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Published in:
S I A M Journal on Discrete Mathematics

Link to article, DOI:
10.1137/15M1040785

Publication date:
2016

Document Version
Peer reviewed version

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Chromatic roots and minor-closed families of graphs

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January 20, 2016

Abstract

Given a minor-closed class of graphs $\mathcal{G}$, what is the infimum of the non-trivial roots of the chromatic polynomial of $G \in \mathcal{G}$? When $\mathcal{G}$ is the class of all graphs, the answer is known to be 32/27. We answer this question exactly for three minor-closed classes of graphs. Furthermore, we conjecture precisely when the value is larger than 32/27.

1   Introduction

The *chromatic polynomial* $P(G, t)$ of a graph G is a polynomial which counts, for each non-negative integer t, the number of proper t-colourings of G. It was introduced by Birkhoff [1] in 1912 for planar graphs and extended to all graphs by Whitney [12, 13] in 1932. More recently, several results have been obtained on the distribution of the real and complex roots of the chromatic polynomial, see for example the survey article [6].

We say that a real number t is a *chromatic root* of a graph G if $P(G, t) = 0$. Since 0 and 1 are chromatic roots of any graph with at least one edge, we say these are *trivial*. Tutte [11] proved that the intervals $(-\infty, 0)$ and $(0, 1)$ contain no chromatic root of any graph, so all non-trivial chromatic roots are greater than 1.

For a class of graphs $\mathcal{G}$, define $\omega(\mathcal{G})$ to be the infimum of the non-trivial chromatic roots of $G \in \mathcal{G}$. We define $\omega(\mathcal{G}) = 2$ if the infimum does not exist. Thus for a class of

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Forbidden minor	$\omega(\mathcal{G})$
K_3	2
$K_4 - e$	2
$K_{2,3}$	2
K_4	32/27
$K_{2,4}$	$\beta \approx 1.43$, chromatic root of $K_{2,3}$
$K_{2,r}$	$32/27 + \varepsilon(r)$

Figure 1: Results of Dong and Koh [4].

graphs $\mathcal{G}$, the interval $(1, \omega(\mathcal{G}))$ contains no chromatic roots of $G \in \mathcal{G}$, and the endpoint $\omega(\mathcal{G})$ can be included if the infimum is not attained. Motivated by what was then the 4-colour conjecture, Birkhoff and Lewis [2] showed that $\omega(\mathcal{G}) = 2$ when $\mathcal{G}$ is the class of planar triangulations. However, since bipartite graphs with an odd number of vertices have a chromatic root in $(1, 2)$, the problem of determining $\omega(\mathcal{G})$ when $\mathcal{G}$ consists of all graphs was open until Jackson proved the following surprising result.

Theorem 1.1. [5] *If $\mathcal{G}$ is the class of all graphs, then $\omega(\mathcal{G}) = 32/27$.*

In the proof of Theorem 1.1, Jackson introduced a class of graphs $\mathcal{K}$, whose elements are called *generalised triangles*, and showed that a sequence of graphs in $\mathcal{K}$ have chromatic roots converging to $32/27$ from above. Later, Thomassen [9] strengthened this by showing that chromatic roots are dense in $(32/27, \infty)$. Since the graphs known to have chromatic roots close to $32/27$ have a very particular structure, it is natural to ask if $\omega(\mathcal{G}) > 32/27$ holds for restricted classes of graphs. This has been studied by several authors, for example Thomassen proved the following.

Theorem 1.2. [10] *If $\mathcal{H}$ is the class of graphs with a Hamiltonian path, then $\omega(\mathcal{H}) = t_0$, where $t_0 \approx 1.296$ is the unique real root of the polynomial $t^3 - 2t^2 + 4t - 4$.*

Dong and Koh suggested the problem of determining $\omega(\mathcal{G})$ for minor-closed classes of graphs. They proved the following theorem, which implies that one need only investigate the graphs in $\mathcal{G}$ which are generalised triangles.

Theorem 1.3. [4] *If $\mathcal{G}$ is a minor-closed class of graphs, then $\omega(\mathcal{G}) = \omega(\mathcal{G} \cap \mathcal{K})$.*

Using Theorem 1.3, they determined the value of $\omega(\mathcal{G})$ for several classes of graphs characterised by forbidding a particular graph as a minor. Their results are summarised by the table in Figure 1.
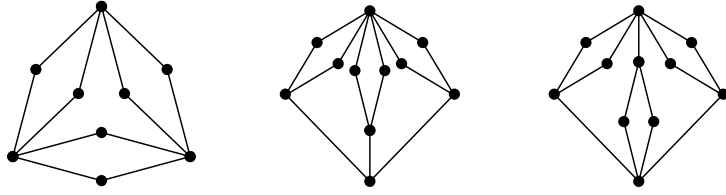
Figure 2: From left to right, the graphs H_0, H_1 and H_2.

In this paper we prove that certain subsets $\mathcal{K}' \subseteq \mathcal{K}$ can be considered minor-closed within $\mathcal{K}$, in the sense that there is a minor-closed class of graphs $\mathcal{G}$ such that $\mathcal{G} \cap \mathcal{K} = \mathcal{K}'$. Using Theorem 1.3, we have $\omega(\mathcal{G}) = \omega(\mathcal{G} \cap \mathcal{K}) = \omega(\mathcal{K}')$, so determining $\omega(\mathcal{K}')$ gives the value $\omega(\mathcal{G})$ for the much larger class $\mathcal{G}$. To illustrate this new technique, we analyse three natural subfamilies of generalised triangles and precisely determine $\omega(\mathcal{G})$ for three minor-closed families of graphs.

Theorem 1.4. *Let H_0, H_1 and H_2 be the graphs in Figure 2.*

(i) *If $\mathcal{G}$ is the class of H_0-minor-free graphs, then $\omega(\mathcal{G}) = 5/4$.*

(ii) *If $\mathcal{G}$ is the class of $\{H_1, H_2\}$-minor-free graphs, then $\omega(\mathcal{G}) = q$, where $q \approx 1.225$ is the real root of $t^4 - 4t^3 + 4t^2 - 4t + 4$ in $(1, 2)$.*

(iii) *If $\mathcal{G}$ is the class of $\{H_0, H_1, H_2\}$-minor-free graphs, then $\omega(\mathcal{G}) = t_0$, where $t_0 \approx 1.296$ is the unique real root of $t^3 - 2t^2 + 4t - 4$.*

For all previously investigated minor-closed classes $\mathcal{G}$, the intersection $\mathcal{G} \cap \mathcal{K}$ is either finite or equal to $\mathcal{K}$. In such cases it is easy to determine $\omega(\mathcal{G})$: If $\mathcal{G} \cap \mathcal{K} = \mathcal{K}$ then $\omega(\mathcal{G}) = 32/27$, while if $\mathcal{G} \cap \mathcal{K}$ is finite, then $\omega(\mathcal{G})$ is the minimum of the non-trivial roots of $G \in \mathcal{G} \cap \mathcal{K}$, which is a finite problem. In contrast to this, each class of graphs $\mathcal{G}$ in the statement of Theorem 1.4 has the property that $\mathcal{G} \cap \mathcal{K}$ is an infinite proper subset of $\mathcal{K}$. Thus, we answer Dong and Koh's question in the first three non-trivial cases. Since the graphs H_0, H_1 and H_2 are some of the smallest generalised triangles, our results also provide evidence for the following conjecture.

Conjecture 1.1. *If $\mathcal{G}$ is a minor-closed class of graphs, then $\omega(\mathcal{G}) > 32/27$ if and only if $\mathcal{G}$ does not contain all generalised triangles.*

Finally, the intervals we find coincide with those obtained or conjectured for other, seemingly unrelated, families of graphs. Notice for example that the interval in Theorem 1.4(iii) is the same as that of Theorem 1.2. This connection will be fully explained.

