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Roots of the Chromatic Polynomial

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Thomas Joseph Perrett

Summary

The chromatic polynomial of a graph G is a univariate polynomial whose evaluation at any positive integer q enumerates the proper q-colourings of G. It was introduced in connection with the famous four colour theorem but has recently found other applications in the field of statistical physics. In this thesis we study the real roots of the chromatic polynomial, termed chromatic roots, and focus on how certain properties of a graph affect the location of its chromatic roots.

Firstly, we investigate how the presence of a certain spanning tree in a graph affects its chromatic roots. In particular we prove a tight lower bound on the smallest non-trivial chromatic root of a graph admitting a spanning tree with at most three leaves. Here, non-trivial means different from 0 or 1. This extends a theorem of Thomassen on graphs with Hamiltonian paths. We also prove similar lower bounds on the chromatic roots of certain minor-closed families of graphs.

Later, we study the Tutte polynomial of a graph, which contains the chromatic polynomial as a specialisation. We discuss a technique of Thomassen using which it is possible to deduce that the roots of the chromatic polynomial are dense in certain intervals. We extend Thomassen's technique to the Tutte polynomial and as a consequence, deduce a density result for roots of the Tutte

polynomial. This partially answers a conjecture of Jackson and Sokal.

Finally, we refocus our attention on the chromatic polynomial and investigate the density of chromatic roots of several graph families. In particular, we show that the chromatic roots of planar graphs are dense in the interval $(3, 4)$, except for a small interval around $\tau + 2 \approx 3.618$, where τ denotes the golden ratio. We also investigate the chromatic roots of related minor-closed classes of graphs and bipartite graphs.

Danish Summary

Det kromatiske polynomium af en graf G er et polynomium, hvis evaluering i ethvert positivt heltal q tæller antallet af q-farvninger af G. Det blev indført i forbindelse med det berømte Fire-Farve-Problem, men har for nylig fundet andre anvendelser inden for statistisk fysik. I denne afhandling undersøger vi de reelle rødder af det kromatiske polynomium, kaldet kromatiske rødder, og fokuserer på, hvordan visse grafegenskaber påvirker placeringen af disse rødder.

Først undersøger vi hvordan tilstedeværelsen af et vis udspændende træ i en graf påvirker dets kromatiske rødder. Nærmere bestemt vi bevise en skarp nedre grænse for den mindste ikke-trivielle kromatiske rod af en graf med et udspændende træ med højst tre blade. Her betyder ikke-triviel forskellig fra 0 og 1. Dette udvider en sætning af Thomassen om grafer med Hamilton-veje. Vi beviser også lignende nedre grænser for de kromatiske rødder af visse familier af grafer som er lukkede under minor-operationer.

Senere studerer vi Tutte-polynomiet af en graf, der indeholder det kromatiske polynomium som en specialisering. Vi diskuterer en teknik af Thomassen som gør det muligt at udlede at rødderne af det kromatiske polynomium er tætte i bestemte intervaller. Vi udvider Thomassens teknik til Tutte-polynomiet og fra dette udleder vi et densitet-resultat for rødderne af Tutte-polynomiet. Dette

besvarer delvist en formodning af Jackson og Sokal.

Endelig koncentrerer vi os om det kromatiske polynomium og undersøger tætheden af kromatiske rødder af nogle familier af grafer. Nærmere bestemt viser vi at de kromatiske rødder af plane grafer er tætte i intervallet $(3, 4)$ med undtagelse af et lille interval omkring $\tau + 2 \approx 3,618$, hvor τ betegner det gyldne snit. Vi undersøger også de kromatiske rødder relateret til todelte grafer og til familier af grafer som er lukkede under minor-operationer.

Preface

This thesis was prepared in the Department of Applied Mathematics and Computer Science at the Technical University of Denmark in fulfilment of the requirements for acquiring a PhD degree in mathematics. The research was carried out under the supervision of Professor Carsten Thomassen, and financed by the ERC Advanced Grant GRACOL (Graph Theory: Colourings, flows, and decompositions), project number 320812. Part of the research was carried out during a three month research stay at the University of Waterloo, Canada, and a one month stay at LaBRI, University of Bordeaux, France.

