\right\rfloor+1\right)} \|v\|_{C^0_{\boldsymbol{\sigma}}(\boldsymbol{\Sigma}(\boldsymbol{\Gamma}^m,\boldsymbol{\tau});X)} \\ &\lesssim g(\tau_k) \sum_{\boldsymbol{\alpha}\in X_{\mathbf{w}_{1:k-1}}(q,k-1)} e^{-h(\tau_k)\left(\left\lfloor\left(q-\sum_{n=1}^{k-1}\alpha_n w_n\right)/w_k\right\rfloor+1\right)-\sum_{n=1}^{k-1}\alpha_n h(\tau_n)} \\ &\qquad \cdot \|v\|_{C^0_{\boldsymbol{\sigma}}(\boldsymbol{\Sigma}(\boldsymbol{\Gamma}^m,\boldsymbol{\tau});X)}. \end{aligned}$$

With the choice $w_n = h(\tau_n)$ for all $n = 1, \ldots, m$, it follows that

$$\begin{aligned}\left\|\left(R(q,k)\bigotimes_{n=k+1}^{m} I^{(n)}\right)v\right\|_X &\lesssim g(\tau_k)\sum_{\boldsymbol{\alpha}\in X_{\mathbf{w}_{1:k-1}}(q,k-1)} e^{-q-\sum_{n=1}^{k-1}\alpha_n w_n+\sum_{n=1}^{k-1}\alpha_n w_n}\|v\|_{C^0_{\boldsymbol{\sigma}}(\boldsymbol{\Sigma}(\boldsymbol{\Gamma}^m,\boldsymbol{\tau});X)} \\ &= g(\tau_k)\sum_{\boldsymbol{\alpha}\in X_{\mathbf{w}_{1:k-1}}(q,k-1)} e^{-q}\|v\|_{C^0_{\boldsymbol{\sigma}}(\boldsymbol{\Sigma}(\boldsymbol{\Gamma}^m,\boldsymbol{\tau});X}.\end{aligned}$$

For $k = 1$, we have that $R(q,1) = I^{(1)} - Q_{G,\lfloor q/w_1\rfloor}$. We thus deduce that

$$\begin{aligned}\left\|\left(R(q,1)\bigotimes_{n=2}^{m} I^{(n)}\right)v\right\|_X &\lesssim g(\tau_1)e^{-h(\tau_1)(\lfloor q/w_1\rfloor+1)}\|v\|_{C^0_{\boldsymbol{\sigma}}(\boldsymbol{\Sigma}(\boldsymbol{\Gamma}^m,\boldsymbol{\tau});X)} \\ &\lesssim g(\tau_1)e^{-q}\|v\|_{C^0_{\boldsymbol{\sigma}}(\boldsymbol{\Sigma}(\boldsymbol{\Gamma}^m,\boldsymbol{\tau});X)}.\end{aligned}$$

Combining our findings yields the estimate (3.18). □

Whenever the sequence $\{g(\tau_k)\}_k$ is summable, we can further estimate the sum in (3.18).

(3.20) **Lemma.** Let the conditions of Lemma (3.17) hold and let the sequence $\{g(\tau_k)\}_k$ be summable. Then, the error estimate (3.18) can be simplified to

(3.21) $$\|(\mathbf{I} - \mathcal{A}_{G,\mathbf{w}}(q,m))v\|_X \lesssim e^{-q}\#X_{\mathbf{w}_{1:m-1}}(q,m-1)\|v\|_{C^0_{\boldsymbol{\sigma}}(\boldsymbol{\Sigma}(\boldsymbol{\Gamma},\boldsymbol{\tau});X)}$$

with a constant which depends on the continuity of $\mathbf{I}$, the summability of $\{g(\tau_k)\}_k$ and on the value of $C(k)$ in (3.18), but which is independent of the dimensionality m.

Proof. The constants $C(k)$ in (3.18) are for each $k = 1, \ldots, m-1$ bounded by a generic constant independent of the dimensionality m, whenever $\{g(\tau_k)\}_k$ is summable. Hence, (3.18) reduces to

$$\|(\mathbf{I}-\mathcal{A}_{G,\mathbf{w}}(q,m))v\|_X \lesssim e^{-q}\left(g(\tau_1)+\sum_{k=1}^{m-1} g(\tau_{k+1})\#X_{\mathbf{w}_{1:k}}(q,k)\right)\|v\|_{C^0_{\boldsymbol{\sigma}}(\boldsymbol{\Sigma}(\boldsymbol{\Gamma},\boldsymbol{\tau});X)}.$$

Furthermore, it holds for $k = 1, \ldots, m-2$ that

$$\#X_{\mathbf{w}_{1:k}}(q,k) \le \#X_{\mathbf{w}_{1:k+1}}(q,k+1).$$

Thus, we can bound the sum in the above inequality by

$$g(\tau_1)+\sum_{k=1}^{m-1} g(\tau_{k+1})\#X_{\mathbf{w}_{1:k}}(q,k) \le \#X_{\mathbf{w}_{1:m-1}}(q,m-1)\sum_{k=1}^{m} g(\tau_k).$$

Then, the summability of $\{g(\tau_k)\}_k$ implies the assertion. □

(3.22) **Remark.** The summability of the sequence $\{g(\tau_k)\}_k$ is an immediate consequence of the summability of $\{\tau_k^{-1}\}_k$, cf. (2.18) and (2.19). For the anisotropic tensor product Gauss-Hermite quadrature as well as for the anisotropic Gauss-Legendre quadrature, the summability of $\{\tau_k^{-1}\}_k$ is deduced from the summability of $\{\gamma_k\}_k$ for the integrands under consideration. Since we diminish the region of analyticity for the anisotropic sparse Gaussian quadrature in order to expand the integrands analytically into a cross product domain $\boldsymbol{\Sigma}(\boldsymbol{\Gamma}, \boldsymbol{\tau})$, see Lemma (3.10), we require that r in (3.11) is greater than 1 for the summability of $\{g(\tau_k)\}_k$. This is the case if γ_k fulfills (III.4.8) with $s_2 > 2$. Hence, we restrict ourselves in the sequel to this situation.

3.4 Cost complexity of the anisotropic sparse Gaussian quadrature

Lemma (3.20) implies that the error of the anisotropic sparse Gaussian quadrature method on level q can be bounded by $\exp(-q)$ times the number of indices which are contained in $X_{\mathbf{w}}(q, m-1)$. To find an error estimate in terms of the number of quadrature points, we additionally have to estimate the cost of the sparse Gaussian quadrature method on level q. Therefore, we establish a bound on the number of quadrature points used in the combination technique formula (3.9). This number is given by

$$\begin{aligned}\operatorname{cost}\big(\mathcal{A}_{G,\mathbf{w}}(q(\varepsilon), m)\big) &= \sum_{\boldsymbol{\alpha}\in Y_{\mathbf{w}}(q,m)} \prod_{k=1}^{m} N_{\alpha_k} = \sum_{\boldsymbol{\alpha}\in Y_{\mathbf{w}}(q,m)} \prod_{k=1}^{m} \left\lceil \frac{1}{2}\Big((\alpha_k+2)\Big)\right\rceil \\ &\leq \sum_{\boldsymbol{\alpha}\in Y_{\mathbf{w}}(q,m)} \prod_{k=1}^{m} (\alpha_k+1).\end{aligned} \tag{3.23}$$

Then, we simply use that $Y_{\mathbf{w}}(q, m) \subset X_{\mathbf{w}}(q, m)$, cf. (3.6) and (3.7), and estimate the maximum value of the summands in (3.23). For this, we have to solve the optimization problem

$$\max_{\boldsymbol{\alpha}\in X_{\mathbf{w}}(q,m)} \prod_{k=1}^{m} (\alpha_k+1).$$

This is equivalent to the problem

$$\max_{\boldsymbol{\alpha}\in \mathbb{N}^m} \prod_{k=1}^{m} (\alpha_k+1) \quad \text{s.t.} \quad \sum_{k=1}^{m} w_k\alpha_k \leq q.$$

We get an upper bound for this optimization problem if we extend the admissible set of multi-indices to arbitrary m-dimensional vectors with positive coefficients

$$\sup_{\boldsymbol{\alpha}\in \mathbb{R}^m} \prod_{k=1}^{m} (\alpha_k+1) \quad \text{s.t.} \quad \sum_{k=1}^{m} w_k\alpha_k \leq q \quad \text{and} \quad \alpha_k \geq 0 \quad \text{for} \quad k = 1, \ldots, m.$$

The problem's solution can be calculated by solving the equivalent optimization problem

$$\sup_{\boldsymbol{\alpha}\in \mathbb{R}^m_+} \sum_{k=1}^{m} \log(\alpha_k+1) \quad \text{s.t.} \quad \sum_{k=1}^{m} w_k\alpha_k \leq q \quad \text{and} \quad \alpha_k \geq 0 \quad \text{for} \quad k = 1, \ldots, m.$$

We solve it by means of Lagrangian multipliers and get the optimal solution

$$\alpha_k = \begin{cases} \frac{q+\sum_{\ell=1}^{n_0} w_\ell}{n_0 w_k} - 1, & \text{if } k \le n_0, \\ 0, & \text{if } k > n_0, \end{cases}$$

where n_0 is determined by

$$n_0 = \operatorname*{argmax}_{n=1,\ldots,m} \left\{ q + \sum_{\ell=1}^{n} w_\ell \ge n w_n \right\}. \tag{3.24}$$

This implies the following lemma on the upper bound for (3.23).

(3.25) **Lemma.** Let the weight vector $\mathbf{w} = [w_1, \ldots, w_m]^\intercal$ be ordered ascendingly. Then, the cost complexity of the anisotropic sparse Gaussian quadrature on level q is, with n_0 from (3.24), bounded by

$$\operatorname{cost}\big(\mathcal{A}_{G,\mathbf{w}}(q,m)\big) \le \# X_{\mathbf{w}}(q,m) \prod_{k=1}^{n_0} \left(\frac{q + \sum_{\ell=1}^{n_0} w_\ell}{n_0 w_k} \right). \tag{3.26}$$

The product on the right-hand side in (3.26) can further be estimated.

(3.27) **Lemma.** Let the weight vector $\mathbf{w} = [w_1, \ldots, w_m]^\intercal$ be ordered ascendingly. Additionally, let n_0 be given according to (3.24). Then, it holds that

$$\prod_{k=1}^{n_0} \left(\frac{q + \sum_{\ell=1}^{n_0} w_\ell}{n_0 w_k} \right) \le \prod_{k=1}^{m} \left(\frac{q}{k w_k} + 1 \right). \tag{3.28}$$

Proof. We show for $k = 1, 2, \ldots, n_0 - 1$ that

$$\begin{aligned} &\left(\frac{q + \sum_{\ell=1}^{n_0-k} w_\ell + k w_{n_0}}{n_0 w_{n_0}} \right) \left(\frac{q + \sum_{\ell=1}^{n_0} w_\ell}{n_0 w_{n_0-k}} \right) \\ &\qquad \le \left(\frac{q + \sum_{\ell=1}^{n_0-k-1} w_\ell + (k+1) w_{n_0}}{n_0 w_{n_0}} \right) \left(\frac{q + \sum_{\ell=1}^{n_0-1} w_\ell}{(n_0-1) w_{n_0-k}} \right). \end{aligned} \tag{3.29}$$

The successive application of this inequality for $k = 1, 2, \ldots, n_0 - 1$ leads to

$$\prod_{k=1}^{n_0} \left(\frac{q + \sum_{\ell=1}^{n_0} w_\ell}{n_0 w_k} \right) \le \left(\frac{q}{n_0 w_{n_0}} + 1 \right) \prod_{k=1}^{n_0-1} \left(\frac{q + \sum_{\ell=1}^{n_0-1} w_\ell}{(n_0-1) w_k} \right).$$

Then, it follows by proceeding in the same way for $n_0 - 1, n_0 - 2, \ldots, 2$ that

$$\begin{aligned} \prod_{k=1}^{n_0} \left(\frac{q + \sum_{\ell=1}^{n_0} w_\ell}{n_0 w_k} \right) &\le \left(\frac{q}{n_0 w_{n_0}} + 1 \right) \left(\frac{q}{(n_0-1) w_{n_0-1}} + 1 \right) \prod_{k=1}^{n_0-2} \left(\frac{q + \sum_{\ell=1}^{n_0-2} w_\ell}{(n_0-2) w_k} \right) \\ &\le \prod_{k=1}^{n_0} \left(\frac{q}{k w_k} + 1 \right). \end{aligned}$$

Since $n_0 < m$, this would immediately imply the assertion.

To prove (3.29), we use the abbreviation $\tilde{q} := q + \sum_{\ell=1}^{n_0-k-1} w_\ell$ and rewrite this inequality by

$$(n_0-1)(\tilde{q}+w_{n_0-k}+kw_{n_0})\left(\tilde{q}+\sum_{\ell=n_0-k}^{n_0} w_\ell\right)-n_0(\tilde{q}+(k+1)w_{n_0})\left(\tilde{q}+\sum_{\ell=n_0-k}^{n_0-1} w_\ell\right)\leq 0.$$

After expanding the products, some of the terms vanish and we can simplify this expression to

$$\begin{aligned}
&n_0\left(\tilde{q}w_{n_0-k} + (k+1)w_{n_0}^2 + (w_{n_0-k} - w_{n_0})\sum_{\ell=n_0-k}^{n_0} w_\ell\right)\\
&\quad - \left(\tilde{q}\left(\tilde{q} + \sum_{\ell=n_0-k}^{n_0} w_\ell\right) + (w_{n_0-k} + kw_{n_0})\left(\tilde{q} + \sum_{\ell=n_0-k}^{n_0} w_\ell\right)\right)\\
&\leq n_0\left(\tilde{q}(w_{n_0-k} - w_{n_0}) + (w_{n_0-k} - w_{n_0})\sum_{\ell=n_0-k}^{n_0} w_\ell\right.\\
&\qquad\qquad \left. + (k+1)w_{n_0}^2 - (w_{n_0-k} + kw_{n_0})w_{n_0}\right)\\
&\leq n_0\Big(n_0 w_{n_0}(w_{n_0-k} - w_{n_0}) - w_{n_0}(w_{n_0-k} - w_{n_0})\Big)\\
&= n_0(n_0-1)(w_{n_0-k} - w_{n_0}) \leq 0.
\end{aligned}$$

Here, the first and second inequality follow from $\tilde{q} + \sum_{\ell=n_0-k}^{n_0} w_\ell = q + \sum_{\ell=1}^{n_0} w_\ell \geq n_0 w_{n_0}$ and from $w_{n_0-k} \leq w_{n_0}$. This completes the proof. □

Now, we can deduce, in view of (3.26) and (3.28), that the complexity of the anisotropic sparse Gaussian quadrature is bounded by

$$\text{cost}(\mathcal{A}_{G,\mathbf{w}}(q,m)) \leq \left(\prod_{k=1}^{m}\left(\frac{q}{kw_k}+1\right)\right)\#X_{\mathbf{w}}(q,m). \tag{3.30}$$

3.5 A sharp estimate on the anisotropic sparse index set

In order to complete the convergence analysis, it remains to estimate the number of indices in the set $X_{\mathbf{w}}(q,m)$. A quite rough estimate is provided by

$$\#X_{\mathbf{w}}(q,m) \leq \prod_{k=1}^{m}\left(\frac{q}{w_k}+1\right) \tag{3.31}$$

which corresponds to the number of indices in the anisotropic full tensor product case. This result was proven in [NTW08b] and is, of course, not sharp. Moreover, with the estimate (3.31), we are obviously not able to improve the convergence results of the anisotropic tensor product Gaussian quadrature. Hence, we provide a sharper bound on $X_{\mathbf{w}}(q,m)$ which is yet only proven for dimensions m up to 5 and is, therefore, a conjecture.

(3.32) **Conjecture.** The cardinality of the set $X_{\mathbf{w}}(q,m)$ in (3.6), where the weight vector $\mathbf{w} = [w_1, \ldots, w_m]$ is ascendingly ordered, i.e. $w_1 \le w_2 \le \ldots \le w_m$, is bounded by

$$\#X_{\mathbf{w}}(q,m) \le \prod_{k=1}^{m} \frac{\frac{q}{w_k} + k}{k}. \tag{3.33}$$

Proof (for $m = 1$ and $m = 2$). The cardinality of the set $X_{\mathbf{w}}(q,m)$ is given by

$$\#X_{\mathbf{w}}(q,m) = \sum_{\alpha_1=0}^{\lfloor \frac{q}{w_1} \rfloor} \sum_{\alpha_2=0}^{\lfloor \frac{q-\alpha_1 w_1}{w_2} \rfloor} \cdots \sum_{\alpha_m=0}^{\left\lfloor \frac{q-\sum_{k=1}^{m-1} \alpha_k w_k}{w_m} \right\rfloor} 1.$$

The assertion is clear for $m = 1$ since

$$\#X_{\mathbf{w}}(q,1) = \sum_{\alpha_1=0}^{\lfloor \frac{q}{w_1} \rfloor} 1 = \left\lfloor \frac{q}{w_1} \right\rfloor + 1.$$

For $m = 2$, we have that

$$\#X_{\mathbf{w}}(q,2) = \sum_{\alpha_1=0}^{\lfloor \frac{q}{w_1} \rfloor} \left(\left\lfloor \frac{q-\alpha_1 w_1}{w_2} \right\rfloor + 1 \right) = \left\lfloor \frac{q}{w_1} \right\rfloor + 1 + \sum_{\alpha_1=0}^{\lfloor \frac{q}{w_1} \rfloor} \left\lfloor \frac{q-\alpha_1 w_1}{w_2} \right\rfloor.$$

Since $w_1 \le w_2$, the term $\lfloor \frac{q-\alpha_1 w_1}{w_2} \rfloor$ vanishes for $\alpha_1 = \lfloor \frac{q}{w_1} \rfloor$. Hence, by rearranging the summation, we can deduce that

$$\begin{aligned}
\#X_{\mathbf{w}}(q,2) &= \left\lfloor \frac{q}{w_1} \right\rfloor + 1 + \sum_{\alpha_1=0}^{\lfloor \frac{q}{w_1} \rfloor - 1} \left\lfloor \frac{q-\alpha_1 w_1}{w_2} \right\rfloor \\
&= \left\lfloor \frac{q}{w_1} \right\rfloor + 1 + \sum_{\alpha_1=1}^{\lfloor \frac{q}{w_1} \rfloor} \left\lfloor \frac{\alpha_1 w_1 + q - \lfloor \frac{q}{w_1} \rfloor w_1}{w_2} \right\rfloor \\
&\le \left\lfloor \frac{q}{w_1} \right\rfloor + 1 + \sum_{\alpha_1=1}^{\lfloor \frac{q}{w_1} \rfloor} \left(\frac{\alpha_1 w_1}{w_2} + \frac{q - \lfloor \frac{q}{w_1} \rfloor w_1}{w_2} \right) \\
&= \left\lfloor \frac{q}{w_1} \right\rfloor + 1 + \left\lfloor \frac{q}{w_1} \right\rfloor \frac{q - \lfloor \frac{q}{w_1} \rfloor w_1}{w_2} + \frac{w_1}{w_2} \frac{\lfloor \frac{q}{w_1} \rfloor (\lfloor \frac{q}{w_1} \rfloor + 1)}{2}.
\end{aligned}$$

Now, we apply the identity

$$\left\lfloor \frac{q}{w_1} \right\rfloor + 1 = \frac{q}{w_1} + 1 - \left(\frac{q}{w_1} - \left\lfloor \frac{q}{w_1} \right\rfloor \right)$$

in the last summand to obtain

$$
\begin{aligned}
&\#X_{\mathbf{w}}(q,2)\\
&\leq \left\lfloor \frac{q}{w_1} \right\rfloor + 1 + \left\lfloor \frac{q}{w_1} \right\rfloor \frac{q - \lfloor \frac{q}{w_1} \rfloor w_1}{w_2} + \frac{w_1}{w_2} \frac{\lfloor \frac{q}{w_1} \rfloor (\frac{q}{w_1} + 1)}{2} - \frac{w_1}{w_2} \frac{\lfloor \frac{q}{w_1} \rfloor (\frac{q}{w_1} - \lfloor \frac{q}{w_1} \rfloor)}{2}\\
&= \left\lfloor \frac{q}{w_1} \right\rfloor + 1 + \left\lfloor \frac{q}{w_1} \right\rfloor \frac{q - \lfloor \frac{q}{w_1} \rfloor w_1}{2w_2} + \frac{w_1}{w_2} \frac{\lfloor \frac{q}{w_1} \rfloor (\frac{q}{w_1} + 1)}{2}\\
&= \left\lfloor \frac{q}{w_1} \right\rfloor + 1 + \frac{w_1}{w_2} \frac{\frac{q}{w_1} (\frac{q}{w_1} + 1)}{2} + \left\lfloor \frac{q}{w_1} \right\rfloor \frac{q - \lfloor \frac{q}{w_1} \rfloor w_1}{2w_2} - \frac{w_1}{w_2} \frac{(\frac{q}{w_1} - \lfloor \frac{q}{w_1} \rfloor)(\frac{q}{w_1} + 1)}{2}\\
&\leq \left\lfloor \frac{q}{w_1} \right\rfloor + 1 + \frac{\frac{q}{w_2} (\frac{q}{w_1} + 1)}{2}\\
&\leq \frac{q}{w_1} + 1 + \frac{\frac{q}{w_2} (\frac{q}{w_1} + 1)}{2} = \left(\frac{q}{w_1} + 1 \right) \left(1 + \frac{q}{2w_2} \right)\\
&= \frac{(\frac{q}{w_2} + 2)(\frac{q}{w_1} + 1)}{2}.
\end{aligned}
$$

The second to last inequality holds since $\frac{q}{w_1} + 1 \geq \lfloor \frac{q}{w_1} \rfloor + 1$. This proves the conjecture also for $m = 2$. □

(3.34) **Remark.** (a) The proof of the conjecture (3.32) is only given for $m = 2$. In the Appendix A, we prove the conjecture for $m = 3, 4, 5$ and provide a strategy for establishing the assertion for general m. Moreover, we reduce the problem to two subproblems which are possibly easier to solve. Nevertheless, the induction step is complicated since one has to handle terms of the form $\sum_{k=1}^{\lfloor q/w_1 \rfloor} k^i$ for $i = 1, \ldots, m$. These terms can be determined by the Faulhaber formulae which involve the Bernoulli numbers and these numbers are not so easy to deal with. So far, we were not able to prove the result in arbitrary dimensions. However, we validated the estimate numerically in various dimensions and with various weights.

(b) We would like to point out that estimate (3.33) is sharp in the isotropic case, i.e. for the weight $\mathbf{w} = \mathbf{1}$. Moreover, the ordering of the weight vector is crucial in this estimate. There are examples where this estimate does not hold if the weights are not in ascending order.

(c) At first glance one might claim that even the estimate

$$\#X_{\mathbf{w}}(q,m) \leq \prod_{k=1}^{m} \frac{\left\lfloor \frac{q}{w_k} \right\rfloor + k}{k}$$

is valid. This is true in a lot of cases which we investigated. Nevertheless, there are examples where this estimate fails. For these reasons, we think that the conjecture is sharp.

(d) There exist a lot of estimates on the cardinality of such index sets in the literature, see e.g. [BD72]. The reason is that this problem is equivalent to the estimation of the number of integer solutions of linear Diophantine inequalities, which is a problem in number theory, or to the calculation of the integer points in a convex polyhedra. Nevertheless, all estimates that we found in the literature were not useful for our specific problem in order to obtain improved results in comparison with the anisotropic tensor product Gauss-Hermite quadrature.

3.6 Convergence in terms of the number of quadrature points

Let us assume that the Conjecture (3.32) holds for arbitrary $m \in \mathbb{N}$. Combining (3.21), (3.30) and (3.33), the findings of the previous four subsections can be summarized to the error estimate of the anisotropic sparse grid quadrature

$$\|(\mathbf{I} - \mathcal{A}_{G,\mathbf{w}}(q,m))v\|_X \lesssim e^{-q}\left(\prod_{k=1}^{m-1}\left(\frac{q}{kw_k}+1\right)\right)\|v\|_{C^0_{\boldsymbol{\sigma}}(\boldsymbol{\Sigma}(\boldsymbol{\Gamma},\boldsymbol{\tau});X)} \tag{3.35}$$

and the complexity estimate

$$\operatorname{cost}(\mathcal{A}_{\mathbf{w}}(q,m)) \leq \left(\prod_{k=1}^{m}\left(\frac{q}{kw_k}+1\right)\right)^2. \tag{3.36}$$

In a similar way as in (2.21), we establish conditions on the decay of $\{\gamma_k\}_k$ such that the convergence rate in terms of the number of quadrature points is dimension-independent and algebraic of arbitrary order.

(3.37) **Theorem.** Let the conditions of Lemma (3.17) and Lemma (3.20) be satisfied and let the Conjecture (3.32) hold. If the sequence $\{(kh(\tau_k))^{-1}\}_k$ is summable, then there exists a constant $C(\delta_1,\delta_2)$ independent of the dimension m for all $\delta_1,\delta_2 > 0$ such that the following estimate holds

$$\|v - \mathcal{A}_{G,\mathbf{w}}(q,m)v\|_X \lesssim C(\delta_1,\delta_2)e^{-q(1-\delta_1-\delta_2)}\|v\|_{C^0_{\boldsymbol{\sigma}}(\boldsymbol{\Sigma}(\mathbb{R}^m,\boldsymbol{\tau});X)}. \tag{3.38}$$

Moreover, the complexity in this case is bounded by

$$\operatorname{cost}(\mathcal{A}_{\mathbf{w}}(q,m)) \leq C(\delta_1,\delta_2)^2 e^{2q(\delta_1+\delta_2)}. \tag{3.39}$$

Hence, the convergence of the anisotropic sparse Gaussian quadrature method is dimension-independent and of arbitrarily algebraic order. More precisely, it holds that

$$\|v - \mathcal{A}_{G,\mathbf{w}}(q,m)v\|_X \lesssim C(\delta_1,\delta_2)^{1+\frac{1-\delta_1-\delta_2}{\delta_1+\delta_2}} N(q)^{-\frac{1-\delta_1-\delta_2}{2(\delta_1+\delta_2)}}\|v\|_{C^0_{\boldsymbol{\sigma}}(\boldsymbol{\Sigma}(\mathbb{R}^m,\boldsymbol{\tau});X)} \tag{3.40}$$

where $N(q)$ denotes the total number of quadrature points in $\mathcal{A}_{GH,\mathbf{w}}(q,m)$.

Proof. The combination of Lemma (3.20), Lemma (3.31) and Conjecture (3.32) leads to

$$\|v - \mathcal{A}_{G,\mathbf{w}}(q,m)v\|_X \lesssim \prod_{k=1}^{m-1}\left(\frac{q}{kw_k}+1\right)e^{-q}\|v\|_{C^0_{\boldsymbol{\sigma}}(\boldsymbol{\Sigma}(\mathbb{R}^m,\boldsymbol{\tau});X)}.$$

From the definition of the weights w_k, cf. Lemma (3.17), we know that $w_k = h(\tau_k)$. Since $\{(kh(\tau_k))^{-1}\}$ is summable, it follows in the same way as for the cost of the tensor product quadrature (2.24) that there exists for each $\delta_1, \delta_2 > 0$ a constant $C(\delta_1, \delta_2)$ independent of m such that

$$(3.41) \qquad \prod_{k=1}^{m-1} \left(\frac{q}{kw_k} + 1 \right) \leq C(\delta_1, \delta_2, r) \exp\left(q(\delta_1 + \delta_2)\right).$$

This implies (3.38). The second estimate (3.39) follows immediately from (3.41) and the third estimate (3.40) is obtained by combining (3.38) and (3.39). □

(3.42) **Remark.** The condition that $\{(kh(\tau_k))^{-1}\}_k$ is summable implies that $h(\tau_k)$ has to increase with a stronger rate than $\log(k)$. In particular, a rate $\log(k)^{1+\delta}$ for arbitrary $\delta > 0$ would be sufficient. Unfortunately, since $h(\tau_k) \approx \log(c\tau_k)$, cf. (2.26) and (2.27), for the Gauss-Hermite and Gauss-Legendre quadrature, any algebraic increase of τ_k is not sufficient for the summability of $\{(kh(\tau_k))^{-1}\}_k$. Nevertheless, if τ_k increases subexponentially with any arbitrary rate, i.e. $\tau_k \approx \exp(k^\delta)$ for arbitrary $\delta > 0$, summability of $\{(kh(\tau_k))^{-1}\}_k$ is guaranteed. This is, for example, the case for analytic kernel functions, like the Gaussian correlation kernel in arbitrary spatial dimensions. Indeed, we know from [ST06] that the sequence $\{\gamma_k\}_k$ decays for analytic correlation kernels subexponentially.

From Remark (3.42), we conclude that we cannot show a dimension-independent convergence rate for the anisotropic sparse Gaussian quadrature method when the sequence $\{\tau_k\}_k$ increases only algebraically. Unfortunately, only an algebraic increase of $\{\tau_k\}_k$ is obtained in the important case of the moment computation of the solution to (III.3.3) or (III.3.5) when the diffusion coefficient is determined by a covariance kernel of the Matèrn class with smoothness parameter $\nu < \infty$. Thus, we investigate how fast the convergence rate deteriorates for an algebraic increase, i.e. $\tau_k \gtrsim k^r$.

(3.43) **Lemma.** Let the sequence $\{h(\tau_k)^{-1}\}_k$ increase as $h(\tau_k) \geq \log(ck^r)$ for some $c > 1$ and $r \in \mathbb{R}_+$. Then, we obtain that the number of indices in the anisotropic sparse grid is bounded by

$$(3.44) \qquad \#X_{\mathbf{w}}(q, m) \lesssim \exp\left(\frac{q}{r} \log(\log(m)) \right)$$

with a constant which is independent of m.

Proof. From Conjecture (3.32), we know that

$$\#X_{\mathbf{w}}(q, m) \leq \prod_{k=1}^{m} \left(\frac{q}{kw_k} + 1 \right).$$

Next, we split the product into

$$(3.45) \qquad \prod_{k=1}^{m} \left(\frac{q}{kw_k} + 1 \right) = \left(\frac{q}{w_1} + 1 \right)\left(\frac{q}{2w_2} + 1 \right)\left(\frac{q}{3w_3} + 1 \right) \prod_{k=4}^{m} \left(\frac{q}{kw_k} + 1 \right).$$

We further estimate that

$$\prod_{k=4}^{m}\left(\frac{q}{kw_k}+1\right) \le \exp\left(\sum_{k=4}^{m}\log\left(\frac{q}{kw_k}+1\right)\right) \le \exp\left(\sum_{k=4}^{m}\frac{q}{kw_k}\right).$$

Next, the sum in the above estimate can be bounded by

$$\begin{aligned}\sum_{k=4}^{m}\frac{q}{kw_k} &\le \int_3^m \frac{q}{x\log(x^r)}\,\mathrm{d}x = \frac{q}{r}\int_3^m \frac{1}{x\log(x)}\,\mathrm{d}x \\ &= \frac{q}{r}\int_{\log(3)}^{\log(m)}\frac{1}{z}\,\mathrm{d}z = \frac{q}{r}\big(\log(\log(m)) - \log(\log(3))\big).\end{aligned} \tag{3.46}$$

The first three factors in (3.45) are estimated by

$$\left(\frac{q}{w_1}+1\right)\left(\frac{q}{2w_2}+1\right)\left(\frac{q}{3w_3}+1\right) \le C\exp\left(\frac{\log(\log(3))}{r}q\right).$$

Hence, we end up with

$$\begin{aligned}\prod_{k=1}^{m}\left(\frac{q}{kw_k}+1\right) &\le C\exp\left(\frac{\log(\log(3))}{r}q\right)\exp\left(\frac{q}{r}\big(\log(\log(m)) - \log(\log(3))\big)\right) \\ &\lesssim \exp\left(\frac{q}{r}\big(\log(\log(m)\big)\right).\end{aligned}$$

□

With Lemma (3.43) at hand, we are able to quantify how the dimensionality m compromises the convergence rate of the anisotropic sparse Gaussian quadrature. In fact, the dimensionality enters only with a factor $\log(\log(m))$ in case of algebraic increasing regions of analyticity.

