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NUMERICAL LIMIT ANALYSIS OF PRECAST CONCRETE STRUCTURES

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ABSTRACT

Design and analysis of precast concrete structures in the ultimate limit state is largely done by simple analytical calculations and linear elastic finite element analysis, which necessarily leads to suboptimal designs. Numerical limit analysis provides a framework well suited for this task; the framework is based on the theory of rigid-plasticity, and the resulting mathematical optimisation problem can be solved efficiently using modern algorithms. This paper gives a brief introduction to convex optimisation and numerical limit analysis. The mathematical formulation of lower bound load optimisation as well as material optimisation is given and a four-storey shear wall is analysed using load optimisation. The analysis yields a capacity more than three times larger than the design load for the critical load case, and the collapse mode and stress distribution are analysed. Finally, numerical limit analysis of three-dimensional precast structures is discussed.

Keywords: precast concrete, limit analysis, rigid-plasticity, finite element, numerical modelling

1 Introduction

Precast concrete elements are widely used in the construction industry. The precast elements are cast and cured in a controlled environment at a factory and then transported to the building site. Generally, the use of precast elements makes the construction phase faster and less labour intensive. The precast panels, slabs and beams, however, need to be connected by in-situ cast joints. Typically, hairpin reinforcement bars (also known as U-bars) or wire loops are extruding from the precast components and a void is filled with a special joint mortar or concrete. In practice, it is difficult to ensure that the entire void has been filled and a sufficient bond between the precast components and the joint concrete has been established; this is especially true for vertical wall joints. These issues as well as shear joints in general have also been treated by fib bulletin 43 (2008) and fib bulletin 74 (2014), which cover the basic design rules for precast structures and describes the difficulties which structural engineers may encounter.

The overall load distribution and capacity of precast concrete structures are in practice assessed by simple analytical calculations or by use of linear elastic finite element analysis. For the ultimate limit state, the analytical methods are typically based on the theory of rigid-plasticity (Drucker et al. 1952, Prager 1952); lower bound models such as strut-and-tie models or stress field methods (Nielsen & Hoang 2010, Muttoni et al. 1997) can be used to obtain a decent estimate of the load carrying capacity, but the accuracy is very dependent on engineering intuition and the complexity of the given structure.
Linear finite element analysis is an excellent tool when it comes to the serviceability limit state, however, for the ultimate limit state the method will necessarily yield a low capacity and a suboptimal design since the plastic behaviour of the concrete and reinforcement is completely disregarded. The method is nevertheless often used in practice, and plastic behaviour is incorporated by manually adjusting the stiffness locally.

The lateral stability of precast structures is typically ensured by shear walls; reinforced precast panels connected by in-situ cast joints. Several researchers have investigated the behaviour and capacity of joints; both experimentally (see e.g. Fauchart & Cortini 1972, Hansen & Olesen 1976, Bljuger 1976, Rizkalla et al. 1989) and analytically (e.g. Christoffersen 1997, fib bulletin 43 2008, Nielsen & Hoang 2010). Despite the extensive work, the shear capacity of the joints is in practice assessed by simple

empirical design formulas (e.g. European Committee for Standardization 2005), which perform poorly when compared to experimental data and more sophisticated models(Herfelt et al. 2016*b*).

Manual limit analysis, e.g. strut-and-tie models or yield line theory, provide excellent tools for analysis of simple structures in the ultimate limit state. The numerical counterpart, known as numerical limit analysis or finite element limit analysis, can handle arbitrary geometries and is capable of modelling architecturally complex structures. The framework is based on the theory of rigid-plasticity coupled with the mathematical discretisation of the finite element method. Anderheggen & Knöpfel (1972) presented the basic foundation of the method as well as slab and solid elements. Since the 1970s several researchers have expanded the theoretical framework and use of the method (see e.g. Christiansen 1986, Sloan 1988, Krenk et al. 1994, Poulsen & Damkilde 2000). Mathematically, numerical limit analysis is formulated as a convex optimisation problem which can be solved remarkably efficiently using interior point methods. Numerical limit analysis is a so-called *direct method*, i.e. the limit load is determined in a single load step. This is a major advantage over non-linear finite element analysis for practical applications, where the ultimate load is the primary result of interest.