The results presented in this thesis can also be found in a number of papers [Per16a, Per16b, PT, OP]. These papers have been published or submitted for publication in various international journals and can also be found online at `arXiv.org` as detailed in the references.

Thomas Joseph Perrett

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Contents

CHAPTER 1

Introduction

In this chapter we introduce the main object of study of this thesis, the chromatic polynomial, and mention some important existing results. For the reader's convenience, we also include an overview of the results contained in the subsequent chapters.

1.1 Preliminaries

This thesis lies within the subject of graph theory, a branch of discrete mathematics. For general notation and terminology we refer the reader to [Die16]. However, we detail some of the most important concepts below.

A **graph** G is an ordered pair (V, E), where V is set whose elements are called **vertices**, and E is a set of unordered 2-tuples of elements from V. The elements of E are called **edges**. For brevity we always write uv in place of (u, v). In

this thesis, we only consider **finite** graphs, that is, where $|V(G)|$ is finite. A graph is said to be **simple** if it contains no **loops**: edges of the form (u, u) for $u \in V(G)$. A **multigraph** is a generalisation of a graph where $E(G)$ is allowed to be a multiset. We will always assume that the edge and vertex sets of a multigraph are finite. In this thesis we only deal with multigraphs in Chapter 5. Thus, for simplicity, we almost always use the term graph to mean both graph and multigraph, and indicate where multigraphs are allowed by a remark. For example, all remaining definitions in this section are valid for multigraphs.

A **subgraph** H of a graph G is a graph such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. If $A \subseteq V(G)$, then $G[A]$ denotes the **induced subgraph** of G where $V(G[A]) = A$, and $E(G[A]) = \{uv \in E(G) : u, v \in A\}$. A sequence of distinct vertices $u_1, \ldots, u_r \in V(G)$ is called a **path** if $u_i u_{i+1} \in E(G)$ for each $i \in \{1, \ldots, r-1\}$. If $u, v \in V(G)$, then a path from u to v is a path $u_1, \ldots, u_r$ such that $u_1 = u$ and $u_r = v$. Similarly the sequence of vertices $u_1, \ldots, u_r$ is called a **cycle** if $r > 2$, $u_1 u_r \in E(G)$ and $u_i u_{i+1} \in E(G)$ for $i \in \{1, \ldots, r-1\}$. We often identify a path or cycle in a graph G with the subgraph of G consisting of the vertices and edges of that path or cycle. A path or cycle is **Hamiltonian** if it contains all vertices of the graph. A graph is said to be **Hamiltonian** if it contains a Hamiltonian cycle.

A graph G is said to be **connected** if for every pair of vertices $u, v \in V(G)$, there exists a path in G from u to v. Otherwise G is **disconnected**. A **component** of G is a maximally connected subgraph. If $S \subseteq V(G)$, then $G - S$ denotes the graph obtained from G by deleting all vertices of S and all edges with an endpoint in S. If $S = \{u\}$, then we write $G - u$ for $G - S$. If $G - S$ has strictly more components than G, then we say that S is a **cut-set**. If $|S| = 1$, then the unique element of S is called a **cut-vertex**. If $|S| = 2$, then we say S is a 2-**cut**. A connected graph G is said to be **separable** if it has a cut-vertex, and **non-separable** otherwise. A **block** of G is a maximal non-separable subgraph of G. We say that a graph G is 2-**connected** if for every pair of vertices $u, v \in V(G)$, there exist two paths P_1 and P_2 from u to v, such that $V(P_1) \cap V(P_2) = \{u, v\}$.

Suppose G is a 2-connected graph and $\{x, y\}$ is a 2-cut of G. Let C be a connected component of $G - \{x, y\}$, and $B = G[V(C) \cup \{x, y\}]$. We say that B is an $\{x, y\}$-**bridge** of G. If $|V(B)| = 3$, then we say B is **trivial**.