(3.47) **Theorem.** Let the conditions of Lemma (3.17) and Lemma (3.20) be satisfied and let the Conjecture (3.32) hold. Moreover, let the assumptions of Lemma (3.43) be fulfilled. Then, the error of the anisotropic sparse Gaussian quadrature $\mathcal{A}_{GH,\mathbf{w}}(q,m)$ is bounded in terms of the total number of quadrature points by

$$\|v - \mathcal{A}_{G,\mathbf{w}}(q,m)v\|_X \lesssim N(q)^{-\frac{r}{2\log(\log(m))}+\frac{1}{2}}\|v\|_{C^0_{\boldsymbol{\sigma}}(\boldsymbol{\Sigma}(\mathbb{R}^m,\boldsymbol{\tau});X)}. \tag{3.48}$$

Proof. Inserting (3.44) into (3.35) and (3.36), respectively, leads to the error estimate

$$\big\|(\mathbf{I} - \mathcal{A}_{G,\mathbf{w}}(q,m))v\big\|_X \lesssim e^{-q\left(1-\frac{\log(\log(m))}{r}\right)}\|v\|_{C^0_{\boldsymbol{\sigma}}(\boldsymbol{\Sigma}(\boldsymbol{\Gamma},\boldsymbol{\tau});X)}$$

and the complexity estimate

$$N(q) = \operatorname{cost}\big(\mathcal{A}_{G,\mathbf{w}}(q,m)\big) \lesssim e^{2q\left(\frac{\log(\log(m))}{r}\right)}.$$

Combining both estimates implies the desired estimate (3.48). □

We analyzed the convergence of anisotropic sparse Gaussian quadrature rules for the moment computation of the solution to (III.3.3) or (III.3.5), respectively. With the new estimate (3.33) on the number of indices $X_{\mathbf{w}}(q, m)$, we are able to get significantly improved results in comparison with the convergence of the anisotropic tensor product Gaussian quadrature. More precisely, we are able to show dimension-independent convergence with an arbitrarily algebraic rate if the regions of analyticity of the integrand grow exponentially like $\tau_k \gtrsim \exp(k^\delta)$ for arbitrary $\delta > 0$. This covers the important case of diffusion coefficients which are derived from Gaussian covariance kernels. In addition, we analyzed the case when τ_k grows algebraically, which covers the case of covariance kernels of the Matèrn class, and obtain that the dimensionality m compromises the convergence rate at most by the term $\log(\log(m))$.

4. Numerical Results

As in the previous chapter, we consider for the numerical validation of our theoretical findings the one-dimensional model problem (IV.3.1). In order to compare the performance of the anisotropic (sparse) Gaussian quadrature to the Monte Carlo or quasi-Monte Carlo quadrature, we employ the same numerical examples as in Chapter IV.

For the anisotropic Gaussian quadrature, we calculate the number of quadrature points N_k for $k = 1, \ldots, m$ in accordance with (2.12), i.e.

$$N_k = \lceil |\log \varepsilon|/2h(\tau_k) + 1/2 \rceil,$$

and successively increase the accuracy $\varepsilon = 10^{-0.5j}$ for $j = 1, 2, 3, \ldots$. The function $h(\tau)$ is given by $h(\tau) = \log(\sqrt{2}\tau)$ in the lognormal case and by $h(\tau) = \log(2\tau + \sqrt{1 + 4\tau^2})$ in the uniformly elliptic case, cf. Remark (2.10). Moreover, the quantity τ_k which describes the region of analyticity is set to $\tau_k = \log(2)/\gamma_k$ in the lognormal case and to $\tau_k = 1/\gamma_k$ in the uniformly elliptic case, respectively. From the regularity results in Sections III.5 and III.6, one might propose to adjust the region of analyticity by a factor $1/p$ for the computation of the p-th moment and by a factor $\underline{a}$ in the uniformly elliptic case. But, as the numerical results demonstrate, the chosen setting provides comparable results for the computation of all moments under consideration.

For the anisotropic sparse Gaussian quadrature, we set the weights w_k according to $w_k = h(\tau_k)$ with the same functions $h(\tau)$ and the same quantities τ_k as for the tensor product quadrature for the lognormal and the uniformly elliptic case, respectively. Hence, our anisotropic sparse Gaussian quadrature is essentially a sparsification of the anisotropic tensor product Gaussian quadrature. To choose the same quantity τ_k for the region of analyticity as for the tensor product quadrature seems to be a violation of Lemma (3.10). Indeed, the assertion of this lemma is that the quantities τ_k, which describes the region of analytic extendability in each direction $\Sigma(\Gamma_k, \tau_k)$, should be scaled by $\tilde{\tau}_k = \tau_k/(C(\delta)k^{1+\delta})$ in order to ensure analytic extendability into the tensor domain $\boldsymbol{\Sigma}(\boldsymbol{\Gamma}, \tilde{\boldsymbol{\tau}})$. Nevertheless, the numerical results suggest that the sparsification of the anisotropic Gaussian quadrature yields an error which is nearly as good as the error of the anisotropic Gaussian quadrature itself. This indicates, additionally to the supposition that the quantity τ_k can be chosen as for the anisotropic tensor product quadrature, that the factor $\#X_{\mathbf{w}_{1:m-1}}(q, m-1)$ in the error estimate (3.21) can be removed or at least be improved. To measure the impact of these considerations, we recall the estimate (3.48)

$$\|v - \mathcal{A}_{G,\mathbf{w}}(q,m)v\|_X \lesssim N^{-\frac{r}{2\log(\log(m))}+\frac{1}{2}} \|v\|_{C^0_{\boldsymbol{\sigma}}(\boldsymbol{\Sigma}(\mathbb{R}^m,\boldsymbol{\tau});X)}.$$

In this estimate, the additive term 1/2 reflects the factor $\#X_{\mathbf{w}_{1:m-1}}(q, m-1)$ from the error estimate (3.21) and r is the rate of algebraic increase of $\tilde{\tau}_k$. Hence, the preceeding considerations suggest that the additive term 1/2 in (3.48) can be reduced and that r behaves rather like s_1 than like $s_1 - 1 - \delta$. This would imply that the convergence rate is better approximated by

$$N^{-\frac{s_1}{2\log(\log(m))}} \tag{4.1}$$

than by (3.48). As we will see, the observed convergence rates are closer to the one predicted by (4.1), particularly for larger values of m.

For nearly all numerical examples, it turns out that the convergence rates slightly decrease from the computation of the mean to the the computation of the second moment and even successively for the higher order moments. Therefore, we state for all examples the actually obtained convergence rate for the mean and for the fourth moment. The convergence rate of the second and third moment is then between these two convergence rates.

4.1 Results for a lognormal diffusion

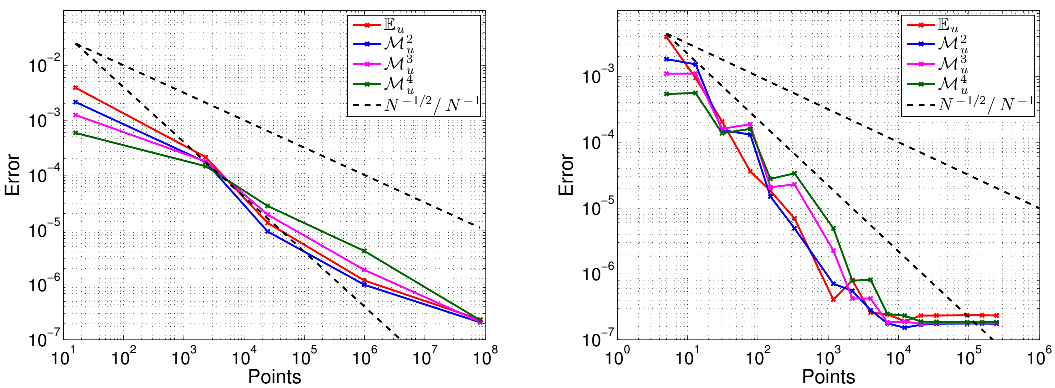


Figure V.3: Errors for $\nu = 7/2$ of the anisotropic Gaussian quadrature (left) and the anisotropic sparse Gaussian quadrature (right) in the lognormal case.

The Matérn kernel for $\nu = 7/2$

In this example, we have to deal with a 30-dimensional integration problem. The convergence rates for the computation of the first four moments of the anisotropic Gaussian quadrature method are depicted on the left-hand side of Figure V.3. From the decay rate

$s_1 = 4$ of γ_k, we can derive the expected convergence rate according to (2.28) and (2.29) as

$$\varepsilon^{-\frac{\mathrm{Li}(30)}{4}} \approx \varepsilon^{-2.9944}.$$

Denoting by N the total number of Gaussian quadrature points, this corresponds approximately to the algebraic convergence rate $N^{-1/2.9944}$. As we see in the error plot, the actually obtained convergence rate is approximately $N^{-1/2}$ for the fourth moment. It increases slightly up to a convergence rate of $N^{-4/7}$ for the mean. Hence, the convergence rate is better than expected.

The associated results for the sparse Gaussian quadrature methods are shown on the right-hand side of Figure V.3. According to (3.48), the predicted convergence rate is

$$N^{-\frac{r}{2\log(\log(m))}+1/2} \approx N^{-\frac{3}{2\log(\log(30))}+1/2} = N^{-0.7254}.$$

In fact, we observe in Figure V.3 a convergence rate of $N^{-1.6}$ for the mean and a convergence rate of $N^{-1.2}$ for the fourth moment. As for the anisotropic tensor product Gaussian quadrature, the results are better than expected. In particular, the anisotropic sparse quadrature achieves the accuracy of the reference solution after only 10^3 to 10^4 points. Especially for the mean, the convergence rate is much better estimated by (4.1) which predicts a convergence rate of

$$N^{-\frac{s_1}{2\log(\log(m))}} \approx N^{-1.6338}.$$

We conclude that the anisotropic sparse Gaussian quadrature exceeds the convergence rates of all other quadrature methods and is therefore the recommend quadrature for this example.

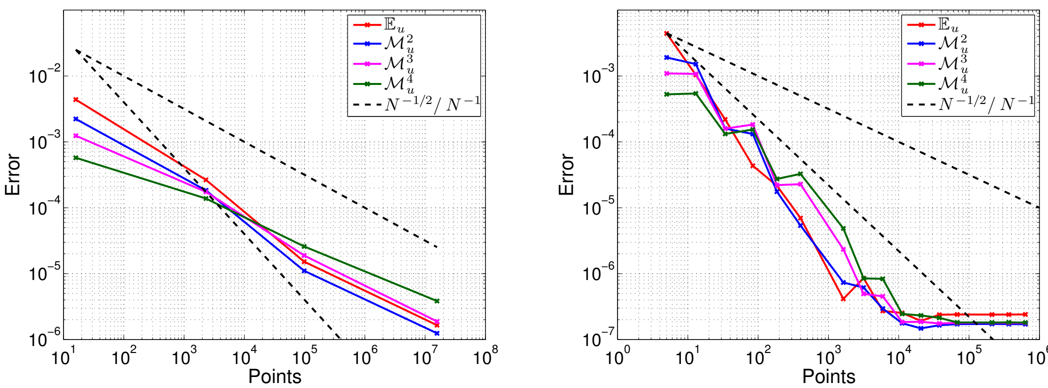


Figure V.4: Errors for $\nu = 5/2$ of the anisotropic Gaussian quadrature (left) and the anisotropic sparse Gaussian quadrature (right) in the lognormal case.

The Matérn kernel for $\nu = 5/2$

For the smoothness parameter $\nu = 5/2$, we end up with a Karhunen-Loève expansion of length $m = 64$. In this case, the expected convergence rate of the anisotropic Gaussian quadrature is given by

$$\varepsilon^{-\frac{\mathrm{Li}(64)}{3}} \approx \varepsilon^{-6.9632}$$

which is associated with the algebraic rate $N^{-0.1436}$. Roughly speaking, this predicts that the method is useless for this kind of problems. From the error plot on the right-hand side of Figure V.4, we, however, see that the convergence rate is much better than predicted. We observe for the computation of the mean a rate of $N^{-1/2}$ and for the fourth moment a rate of $N^{-1/3}$. This means that the anisotropic Gaussian tensor product quadrature performs comparable to the Monte Carlo method for this 64-dimensional integration problem.

For the anisotropic sparse Gaussian quadrature, the convergence rate is much better predicted by (4.1) which leads to

$$N^{-\frac{3}{2\log(\log(64))}} \approx N^{-1.0524}$$

than with (3.48) which yields

$$N^{-\frac{2}{2\log(\log(64))}+1/2} = N^{-0.2016}.$$

Indeed, the actually observed rate in the plot on the right-hand side of Figure V.4 is approximately $N^{-1.25}$ for the mean and N^{-1} for the fourth moment. This shows that the sparse Gaussian quadrature yields good results even for a quite large dimensionality $m = 64$. Moreover, the formula (4.1) leads to quite accurate results for the approximation of the convergence rate in the lognormal case. In addition, the results validate that the impact of the dimenisonality m on the algebraic convergence rate is not more than a factor $\log(\log(m))$.

4.2 Results for a uniformly elliptic diffusion

For the uniformly elliptic case, we expect similar results in comparison with the results in the lognormal case due to the similarity of the one-dimensional error estimates of the Gauss-Hermite and the Gauss-Legendre quadrature. Recall that we set the expectation of the diffusion coefficient for the uniformly elliptic case to $\mathbb{E}_a(x) \equiv 2.5$.

The Matérn kernel for $\nu = 7/2$

The numerical results for the Matérn kernel with smoothness parameter $\nu = 7/2$, correlation length $\ell = 1/2$ and variance $\sigma^2 = 1/4$ are depicted in Figure V.5.

We observe for the anisotropic Gauss-Legendre quadrature a convergence rate of about $N^{-1/2}$ for the mean and $N^{-1/3}$ for the fourth moment. Surprisingly, this coincides with the convergence rate of the anisotropic Gauss-Hermite quadrature for the rougher Matérn kernel with $\nu = 5/2$. Moreover, from the convergence plot, one might deduce that the convergence rates stagnate after 10^4 quadrature points. Nevertheless, the employed anisotropic sparse Gauss-Legendre is just a sparsification of the anisotropic Gauss-Legendre quadrature and the former quadrature converges even after some stagnation, for example from 70 to 300 quadrature points. Hence, it can be expected that the error of the anisotropic Gauss-Legendre quadrature would decrease when employing the next smaller ε. Since the number of quadrature points of the anisotropic Gauss-Legendre

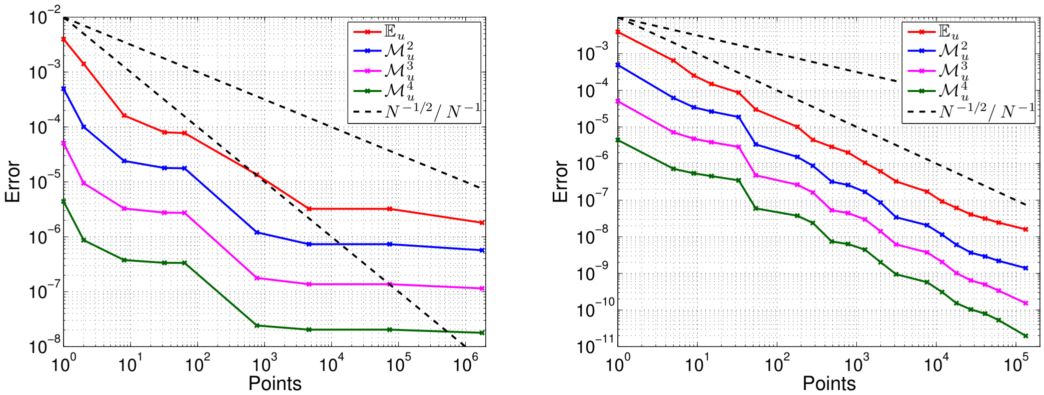


Figure V.5: Errors for $\nu = 7/2$ of the anisotropic Gaussian quadrature (left) and the anisotropic sparse Gaussian quadrature (right) in the uniformly elliptic case.

quadrature drastically increases with the accuracy, this next smaller ε is not feasible for computation anymore. Notice that the number of quadrature points of the anisotropic Gauss-Legendre quadrature in this example for the smallest value of ε is already about $4 \cdot 10^6$.

The convergence rate of the anisotropic sparse Gauss-Legendre quadrature is visualized on the right hand side of Figure V.5. In contrast to the lognormal case, we obtain a convergence rate which is essentially the same for the computation of all considered moments and of order N^{-1}. The results are slightly worse in comparison with the associated results for the lognormal case. Nevertheless, the anisotropic sparse Gauss-Legendre quadrature performs in this example still slightly better than the quasi-Monte Carlo method (rate $N^{-0.91}$) and outperforms the Monte Carlo quadrature or the anisotropic tensor product Gauss-Legendre quadrature.

The Matérn kernel for $\nu = 5/2$

At first glance, the convergence plots of the anisotropic tensor as well as the anisotropic sparse Gauss-Legendre quadrature for the Matérn kernel with $\nu = 5/2$ look identically to those for the Matérn kernel with $\nu = 7/2$. This can be explained since the observed error is essentially the same for the same choice of ε in (2.12). Nevertheless, a closer look on the axis of abscissae confirms that the number of quadrature points in this example is, as expected, higher compared to the number of points for the Matérn kernel with $\nu = 7/2$. For the anisotropic Gauss-Legendre quadrature, the convergence rate for the mean reduces from $N^{-1/2}$ for the Matérn kernel with $\nu = 7/2$ to $N^{-3/7}$ and the convergence rate for the fourth moment reduces from $N^{-1/3}$ to $N^{-2/7}$. This reflects the behaviour described before. Nevertheless, the rates are still much better than the expected rate $N^{-0.1436}$. This suggests that the dimensionality m has less impact on the deterioration of the convergence rate than expected by (2.29).

For the anisotropic sparse Gauss-Legendre quadrature, the convergence rate decreases only slightly from N^{-1} to $N^{-0.91}$. This convergence rate is exactly the convergence rate obtained by the quasi-Monte Carlo method. Hence, we can recommend the use of the

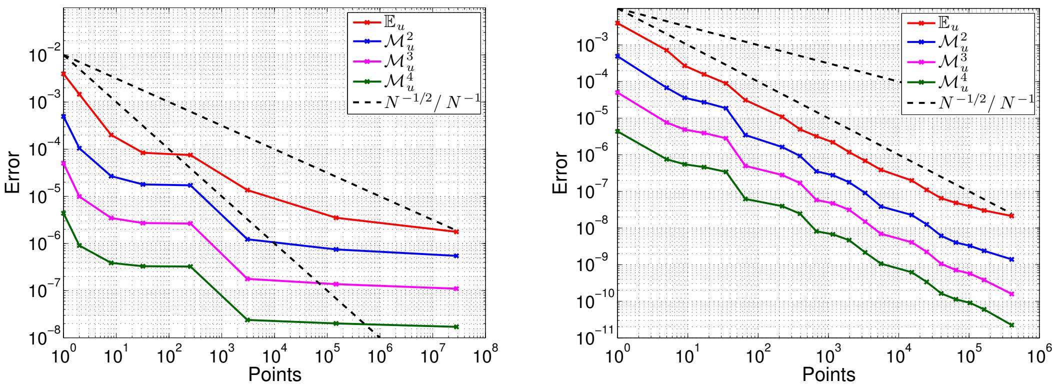


Figure V.6: Errors for $\nu = 5/2$ of the anisotropic Gaussian quadrature (left) and the anisotropic sparse Gaussian quadrature (right) in the uniformly elliptic case.

quasi-Monte Carlo quadrature based on Halton points or the application of the anisotropic sparse Gauss-Legendre quadrature for this example.

The numerical experiments validate that the influence of the dimensionality for anisotropic Gaussian quadratures is drastically reduced in comparison with the corresponding isotropic Gaussian quadratures. Indeed, isotropic Gaussian quadrature rules or even isotropic sparse Gaussian quadrature rules are not able to deal with a 64-dimensional integration problem. Moreover, the results corroborate that m influences the convergence rates at most by a factor $m/\log(m)$ for the anisotropic tensor product Gaussian quadrature and at most by a factor $\log(\log(m))$ for the anisotropic sparse Gaussian quadratures and, hence, our theoretical findings.

Chapter VI

MULTILEVEL QUADRATURE

Up to now, we were only concerned with the approximation error of the solution which is caused by the truncation of the Karhunen-Loève expansion of the diffusion coefficient and the quadrature error of the Bochner integral in the parametric variable $\mathbf{y}$. We neglected so far that we have to solve a deterministic elliptic boundary value problem for each quadrature point $\boldsymbol{\xi}_k \in \mathbf{\Gamma}$. In general, we cannot solve this problem analytically and, therefore, we need to approximate the solution $u_m(\mathbf{x}, \boldsymbol{\xi}_k)$ by e.g. finite elements. In this chapter, we hence discuss at first the error of a finite element discretization. Of course, one can solve all of the occurring deterministic elliptic boundary value problems on a high refinement scale, which is chosen in such a way that the discretization error in the finite element space is of the same order as the truncation error and the quadrature error. This means that the computational effort is given by the computational effort for a single solve of a deterministic elliptic boundary value problem times the number of quadrature points needed for the approximation of the Bochner integral. Another approach, the multilevel quadrature method, does not treat the approximation error of the Bochner integral and the finite element error separately, but instead combines these two error sources in a sparse grid like fashion. To that end, one defines a nested sequence of finite element spaces and a sequence of quadrature rules with increasing accuracy. Then, one combines these two scales of refinement in such a way that only a few quadrature points are used when the resulting PDEs are solved on the finest finite element scale while successively more quadrature points are spent when the resulting PDEs are solved on a coarser refinement level. This, of course, leads to a reduction of the computational complexity, but requires additional regularity results on the integrand under consideration. These regularity results can be derived by essentially the same techniques in the uniformly elliptic and in the lognormal case except for the additional problems which are caused due to the lack of uniformly boundedness for lognormal diffusion coefficients. Hence, we consider throughout this chapter only the lognormal case in detail and refer to some regularity results from [CDS10] in the uniformly elliptic case.

1. Finite element approximation in the spatial variable

For the spatial discretization of the diffusion problem under consideration, we will employ multilevel finite elements. This constitutes the key ingredient for the multilevel quadrature idea. Therefore, we consider a coarse grid triangulation $\mathcal{T}_0 = \{\tau_{0,k}\}$ of the domain D.

Then, for $\ell \geq 1$, a uniform and shape regular triangulation $\mathcal{T}_\ell = \{\tau_{\ell,k}\}$ is recursively obtained by uniformly refining each simplex $\tau_{\ell-1,k}$ into 2^d simplices with diameter $h_\ell \eqsim 2^{-\ell}$. For $n \geq 1$, we define the finite element spaces on level ℓ by

$$\mathcal{S}_\ell^n(D) := \{v \in C(D) : v|_{\partial D} = 0 \text{ and } v|_\tau \in \mathcal{P}_n \text{ for all } \tau \in \mathcal{T}_\ell\} \subset H_0^1(D),$$

where $\mathcal{P}_n$ denotes the space of all polynomials of total degree n. We restrict ourselves in the sequel to the case $n = 1$, i.e. linear finite elements. Then, for given $\mathbf{y} \in \mathbf{\Gamma}$, we shall introduce the Galerkin projection $G_\ell(\mathbf{y}) : H_0^1(D) \to \mathcal{S}_\ell^1(D)$ to discretize the spatial variable. It is defined via the Galerkin orthogonality

$$\int_D a_m(\mathbf{y})\nabla(v - G_\ell(\mathbf{y})v)\nabla w \,\mathrm{d}\mathbf{x} = 0 \text{ for all } w \in \mathcal{S}_\ell^1(D).$$

Moreover, we set $G_{-1}(\mathbf{y}) := 0$ for all $\mathbf{y} \in \mathbf{\Gamma}$. In the sequel, letters in the German type setting will always refer to a Galerkin projection, i.e.

$$\mathfrak{v}_\ell(\mathbf{y}) := G_\ell(\mathbf{y})v \in \mathcal{S}_\ell^1(D).$$

The Galerkin projection $\mathfrak{u}_{m,\ell}(\mathbf{y})$ of the solution $u_m(\mathbf{y})$ to the diffusion problem (III.3.5) is known to fulfill the following error estimate.[1]

(1.1) **Lemma.** Let the domain D be convex or sufficiently smooth, $f \in L^2(D)$ and $a_m(\mathbf{y}) \in W^{1,\infty}(D)$. Then, the Galerkin projection $\mathfrak{u}_{m,\ell}(\mathbf{y}) \in \mathcal{S}_\ell^1(D)$ of the lognormal diffusion problem (III.3.5) satisfies the error estimate

$$\|u_m(\mathbf{y}) - \mathfrak{u}_{m,\ell}(\mathbf{y})\|_{H_0^1(D)} \lesssim 2^{-\ell}\sqrt{\kappa_m(\mathbf{y})}\|u_m(\mathbf{y})\|_{H^2(D)}, \tag{1.2}$$

where $\kappa_m(\mathbf{y})$ is given by (III.3.13). Moreover, if $f \in L^p(D)$ for given $p > 2$, then $u_m(\mathbf{y}) \in W^{2,p}(D)$ and it holds $\mathfrak{u}_{m,\ell}^p(\mathbf{y}) \in \mathcal{S}_\ell^p(D)$ with

$$\|(u_m^p - \mathfrak{u}_{m,\ell}^p)(\mathbf{y})\|_{W_0^{1,1}(D)} \lesssim 2^{-\ell}\kappa_m(\mathbf{y})^p\|u_m(\mathbf{y})\|_{W^{2,p}(D)}^p. \tag{1.3}$$

Here, the constants hidden in (1.2) depend on D and in (1.3) additionally on p but not on $\mathbf{y} \in \mathbb{R}^m$.

Proof. The parametric diffusion problem (III.3.5) is H^2-regular for each fixed $\mathbf{y} \in \mathbb{R}^m$ since D is convex or sufficiently smooth and $f \in L^2(D)$. Hence, the first error estimate immediately follows from the standard finite element theory.

For $p > 2$, it follows from [Gri11] that the solution $u_m(\mathbf{y})$ belongs to $W^{2,p}(D)$ for each fixed $\mathbf{y} \in \mathbb{R}^m$. Then, we apply Lemma (II.2.10) to obtain

$$\begin{aligned}\|(u_m^p - \mathfrak{u}_{m,\ell}^p)(\mathbf{y})\|_{W_0^{1,1}(D)} &= \left\|(u_m - \mathfrak{u}_{m,\ell})(\mathbf{y})\sum_{i=0}^{p-1} u_m^i(\mathbf{y})\mathfrak{u}_{m,\ell}^{p-1-i}(\mathbf{y})\right\|_{W_0^{1,1}(D)} \\ &\leq \sum_{i=0}^{p-1}\|(u_m - \mathfrak{u}_{m,\ell})(\mathbf{y})\|_{W_0^{1,p}(D)}\|u_m(\mathbf{y})\|_{W_0^{1,p}(D)}^i\|\mathfrak{u}_{m,\ell}(\mathbf{y})\|_{W_0^{1,p}(D)}^{p-i-1}.\end{aligned} \tag{1.4}$$

[1] Error estimates in respectively $L^2(D)$ and $L^1(D)$ are derived by straightforward modifications, yielding the convergence rate $4^{-\ell}$. Then, the error analysis of the multilevel quadrature can be performed with respect to these norms, provided that the precision of the underlying quadrature rule, see (2.3), is also chosen as $\varepsilon_\ell = 4^{-\ell}$.

By using the estimate $\|(u_m - \mathfrak{u}_{m,\ell})(\mathbf{y})\|_{W_0^{1,p}(D)} \lesssim 2^{-\ell}\kappa_m(\mathbf{y})\|u_m(\mathbf{y})\|_{W^{2,p}(D)}$, cf. [BS08], it follows that

$$\begin{aligned}\|\mathfrak{u}_{m,\ell}(\mathbf{y})\|_{W_0^{1,p}(D)} &\leq \|u_m(\mathbf{y})\|_{W_0^{1,p}(D)} + \|(u_m - \mathfrak{u}_{m,\ell})(\mathbf{y})\|_{W_0^{1,p}(D)} \\ &\lesssim \Big(1 + \kappa_m(\mathbf{y})2^{-\ell}\Big)\|u_m(\mathbf{y})\|_{W^{2,p}(D)}.\end{aligned}$$

Inserting this into the previous estimate (1.4), we finally arrive at (1.3). □

(1.5) **Remark.** Of course, there are similar results available for the solution u_m to the uniformly elliptic problem (III.3.3). In this case, the factor $\kappa_m(\mathbf{y})$ is bounded uniformly in the parametric variable $\mathbf{y}$. More precisely, it holds that

$$\|(u_m - \mathfrak{u}_{m,\ell})(\mathbf{y})\|_{H_0^1(D)} \lesssim 2^{-\ell}\|u_m(\mathbf{y})\|_{H^2(D)}$$

and

$$\|(u_m^p - \mathfrak{u}_{m,\ell}^p)(\mathbf{y})\|_{W_0^{1,1}(D)} \lesssim 2^{-\ell}\|u_m(\mathbf{y})\|_{W^{2,p}(D)}^p$$

with a constant depending on p, the domain D, $\underline{a}$ and $\overline{a}$ but not on $\mathbf{y}$.

(1.6) **Remark.** All regularity results established in Section III.5, i.e. the regularity results with respect to the parametric variable $\mathbf{y}$, remain valid if one replaces the solution u_m by its Galerkin projection $\mathfrak{u}_{m,\ell}$. This issues from the fact that the proofs can be performed in the same way with the Galerkin projection instead of the solution itself.

2. Multilevel quadrature

Let us assume that we want to approximate the moments of the solution u_m to (III.3.3) or (III.3.5) up to an accuracy $\varepsilon_j = 2^{-j}$. This implies that the refinement level of the finite element discretization is given by j. The crucial idea of the multilevel quadrature is a representation of the Galerkin projection $G_j(\mathbf{y})$ on the refinement level j as a telescoping sum

(2.1) $$G_j(\mathbf{y}) = \sum_{\ell=0}^{j} G_\ell(\mathbf{y}) - G_{\ell-1}(\mathbf{y}).$$

Notice that each summand in (2.1) corresponds to the difference between two Galerkin projections on two consecutive levels of spatial refinement $\ell-1$ and ℓ. Since the expectation is a linear operator, we observe from (2.1) that

(2.2) $$\mathbf{I}(G_j(\mathbf{y})v(\mathbf{y})) = \sum_{\ell=0}^{j} \mathbf{I}(G_\ell(\mathbf{y})v(\mathbf{y}) - G_{\ell-1}(\mathbf{y})v(\mathbf{y})).$$

In contrast to the single level quadrature, which uses only one quadrature rule on the finest spatial refinement level j, the multilevel quadrature uses quadrature rules with different accuracies for each summand in the telescoping sum (2.2). The accuracy is chosen contradirectional to the approximation power of the finite element level for the spatial domain. For the approximation in the stochastic variable $\mathbf{y}$, we shall hence provide a sequence of quadrature formulae $\{\mathbf{Q}_\ell\}_\ell$ for $\ell = 0, \ldots, j$ for the Bochner integral $(\mathbf{I}v)(\mathbf{x}) = \int_{\mathbf{\Gamma}} v(\mathbf{x}, \mathbf{y})\boldsymbol{\rho}(\mathbf{y})\, \mathrm{d}\mathbf{y}$ of the form

$$\mathbf{Q}_\ell : C^0_{\boldsymbol{\sigma}_s}(\mathbf{\Gamma}; X) \to X, \quad v \mapsto \mathbf{Q}_\ell v = \sum_{i=1}^{N_\ell} \boldsymbol{\omega}_{\ell,i} v(\cdot, \boldsymbol{\xi}_{\ell,i}),$$

where $\boldsymbol{\sigma}_s$ is given by (III.5.23) in the lognormal case and $\boldsymbol{\sigma}_s(\mathbf{y}) \equiv 1$ in the uniformly elliptic case. This can be any quadrature rule analyzed in the preceeding chapters. For our purposes, we assume that the number of points N_ℓ of the quadrature formula $\mathbf{Q}_\ell$ is chosen such that the corresponding accuracy is $\mathcal{O}(\varepsilon_\ell)$ with

$$\varepsilon_\ell = 2^{-\ell}. \tag{2.3}$$

The single level quadrature to determine the expectation of a function $v \in C^0_{\boldsymbol{\sigma}_s}(\mathbf{\Gamma}; X)$ up to an accuracy of order ε_j is then given by

$$\mathbb{E}_v(\mathbf{x}) \approx \mathbf{Q}_j \mathfrak{v}_j = \sum_{i=1}^{N_j} \boldsymbol{\omega}_{j,i} \mathfrak{v}_j(\cdot, \boldsymbol{\xi}_{j,i}) \tag{2.4}$$

and for the higher order moments by

$$\mathcal{M}^p_v(\mathbf{x}) \approx \mathbf{Q}_j \mathfrak{v}^p_j = \sum_{i=1}^{N_j} \boldsymbol{\omega}_{j,i} \mathfrak{v}^p_j(\cdot, \boldsymbol{\xi}_{j,i}). \tag{2.5}$$

Hence, the quadrature rule with the highest accuracy is combined with Galerkin approximations on the finest refinement level for the resulting deterministic boundary value problem associated with each quadrature point. Compared with this, the multilevel quadrature equilibrates the accuracies appropriately. For the approximation of the expectation of a function $v \in C^0_{\boldsymbol{\sigma}_s}(\mathbf{\Gamma}; X)$, the multilevel quadrature is defined by

$$\mathbb{E}_v(\mathbf{x}) \approx \sum_{\ell=0}^{j} \mathbf{Q}_{j-\ell}(\mathfrak{v}_\ell - \mathfrak{v}_{\ell-1})(\mathbf{x}, \mathbf{y}) = \sum_{\ell=0}^{j} \sum_{i=0}^{N_{j-\ell}} \boldsymbol{\omega}_{j-\ell,i}(\mathfrak{v}_\ell - \mathfrak{v}_{\ell-1})(\mathbf{x}, \boldsymbol{\xi}_{j-\ell,i}). \tag{2.6}$$

The higher order moments are approximated in complete analogy by

$$\mathcal{M}^p_v(\mathbf{x}) \approx \sum_{\ell=0}^{j} \mathbf{Q}_{j-\ell}(\mathfrak{v}^p_\ell - \mathfrak{v}^p_{\ell-1})(\mathbf{x}, \mathbf{y}) = \sum_{\ell=0}^{j} \sum_{i=0}^{N_{j-\ell}} \boldsymbol{\omega}_{j-\ell,i}(\mathfrak{v}^p_\ell - \mathfrak{v}^p_{\ell-1})(\mathbf{x}, \boldsymbol{\xi}_{j-\ell,i}). \tag{2.7}$$

Since the multilevel quadrature (2.6) and (2.7) can be interpreted as a sparse-grid approach, cf. [HPS13a], it is known that mixed regularity results of the integrand have to be provided. Thus, we present the necessary regularity estimates of the solution u_m to (III.3.3) or to (III.3.5) and its powers in the following section. Afterwards, we investigate the regularity of the differences between two successive Galerkin approximations $\mathfrak{u}_{m,\ell}$ and $\mathfrak{u}_{m,\ell-1}$ and their powers $\mathfrak{u}^p_{m,\ell}$ and $\mathfrak{u}^p_{m,\ell-1}$ which then allows us to apply the error estimates established in the previous chapters.