Herfelt et al. (2016*b*) presented a detailed numerical model of keyed shear joints based on numerical limit analysis. The model produced a decent estimate of the shear capacity when compared to experimental data. Moreover, the model was capable of representing the local mechanisms in the concrete core of the joint, which are not accounted for by previous models.

This paper will present the basic mathematical formulation of numerical limit analysis; more specific lower bound load optimisation and material optimisation. A brief review of convex optimisation and solution strategies will be presented and the current progress and development within the topic of numerical limit analysis will be discussed. The plate element subjected to in-plane forces (Sloan 1988, Poulsen & Damkilde 2000) will be used to analysed a four-storey precast wall subjected to horizontal load and dead load. The in-situ cast joints will be modelled using a specialised joint element (Herfelt et al. 2016*a*). The analysis will clearly display the benefits of using numerical limit analysis over the conventional methods.

2   Convex optimisation and numerical limit analysis

Convex optimisation problems (also called convex programs) can be found within several engineering applications, e.g. antenna array weight design and truss optimisation (Lobo et al. 1998). Convex programs have exactly one minimum, thus, they can be solved efficiently using a steepest-descend method (Newton's method). In this paper, two subclasses of convex optimisation will be presented, namely second-order cone programming (SOCP) and semidefinite programming (SDP).

In SOCP, a linear object function is minimised over the intersection of an affine set and the Cartesian product of second-order (quadratic) cones. The standard form of SOCP can be stated as (Andersen et al. 2003):

$$\begin{aligned} \max \quad & g^T x \\ \text{subject to} \quad & A\,x = b, \\ & x \in \mathcal{Q} \end{aligned} \tag{1}$$

where $x \in \mathcal{Q}$ indicate that the vector x should be in the Cartesian product of second order cones. All second-order conic optimisation problems can be recast to fit the standard form (1). The most common quadratic cone is the second-order cone also known as the Lorentz cone or ice-cream cone, which can be stated as the following set:

$$\left\{ x \; : \; x_1^2 \geq \sum_{j=2}^{n} x_j^2, \; x_1 \geq 0 \right\} \tag{2}$$

Every quadratic constraint can be transformed to fit the format of the second-order cone (2).

The semidefinite optimization program can be stated in several different ways. Vandenberghe & Boyd (1996) uses the following, very compact, form:

$$\begin{aligned} \min \quad & c^T x \\ \text{subject to} \quad & F(x) \succeq 0 \end{aligned} \tag{3}$$

where $F(x) \succeq 0$ is called a linear matrix inequality and indicates that $F(x)$ should be semidefinite positive, i.e. that the smallest eigenvalue should be non-negative. $F(x)$ is a linear function of the variable vector x and defined as:

$$F(x) = F_0 + \sum_{i=1}^{m} x_i F_i$$

where F_i are symmetric matrices.

Both SOCP and SDP can be solved by interior point methods, a class of algorithms developed from the polynomial time algorithm proposed by Karmarkar (1984). Modern solvers are capable of solving large-scale problems in a matter of minutes on a standard laptop, which again is a major advantage over non-linear finite element analysis. For an in-depth description of convex optimisation and solvers the reader is referred to the work of Boyd & Vandenberghe (2004) and Andersen et al. (2003).

2.1 Numerical lower bound limit analysis

The scope of lower bound limit analysis is either to maximise the variable load acting on the structure or to minimise the material usage. A statically admissible stress field, i.e. a stress field which satisfies equilibrium and does not violate the yield criterion in any point, is determined from the analysis together with either a load safety factor, denoted λ, or material parameters, denoted d. Every lower bound problem has a corresponding upper bound problem, which is solved simultaneously. The analysis therefore also yields the plastic strains which can be used to identify the collapse mode (Krenk et al. 1994).