Suppose G is a graph and $u, v \in V(G)$. We denote by $G + uv$ the graph formed from G by adding an edge uv. If uv is an edge of G with multiplicity r, then in $G + uv$ the edge uv has multiplicity $r + 1$. We let G_{uv} denote the graph formed from G by identifying the vertices u and v. All edges between u and v in G become loops in G_{uv}. Multiple edges may also arise for example if there is a vertex $w \in V(G)$ such that $wu, wv \in E(G)$. Thus, we have $|E(G)| = |E(G_{uv})|$. We let G/uv denote the simple graph formed from G_{uv} by deleting all loops and all but one copy of each edge. This operation is referred to as the **contraction** of uv. If G_1 and G_2 are graphs then $G_1 \cup G_2$ denotes the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. Similarly, $G_1 \cap G_2$ denotes the graph with vertex set $V(G_1) \cap V(G_2)$ and edge set $E(G_1) \cap E(G_2)$.

We let $\overline{A}$ denote the topological closure of a set $A \subseteq \mathbb{R}^n$. We also let $\mathbb{N}$ denote the set $\{1, 2, \dots\}$ and $\mathbb{N}_0$ be the set $\mathbb{N} \cup \{0\}$. Finally, a graph is said to be **planar** if it can be embedded on the sphere such that its edges intersect only at their endpoints.

1.2   Graph Colouring

Many problems in graph theory involve assigning labels to the vertices of a graph. In graph colouring, we do so under the additional restriction that neighbouring vertices receive different labels.

DEFINITION 1.1 A **proper** q-**colouring** of a graph G is a map $\phi : V(G) \to \{1, \dots, q\}$ such that for all vertices $u, v \in V(G)$, we have $\phi(u) \neq \phi(v)$ whenever $uv \in E(G)$.

We say that a graph G is q-**colourable** if there exists a proper q-colouring of G. Since we shall always deal with this type of colouring, we will drop the word proper in what follows. The smallest natural number q such that G has a q-colouring is called the **chromatic number** of G and is denoted by $\chi(G)$.

The study of graph colouring began with the four colour conjecture which seems to have been posed by Guthrie around 1852. The conjecture, which says that every planar graph is 4-colourable, was finally proved in 1976 by Appel and Haken.

1.3   The Chromatic Polynomial

The **chromatic polynomial** $P(G, q)$ of a graph G is a univariate polynomial which contains all the quantitative information about the colourings of G. It was introduced by Birkhoff [Bir12] in 1912 for planar graphs and extended to all graphs by Whitney [Whi32a, Whi32b] in 1932. The initial hope was that its study might lead to an analytic proof of the four colour theorem, but this has not yet been realised.

Perhaps most intuitively, one may derive the chromatic polynomial as follows. For a graph G, first define $P(G, q)$ to be the function of the non-negative integers, such that for $q \in \mathbb{N}_0$, the number of proper q-colourings of G is precisely $P(G, q)$. Next, note that for every $q \in \mathbb{N}_0$ and for every pair of non-adjacent vertices $x, y \in V(G)$, this function satisfies the equality

$$P(G, q) = P(G + xy, q) + P(G/xy, q), \tag{1.1}$$

because the q-colourings of G come in one of two types: those that assign different colours to x and y, and those that assign the same colours to x and y. These correspond precisely to the q-colourings of $G + xy$ and G/xy respectively.

LEMMA 1.2 *For a graph G, the function $P(G, q)$ is a polynomial in q.*

SKETCH OF PROOF. We proceed by induction, firstly on the number of vertices n and secondly on the number of missing edges $m(G) = \binom{n}{2} - |E(G)|$. If $n = 1$ or $m(G) = 0$, then G is a clique on n vertices, and a simple counting argument shows that $P(G, q) = q(q-1)(q-2)\cdots(q-n+1)$. This is a polynomial in q, so we proceed to the induction step. If $m(G) > 0$, then we choose a non-adjacent pair of vertices and apply equality (1.1). By induction, and since the sum of two polynomials is a polynomial, the result follows. $\qquad\square$

One should be careful to check that the derivation of the chromatic polynomial does not depend on the order of the non-adjacent pairs chosen in the proof of Lemma 1.2. This is easy to see. Indeed, since the evaluation of $P(G, q)$ was determined at infinitely many points, there is a unique polynomial which interpolates them all.