3. Mixed regularity in the spatial and the parametric variable

In Section III.5, we provided bounds on the derivatives of the solution u_m and of its powers u_m^p when the spatial regularity is measured in $H_0^1(D)$ and in $W_0^{1,1}(D)$ for $p \geq 2$, respectively. We now present a result from [HS14] in the lognormal case and from [CDS10] in the uniformly elliptic case which establishes estimates on $\partial_{\mathbf{y}}^{\boldsymbol{\alpha}} u_m$ when the spatial regularity is measured in the space $\mathcal{W} := H^2(D) \cap H_0^1(D)$. These results guarantee the mixed regularity which is necessary for the sparse-grid construction between the spatial and the stochastic variable. To that end, we shall recall that the corresponding eigenfunctions of the Karhunen-Loève expansion (III.2.2) belong to $W^{1,\infty}(D)$ by (III.2.5).

We start now with the lognormal case and define in analogy to (III.3.10) and (III.3.13)

$$\tag{3.1} \underline{\tilde{a}}_m(\mathbf{y}) := \exp\left(-\sum_{k=1}^{m} \tilde{\gamma}_k |y_k|\right), \quad \overline{\tilde{a}}_m(\mathbf{y}) := \exp\left(\sum_{k=1}^{m} \tilde{\gamma}_k |y_k|\right), \quad \tilde{\kappa}_m(\mathbf{y}) := \frac{\overline{\tilde{a}}_m(\mathbf{y})}{\underline{\tilde{a}}_m(\mathbf{y})}.$$

Furthermore, we will employ the spaces $C^0_{\tilde{\boldsymbol{\sigma}}_s}(\mathbb{R}^m; X)$, see Definition (III.5.22) and also Definition (III.5.23), with the auxiliary weight $\tilde{\boldsymbol{\sigma}}_s$ defined with respect to $\tilde{\boldsymbol{\gamma}}$ instead of $\boldsymbol{\gamma}$. Since for convex or sufficiently smooth domains a norm on $\mathcal{W}$ is given by

$$\|v\|_{\mathcal{W}} := \|\nabla v\|_{L^2(D)} + \|\Delta v\|_{L^2(D)},$$

cf. [CDS10, HS14], it only remains, in view of the results of Section (III.5.3), to analyze the term $\|\Delta \partial_{\mathbf{y}}^{\boldsymbol{\alpha}} u_m(\mathbf{y})\|_{L^2(D)}$ in order to establish estimates on the derivatives of $u_m(\mathbf{y})$ with respect to the stochastic variable $\mathbf{y}$ measured in $H^2(D)$. Along the lines of [HS14], we have the following result.

(3.2) **Proposition ([HS14]).** For all $\mathbf{y} \in \mathbb{R}^m$, the solution $u_m(\mathbf{y}) \in H_0^1(D)$ to problem (III.3.5) satisfies

$$\begin{aligned} &\left\|\sqrt{a_m(\mathbf{y})}\Delta \partial_{\mathbf{y}}^{\boldsymbol{\alpha}} u_m(\mathbf{y})\right\|_{L^2(D)} \\ &\quad \lesssim |\boldsymbol{\alpha}|!\left(\frac{2\tilde{\gamma}}{\log 2}\right)^{\boldsymbol{\alpha}}\left(\left\|\sqrt{a_m(\mathbf{y})^{-1}}f\right\|_{L^2(D)} + 2g(\mathbf{y})\left\|\sqrt{a_m(\mathbf{y})}\nabla u_m(\mathbf{y})\right\|_{L^2(D)}\right) \end{aligned}$$

with $g(\mathbf{y}) := 1 + 2\sum_{k=1}^{m} |y_k|\sqrt{\lambda_k}\|\nabla \phi_k\|_{L^\infty(D)} < \infty$.

Proposition (3.2) implies the estimate

$$\|\Delta \partial_{\mathbf{y}}^{\boldsymbol{\alpha}} u_m(\mathbf{y})\|_{L^2(D)} \lesssim \sqrt{\kappa_m(\mathbf{y})}|\boldsymbol{\alpha}|!\left(\frac{2\tilde{\gamma}}{\log 2}\right)^{\boldsymbol{\alpha}}\left(\|f\|_{L^2(D)} + 2g(\mathbf{y})\|u_m(\mathbf{y})\|_{H_0^1(D)}\right),$$

which can be further bounded by

$$\|\Delta \partial_{\mathbf{y}}^{\boldsymbol{\alpha}} u_m(\mathbf{y})\|_{L^2(D)} \lesssim \kappa_m(\mathbf{y})g(\mathbf{y})|\boldsymbol{\alpha}|!\left(\frac{2\tilde{\gamma}}{\log 2}\right)^{\boldsymbol{\alpha}}\|f\|_{L^2(D)}$$

due to (III.3.10) and (III.3.11). It follows together with Lemma (III.5.3) and with

(3.3)
$$\begin{aligned}\kappa_m(\mathbf{y})g(\mathbf{y}) &= \exp\left(2\sum_{k=1}^m \gamma_k|y_k|\right)\left(1+2\sum_{k=1}^m |y_k|\sqrt{\lambda_k}\|\nabla\phi_k\|_{L^\infty(D)}\right)\\ &\le \exp\left(2\sum_{k=1}^m \gamma_k|y_k|\right)\exp\left(2\sum_{k=1}^m |y_k|\sqrt{\lambda_k}\|\nabla\phi_k\|_{L^\infty(D)}\right) = \tilde{\kappa}_m(\mathbf{y})\end{aligned}$$

that

(3.4)
$$\|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}} u_m(\mathbf{y})\|_{H^2(D)} \lesssim \|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}} u_m(\mathbf{y})\|_{\mathcal{W}} \lesssim \tilde{\kappa}_m(\mathbf{y})|\boldsymbol{\alpha}|!\left(\frac{2\tilde{\gamma}}{\log 2}\right)^{\boldsymbol{\alpha}}\|f\|_{L^2(D)}.$$

This establishes the following proposition:

(3.5) **Proposition.** The solution u_m to (III.3.5) is contained in $C^0_{\tilde{\boldsymbol{\sigma}}_s}(\mathbb{R}^m; H^2(D))$ for all $s \ge 2$ and it holds for any multi-index $|\boldsymbol{\alpha}| \ge 0$ that

(3.6)
$$\|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}} u_m\|_{C^0_{\tilde{\boldsymbol{\sigma}}_s}(\mathbb{R}^m;H^2(D))} \lesssim |\boldsymbol{\alpha}|!\left(\frac{2\tilde{\gamma}}{\log 2}\right)^{\boldsymbol{\alpha}}\|f\|_{L^2(D)}.$$

In the uniformly elliptic case, we refer to [CDS10] where the next result is proven.

(3.7) **Proposition ([CDS10]).** The solution u_m to (III.3.3) belongs to the space $C^0([-1/2,1/2]^m; H^2(D))$ and satisfies for any multi-index $|\boldsymbol{\alpha}| \ge 0$ that

$$\|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}} u_m\|_{C^0([-1/2,1/2]^m;H^2(D))} \lesssim |\boldsymbol{\alpha}|!\left(\frac{2\tilde{\gamma}}{\underline{a}}\right)^{\boldsymbol{\alpha}}\|f\|_{L^2(D)}$$

with a constant which depends on $\underline{a}$ and on $\sup_{\mathbf{y}\in[-1/2,1/2]^m}\|\nabla a(\mathbf{y})\|_{L^\infty(D)}$.

(3.8) **Remark.** In [CDS10], even a stronger result on the derivatives is proven, namely that

$$\|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}} u_m\|_{C^0([-1/2,1/2]^m;H^2(D))} \lesssim |\boldsymbol{\alpha}|!\mathbf{b}(\varepsilon)^{\boldsymbol{\alpha}}\|f\|_{L^2(D)},$$

where $b_k(\varepsilon) = \gamma_k/\underline{a} + \varepsilon\left(\sqrt{\lambda_k}\|\nabla\varphi_k\|_{L^\infty(D)} + c\|\varphi\|_{L^\infty(D)}\right)$. Thus, the involved constant depends on ε and $\underline{a}$. Since the focus of our regularity results is on the lognormal case, we will not work with this stronger result in the sequel.

With the previous two propositions, we can establish error estimates for the multilevel quadrature of the first and second moment, see Section 4. For the higher order moments, we have additionally to establish estimates of the derivatives $\partial_{\mathbf{y}}^{\boldsymbol{\alpha}} u_m(\mathbf{y})$ with respect to the Sobolev space $W^{2,p}(D)$ which are the L^p-extensions of Proposition (3.4) and Proposition (3.7), respectively. The proof of the L^p-extension of (3.4) is very similar to that of (3.2) in [HS14]. We will exploit the norm equivalence

(3.9)
$$\|v\|_{W^{2,p}(D)} \eqsim \|v\|_{W_0^{1,p}(D)} + \|\Delta v\|_{L^p(D)}$$

for functions $v \in W^{2,p}(D) \cap W_0^{1,p}(D)$, where D is a sufficiently smooth or convex domain. Since estimates for the first term on the right-hand side in (3.9) are provided in Section (III.5.3), it suffices, analogous to the case $p = 2$, to consider bounds of $\|\Delta\partial_{\mathbf{y}}^{\boldsymbol{\alpha}} u\|_{L^p(D)}$.

(3.10) **Lemma.** Let $f \in L^p(D)$ and let D be a sufficiently smooth or convex domain. Then, the derivatives of the solution u_m to (III.3.5) satisfy

$$\|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}} u_m(\mathbf{y})\|_{W^{2,p}(D)} \lesssim |\boldsymbol{\alpha}|! \left(\frac{2C(p,D)\tilde{\gamma}}{\log 2}\right)^{\boldsymbol{\alpha}} \tilde{\kappa}(\mathbf{y})^{3/2} \|f\|_{L^p(D)}. \tag{3.11}$$

Proof. It holds with the definition $v_{\boldsymbol{\alpha}}(\mathbf{x},\mathbf{y}) := \operatorname{div}\left(a_m(\mathbf{x},\mathbf{y})\nabla_{\mathbf{x}}\partial_{\mathbf{y}}^{\boldsymbol{\alpha}} u_m(\mathbf{x},\mathbf{y})\right)$ that

$$a_m(\mathbf{x},\mathbf{y})\Delta_{\mathbf{x}}\partial_{\mathbf{y}}^{\boldsymbol{\alpha}} u_m(\mathbf{x},\mathbf{y}) = v_{\boldsymbol{\alpha}}(\mathbf{x},\mathbf{y}) - \nabla_{\mathbf{x}} a_m(\mathbf{x},\mathbf{y})\nabla_{\mathbf{x}}\partial_{\mathbf{y}}^{\boldsymbol{\alpha}} u_m(\mathbf{x},\mathbf{y}).$$

Additionally, we have that

$$\nabla_{\mathbf{x}} a_m(\mathbf{x},\mathbf{y}) = \left(\sum_{k=1}^m \sqrt{\lambda_k}\nabla_{\mathbf{x}}\varphi_k(\mathbf{x}) y_k\right) a_m(\mathbf{x},\mathbf{y}).$$

Hence, we get for all $\mathbf{y} \in \mathbb{R}^m$ that

$$\begin{aligned} \left\|a_m(\mathbf{y})\Delta\partial_{\mathbf{y}}^{\boldsymbol{\alpha}} u_m(\mathbf{y})\right\|_{L^p(D)} &\leq \|v_{\boldsymbol{\alpha}}(\mathbf{y})\|_{L^p(D)} \\ &\quad + \sum_{k=1}^m \sqrt{\lambda_k}|y_k|\|\nabla\varphi_k\|_{L^\infty(D)} \left\|a_m(\mathbf{y})\nabla\partial_{\mathbf{y}}^{\boldsymbol{\alpha}} u_m(\mathbf{y})\right\|_{[L^p(D)]^d}. \end{aligned} \tag{3.12}$$

In view of (III.5.5), we obtain that

$$\begin{aligned} v_{\boldsymbol{\alpha}}(\mathbf{x},\mathbf{y}) = -\sum_{\boldsymbol{\beta}<\boldsymbol{\alpha}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} \Big[&\nabla_{\mathbf{x}}\partial_{\mathbf{y}}^{\boldsymbol{\alpha}-\boldsymbol{\beta}} a_m(\mathbf{x},\mathbf{y})\nabla_{\mathbf{x}}\partial_{\mathbf{y}}^{\boldsymbol{\beta}} u_m(\mathbf{x},\mathbf{y}) \\ &+ \partial_{\mathbf{y}}^{\boldsymbol{\alpha}-\boldsymbol{\beta}} a_m(\mathbf{x},\mathbf{y})\Delta_{\mathbf{x}}\partial_{\mathbf{y}}^{\boldsymbol{\beta}} u_m(\mathbf{x},\mathbf{y})\Big]. \end{aligned}$$

With $\boldsymbol{\Phi}(\mathbf{x}) := [\sqrt{\lambda_1}\varphi_1(\mathbf{x}),\ldots,\sqrt{\lambda_m}\varphi_m(\mathbf{x})]$, we derive that

$$\begin{aligned} \nabla_{\mathbf{x}}(\partial_{\mathbf{y}}^{\boldsymbol{\alpha}} a_m(\mathbf{x},\mathbf{y})) &= \nabla_{\mathbf{x}}(\Phi(\mathbf{x})^{\boldsymbol{\alpha}} a_m(\mathbf{x},\mathbf{y})) \\ &= \nabla_{\mathbf{x}}(\Phi(\mathbf{x})^{\boldsymbol{\alpha}}) a_m(\mathbf{x},\mathbf{y}) + \Phi(\mathbf{x})^{\boldsymbol{\alpha}} \sum_{k=1}^m y_k\sqrt{\lambda_k}\nabla_{\mathbf{x}}\varphi_k(\mathbf{x}) a_m(\mathbf{x},\mathbf{y}). \end{aligned} \tag{3.13}$$

We deduce from (3.13), from $\nabla_{\mathbf{x}}(\Phi(\mathbf{x})^{\boldsymbol{\alpha}}) = \sum_{k=1}^m \alpha_k\sqrt{\lambda_k}\nabla_{\mathbf{x}}\varphi_k(\mathbf{x})\Phi(\mathbf{x})^{\boldsymbol{\alpha}-\mathbf{e}_k}$, where $\mathbf{e}_k$ denotes the k-th unit vector, and from the estimate

$$\begin{aligned} \tilde{\gamma}^{\boldsymbol{\alpha}} &= \sum_{\boldsymbol{\beta}\leq\boldsymbol{\alpha}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} \boldsymbol{\gamma}^{\boldsymbol{\alpha}-\boldsymbol{\beta}} \prod_{k=1}^m \left(\sqrt{\lambda_k}\|\nabla\varphi_k\|_{L^\infty(D)}\right)^{\beta_k} \\ &\leq \boldsymbol{\gamma}^{\boldsymbol{\alpha}} + \sum_{k=1}^m \binom{\alpha_k}{1} \boldsymbol{\gamma}^{\boldsymbol{\alpha}-\mathbf{e}_k}\sqrt{\lambda_k}\|\nabla\varphi_k\|_{L^\infty(D)} \end{aligned}$$

that

$$\left\|a_m(\mathbf{y})^{-1}\nabla_{\mathbf{x}}(\partial_{\mathbf{y}}^{\boldsymbol{\alpha}-\boldsymbol{\beta}} a_m(\mathbf{y}))\right\|_{L^\infty(D)} \leq \tilde{\gamma}^{\boldsymbol{\alpha}-\boldsymbol{\beta}}\left(1 + \sum_{k=1}^m \sqrt{\lambda_k}|y_k|\|\nabla\varphi_k\|_{L^\infty(D)}\right).$$

Then, it follows that

$$\|v_{\boldsymbol{\alpha}}(\mathbf{y})\|_{L^p(D)} \leq \sum_{\boldsymbol{\beta}<\boldsymbol{\alpha}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} \bigg[\tilde{\boldsymbol{\gamma}}^{\boldsymbol{\alpha}-\boldsymbol{\beta}}\bigg(1+\sum_{k=1}^{m}\sqrt{\lambda_k}|y_k|\|\nabla\varphi_k\|_{L^\infty(D)}\bigg) \\ \cdot \big\|a_m(\mathbf{y})\nabla\partial_{\mathbf{y}}^{\boldsymbol{\beta}}u_m(\mathbf{y})\big\|_{[L^p(D)]^d} + \tilde{\boldsymbol{\gamma}}^{\boldsymbol{\alpha}-\boldsymbol{\beta}}\big\|a_m(\mathbf{y})\Delta\partial_{\mathbf{y}}^{\boldsymbol{\beta}}u_m(\mathbf{y})\big\|_{L^p(D)}\bigg].$$

Inserting (3.12) into the above inequality yields

$$\|v_{\boldsymbol{\alpha}}(\mathbf{y})\|_{L^p(D)} \leq \sum_{\boldsymbol{\beta}<\boldsymbol{\alpha}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} \tilde{\boldsymbol{\gamma}}^{\boldsymbol{\alpha}-\boldsymbol{\beta}}\Big[g(\mathbf{y})\|a_m(\mathbf{y})\nabla\partial_{\mathbf{y}}^{\boldsymbol{\beta}}u_m(\mathbf{y})\|_{[L^p(D)]^d} + \|v_{\boldsymbol{\beta}}(\mathbf{y})\|_{L^p(D)}\Big],$$

where $g(\mathbf{y}) = 1 + 2\sum_{k=1}^{m}\sqrt{\lambda_k}|y_k|\|\nabla\varphi_k\|_{L^\infty(D)}$ is given as in Proposition (3.2). From the proof of (III.5.18), we deduce that

$$\|a_m(\mathbf{y})\nabla\partial_{\mathbf{y}}^{\boldsymbol{\alpha}}u_m(\mathbf{y})\|_{[L^p(D)]^d} \leq C(p,D)\sum_{\boldsymbol{\beta}<\boldsymbol{\alpha}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} \boldsymbol{\gamma}^{\boldsymbol{\alpha}-\boldsymbol{\beta}}\|a_m(\mathbf{y})\nabla\partial_{\mathbf{y}}^{\boldsymbol{\beta}}u_m(\mathbf{y})\|_{[L^p(D)]^d}.$$

Thus, we observe from $C(p,D)\geq 1$ and $\gamma_k \leq \tilde{\gamma}_k$ that

$$g(\mathbf{y})^{-1}\|v_{\boldsymbol{\alpha}}(\mathbf{y})\|_{L^p(D)} + \|a_m(\mathbf{y})\nabla\partial_{\mathbf{y}}^{\boldsymbol{\alpha}}u_m(\mathbf{y})\|_{[L^p(D)]^d} \\ \leq \sum_{\boldsymbol{\beta}<\boldsymbol{\alpha}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} 2C(p,D)\tilde{\boldsymbol{\gamma}}^{\boldsymbol{\alpha}-\boldsymbol{\beta}}\Big[\|a_m(\mathbf{y})\nabla\partial_{\mathbf{y}}^{\boldsymbol{\beta}}u_m(\mathbf{y})\|_{[L^p(D)]^d} + g(\mathbf{y})^{-1}\|v_{\boldsymbol{\beta}}(\mathbf{y})\|_{L^p(D)}\Big].$$

Now, one can show by the same arguments as in (III.5.3) that

$$g(\mathbf{y})^{-1}\|v_{\boldsymbol{\alpha}}(\mathbf{y})\|_{L^p(D)} + \|a_m(\mathbf{y})\nabla\partial_{\mathbf{y}}^{\boldsymbol{\alpha}}u_m(\mathbf{y})\|_{[L^p(D)]^d} \\ \leq |\boldsymbol{\alpha}|!\bigg(\frac{2C(p,D)\tilde{\boldsymbol{\gamma}}}{\log 2}\bigg)^{\boldsymbol{\alpha}}\Big(g(\mathbf{y})^{-1}\|v_0(\mathbf{y})\|_{L^p(D)} + \|a_m(\mathbf{y})\nabla u_m(\mathbf{y})\|_{[L^p(D)]^d}\Big).$$

Especially, this implies in combination with $v_0 = f$ and (III.5.18) that

$$\|v_{\boldsymbol{\alpha}}(\mathbf{y})\|_{L^p(D)} \leq |\boldsymbol{\alpha}|!\bigg(\frac{2C(p,D)\tilde{\boldsymbol{\gamma}}}{\log 2}\bigg)^{\boldsymbol{\alpha}}\Big(\|f\|_{L^p(D)} + g(\mathbf{y})\|a_m(\mathbf{y})\nabla u_m(\mathbf{y})\|_{[L^p(D)]^d}\Big). \tag{3.14}$$

Inserting this estimate into (3.12) leads to

$$\|a_m(\mathbf{y})\Delta\partial_{\mathbf{y}}^{\boldsymbol{\alpha}}u_m(\mathbf{y})\|_{L^p(D)} \\ \leq |\boldsymbol{\alpha}|!\bigg(\frac{2C(p,D)\tilde{\boldsymbol{\gamma}}}{\log 2}\bigg)^{\boldsymbol{\alpha}}\Big(\|f\|_{L^p(D)} + 2g(\mathbf{y})\|a_m(\mathbf{y})\nabla u_m(\mathbf{y})\|_{[L^p(D)]^d}\Big).$$

With (3.3) and (III.5.14), it follows that

$$\|\Delta\partial_{\mathbf{y}}^{\boldsymbol{\alpha}}u_m(\mathbf{y})\|_{L^p(D)} \lesssim |\boldsymbol{\alpha}|!\bigg(\frac{2C(p,D)\tilde{\boldsymbol{\gamma}}}{\log 2}\bigg)^{\boldsymbol{\alpha}}\kappa_m(\mathbf{y})\frac{g(\mathbf{y})}{\underline{a}_m(\mathbf{y})}\|f\|_{L^p(D)}.$$

Thus, we conclude with (3.3) that

$$\|\Delta\partial_{\mathbf{y}}^{\boldsymbol{\alpha}}u_m(\mathbf{y})\|_{L^p(D)} \lesssim |\boldsymbol{\alpha}|!\bigg(\frac{2C(p,D)\tilde{\boldsymbol{\gamma}}}{\log 2}\bigg)^{\boldsymbol{\alpha}}\frac{\tilde{\kappa}_m(\mathbf{y})}{\underline{a}_m(\mathbf{y})}\|f\|_{L^p(D)}.$$

The assertion is finally obtained from $\underline{a}_m(\mathbf{y}) \geq \underline{\tilde{a}}_m(\mathbf{y})$ and from estimate (III.5.15). □

From Lemma (3.10), we derive that $\partial_{\mathbf{y}}^{\boldsymbol{\alpha}} u_m$ is an element of $C^0_{\tilde{\boldsymbol{\sigma}}_s}(\mathbb{R}^m; W^{2,p}(D))$ for all $s \geq 3$. Especially, it holds that

$$\|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}} u_m\|_{C^0_{\tilde{\boldsymbol{\sigma}}_s}(\mathbb{R}^m;W^{2,p}(D))} \lesssim |\boldsymbol{\alpha}|! \left(\frac{2C(p,D)\tilde{\boldsymbol{\gamma}}}{\log 2}\right)^{\boldsymbol{\alpha}} \|f\|_{L^p(D)}. \tag{3.15}$$

4. Analysis of the multilevel quadrature

We shall next have a closer look at the different impacts of the error of the moment computation. In the single level quadrature method, we can split the error into three parts

$$\|\mathcal{M}_u^p - \mathbf{Q}_j \mathfrak{u}_{m,j}^p\|_X \leq \underbrace{\|\mathcal{M}_u^p - \mathcal{M}_{u_m}^p\|_X}_{\mathrm{I}} + \underbrace{\|\mathcal{M}_{u_m}^p - \mathcal{M}_{\mathfrak{u}_{m,j}}^p\|_X}_{\mathrm{II}} + \underbrace{\|\mathcal{M}_{\mathfrak{u}_{m,j}}^p - \mathbf{Q}_j \mathfrak{u}_{m,j}\|_X}_{\mathrm{III}}, \tag{4.1}$$

where the truncation error, the first term on the right-hand side of (4.1), can be controlled by the number of terms in the Karhunen-Loève expansion, see Theorems (III.2.12), (III.2.10) and (III.2.14). The second term on the right-hand side of (4.1) describes the finite element discretization error and can be estimated by Lemma (1.1). In the Chapters IV and V, the third term on the right-hand side of (4.1), which characterizes the quadrature error, is analyzed for various quadrature rules. The Galerkin projection provides the required regularity with respect to the parametric variable $\mathbf{y}$, see also Remark (1.6). In case of the Monte Carlo quadrature, we have of course to consider the RMSE instead of the error measured in X. Notice that this affects only the error term III, since the terms I and II are independent of the integration error in the parametric variable $\mathbf{y}$. We perform here the further error analysis only for the lognormal case. In the uniformly elliptic case, the proofs can be straightforwardly transferred from the lognormal case. In particular, the appearing constants are easier to handle since they are independent of the parameter $\mathbf{y}$.

In the multilevel quadrature, we aim at equilibrating the approximation error in the spatial and in the stochastic variable. Thus, the error contributions II and III cannot be estimated separately and we have to investigate

$$\begin{aligned} &\left\|\mathcal{M}_{u_m}^p - \sum_{\ell=0}^{j} \mathbf{Q}_{j-\ell}(\mathfrak{u}_{m,\ell}^p - \mathfrak{u}_{m,\ell-1}^p)\right\|_X \\ &\qquad \leq \|\mathcal{M}_{u_m}^p - \mathcal{M}_{\mathfrak{u}_{m,j}}^p\|_X + \sum_{\ell=0}^{j} \underbrace{\|(\mathbf{I} - \mathbf{Q}_{j-\ell})(\mathfrak{u}_{m,\ell}^p - \mathfrak{u}_{m,\ell-1}^p)\|_X}_{\mathrm{IV}}. \end{aligned} \tag{4.2}$$

The first part here coincides with the finite element error term II in (4.1). Hence, we have to estimate each summand IV in (4.2) for the different types of quadrature formulae. For equilibrating the approximations in the spatial and parametric variable, the aim is to obtain the error bound

$$\|(\mathbf{I} - \mathbf{Q}_{j-\ell})(\mathfrak{u}_\ell^p - \mathfrak{u}_{\ell-1}^p)(\mathbf{y})\|_X \lesssim \varepsilon_{j-\ell} 2^{-\ell} \|f\|_{L^{p+\delta_{1,p}}(D)}^p \quad \text{with} \quad \delta_{1,p} = \begin{cases} 1, & \text{if } p = 1, \\ 0, & \text{if } p \geq 2. \end{cases} \tag{4.3}$$

Herein, we have $X = H_0^1(D)$ if $p = 1$ and $X = W_0^{1,1}(D)$ if $p \geq 2$, respectively.

For the Monte Carlo quadrature, we know that (4.3) is bounded in the root mean square sense by

(4.4) $$\begin{aligned}\sqrt{\mathbb{E}\|(\mathbf{I}-\mathbf{Q}_{j-\ell})(\mathfrak{u}_\ell^p-\mathfrak{u}_{\ell-1}^p)(\mathbf{y})\|_X^2} &\lesssim \varepsilon_{j-\ell}\|(\mathfrak{u}_\ell^p-\mathfrak{u}_{\ell-1}^p)(\mathbf{y})\|_{L_\rho^2(\mathbb{R}^m;X)}\\ &\lesssim \varepsilon_{j-\ell}2^{-\ell}\|f\|_{L^{p+\delta_{1,p}}(D)}^p.\end{aligned}$$

The last inequality is obtained by using the estimate (1.2) if $p = 1$ and by using the estimate (1.3) if $p \geq 2$. Then, we apply for $|\boldsymbol{\alpha}| = 0$ the estimate (3.4) if $p = 1$ and the estimate (3.11) if $p = 2$. By the integrability of $\tilde{\kappa}_m(\mathbf{y})^s$ for arbitrary $s > 0$, we arrive at (4.4). Notice that it is important here, for the dimension-independent convergence, that the value of the integral of $\tilde{\kappa}_m(\mathbf{y})^s$ is independent of the dimensionality m, see (III.5.26). With the bound (4.4), we can estimate the error of the multilevel Monte Carlo quadrature.

(4.5) **Theorem.** Let u_m be the solution to (III.3.5) and let $\{\mathbf{Q}_\ell\}_\ell$ be a sequence of Monte Carlo quadrature rules where the number of quadrature points N_ℓ of $\mathbf{Q}_\ell$ is chosen in accordance with (2.3).

Then, the error of the multilevel Monte Carlo method (MLMC) defined in (2.6) for the expectation and in (2.7) for the moments is bounded in the root mean square sense by

(4.6) $$\sqrt{\mathbb{E}\left\|\mathcal{M}_{u_m}^p-\sum_{\ell=0}^{j}\mathbf{Q}_{j-\ell}(\mathfrak{u}_{m,\ell}^p-\mathfrak{u}_{m,\ell-1}^p)\right\|_X^2} \lesssim 2^{-j}j\|f\|_{L^{p+\delta_{1,p}}(D)}^p,$$

where $X = H_0^1(D)$ if $p = 1$ and $X = W_0^{1,1}(D)$ if $p \geq 2$.

Proof. For MLMC, we have the multilevel splitting of the error

(4.7) $$\begin{aligned}&\sqrt{\mathbb{E}\left\|\mathcal{M}_{u_m}^p-\sum_{\ell=0}^{j}\mathbf{Q}_{j-\ell}(\mathfrak{u}_{m,\ell}^p-\mathfrak{u}_{m,\ell-1}^p)\right\|_X^2}\\ &\quad\leq \sqrt{\mathbb{E}\|\mathcal{M}_{u_m}^p-\mathcal{M}_{\mathfrak{u}_{m,j}}^p\|_X^2}+\sum_{\ell=0}^{j}\sqrt{\mathbb{E}\|(\mathbf{I}-\mathbf{Q}_{j-\ell})(\mathfrak{u}_{m,\ell}^p-\mathfrak{u}_{m,\ell-1}^p)\|_X^2}\\ &\quad= \|\mathcal{M}_{u_m}^p-\mathcal{M}_{\mathfrak{u}_{m,j}}^p\|_X+\sum_{\ell=0}^{j}\sqrt{\mathbb{E}\|(\mathbf{I}-\mathbf{Q}_{j-\ell})(\mathfrak{u}_{m,\ell}^p-\mathfrak{u}_{m,\ell-1}^p)\|_X^2}.\end{aligned}$$

The inequality in (4.7) is valid since the Monte Carlo quadrature is unbiased and the cross terms therefore vanish. The equality follows since $\mathcal{M}_{u_m}^p - \mathcal{M}_{\mathfrak{u}_{m,j}}^p$ is a function of $\mathbf{x}$ which is independent of the random parameter $\mathbf{y}$.