The mathematical problem of lower bound load optimisation can be stated as follows (see e.g. Poulsen & Damkilde 2000, Krabbenhoft & Damkilde 2002):

$$
\begin{aligned}
&\max && \lambda \\
&\text{subject to} && H\beta = R\lambda + R_0 \\
& && f(\beta_i) \leq 0, \qquad i = 1, 2, \ldots, m
\end{aligned}
\tag{4}
$$

The load factor λ is maximised while equilibrium of the structure is ensured by the linear equations, where H is the global equilibrium matrix (or flexibility matrix) and β is the stress vector. The load acting on the structure consists of a scalable part $R\lambda$ and a fixed part R. The yield criterion $f(\beta_i) \leq 0$ ensures that all points are inside or at the yield envelope. The yield function f is generally non-linear, but convex, hence, the problem (4) will be a convex optimisation problem. For plane problems, the modified Mohr-Coulomb yield criterion can be cast as second-order cones, thus, (4) will be a second-order cone program. For the modified Mohr-Coulomb yield criterion for triaxial stress, it is necessary to use semidefinite programming which allows constraints on the eigenvalues of a matrix, e.g. the stress tensor.

The formulation of material optimisation leads to a similar optimisation problem:

$$
\begin{aligned}
&\min && w^T d \\
&\text{subject to} && H\beta = R_0 \\
& && f(\beta_i, d) \leq 0, \quad i = 1, 2, \ldots, m
\end{aligned}
\tag{5}
$$

In this case, a weighted combination of the material parameters, d, is sought to be minimised. Again, equilibrium is ensured by a set of linear equations, and the yield function $f(\beta_i, d) \leq 0$ ensure that no points violate the yield criterion. The material parameters d is now problem variables and input to the yield function.

The equilibrium matrix H depends on the chosen discretisation as well as the chosen elements. Plate elements for both bending and in-plane forces as well as shell elements have been developed (see Sloan 1988, Poulsen & Damkilde 2000, Larsen 2010). An interface element representing concrete-to-concrete interfaces was presented by Herfelt et al. (2016b), while a specialised joint element was proposed by Herfelt et al. (2016a). In this work, the plate element subjected to in-plane forces coupled with the joint and interface elements will be used to model a four-storey shear wall subjected to wind and dead load.

3 Analysis of shear wall

The work presented in this paper is based on lower bound load optimisation. A four-storey shear wall is analysed using the present framework and the results are compared to the original design of the wall, which was done by hand calculations.

The shear wall comprise 12 precast panels of which three have door openings. The panels are connected by in-situ cast joints; the vertical joints are keyed and reinforced with loop reinforcement, while the horizontal joints are reinforced with reinforcement rods extruding from the bottom panel. The interfaces of the horizontal joints are plane with a rough surface treatment which increase the friction coefficient moderately (European Committee for Standardization 2005). A sketch of the shear wall is seen in Fig. 1. The wall has a total height of 16.7 metres and a width of 7.6 metres. All door openings have a height of 2.1 metres and a width of 0.9 metres as indicated in the figure.
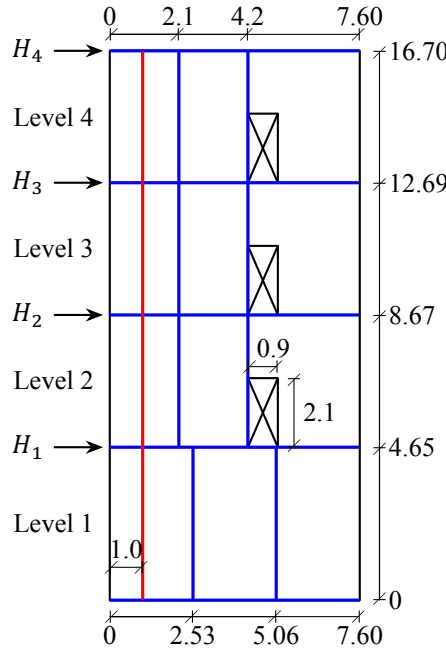

Fig. 1: Geometry of the four-storey shear wall with door openings: Blue lines indicate joints, while the red line indicate a reinforcement stringer. All measurements are given in metres.