Equality (1.1) is called the **addition-contraction identity**, and by the same reasoning as above, can now be seen to hold for all $q \in \mathbb{R}$.

PROPOSITION 1.3 *If G is a graph and $x, y \in V(G)$ such that $xy \notin E(G)$, then $P(G, q) = P(G + xy, q) + P(G/xy, q)$ for all $q \in \mathbb{R}$.*

By rearranging and renaming the graphs, we obtain the following **deletion-contraction identity**, which will sometimes be more convenient to work with.

PROPOSITION 1.4 *If G is a graph and $x, y \in V(G)$ such that $xy \in E(G)$, then $P(G, q) = P(G - xy, q) - P(G/xy, q)$ for all $q \in \mathbb{R}$.*

Propositions 1.3 and 1.4 are central tools in studying all aspects of the chromatic polynomial. From them, one can easily extract an algorithm for computing the chromatic polynomial of a given graph, however we mention that since the chromatic polynomial contains all information about the number of colourings, it is necessarily #P-hard to compute.

It is also possible to define the chromatic polynomial explicitly. The following formula was found independently by Birkhoff [Bir12] and Whitney [Whi32b].

DEFINITION 1.5 The chromatic polynomial $P(G, q)$ of a graph G is the polynomial

$$\sum_{A \subseteq E(G)} (-1)^{|A|} q^{k(A)}, \tag{1.2}$$

where q is an indeterminate and $k(A)$ denotes the number of components of the graph with vertex set $V(G)$ and edge set A.

We now note two further identities. The first allows us to express the chromatic polynomial of a graph as the product of the chromatic polynomials of two smaller graphs.

PROPOSITION 1.6 *If G is a graph such that $G = G_1 \cup G_2$ and $G_1 \cap G_2 = K_k$ for some $k \in \mathbb{N}$, then*

$$P(G, q) = \frac{P(G_1, q)P(G_2, q)}{P(K_k, q)} = \frac{P(G_1, q)P(G_2, q)}{q(q-1)\cdots(q-k+1)}.$$

If $G = G_1 \cup G_2$ and $G_1 \cap G_2 = \emptyset$, then $P(G, q) = P(G_1, q)P(G_2, q)$.

The second identity regards an operation studied by Whitney [Whi33], and which we call a **Whitney 2-switch**. Let G be a graph, $\{x, y\}$ be a 2-cut of G, and C be a component of $G - \{x, y\}$. Define G' to be the graph obtained from the disjoint union of $G - C$ and C by adding for all $z \in V(C)$ the edge xz (respectively yz) if and only if yz (respectively xz) is an edge of G.

PROPOSITION 1.7 *If G is a graph and G' is obtained from G by a Whitney 2-switch, then $P(G, q) = P(G', q)$.*

SKETCH OF PROOF. If $xy \in E(G)$, then apply Proposition 1.6 to the 2-cut $\{x, y\}$ in both G and G'. The resulting expressions for $P(G, q)$ and $P(G', q)$ are the same. If $xy \notin E(G)$, then first apply Proposition 1.3 to the pair $\{x, y\}$

in both G and G'. Next, apply Proposition 1.6 to $G + xy, G/xy, G' + xy$ and G'/xy. Again the resulting expressions for $P(G, q)$ and $P(G', q)$ are the same.

$\square$

If G' can be obtained from G by a sequence of Whitney 2-switches, then $P(G, q) = P(G', q)$, and we say that G and G' are **Whitney-equivalent**.

1.4 Chromatic Roots

Amongst the most basic attributes of a polynomial P are its **roots** or **zeros**: the complex numbers $q \in \mathbb{C}$ which satisfy $P(q) = 0$. In this thesis we will be interested exclusively in real roots of the chromatic polynomial. This motivates the following definition.

DEFINITION 1.8 Let G be a graph and $q \in \mathbb{R}$. We say that q is a **chromatic root** of G if $P(G, q) = 0$.