The accuracy of the quadrature method $\mathbf{Q}_\ell$ is, according to (2.3), given by $\varepsilon_\ell = 2^{-\ell}$. Hence, the sum on the right-hand side of (4.7) is, with the help of (4.4), estimated by

(4.8) $$\begin{aligned}\sum_{\ell=0}^{j}\sqrt{\mathbb{E}\|(\mathbf{I}-\mathbf{Q}_{j-\ell})(\mathfrak{u}_{m,\ell}^p-\mathfrak{u}_{m,\ell-1}^p)\|_X^2} &\lesssim \sum_{\ell=0}^{j}2^{-(j-\ell)}2^{-\ell}\|f\|_{L^{p+\delta_{1,p}}(D)}^p\\ &\lesssim j2^{-j}\|f\|_{L^{p+\delta_{1,p}}(D)}^p.\end{aligned}$$

The first term in (4.7) is bounded in the following way. For $p = 1$, due to (1.2) and the continuity, with continuity constant independent of m, of $\mathbf{I}$ in $C^0_{\tilde{\boldsymbol{\sigma}}_s}(\mathbb{R}^m, H^2(D))$ for $\tilde{\boldsymbol{\sigma}}_s$ with $s \geq 3$, there holds

(4.9) $$\|\mathbb{E}_{u_m} - \mathbb{E}_{\mathrm{u}_{m,j}}\|_{H^1_0(D)} \lesssim 2^{-j} \sup_{\mathbf{y}\in\mathbb{R}^m} \tilde{\boldsymbol{\sigma}}_s(\mathbf{y})\sqrt{\kappa_m(\mathbf{y})}\|u_m(\mathbf{y})\|_{H^2(D)} \lesssim 2^{-j}\|f\|_{L^2(D)}.$$

For $p \geq 2$, we use (1.3) and the continuity, with dimension-independent continuity constant, of $\mathbf{I}$ in $C^0_{\tilde{\boldsymbol{\sigma}}_s}(\mathbb{R}^m, W^{2,p}(D))$ for $\tilde{\boldsymbol{\sigma}}_s$ with $s \geq 5p$ to obtain

(4.10) $$\|\mathcal{M}^p_{u_m} - \mathcal{M}^p_{\mathrm{u}_{m,j}}\|_{W^{1,1}_0(D)} \lesssim 2^{-j} \sup_{\mathbf{y}\in\mathbb{R}^m} \tilde{\boldsymbol{\sigma}}_s(\mathbf{y})\kappa_m(\mathbf{y})^p\|u_m(\mathbf{y})\|^p_{W^{2,p}(D)} \lesssim 2^{-j}\|f\|^p_{L^p(D)}.$$

Inserting (4.9) or (4.10) and (4.8) into (4.7) yields the desired estimate

$$\begin{aligned}\sqrt{\mathbb{E}\left\|\mathcal{M}^p_{u_m} - \sum_{\ell=0}^{j}\mathbf{Q}_{j-\ell}(\mathrm{u}^p_{m,\ell} - \mathrm{u}^p_{m,\ell-1})\right\|^2_X} &\lesssim 2^{-j}\|f\|^p_{L^{p+\delta_{1,p}}(D)} + j2^{-j}\|f\|^p_{L^{p+\delta_{1,p}}(D)}\\ &\lesssim j2^{-j}\|f\|^p_{L^{p+\delta_{1,p}}(D)}.\end{aligned}$$ □

For deterministic multilevel quadrature rules which satisfy (4.3), we can establish analogously the following error estimate.

(4.11) **Theorem.** Let u_m be the solution to (III.3.5) and let $\{\mathbf{Q}_\ell\}$ be a sequence of quadrature rules which satisfy (4.3). Moreover, let the number of quadrature points N_ℓ of $\mathbf{Q}_\ell$ be chosen in accordance with (2.3), which means that $\varepsilon_\ell = 2^{-\ell}$.

Then, the error of the multilevel quadrature method defined in (2.6) for the expectation and in (2.7) for the moments fulfills the estimate

(4.12) $$\left\|\mathcal{M}^p_{u_m} - \sum_{\ell=0}^{j}\mathbf{Q}_{j-\ell}(\mathrm{u}^p_{m,\ell} - \mathrm{u}^p_{m,\ell-1})\right\|_X \lesssim 2^{-j}j\|f\|^p_{L^{p+\delta_{1,p}}(D)}$$

where $X = H^1_0(D)$ if $p = 1$ and $X = W^{1,1}_0(D)$ if $p \geq 2$.

Proof. We apply the multilevel splitting (4.2) of the error for deterministic quadrature rules. The first term on the right-hand side of (4.2) is estimated in (4.9) for the mean or in (4.10) for the higher order moments. In addition, the sum on the right-hand side of (4.2) is bounded with (4.3) and with the choice $\varepsilon_\ell = 2^{-\ell}$. This implies the assertion in the same way as in the proof of (4.5). □

(4.13) **Remark.** The logarithmic factor j in (4.6) and (4.12) can be removed, if we choose the accuracy of the quadrature rule on level $j - \ell$ in such a way that it has an accuracy $\ell^{-1-\eta}2^{-(j-\ell)}$ for some $\eta > 0$, see [BSZ11]. In that case, the ℓ-th summand in (4.2) and (4.7) is scaled by $\ell^{-1-\eta}$ and, thus, we get a summable series times $2^{-\ell}$ in the error estimate.

It remains to establish the estimate (4.3) for the deterministic quadrature rules under consideration. As we have seen, the error analysis in case of the quasi-Monte Carlo quadrature, the anisotropic Gaussian quadrature and the sparse anisotropic Gaussian quadrature is based on the derivatives of the integrand. Hence, we shall show that the derivatives of the term $(\mathfrak{u}_{m,\ell}^p - \mathfrak{u}_{m,\ell-1}^p)(\mathbf{y})$ exhibit a behaviour similar to the derivatives of $u_m^p(\mathbf{y})$ and provide in addition the factor $2^{-\ell}$. This will then lead to the estimate (4.3) for the quasi-Monte Carlo quadrature, the Gaussian quadrature and the sparse anisotropic Gaussian quadrature, respectively.

(4.14) **Lemma.** For the error $\delta_{m,\ell}(\mathbf{y}) := (\mathfrak{u}_{m,\ell} - u_m)(\mathbf{y})$ of the Galerkin projection of the solution u_m of (III.3.5), there holds the estimate

$$\|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}} \delta_{m,\ell}(\mathbf{y})\|_{H_0^1(D)} \lesssim 2^{-\ell} |\boldsymbol{\alpha}|! \tilde{\kappa}_m(\mathbf{y})^2 \left(\frac{3\tilde{\boldsymbol{\gamma}}}{\log 2}\right)^{\boldsymbol{\alpha}} \|f\|_{L^2(D)} \quad \text{for all } |\boldsymbol{\alpha}| \geq 0. \tag{4.15}$$

Therefore, we have for the *detail projections* $\theta_{m,\ell}(\mathbf{y}) := (\mathfrak{u}_{m,\ell} - \mathfrak{u}_{m,\ell-1})(\mathbf{y})$ the estimate

$$\|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}} \theta_{m,\ell}(\mathbf{y})\|_{H_0^1(D)} \lesssim 3 \cdot 2^{-\ell} |\boldsymbol{\alpha}|! \tilde{\kappa}_m(\mathbf{y})^2 \left(\frac{3\tilde{\boldsymbol{\gamma}}}{\log 2}\right)^{\boldsymbol{\alpha}} \|f\|_{L^2(D)} \quad \text{for all } |\boldsymbol{\alpha}| \geq 0. \tag{4.16}$$

Proof. Since the Galerkin projection satisfies $\big(a_m(\mathbf{y})\nabla_{\mathbf{x}}\delta_{m,\ell}(\mathbf{y}), \nabla_{\mathbf{x}} v\big)_{L^2(D)} = 0$ for all $v \in \mathcal{S}_\ell^1(D)$, it follows by differentiation that

$$-\int_D a_m(\mathbf{y}) \nabla_{\mathbf{x}} \partial_{\mathbf{y}}^{\boldsymbol{\alpha}} \delta_{m,\ell}(\mathbf{y}) \nabla_{\mathbf{x}} v \,\mathrm{d}\mathbf{x} = \sum_{\mathbf{0} \neq \boldsymbol{\beta} \leq \boldsymbol{\alpha}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} \int_D \partial_{\mathbf{y}}^{\boldsymbol{\beta}} a_m(\mathbf{y}) \nabla_{\mathbf{x}} \partial_{\mathbf{y}}^{\boldsymbol{\alpha}-\boldsymbol{\beta}} \delta_{m,\ell}(\mathbf{y}) \nabla_{\mathbf{x}} v \,\mathrm{d}\mathbf{x}$$

for all $v \in \mathcal{S}_\ell^1(D)$. For an arbitrary function $v \in \mathcal{S}_\ell^1(D)$, we therefore obtain that

$$\begin{aligned}
\Big\|\sqrt{a_m(\mathbf{y})} \nabla \partial_{\mathbf{y}}^{\boldsymbol{\alpha}} \delta_{m,\ell}(\mathbf{y})\Big\|_{L^2(D)}^2 &= \int_D a_m(\mathbf{y}) \big|\nabla_{\mathbf{x}} \partial_{\mathbf{y}}^{\boldsymbol{\alpha}} \delta_{m,\ell}(\mathbf{y})\big|^2 \,\mathrm{d}\mathbf{x} \\
&= \int_D a_m(\mathbf{y}) \nabla_{\mathbf{x}} \partial_{\mathbf{y}}^{\boldsymbol{\alpha}} \delta_{m,\ell}(\mathbf{y}) \big[\nabla_{\mathbf{x}} \partial_{\mathbf{y}}^{\boldsymbol{\alpha}} \delta_{m,\ell}(\mathbf{y}) - \nabla_{\mathbf{x}} v\big] \,\mathrm{d}\mathbf{x} \\
&\quad + \sum_{\mathbf{0} \neq \boldsymbol{\beta} \leq \boldsymbol{\alpha}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} \int_D \partial_{\mathbf{y}}^{\boldsymbol{\beta}} a_m(\mathbf{y}) \nabla_{\mathbf{x}} \partial_{\mathbf{y}}^{\boldsymbol{\alpha}-\boldsymbol{\beta}} \delta_{m,\ell}(\mathbf{y}) \big[\nabla_{\mathbf{x}} \partial_{\mathbf{y}}^{\boldsymbol{\alpha}} \delta_{m,\ell}(\mathbf{y}) - \nabla_{\mathbf{x}} v\big] \,\mathrm{d}\mathbf{x} \\
&\quad - \sum_{\mathbf{0} \neq \boldsymbol{\beta} \leq \boldsymbol{\alpha}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} \int_D \partial_{\mathbf{y}}^{\boldsymbol{\beta}} a_m(\mathbf{y}) \nabla_{\mathbf{x}} \partial_{\mathbf{y}}^{\boldsymbol{\alpha}-\boldsymbol{\beta}} \delta_{m,\ell}(\mathbf{y}) \nabla_{\mathbf{x}} \partial_{\mathbf{y}}^{\boldsymbol{\alpha}} \delta_{m,\ell}(\mathbf{y}) \,\mathrm{d}\mathbf{x}.
\end{aligned}$$

With $\|\partial_{\mathbf{y}}^{\boldsymbol{\beta}} a_m(\mathbf{y}) / a_m(\mathbf{y})\|_{L^\infty(D)} \leq \boldsymbol{\gamma}^{\boldsymbol{\beta}}$, we can further estimate that

$$\begin{aligned}
&\Big\|\sqrt{a_m(\mathbf{y})} \nabla \partial_{\mathbf{y}}^{\boldsymbol{\alpha}} \delta_{m,\ell}(\mathbf{y})\Big\|_{L^2(D)}^2 \\
&\quad \leq \int_D a_m(\mathbf{y}) \nabla_{\mathbf{x}} \partial_{\mathbf{y}}^{\boldsymbol{\alpha}} \delta_{m,\ell}(\mathbf{y}) \big[\nabla_{\mathbf{x}} \partial_{\mathbf{y}}^{\boldsymbol{\alpha}} \delta_{m,\ell}(\mathbf{y}) - \nabla_{\mathbf{x}} v\big] \,\mathrm{d}\mathbf{x} \\
&\qquad + \sum_{\mathbf{0} \neq \boldsymbol{\beta} \leq \boldsymbol{\alpha}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} \boldsymbol{\gamma}^{\boldsymbol{\beta}} \int_D a_m(\mathbf{y}) \big|\nabla_{\mathbf{x}} \partial_{\mathbf{y}}^{\boldsymbol{\alpha}-\boldsymbol{\beta}} \delta_{m,\ell}(\mathbf{y}) \big[\nabla_{\mathbf{x}} \partial_{\mathbf{y}}^{\boldsymbol{\alpha}} \delta_{m,\ell}(\mathbf{y}) - \nabla_{\mathbf{x}} v\big]\big| \,\mathrm{d}\mathbf{x} \\
&\qquad + \sum_{\mathbf{0} \neq \boldsymbol{\beta} \leq \boldsymbol{\alpha}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} \boldsymbol{\gamma}^{\boldsymbol{\beta}} \int_D a_m(\mathbf{y}) \big|\nabla_{\mathbf{x}} \partial_{\mathbf{y}}^{\boldsymbol{\alpha}-\boldsymbol{\beta}} \delta_{m,\ell}(\mathbf{y}) \nabla_{\mathbf{x}} \partial_{\mathbf{y}}^{\boldsymbol{\alpha}} \delta_{m,\ell}(\mathbf{y})\big| \,\mathrm{d}\mathbf{x}.
\end{aligned}$$

The Cauchy-Schwarz inequality yields that

$$
\begin{aligned}
&\left\|\sqrt{a_m(\mathbf{y})}\nabla\partial_{\mathbf{y}}^{\boldsymbol{\alpha}}\delta_{m,\ell}(\mathbf{y})\right\|_{L^2(D)}^2\\
&\quad\le \left\|\sqrt{a_m(\mathbf{y})}\nabla\partial_{\mathbf{y}}^{\boldsymbol{\alpha}}\delta_{m,\ell}(\mathbf{y})\right\|_{L^2(D)}\left\|\sqrt{a_m(\mathbf{y})}\nabla\big(\partial_{\mathbf{y}}^{\boldsymbol{\alpha}}\delta_{m,\ell}(\mathbf{y})-v\big)\right\|_{L^2(D)}\\
&\qquad+\sum_{\mathbf{0}\neq\boldsymbol{\beta}\le\boldsymbol{\alpha}}\binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}}\boldsymbol{\gamma}^{\boldsymbol{\beta}}\left\|\sqrt{a_m(\mathbf{y})}\nabla\partial_{\mathbf{y}}^{\boldsymbol{\alpha}-\boldsymbol{\beta}}\delta_{m,\ell}(\mathbf{y})\right\|_{L^2(D)}\left\|\sqrt{a_m(\mathbf{y})}\nabla\big(\partial_{\mathbf{y}}^{\boldsymbol{\alpha}}\delta_{m,\ell}(\mathbf{y})-v\big)\right\|_{L^2(D)}\\
&\qquad+\sum_{\mathbf{0}\neq\boldsymbol{\beta}\le\boldsymbol{\alpha}}\binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}}\boldsymbol{\gamma}^{\boldsymbol{\beta}}\left\|\sqrt{a_m(\mathbf{y})}\nabla\partial_{\mathbf{y}}^{\boldsymbol{\alpha}-\boldsymbol{\beta}}\delta_{m,\ell}(\mathbf{y})\right\|_{L^2(D)}\left\|\sqrt{a_m(\mathbf{y})}\nabla\partial_{\mathbf{y}}^{\boldsymbol{\alpha}}\delta_{m,\ell}(\mathbf{y})\right\|_{L^2(D)}.
\end{aligned}
\tag{4.17}
$$

Since $v\in\mathcal{S}_\ell^1(D)$ can be chosen arbitrarily, the bound holds also for the infimum

$$
\begin{aligned}
&\left\|\sqrt{a_m(\mathbf{y})}\nabla\partial_{\mathbf{y}}^{\boldsymbol{\alpha}}\delta_{m,\ell}(\mathbf{y})\right\|_{L^2(D)}^2\\
&\quad\le \inf_{v\in\mathcal{S}_\ell^1(D)}\Bigg(\left\|\sqrt{a_m(\mathbf{y})}\nabla\partial_{\mathbf{y}}^{\boldsymbol{\alpha}}\delta_{m,\ell}(\mathbf{y})\right\|_{L^2(D)}\left\|\sqrt{a_m(\mathbf{y})}\nabla\big(\partial_{\mathbf{y}}^{\boldsymbol{\alpha}}\delta_{m,\ell}(\mathbf{y})-v\big)\right\|_{L^2(D)}\\
&\quad+\sum_{\mathbf{0}\neq\boldsymbol{\beta}\le\boldsymbol{\alpha}}\binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}}\boldsymbol{\gamma}^{\boldsymbol{\beta}}\left\|\sqrt{a_m(\mathbf{y})}\nabla\partial_{\mathbf{y}}^{\boldsymbol{\alpha}-\boldsymbol{\beta}}\delta_{m,\ell}(\mathbf{y})\right\|_{L^2(D)}\left\|\sqrt{a_m(\mathbf{y})}\nabla\big(\partial_{\mathbf{y}}^{\boldsymbol{\alpha}}\delta_{m,\ell}(\mathbf{y})-v\big)\right\|_{L^2(D)}\Bigg)\\
&\quad+\sum_{\mathbf{0}\neq\boldsymbol{\beta}\le\boldsymbol{\alpha}}\binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}}\boldsymbol{\gamma}^{\boldsymbol{\beta}}\left\|\sqrt{a_m(\mathbf{y})}\nabla\partial_{\mathbf{y}}^{\boldsymbol{\alpha}-\boldsymbol{\beta}}\delta_{m,\ell}(\mathbf{y})\right\|_{L^2(D)}\left\|\sqrt{a_m(\mathbf{y})}\nabla\partial_{\mathbf{y}}^{\boldsymbol{\alpha}}\delta_{m,\ell}(\mathbf{y})\right\|_{L^2(D)}.
\end{aligned}
$$

The expression containing the infimum is now estimated in two different ways. On the one hand, the approximation property of the finite element space $\mathcal{S}_\ell^1(D)$ implies that

$$
\inf_{v\in\mathcal{S}_\ell^1(D)}\left\|\sqrt{a_m(\mathbf{y})}\nabla\big(\partial_{\mathbf{y}}^{\boldsymbol{\alpha}}\delta_{m,\ell}(\mathbf{y})-v\big)\right\|_{L^2(D)}\le c2^{-\ell}\sqrt{\overline{a}_m(\mathbf{y})}\|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}}u_m(\mathbf{y})\|_{H^2(D)}.
$$

On the other hand, due to $0\in\mathcal{S}_\ell^1(D)$, we find that

$$
\inf_{v\in\mathcal{S}_\ell^1(D)}\left\|\sqrt{a_m(\mathbf{y})}\nabla\big(\partial_{\mathbf{y}}^{\boldsymbol{\alpha}}\delta_{m,\ell}(\mathbf{y})-v\big)\right\|_{L^2(D)}\le\left\|\sqrt{a_m(\mathbf{y})}\nabla\partial_{\mathbf{y}}^{\boldsymbol{\alpha}}\delta_{m,\ell}(\mathbf{y})\right\|_{L^2(D)}.
$$

Dividing (4.17) by $\left\|\sqrt{a_m(\mathbf{y})}\nabla\partial_{\mathbf{y}}^{\boldsymbol{\alpha}}\delta_{m,\ell}(\mathbf{y})\right\|_{L^2(D)}$ leads in combination with both estimates to

$$
\begin{aligned}
\left\|\sqrt{a_m(\mathbf{y})}\nabla\partial_{\mathbf{y}}^{\boldsymbol{\alpha}}\delta_{m,\ell}(\mathbf{y})\right\|_{L^2(D)}&\le c2^{-\ell}\sqrt{\overline{a}_m(\mathbf{y})}\|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}}u_m(\mathbf{y})\|_{H^2(D)}\\
&\quad+2\sum_{\mathbf{0}\neq\boldsymbol{\beta}\le\boldsymbol{\alpha}}\binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}}\boldsymbol{\gamma}^{\boldsymbol{\beta}}\left\|\sqrt{a_m(\mathbf{y})}\nabla\partial_{\mathbf{y}}^{\boldsymbol{\alpha}-\boldsymbol{\beta}}\delta_{m,\ell}(\mathbf{y})\right\|_{L^2(D)}.
\end{aligned}
$$

In view of (3.4), we obtain that

$$
\begin{aligned}
\left\|\sqrt{a_m(\mathbf{y})}\nabla\partial_{\mathbf{y}}^{\boldsymbol{\alpha}}\delta_{m,\ell}(\mathbf{y})\right\|_{L^2(D)}&\le \tilde{c}2^{-\ell}\sqrt{\overline{a}_m(\mathbf{y})}\tilde{\kappa}_m(\mathbf{y})|\boldsymbol{\alpha}|!\left(\frac{2\tilde{\boldsymbol{\gamma}}}{\log 2}\right)^{\boldsymbol{\alpha}}\|f\|_{L^2(D)}\\
&\quad+2\sum_{\mathbf{0}\neq\boldsymbol{\beta}\le\boldsymbol{\alpha}}\binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}}\boldsymbol{\gamma}^{\boldsymbol{\beta}}\left\|\sqrt{a_m(\mathbf{y})}\nabla\partial_{\mathbf{y}}^{\boldsymbol{\alpha}-\boldsymbol{\beta}}\delta_{m,\ell}(\mathbf{y})\right\|_{L^2(D)}.
\end{aligned}
\tag{4.18}
$$

We conclude now by induction that

$$\left\|\sqrt{a_m(\mathbf{y})}\nabla\partial_{\mathbf{y}}^{\boldsymbol{\alpha}}\delta_{m,\ell}(\mathbf{y})\right\|_{L^2(D)} \le 2\tilde{c}2^{-\ell}\frac{\overline{a}_m(\mathbf{y})}{\sqrt{\underline{a}_m(\mathbf{y})}}\tilde{\kappa}_m(\mathbf{y})(3\tilde{\boldsymbol{\gamma}})^{\boldsymbol{\alpha}}B_{|\boldsymbol{\alpha}|}\|f\|_{L^2(D)}. \tag{4.19}$$

Herein, B_k denotes the k-th ordered Bell-number as defined in (III.5.8). For $|\boldsymbol{\alpha}| = 0$, the inequality (4.19) is simply obtained by estimate (3.4) and (1.2). Let now the induction hypothesis holds for all $\boldsymbol{\beta} < \boldsymbol{\alpha}$. Inserting this hypothesis into (4.18) yields that

$$\begin{aligned}\left\|\sqrt{a_m(\mathbf{y})}\nabla\partial_{\mathbf{y}}^{\boldsymbol{\alpha}}\delta_{m,\ell}(\mathbf{y})\right\|_{L^2(D)} &\le \tilde{c}2^{-\ell}\frac{\overline{a}_m(\mathbf{y})}{\sqrt{\underline{a}_m(\mathbf{y})}}\tilde{\kappa}_m(\mathbf{y})\|f\|_{L^2(D)}\\ &\quad\cdot\left(|\boldsymbol{\alpha}|!\left(\frac{2\tilde{\boldsymbol{\gamma}}}{\log 2}\right)^{\boldsymbol{\alpha}} + 4\tilde{\boldsymbol{\gamma}}^{\boldsymbol{\alpha}}\sum_{\mathbf{0}\ne\boldsymbol{\beta}\le\boldsymbol{\alpha}}\binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}}3^{|\boldsymbol{\alpha}-\boldsymbol{\beta}|}B_{|\boldsymbol{\alpha}-\boldsymbol{\beta}|}\right).\end{aligned}$$

With

$$\begin{aligned}\sum_{\mathbf{0}\ne\boldsymbol{\beta}\le\boldsymbol{\alpha}}\binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}}3^{|\boldsymbol{\alpha}-\boldsymbol{\beta}|}B_{|\boldsymbol{\alpha}-\boldsymbol{\beta}|} &\le 3^{|\boldsymbol{\alpha}|-1}\sum_{k=0}^{|\boldsymbol{\alpha}|-1}B_k\sum_{\substack{\boldsymbol{\beta}<\boldsymbol{\alpha}\\|\boldsymbol{\beta}|=k}}\binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} = 3^{|\boldsymbol{\alpha}|-1}\sum_{k=0}^{|\boldsymbol{\alpha}|-1}B_k\binom{|\boldsymbol{\alpha}|}{k}\\ &= 3^{|\boldsymbol{\alpha}|-1}B_{|\boldsymbol{\alpha}|}\end{aligned}$$

and the estimate $B_k \le k!(\log 2)^{-k}$, cf. [BNTT12], we get that

$$\begin{aligned}&\left\|\sqrt{a_m(\mathbf{y})}\nabla\partial_{\mathbf{y}}^{\boldsymbol{\alpha}}\delta_{m,\ell}(\mathbf{y})\right\|_{L^2(D)}\\ &\qquad\le \tilde{c}2^{-\ell}\frac{\overline{a}_m(\mathbf{y})}{\sqrt{\underline{a}_m(\mathbf{y})}}\tilde{\kappa}_m(\mathbf{y})\|f\|_{L^2(D)}|\boldsymbol{\alpha}|!\left(\frac{\tilde{\boldsymbol{\gamma}}}{\log 2}\right)^{\boldsymbol{\alpha}}\left(2^{|\boldsymbol{\alpha}|} + 4\cdot 3^{|\boldsymbol{\alpha}|-1}\right).\end{aligned}$$

Additionally, since $2^{|\boldsymbol{\alpha}|}+4\cdot 3^{|\boldsymbol{\alpha}|-1} \le 2\cdot 3^{|\boldsymbol{\alpha}|}$ for $|\boldsymbol{\alpha}| \ge 1$, $\overline{a}_m(\mathbf{y}) \le \tilde{\overline{a}}_m(\mathbf{y})$ and $\underline{a}_m(\mathbf{y}) \ge \tilde{\underline{a}}_m(\mathbf{y})$, we conclude that

$$\left\|\sqrt{a_m(\mathbf{y})}\nabla\partial_{\mathbf{y}}^{\boldsymbol{\alpha}}\delta_{m,\ell}(\mathbf{y})\right\|_{L^2(D)} \le 2\tilde{c}|\boldsymbol{\alpha}|!2^{-\ell}\frac{\big(\tilde{\overline{a}}_m(\mathbf{y})\big)^2}{\big(\tilde{\underline{a}}_m(\mathbf{y})\big)^{3/2}}\left(\frac{3\tilde{\boldsymbol{\gamma}}}{\log 2}\right)^{\boldsymbol{\alpha}}\|f\|_{L^2(D)},$$

which implies (4.19) and finally, by dividing by $\sqrt{\underline{a}_m(\mathbf{y})}$, the assertion. □

To get an error estimate for the multilevel computation of the p-th moment, we also need the $L^p(D)$-extension of the previous lemma. The proof is performed in a similar way to that of Lemma (4.14). Hence, we present only the changes in detail and refer to the combinatorial estimates in Lemma (4.14).

(4.20) **Lemma.** Let $f \in L^p(D)$ for $p > 2$ and u_m be the solution of (III.3.5). Then, the error $\delta_{m,\ell}(\mathbf{y})$ of the Galerkin projection, fulfills the estimate

$$\|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}}\delta_{m,\ell}(\mathbf{y})\|_{W_0^{1,p}(D)} \lesssim 2^{-\ell}|\boldsymbol{\alpha}|!\tilde{\kappa}_m(\mathbf{y})^2\left(\frac{3C(p,D)c_1\tilde{\boldsymbol{\gamma}}}{\log 2}\right)^{\boldsymbol{\alpha}}\|f\|_{L^p(D)} \quad \text{for all } |\boldsymbol{\alpha}| \ge 0. \tag{4.21}$$

Furthermore, the *detail projections* $\theta_{m,\ell}(\mathbf{y})$ satisfy the estimate

$$\|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}}\theta_{m,\ell}(\mathbf{y})\|_{W_0^{1,p}(D)} \lesssim 3\cdot 2^{-\ell}|\boldsymbol{\alpha}|!\tilde{\kappa}_m(\mathbf{y})^2\left(\frac{3C(p,D)c_1\tilde{\boldsymbol{\gamma}}}{\log 2}\right)^{\boldsymbol{\alpha}}\|f\|_{L^p(D)} \quad \text{for all } |\boldsymbol{\alpha}| \ge 0. \tag{4.22}$$

Proof. Since the Galerkin projection satisfies $\big(a_m(\mathbf{y})\nabla_{\mathbf{x}}\delta_{m,\ell}(\mathbf{y}),\nabla_{\mathbf{x}}v\big)_{L^2(D)} = 0$ for all $v\in\mathcal{S}_\ell^1(D)$, it follows by differentiation as in the proof of (4.14) that

$$-\mathcal{B}_{m,\mathbf{y}}(\partial_{\mathbf{y}}^{\boldsymbol{\alpha}}\delta_{m,\ell}(\mathbf{y}),v) = \sum_{\mathbf{0}\neq\boldsymbol{\beta}\leq\boldsymbol{\alpha}}\binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}}\int_D \partial_{\mathbf{y}}^{\boldsymbol{\beta}}a_m(\mathbf{y})\nabla_{\mathbf{x}}\partial_{\mathbf{y}}^{\boldsymbol{\alpha}-\boldsymbol{\beta}}\delta_{m,\ell}(\mathbf{y})\nabla_{\mathbf{x}}v\,\mathrm{d}\mathbf{x}$$

for all $v\in\mathcal{S}_\ell^1(D)$. Moreover, there exists for all $q\geq 1$ a Clément interpolation operator $P_\ell: W_0^{1,q}(D)\to S^{\ell,d}(D)$, cf. [BS08], such that

$$\|P_\ell v\|_{W_0^{1,q}(D)} \leq c_1\|v\|_{W_0^{1,q}(D)},\quad \|v-P_\ell v\|_{L^q(D)}\leq c_2 2^{-\ell}\|v\|_{W_0^{1,q}(D)} \tag{4.23}$$

with constants $c_1,c_2\geq 1$. We show now by induction that

$$\|a_m(\mathbf{y})\nabla\partial_{\mathbf{y}}^{\boldsymbol{\alpha}}\delta_{m,\ell}(\mathbf{y})\|_{[L^p(D)]^d}\leq 8\cdot 2^{-\ell}C(p,D)^2c_2B_{|\boldsymbol{\alpha}|}\big(3C(p,D)c_1\tilde{\boldsymbol{\gamma}}\big)^{\boldsymbol{\alpha}}\tilde{\kappa}_m(\mathbf{y})\|f\|_{L^p(D)} \tag{4.24}$$

with the ordered Bell numbers B_k as in the proof of Lemma (4.14). The assertion is obvious for $|\boldsymbol{\alpha}|=0$, see the proof of inequality (III.5.14). For $|\boldsymbol{\alpha}|>0$, we calculate in the same way as in the proof of (III.5.15) that

$$\begin{aligned}
&\|a_m(\mathbf{y})\nabla\partial_{\mathbf{y}}^{\boldsymbol{\alpha}}\delta_{m,\ell}(\mathbf{y})\|_{[L^p(D)]^d}\\
&\quad\leq C(p,D)\sup_{0\neq v\in W_0^{1,p'}(D)}\frac{\mathcal{B}_{m,\mathbf{y}}(\partial_{\mathbf{y}}^{\boldsymbol{\alpha}}\delta_{m,\ell}(\mathbf{y}),v)}{\|v\|_{W_0^{1,p'}(D)}}\\
&\quad= C(p,D)\sup_{0\neq v\in W_0^{1,p'}(D)}\frac{1}{\|v\|_{W_0^{1,p'}(D)}}\bigg[\mathcal{B}_{m,\mathbf{y}}(\partial_{\mathbf{y}}^{\boldsymbol{\alpha}}\delta_{m,\ell}(\mathbf{y}),v-P_\ell v)\\
&\qquad\qquad+\sum_{\mathbf{0}\neq\boldsymbol{\beta}\leq\boldsymbol{\alpha}}\binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}}\int_D\partial_{\mathbf{y}}^{\boldsymbol{\beta}}a_m(\mathbf{x},\mathbf{y})\nabla_{\mathbf{x}}\partial_{\mathbf{y}}^{\boldsymbol{\alpha}-\boldsymbol{\beta}}\delta_{m,\ell}(\mathbf{x},\mathbf{y})\nabla_{\mathbf{x}}P_\ell v(\mathbf{x})\,\mathrm{d}\mathbf{x}\bigg].
\end{aligned}\tag{4.25}$$