The shear wall is subjected to horizontal loads from the wind as well as vertical load from the self-weight of the structure and additional imposed loads. Three load cases are considered as seen in Tab. 1, where L1, L2, L3, and L4 refer to the four levels of the wall. The vertical load for e.g. level 1 is applied on top of level 1, while the vertical load for from the staircase for e.g. level 2 is applied on top of the door opening in level 2. All precast panels and joints have a thickness of 240 mm. The vertical joints have width of 60 mm, while the horizontal joints have a width of 200 mm.

The variable load ($R\lambda$ in Eq. 4) is the horizontal forces H_i, while the vertical load is fixed (R_0 in Eq. 4); hence, the scope of the analysis is to determine the largest horizontal forces the structure can sustain.

Table 1: Design loads acting on the shear wall.

Load case:	Vertical loads [kN/m]			Vertical loads, stairs [kN/m]		Horizontal loads [kN]		
	L1	L2, L3	L4	L2, L3	L4	L1	L2, L3	L4
1	65.2	61.8	67.6	68.4	70.3	45.0	42.0	73.0
2	33.4	31.0	31.7	24.2	24.2	45.0	42.0	73.0
3	74.2	70.2	74.2	65.9	76.9	46.0	46.0	58.0

All precast panels are reinforced with Ø8 rebars per 150 mm in both directions. The reinforcement has a design yield strength f_{yd} of 350 MPa, and the concrete has a design compressive strength f_{cd} of 22 MPa.

The tensile strength of the concrete is neglected. Moreover, the compressive strength of the reinforcement is neglected as well.

The reinforcement stringer seen in Fig. 1 is crucial to balancing the overturning moment from the horizontal loads. For level 1, 2 high strength Ø25 rebars with a design yield strength of 664 MPa is used, while a single Ø25 rebar with a design yield strength of 385 MPa is used for the three top levels.

Table 2: Transverse and longitudinal reinforcement of the in-situ cast joints.

	Horizontal joints [kN/m]		Vertical joints		
	L0, L1	L2, L3, L4	L1	L2	L3, L4
Top side	-	-	Ø10	Ø10	Ø10
Bottom side	Ø10	Ø10	Ø10	Ø10	Ø10
s [mm]	200	200	150	450	600
Locking bar	4 Ø25	2 Ø20	-	-	-

The joint concrete has a design compressive strength of 22 MPa and tensile strength is also neglected in this cases. The joint reinforcement has a design yield strength of 385 MPa. The amount of reinforcement and the distance s between the U-bars or rods are given in Tab. 2; the joints in the bottom part of the wall are heavily reinforced compared to the joints in the top. The vertical joints are reinforced with U-bars in the transverse direction but no locking bar in the longitudinal direction. The horizontal joints are reinforced with steel rods from the bottom precast panel and several rebars in the longitudinal direction to distribute the horizontal loads. The friction coefficients are taken in accordance with the Eurocode 2 (European Committee for Standardization 2005), namely $\mu = 0.9$ for the keyed interfaces and $\mu = 0.7$ for the rough interfaces. The cohesion is neglected.

The joint element and submodel presented by Herfelt et al. (2016a) makes it possible to account for local failures within the concrete core of the joint. Ideally, the U-bars of the vertical joints should be placed closely together, yet this is not always the case in practice. An offset between two U-bars of a pair can be defined, and for this analysis the worst case scenario is assumed, which will decrease the shear capacity of the vertical joints significantly for the joints in the top part of the wall, but only a small amount for the joints in the bottom part of the wall where the U-bars are placed closely together. The local mechanisms will, however, only have little impact in the present work, since the failures occur at the bottom level.

The three load cases are analysed using a rather fine mesh of 6834 plate elements, 380 joint elements, 760 interface elements (two for each joint element), and 87 bar elements representing the stringer reinforcement. The mesh can be seen in Fig. 2 which shows the failure modes for the three load cases. The analysis yields a load factor λ of 5.05 for load case 1, 3.45 for load case 2, and 6.23 for load case 3. Load case 2 features large horizontal loads combined with low vertical loads which makes it the critical load case.