Furthermore, the intervals in parts (i) and (ii) of Theorem 1.4 coincide precisely with those in important conjectures of Dong and Jackson. These conjectures would have implications for the chromatic roots of 3-connected graphs, about which very little is currently known. We describe how our results suggest that it might be fruitful to attack a relaxed version of these conjectures.

The structure of the paper is as follows. In Section 2 we make several observations about generalised triangles and minors. We apply these results in Section 3 to obtain Theorem 1.4. Two lengthier proofs are deferred to Section 4.

2   Generalised triangles and minors

All graphs in this paper are finite and simple, that is they have no loops or multiple edges. A *cut-set* S of a graph G is a set of vertices whose removal increases the number of components of G. If $S = \{u\}$, then we say u is a *cut-vertex*. If $|S| = 2$, then we refer to S as a *2-cut*. Suppose G is a 2-connected graph and $\{x, y\}$ is a 2-cut of G. Let C be a connected component of $G - \{x, y\}$ and $B = G[V(C) \cup \{x, y\}]$. We say that B is an $\{x, y\}$-*bridge* of G. If $|V(B)| = 3$, then we say B is *trivial*.

Jackson [5] defined the following operation on a graph G called *double subdivision*: choose an edge uv of G and construct a new graph from $G - uv$ by adding two new vertices and joining both of them to u and v. A *generalised triangle* is either K_3 or any graph which can be obtained from K_3 by a sequence of double subdivisions. We denote the class of generalised triangles by $\mathcal{K}$.

We shall require the following properties of generalised triangles which were given by Dong and Koh, see also [5].

Proposition 2.1. [4] *A graph G is a generalised triangle if and only if it satisfies both of the following conditions.*

(i) *G is 2-connected but not 3-connected.*

(ii) *For every 2-cut $\{x, y\}$, we have $xy \notin E(G)$ and there are precisely three $\{x, y\}$-bridges, none of which is 2-connected.*

Additionally, the following observations will be useful.

Proposition 2.2. *Suppose that G is a generalised triangle, $\{x, y\}$ is a 2-cut of G, and B is an $\{x, y\}$-bridge of G. The following hold.*

(i) *$B + xy$ is a generalised triangle.*

(ii) If B is non-trivial, then there is a 2-cut $\{u, v\} \subseteq V(B)$, such that the two $\{u, v\}$-bridges of G which are contained in B are trivial.

Proof. To verify (i), it is easy to check that the conditions in Proposition 2.1 hold for the graph $B + xy$. To prove part (ii), let z be a vertex of G not in B. Choose a 2-cut $\{u, v\} \subseteq V(B)$ such that the $\{u, v\}$-bridge of G containing z has as many vertices as possible. Suppose some $\{u, v\}$-bridge B' contained in B is not trivial. Proposition 2.1 implies that B' has a cut-vertex w. Since B' is not trivial, one of $\{u, w\}$ and $\{v, w\}$ is a 2-cut of G and the bridge of this 2-cut containing z is larger, a contradiction. □

The double subdivision operation defines a partial order on the class of generalised triangles. More precisely, for $G, H \in \mathcal{K}$ we define $H \leq G$ if G can be obtained from H by a sequence of double subdivisions. The key observation of this paper is that the minor operation gives rise to the same partial order on $\mathcal{K}$.

Lemma 2.1. *If $G, H \in \mathcal{K}$, then H is a minor of G if and only if G can be obtained from H by a sequence of double subdivisions.*

Proof. If G can be obtained from H by a sequence of double subdivisions, then clearly H is a minor of G. To prove the forward implication we proceed by induction on $|V(H)|$. If $|V(H)| = 3$, then $H = K_3$ and the result follows from the definition of $\mathcal{K}$. So suppose $|V(H)| > 3$ and the result holds for all generalised triangles on fewer vertices. Let $G \in \mathcal{K}$ such that H is a minor of G. Since $G \in \mathcal{K}$, we may fix a sequence of graphs $G_0, G_1, \ldots, G_r$, such that $G_0 = K_3$, $G_r = G$, and for each $i \in \{1, \ldots, r\}$, G_i is obtained from G_{i-1} by a double subdivision operation. Let $i \in \{1, \ldots, r\}$ be minimal so that G_i has an H-minor, but G_{i-1} does not. It suffices to show that G_i can be obtained from H by a sequence of double subdivisions.

Let uv be the edge of G_{i-1} which is double subdivided to form G_i, and let $x, y \in V(G_i)$ be the new vertices created. Also, let B be the $\{u, v\}$-bridge of G_i not containing x or y. Since G_i is a generalised triangle, Proposition 2.1 implies that B has a cut-vertex w which separates u from v. We let L_u and L_v denote the blocks of B containing u and v respectively, see Figure 3. Finally, let J be a fixed H-minor of G_i.

Claim: $ux, xv, vy, yu \in E(J)$.

Since H is 2-connected and a minor of G_i but not G_{i-1}, the vertices u and v are not identified to form J. Furthermore, at least one of x and y, say x, has neither of its adjacent edges deleted or contracted. Since $|V(H)| > 3$, we have that $\{u, v\}$ is a 2-cut of J with precisely three $\{u, v\}$-bridges B_1^J, B_2^J and B_3^J, one of which, say B_1^J, is the path uxv. It remains to show that one of B_2^J or B_3^J is the path uyv, so suppose for a
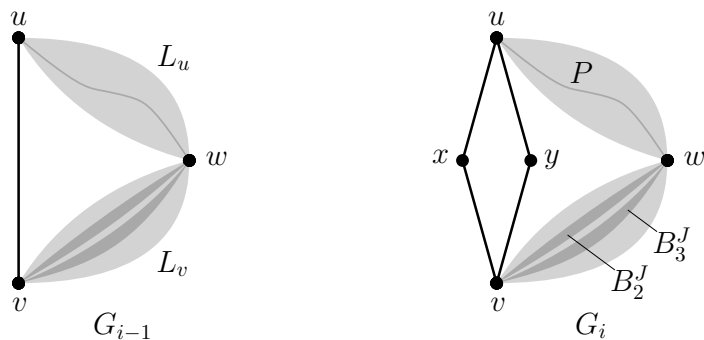
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Figure 3: The structure of G_{i-1} and G_i in the proof of Lemma 2.1.

contradiction that this is not the case, and that $B_2^J \cup B_3^J$ is a minor of B. Every path in B from u to v must go through w. However, since B_2^J and B_3^J are distinct $\{u, v\}$-bridges, they each contain a path from u to v, and these paths are internally disjoint. It follows that to form J, the vertex w must be identified with either u or v, say u. In fact, since H is 2-connected, J is a minor of the graph formed from G_i by contracting the whole of L_u to a single vertex. Thus $B_2^J \cup B_3^J$ is a minor of L_v, see Figure 3. Now let P be a path from u to w in L_u. Since P has at least one edge and B_1^J is a trivial bridge, it follows that B_1^J is a minor of the graph $P + uv$. But now J is a minor of the graph $L_v \cup P + uv$, which is a subgraph of G_{i-1}. This contradicts the fact that H is not a minor of G_{i-1}, completing the proof of the claim.