Herein, we denote the dual exponent to p by $p'\geq 1$. The first term on the right-hand side of (4.25) can be estimated as follows. From Green's formula and since $v-P_\ell v$ vanishes at the boundary of D, we obtain that

$$\mathcal{B}_{m,\mathbf{y}}(\partial_{\mathbf{y}}^{\boldsymbol{\alpha}}\delta_{m,\ell}(\mathbf{y}),v-P_\ell v) = -\int_D \operatorname{div}_{\mathbf{x}}\big(a_m(\mathbf{x},\mathbf{y})\nabla_{\mathbf{x}}\partial_{\mathbf{y}}^{\boldsymbol{\alpha}}\delta_{m,\ell}(\mathbf{x},\mathbf{y})\big)(v-P_\ell v)(\mathbf{x})\,\mathrm{d}\mathbf{x}.$$

As in the proof of Lemma (3.10), we use the notation $v_{\boldsymbol{\alpha}}=\operatorname{div}(a_m\nabla\partial_{\mathbf{y}}^{\boldsymbol{\alpha}}u_m)$ and recall that

$$\|v_{\boldsymbol{\alpha}}(\mathbf{y})\|_{L^p(D)}\leq|\boldsymbol{\alpha}|!\bigg(\frac{2C(p,D)\tilde{\boldsymbol{\gamma}}}{\log 2}\bigg)^{\boldsymbol{\alpha}}\Big(\|f\|_{L^p(D)}+g(\mathbf{y})\|a_m(\mathbf{y})\nabla u_m(\mathbf{y})\|_{[L^p(D)]^d}\Big).$$

With the estimate

$$\|a_m(\mathbf{y})\nabla u_m(\mathbf{y})\|_{[L^p(D)]^d}\leq C(p,D)\|f\|_{L^p(D)},$$

which can be obtained from the proof of (III.5.14), and with $g(\mathbf{y})\leq\tilde{\kappa}_m(\mathbf{y})$, we arrive at

$$\|v_{\boldsymbol{\alpha}}(\mathbf{y})\|_{L^p(D)}\leq 2|\boldsymbol{\alpha}|!C(p,D)\bigg(\frac{2C(p,D)\tilde{\boldsymbol{\gamma}}}{\log 2}\bigg)^{\boldsymbol{\alpha}}\tilde{\kappa}_m(\mathbf{y})\|f\|_{L^p(D)}.$$

Since u_m and $\mathfrak{u}_m$ have the same regularity with respect to the parametric variable $\mathbf{y}$, we obtain that

$$\| \operatorname{div}\left(a_m(\mathbf{y})\nabla\partial_{\mathbf{y}}^{\boldsymbol{\alpha}}\delta_{\ell,m}(\mathbf{y})\right)\|_{L^p(D)} \leq 4|\boldsymbol{\alpha}|! C(p,D)\left(\frac{2C(p,D)\tilde{\boldsymbol{\gamma}}}{\log 2}\right)^{\boldsymbol{\alpha}}\tilde{\kappa}_m(\mathbf{y})\|f\|_{L^p(D)}.$$

Furthermore, we establish from

$$\partial_{\mathbf{y}}^{\boldsymbol{\beta}} a_m(\mathbf{x},\mathbf{y}) = \prod_{k=1}^{m}\left(\sqrt{\lambda_k}\varphi_k(\mathbf{x})\right)^{\beta_k} a_m(\mathbf{x},\mathbf{y})$$

and the application of the Hölder inequality that the second term on the right-hand side of (4.25) is bounded by

$$\sum_{\mathbf{0}\neq\boldsymbol{\beta}\leq\boldsymbol{\alpha}}\binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}}\int_D \partial_{\mathbf{y}}^{\boldsymbol{\beta}} a_m(\mathbf{x},\mathbf{y})\nabla_{\mathbf{x}}\partial_{\mathbf{y}}^{\boldsymbol{\alpha}-\boldsymbol{\beta}}\delta_{m,\ell}(\mathbf{x},\mathbf{y})\nabla_{\mathbf{x}} P_\ell v(\mathbf{x})\,\mathrm{d}\mathbf{x}$$
$$\leq \sum_{\mathbf{0}\neq\boldsymbol{\beta}\leq\boldsymbol{\alpha}}\binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}}\gamma^{\boldsymbol{\beta}}\|a_m(\mathbf{y})\nabla\partial_{\mathbf{y}}^{\boldsymbol{\alpha}-\boldsymbol{\beta}}\delta_{m,\ell}(\mathbf{y})\|_{[L^p(D)]^d}\|P_\ell v\|_{W_0^{1,p'}(D)}.$$

Using the properties of P_ℓ in (4.23), we deduce that

$$\|a_m(\mathbf{y})\nabla\partial_{\mathbf{y}}^{\boldsymbol{\alpha}}\delta_{m,\ell}(\mathbf{y})\|_{[L^p(D)]^d}$$
$$\leq C(p,D)\Bigg[2^{-\ell}|\boldsymbol{\alpha}|!4C(p,D)\left(\frac{2C(p,D)\tilde{\gamma}}{\log 2}\right)^{\boldsymbol{\alpha}}\tilde{\kappa}_m(\mathbf{y})\|f\|_{L^p(D)}$$
$$+\sum_{\mathbf{0}\neq\boldsymbol{\beta}\leq\boldsymbol{\alpha}}\binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}}c_1\boldsymbol{\gamma}^{\boldsymbol{\beta}}\|a_m(\mathbf{y})\nabla_{\mathbf{x}}\partial_{\mathbf{y}}^{\boldsymbol{\alpha}-\boldsymbol{\beta}}\delta_{m,\ell}(\mathbf{x},\mathbf{y})\|_{[L^p(D)]^d}\Bigg].$$

Now, inserting the induction hypothesis (4.24) leads to

$$\|a_m(\mathbf{y})\nabla\partial_{\mathbf{y}}^{\boldsymbol{\alpha}}\delta_{m,\ell}(\mathbf{y})\|_{[L^p(D)]^d}$$
$$\leq 2^{-\ell}4C(p,D)^2c_2\|f\|_{L^p(D)}\tilde{\kappa}_m(\mathbf{y})\Bigg[|\boldsymbol{\alpha}|!\left(\frac{2C(p,D)\tilde{\gamma}}{\log 2}\right)^{\boldsymbol{\alpha}}$$
$$+2\big(C(p,D)c_1\tilde{\gamma}\big)^{\boldsymbol{\alpha}}\sum_{\mathbf{0}\neq\boldsymbol{\beta}\leq\boldsymbol{\alpha}}\binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}}3^{|\boldsymbol{\alpha}-\boldsymbol{\beta}|}B_{|\boldsymbol{\alpha}-\boldsymbol{\beta}|}\Bigg]$$

The inequality (4.24) follows now with similar calculations as in the proof of Lemma (4.14). Then, employing (4.24) establishes together with the estimate $B_n \leq n!\log(2)^{-n}$ and with

$$\underline{a}_m(\mathbf{y})\|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}}\delta_{m,\ell}(\mathbf{y})\|_{W_0^{1,p}(D)} \leq \|a_m(\mathbf{y})\nabla\partial_{\mathbf{y}}^{\boldsymbol{\alpha}}\delta_{m,\ell}(\mathbf{y})\|_{[L^p(D)]^d}$$

the inequality (4.21). This finally completes the proof. □

To prove the convergence of the multilevel quadrature in case of the higher order moments, we need a regularity result for the derivatives of $\mathfrak{u}_{m,\ell}^p - \mathfrak{u}_{m,\ell-1}^p$. With the previous two lemmata at hand, we can obtain such a result similar to the proof of Lemma (III.5.10) for the case $p = 2$ and the proof of the estimate (III.5.16) for the case $p > 2$.

(4.26) **Lemma.** The derivatives of the difference $\mathfrak{u}_{m,\ell}^2 - \mathfrak{u}_{m,\ell-1}^2$ satisfy the estimate

$$\|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}}(\mathfrak{u}_{m,\ell}^2 - \mathfrak{u}_{m,\ell-1}^2)(\mathbf{y})\|_{W_0^{1,1}(D)} \lesssim 2^{-\ell}|\boldsymbol{\alpha}|!\left(\frac{6\tilde{\gamma}}{\log 2}\right)^{\boldsymbol{\alpha}} \tilde{\kappa}_m(\mathbf{y})^3\|f\|_{L^2(D)}^2. \tag{4.27}$$

Moreover, let $f \in L^p(D)$ for $p > 2$. Then, for the p-th powers of two successive Galerkin projections $\mathfrak{u}_{m,\ell}^p$ and $\mathfrak{u}_{m,\ell-1}^p$, there holds the estimate

$$\|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}}(\mathfrak{u}_{m,\ell}^p - \mathfrak{u}_{m,\ell-1}^p)(\mathbf{y})\|_{W_0^{1,1}(D)} \lesssim 2^{-\ell}|\boldsymbol{\alpha}|!\left(\frac{3pC(p,D)c_1\tilde{\gamma}}{\log 2}\right)^{\boldsymbol{\alpha}} \tilde{\kappa}_m(\mathbf{y})^{3/2p+1/2}\|f\|_{L^p(D)}^p. \tag{4.28}$$

Proof. We start with the case $p = 2$. It holds that

$$\begin{aligned}
&\|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}}(\mathfrak{u}_{m,\ell}^2 - \mathfrak{u}_{m,\ell-1}^2)(\mathbf{y})\|_{W_0^{1,1}(D)} \\
&\quad \le \sum_{\boldsymbol{\beta}\le\boldsymbol{\alpha}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} \left\|\partial_{\mathbf{y}}^{\boldsymbol{\beta}}\theta_{m,\ell}(\mathbf{y})\partial_{\mathbf{y}}^{\boldsymbol{\alpha}-\boldsymbol{\beta}}(\mathfrak{u}_{m,\ell} + \mathfrak{u}_{m,\ell-1})(\mathbf{y})\right\|_{W_0^{1,1}(D)} \\
&\quad \lesssim \sum_{\boldsymbol{\beta}\le\boldsymbol{\alpha}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} \|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}}\theta_{m,\ell}(\mathbf{y})\|_{H_0^1(D)} \left\|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}-\boldsymbol{\beta}}(\mathfrak{u}_{m,\ell} + \mathfrak{u}_{m,\ell-1})(\mathbf{y})\right\|_{H_0^1(D)}.
\end{aligned}$$

Using the estimate (4.16), the fact that the Galerkin projection $\mathfrak{u}_{m,\ell}(\mathbf{y})$ has the same regularity with respect to the parametric variable as the solution itself, see Remark (1.6), and Lemma (III.5.3), we obtain that

$$\begin{aligned}
&\|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}}(\mathfrak{u}_{m,\ell}^2 - \mathfrak{u}_{m,\ell-1}^2)(\mathbf{y})\|_{W_0^{1,1}(D)} \\
&\quad \lesssim \sum_{\boldsymbol{\beta}\le\boldsymbol{\alpha}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} 2^{-\ell}|\boldsymbol{\beta}|!\tilde{\kappa}_m(\mathbf{y})^2\left(\frac{3\tilde{\gamma}}{\log 2}\right)^{\boldsymbol{\beta}}(\boldsymbol{\alpha}-\boldsymbol{\beta})!\left(\frac{2\gamma}{\log 2}\right)^{\boldsymbol{\alpha}-\boldsymbol{\beta}}\kappa_m(\mathbf{y})\|f\|_{L^2(D)}^2 \\
&\quad \lesssim 2^{-\ell}|\boldsymbol{\alpha}|!\left(\frac{3\tilde{\gamma}}{\log 2}\right)^{\boldsymbol{\alpha}}\tilde{\kappa}_m(\mathbf{y})^3\|f\|_{L^2(D)}^2 \sum_{\boldsymbol{\beta}\le\boldsymbol{\alpha}}\binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} \\
&\quad = 2^{-\ell}|\boldsymbol{\alpha}|!\left(\frac{3\tilde{\gamma}}{\log 2}\right)^{\boldsymbol{\alpha}}\tilde{\kappa}_m(\mathbf{y})^3\|f\|_{L^2(D)}^2 2^{|\boldsymbol{\alpha}|},
\end{aligned}$$

which yields the assertion for $p = 2$.

For the case $p > 2$, we proceed in a similar way as in the proof of (III.5.16), but apply here the multivariate extension of the generalized Leibniz formula for products of more than two factors. In order to apply this formula, we first rewrite as in Lemma (1.1)

$$\mathfrak{u}_{m,\ell}^p - \mathfrak{u}_{m,\ell-1}^p = (\mathfrak{u}_{m,\ell} - \mathfrak{u}_{m,\ell-1})(\mathbf{y})\sum_{i=0}^{p-1}\mathfrak{u}_{m,\ell}^i(\mathbf{y})\mathfrak{u}_{m-1,\ell}^{p-1-i}(\mathbf{y}).$$

Then, it holds by the linearity of the differential operator $\partial_{\mathbf{y}}^{\boldsymbol{\alpha}}$ and by the triangle inequality that

$$\begin{aligned}
\|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}}(\mathfrak{u}_{m,\ell}^p - \mathfrak{u}_{m,\ell-1}^p)(\mathbf{y})\|_{W_0^{1,1}(D)} &= \left\|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}}\left(\theta_{m,\ell}(\mathbf{y})\sum_{i=0}^{p-1}\mathfrak{u}_{m,\ell}^i(\mathbf{y})\mathfrak{u}_{m-1,\ell}^{p-1-i}(\mathbf{y})\right)\right\|_{W_0^{1,1}(D)} \\
&\le \sum_{i=0}^{p-1}\left\|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}}\left(\theta_{m,\ell}(\mathbf{y})\mathfrak{u}_{m,\ell}^i(\mathbf{y})\mathfrak{u}_{m-1,\ell}^{p-1-i}(\mathbf{y})\right)\right\|_{W_0^{1,1}(D)}.
\end{aligned}$$

The application of the generalized Leibniz formula yields then that

$$\partial_{\mathbf{y}}^{\boldsymbol{\alpha}}\Big(\theta_{m,\ell}\mathfrak{u}_{m,\ell}^{i}\mathfrak{u}_{m-1,\ell}^{p-1-i}\Big) = \sum_{\substack{\boldsymbol{\beta}_1,\boldsymbol{\beta}_2,\ldots,\boldsymbol{\beta}_p\leq\boldsymbol{\alpha}\\ \boldsymbol{\beta}_1+\boldsymbol{\beta}_2+\cdots+\boldsymbol{\beta}_p=\boldsymbol{\alpha}}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}_1,\ldots,\boldsymbol{\beta}_p} \cdot\Big[\partial_{\mathbf{y}}^{\boldsymbol{\beta}_1}\theta_{m,\ell}\partial_{\mathbf{y}}^{\boldsymbol{\beta}_2}\mathfrak{u}_{m,\ell}\cdots\partial_{\mathbf{y}}^{\boldsymbol{\beta}_{i+1}}\mathfrak{u}_{m,\ell}\partial_{\mathbf{y}}^{\boldsymbol{\beta}_{i+2}}\mathfrak{u}_{m,\ell-1}\cdots\partial_{\mathbf{y}}^{\boldsymbol{\beta}_p}\mathfrak{u}_{m,\ell-1}\Big]. \tag{4.29}$$

For the sake of clarity, we omit in (4.29) the dependencies of the functions $\theta_{m,\ell}$, $\mathfrak{u}_{m,\ell}$ and $\mathfrak{u}_{m,\ell-1}$ on $\mathbf{y}$. The Galerkin projections $\mathfrak{u}_{m,\ell}$ and $\mathfrak{u}_{m,\ell-1}$ fulfill the regularity estimate (III.5.15). This implies, in combination with Lemma (II.2.10) and estimate (4.22), that

$$\begin{aligned}
&\Big\|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}}\Big(\theta_{m,\ell}(\mathbf{y})\mathfrak{u}_{m,\ell}^{i}(\mathbf{y})\mathfrak{u}_{m-1,\ell}^{p-1-i}(\mathbf{y})\Big)\Big\|_{W_0^{1,1}(D)}\\
&\lesssim \sum_{\substack{\boldsymbol{\beta}_1,\boldsymbol{\beta}_2,\ldots,\boldsymbol{\beta}_p\leq\boldsymbol{\alpha}\\ \boldsymbol{\beta}_1+\boldsymbol{\beta}_2+\cdots+\boldsymbol{\beta}_p=\boldsymbol{\alpha}}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}_1,\ldots,\boldsymbol{\beta}_p}\Big[\Big\|\partial_{\mathbf{y}}^{\boldsymbol{\beta}_1}\theta_{m,\ell}(\mathbf{y})\Big\|_{W_0^{1,p}(D)}\Big\|\partial_{\mathbf{y}}^{\boldsymbol{\beta}_2}\mathfrak{u}_{m,\ell}(\mathbf{y})\Big\|_{W_0^{1,p}(D)}\cdots\\
&\qquad\cdot\Big\|\partial_{\mathbf{y}}^{\boldsymbol{\beta}_{i+1}}\mathfrak{u}_{m,\ell}(\mathbf{y})\Big\|_{W_0^{1,p}(D)}\Big\|\partial_{\mathbf{y}}^{\boldsymbol{\beta}_{i+2}}\mathfrak{u}_{m,\ell-1}(\mathbf{y})\Big\|_{W_0^{1,p}(D)}\cdots\Big\|\partial_{\mathbf{y}}^{\boldsymbol{\beta}_p}\mathfrak{u}_{m,\ell-1}(\mathbf{y})\Big\|_{W_0^{1,p}(D)}\Big]\\
&\lesssim \|f\|_{L^p(D)}^p 2^{-\ell}\tilde{\kappa}_m(\mathbf{y})^2\bigg(\frac{\kappa_m(\mathbf{y})}{\underline{a}_m(\mathbf{y})}\bigg)^{p-1}\sum_{\substack{\boldsymbol{\beta}_1,\boldsymbol{\beta}_2,\ldots,\boldsymbol{\beta}_p\leq\boldsymbol{\alpha}\\ \boldsymbol{\beta}_1+\boldsymbol{\beta}_2+\cdots+\boldsymbol{\beta}_p=\boldsymbol{\alpha}}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}_1,\ldots,\boldsymbol{\beta}_p}\cdot\\
&\qquad\cdot\bigg[|\boldsymbol{\beta}_1|!\cdots|\boldsymbol{\beta}_p|!\bigg(\frac{3C(p,D)c_1\tilde{\boldsymbol{\gamma}}}{\log 2}\bigg)^{\boldsymbol{\beta}_1}\bigg(\frac{C(p,D)\boldsymbol{\gamma}}{\log 2}\bigg)^{\sum_{k=2}^p\boldsymbol{\beta}_k}\bigg]\\
&\leq \|f\|_{L^p(D)}^p 2^{-\ell}\tilde{\kappa}_m(\mathbf{y})^{3/2p+1/2}\bigg(\frac{3C(p,D)c_1\tilde{\boldsymbol{\gamma}}}{\log 2}\bigg)^{\boldsymbol{\alpha}}\cdot\\
&\qquad\cdot\sum_{\substack{\boldsymbol{\beta}_1,\boldsymbol{\beta}_2,\ldots,\boldsymbol{\beta}_p\leq\boldsymbol{\alpha}\\ \boldsymbol{\beta}_1+\boldsymbol{\beta}_2+\cdots+\boldsymbol{\beta}_p=\boldsymbol{\alpha}}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}_1,\ldots,\boldsymbol{\beta}_p}|\boldsymbol{\beta}_1|!\cdots|\boldsymbol{\beta}_p|!.
\end{aligned}$$

The assertion follows then with $|\boldsymbol{\beta}_1|!\cdots|\boldsymbol{\beta}_p|!\leq|\boldsymbol{\alpha}|!$ and the multivariate multinomial theorem. From this theorem, we have the identity

$$\sum_{\substack{\boldsymbol{\beta}_1,\boldsymbol{\beta}_2,\ldots,\boldsymbol{\beta}_p\leq\boldsymbol{\alpha}\\ \boldsymbol{\beta}_1+\boldsymbol{\beta}_2+\cdots+\boldsymbol{\beta}_p=\boldsymbol{\alpha}}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}_1,\ldots,\boldsymbol{\beta}_p} = p^{|\boldsymbol{\alpha}|}.$$

This finally concludes the proof. □

Employing Lemma (4.14), we immediately derive that

$$\|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}}\theta_{m,\ell}\|_{C^0_{\tilde{\boldsymbol{\sigma}}_s}(\mathbb{R}^m;H_0^1(D))}\lesssim 2^{-\ell}|\boldsymbol{\alpha}|!\bigg(\frac{3\tilde{\boldsymbol{\gamma}}}{\log 2}\bigg)^{\boldsymbol{\alpha}}\|f\|_{L^2(D)} \tag{4.30}$$

holds for all weights $\tilde{\boldsymbol{\sigma}}_s$ with $s\geq 4$. Furthermore, from estimate (4.27), we deduce that

$$\|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}}(\mathfrak{u}_{m,\ell}^2-\mathfrak{u}_{m,\ell-1}^2)(\mathbf{y})\|_{C^0_{\tilde{\boldsymbol{\sigma}}_s}(\mathbb{R}^m;W_0^{1,1}(D))}\lesssim 2^{-\ell}|\boldsymbol{\alpha}|!\bigg(\frac{6\tilde{\boldsymbol{\gamma}}}{\log 2}\bigg)^{\boldsymbol{\alpha}}\|f\|_{L^2(D)}^2 \tag{4.31}$$

holds for all $s \geq 6$. Finally, for the powers of higher order, estimate (4.28) implies that

$$\|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}}(\mathrm{u}_{m,\ell}^{p} - \mathrm{u}_{m,\ell-1}^{p})(\mathbf{y})\|_{C^0_{\tilde{\boldsymbol{\sigma}}_s}(\mathbb{R}^m;W_0^{1,1}(D))} \lesssim 2^{-\ell}|\boldsymbol{\alpha}|!\left(\frac{3C(p,D)c_1 p\tilde{\boldsymbol{\gamma}}}{\log 2}\right)^{\boldsymbol{\alpha}}\|f\|_{L^p(D)}^p \tag{4.32}$$

for all $s \geq 3p+1$. Hence, we can provide the regularity results which are necessary to perform the error analysis from Chapters IV and V for the quasi-Monte Carlo quadrature, the anisotropic Gaussian quadrature and the sparse anisotropic Gaussian quadrature with the modified sequences $\{3p\tilde{\gamma}_k/\log 2\}_k$ instead of $\{p\gamma_k/\log 2\}_k$ for $p = 1, 2$ and $\{3pC(p,D)c_1\tilde{\gamma}_k/\log 2\}_k$ instead of $\{C(p,D)p\gamma_k/\log 2\}_k$ for $p > 2$.

The convergence analysis of the mentioned quadrature rules depends on the asymptotic decay behaviour of the above sequences which is for fixed p either determined by the decay of $\{\gamma_k\}_k$ in the single level case or by the decay of $\{\tilde{\gamma}_k\}_k$ in the multilevel case. Thus, the only change in comparison with the single level case is that we have to deal with the asymptotic decay behaviour of $\{\tilde{\gamma}_k\}_k$ instead of $\{\gamma_k\}_k$. Therefore, the estimate (4.3) follows if we choose the number of quadrature points of $\mathbf{Q}_{j-\ell}$ with respect to the modified sequences above and in such a way that the accuracy $\varepsilon_{j-\ell}$ is achieved. This yields, as can be seen from Theorem (4.11), that the error of the multilevel quadrature has essentially the same rate as the error of the according single level quadrature. The gain of the multilevel quadrature can then simply be expressed by the comparison of the computational cost given in Section 6.

5. Computation of the stiffness matrix

In this section, we discuss the computation of the stiffness matrix on the different levels of spatial refinement for the lognormal diffusion coefficient. To this end, we employ piecewise linear finite elements. For the complexity analysis in the next section, we will assume that each deterministic PDE on level ℓ can be solved with cost $\mathcal{O}(2^{d\ell})$, see Remark (6.1). Since the stiffness matrix on level ℓ with piecewise linear ansatz functions has already $\mathcal{O}(2^{d\ell})$ non-zero entries, each entry has to be assembled in constant time.

Therefore, we recall that $\mathcal{S}_\ell^1(D)$ is the finite element space of piecewise linear functions on the triangulation $\mathcal{T}_\ell$ with meshwidth $h_\ell \eqsim 2^{-\ell}$. Additionally, we denote by $w_i, w_j \in \mathcal{S}_\ell^1(D)$ two elements of the nodal basis of the finite element space on level ℓ and define the common support of w_i and w_j by $\operatorname{supp}(w_i, w_j) := \operatorname{supp} w_i \cap \operatorname{supp} w_j$. Since the gradients of w_i and w_j are constant on each element $T \in \mathcal{T}_\ell$, the entry $a_{i,j}(\boldsymbol{\xi})$ of the stiffness matrix $\mathbf{A}_\ell(\boldsymbol{\xi})$ is for each quadrature point $\boldsymbol{\xi} \in \mathbb{R}^m$ determined by

$$\begin{aligned} a_{i,j}(\boldsymbol{\xi}) &= \int_D a_m(\mathbf{x},\boldsymbol{\xi})\nabla w_i(\mathbf{x})\nabla w_j(\mathbf{x})\,\mathrm{d}\mathbf{x} \\ &= \sum_{T\in\operatorname{supp}(w_i,w_j)} \int_T a_m(\mathbf{x},\boldsymbol{\xi})\nabla w_i(\mathbf{x})\nabla w_j(\mathbf{x})\,\mathrm{d}\mathbf{x} \\ &= \sum_{T\in\operatorname{supp}(w_i,w_j)} \nabla w_i(\mathbf{x})\nabla w_j(\mathbf{x}) \int_T a_m(\mathbf{x},\boldsymbol{\xi})\,\mathrm{d}\mathbf{x}. \end{aligned}$$

The index set which contains the elements $T \in \mathcal{T}_\ell$ in the common support of w_i and w_j has, due to the compact support of w_i and w_j, on each level ℓ only $\mathcal{O}(1)$ elements.

Nevertheless, the calculation of the integral over T of the diffusion coefficient a_m needs to be taken into account. Since the diffusion coefficient depends on $\boldsymbol{\xi} \in \mathbb{R}^m$, each evaluation of the diffusion coefficient needs $\mathcal{O}(m)$ operations, at least when the eigenfunctions of the Karhunen-Loève expansion are globally supported. We neglect this factor m here and assume that an evaluation of the diffusion coefficient in $(\mathbf{x}, \boldsymbol{\xi}) \in D \times \mathbb{R}^m$ can be performed in constant time, see Remark (6.1). Then, it remains to show that for each element $T \in \mathcal{T}_\ell$ only a constant number of function evaluations of a_m is required to approximate the integral over T in order to sustain the overall accuracy of the multilevel quadrature method. More precisely, we show that an evaluation of the diffusion coefficient at the midpoint of each element is sufficient for this purpose. Hence, we will at first have a look which additional error has to be estimated when we replace the diffusion coefficient a_m by a piecewise constant approximation a_{m,h_ℓ}, where the value on each element $T \in \mathcal{T}_\ell$ is given by the evaluation at the midpoint of T.

We consider the multilevel quadrature of the solution u_m to (III.3.5) which is given by

$$\mathbb{E}_{u_m} \approx \sum_{\ell=0}^{j} \mathbf{Q}_{j-\ell}(\mathfrak{u}_{m,\ell} - \mathfrak{u}_{m,\ell-1})(\mathbf{x}, \mathbf{y}). \tag{5.1}$$

Applying now for the assembly of the stiffness matrix on level ℓ a midpoint rule yields that we calculate a disturbed solution $\mathfrak{u}_{m,\ell,h_\ell}$ instead of $\mathfrak{u}_{m,\ell}$. Hence, our actual multilevel quadrature method reads

$$\mathbb{E}_{u_m} \approx \sum_{\ell=0}^{j} \mathbf{Q}_{j-\ell}(\mathfrak{u}_{m,\ell,h_\ell} - \mathfrak{u}_{m,\ell-1,h_{\ell-1}})(\mathbf{x}, \mathbf{y}). \tag{5.2}$$

This leads to the multilevel error splitting

$$\begin{aligned}
&\left\| \mathbf{I}(u_m) - \sum_{\ell=0}^{j} \mathbf{Q}_{j-\ell}(\mathfrak{u}_{m,\ell,h_\ell} - \mathfrak{u}_{m,\ell-1,h_{\ell-1}})(\mathbf{x}, \mathbf{y}) \right\|_{H_0^1(D)} \\
&\quad \leq \|\mathbf{I}(u_m) - \mathbf{I}(\mathfrak{u}_{m,j,h_j})\|_{H_0^1(D)} + \sum_{\ell=0}^{j} \underbrace{\|(\mathbf{I} - \mathbf{Q}_{j-\ell})(\mathfrak{u}_{m,\ell,h_\ell} - \mathfrak{u}_{m,\ell-1,h_{\ell-1}})(\mathbf{y})\|_{H_0^1(D)}}_{\text{IV}}.
\end{aligned}$$

Next, we use the triangle inequality and the linearity of $\mathbf{I}$ and $\mathbf{Q}_{j-\ell}$ in order to estimate IV by

$$\begin{aligned}
&\|(\mathbf{I} - \mathbf{Q}_{j-\ell})(\mathfrak{u}_{m,\ell,h_\ell} - \mathfrak{u}_{m,\ell-1,h_{\ell-1}})(\mathbf{y})\|_{H_0^1(D)} \\
&\quad \leq \|(\mathbf{I} - \mathbf{Q}_{j-\ell})(\mathfrak{u}_{m,\ell} - \mathfrak{u}_{m,\ell-1})(\mathbf{y})\|_{H_0^1(D)} + \|(\mathbf{I} - \mathbf{Q}_{j-\ell})(\mathfrak{u}_{m,\ell} - \mathfrak{u}_{m,\ell,h_\ell})(\mathbf{y})\|_{H_0^1(D)} \\
&\qquad + \|(\mathbf{I} - \mathbf{Q}_{j-\ell})(\mathfrak{u}_{m,\ell-1} - \mathfrak{u}_{m,\ell-1,h_{\ell-1}})(\mathbf{y})\|_{H_0^1(D)}.
\end{aligned} \tag{5.3}$$

The first term on the right-hand side has already been estimated in (4.3). Hence, it remains to bound the second and the third term on the right-hand side of (5.3). In particular, we have to show that both terms are, as the first one, of order 2^{-j}. Hence, also in these terms, the spatial and stochastic accuracy need to multiply, i.e. we need mixed regularity here. By the application of Strang's Lemma, it follows that

$$\|u_{m,\ell} - u_{m,\ell,h_\ell}\|_{L^2_{\boldsymbol{\rho}}(\mathbb{R}^m;H^1_0(D))} \lesssim h_\ell \|f\|_{L^2(D)}$$

which is proven in [CST13] and sufficient for the convergence of the multilevel Monte Carlo method. For the multilevel quasi-Monte Carlo quadrature or the multilevel Gaussian quadrature, we require a stronger result which also takes into account the derivatives of $u_{m,\ell} - u_{m,\ell,h_\ell}$ with respect to $\mathbf{y}$. To this end, we prove the following estimate for the derivatives of $a_m - a_{m,h_\ell}$.

(5.4) **Lemma.** Let the diffusion coefficient $a_m(\mathbf{y})$ belong to $W^{1,\infty}(D)$ for every $\mathbf{y} \in \boldsymbol{\Gamma}$. Then, we can bound the derivatives of the difference between the diffusion coefficient a_m and its piecewise constant approximation a_{m,h_ℓ} by

$$\|\partial^{\boldsymbol{\alpha}}_{\mathbf{y}}(a_m(\mathbf{y}) - a_{m,h_\ell}(\mathbf{y}))\|_{L^\infty(D)} \leq h_\ell (2\tilde{\boldsymbol{\gamma}})^{\boldsymbol{\alpha}} \|a_m(\mathbf{y})\|_{W^{1,\infty}(D)}. \tag{5.5}$$

Proof. Let us denote by $\boldsymbol{\Phi} = [\varphi_1(\mathbf{x}), \ldots, \varphi_m(\mathbf{x})]^\intercal$ the collection of eigenfunctions in the Karhunen-Loève expansion of $\log(a_m)$ and by $\boldsymbol{\Phi}_{h_\ell} = [\varphi_{1,h_\ell}(\mathbf{x}), \ldots, \varphi_{m,h_\ell}(\mathbf{x})]^\intercal$ the vector of the piecewise constant function such that φ_k and φ_{k,h_ℓ} coincide in the midpoints of each element of the triangulation $\mathcal{T}_\ell$.