For load case 1 and 3, the failure occurs in the bottom level of precast panels; a diagonal yield line is formed through the three panels and the top part of the wall simply starts rotating. Sliding and separation is also observed in the joint above level 1, but this behaviour is more pronounced for load case 3, see Fig. 2(c). For the critical load case, load case 2, Fig. 2(b), a bending failure occurs: The horizontal joints have no tensile strength, hence, the structure relies entirely on the reinforcement stringer to carry the tensile force from the overturning moment. This failure mode will produce a ductile failure since the reinforcement stringer yields in tension.

Fig. 3 shows how the horizontal forces are transferred via strut action to the foundation of the shear wall. This pattern can be observed for all three load cases, but it is struts are most pronounced for load case 1 and 3, where a wide compression strut can be seen in the bottom level which is the cause of the collapse. It is also observed in Fig. 3 that stress concentrations occur near the corners of the door openings, however, neither crushing of the concrete nor yielding of the reinforcement happen in those regions.

The collapse mode and stress distribution, see Fig. 2 and 3, provide excellent tools for validating the calculations. The collapse mode will often be rather simple, hence, it can easily be validated by use of manual limit analysis, i.e. the yield line method. This is a clear advantage over general non-linear
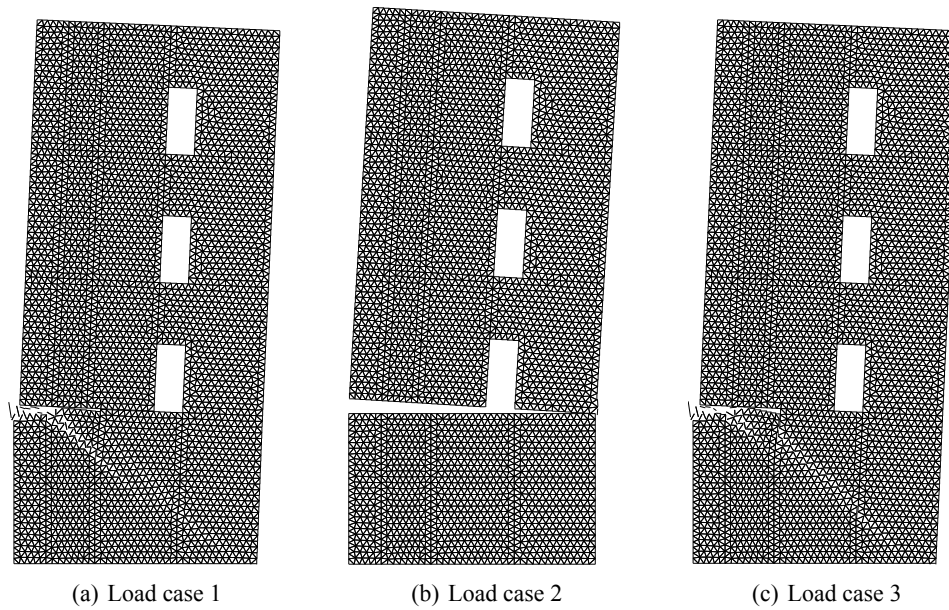
(a) Load case 1	(b) Load case 2	(c) Load case 3

Fig. 2: Collapse mode of the four-storey precast shear wall extracted from the corresponding upper bound problems.