Define J' to be the graph formed from J by deleting x and y and adding the edge uv. Note that J is formed from J' by applying the double subdivision operation to uv and Proposition 2.2(i) implies that J' is a generalised triangle. Clearly G_{i-1} contains J' as a minor. By induction, G_{i-1} can be formed from J' by a sequence of double subdivisions. That is, there exists a sequence of graphs $J'_0, J'_1, \ldots, J'_s$ such that $J'_0 = J'$, $J'_s = G_{i-1}$ and for each $i \in \{1, \ldots, s\}$, J'_i is obtained from J'_{i-1} by a double subdivision operation. If, for some $i \in \{1, \ldots, s\}$, the edge $uv \in E(J'_{i-1})$ is double subdivided to form J'_i, then G_{i-1} contains H as a minor, a contradiction. Thus the double subdivision operation is never applied to the edge uv. For $i \in \{0, \ldots, s\}$, let J_i be the graph obtained from J'_i by applying the double subdivision operation to uv. Then $J_0 = J$, $J_s = G_i$ and for $i \in \{1, \ldots, s\}$, J_i is obtained from J_{i-1} by a double subdivision operation. Thus G_i can be obtained from H by a sequence of double subdivisions as required. $\square$

Let $\mathcal{A} \subseteq \mathcal{K}$. We say $\mathcal{A}$ is a *downward-closed* subset of $\mathcal{K}$ if for all $G \in \mathcal{A}$ and $H \in \mathcal{K}$, we have that $H \leq G$ implies $H \in \mathcal{A}$. By Lemma 2.1, such subsets behave as minor-closed classes within $\mathcal{K}$, and so have a forbidden minor characterisation within $\mathcal{K}$. This is made

precise in the following lemma.

Lemma 2.2. *Let $G \in \mathcal{K}$, and suppose $\mathcal{K}'$ is a downward-closed subset of $\mathcal{K}$. If $\mathcal{F} = \mathcal{F}(\mathcal{K}')$ is the set of minimal elements of $\mathcal{K} \setminus \mathcal{K}'$, then $G \in \mathcal{K}'$ if and only if G is $\mathcal{F}$-minor-free.*

In practice, if $\mathcal{K}'$ is a class of generalised triangles defined by some graph property, then $\mathcal{K}'$ is frequently downwards-closed. It is often possible to exploit this and determine the value of $\omega(\mathcal{K}')$. When combined with the observations above, one obtains the value of $\omega(\mathcal{G})$ for a much larger class $\mathcal{G}$.

Theorem 2.1. *Let $\mathcal{K}'$ be a downward-closed subset of $\mathcal{K}$ and let $\mathcal{F} = \mathcal{F}(\mathcal{K}')$ be the set of minimal elements of $\mathcal{K} \setminus \mathcal{K}'$. If $\mathcal{G}$ is the class of $\mathcal{F}$-minor-free graphs, then $\omega(\mathcal{G}) = \omega(\mathcal{K}')$.*

Proof. Since $\mathcal{G}$ is minor-closed, Theorem 1.3 gives that $\omega(\mathcal{G}) = \omega(\mathcal{G} \cap \mathcal{K})$. By Lemma 2.2 and the definition of $\mathcal{F}$, we have $\mathcal{G} \cap \mathcal{K} = \mathcal{K}'$, whence $\omega(\mathcal{G}) = \omega(\mathcal{G} \cap \mathcal{K}) = \omega(\mathcal{K}')$ as claimed. $\square$

3   Restricted families of generalised triangles

In this section we apply the the method described in Theorem 2.1 to obtain the value of $\omega(\mathcal{G})$ for three minor-closed classes of graphs. We do this by investigating classes of generalised triangles defined by properties of their 2-cuts.

Definition 3.1. *Let G be a graph.*

1. *A 2-cut $\{x, y\}$ of G has property P_1 if for every $\{x, y\}$-bridge B, at least one of x and y has degree 1 in B.*

2. *A 2-cut $\{x, y\}$ of G has property P_2 if at least one $\{x, y\}$-bridge is trivial.*

For $i \in \{1, 2\}$, define $\mathcal{K}_i$ to be the family of generalised triangles satisfying property P_i at every 2-cut.

Lemma 3.1. *$\mathcal{K}_1$ and $\mathcal{K}_2$ are downward-closed subsets of $\mathcal{K}$.*

Sketch of proof. Let $i \in \{1, 2\}$. It suffices to show that if $G \in \mathcal{K}_i$ and G is formed from G' by a single double subdivision, then $G' \in \mathcal{K}_i$. The contrapositive of this statement is much easier to see. Indeed if $G' \notin \mathcal{K}_i$, then there is some 2-cut $\{x, y\}$ of G' which does not satisfy property P_i. The same vertices form a 2-cut of G which does not satisfy property P_i. $\square$

3.1 The family $\mathcal{K}_1$

The aim of this section is to prove Theorem 1.4(i). To do this we must show that $\omega(\mathcal{K}_1) = 5/4$, and that $\{H_0\}$ is the family of forbidden minors which characterises the class $\mathcal{K}_1$ within $\mathcal{K}$. The result then follows from Theorem 2.1.

Lemma 3.2. *If $G \in \mathcal{K}_1$, then $(-1)^{|V(G)|}P(G,t) > 0$ for $t \in (1, 5/4]$.*

The proof of this lemma is simple but fairly lengthy and can be found in Section 4.1. The idea is to prove several inequalities simultaneously by induction, one of which is the statement above.

Lemma 3.3. $\omega(\mathcal{K}_1) = 5/4$.

Proof. Let $J_0 = K_3$ and $x \in V(J_0)$. For $i \in \mathbb{N}$, let J_i be obtained from J_{i-1} by applying the double subdivision operation to each edge of J_{i-1} incident with x. Dong and Jackson [3] say the graphs in this sequence have chromatic roots converging to $5/4$ from above. We shall show that $J_i \in \mathcal{K}_1$ for each $i \in \mathbb{N}_0$. It then follows that $\omega(\mathcal{K}_1) \leq 5/4$, which together with Lemma 3.2 implies that $\omega(\mathcal{K}_1) = 5/4$.

Let $i \in \mathbb{N}_0$ and note that, by construction, every 2-cut of J_i contains the vertex x. Consider a 2-cut $\{x, y\}$ and let B be an $\{x, y\}$-bridge of J_i. Since J_i is a generalised triangle, Proposition 2.1(ii) gives that B has a cut-vertex z which separates x from y. If y has degree at least 2 in B, then $\{y, z\}$ is a 2-cut of G, contradicting the fact that each 2-cut contains x. Thus, for each 2-cut $\{x, y\}$, the vertex y has degree 1 in each $\{x, y\}$-bridge. We conclude that each 2-cut has property P_1, so $J_i \in \mathcal{K}_1$ as desired. □

Recall that H_0 is the graph depicted in Figure 2. It is formed from K_3 by applying the double subdivision operation to each edge.

Lemma 3.4. *If $G \in \mathcal{K}$, then $G \in \mathcal{K}_1$ if and only if G is H_0-minor-free.*

Proof. By Lemma 3.1, $\mathcal{K}_1$ is a downward-closed subset of $\mathcal{K}$. Thus, by Lemma 2.2, we need only determine that $\{H_0\}$ is the subset $\mathcal{F}$ of minimal elements of $\mathcal{K} \setminus \mathcal{K}_1$. So suppose $H \in \mathcal{F}$. Since $H \notin \mathcal{K}_1$, there is some 2-cut $\{x, y\}$ with $\{x, y\}$-bridges B_1, B_2 and B_3 such that both x and y have degree at least 2 in B_1 say. We claim that the bridges B_2 and B_3 are trivial, so suppose for a contradiction that B_2, say, is not. By Proposition 2.2(ii), we can find a 2-cut $\{u, v\} \subseteq V(B_2)$, such that the two $\{u, v\}$-bridges contained in B_2 are trivial. By Proposition 2.2(i), replacing these two bridges by a single edge uv yields a generalised triangle H', and H can be obtained from H' by a double subdivision. Because

of B_1, the 2-cut $\{x, y\}$ also does not satisfy P_1 in H', so $H' \in \mathcal{K} \setminus \mathcal{K}_1$, which contradicts the minimality of H. Thus B_2 and B_3 are trivial.