We consider at first only a single element $T \in \mathcal{T}_\ell$. It holds for each $\mathbf{x} \in T$ that

$$\begin{aligned}
&\partial^{\boldsymbol{\alpha}}_{\mathbf{y}}(a_m(\mathbf{y}) - a_{m,h_\ell}(\mathbf{y}))\\
&\qquad = \left(\sqrt{\boldsymbol{\lambda}}\boldsymbol{\Phi}\right)^{\boldsymbol{\alpha}} a_m(\mathbf{y}) - \left(\sqrt{\boldsymbol{\lambda}}\boldsymbol{\Phi}_{h_\ell}\right)^{\boldsymbol{\alpha}} a_{m,h_\ell}(\mathbf{y})\\
&\qquad = \left(\sqrt{\boldsymbol{\lambda}}\boldsymbol{\Phi}\right)^{\boldsymbol{\alpha}} (a_m(\mathbf{y}) - a_{m,h_\ell}(\mathbf{y})) + \left(\left(\sqrt{\boldsymbol{\lambda}}\boldsymbol{\Phi}\right)^{\boldsymbol{\alpha}} - \left(\sqrt{\boldsymbol{\lambda}}\boldsymbol{\Phi}_{h_\ell}\right)^{\boldsymbol{\alpha}}\right) a_{m,h_\ell}(\mathbf{y})\\
&\qquad = \left(\sqrt{\boldsymbol{\lambda}}\boldsymbol{\Phi}\right)^{\boldsymbol{\alpha}} (a_m(\mathbf{y}) - a_{m,h_\ell}(\mathbf{y})) + \sum_{k=1}^{m} \prod_{i=1}^{k-1} \left(\sqrt{\lambda_i}\varphi_{i,h_\ell}\right)^{\alpha_i} \left(\sqrt{\lambda_k}(\varphi_k - \varphi_{k,h_\ell})\right)\\
&\qquad\qquad \cdot \sum_{j=0}^{\alpha_k - 1} \left(\sqrt{\lambda_k}\varphi_{k,h_\ell}\right)^{j} \left(\sqrt{\boldsymbol{\lambda}}\boldsymbol{\Phi}\right)^{\boldsymbol{\alpha} - \sum_{i=1}^{k-1}\alpha_i \mathbf{e}_i - (j+1)\mathbf{e}_k} a_{m,h_\ell}(\mathbf{y}).
\end{aligned} \tag{5.6}$$

The last inequality follows from successively inserting summands of the form

$$\pm \prod_{i=1}^{k-1} \left(\sqrt{\lambda_i}\varphi_{i,h_\ell}\right)^{\alpha_i} \left(\sqrt{\lambda_k}\varphi_{k,h_\ell}\right)^{j} \left(\sqrt{\boldsymbol{\lambda}}\boldsymbol{\Phi}\right)^{\boldsymbol{\alpha} - \sum_{i=1}^{k-1}\alpha_i \mathbf{e}_i - j\mathbf{e}_k} a_{m,h_\ell}(\mathbf{y})$$

for $k = 1, \ldots, m$ and $j = 0, \ldots, \alpha_k - 1$.

From

$$\begin{aligned}
\|a_m(\mathbf{y}) - a_{m,h_\ell}(\mathbf{y})\|_{L^\infty(T)} &\leq h_\ell \|a_m(\mathbf{y})\|_{W^{1,\infty}(T)},\\
\|a_{m,h_\ell}(\mathbf{y})\|_{L^\infty(T)} &\leq \|a_m(\mathbf{y})\|_{L^\infty(T)},
\end{aligned}$$

as well as

$$\begin{aligned}
\|\varphi_k(\mathbf{y}) - \varphi_{k,h_\ell}(\mathbf{y})\|_{L^\infty(T)} &\leq h_\ell \|\varphi_k(\mathbf{y})\|_{W^{1,\infty}(T)},\\
\|\varphi_{k,h_\ell}(\mathbf{y})\|_{L^\infty(T)} &\leq \|\varphi_k(\mathbf{y})\|_{L^\infty(T)},
\end{aligned}$$

it follows by inserting these estimates into (5.6) that

$$
\begin{aligned}
&\|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}}(a_m(\mathbf{y}) - a_{m,h_\ell}(\mathbf{y}))\|_{L^\infty(T)} \\
&\qquad \leq h_\ell \tilde{\boldsymbol{\gamma}}^{\boldsymbol{\alpha}} \|a_m(\mathbf{y})\|_{W^{1,\infty}(T)} + h_\ell \tilde{\boldsymbol{\gamma}}^{\boldsymbol{\alpha}} \|a_m(\mathbf{y})\|_{L^\infty(T)} \sum_{k=1}^m \alpha_k \\
&\qquad \leq (|\boldsymbol{\alpha}|+1) h_\ell \tilde{\boldsymbol{\gamma}}^{\boldsymbol{\alpha}} \|a_m(\mathbf{y})\|_{W^{1,\infty}(T)} \leq h_\ell (2\tilde{\boldsymbol{\gamma}})^{\boldsymbol{\alpha}} \|a_m(\mathbf{y})\|_{W^{1,\infty}(T)}.
\end{aligned}
$$

This establishes the assertion since $T \in \mathcal{T}_h$ can be chosen arbitrarily. □

Note that, with the identity

$$
\nabla a_m(\mathbf{x}, \mathbf{y}) = a_m(\mathbf{x}, \mathbf{y}) \sum_{k=1}^m \sqrt{\lambda_k} \nabla \varphi_k(\mathbf{x}) y_k,
$$

the norm $\|a_m(\mathbf{y})\|_{W^{1,\infty}(D)}$ is easily calculated by

(5.7) $$\|a_m(\mathbf{y})\|_{W^{1,\infty}(D)} \leq \overline{a}(\mathbf{y}) + \overline{a}(\mathbf{y}) \sum_{k=1}^m \tilde{\gamma}_k |y_k|.$$

Hence, $\|a_m(\mathbf{y})\|_{W^{1,\infty}(D)}$ can be bounded when it is multiplied by $\tilde{\boldsymbol{\sigma}}_s(\mathbf{y})$ for all $s > 1$.

With Lemma (5.4) at hand, we are able to bound the derivatives of $\mathfrak{u}_{m,\ell} - \mathfrak{u}_{m,\ell,h_\ell}$ with respect to the random parameter $\mathbf{y}$.

(5.8) **Lemma.** The derivatives of $\mathfrak{u}_{m,\ell} - \mathfrak{u}_{m,\ell,h_\ell}$ with respect to $\mathbf{y}$ fulfill the estimate

(5.9) $$
\begin{aligned}
&\big\|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}}(\mathfrak{u}_{m,\ell} - \mathfrak{u}_{m,\ell,h_\ell})\big\|_{H_0^1(D)} \\
&\qquad \leq 2h_\ell |\boldsymbol{\alpha}|! \left(\frac{3\tilde{\boldsymbol{\gamma}}}{\log 2}\right)^{\boldsymbol{\alpha}} \tilde{\kappa}(\mathbf{y}) \underline{a}(\mathbf{y}) \|a_m(\mathbf{y})\|_{W^{1,\infty}(D)} \|u_m(\mathbf{y})\|_{H_0^1(D)}.
\end{aligned}
$$

Proof. We prove the following hypothesis

(5.10) $$
\begin{aligned}
&\Big\|\sqrt{a_m(\mathbf{y})} \nabla \partial_{\mathbf{y}}^{\boldsymbol{\alpha}}(\mathfrak{u}_{m,\ell} - \mathfrak{u}_{m,\ell,h_\ell})\Big\|_{L^2(D)} \\
&\qquad \leq 2h_\ell B_{|\boldsymbol{\alpha}|} (3\tilde{\boldsymbol{\gamma}})^{\boldsymbol{\alpha}} \tilde{\kappa}(\mathbf{y}) \|a_m(\mathbf{y})\|_{W^{1,\infty}(D)} \|u_m(\mathbf{y})\|_{H_0^1(D)}
\end{aligned}
$$

which implies the assertion with the help of the bound $B_k \leq k!(\log 2)^{-k}$ on the ordered Bell numbers.

The estimate (5.10) is proven in [CST13] for $|\boldsymbol{\alpha}| = 0$ with Strang's lemma. Let now the hypothesis be fulfilled for all $\boldsymbol{\beta}$ with $|\boldsymbol{\beta}| \leq n-1$. For the induction step, we consider that $\mathfrak{u}_{m,\ell}$ and $\mathfrak{u}_{m,\ell,h_\ell}$ solve the variational formulations

$$
\begin{aligned}
\int_D a_m(\mathbf{y}) \nabla \mathfrak{u}_{m,\ell}(\mathbf{y}) \nabla v \, \mathrm{d}\mathbf{x} &= \int_D f v \, \mathrm{d}\mathbf{x} \qquad \forall v \in \mathcal{S}_\ell^1(D) \\
\int_D a_{m,h_\ell}(\mathbf{y}) \nabla \mathfrak{u}_{m,\ell,h_\ell}(\mathbf{y}) \nabla v \, \mathrm{d}\mathbf{x} &= \int_D f v \, \mathrm{d}\mathbf{x} \qquad \forall v \in \mathcal{S}_\ell^1(D).
\end{aligned}
$$

Hence, it follows that

$$\int_D a_m(\mathbf{y})\nabla(\mathfrak{u}_{m,\ell}(\mathbf{y}) - \mathfrak{u}_{m,\ell,h_\ell}(\mathbf{y}))\nabla v \,\mathrm{d}\mathbf{x} = -\int_D (a_m(\mathbf{y}) - a_{m,h_\ell}(\mathbf{y}))\nabla \mathfrak{u}_{m,\ell,h_\ell}(\mathbf{y})\nabla v \,\mathrm{d}\mathbf{x}.$$

Computing the derivative $\partial_{\mathbf{y}}^{\boldsymbol{\alpha}}(\cdot)$ via the Leibniz formula leads to

$$\sum_{\boldsymbol{\beta}\leq\boldsymbol{\alpha}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} \int_D \partial_{\mathbf{y}}^{\boldsymbol{\beta}} a_m(\mathbf{y})\nabla\partial_{\mathbf{y}}^{\boldsymbol{\alpha}-\boldsymbol{\beta}}(\mathfrak{u}_{m,\ell}(\mathbf{y}) - \mathfrak{u}_{m,\ell,h_\ell}(\mathbf{y}))\nabla v \,\mathrm{d}\mathbf{x} = -\sum_{\boldsymbol{\beta}\leq\boldsymbol{\alpha}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} \int_D \partial_{\mathbf{y}}^{\boldsymbol{\beta}}(a_m(\mathbf{y}) - a_{m,h_\ell}(\mathbf{y}))\partial_{\mathbf{y}}^{\boldsymbol{\alpha}-\boldsymbol{\beta}}\nabla \mathfrak{u}_{m,\ell,h_\ell}(\mathbf{y})\nabla v \,\mathrm{d}\mathbf{x}.$$

This equation is rewritten such that all terms appear on the right-hand side except for the summand involving $\partial_{\mathbf{y}}^{\boldsymbol{\alpha}}(\mathfrak{u}_{m,\ell}(\mathbf{y}) - \mathfrak{u}_{m,\ell,h_\ell}(\mathbf{y}))$. This leads to

$$\begin{aligned}\int_D a_m(\mathbf{y})\nabla\partial_{\mathbf{y}}^{\boldsymbol{\alpha}}(\mathfrak{u}_{m,\ell}(\mathbf{y}) - \mathfrak{u}_{m,\ell,h_\ell}(\mathbf{y}))\nabla v \,\mathrm{d}\mathbf{x} \\ = -\sum_{\mathbf{0}\neq\boldsymbol{\beta}\leq\boldsymbol{\alpha}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} \int_D \partial_{\mathbf{y}}^{\boldsymbol{\beta}} a_m(\mathbf{y})\nabla\partial_{\mathbf{y}}^{\boldsymbol{\alpha}-\boldsymbol{\beta}}(\mathfrak{u}_{m,\ell}(\mathbf{y}) - \mathfrak{u}_{m,\ell,h_\ell}(\mathbf{y}))\nabla v \,\mathrm{d}\mathbf{x} \\ - \sum_{\boldsymbol{\beta}\leq\boldsymbol{\alpha}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} \int_D \partial_{\mathbf{y}}^{\boldsymbol{\beta}}(a_m(\mathbf{y}) - a_{m,h_\ell}(\mathbf{y}))\partial_{\mathbf{y}}^{\boldsymbol{\alpha}-\boldsymbol{\beta}}\nabla \mathfrak{u}_{m,\ell,h_\ell}(\mathbf{y})\nabla v \,\mathrm{d}\mathbf{x}.\end{aligned}$$

Next, we set $v = \partial_{\mathbf{y}}^{\boldsymbol{\alpha}}(\mathfrak{u}_{m,\ell}(\mathbf{y}) - \mathfrak{u}_{m,\ell,h_\ell}(\mathbf{y}))$ and exploit the bounds on the derivatives of $\partial_{\mathbf{y}}^{\boldsymbol{\beta}} a_m(\mathbf{y})$ and Lemma (5.5). As in the proof of Lemma (4.14), we employ additionally Hölder's inequality and divide the resulting inequality afterwards by the term $\|\sqrt{a_m(\mathbf{y})}\nabla\partial_{\mathbf{y}}^{\boldsymbol{\alpha}}(\mathfrak{u}_{m,\ell} - \mathfrak{u}_{m,\ell,h_\ell})\|_{L^2(D)}$ to conclude that

$$\begin{aligned}\Big\|\sqrt{a_m(\mathbf{y})}\nabla\partial_{\mathbf{y}}^{\boldsymbol{\alpha}}(\mathfrak{u}_{m,\ell} - \mathfrak{u}_{m,\ell,h_\ell})\Big\|_{L^2(D)} \\ \leq \sum_{\mathbf{0}\neq\boldsymbol{\beta}\leq\boldsymbol{\alpha}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} \boldsymbol{\gamma}^{\boldsymbol{\alpha}} \Big\|\sqrt{a_m(\mathbf{y})}\nabla\partial_{\mathbf{y}}^{\boldsymbol{\alpha}-\boldsymbol{\beta}}(\mathfrak{u}_{m,\ell} - \mathfrak{u}_{m,\ell,h_\ell})\Big\|_{L^2(D)} \\ + \sum_{\boldsymbol{\beta}\leq\boldsymbol{\alpha}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} h_\ell (2\tilde{\boldsymbol{\gamma}})^{\boldsymbol{\beta}} \|a_m(\mathbf{y})\|_{W^{1,\infty}(D)} \underline{a}^{-1}(\mathbf{y}) \Big\|\sqrt{a_m(\mathbf{y})}\nabla\partial_{\mathbf{y}}^{\boldsymbol{\alpha}-\boldsymbol{\beta}} \mathfrak{u}_{m,\ell,h_\ell}\Big\|_{L^2(D)}.\end{aligned}$$

The derivatives of the approximate solution $\mathfrak{u}_{m,\ell,h_\ell}$ on level ℓ with respect to the parameter $\mathbf{y}$ fulfill estimate (III.5.7), which bounds the derivatives of u_m. This is due to the fact that the proof of (III.5.7) can be performed in a similar way when u_m is replace by $\mathfrak{u}_{m,\ell,h_\ell}$. The only change is that the $L^\infty(D)$-norm of the eigenfunctions φ_k has to be replaced by the $L^\infty(D)$-norm of their piecewise constant approximations φ_{k,h_ℓ}. Since $\|\varphi_{k,h_\ell}\|_{L^\infty(D)} \leq \|\varphi_k\|_{L^\infty(D)}$, the bound (III.5.7) remains valid for $\mathfrak{u}_{m,\ell,h_\ell}$. Inserting additionally the induction hypothesis leads to

$$\begin{aligned}
&\left\|\sqrt{a_m(\mathbf{y})}\nabla\partial_{\mathbf{y}}^{\boldsymbol{\alpha}}(\mathfrak{u}_{m,\ell}-\mathfrak{u}_{m,\ell,h_\ell})\right\|_{L^2(D)}\\
&\qquad\leq h_\ell\tilde{\gamma}^{\boldsymbol{\alpha}}\tilde{\kappa}(\mathbf{y})\|a_m(\mathbf{y})\|_{W^{1,\infty}(D)}\|u_m(\mathbf{y})\|_{H_0^1(D)}\\
&\qquad\qquad\cdot\Bigg(\sum_{\mathbf{0}\neq\boldsymbol{\beta}\leq\boldsymbol{\alpha}}\binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}}2B_{|\boldsymbol{\alpha}-\boldsymbol{\beta}|}3^{\boldsymbol{\alpha}-\boldsymbol{\beta}}+\sum_{\boldsymbol{\beta}\leq\boldsymbol{\alpha}}\binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}}2^{\boldsymbol{\beta}}B_{|\boldsymbol{\alpha}-\boldsymbol{\beta}|}\Bigg).
\end{aligned}$$

The hypothesis (5.10) follows with the identity $\sum_{\mathbf{0}\neq\boldsymbol{\beta}\leq\boldsymbol{\alpha}}\binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}}B_{|\boldsymbol{\alpha}-\boldsymbol{\beta}|}=B_{|\boldsymbol{\alpha}|}$, cf. (III.5.9), by

$$\begin{aligned}
&\sum_{\mathbf{0}\neq\boldsymbol{\beta}\leq\boldsymbol{\alpha}}\binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}}2B_{|\boldsymbol{\alpha}-\boldsymbol{\beta}|}3^{\boldsymbol{\alpha}-\boldsymbol{\beta}}+\sum_{\boldsymbol{\beta}\leq\boldsymbol{\alpha}}\binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}}2^{\boldsymbol{\beta}}B_{|\boldsymbol{\alpha}-\boldsymbol{\beta}|}\\
&\qquad\leq 2\cdot 3^{|\boldsymbol{\alpha}|-1}B_{|\boldsymbol{\alpha}|}+B_{|\boldsymbol{\alpha}|}+\sum_{0\neq\boldsymbol{\beta}\leq\boldsymbol{\alpha}}\binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}}2^{\boldsymbol{\beta}}B_{|\boldsymbol{\alpha}-\boldsymbol{\beta}|}\\
&\qquad\leq(2^{|\boldsymbol{\alpha}|}+1+2\cdot 3^{|\boldsymbol{\alpha}|-1})B_{|\boldsymbol{\alpha}|}\leq 2\cdot 3^{|\boldsymbol{\alpha}|}B_{|\boldsymbol{\alpha}|}.
\end{aligned}$$

□

From the estimate (5.9) for the derivatives of $\mathfrak{u}_{m,\ell}-\mathfrak{u}_{m,\ell,h_\ell}$, we deduce the estimate

$$\|(\mathbf{I}-\mathbf{Q}_{j-\ell})(\mathfrak{u}_{m,\ell}-\mathfrak{u}_{m,\ell,h_\ell})(\mathbf{y})\|_{H_0^1(D)}\lesssim h_\ell 2^{-(j-\ell)}\|f\|_{L^2(D)}$$

if $\mathbf{Q}_{j-\ell}$ denotes a quasi-Monte Carlo or Gaussian quadrature method which provides a precision of $\varepsilon_{j-\ell}=2^{-(j-\ell)}$, cf. (4.3). Hence, the multilevel quadrature in (5.2) provides up to a constant the same approximation error as the multilevel quadrature (5.1) itself.

Using the same techniques as in Section 4, the result can, with the help of Lemma (5.5), also be transferred to the moment computation.

6. Complexity

We estimate the overall complexity of the different single and multilevel quadrature methods for the moment computation of the solution u_m of (III.3.5). For these estimates, we assume that each quadrature point can be generated in constant time and that for each quadrature point the computational cost is $\mathcal{O}(2^{d\ell})$ for solving the resulting elliptic partial differential equation on level ℓ. Notice that the space $\mathcal{S}_\ell^1(D)$ for $D\subset\mathbb{R}^d$ already has $\mathcal{O}(2^{d\ell})$ basis elements. Hence, we assume that we are able to solve a deterministic and elliptic partial differential equation on level ℓ in linear complexity.

(6.1) **Remark.** The construction cost of one single diffusion coefficient corresponds to the evaluation of the discretized Karhunen-Loève expansion which cost at least m times the number of spatial grid points. From Section 5, we know that a midpoint rule for the computation of the entrays of the stiffness matrix is sufficient. Hence, each entry of the stiffness matrix can be calculated in $\mathcal{O}(m)$ operations. This result implies that the stiffness matrix on level ℓ can be assembled in $\mathcal{O}(m2^{d\ell})$. The linear factor m in the construction cost of the diffusion coefficient occurs for the single level quadrature as well as for the multilevel quadrature. Since we are interested in the comparison between the complexity of single level quadrature and multilevel quadrature methods, this factor is neglected for our further considerations.

The Monte Carlo quadrature with N_ℓ sample points has a precision of order $\mathcal{O}(N_\ell^{-1/2})$. Thus, to reach an accuracy of $2^{-\ell}$, we need $N_\ell = \mathcal{O}(2^{2\ell})$ sample points. Therefore, the over-all cost of the single level Monte Carlo quadrature to achieve the RMSE $\varepsilon = 2^{-j}$ is bounded by

$$\mathrm{cost}_{\mathrm{MC}}(2^{-j}) \lesssim 2^{2j} 2^{dj} = 2^{(d+2)j}.$$

With the multilevel quadrature, we are able to reach an accuracy $j2^{-j}$ when employing quadrature methods with an accuracy of $2^{\ell-j}$ on the level ℓ of spatial refinement. Hence, the cost complexity of the MLMC can be estimated by

$$\mathrm{cost}_{\mathrm{MLMC}}(j2^{-j}) \lesssim \sum_{\ell=0}^{j} 2^{2(j-\ell)} 2^{d\ell} = 2^{2j} \sum_{\ell=0}^{j} 2^{(d-2)\ell} = \begin{cases} \mathcal{O}(2^{2j}), & \text{if } d = 1, \\ \mathcal{O}(j2^{2j}), & \text{if } d = 2, \\ \mathcal{O}(2^{dj}), & \text{if } d > 2. \end{cases}$$

For the quasi-Monte Carlo quadrature, we employ the Halton sequence and estimate the complexity under the assumption that $\tilde{\gamma}_k \lesssim k^{-4-\eta}$ for some $\eta > 0$. Then, we have by Theorem (IV.2.25) that the accuracy of the quasi-Monte Carlo quadrature is bounded by $\mathcal{O}(mN^{-1+\delta})$ for arbitrary $\delta > 0$. Hence, we have at first to estimate the dimensionality m of the domain of integration in dependence of the accuracy ε. This dimensionality corresponds to the number of terms which are needed in the Karhunen-Loève expansion of the random diffusion coefficient to guarantee a truncation error of order ε. In case of the computation of the mean, the variance or higher order moments, the truncation error can be estimated by Theorem (III.2.12). From this theorem, we know that the truncation error is bounded by

$$\|\mathcal{M}_u^p - \mathcal{M}_{u_m}^p\|_{C^{1,\alpha}(D)} \lesssim \sum_{k=m+1}^{\infty} \lambda_k \|\varphi_k\|_{C^{0,\alpha}(D)}^2.$$

Since this holds for all sufficiently small $\alpha > 0$, it follows that

$$\begin{aligned} \|\mathcal{M}_u^p - \mathcal{M}_{u_m}^p\|_{H_0^1(D)} \lesssim \|\mathcal{M}_u^p - \mathcal{M}_{u_m}^p\|_{C^{1,\alpha}(D)} &\lesssim \sum_{k=m+1}^{\infty} \lambda_k \|\varphi_k\|_{C^{0,\alpha}(D)}^2 \\ &\leq \sum_{k=m+1}^{\infty} \lambda_k \|\varphi_k\|_{W^{1,\infty}(D)}^2 \leq \sum_{k=m+1}^{\infty} \tilde{\gamma}_k^2. \end{aligned}$$

The aim is to calculate m in such a way that the truncation error is of order $\varepsilon = \mathcal{O}(2^{-j})$. To that end, the above inequalities lead in combination with the decay assumption $\tilde{\gamma}_k \lesssim k^{-4-\eta}$ to

$$\sum_{k=m+1}^{\infty} \tilde{\gamma}_k^2 \lesssim \sum_{k=m+1}^{\infty} k^{-8} \lesssim m^{-\frac{1}{7}} \overset{!}{\lesssim} 2^{-j} \quad \Longleftrightarrow \quad m = \mathcal{O}(2^{\frac{j}{7}}).$$

Moreover, we can estimate the number of quadrature points which are needed to achieve an accuracy of 2^{-j} by $N_j = \mathcal{O}\big(2^{j/(1-\delta)} m^{1/(1-\delta)}\big)$. Therefore, the complexity of the single level quasi-Monte Carlo quadrature is bounded by

$$\mathrm{cost}_{\mathrm{QMC,log}}(2^{-j}) \lesssim 2^{\frac{j}{1-\delta}} m^{\frac{1}{1-\delta}} 2^{jd} = 2^{j\left(d + \frac{8}{7(1-\delta)}\right)}.$$

The truncation length of the Karhunen-Loève expansion in the multilevel quadrature method is calculated according to the accuracy of the finest grid and, thus, given as in the single level case. There are possibilities to work with different lengths of the Karhunen-Loève expansion on different levels of spatial refinement at least for the MLMC, see e.g. [TSGU13]. However, it is not clear if these constructions simply transfer to other multilevel quadrature methods. Thus, we use the same length of the Karhunen-Loève expansion for the multilevel quadrature methods on each spatial level of refinement. Therefore, in accordance to the complexity estimate of the single level quasi-Monte Carlo quadrature, we determine the overall cost of the MLQMC by

$$\mathrm{cost}_{\mathrm{MLQMC,log}}(j2^{-j}) = 2^{j/7}\sum_{\ell=0}^{j} 2^{\frac{j-\ell}{1-\delta}} 2^{d\ell} = \begin{cases} \mathcal{O}\Big(2^{j\frac{8}{7(1-\delta)}}\Big), & \text{if } d = 1,\\ \mathcal{O}\Big(2^{j\left(d+\frac{1}{7(1-\delta)}\right)}\Big), & \text{if } d \geq 2. \end{cases}$$

Herein, we tacitly assume that $\delta < 1/2$.

For the anisotropic Gauss-Hermite quadrature, we obtain dimension-independent convergence rates only if the sequence $\{\tilde{\gamma}_k\}_k$ satisfies $\tilde{\gamma}_k \lesssim \exp(k^{-1-\eta})$ for arbitrary $\eta > 0$, cf. Lemma (V.2.21). In this case, the number of quadrature points is bounded by

$$\mathrm{cost}(\mathbf{Q}_{G,\mathbf{N}(\varepsilon)}, m) \leq C(\delta_1, \delta_2)\varepsilon^{-\delta_1-\delta_2}. \tag{6.2}$$

Herein, $\delta_1, \delta_2 > 0$ can be chosen arbitrarily, but the constant in (6.2) tends to infinity for $\delta_1 \to 0$ or $\delta_2 \to 0$. This leads to the cost complexity of the single level anisotropic Gauss-Hermite quadrature

$$\mathrm{cost}_{\mathrm{GQ}}(2^{-j}) \lesssim \mathcal{O}\Big(2^{j(d+\delta_1+\delta_2)}\Big).$$

Likewise, we apply the estimate (6.2) in order to calculate the complexity of the MLGQ. This yields that

$$\mathrm{cost}_{\mathrm{MLGQ}}(j2^{-j}) \lesssim \sum_{\ell=0}^{m} 2^{(j-\ell)(\delta_1+\delta_2)} 2^{\ell d} = \begin{cases} \mathcal{O}(2^{dj}), & \text{if } d > \delta_1+\delta_2,\\ \mathcal{O}(j2^{dj}), & \text{if } d = \delta_1+\delta_2,\\ \mathcal{O}\Big(2^{j(\delta_1+\delta_2)}\Big), & \text{if } d < \delta_1+\delta_2. \end{cases}$$

For the anisotropic sparse Gauss-Hermite quadrature, we obtain with the help of Conjecture (V.3.32) dimension-independent convergence when $\tilde{\gamma}_k \lesssim \exp(k^{-\eta})$ for arbitrary $\eta > 0$, cf. Theorem (V.3.37). In this case, the estimate (V.3.40) in Theorem (V.3.37) leads to the bound of the number of quadrature points

$$\mathrm{cost}\big(\mathcal{A}_{GH,\mathbf{w}}(q(\varepsilon), m)\big) = \mathcal{O}\Big(\varepsilon^{-\frac{2(\delta_1+\delta_2)}{1-\delta_1-\delta_2}}\Big)$$

with arbitrary $\delta_1, \delta_2 > 0$. The estimation of the cost complexity of the anisotropic sparse Gauss-Hermite quadrature follows then in the same way as for the anisotropic Gauss-Hermite quadrature. Thus, we have the complexity estimate

$$\mathrm{cost}_{\mathrm{sGQ}}(2^{-j}) \lesssim \mathcal{O}\Big(2^{j\left(d+\frac{2(\delta_1+\delta_2)}{1-\delta_1-\delta_2}\right)}\Big)$$

for the single level anisotropic sparse Gauss-Hermite quadrature and the complexity estimate

$$\mathrm{cost}_{\mathrm{MLsGQ}}\big(j2^{-j}\big) \lesssim \sum_{\ell=0}^{m} 2^{(j-\ell)\left(\frac{2(\delta_1+\delta_2)}{1-\delta_1-\delta_2}\right)} 2^{\ell n} = \begin{cases} \mathcal{O}(2^{dj}), & \text{if } d > \frac{2(\delta_1+\delta_2)}{1-\delta_1-\delta_2}, \\ \mathcal{O}(j2^{dj}), & \text{if } d = \frac{2(\delta_1+\delta_2)}{1-\delta_1-\delta_2}, \\ \mathcal{O}\Big(2^{j\left(\frac{2(\delta_1+\delta_2)}{1-\delta_1-\delta_2}\right)}\Big), & \text{if } d < \frac{2(\delta_1+\delta_2)}{1-\delta_1-\delta_2}, \end{cases}$$

for the MLsGQ.

Since the variables δ_1 and δ_2 can be chosen arbitrarily close to 0, only the first cases in the complexity estimates of the MLGQ and the MLsGQ are asymptotically relevant. Nevertheless, the choices of these variables have an immense impact on the constants. Therefore, it seems reasonable to maintain these variables in the estimates.

decay condition γ_k	quadrature method	spatial dimension $d = 1$	$d = 2$	$d = 3$
$k^{-1-\eta}$	MC	2^{3j}	2^{4j}	2^{5j}
$k^{-4-\eta}$	QMC	$2^{j\left(1+\frac{8}{7(1-\delta)}\right)}$	$2^{j\left(2+\frac{8}{7(1-\delta)}\right)}$	$2^{j\left(3+\frac{8}{7(1-\delta)}\right)}$
$\exp\big(-k^{-1-\eta}\big)$	GLQ	$2^{j(1+\delta_1+\delta_2)}$	$2^{j(2+\delta_1+\delta_2)}$	$2^{j(3+\delta_1+\delta_2)}$
$\exp\big(-k^{-\eta}\big)$	sGLQ	$2^{j\left(1+\frac{2(\delta_1+\delta_2)}{1-\delta_1-\delta_2}\right)}$	$2^{j\left(2+\frac{2(\delta_1+\delta_2)}{1-\delta_1-\delta_2}\right)}$	$2^{j\left(3+\frac{2(\delta_1+\delta_2)}{1-\delta_1-\delta_2}\right)}$

Table VI.1: Cost complexities of the different single level quadrature methods to get the accuracy 2^{-j}.

The cost complexities of the different methods are summarized for the single level quadratures in Table VI.1. Accordingly, the results for the complexities of the multilevel quadratures are visualized in Table VI.2. Here, we choose δ_1 and δ_2 for the anisotropic

decay condition $\tilde{\gamma}_k$	quadrature method	spatial dimension $d = 1$	$d = 2$	$d = 3$
$k^{-1-\eta}$	MLMC	2^{2j}	$j2^{2j}$	2^{3j}
$k^{-4-\eta}$	MLQMC	$2^{j\frac{8}{7(1-\delta)}}$	$2^{j\left(2+\frac{1}{7(1-\delta)}\right)}$	$2^{j\left(3+\frac{1}{7(1-\delta)}\right)}$
$\exp\big(-k^{-1-\eta}\big)$	MLGHQ	2^{j}	2^{2j}	2^{3j}
$\exp\big(-k^{-\eta}\big)$	MLsGHQ	2^{j}	2^{2j}	2^{3j}

Table VI.2: Cost complexities of the different multilevel quadrature methods to get the accuracy $j2^{-j}$ for the lognormal case.

Gaussian quadrature in such a way that $\delta_1 + \delta_2 < 1$. For the anisotropic sparse Gaussian quadrature, δ_1 and δ_2 are chosen with $2(\delta_1 + \delta_2)/(1 - \delta_1 - \delta_2) < 1$. This is possible since $\delta_1 > 0$ and $\delta_2 > 0$ are arbitrary parameters. But as mentioned before, the constants hidden in Table VI.2 are influenced by these choices. In both tables, the required assumptions on the decay of γ_k or $\tilde{\gamma}_k$, respectively, are written in the first columns.