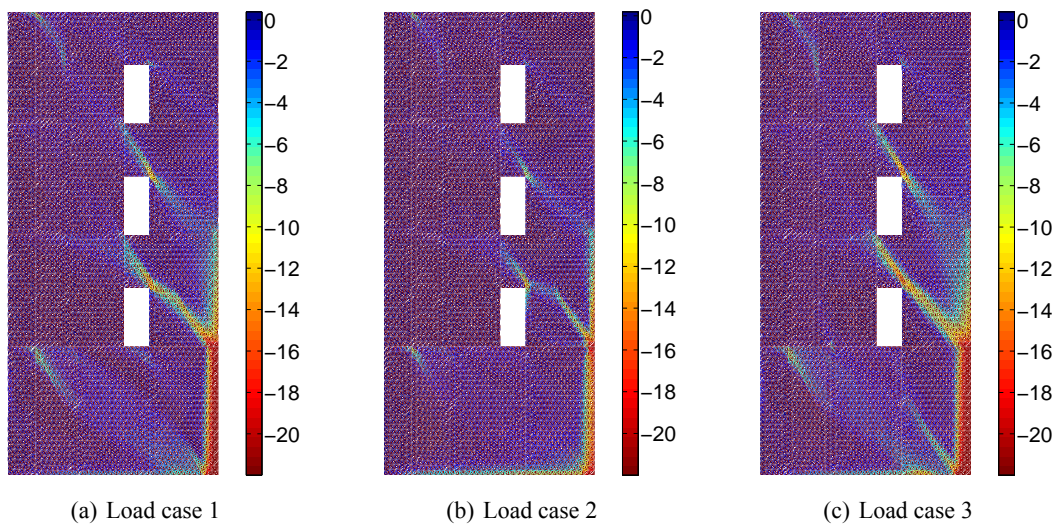|		
(a) Load case 1	(b) Load case 2	(c) Load case 3

Fig. 3: Smallest principal stress for the three load cases; the stresses are given in MPa.

finite element analysis, where the non-linear behaviour can cause non-intuitive effects and make it near impossible to validate by hand.

The models presented in this sections lead to rather large optimisation problems of approximately 600,000 variables, 612,000 linear constraints and 65,000 second-order constraints. The optimisation problems are solved in approximately 45 seconds and 50 iterations on an Lenovo T530 laptop with an Intel Core i5-3320M (4 CPUs and 2.6 GHz). This displays, once again, the strength of numerical limit analysis.

4   Future work

The example presented in the previous section clearly displays the strength of the framework; a capacity many times larger than the design load was calculated, which ultimately can lead to an optimised and cheaper structure. An important aspect is also the fact that a lower bound solution is calculated, which is preferable over upper bound solutions e.g. the yield line method. The largest potential of the method and framework, however, lies in material optimisation of three-dimensional structures.

Analysis of two-dimensional structures rather than three-dimensional is often a necessary simplification in the design process. The complexity of the model increases manifold when going from 2D to 3D, however, three-dimensional analysis of precast concrete structures is possible within the presented framework. In the current practice, the structures are typically divided into two-dimensional substructures, which necessarily leads to a suboptimal structure since the interaction between the substructures are neglected.

The joint element presented by Herfelt et al. (2016*a*) is crucial for modelling of in-situ cast joints in two-dimensional precast concrete structures. The authors are currently working on a three-dimensional counterpart, which will be capable of handling transfer of shear stresses from slabs to shear walls; a mechanism which is not considered by the current practice. Moreover, suitable finite elements are needed for modelling of slabs, shear walls, beams and so on. Several elements have already been developed as mentioned earlier, however, an adequate yield criteria are needed to represent the precast elements, e.g. hollow core slabs.

5 Conclusion

The mathematical framework and background of numerical limit analysis have been presented. The method is based on the theory of rigid-plasticity and the problems are cast as convex optimisation problems, which can be solved efficiently using interior point methods. Two versions of numerical lower bound limit analysis are presented, namely load optimisation where a load factor is optimised for a given structure, and material optimisation where the material usage is optimised for a given geometry and load case.

To display the strength of the presented framework, a four-storey shear wall is analysed for three different load cases. The wall is subjected to horizontal wind load, which is sought to be maximised, as well as vertical dead load, which is treated as constant. The analysis shows that the four-storey shear wall can sustain more than three times the design wind load for the critical load case. The collapse mode shows that the failure occurs in the horizontal joints as separation for the critical load case, while a diagonal yield line in the bottom level is observed for the two remaining load cases. The stress distribution shows that the forces are transferred to the foundations via strut actions. It also shows that stress concentrations are formed near the door openings, however, none of these are critical.

The example of the four-storey shear wall illustrates that the method and framework can provide the design engineers with an excellent tool for design and analysis of complex structures. The engineer can easily analyse the solution based on the collapse mode and stress distribution and, for simple failure modes, even verify the solution by analytical hand calculations. The presented framework also makes it possible to analyse complex precast structures in three dimensions. The necessary finite elements are currently being developed by the authors.

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