Now B_1 has more than three vertices and as such has a cut-vertex z. Since both x and y have degree at least 2 in B_1, both $\{x, z\}$ and $\{y, z\}$ are 2-cuts of H. Let X_1, X_2 be the $\{x, z\}$-bridges not containing y, and let Y_1, Y_2 be the $\{y, z\}$-bridges not containing x. Since H is minimal, the same reasoning as above implies that X_1, X_2, Y_1, Y_2 are trivial bridges. We conclude that $H = H_0$ and thus $\mathcal{F} = \{H_0\}$. $\square$

Let $\mathcal{G}_1$ be the class of graphs such that some vertex is contained in every 2-cut. In [3] Dong and Jackson conjecture that $\omega(\mathcal{G}_1) = 5/4$. This conjecture is important, since $\mathcal{G}_1$ contains the class of 3-connected graphs, and so a positive solution would give a lower bound on the non-trivial roots of 3-connected graphs. While it can be shown that $\mathcal{G}_1 \cap \mathcal{K} \subset \mathcal{K}_1$, this does not prove the conjecture since it is not known if $\omega(\mathcal{G}_1) = \omega(\mathcal{G}_1 \cap \mathcal{K})$. In particular $\mathcal{G}_1$ is not minor-closed so Theorem 1.3 does not apply.

The fact that $\mathcal{G}_1 \cap \mathcal{K}$ is not the largest class of generalised triangles $\mathcal{K}'$ such that $\omega(\mathcal{K}') = 5/4$ suggests that a well chosen weaker property could be used to make progress on Dong and Jackson's conjecture.

Problem 3.1. *Find a class of graphs $\mathcal{G}$ such that $\mathcal{G}_1 \subseteq \mathcal{G}$, $\omega(\mathcal{G}) = \omega(\mathcal{G} \cap \mathcal{K})$ and $\mathcal{G} \cap \mathcal{K} = \mathcal{K}_1$.*

3.2 The family $\mathcal{K}_2$

In this section we prove Theorem 1.4(ii). To do this we must show that $\omega(\mathcal{K}_2) = q$, where $q \approx 1.225$ is the unique real root of the polynomial $t^4 - 4t^3 + 4t^2 - 4t + 4$ in the interval $(1, 2)$. We must also show that $\{H_1, H_2\}$ is the family of forbidden minors which characterises the class $\mathcal{K}_2$ within $\mathcal{K}$. The result then follows from Theorem 2.1.

Lemma 3.5. *If $G \in \mathcal{K}_2$, then $(-1)^{|V(G)|} P(G, t) > 0$ for $t \in (1, q]$.*

The proof of this lemma is also simple but fairly lengthy and can be found in Section 4.2. The idea is the same as that of Lemma 3.2.

Lemma 3.6. $\omega(\mathcal{K}_2) = q$.

Proof. Define $J_0 = K_3$ and consider an embedding of J_0 in the plane. For $i \in \mathbb{N}$, let J_i be formed from J_{i-1} by applying the double subdivision operation to each edge of J_{i-1} on the outer face. In [3], Dong and Jackson say this sequence of graphs has chromatic

roots converging to q from above. The following claim implies that $J_i \in \mathcal{K}_2$ for $i \in \mathbb{N}_0$. It then follows that $\omega(\mathcal{K}_2) \leq q$, which together with Lemma 3.5 gives that $\omega(\mathcal{K}_2) = q$.

Claim: For $i \in \mathbb{N}_0$, every 2-cut of J_i has a trivial bridge which lies inside the outer cycle of J_i.

We prove the claim by induction on i. The result holds vacuously for $i = 0$ and is easily checked for $i = 1$, so suppose the result is true for $k \in \mathbb{N}_0$. For the induction step, note that the 2-cuts of J_{k+1} consist of the 2-cuts of J_k, and $\{u, v\}$ for every edge uv of the outer cycle of J_k. If $\{x, y\}$ is a 2-cut of J_k, then by the induction hypothesis there is a trivial $\{x, y\}$-bridge which lies inside the outer cycle of J_k. This bridge is left unchanged in J_{k+1} so $\{x, y\}$ still satisfies the hypothesis. Alternatively, if uv is an edge of the outer cycle of J_k, then in J_{k+1}, the edge uv is replaced with two trivial $\{u, v\}$-bridges. One of these bridges forms part of the outer cycle of J_{k+1}, whilst the other bridge lies inside the new outer cycle as required. $\qquad\square$

Let $H = K_{2,3}$ with vertex partition $\{\{x, y\}, \{u, v, w\}\}$. The graph H_1 is formed from H by applying the double subdivision operation to each edge adjacent to x. The graph H_2 is formed from H by applying the double subdivision operation to the edges xu, xv and yw, see Figure 2.

Lemma 3.7. *If $G \in \mathcal{K}$, then $G \in \mathcal{K}_2$ if and only if G is $\{H_1, H_2\}$-minor-free.*

Proof. By Lemma 3.1, $\mathcal{K}_2$ is a downward-closed subset of $\mathcal{K}$. Thus by Lemma 2.2 we need only determine that $\mathcal{F}(\mathcal{K}_2) = \{H_1, H_2\}$. To this end, let G be a minimal element of $\mathcal{K} \setminus \mathcal{K}_2$. Since $G \notin \mathcal{K}_2$, there is a 2-cut $\{x, y\}$ of G with $\{x, y\}$-bridges B_1, B_2, and B_3, none of which is trivial. Suppose that, for some $i \in \{1, 2, 3\}$, the bridge B_i has more than five vertices. By Proposition 2.2(ii), there is a 2-cut $\{u, v\} \subseteq V(B_i)$ such that the two $\{u, v\}$-bridges contained in B_i are trivial. Let B' denote the third $\{u, v\}$-bridge, and let G' be the graph formed from B' by adding the edge uv. Note that $G' \in \mathcal{K}$ by Proposition 2.2(i). Let B'_i be the $\{x, y\}$-bridge of G' corresponding to B_i in G. The other $\{x, y\}$-bridges of G are left unchanged in G'. Since B_i has more than 5 vertices, B'_i is not trivial. Thus $G' \in \mathcal{K} \setminus \mathcal{K}_2$. Since G can be formed from G' by applying the double subdivision operation to uv, this contradicts the minimality of G. Therefore each of B_1, B_2 and B_3 has precisely five vertices and so is formed from a trivial $\{x, y\}$-bridge by precisely one double subdivision. We conclude that $G \in \{H_1, H_2\}$. $\qquad\square$

Let $\mathcal{G}_2$ be the class of 2-connected plane graphs such that every 2-cut is contained in the outer-cycle. In [3], Dong and Jackson conjecture that $\omega(\mathcal{G}_2) = q$. Once again, this is an important conjecture since $\mathcal{G}_2$ contains the class of 3-connected planar graphs. Whilst

it can be shown that $\mathcal{G}_2 \cap \mathcal{K} \subset \mathcal{K}_2$, this does not prove the conjecture since it is not known if $\omega(\mathcal{G}_2) = \omega(\mathcal{G}_2 \cap \mathcal{K})$. In particular $\mathcal{G}_2$ is not minor-closed so Theorem 1.3 does not apply.