Since the evaluations of the chromatic polynomial count the number of colourings of a graph, it follows that the numbers $0, 1, \ldots, \chi(G) - 1$ are always chromatic roots of a graph G. In particular, 0 and 1 are chromatic roots of any graph with at least one edge. Thus we say that a chromatic root q is **trivial** if $q \in \{0, 1\}$, and **non-trivial** otherwise. If $\mathcal{G}$ is a class of graphs, then we denote the set of real chromatic roots of $G \in \mathcal{G}$ by $\boldsymbol{R(\mathcal{G})}$. Let $\mathcal{G}$ be a class of graphs and $I \subseteq \mathbb{R}$ be an interval. We say that I is **zero-free** for $\mathcal{G}$ if $R(\mathcal{G}) \cap I = \emptyset$. If I is zero-free for the class of all graphs, then we simply say that I is zero-free.

Using deletion-contraction and induction, it can easily be shown that the coefficients of the chromatic polynomial have alternating signs, that is $P(G, q) = \sum_{i=0}^{n} (-1)^i a_i q^{n-i}$, where $a_i \in \mathbb{N}$ and $n = |V(G)|$. From this fact it follows easily that the interval $(-\infty, 0)$ is zero-free. Tutte [Tut74] showed that the interval $(0, 1)$ is also zero-free. Thus, all non-trivial chromatic roots are greater than 1.

For a class of graphs $\mathcal{G}$, we define $\boldsymbol{\omega(\mathcal{G})}$ to be the infimum of the non-trivial chromatic roots of the graphs $G \in \mathcal{G}$. For technical reasons, we define $\omega(\emptyset) = \infty$. In 1993, Jackson proved the following surprising result, which is a central result in the study of chromatic roots and the starting point of this thesis.

THEOREM 1.9 [Jac93] *If $\mathcal{G}$ denotes the class of all graphs, then $\omega(\mathcal{G}) = 32/27$.*

Theorem 1.9 incorporates two results. The first is that the interval $(1, 32/27)$ is zero-free. The second is that $32/27$ is a limit point of the set of chromatic roots of all graphs. The fact that $32/27$ cannot itself be a chromatic root follows from the well known rational root theorem together with the observation that all chromatic polynomials are **monic**, that is the coefficient of $q^{|V(G)|}$ is 1.

Later, Thomassen [Tho97] proved a counterpart to Jackson's result.

THEOREM 1.10 [Tho97] *The set of chromatic roots of all graphs contains 0, 1 and a dense subset of the interval $[32/27, \infty)$.*

Theorems 1.9 and 1.10 are the main coordinates from which this thesis begins. If $\mathcal{G}$ denotes the class of all graphs, then taken together, they imply that $\overline{R(\mathcal{G})} = \{0, 1\} \cup [32/27, \infty)$ and that the chromatic roots of all graphs exhibit a striking dichotomy. However, for restricted classes of graphs our knowledge is much less complete. The classes of planar graphs and 3-connected graphs are prominent examples. Indeed, despite the fact that the chromatic polynomial was initially introduced to study planar graphs, we still do not know the closure of their chromatic roots. These problems motivate the results of this thesis.

1.5   Outline of the Thesis

This thesis is organised as follows. In Chapter 2 we introduce the class of generalised triangles and show how these graphs play a key role in determining

the zero-free intervals for various graph classes.

In Chapter 3 we investigate how the presence of a certain spanning tree in a graph affects its chromatic roots. In particular we employ the techniques from Chapter 2 to find a zero-free interval for graphs admitting a spanning tree with at most three leaves. This extends a theorem of Thomassen on graphs with Hamiltonian paths.

Chapter 4 concerns the chromatic roots of minor-closed classes of graphs. We first note a relationship between graph minors and certain classes of generalised triangles. Using this observation, we derive zero-free intervals for several classes of graphs characterised by excluding a generalised triangle as a minor.

In Chapter 5 we introduce the Tutte polynomial of a graph, which contains the chromatic polynomial as a specialisation. We discuss a technique of Thomassen using which it is possible to deduce that the roots of the chromatic polynomial are dense in certain intervals. We extend Thomassen's technique to the Tutte polynomial and as a consequence, deduce a density result for roots of the Tutte polynomial. This partially answers a conjecture of Jackson and Sokal.