(6.3) **Remark.** For the complexity estimates of MLGHQ and MLsGHQ, we require a strong decay behaviour of the sequence $\{\tilde{\gamma}_k\}_k$. If this decay behaviour is violated, the convergence rate deteriorates for $\varepsilon \to 0$ since the dimensionality m tends to ∞. Hence, the cost complexity will be dominated for $\varepsilon \to 0$ by the quadrature complexity. Nevertheless, within the realms of computability, algebraic convergence rates of GHQ and sGHQ are obtained even for problems where the decay requirements on $\{\tilde{\gamma}_k\}_k$ are not fulfilled, see Section V.4. We would like to mention that the benefit of multilevel methods could then be calculated dependent on the actually obtained algebraic rates. Additionally, we would like to emphasize that, regardless of the convergence rate of the single level quadrature, multilevel quadrature methods remarkably improve the performance of the corresponding single level quadrature method.

The findings in this Chapter can immediately be transferred to the moment computation of the solution u_m to (III.3.3). Of course, the better convergence result for the quasi-Monte Carlo quadrature in the uniformly elliptic case also leads to an improved complexity result for the MLQMC.

7. Numerical results

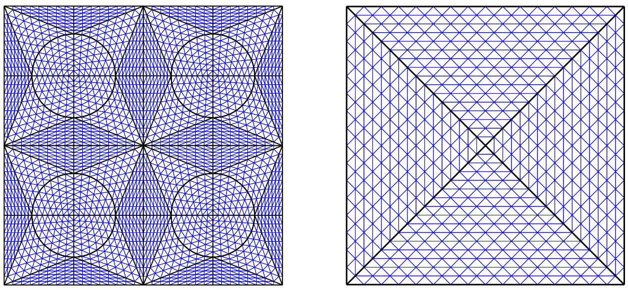


Figure VI.1: Computational domains with inscribed coarse grids.

In this section, we present numerical examples to validate the proposed analysis. Since most of the analysis in this chapter was only concerned with the lognormal case, we restrict ourselves also in the numerical examples to this setting. We study the convergence for the mean and for the second moment of the solution to (III.3.5). To that end, we consider two different settings. In the first case, the diffusion coefficient is represented exactly by a Karhunen-Loève expansion of finite rank. In the second case, the diffusion coefficient is described by a Gaussian correlation function. Hence, the Karhunen-Loève expansion has to be appropriately truncated, where the truncation rank m tends to ∞ as the over-all accuracy increases. The domain of the spatial variable is the unit square in both examples. For the approximation of the Karhunen-Loève expansion, we employ the pivoted Cholesky expansion as described in [HPS12, HPS15] together with a piecewise constant finite element discretization of the two-point correlation.

The finite element method we use is based on continuous piecewise linear ansatz functions. The grid transfer for the solution's second moment is thus performed via a quadratic prolongation.

7.1 An example with a covariance function of finite rank

In our first numerical example, we focus on a diffusion coefficient with four stochastical sources on discs $B_1, \ldots, B_4$ of diameter 0.3 equispaced in $D = (0,1)^2$. The associated covariance function is given by $\operatorname{Cov}_b(\mathbf{x}, \mathbf{x}') = \sum_{i=1}^{4} \mathbb{1}_{B_i}(\mathbf{x})\mathbb{1}_{B_i}(\mathbf{x}')$ and as a loading we consider $f \equiv 1$. A visualization of the associated triangulation can be seen in the left picture of Figure VI.1. Since the reference solution is not analytically known, we compute it

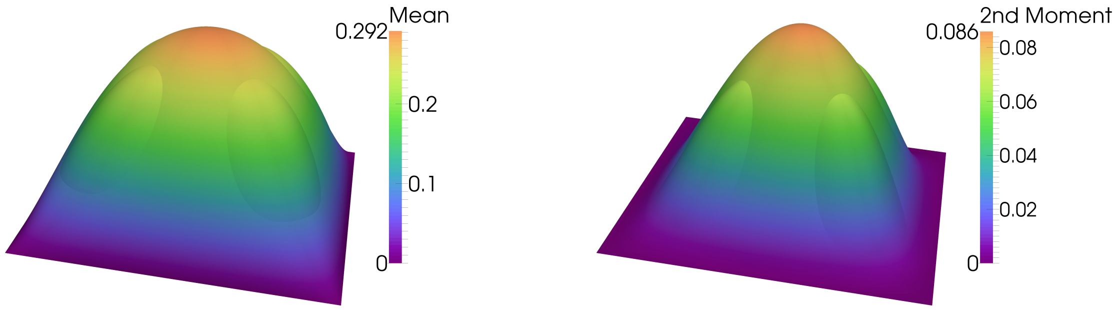


Figure VI.2: Solution's mean (left) and solution's second moment in the case of finite dimensional stochastics.

numerically. To that end, we employ a single level quadrature, namely a quasi-Monte Carlo quadrature on a finer spatial grid with 1 048 576 finite elements and 10^6 samples based on the Halton sequence, as described in Section IV.2.3. Figure VI.2 shows the mean (on the left) and the second moment (on the right) of this reference solution. For the multilevel methods, we choose the respective number of samples as follows. We set $N_\ell = 10 \cdot 4^{j-\ell}$ for MLMC and $N_\ell = 10 \cdot 2^{(j-\ell)/(1-\delta)}$ for MLQMC where $\delta = 0, 0.25$. As quadrature method in the quasi-Monte Carlo case, we employ the quasi Monte-Carlo method based on the Halton sequence with respect to the auxiliary density function $\sqrt{\rho(y)}/\overline{\rho}$ as presented in Section IV.2.3. The number of samples for MLGQ is controlled by the polynomial degrees determined from (V.1.8). We employ this estimate since each dimension is equally weighted and the region of analyticity is described by a small $\tau_k = 1$ for $k = 1, \ldots, 4$, cf. Remark (V.1.7). Hence, the quadrature corresponds to an isotropic tensor product quadrature. In order to guarantee the over-all precision of ε, we scale the precision of each stochastic dimension by the factor $1/4$. The MLsGQ is computed with the combination technique formula (V.3.5) and q is chosen in such a way that the maximal univariate quadrature rule in each direction coincides with the quadrature rule in the tensor product formula in the according direction. Thus, it is simply a sparsification of the full grid quadrature formula.

A comparison of the number of samples for each of the methods is depicted on the left hand side of Figure VI.4. Here, the different colors refer to the samples on each particular level sorted in decreasing order from left to right, i.e. $\ell = 6, 5, \ldots, 0$. Note that, for the MLMC, we compute the RMSE based on five realizations of the Multilevel Monte-Carlo estimator. This fact is not taken into account in the visualization in Figure VI.4 where the number of quadrature points is presented.

The error plots found in Figure VI.3 indicate that the four methods, i.e. MLGQ, MLsGQ, MLMC, and MLQMC, provide the theoretic order of convergence of $j2^{-j}$ of the

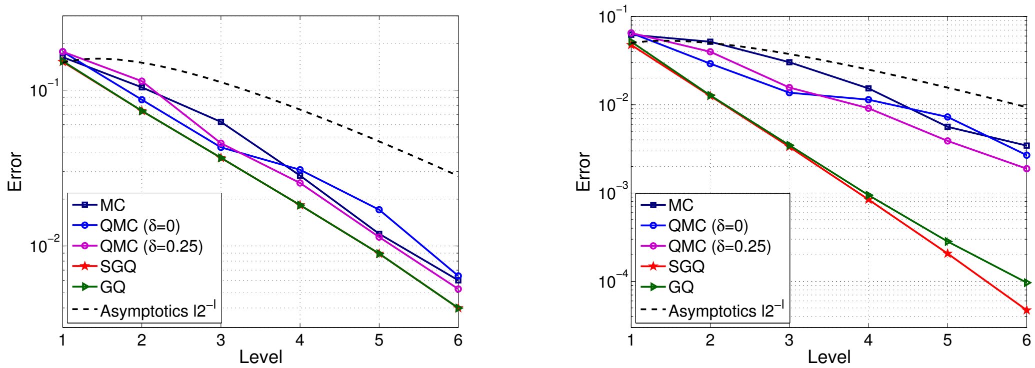
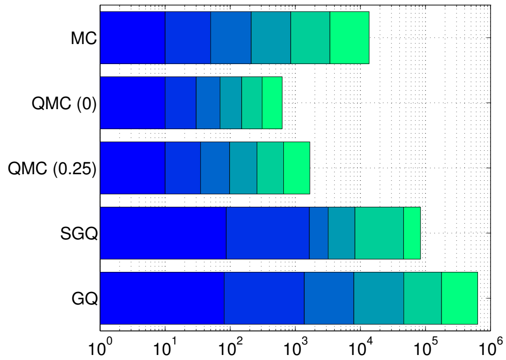


Figure VI.3: Error of the mean (left) and error of the second moment (right) in the finite dimensional stochastics case.

approximate mean with respect to the H^1-norm (picture on the left) and of the approximate second moment with respect to the $W^{1,1}$-norm (picture on the right). Indeed, it seems that the logarithmic term j which stems from the summation of the multilevel error does not occur. Moreover, the choice of $\delta = 0$, i.e. a linearly increasing number of samples, seems to be sufficient in the MLQMC to maintain the optimal rate of convergence.

In Table VI.3, the total cost of the different methods are stated in terms of the cost for a sample on the finest grid. We determine these numbers by summing up the scaled number of samples in Figure VI.4 on the different levels. According to Section 6, the scale factor on level ℓ is $2^{-d(j-\ell)}$. As we see from the table, the MLQMC with $\delta = 0$ or $\delta = 0.25$ requires only a cost of 20 or 25 fine grid samples, respectively, to recover the convergence rate. It is, therefore, superior to the MLMC method which yields essentially the same convergence rate with a cost of 60 fine grid samples. We recommend the use of MLQMC with $\delta = 0.25$ for this example since the overhead of the MLQMC with $\delta = 0.25$ in comparison to MLQMC with $\delta = 0$ is low and the error plots of the MLQMC with $\delta = 0.25$ are preferable to that of the MLQMC with $\delta = 0$. In comparison, the cost of the MLGQ are much higher, approximately eight times as high as the cost of the MLMC and 95 times as high as the cost of the MLQMC. Since the method provides a higher convergence rate for the second moment, it seems that the finite element error for the computation of the second moment converges with a higher rate than expected in Lemma (1.1). In addition, this suggests that the MLGQ heavily overestimates the quadrature error. The same effect occurs for the MLsGQ. This method preserves essentially the same convergence plot as the MLGQ while the complexity is considerably reduced. Indeed the cost of MLsGQ is only 835 fine grid samples instead of 2371 for the MLGQ. Nevertheless, the cost still significantly surpasses the cost of the MLQMC. This can be explained since the a priori knowledge, which is used to determine each order $q_{j-\ell}$ for the MLGQ and the MLsGQ, substantially underestimates the performance of the Gaussian quadrature method. Hence, a lot of quadrature points are used for MLGQ and MLsGQ which would not be necessary to cover the proven convergence rates.



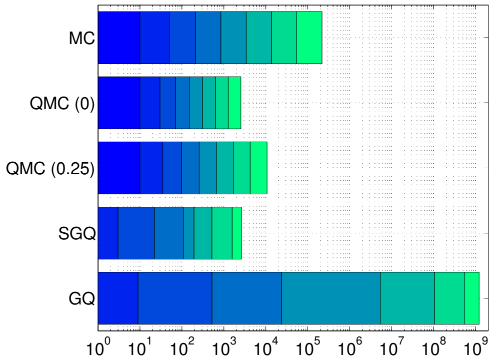


Figure VI.4: Number of samples for each of the methods for the first example (left) and for the second example (right).

	MLMC	MLQMC($\delta = 0.25$)	MLQMC($\delta = 0$)	MLGQ	MLsGQ
Example 1	60	25	20	2371	837
Example 2	80	26	19	267749	5

Table VI.3: Cost of each method for both examples in terms of fine-grid samples.

7.2 An example with a covariance function of infinite rank

For our second example, we consider the covariance function

$$\operatorname{Cov}_b(\mathbf{x}, \mathbf{x}') = 0.5 \exp\big(-2\|\mathbf{x} - \mathbf{x}'\|_2^2\big).$$

The loading is again $f \equiv 1$ on the computational domain $D = (0, 1)^2$. The unit square is triangulated as seen in the right picture of Figure VI.1. The Karhunen-Loève expansion is approximated with a trace error of 10^{-6} which results in $m = 72$ terms. As a reference solution, we compute again a single level quasi-Monte Carlo solution on a finer spatial grid with 1 048 576 finite elements and 10^6 samples based on Halton points. Instead of working with an auxiliary density function as in the previous example, we exploit the results of Subsection IV.2.4 and map the points of the Halton sequence from the unit cube $[0, 1]^m$ to $\mathbb{R}^m$ by the inverse Gaussian distribution function. Since the sequence $\{\gamma_k\}_k$ decays sufficiently fast for this example, we expect nearly dimension-independent convergence. Nevertheless, we emphasize that, in case of a relatively low-dimensional stochastics where the different dimensions are equally important like in the previous example, the QMC approach with auxiliary density performs better. In Figure VI.5 the mean (picture on the left) and the second moment (picture on the right) of the reference solution are visualized.

In order to preserve the approximation order of 2^{-j}, the Karhunen-Loève expansion is truncated after $m = 21$ terms on the finest level of computation $j = 8$. The number of samples for MLMC and MLQMC is chosen as in the previous example. The degrees in MLGQ are computed with respect to formula (V.2.12). Although, the results of Section 4 imply that we have to choose τ_k with respect to $3\tilde{\gamma}_k/\log 2$, for reasons of computability

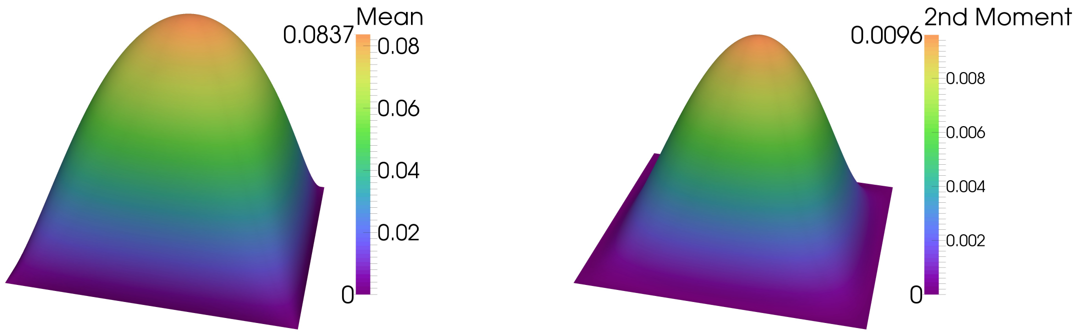


Figure VI.5: Solution's mean (left) and solution's second moment in the case of infinite dimensional stochastics.

of the tensor-product Gaussian formula, we set $\tau_k = \log 2/\gamma_k$ and scale the function h in (V.2.12) additionally by $\sqrt{2}$. The anisotropic sparse Gaussian quadrature is chosen as a sparsification of the respective tensor product quadrature.

The number of samples for the different methods are shown on the right-hand side of Figure VI.4. The related computational errors are depicted in Figure VI.6.

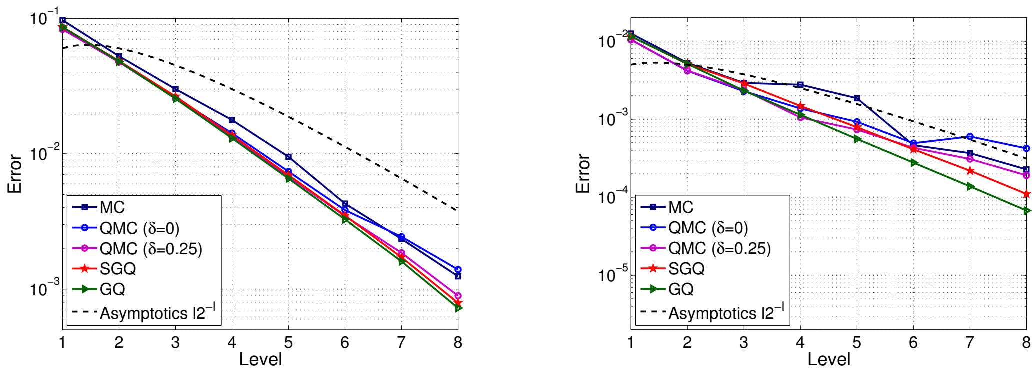


Figure VI.6: Error of the mean (left) and error of the second moment (right) in the infinite dimensional stochastics case.

We achieve similar results to the example in Subsection 7.1 for the computation of the mean. All methods provide the expected error rates and the logarithmic factor j does not occur. For the computation of the second moment, the results differ to those obtained for the example in Subsection 7.1. Except for the MLQMC with $\delta = 0$, whose convergence rate for the second moment stagnates on level $6, 7$ and 8, all methods provide the expected error rates. In contrast to the previous example, the rate for the second moment for the MLGQ and the MLsGQ is not significantly higher anymore. This implies that the chosen quadrature settings on level ℓ of the MLGQ and the MLsGQ yield the required accuracy $\mathcal{O}(2^{-\ell})$, but not drastically overestimate the quadrature error anymore. Nevertheless, we observe that the MLGQ and the MLsGQ still provide slightly better convergence rates than the MLMC or the MLQMC.

The computational cost in terms of fine grid samples, cf. Table VI.3, are essentially the same for QMC with $\delta = 0$ and $\delta = 0.25$, respectively, and multiplied by a factor 4/3 for MLMC compared to the cost in the previous example. In contrast to this, for the MLsGQ, the computational cost for this example are much lower than in the previous example. The MLsGQ requires only 5 fine grid samples for this example instead of 837 fine grid samples which are required for the example in Subsection 7.1. The cost of the MLGQ, however, are much higher as in the previous example. This is not surprisingly since a tensor product quadrature in 21 dimensions is not competitive at all.

Hence, we recommend for this example the application of MLsGQ since the performance is outstanding. With computational cost of just five fine grid samples, the MLsGQ is able to approximate the mean with an error of $7.9 \cdot 10^{-4}$ measured in the $H^1(D)$-Norm and the variance with an error of $1.1 \cdot 10^{-4}$ measured in the $W^{1,1}(D)$-Norm. Nevertheless, the MLQMC with $\delta = 0.25$ works also very well for this example. This method requires only 26 fine grid samples to obtain an error of $8.9 \cdot 10^{-4}$ for the computation of the mean and an error of $1.9 \cdot 10^{-4}$ for the computation of the variance.

To summarize, the numerical results corroborate our theoretical findings and demonstrate that arbitrary quadrature methods can significantly be accelerated by multilevel techniques.

Appendix

We provide in this appendix the proof of Conjecture (V.3.32) for $m = 3, 4, 5$. To that end, we first recall the statement of the conjecture:

(0.1) **Conjecture.** For an ascendingly ordered weight vector $\mathbf{w} = [w_1, \ldots, w_m] \in \mathbb{R}_+^m$ and $q \in \mathbb{N}$, the cardinality of the set

$$X_{\mathbf{w}}(q,m) = \left\{ \mathbf{0} \le \boldsymbol{\alpha} \in \mathbb{N}^m \colon \sum_{n=1}^{m} \alpha_n w_n \le q \right\}$$

is bounded by

$$\#X_{\mathbf{w}}(q,m) \le \prod_{k=1}^{m} \frac{\frac{q}{w_k} + k}{k}. \tag{0.2}$$

The conjecture is obviously fulfilled for $m = 1$ and it was shown in Section V.3.5 that it is also true for $m = 2$. The cardinality of the set $X_{\mathbf{w}}(q,m)$ can recursively be calculated by

$$\begin{aligned}
\#X_{\mathbf{w}}(q,m) &= \sum_{k=0}^{\left\lfloor \frac{q}{w_1} \right\rfloor} X_{\mathbf{w}_{2:m}}(q - kw_1, m-1) \\
\text{(since } w_1 \le w_i\text{)} \qquad &= 1 + \sum_{k=0}^{\left\lfloor \frac{q}{w_1} \right\rfloor - 1} X_{\mathbf{w}_{2:m}}(q - kw_1, m-1) \\
\text{(by induction)} \qquad &\le 1 + \sum_{k=0}^{\left\lfloor \frac{q}{w_1} \right\rfloor - 1} \frac{1}{(m-1)! \prod_{j=2}^{m} w_j} \prod_{\ell=1}^{m-1} (q - kw_1 + \ell w_{\ell+1}) \\
\left(\delta := q - \left\lfloor \frac{q}{w_1} \right\rfloor\right) \qquad &= 1 + \sum_{k=1}^{\left\lfloor \frac{q}{w_1} \right\rfloor} \frac{1}{(m-1)! \prod_{j=2}^{m} w_j} \prod_{\ell=1}^{m-1} (kw_1 + \delta + \ell w_{\ell+1}).
\end{aligned}$$

With $\theta_\ell := \delta + \ell w_{\ell+1}$ for $\ell = 1, \dots, m-1$, it follows that

$$(\#X_{\mathbf{w}}(q,m)-1)m!\prod_{j=1}^{m} w_j \leq \sum_{k=1}^{\left\lfloor \frac{q}{w_1}\right\rfloor} mw_1 \prod_{\ell=1}^{m-1}(kw_1+\theta_\ell)$$
$$= \sum_{k=1}^{\left\lfloor \frac{q}{w_1}\right\rfloor} mw_1 \sum_{\ell=0}^{m-1}(kw_1)^{m-1-\ell}\sum_{\substack{\|\boldsymbol{\alpha}\|_\infty\leq 1,\\ |\boldsymbol{\alpha}|=\ell}}\theta^{\boldsymbol{\alpha}}.$$

Herein, $\boldsymbol{\alpha}\in\mathbb{N}^{m-1}$ denotes a multi-index. Moreover, it holds with $\boldsymbol{\beta}\subset\mathbb{N}^m$ that

$$\prod_{k=1}^{m}(q+kw_k) = m!\prod_{k=1}^{m} w_k + \sum_{\ell=0}^{m-1} q^{m-\ell}\sum_{\substack{\|\boldsymbol{\beta}\|_\infty\leq 1,\\ |\boldsymbol{\beta}|=\ell}}\prod_{j=1}^{m}(jw_j)^{\beta_j}.$$

Thus, it remains to show

$$\sum_{k=1}^{\left\lfloor \frac{q}{w_1}\right\rfloor} mw_1 \sum_{\ell=0}^{m-1}(kw_1)^{m-1-\ell}\sum_{\substack{\|\boldsymbol{\alpha}\|_\infty\leq 1,\\ |\boldsymbol{\alpha}|=\ell}}\theta^{\boldsymbol{\alpha}} \leq \sum_{\ell=0}^{m-1} q^{m-\ell}\sum_{\substack{\|\boldsymbol{\beta}\|_\infty\leq 1,\\ |\boldsymbol{\beta}|=\ell}}\prod_{j=1}^{m}(jw_j)^{\beta_j}. \tag{0.3}$$

With the notation

$$E_\ell^{(m)} := \sum_{\substack{\|\boldsymbol{\beta}\|_\infty\leq 1,\\ |\boldsymbol{\beta}|=\ell}}\prod_{j=1}^{m}(jw_j)^{\beta_j} \quad\text{and}\quad F_\ell^{(m-1)} := \sum_{\substack{\|\boldsymbol{\alpha}\|_\infty\leq 1,\\ |\boldsymbol{\alpha}|=\ell}}\theta^{\boldsymbol{\alpha}},$$

the left-hand side in (0.3) can be rewritten by

$$\sum_{k=1}^{\left\lfloor \frac{q}{w_1}\right\rfloor} mw_1 \sum_{\ell=0}^{m-1} k^{m-1-\ell} w_1^{m-1-\ell} F_\ell^{(m-1)} = \sum_{\ell=0}^{m-1} mw_1^{m-\ell}F_\ell^{(m-1)}\sum_{k=1}^{\left\lfloor \frac{q}{w_1}\right\rfloor} k^{m-1-\ell}.$$

Now, it holds with the Faulhaber formula and the Bernoulli numbers B_j, cf. [Knu93], that

$$\sum_{\ell=0}^{m-1} mw_1^{m-\ell}F_\ell^{(m-1)}\sum_{k=1}^{\left\lfloor \frac{q}{w_1}\right\rfloor} k^{m-1-\ell}$$
$$= \sum_{\ell=0}^{m-1} mw_1^{m-\ell}F_\ell^{(m-1)}\left[\frac{1}{m-\ell}\sum_{j=0}^{m-1-\ell}(-1)^j\binom{m-\ell}{j}B_j\left\lfloor\frac{q}{w_1}\right\rfloor^{m-\ell-j}\right]$$
$$= \sum_{k=0}^{m-1}\left(\left\lfloor\frac{q}{w_1}\right\rfloor\right)^{m-k}\sum_{j=0}^{k} w_1^{m-k+j}\frac{m}{m-k+j}\binom{m-k+j}{j}(-1)^jB_jF_{k-j}^{(m-1)}$$
$$= \sum_{k=0}^{m-1}(q-\delta)^{m-k}\sum_{j=0}^{k}\frac{mw_1^j}{m-k+j}\binom{m-k+j}{j}(-1)^jB_jF_{k-j}^{(m-1)}$$
$$= \sum_{k=0}^{m-1}\sum_{\ell=0}^{m-k}\binom{m-k}{\ell}(-1)^\ell q^{m-k-\ell}\delta^\ell\sum_{j=0}^{k}\frac{mw_1^j}{m-k+j}\binom{m-k+j}{j}(-1)^jB_jF_{k-j}^{(m-1)}.$$

We would like to consider the parts which contain δ and the parts without δ separately. Therefore, we split up the first summand in the second summation and obtain

$$
\begin{aligned}
&\sum_{\ell=0}^{m-1} m w_1^{m-\ell} F_\ell^{(m-1)} \sum_{k=1}^{\left\lfloor \frac{q}{w_1} \right\rfloor} k^{m-1-\ell} \\
&= \sum_{k=0}^{m-1} \sum_{\ell=1}^{m-k} \binom{m-k}{\ell} (-1)^\ell q^{m-k-\ell} \delta^\ell \sum_{j=0}^{k} \frac{m w_1^j}{m-k+j} \binom{m-k+j}{j} (-1)^j B_j F_{k-j}^{(m-1)} \\
&\quad + \sum_{k=0}^{m-1} q^{m-k} \sum_{j=0}^{k} \frac{m w_1^j}{m-k+j} \binom{m-k+j}{j} (-1)^j B_j \left(F_{k-j}^{(m-1)} - G_{k-j}^{(m-1)} \right) \\
&\quad + \sum_{k=0}^{m-1} q^{m-k} \sum_{j=0}^{k} \frac{m w_1^j}{m-k+j} \binom{m-k+j}{j} (-1)^j B_j G_{k-j}^{(m-1)},
\end{aligned} \tag{0.4}
$$

where

$$G_{k-j}^{(m-1)} = \sum_{\substack{\|\boldsymbol{\alpha}\|_\infty \le 1, \\ |\boldsymbol{\alpha}| = k-j}} \prod_{i=1}^{m-1} (i w_{i+1})^{\alpha_i}.$$

In view of (0.3), the assertion follows if we can show that the first two summands in (0.4) are non-negative, and if it holds

$$\sum_{k=0}^{m-1} q^{m-k} \sum_{j=0}^{k} \frac{m w_1^j}{m-k+j} \binom{m-k+j}{j} (-1)^j B_j G_{k-j}^{(m-1)} \overset{!}{\le} \sum_{k=0}^{m-1} q^{m-k} E_k^{(m)}.$$

Since q and $E_k^{(m)}$ are positive, it suffices to compare the coefficients for $k = 0, \ldots, m-1$, i.e.

$$\sum_{j=0}^{k} \frac{m w_1^j}{m-k+j} \binom{m-k+j}{j} (-1)^j B_j G_{k-j}^{(m-1)} \overset{!}{\le} E_k^{(m)}. \tag{0.5}$$

Both sides of this equation are equal to 1 for $k = 0$ and every $m \in \mathbb{N}$ since $B_0 = 1$. Furthermore, the inequality reads for $k = 1$ with $B_1 = -1/2$:

$$\frac{m}{m-1} \sum_{i=1}^{m-1} i w_{i+1} + \frac{m}{2} w_1 \overset{!}{\le} \sum_{i=1}^{m} i w_i \iff \sum_{i=1}^{m-1} \left(i + 1 - \frac{m}{m-1} i \right) w_{i+1} \overset{!}{\ge} \left(\frac{m}{2} - 1 \right) w_1.$$

From $w_i \ge w_1$, we conclude

$$\sum_{i=1}^{m-1} \left(i + 1 - \frac{m}{m-1} i \right) w_{i+1} \ge w_1 \left(m - 1 - \sum_{i=1}^{m-1} \frac{i}{m-1} \right) = \left(\frac{m}{2} - 1 \right) w_1.$$

Thus, we have only to elaborate on (0.5) for $k = 2, \ldots, m-1$.