Again, the fact that $\mathcal{G}_2 \cap \mathcal{K}$ is not the largest class of generalised triangles $\mathcal{K}'$ such that $\omega(\mathcal{K}') = q$ suggests that a well chosen weaker property could be used to make progress on Dong and Jackson's conjecture.

Problem 3.2. *Find a class of graphs $\mathcal{G}$ such that $\mathcal{G}_2 \subseteq \mathcal{G}$, $\omega(\mathcal{G}) = \omega(\mathcal{G} \cap \mathcal{K})$ and $\mathcal{G} \cap \mathcal{K} = \mathcal{K}_2$.*

3.3 The family $\mathcal{K}_1 \cap \mathcal{K}_2$

In this section we show that $\omega(\mathcal{K}_1 \cap \mathcal{K}_2) = t_0$, where $t_0 \approx 1.296$ is the unique real root of the polynomial $t^3 - 2t^2 + 4t - 4$. Theorem 1.4(iii) then follows from Lemma 3.4, Lemma 3.7 and Theorem 2.1.

We require the following proposition regarding an operation called a *Whitney 2-switch*.

Proposition 3.1. *Let G be a graph and $\{x, y\}$ be a 2-cut of G. Let C denote a component of $G - \{x, y\}$. Define G' to be the graph obtained from the disjoint union of $G - C$ and C by adding for all $z \in V(C)$ the edge xz (respectively yz) if and only if yz (respectively xz) is an edge of G. Then we have $P(G, t) = P(G', t)$.*

Sketch of proof. If $xy \in E(G)$, then apply Proposition 4.2 to the 2-cut $\{x, y\}$ in both G and G'. The resulting expressions for $P(G, t)$ and $P(G', t)$ are the same. If $xy \notin E(G)$, then first apply Proposition 4.1(ii) to the pair $\{x, y\}$ in both G and G'. Next, apply Proposition 4.2 to $G + xy, G/xy, G' + xy$ and G'/xy. Again the resulting expressions for $P(G, t)$ and $P(G', t)$ are the same. $\qquad\square$

Let $\mathcal{H}$ denote the family of graphs which have a Hamiltonian path. To prove Theorem 1.2, Thomassen implicitly proved the following lemma.

Lemma 3.8. [10] $\omega(\mathcal{H}) = \omega(\mathcal{H} \cap \mathcal{K})$.

The next two lemmas show that $\{P(H, t) : H \in \mathcal{H} \cap \mathcal{K}\} = \{P(G, t) : G \in \mathcal{K}_1 \cap \mathcal{K}_2\}$, whence $\omega(\mathcal{H} \cap \mathcal{K}) = \omega(\mathcal{K}_1 \cap \mathcal{K}_2)$. The fact that $\omega(\mathcal{K}_1 \cap \mathcal{K}_2) = t_0$ then follows from Theorem 1.2 and Lemma 3.8.

Lemma 3.9. $\mathcal{H} \cap \mathcal{K} \subseteq \mathcal{K}_1 \cap \mathcal{K}_2$.

Proof. Suppose $G \in \mathcal{H} \cap \mathcal{K}$ and let P denote the Hamiltonian path of G. If $G = K_3$ we are done so we may assume that G contains a 2-cut. Let $\{x, y\}$ be an arbitrary 2-cut of
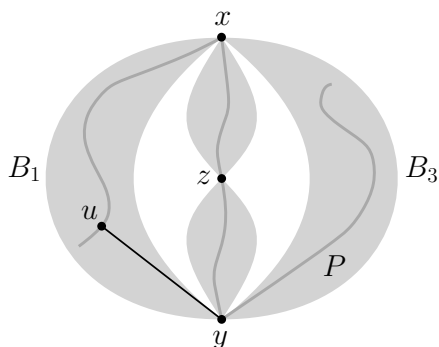
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Figure 4: The structure of a graph in $\mathcal{H} \cap \mathcal{K}$.

G. We shall show that $\{x, y\}$ has properties P_1 and P_2. Since G is a generalised triangle, there are precisely three $\{x, y\}$-bridges B_1, B_2 and B_3, none of which is 2-connected. Without loss of generality, assume that P begins in B_1, visits x before y, and ends in B_3, see Figure 4. Finally, let z be the cut-vertex of B_2.

Claim: $\{x, z\}$ and $\{y, z\}$ are not 2-cuts of G.

Suppose for a contradiction that $\{x, z\}$ is a 2-cut. Let uy be an edge from y to a vertex $u \in V(B_1)$. The graph $G' = P \cup uy$ is a spanning subgraph of G such that $G' - \{x, z\}$ has at most 2-components, see Figure 4. It follows that $G - \{x, z\}$ can have at most two components, which contradicts Proposition 2.1(ii). The situation is symmetric so the same proof shows that $\{y, z\}$ is not a 2-cut of G. This completes the proof of the claim.

Since z is a cut-vertex of B_2, the claim implies that B_2 is the path xzy, a trivial $\{x, y\}$-bridge. Thus $\{x, y\}$ has property P_2.

We now show that $\{x, y\}$ satisfies P_1, so suppose for a contradiction that this is not the case. Thus, there is an $\{x, y\}$-bridge B_1, say, such that both x and y have degree at least 2 in B_1. Since G is a generalised triangle, B_1 has a cut vertex w which separates x from y. Now, because x and y have degree at least 2 in B_1, both $\{x, w\}$ and $\{y, w\}$ are 2-cuts of G, and since G is a generalised triangle, both $\{x, w\}$ and $\{y, w\}$ have precisely three bridges. Two of the $\{x, w\}$-bridges lie in B_1, and the same is true for $\{y, w\}$. Therefore, $G - \{x, y, w\}$ has six components. However, since G has a Hamiltonian path, deleting r vertices from G can leave at most $r + 1$ components. This gives the required contradiction. $\qquad\square$

Lemma 3.10. *If $G \in \mathcal{K}_1 \cap \mathcal{K}_2$, then there is $H \in \mathcal{H} \cap \mathcal{K}$ such that $P(G, t) = P(H, t)$.*

Proof. Let $G \in \mathcal{K}_1 \cap \mathcal{K}_2$. By the characterisation of generalised triangles in Proposition 2.1, it is easy to see that $\mathcal{K}$ is invariant under Whitney 2-switches. Thus we need only prove that G can be transformed into a graph with a Hamiltonian path by a se-
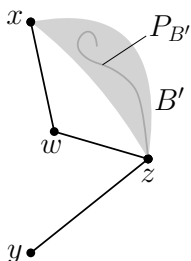
Figure 5: The $\{x, y\}$-bridge B in the proof of Lemma 3.10.

quence of Whitney 2-switches. The result clearly holds if $G = K_3$ so we may suppose that $|V(G)| > 3$. We first prove the following claim.

Claim: Let $\{x, y\}$ be a 2-cut of G and B be an $\{x, y\}$-bridge of G. If y has degree 1 in B, then there is a sequence of Whitney 2-switches in G such that in the resulting graph, the $\{x, y\}$-bridge corresponding to B contains a path P_B starting at x, and such that $V(P_B) = V(B) \setminus \{y\}$.

We proceed by induction on $|V(B)|$. If $|V(B)| = 3$, then B is trivial and the result is clear. Thus we may suppose $|V(B)| > 3$. Let z be the neighbour of y in B. Since $|V(B)| > 3$, we have that $\{x, z\}$ is a 2-cut of G with precisely three $\{x, z\}$-bridges, two of which, B' and B'', are contained in B. Since $G \in \mathcal{K}_2$, one of B' and B'' is trivial, say B'' is the path xwz, see Figure 5. Furthermore, since $G \in \mathcal{K}_1$, at least one of x and z has degree 1 in B'. If necessary, we perform a Whitney 2-switch of B' about $\{x, z\}$ so that this vertex is x, and call the resulting graph G'. Now $|V(B')| < |V(B)|$, so by induction, we have that there is a sequence of Whitney 2-switches in G' such that in the resulting graph G'', the bridge corresponding to B' contains a path $P_{B'}$, starting at z, and such that $V(P_{B'}) = V(B') \setminus \{x\}$. Now $xwz \cup P_{B'}$ is the desired path P_B in G'', see Figure 5. This completes the proof of the claim.

