Technical University of Denmark

# Comment on “Temporal Correlations of the Running Maximum of a Brownian Trajectory”

**Mortensen, Kim; Pedersen, Jonas Nyvold**

*Published in:*
Physical Review Letters

*Link to article, DOI:*
10.1103/PhysRevLett.117.248901

*Publication date:*
2016

*Document Version*
Publisher's PDF, also known as Version of record

## Comment on "Temporal Correlations of the Running Maximum of a Brownian Trajectory"

Bénichou *et al.* [1] use the running maximum (RM) position in a single experimental trajectory of a particle exhibiting 1D Brownian motion (BM) to estimate its diffusion coefficient. This is unreliable: While the estimator's precision (reproducibility) increases with the suggested parameter tuning, so does its *inaccuracy* (bias), as increasing emphasis is put on the RM's maximum value.

In the mathematical idealization for BM used in Ref. [1], $B_t$ is the position of a particle diffusing with coefficient $D$. However, $B_t = \sqrt{2D}W_t$, where $W_t$ is the Wiener process. In this model, BM is a scale-free process.

Experimentally, one samples positions $x_{i=1,\ldots,N}$ at time points $t_{i=1,\ldots,N}$ [1]. Typically, constant time lapse $\Delta t$ is used, such that $t_i = i\Delta t$ and $T = N\Delta t$. For BM, measured positions relate as $x_{i+1} = x_i + \sqrt{2D}\eta_i$, where $\eta_i = W_{t_{i+1}} - W_{t_i}$ is a Gaussian white noise with $\langle \eta_i \rangle = 0$ and $\langle \eta_i \eta_j \rangle = \Delta t \delta_{i,j}$ for all $i$, $j$. Each of the $N-1$ displacements $\Delta x_i = x_{i+1} - x_i$ contains information about $D$; hence, variances of estimators in this discrete case are limited by $N$, not $T$, due to the scale invariance of BM.

A reasonable estimator $\hat{D}$ for $D$ should (i) be unbiased, i.e., $\langle \hat{D} \rangle = D$, and (ii) have a variance that decreases as $1/N$, for sufficiently large but practically relevant $N$. The discretized version $\hat{D}_{\text{msd}}^{(N)}$ of $D_{\text{msd}}$ [1] with $\tau = \Delta t$, i.e., $\hat{D}_{\text{msd}}^{(N)} = \sum_{i=1}^{N-1} (\Delta x_i)^2 / [2(N-1)\Delta t]$, complies with (i) and (ii) for $N \geq 2$ in the present case of instantaneous recording of positions and in the absence of measurement noise. It is even optimal: It achieves the Cramér-Rao lower bound [2,3] and thus has the lowest possible variance among unbiased estimators.

With discrete sampling, the RM is $M_i = \max_{j=1,\ldots,i} x_j$, and thus the RM-based estimator of Ref. [1] must read $\hat{D}_{\text{es}}^{(N,k)} = [C(k) \sum_{i=1}^{N} M_i^k]^{2/k}$, with $C(k) \equiv ([\Delta t \sqrt{\pi}(k/2+1)] / \{2^k \Gamma[(k+1)/2] T^{k/2+1}\})$ and $k > 0$. As a function of $N$, the information available to $\hat{D}_{\text{es}}^{(N,k)}$ increases so slowly that its variance approaches a constant value [1]. This is in conflict with (ii). The variance can be made arbitrarily small, however, by increasing $k$ [1]; thus it is argued that $\hat{D}_{\text{es}}^{(N,k)}$ is superior to $\hat{D}_{\text{msd}}^{(N)}$ for small $T$ [1].

Application of both estimators to Monte Carlo (MC) simulated BM shows, however, that the estimates of $\hat{D}_{\text{msd}}^{(N)}$ scatter with a normal distribution around $D$, while the estimates of $\hat{D}_{\text{es}}^{(N,k)}$ are skewed [Figs. 1(a) and 1(b)]. This results in a bias, $\langle \hat{D}_{\text{es}}^{(N,k)} \rangle \neq D$, which is in conflict with (i). The bias becomes worse with increasing $k$ [Fig. 1(c)], while the variance indeed decreases [Fig. 1(d)]. The bias of $\hat{D}_{\text{es}}^{(N,k)}$ vanishes too slowly with $N$ to ensure any practical relevance of $\hat{D}_{\text{es}}^{(N,k)}$ relative to $\hat{D}_{\text{msd}}^{(N)}$ [Figs. 1(c) and 1(d)].

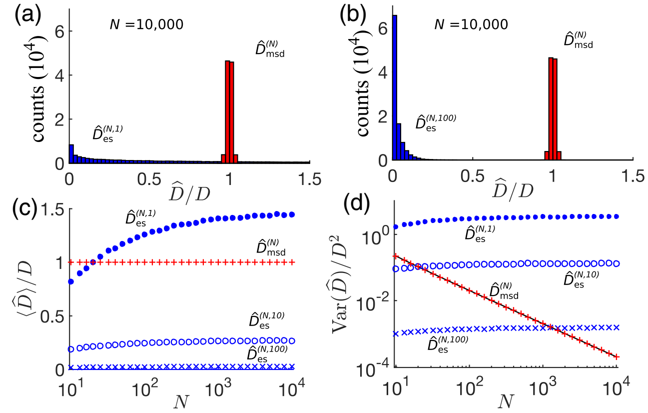


FIG. 1. (a) Histograms of estimates obtained from application of, respectively, $\hat{D}_{\text{msd}}^{(N)}$ (red) and $\hat{D}_{\text{es}}^{(N,k)}$ with $k=1$ (blue) to $10^5$ MC simulated, discretely sampled BM trajectories using $D = 0.25$, $\Delta t = 1$, and $N = 10^4$. (b) The same as (a) for $k = 100$. (c) Mean values of estimates obtained as in (a) for various values of $N$. Results are shown for $\hat{D}_{\text{msd}}^{(N)}$ (pluses) and $\hat{D}_{\text{es}}^{(N,k)}$ with, respectively, $k$ values of 1 (full circles), 10 (open circles), and 100 (crosses). (d) The same as (c) for the variances of the estimates. The theoretical variance $2D^2/(N-1)$ for $\hat{D}_{\text{msd}}^{(N)}$, the Cramér-Rao lower bound, is indicated (full line).

In summary, the estimator suggested by Bénichou *et al.* [1] unfortunately yields biased values for the diffusion coefficient, while optimal, plug-and-play alternatives already exist [2,3].

The Danish Council for Strategic Research (Grant No. 10-092322/DSF) and the European Union's Horizon 2020 research and innovation programme (Grant No. 696656) are acknowledged.

Kim I. Mortensen* and Jonas N. Pedersen†
Department of Micro-and Nanotechnology
Technical University of Denmark
DK-2800 Kongens Lyngby, Denmark

Received 25 August 2016; published 6 December 2016
DOI: 10.1103/PhysRevLett.117.248901

*kim.mortensen@nanotech.dtu.dk
†jonas.pedersen@nanotech.dtu.dk

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