Next, we simplify the first two summands in (0.4). At first, we notice that

$$F_0^{(m-1)} = G_0^{(m-1)} = 1.$$

Hence, the two summands can be rewritten by

$$\begin{aligned}&\sum_{k=0}^{m-1}\sum_{\ell=1}^{m-k}\binom{m-k}{\ell}(-1)^{\ell}q^{m-k-\ell}\delta^{\ell}\sum_{j=0}^{k}\frac{mw_1^j}{m-k+j}\binom{m-k+j}{j}(-1)^jB_jF_{k-j}^{(m-1)}\\&\quad+\sum_{k=1}^{m-1}q^{m-k}\sum_{j=0}^{k-1}\frac{mw_1^j}{m-k+j}\binom{m-k+j}{j}(-1)^jB_j\left(F_{k-j}^{(m-1)}-G_{k-j}^{(m-1)}\right)\overset{!}{\leq}0.\end{aligned}\tag{0.6}$$

The first term herein is rearranged according to the transform $k\mapsto k+\ell$ and $\ell\mapsto k$ in order to compare the first and the second term in (0.6). This means that the ℓ in (0.6) is now represented by $k-\ell$ and k by ℓ. This leads to

$$\sum_{k=1}^{m}q^{m-k}\sum_{\ell=0}^{k-1}\binom{m-\ell}{k-\ell}(-1)^{k-\ell}\delta^{k-\ell}\sum_{j=0}^{\ell}\frac{mw_1^j}{m-\ell+j}\binom{m-\ell+j}{j}(-1)^jB_jF_{\ell-j}^{(m-1)}.\tag{0.7}$$

We show next that it holds for $k\geq 1$ that

$$\begin{aligned}&-\sum_{\ell=0}^{k-1}\binom{m-\ell}{k-\ell}(-1)^{k-\ell}\delta^{k-\ell}\sum_{j=0}^{\ell}\frac{mw_1^j}{m-\ell+j}\binom{m-\ell+j}{j}(-1)^jB_jF_{\ell-j}^{(m-1)}\\&\quad=\sum_{j=0}^{k-1}\frac{mw_1^j}{m-k+j}\binom{m-k+j}{j}(-1)^jB_j\left(F_{k-j}^{(m-1)}-G_{k-j}^{(m-1)}\right).\end{aligned}\tag{0.8}$$

Thus, all terms in (0.6) would vanish except for the term in (0.7) where $k=m$. This would imply that the expression simplifies to

$$\sum_{\ell=0}^{m-1}(-1)^{m-\ell}\delta^{m-\ell}\sum_{j=0}^{\ell}\frac{mw_1^j}{m-\ell+j}\binom{m-\ell+j}{j}(-1)^jB_jF_{\ell-j}^{(m-1)}\overset{!}{\leq}0.\tag{0.9}$$

The left-hand side of (0.8) is rewritten by

$$-\sum_{j=0}^{k-1}mw_1^j(-1)^jB_j\sum_{\ell=j}^{k-1}\binom{m-\ell}{k-\ell}(-1)^{k-\ell}\delta^{k-\ell}\frac{1}{m-\ell+j}\binom{m-\ell+j}{j}F_{\ell-j}^{(m-1)}.$$

Hence, it remains to show that

$$\begin{aligned}&-\sum_{\ell=j}^{k-1}\binom{m-\ell}{k-\ell}(-1)^{k-\ell}\delta^{k-\ell}\frac{1}{m-\ell+j}\binom{m-\ell+j}{j}F_{\ell-j}^{(m-1)}\\&\qquad=\frac{1}{m-k+j}\binom{m-k+j}{j}\left(F_{k-j}^{(m-1)}-G_{k-j}^{(m-1)}\right).\end{aligned}\tag{0.10}$$

To that end, we exploit the identity

$$F_s^{(m-1)}=\sum_{\ell=0}^{s}\binom{m-1-\ell}{s-\ell}\delta^{s-\ell}G_\ell^{(m-1)}.\tag{0.11}$$

Inserting this identity into the right-hand side of (0.10) yields on the one hand that

$$
\begin{aligned}
&\frac{1}{m-k+j}\binom{m-k+j}{j}\left(F_{k-j}^{(m-1)}-G_{k-j}^{(m-1)}\right)\\
&\quad=\frac{1}{m-k+j}\binom{m-k+j}{j}\left(\left[\sum_{\ell=0}^{k-j}\binom{m-1-\ell}{k-j-\ell}\delta^{k-j-\ell}G_\ell^{(m-1)}\right]-G_{k-j}^{(m-1)}\right)\\
&\quad=\frac{1}{m-k+j}\binom{m-k+j}{j}\sum_{\ell=0}^{k-j-1}\binom{m-1-\ell}{k-j-\ell}\delta^{k-j-\ell}G_\ell^{(m-1)}\\
&\quad\overset{r=k-j}{=}\frac{1}{m-r}\binom{m-r}{j}\sum_{\ell=0}^{r-1}\binom{m-1-\ell}{r-\ell}\delta^{r-\ell}G_\ell^{(m-1)}.
\end{aligned}
$$

On the other hand, inserting identity (0.11) into the left-hand side of (0.10), gives

$$
\begin{aligned}
&-\sum_{\ell=j}^{k-1}\binom{m-\ell}{k-\ell}(-1)^{k-\ell}\delta^{k-\ell}\frac{1}{m-\ell+j}\binom{m-\ell+j}{j}F_{\ell-j}^{(m-1)}\\
&\quad=-\sum_{\ell=0}^{r-1}\binom{m-\ell-j}{r-\ell}(-1)^{r-\ell}\frac{\delta^{r-\ell}}{m-\ell}\binom{m-\ell}{j}\sum_{i=0}^{\ell}\binom{m-1-i}{\ell-i}\delta^{\ell-i}G_i^{(m-1)}\\
&\quad=\sum_{i=0}^{r-1}\delta^{r-i}G_i^{(m-1)}\sum_{\ell=i}^{r-1}\frac{(-1)^{r-\ell-1}}{m-\ell}\binom{m-\ell-j}{r-\ell}\binom{m-\ell}{j}\binom{m-1-i}{\ell-i}.
\end{aligned}
\tag{0.12}
$$

The identity (0.8) follows then from

$$
\begin{aligned}
&\sum_{\ell=i}^{r-1}\frac{(-1)^{r-\ell-1}}{m-\ell}\binom{m-\ell-j}{r-\ell}\binom{m-\ell}{j}\binom{m-1-i}{\ell-i}\\
&\quad=\binom{m-r}{j}\frac{1}{m-r}\sum_{\ell=i}^{r-1}\frac{(-1)^{r-\ell-1}}{m-\ell}(m-r)\frac{(m-\ell)!}{(r-\ell)!(m-r)!}\binom{m-i-1}{\ell-i}\\
&\quad=\binom{m-r}{j}\frac{1}{m-r}\sum_{\ell=i}^{r-1}\frac{(-1)^{r-\ell-1}(m-1-i)!}{(r-\ell)!(m-r-1)!(\ell-i)!}\\
&\quad=\binom{m-r}{j}\frac{1}{m-r}\binom{m-i-1}{r-i}\sum_{\ell=i}^{r-1}(-1)^{r-\ell-1}\binom{r-i}{\ell-i}
\end{aligned}
$$

together with the identity

$$
\sum_{\ell=i}^{r-1}(-1)^{r-\ell-1}\binom{r-i}{\ell-i}=-\sum_{\ell=0}^{r-i-1}(-1)^{r-i-\ell}\binom{r-i}{\ell}=-(-1)=1.
$$

It remains to successively show for $m=3,4,5$ that inequality (0.5) holds for $k=2,\dots,m-1$ and that (0.9) is not positive. The second assertion is satisfied for

$m = 3, 4, 5$ since we can rearrange (0.9) with the same arguments as above in (0.12) by

$$\begin{aligned} &\sum_{\ell=0}^{m-1} (-1)^{m-\ell} \delta^{m-\ell} \sum_{j=0}^{\ell} \frac{m w_1^j}{m-\ell+j} \binom{m-\ell+j}{j} (-1)^j B_j F_{\ell-j}^{(m-1)} \\ &= - \sum_{j=0}^{m-1} m w_1^j (-1)^j B_j \sum_{i=0}^{m-j-1} \delta^{m-j-i} G_i^{(m-i)} \binom{m-1-i}{j} \frac{1}{m-j-i} \overset{!}{\leq} 0. \end{aligned} \tag{0.13}$$

This inequality is fulfilled for all $m \leq 6$ since $(-1)^j B_j \geq 0$ for $j = 1, \ldots, 5$ and all other quantities in the sum in (0.5) are positive as well.

Due to $B_2 = \frac{1}{6}$, the expression (0.5) reads for $k = 2$ as

$$\frac{m}{m-2} \sum_{i=1}^{m-2} \sum_{j=i+1}^{m-1} ij w_{i+1} w_{j+1} + \frac{m}{2} w_1 \sum_{i=1}^{m-1} i w_{i+1} + \frac{m(m-1)}{12} w_1^2 \overset{!}{\leq} \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} ij w_i w_j. \tag{0.14}$$

In view of $w_1 \leq w_2$, this leads for $m = 3$ to

$$\begin{aligned} 6w_2w_3 + \frac{3}{2} w_1(w_2 + 2w_3) + \frac{1}{2} w_1^2 &\overset{!}{\leq} 2w_1w_2 + 3w_1w_3 + 6w_2w_3 \\ \iff 2w_1^2 &\overset{!}{\leq} 2w_1w_2. \end{aligned}$$

Analogously, for $m = 4$, we arrive at

$$\begin{aligned} &2(2w_2w_3 + 3w_2w_4 + 6w_3w_4) + 2w_1(w_2 + 2w_3 + 3w_4) + w_1^2 \\ &\quad \overset{!}{\leq} 2w_1w_2 + 3w_1w_3 + 4w_1w_4 + 6w_2w_3 + 8w_2w_4 + 12w_3w_4 \\ &\qquad \iff w_1w_3 + 2w_1w_4 + w_1^2 \overset{!}{\leq} 2w_2w_3 + 2w_2w_4. \end{aligned}$$

The latter inequality is fulfilled due to $w_1w_4 \leq w_2w_4$ and $w_1^2 \leq w_1w_3$. Inserting $m = 5$ into (0.14), leads to

$$\begin{aligned} &\frac{5}{3}(2w_2w_3 + 3w_2w_4 + 4w_2w_5 + 6w_3w_4 + 8w_3w_5 + 12w_4w_5) \\ &\quad + \frac{5}{2} w_1(w_2 + 2w_3 + 3w_4 + 4w_5) + \frac{5}{3} w_1^2 \\ &\quad \overset{!}{\leq} 2w_1w_2 + 3w_1w_3 + 4w_1w_4 + 5w_1w_5 + 6w_2w_3 + 8w_2w_4 \\ &\qquad + 10w_2w_5 + 12w_3w_4 + 15w_3w_5 + 20w_4w_5 \\ \iff &\qquad \frac{1}{2} w_1w_2 + 2w_1w_3 + \frac{7}{2} w_1w_4 + 5w_1w_5 + \frac{5}{3} w_1^2 \\ &\qquad\quad \overset{!}{\leq} \frac{8}{3} w_2w_3 + 3w_2w_4 + \frac{10}{3} w_2w_5 + 2w_3w_4 + \frac{5}{3} w_3w_5. \end{aligned}$$

This is valid since $5w_1w_5 \leq 10w_2w_5/3 + 5w_3w_5/3$, $7w_1w_4/2 \leq 3w_2w_4 + w_3w_4/2$ and $2w_1w_3 + w_1w_2/2 + 5w_1^2/3 \leq 3w_3w_4/2 + 8w_2w_3/3$.

With the Bernoulli number $B_3 = 0$, inserting $k = 3$ into (0.5) leads to

$$\frac{m}{m-3} G_3^{(m-1)} + \frac{m}{2} G_2^{(m-1)} + \frac{m(m-2)}{12} w_1^2 G_1^{(m-1)} \overset{!}{\leq} E_3^{(m)}.$$

Accordingly, we obtain for $k = 4$ and $B_4 = -1/30$ the inequality

$$\text{(0.15)} \qquad \frac{m}{m-4} G_4^{(m-1)} + \frac{m}{2} w_1 G_3^{(m-1)} + \frac{m(m-3)}{12} w_1^2 G_2^{(m-1)} - \binom{m}{4} \frac{w_1^4}{30} \overset{!}{\leq} E_4^{(m)}.$$

The calculations for $k = 3$ are very similar to the calculations for $k = 2$ and therefore omitted here. The case $k = 4$ is a bit different since there is for the first time a negative summand on the left-hand side in (0.15). Thus, we also present the verification of inequality (0.15) for $m = 5$:

$$\begin{aligned}
&5G_4^{(4)} + \frac{5w_1}{2} G_3^{(4)} + \frac{5w_1^2}{6} G_2^{(4)} - \frac{5w_1^4}{30} \\
&\quad = 120 w_2 w_3 w_4 w_5 + \frac{5w_1}{2}(6w_2w_3w_4 + 8w_2w_3w_5 + 12w_2w_4w_5 + 24w_3w_4w_5) \\
&\qquad + \frac{5w_1^2}{6}(2w_2w_3 + 3w_2w_4 + 4w_2w_5 + 6w_3w_4 + 8w_3w_5 + 12w_4w_5) - \frac{w_1^4}{6} \\
&\quad \overset{!}{\leq} 24w_1w_2w_3w_4 + 30w_1w_2w_3w_5 + 40w_1w_2w_4w_5 + 60w_1w_3w_4w_5 + 120w_2w_3w_4w_5.
\end{aligned}$$

This is equivalent to

$$\begin{aligned}
&w_1^2\left(\frac{5}{3}w_2w_3 + \frac{5}{2}w_2w_4 + \frac{10}{3}w_2w_5 + 5w_3w_4 + \frac{10}{3}w_3w_5 + 10w_4w_5\right) - \frac{1}{6}w_1^4 \\
&\quad \overset{!}{\leq} 9w_1w_2w_3w_4 + 10w_1w_2w_3w_5 + 10w_1w_2w_4w_5 \\
\iff &\frac{5}{2}w_1w_2w_4(w_3-w_1) + 5w_1w_3w_4(w_2-w_1) + \frac{3}{2}w_1w_2w_3(w_4-w_1) + \frac{10}{3}w_1w_2w_5(w_3-w_1) \\
&\quad + \frac{20}{3}w_1w_3w_5(w_2-w_1) + 10w_1w_4w_5(w_2-w_1) - \frac{1}{6}w_1^2(w_2w_3 - w_1^2) \overset{!}{\geq} 0.
\end{aligned}$$

Since the weights are positive and ordered ascendingly, we have that the second, third, fifth and sixth term in the last inequality are not negative. Moreover, we deduce from $\frac{1}{6}w_1^2(w_2w_3 - w_1^2) \leq \frac{1}{6}w_1^2(w_3 - w_1)(w_3 + w_1)$ that the left hand side of the last inequality is bounded from below by

$$\begin{aligned}
&\frac{5}{2}w_1w_2w_4(w_3 - w_1) + \frac{10}{3}w_1w_2w_5(w_3 - w_1) - \frac{1}{6}w_1^2(w_3 - w_1)(w_3 + w_1) \geq \\
&\qquad (w_3 - w_1)\left(\frac{5}{2}w_1w_2w_4 + \frac{10}{3}w_1w_2w_5 - \frac{1}{6}w_1^2(w_3 + w_1)\right) \geq 0.
\end{aligned}$$

This establishes (V.3.32) for $m = 3, 4, 5$.

(0.16) **Remark.** In this section, we have validated Conjecture (V.3.32) for $m = 3, 4, 5$. Moreover, we have shown that the successive verification of the inequalities (0.5) and (0.13) is sufficient to prove the conjecture. Unfortunately, the induction step is still not trivial. Nevertheless, as mentioned in Remark (V.3.34), the conjecture is not only of interest for the estimation of index sets in anisotropic sparse grids, but also since it can be applied in various other mathmatical research fields. Especially, since the conjecture is, to the best of our knowledge, superior to all existing estimates in the literature for this kind of problem, it would be important to prove it in arbitrary dimension and, hopefully, some of the ideas in this appendix can be useful for that.

BIBLIOGRAPHY

[Ada75] R. A. Adams. *Sobolev Spaces.* Academic Press, New York, 1975.

[AE08] H. Amann and J. Escher. *Analysis III.* Birkhäuser, Basel, 2008.

[AF03] R. A. Adams and J. J. F. Fournier. *Sobolev Spaces.* Academic Press, New York, 2003.

[Alt07] H. W. Alt. *Lineare Funktionalanalysis.* Springer, London, 4th edition, 2007.

[AS64] M. Abramowitz and I. A. Stegun. *Handbook of Mathematical Functions: With Formulas, Graphs, and Mathematical Tables.* Applied mathematics series. Dover Publications, N. Chemsford, MA, 1964.

[Ata04] E. Atanassov. On the discrepancy of the Halton sequences. *Mathematica Balkanica. New Series*, 18(1-2):15–32, 2004.

[BD72] A. G. Beged-Dov. Lower and upper bounds for the number of lattice points in a simplex. *SIAM Journal on Applied Mathematics*, 22(1):106–108, 1972.

[Bel34] E. T. Bell. Exponential polynomials. *Annals of Mathematics*, 35(2):258–277, 1934.

[BG04] H.-J. Bungartz and M. Griebel. Sparse grids. *Acta Numerica*, 13:147–269, 2004.

[Bie09] Marcel Bieri. *Sparse Tensor Discretizations of Elliptic PDEs with Random Input Data.* PhD thesis, ETH Zürich, 2009.

[Bie11] M. Bieri. A sparse composite collocation finite element method for elliptic sPDEs. *SIAM Journal on Numerical Analysis*, 49(6):2277–2301, 2011.

[BJ05] F. R. Bach and M. I. Jordan. Predictive low-rank decomposition for kernel methods. In L. De Raedt and S. Wrobel, editors, *Proceedings of the 22Nd International Conference on Machine Learning*, ICML '05, pages 33–40. ACM, New York, NY, USA, 2005.

[BL76] J. Bergh and J. Löfström. *Interpolation Spaces. An Introduction.* Springer, Berlin, 1976.

[BNR00] V. Barthelmann, E. Novak, and K. Ritter. High dimensional polynomial interpolation on sparse grids. *Advances in Computational Mathematics*, 12(4):273–288, 2000.

[BNT07] I. Babuška, F. Nobile, and R. Tempone. A stochastic collocation method for elliptic partial differential equations with random input data. *SIAM Journal on Numerical Analysis*, 45(3):1005–1034, 2007.

[BNTT11] J. Beck, F. Nobile, L. Tamellini, and R. Tempone. Implementation of optimal Galerkin and collocation approximations of PDEs with random coefficients. *ESAIM:Proceedings*, 33:10–21, 2011.

[BNTT12] J. Beck, F. Nobile, L. Tamellini, and R. Tempone. On the optimal polynomial approximation of stochastic PDEs by Galerkin and collocation methods. *Mathematical Models and Methods in Applied Sciences*, 22(9):1250023, 2012.

[BNTT14] J. Beck, F. Nobile, L. Tamellini, and R. Tempone. A quasi-optimal sparse grids procedure for groundwater flows. In M. Azaïez, H. El Fekih, and J. S. Hesthaven, editors, *Spectral and High Order Methods for Partial Differential Equations - ICOSAHOM 2012*, volume 95 of *Lecture Notes in Computational Science and Engineering*, pages 1–16. Springer International Publishing, 2014.

[Boc33] S. Bochner. Integration von Funktionen, deren Werte die Elemente eines Vektorraumes sind. *Fundamenta Mathematicae*, 20(1):262–176, 1933.

[BS08] S. C. Brenner and L. R. Scott. *The Mathematical Theory of Finite Element Methods*. Springer, Berlin, 3rd edition, 2008.

[BSZ11] A. Barth, C. Schwab, and N. Zollinger. Multi-level Monte Carlo finite element method for elliptic PDEs with stochastic coefficients. *Numerische Mathematik*, 119(1):123–161, 2011.

[BTZ04] I. Babuška, R. Tempone, and G. Zouraris. Galerkin finite element approximations of stochastic elliptic partial differential equations. *SIAM Journal on Numerical Analysis*, 42(2):800–825, 2004.

[Caf98] R. Caflisch. Monte Carlo and quasi-Monte Carlo methods. *Acta Numerica*, 7:1–49, 1998.

[CCS15] A. Chkifa, A. Cohen, and C. Schwab. Breaking the curse of dimensionality in sparse polynomial approximation of parametric PDEs. *Journal de Mathématiques Pures et Appliquées*, 103(2):400–428, 2015.

[CD13] J. Charrier and A. Debussche. Weak truncation error estimates for elliptic PDEs with lognormal coefficients. *Stochastic Partial Differential Equations: Analysis and Computations*, 1(1):63–93, 2013.

[CDS10] A. Cohen, R. DeVore, and C. Schwab. Convergence rates of best N-term Galerkin approximations for a class of elliptic sPDEs. *Foundations of Computational Mathematics*, 10(6):615–646, 2010.

[CDS11] A. Cohen, R. DeVore, and C. Schwab. Analytic regularity and polynomial approximation of parametric and stochastic elliptic PDEs. *Analysis and Applications*, 09(01):11–47, 2011.

[CGSS00] K. A. Cliffe, I. G. Graham, R. Scheichl, and L. Stals. Parallel computation of flow in heterogeneous media modelled by mixed finite elements. *Journal of Computational Physics*, 164(2):258–282, 2000.

[Cha12] J. Charrier. Strong and weak error estimates for elliptic partial differential equations with random coefficients. *SIAM Journal on Numerical Analysis*, 50(1):216–246, 2012.

[CS96] G. M. Constantine and T. H. Savits. A multivariate Faà di Bruno formula with applications. *Transaction of the American Mathematical Society*, 248:503–520, 1996.

[CST13] J. Charrier, R. Scheichl, and A. Teckentrup. Finite element error analysis of elliptic PDEs with random coefficients and its application to multilevel Monte Carlo methods. *SIAM Journal on Numerical Analysis*, 51(1):322–352, 2013.

[Dar56] H. Darcy. *Les Fontaines Publiques de la Ville de Dijon*. Victor Dalmont, Paris, 1856.

[DD12] F. Demengel and G. Demengel. *Functional Spaces for the Theory of Elliptic Partial Differential Equations*. Springer, London, 2012.

[Del79] J. P. Delhomme. Spatial variability and uncertainty in groundwater flow parameters: A geostatistical approach. *Water Resources Research*, 15(2):269–280, 1979.

[DHS14] J. Dölz, H. Harbrecht, and C. Schwab. Covariance regularity and $\mathcal{H}$-matrix approximation for rough random fields. *Preprint 2014-11 Mathematisches Institut Universität Basel*, 2014. to appear in Numerische Mathematik.

[DKLG+14] J. Dick, F. Y. Kuo, Q. Le Gia, D. Nuyens, and C. Schwab. Higher order QMC Petrov–Galerkin discretization for affine parametric operator equations with random field inputs. *SIAM Journal on Numerical Analysis*, 52(6):2676–2702, 2014.

[dM86] G. de Marsily. *Quantitative Hydrogeology: Groundwater Hydrology for Engineers*. Academic Press, New York, 1986.

[dMDG+05] G. de Marsily, F. Delay, J. Gonçalvès, P. Renard, V. Teles, and S. Violette. Dealing with spatial heterogeneity. *Hydrogeology Journal*, 13(1):161–183, 2005.

[Fan13] P. Fan. New inequalities of Mill's ratio and its application to the inverse Q-function approximation. *The Australian Journal of Mathematical Analysis and Applications*, 10(1):11, 2013.

[FST05] P. Frauenfelder, C. Schwab, and R. Todor. Finite elements for elliptic problems with stochastic coefficients. *Computer Methods in Applied Mechanics and Engineering*, 194(2-5):205–228, 2005.

[GG98] T. Gerstner and M. Griebel. Numerical integration using sparse grids. *Numerical Algorithms*, 18:209–232, 1998.

[GH14] M. Griebel and H. Harbrecht. Approximation of bi-variate functions: singular value decomposition versus sparse grids. *IMA Journal of Numerical Analysis*, 34(1):28–54, 2014.

[Gil08] M. Giles. Multilevel Monte Carlo path simulation. *Operations Research*, 56(3):607–617, 2008.

[Git10] C. J. Gittelson. Stochastic Galerkin discretization of the log-normal isotropic diffusion problem. *Mathematical Models and Methods in Applied Science*, 20(2):237–263, 2010.

[GKN$^+$14] I. G. Graham, F. Y. Kuo, J. A. Nichols, R. Scheichl, C. Schwab, and I. H. Sloan. Quasi-Monte Carlo finite element methods for elliptic PDEs with lognormal random coefficients. *Numerische Mathematik*, 128(4), 2014.

[Gri11] P. Grisvard. *Elliptic Problems in Nonsmooth Domains*. Classics in applied mathematics. Society for Industrial and Applied Mathematics, Philadelphia, PA, 2011.

[GS91] R. Ghanem and P. Spanos. *Stochastic Finite Elements: A Spectral Approach*. Springer, New York, 1991.

[GSZ92] M. Griebel, M. Schneider, and C. Zenger. A combination technique for the solution of sparse grid problems. In P. de Groen and R. Beauwens, editors, *Iterative Methods in Linear Algebra*, pages 263–281. IMACS, Elsevier, North Holland, 1992.

[Hal60] J. H. Halton. On the efficiency of certain quasi-random sequences of points in evaluating multi-dimensional integrals. *Numerische Mathematik*, 2(1):84–90, 1960.

[HB09] M. Hanke-Bourgeois. *Grundlagen der Numerischen Mathematik und des Wissenschaftlichen Rechnens*. Vieweg+Teubner, Wiesbaden, 3rd edition, 2009.

[Hei00] S. Heinrich. The multilevel method of dependent tests. In N. Balakrishnan, V.B. Melas, and S. Ermakov, editors, *Advances in Stochastic Simulation Methods*, Statistics for Industry and Technology, pages 47–61. Birkhäuser Boston, 2000.

[Hei01] S. Heinrich. Multilevel Monte Carlo methods. In S. Margenov, J. Waśniewski, and P. Yalamov, editors, *Large-Scale Scientific Computing*, volume 2179 of *Lecture Notes in Computer Science*, pages 58–67. Springer Berlin-Heidelberg, 2001.

[HH64] J. M. Hammersley and D. C. Handscomb. *Monte Carlo methods.* Methuen, London, 1964.

[Hic98] F. J. Hickernell. A generalized discrepancy and quadrature error bound. *Mathematics of Computation*, 67:299–322, 1998.

[Hla61] E. Hlawka. Über die Diskrepanz mehrdimensionaler Folgen mod. 1. *Mathematische Zeitschrift*, 77(1):273–284, 1961.

[HNT16] A.-L. Haji-Ali, F. Nobile, and R. Tempone. Multi-index Monte Carlo: When sparsity meets sampling. *Numerische Mathematik*, 132(4):767–806, 2016.

[HPS12] H. Harbrecht, M. Peters, and R. Schneider. On the low-rank approximation by the pivoted Cholesky decomposition. *Applied Numerical Mathematics*, 62:28–440, 2012.

[HPS13a] H. Harbrecht, M. Peters, and M. Siebenmorgen. On multilevel quadrature for elliptic stochastic partial differential equations. In J. Garcke and M. Griebel, editors, *Sparse Grids and Applications*, volume 88 of *Lecture Notes in Computational Science and Engineering*, pages 161–179, Berlin-Heidelberg, 2013. Springer.

[HPS13b] H. Harbrecht, M. Peters, and M. Siebenmorgen. On the quasi-Monte Carlo quadrature with Halton points for elliptic PDEs with log-normal diffusion. *Preprint 2013-28 Mathematisches Institut Universität Basel*, 2013. to appear in Mathematics of Computation.

[HPS15] H. Harbrecht, M. Peters, and M. Siebenmorgen. Efficient approximation of random fields for numerical applications. *Numerical Linear Algebra with Applications*, 22(4):596–617, 2015.

[HPS16] H. Harbrecht, M. Peters, and M. Siebenmorgen. Multilevel accelerated quadrature for pdes with log-normally distributed diffusion coefficient. *SIAM/ASA Journal on Uncertainty Quantification*, 4(1):520–551, 2016.

[HS14] V. A. Hoang and C. Schwab. N-term Wiener chaos approximation rates for elliptic PDEs with lognormal Gaussian random inputs. *Mathematical Models and Methods in Applied Sciences*, 24(4):797–826, 2014.

[HSW04] F. J. Hickernell, I. H. Sloan, and G. W. Wasilkowski. On tractability of weighted integration over bounded and unbounded regions in reals. *Mathematics of Computations*, 73(248):1885–1902, 2004.

[HW02] F. J. Hickernell and X. Wang. The error bounds and tractability of quasi-Monte Carlo algorithms in infinite dimension. *Mathematics of Computations*, 71(240):1641–1661, 2002.

[Kit97] P. K. Kitanidis. *Introduction to Geostatistics: Applications in Hydrogeology.* Stanford-Cambridge program. Cambridge University Press, Cambridge, 1997.

[Knu93] D. E. Knuth. Johann Faulhaber and sums of powers. *Mathematics of Computation*, 61(203):277–294, 1993.

[KSS12a] F. Y. Kuo, C. Schwab, and I. H. Sloan. Quasi-Monte Carlo finite element methods for a class of elliptic partial differential equations with random coefficients. *SIAM Journal of Numerical Analysis*, 50(6):3351–3374, 2012.

[KSS12b] F. Y. Kuo, C. Schwab, and I. H. Sloan. Quasi-Monte Carlo methods for high-dimensional integration: the standard (weighted Hilbert space) setting and beyond. *The ANZIAM Journal*, 53(0):1–37, 2012.

[KSS15] F. Y. Kuo, C. Schwab, and I. H. Sloan. Multi-level quasi-Monte Carlo finite element methods for a class of elliptic partial differential equations with random coefficients. *Foundations of Computational Mathematics*, 15(2):411–449, 2015.

[KX02a] G. Karniadakis and D. Xiu. Modeling uncertainty in steady state diffusion problems via generalized polynomial chaos. *Computer Methods in Applied Mechanics and Engineering*, 191(43):4927–4948, 2002.

[KX02b] G. Karniadakis and D. Xiu. The Wiener–Askey polynomial chaos for stochastic differential equations. *SIAM Journal on Scientific Computing*, 24(2):619–644, 2002.

[LC85] W. A. Light and E. W. Cheney. *Approximation Theory in Tensor Product Spaces.* Lecture notes in mathematics Volume 1169. Springer, Berlin-Heidelberg, 1985.

[Loè77] M. Loève. *Probability Theory. I+II.* Number 45 in Graduate Texts in Mathematics. Springer, New York, fourth edition, 1977.

[Mat86] B. Matérn. *Spatial Variation.* Springer Lecture Notes in Statistics. Springer, New York, 1986.

[MK05] H. Matthies and A. Keese. Galerkin methods for linear and nonlinear elliptic stochastic partial differential equations. *Computer Methods in Applied Mechanics and Engineering*, 194(12-16):1295–1331, 2005.

[Mor95] B. Moro. The full Monte. *Risk Magazine*, 8(2):57–58, 1995.

[NC06] D. Nuyens and R. Cools. Fast component-by-component construction of rank-1 lattice rules with a non-prime number of points. *Journal of Complexity*, 22(1):4–28, 2006.

[Nie92] H. Niederreiter. *Random Number Generation and Quasi-Monte Carlo Methods.* Society for Industrial and Applied Mathematics, Philadelphia, PA, 1992.

[NTW08a] F. Nobile, R. Tempone, and C. Webster. A sparse grid stochastic collocation method for partial differential equations with random input data. *SIAM Journal on Numerical Analysis*, 46(5):2309–2345, 2008.

[NTW08b] F. Nobile, R. Tempone, and C. G. Webster. An anisotropic sparse grid stochastic collocation method for partial differential equations with random input data. *SIAM Journal on Numerical Analysis*, 46(5):2411–2442, 2008.

[NW08] E. Novak and H. Woźniakowski. *Tractability of Multivariate Problems: Linear Information.* EMS Tracts in Mathematics. European Mathematical Society, Zürich, 2008.

[NW10] E. Novak and H. Woźniakowski. *Tractability of Multivariate Problems: Standard Information for Functionals.* EMS Tracts in Mathematics. European Mathematical Society, Zürich, 2010.

[NW12] E. Novak and H. Woźniakowski. *Tractability of Multivariate Problems: Standard Information for Operators.* EMS Tracts in Mathematics. European Mathematical Society, Zürich, 2012.

[Owe06] A. B. Owen. Halton sequences avoid the origin. *SIAM Review*, 48(3):487–503, 2006.

[PR96] J. K. Patel and C. B. Read. *Handbook of the Normal Distribution.* CRC Press, Boca Raton, 1996.

[RW05] C. E. Rasmussen and C. K. I. Williams. *Gaussian Processes for Machine Learning (Adaptive Computation and Machine Learning).* The MIT Press, Cambridge, 2005.

[SG11] C. Schwab and C. J. Gittelson. Sparse tensor discretizations of high-dimensional parametric and stochastic PDEs. *Acta Numerica*, 20:291–467, 2011.

[Sim72] C. G. Simader. *On Dirichlet's Boundary Value Problem: LP-Theory based on a Generalization of Garding's Inequality.* Lecture Notes in Mathematics. Springer, Berlin-Heidelberg-New York, 1972.

[SKJ02] I. H. Sloan, F. Y. Kuo, and S. Joe. Constructing randomly shifted lattice rules in weighted Sobolev spaces. *SIAM Journal on Numerical Analysis*, 40(5):1650–1665, 2002.

[SMC95] J. Sándor, D. S. Mitrinovic, and B. Crstici. *Handbook of Number Theory I.* Mathematics and Its Applications. Springer, Berlin-Heidelberg-New York, 1995.

[Smo63] S. Smolyak. Quadrature and interpolation formulas for tensor products of certain classes of functions. *Doklady Akademii Nauk SSSR*, 4:240–243, 1963.

[ST06] C. Schwab and R. Todor. Karhunen-Loève approximation of random fields by generalized fast multipole methods. *Journal of Computational Physics*, 217:100–122, 2006.

[Ste03] O. Steinbach. *Numerische Näherungsverfahren für elliptische Randwertprobleme: Finite Elemente und Randelemente.* Advances in Numerical Mathematics. Vieweg+Teubner, Stuttgart-Leipzig-Wiesbaden, 2003.

[STZ01] A. Szepessy, R. Tempone, and G. E. Zouraris. Adaptive weak approximation of stochastic differential equations. *Communications on Pure and Applied Mathematics*, 54(10):1169–1214, 2001.

[SW97] I. H. Sloan and H. Woźniakowski. When are quasi-Monte Carlo algorithms efficient for high dimensional integrals? *Journal of Complexity*, 14:1–33, 1997.

[TJWG15] A. L. Teckentrup, P. Jantsch, C. G. Webster, and M. Gunzburger. A multilevel stochastic collocation method for partial differential equations with random input data. *SIAM/ASA Journal on Uncertainty Quantification*, 3(1):1046–1074, 2015.

[TSGU13] A. Teckentrup, R. Scheichl, M.B. Giles, and E. Ullmann. Further analysis of multilevel Monte Carlo methods for elliptic PDEs with random coefficients. *Numerische Mathematik*, 125(3):569–600, 2013.

[Usp28] J. V. Uspensky. On the convergence of quadrature formulas related to an infinite interval. *Transactions of the American Mathematical Society*, 30(3):542–559, 1928.

[VW14] H.-W. Van Wyk. Multilevel sparse grid methods for elliptic partial differential equations with random coefficients. *ArXiv e-prints*, 1404.0963, 2014.

[Wan02] X. Wang. A constructive approach to strong tractability using quasi-Monte Carlo algorithms. *Journal of Complexity*, 18:683–701, 2002.

[Woź94] H. Woźniakowski. Tractability and strong tractability of linear multivariate problems. *Journal of Complexity*, 10(1):96–128, 1994.

[Zar68] S. K. Zaremba. Some applications of multidimensional integration by parts. *Annales Polonici Mathematici*, 21(2):85–96, 1968.

[Zen91] C. Zenger. Sparse grids. In W. Hackbusch, editor, *Parallel Algorithms for Partial Differential Equations, Proceedings of the Sixth GAMM-Seminar, Kiel, 1990*, pages 263–281. Vieweg, Wiesbaden, 1991.

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