To prove the lemma, let $\{x, y\}$ be a 2-cut of G such that two of the $\{x, y\}$-bridges B_1 and B_2 are trivial with vertex-sets $\{x, y, u\}$ and $\{x, y, v\}$ respectively. Such a 2-cut can easily be found by considering the construction of G from a triangle by double subdivisions. Let B be the remaining $\{x, y\}$-bridge. By the claim above, there is a sequence of Whitney 2-switches in G such that the resulting graph has a path P_B starting at x and covering all vertices of B except for y. In the resulting graph, $uyvx \cup P_B$ is a Hamiltonian path. $\qquad\square$

We remark that in [7], the present author proved an analogue of Thomassen's result for a slightly more general class of graphs.

Theorem 3.1. *If $\mathcal{H}'$ is the class of graphs containing a spanning tree with at most three leaves, then $\omega(\mathcal{H}') = t_1$ where $t_1 \approx 1.290$ is the smallest real root of the polynomial $t^6 - 8t^5 + 27t^4 - 56t^3 + 82t^2 - 76t + 31$.*

To prove Theorem 3.1 it was shown that $\omega(\mathcal{H}') = \omega(\mathcal{H}' \cap \mathcal{K})$ and $\omega(\mathcal{H}' \cap \mathcal{K}) = t_1$. Using the method detailed in this section, one can similarly find a minor-closed family $\mathcal{G}$, such that $\omega(\mathcal{G} \cap \mathcal{K}) = \omega(\mathcal{H}') = t_1$. For details we refer the reader to [8].

4 Proofs of the lemmas

In this section we prove Lemma 3.2 and Lemma 3.5. The proofs are similar to Jackson's proof in [5] of the result that $\omega(\mathcal{K}) \geq 32/27$, except that the additional structure of the classes $\mathcal{K}_1$ and $\mathcal{K}_2$ allows us to make some savings and get a larger value for $\omega(\mathcal{K}_1)$ and $\omega(\mathcal{K}_2)$. The proofs are fairly long, but nevertheless rely only on the basic identities introduced in Proposition 4.1 and Proposition 4.2.

For $t \in (-\infty, 32/27]$, the sign of the chromatic polynomial of a 2-connected graph is completely determined and depends on the number of vertices, see [5]. For this reason we will find it useful to work with the function $Q(G, t) = (-1)^{|V(G)|} P(G, t)$. We shall make repeated use of equalities (i) and (ii) in Proposition 4.1 which will be referred to as *deletion-contraction* and *addition-contraction* respectively. If xy is an edge of a graph G, then $G - xy$ denotes the graph formed by deleting xy. We denote by G/xy the graph formed from G by identifying the vertices x and y, and deleting all loops and multiple edges created. In this case xy need not be an edge of the graph.

Proposition 4.1. *Let G be a graph and $x, y \in V(G)$.*

(i) If $xy \in E(G)$, then $Q(G, t) = Q(G - xy, t) + Q(G/xy, t)$.

(ii) If $xy \notin E(G)$, then $Q(G, t) = Q(G + xy, t) - Q(G/xy, t)$.

The following proposition will be used frequently for $r = 1$ and $r = 2$.

Proposition 4.2. *If G is a graph such that $G = G_1 \cup G_2$ and $G_1 \cap G_2 = K_r$ for some $r \in \mathbb{N}$, then*

$$Q(G, t) = \frac{Q(G_1, t) Q(G_2, t)}{Q(K_r, t)} = \frac{Q(G_1, t) Q(G_2, t)}{(-1)^r t(t-1) \cdots (t-r+1)}.$$

For use in the following proofs, we define a *generalised edge* to be either K_2 or any graph obtained from K_2 by a sequence of double subdivisions. When $V(K_2) = \{u, v\}$ we

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shall refer to a generalised edge obtained from this K_2 as a *generalised uv-edge*. Let G be a generalised uv-edge with $|V(G)| \geq 4$, and let B_1 and B_2 be the $\{u, v\}$-bridges of G. Recall the properties of a 2-cut defined in Definition 3.1. For $i \in \{1, 2\}$, we say G has property P_i if every 2-cut $\{x, y\}$ such that $\{x, y\} \subseteq V(B_j)$ for some $j \in \{1, 2\}$ has property P_i.

The following extra property of generalised triangles will be useful in this analysis.

Proposition 4.3. [4] *If G is a generalised triangle, then for every edge uv, $G - uv = G_1 \cup G_2$, where G_1 is a generalised uz-edge, G_2 is a generalised vz-edge, and $G_1 \cap G_2 = \{z\}$.*

4.1 Proof of Lemma 3.2

Lemma 3.2 is statement (e) in the following lemma.

Lemma 4.1. *Let G be a graph and let $t \in (1, 5/4]$.*

(a) *If $G \in \mathcal{K}_1$ and v is a vertex of degree 2 in G with neighbours u and w, then $Q(G, t) \geq \frac{1}{2}Q(G/uv, t)$.*

(b) *If G is a generalised uw-edge with property P_1 and $|V(G)| \geq 4$, then $Q(G + uw, t) \geq \frac{1}{2}Q(G, t)$.*

(c) *If $G \in \mathcal{K}_1$ and v is a vertex of degree 2 in G with neighbours u and w, then $Q(G/uv, t) > 0$.*

(d) *If G is a generalised uw-edge with property P_1 then $Q(G, t) > 0$.*

(e) *If $G \in \mathcal{K}_1$, then $Q(G, t) > 0$.*

Proof. We prove the results simultaneously by induction on $|V(G)|$. If $|V(G)| \leq 4$ then either $G = K_3$ if $G \in \mathcal{K}_1$ or $G = C_4$ if G is a generalised edge with property P_1. Thus (c), (d) and (e) are easily verified. Part (a) also holds since $Q(K_3, t) = (2 - t)Q(K_2, t) \geq \frac{3}{4}Q(K_2, t) > \frac{1}{2}Q(K_2, t)$. Finally (b) holds when $G = C_4$ since $Q(C_4 + uw, t) - \frac{1}{2}Q(C_4, t) = \frac{1}{2}t(t - 1)((t - 2)^2 - (t - 1)) > 0$. Thus we may suppose $|V(G)| > 4$ and that (a) to (e) hold for all graphs with fewer vertices.