Finally, in Chapter 6, we refocus our attention on the chromatic polynomial and apply the methods of Chapter 5 to investigate the density of chromatic roots of planar graphs. In particular, we show that the chromatic roots of planar graphs are dense in the interval $(3, 4)$, except for a small interval around $\tau + 2 \approx 3.618$, where τ denotes the golden ratio. We also investigate the chromatic roots of related minor-closed classes of graphs and the class of bipartite graphs.

CHAPTER 2

Generalised Triangles

2.1 Introduction

Given a class of graphs $\mathcal{G}$, what is the infimum of the non-trivial chromatic roots of $G \in \mathcal{G}$? In this chapter we introduce a method which has been used to attack this problem. Loosely speaking, the method shows that to determine $\omega(\mathcal{G})$ for certain classes of graphs $\mathcal{G}$, one only needs to investigate a class of graphs $\mathcal{K}$ whose elements are called generalised triangles. More precisely, we show that $\omega(\mathcal{G}) = \omega(\mathcal{G} \cap \mathcal{K})$ when $\mathcal{G}$ satisfies certain conditions. Of course it still remains to determine $\omega(\mathcal{G} \cap \mathcal{K})$, but the additional structure of the generalised triangles makes this a tractable problem, normally solvable by Propositions 1.3, 1.4, 1.6 and some elementary analysis.

The generalised triangles were first introduced by Jackson [Jac93] when proving Theorem 1.9. To define them, we first define the following operation on a graph G called **double subdivision**: choose an edge uv of G and construct a new

graph from $G - uv$ by adding two new vertices and joining both of them to u and v. A **generalised triangle** is either K_3 or any graph which can be obtained from K_3 by a sequence of double subdivisions. We denote the class of generalised triangles by $\mathcal{K}$. The following alternative characterisation of generalised triangles can be found in Dong and Koh [DK10], see also [Jac93].

PROPOSITION 2.1 [DK10] *A graph G is a generalised triangle if and only if it satisfies the following conditions.*

(GT1) *G is connected and non-separable.*

(GT2) *G is not 3-connected.*

(GT3) *For every 2-cut $\{x, y\}$, we have $xy \notin E(G)$.*

(GT4) *For every 2-cut $\{x, y\}$, G has precisely three $\{x, y\}$-bridges.*

(GT5) *For every 2-cut $\{x, y\}$, each $\{x, y\}$-bridge is separable.*

For a graph G, let $\Diamond(G)$ denote the graph obtained from G by applying the double subdivision operation to every edge of G. For $k \in \mathbb{N}$, we define $\Diamond^k(G)$ to be the graph obtained from G by repeating this operation k times recursively. That is, we define $\Diamond^0(G) = G$ and $\Diamond^k(G) = \Diamond^{k-1}(\Diamond(G))$ for $k \in \mathbb{N}$. We have already seen the surprising result of Jackson [Jac93] that the interval $(1, 32/27]$ is zero-free for the class of all graphs. The following lemma demonstrates immediately that the double subdivision operation is of great importance in this context.

LEMMA 2.2 [JS09] *Let G be a non-separable graph with an odd number of vertices. For every $\varepsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that for every $k \geq k_0$, the graph $\Diamond^k(G)$ has a chromatic root in the interval $(32/27, 32/27 + \varepsilon)$.*

Since K_3 is the smallest non-separable graph with an odd number of vertices, the class of generalised triangles is in some sense representative of this behaviour.

In particular it follows from Lemma 2.2 that if $\mathcal{G}$ is a class of graphs such that $\mathcal{K} \subseteq \mathcal{G}$, then $\omega(\mathcal{G}) = 32/27$.

2.2 The Generalised Triangle Method

In this section we present the generalised triangle method and describe the classes of graphs to which it can be applied. The method was first used implicity by Jackson [Jac93] to prove Theorem 1.9. Later it was again used implicitly by Thomassen [Tho00] who determined $\omega(\mathcal{H})$ where $\mathcal{H}$ denotes the class of graphs with a Hamiltonian path. The method was extracted and abstracted by Dong and Koh [DK08a, DK10], and the remainder of this section can largely be found in those papers. Nevertheless, we give a slightly different presentation intended to ease the readers understanding of the subsequent results in this thesis.