(a) Set $H = G - v$. Note that H is a generalised uw-edge with property P_1 and $G/uv = H + uw$. By deletion-contraction and Proposition 4.2 we have

$$
\begin{aligned}
Q(G, t) &= Q(G - uv, t) + Q(G/uv, t) \\
&= (1 - t)Q(H, t) + Q(H + uw, t).
\end{aligned}
\tag{1}
$$

By the induction hypothesis of (d) on H, we have $Q(H, t) > 0$. Furthermore, by the induction hypothesis of (b) on H, we have $Q(H + uw, t) \geq \frac{1}{2}Q(H, t)$. Using the fact that $t \in (1, 5/4]$, equation (1) becomes

$$
\begin{aligned}
Q(G, t) &\geq 2(1 - t)Q(H + uw, t) + Q(H + uw, t) \\
&= (3 - 2t)Q(H + uw, t) \\
&\geq \tfrac{1}{2}Q(H + uw, t) \\
&= \tfrac{1}{2}Q(G/uv, t).
\end{aligned}
$$

(b) Let $s = Q(G+uw, t) - \frac{1}{2}Q(G, t)$. Also let H_1 and H_2 be the $\{u, w\}$-bridges of the graph $G + uw$ and note that $H_1, H_2 \in \mathcal{K}_1$. By addition-contraction and Proposition 4.2,

$$
\begin{aligned}
s &= Q(G + uw, t) - \tfrac{1}{2}[Q(G + uw, t) - Q(G/uw, t)] \\
&= \tfrac{1}{2}Q(G + uw, t) + \tfrac{1}{2}Q(G/uw, t) \\
&= \tfrac{1}{2}t^{-1}(t - 1)^{-1}Q(H_1, t)Q(H_2, t) - \tfrac{1}{2}t^{-1}Q(H_1/uw, t)Q(H_2/uw, t).
\end{aligned}
$$

By the induction hypotheses of (c) and (e), we have $Q(H_i/uw, t) > 0$ and $Q(H_i, t) > 0$ for $i \in \{1, 2\}$. Since the 2-cut $\{u, w\}$ of G has property P_1, in each of H_1 and H_2 at least one of the vertices u and w has degree 2. Therefore, for $i \in \{1, 2\}$, the induction hypothesis of (a) on the edge uw of H_i implies that $Q(H_i, t) \geq \frac{1}{2}Q(H_i/uw, t)$. Now since $t \in (1, 5/4]$,

$$
\begin{aligned}
2st(t - 1) &= Q(H_1, t)Q(H_2, t) - (t - 1)Q(H_1/uw, t)Q(H_2/uw, t) \\
&\geq Q(H_1/uw, t)Q(H_2/uw, t)[(\tfrac{1}{2})^2 - \tfrac{1}{4}] = 0.
\end{aligned}
$$

(c) Since v has degree 2, the set $\{u, w\}$ is a 2-cut of G/uv and $uw \in E(G/uv)$. Thus the $\{u, w\}$-bridges H_1 and H_2 of G/uv are members of $\mathcal{K}_1$ and so $Q(H_i, t) > 0$ for $i \in \{1, 2\}$ by the induction hypothesis of (e). Finally, since H_1 and H_2 intersect in a complete subgraph,

$$
Q(G/uv, t) = t^{-1}(t - 1)^{-1}Q(H_1, t)Q(H_2, t) > 0.
$$

(d) Let H_1 and H_2 be the uw-bridges of $G + uw$ and note that $H_1, H_2 \in \mathcal{K}_1$. By the induction hypothesis of (e), we have $Q(H_i, t) > 0$ for $i \in \{1, 2\}$. Since G is a generalised edge with property P_1, the 2-cut $\{u, w\}$ of G has property P_1 and so

in each of H_1, H_2, at least one of u or w has degree 2. Thus, by the induction hypothesis of (c), we have $Q(H_i/uw, t) > 0$ for $i \in \{1, 2\}$. Now addition-contraction and Proposition 4.2 give,

$$Q(G, t) = Q(G + uw, t) - Q(G/uw, t)$$
$$= t^{-1}(t-1)^{-1}Q(H_1, t)Q(H_2, t) + t^{-1}Q(H_1/uw, t)Q(H_2/uw, t) > 0.$$

(e) Firstly, note that (a) and (c) have now been proven for a graph on $|V(G)|$ vertices. Let v be a vertex of degree 2 with neighbours u and w. By (a), $Q(G, t) \geq \frac{1}{2}Q(G/uv, t)$, and by (c), $Q(G/uv, t) > 0$. Therefore $Q(G, t) > 0$.

$\square$

4.2 Proof of Lemma 3.5

To prove Lemma 3.5 we require a few preliminary results. Recall that $q \approx 1.225$ is the unique real root of the polynomial $t^4 - 4t^3 + 4t^2 - 4t + 4$ in $(1, 2)$ and define the constants

$$\gamma = \tfrac{1}{4}(q-2)(q^2 - 2q - 2) \approx 0.571,$$
$$\alpha = (1-\gamma)(2-q)(2-q-\gamma)^{-1} \approx 1.632,$$
$$\beta = 1 - \alpha^{-1} = \gamma(q-1)(1-\gamma)^{-1}(2-q)^{-1} \approx 0.387.$$

Lemma 4.2. *For all $t \in (1, q]$ we have*

(i) $t(t-1)^{-1}\gamma^2 - 2\gamma + 1 \geq \alpha$

(ii) $(1-t)\gamma^{-1} + 1 \geq \beta$

(iii) $(1-\gamma)(2-t)\beta - (t-1)\gamma \geq 0$.

Proof. For $t \in (1, q]$, the left hand sides of the three inequalities are decreasing functions of t. Thus we need only verify them for $t = q$. Now (i) and (ii) can be verified by lengthy substitution using the expression for γ. Part (iii) follows immediately from the definition of β. $\square$

The following useful reduction lemma is due to Jackson.

Lemma 4.3. [5] *Let G be a 2-connected graph and $\{u, v\}$ be a 2-cut such that uv is not an edge of G. If G_1 and G_2 are subgraphs of G such that $G_1 \cup G_2 = G$, $V(G_1) \cap V(G_2) =$*

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$\{u, v\}$, $|V(G_1)| \geq 3$ and $|V(G_2)| \geq 3$, then

$$t(t-1)Q(G,t) = tQ(G_1 + uv, t)Q(G_2 + uv, t)$$
$$+ (t-1)\left[Q(G_1, t)Q(G_2, t) - Q(G_1 + uv, t)Q(G_2, t) - Q(G_1, t)Q(G_2 + uv, t)\right].$$

Now Lemma 3.5 is statement (e) in the following result.

Lemma 4.4. *Let G be a graph and let $t \in (1, q]$.*

(a) *Suppose $G = G_1 \cup G_2$ where G_1 and G_2 are generalised uv-edges such that $G_1 \cap G_2 = \{u, v\}$, $|V(G_1)| \geq 4$ and $|V(G_2)| \geq 4$. If G_1 and G_2 have property P_2, then*

$$Q(G, t) \geq \alpha t^{-1} Q(G_1, t) Q(G_2, t).$$

(b) *Suppose $G = G_1 \cup G_2 + uv$ where G_1 is a generalised uw-edge, G_2 is a generalised vw-edge and $G_1 \cap G_2 = \{w\}$. If G_1 and G_2 have property P_2, then*

$$Q(G, t) \geq \beta Q(G/uv, t).$$

(c) *If G is a generalised uv-edge with property P_2 and $|V(G)| \geq 4$, then*

$$Q(G + uv, t) \geq \gamma Q(G, t).$$

(d) *Suppose $G = G_1 \cup G_2$ where G_1 and G_2 are generalised uw-edges such that $G_1 \cap G_2 = \{u, w\}$. If G_1 and G_2 have property P_2, then $Q(G, t) > 0$.*

(e) *If $G \in \mathcal{K}_2$ then $Q(G, t) > 0$.*

(f) *If G is a generalised uw-edge with property P_2 then $Q(G, t) > 0$.*

Proof. We prove the results simultaneously by induction on $|V(G)|$. If $|V(G)| \leq 4$ then either $G = K_3$ if $G \in \mathcal{K}_2$ or $G = C_4$ if G is a generalised edge with property P_2. Thus (e) and (f) are easily verified. Part (b) also holds since $Q(K_3, t) = (2 - t)Q(K_2, t) > \frac{3}{4}Q(K_2, t) > \beta Q(K_2)$. Part (c) holds when $G = C_4$ since

$$Q(C_4 + uw, t) - \gamma Q(C_4, t) = t(t-1)((1 - \gamma)(t-2)^2 - \gamma(t-1)) > 0.$$

Parts (a) and (d) are vacuously true, thus we may suppose $|V(G)| > 4$ and that (a) to (f) hold for all graphs with fewer vertices.

(a) Applying Lemma 4.3 to G and rearranging, we have

$$
\begin{aligned}
tQ(G,t) =&Q(G_1 + uv, t)\left[\tfrac{1}{2}t(t-1)^{-1}Q(G_2 + uv, t) - Q(G_2, t)\right] \\
&+Q(G_2 + uv, t)\left[\tfrac{1}{2}t(t-1)^{-1}Q(G_1 + uv, t) - Q(G_1, t)\right] \\
&+Q(G_1, t)Q(G_2, t). \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (2)
\end{aligned}
$$

By the induction hypothesis of (c), we have $Q(G_i + uv, t) \geq \gamma Q(G_i, t)$ for $i \in \{1, 2\}$. Also, by the induction hypothesis of (e), $Q(G_i, t) > 0$ for $i \in \{1, 2\}$. Substituting into (2) and using Lemma 4.2(i) now gives

$$
\begin{aligned}
tQ(G,t) &\geq Q(G_1, t)Q(G_2, t)\left[t(t-1)^{-1}\gamma^2 - 2\gamma + 1\right] \\
&\geq \alpha Q(G_1, t)Q(G_2, t).
\end{aligned}
$$

(b) If one of G_1 and G_2 is a single edge, say $G_1 = uw$, then $G/uv = G_2 + vw$. By deletion-contraction on uv and Proposition 4.2,

$$
Q(G, t) = (1 - t)Q(G_2, t) + Q(G_2 + vw, t). \qquad\qquad (3)
$$

The induction hypothesis of (f) gives that $Q(G_2, t) > 0$. Moreover the induction hypothesis of (c) on G_2 gives $Q(G_2 + vw, t) \geq \gamma Q(G_2, t)$. Substituting into (3) and using Lemma 4.2(ii) we get

$$
Q(G, t) \geq ((1 - t)\gamma^{-1} + 1)Q(G_2 + vw) \geq \beta Q(G_2 + vw, t) = \beta Q(G/uv, t). \qquad (4)
$$

So suppose that both G_1 and G_2 have at least 4 vertices. Deletion-contraction on uv and Proposition 4.2 yield

$$
Q(G, t) = -t^{-1}Q(G_1, t)Q(G_2, t) + Q(G/uv, t). \qquad\qquad (5)
$$

The induction hypothesis of (f) gives $Q(G_i, t) > 0$ for $i \in \{1, 2\}$, and the induction hypothesis of (a) gives $t^{-1}Q(G_1, t)Q(G_2, t) \leq \alpha^{-1}Q(G/uv, t)$. Substituting into (5) we have

$$
Q(G, t) \geq (1 - \alpha^{-1})Q(G/uv, t) = \beta Q(G/uv, t). \qquad\qquad (6)
$$

(c) Let $s = Q(G + uv, t) - \gamma Q(G, t)$. Since the 2-cut $\{u, v\}$ of G has property P_2, one $\{u, v\}$-bridge of G is trivial. Let H be the other $\{u, v\}$-bridge of G and notice that

$H + uv \in \mathcal{K}_2$. By addition-contraction on G, and using Proposition 4.2 we get

$$\begin{aligned}
s &= Q(G + uv, t) - \gamma[Q(G + uv, t) - Q(G/uv, t)] \\
&= (1 - \gamma)Q(G + uv, t) + \gamma Q(G/uv, t) \qquad\qquad\qquad (7) \\
&= (1 - \gamma)(2 - t)Q(H + uv, t) - \gamma(t - 1)Q(H/uv, t).
\end{aligned}$$

Note that $H = H_1 \cup H_2$ where H_1 is a generalised uw-edge with property P_2, H_2 is a generalised vw-edge with property P_2, and $H_1 \cap H_2 = \{w\}$. Thus by the induction hypothesis of (d), $Q(H/uv, t) > 0$. Now by the induction hypothesis of (b), we have $Q(H + uv, t) \geq \beta Q(H/uv, t)$. Substituting into (7) and using Lemma 4.2(iii) gives

$$s \geq [(1 - \gamma)(2 - t)\beta - \gamma(t - 1)]Q(H/uv, t) \geq 0.$$

(d) If one of G_1, G_2 is a single edge, then G is either a single edge, or $G = H_1 \cup H_2$, where $H_1, H_2 \in \mathcal{K}_2$, and $H_1 \cap H_2$ is the edge uw. By the induction hypothesis of (e) and Proposition 4.2, we conclude that $Q(G, t) > 0$. So suppose both G_1 and G_2 have at least 4 vertices. By (a), which has now been proven for a graph on $|V(G)|$ vertices, we conclude that $Q(G, t) \geq \alpha t^{-1} Q(G_1, t) Q(G_2, t)$. By the induction hypothesis of (f), $Q(G_i, t) > 0$ for $i \in \{1, 2\}$. Therefore $Q(G, t) > 0$.

(e) Let $\{u, w\}$ be a 2-cut of G so that two of the $\{u, w\}$-bridges are trivial. Such a 2-cut is easily found by considering the construction of G from K_3 by the double subdivision operation. Let v be a vertex of degree 2 in G with neighbours u and w. By Proposition 4.3, we may write $G = G_1 \cup G_2 + uv$ where G_1 is a generalised vw-edge, G_2 is a generalised uw-edge, and $G_1 \cap G_2 = \{w\}$. By the choice of $\{u, w\}$ we have in particular that G_1 is the edge vw, and G_2 is a generalised uw-edge with property P_2.

Now we may apply (b) to deduce $Q(G, t) \geq \beta Q(G/uv, t)$. Note that $G/uv = H_1 \cup H_2$ where $H_1, H_2 \in \mathcal{K}_2$ and $H_1 \cap H_2$ is the edge uw. By the induction hypothesis of (e) we have that $Q(H_i, t) > 0$ for $i \in \{1, 2\}$. Now finally Proposition 4.2 gives $Q(G/uv, t) = t^{-1}(t - 1)^{-1} Q(H_1, t) Q(H_2, t) > 0$, whence $Q(G, t) > 0$.

(f) Let v be a vertex of degree 2 with neighbours u and w. Let $H = G - v$ and z be a cut-vertex of H. Note that $H = H_1 \cup H_2$ where H_1 is a generalised uz-edge with property P_2, H_2 is a generalised wz-edge with property P_2, and $H_1 \cap H_2 = \{z\}$. Note

also that this implies $H + uw \in \mathcal{K}_2$. By addition-contraction and Proposition 4.2,

$$Q(G, t) = Q(G + uw, t) - Q(G/uw, t)$$
$$= (2 - t)Q(H + uw, t) + (t - 1)Q(H/uw, t).$$

By the induction hypothesis of (e), we have $Q(H + uw, t) > 0$. If one of H_1 or H_2 is a single edge, then H/uw is either a single edge or an element of $\mathcal{K}_2$. In either case $Q(H/uw, t) > 0$. Thus we may suppose both H_1 and H_2 have at least 4 vertices. By the induction hypothesis of (f), $Q(H_i, t) > 0$ for $i \in \{1, 2\}$. Now we apply the induction hypothesis of (a) to get

$$Q(H/uw, t) \geq \alpha t^{-1} Q(H_1, t) Q(H_2, t) > 0.$$

$\square$

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