We have already seen the results of Tutte and Jackson which imply that $\omega(\mathcal{G}) = 32/27$ when $\mathcal{G}$ is the class of all graphs. From this fact, it is clear that for any graph G, the sign of $P(G, q)$ remains constant for $q \in (1, 32/27)$. In fact, it can be easily determined because of the following theorem.

THEOREM 2.3 [Jac93] *Let G be a graph with n vertices and c components. If b denotes the number of blocks of G with at least two vertices, then*

(i) $(-1)^n P(G, q) > 0$ *for $q \in (-\infty, 0)$.*

(ii) $P(G, q)$ *has a zero of multiplicity c at $q = 0$.*

(iii) $(-1)^{n+c} P(G, q) > 0$ *for $q \in (0, 1)$.*

(iv) $P(G, q)$ *has a zero of multiplicity b at $q = 1$.*

(v) $(-1)^{n+c+b} P(G, q) > 0$ *for $q \in (1, 32/27]$.*

For simplicity, define $Q(G, q) = (-1)^{n+c+b} P(G, q)$, where n, c and b are the

quantities defined in Theorem 2.3 with respect to the graph G. We briefly note how $Q(G, q)$ interacts with Propositions 1.3, 1.4 and 1.6.

PROPOSITION 2.4 *Let G be a non-separable graph such that $G \neq K_2$, and let $x, y \in V(G)$. Suppose that G/xy has r blocks and that if $xy \in E(G)$, then $G - xy$ has s blocks. For all $q \in \mathbb{R}$, we have the following.*

(i) *If $xy \notin E(G)$, then $Q(G, q) = Q(G + xy, q) + (-1)^r Q(G/xy, q)$.*

(ii) *If $xy \in E(G)$, then $Q(G, q) = (-1)^{s+1} Q(G - xy, q) - (-1)^r Q(G/xy, q)$.*

PROPOSITION 2.5 *Let G, G_1 and G_2 be graphs such that $G = G_1 \cup G_2$. If $G_1 \cap G_2 = \emptyset$, then $Q(G, q) = Q(G_1, q)Q(G_2, q)$. Similarly, if $G_1 \cap G_2 = K_k$ for some $k \in \{1, 2\}$, then*

$$Q(G, q) = P(K_k, q)^{-1} Q(G_1, q) Q(G_2, q). \tag{2.1}$$

PROOF. Let n, c and b be the number of vertices, components and blocks with at least two vertices of G. Similarly, for $i \in \{1, 2\}$, let n_i, c_i and b_i denote the corresponding quantities for G_i. Clearly, if $G_1 \cap G_2 = \emptyset$, then $n + c + b = \sum_{i \in \{1,2\}} (n_i + c_i + b_i)$. Hence, Proposition 1.6 gives $Q(G, q) = Q(G_1, q)Q(G_2, q)$.

So suppose $G_1 \cap G_2 = K_k$ for $k \in \{1, 2\}$. Note that $n_1 + n_2 = n + k$ and $c_1 + c_2 = c + 1$. Finally, if $k = 1$, then we have $b_1 + b_2 = b$. On the other hand, if $k = 2$, then $b_1 + b_2 = b + 1$. To see this, let e be the unique edge of $G_1 \cap G_2$. Note that every block of G_1 which has at least two vertices and does not contain e is a block of G. The same statement holds for G_2. Finally, the unique blocks of G_1 and G_2 which contain e become one in G. This is because if B_1 and B_2 are non-separable graphs, then the graph formed by identifying an edge of B_1 with an edge of B_2 is a non-separable graph. It can now be checked that in both cases, we have $n + c + b \equiv \sum_{i \in \{1,2\}} (n_i + c_i + b_i) \mod 2$. Hence (2.1) follows from Proposition 1.6. $\square$

Note that parts (i) - (iv) of Theorem 2.3 imply that if $\mathcal{G}$ is a class of graphs,

then $Q(G, q) > 0$ for $q \in (1, \omega(\mathcal{G}))$ and $G \in \mathcal{G}$. This stronger "determined sign" formulation is often easier to work with. As an example, suppose we are investigating a class of graphs $\mathcal{G}$ and we conjecture that there is $q_0 \in (1, \infty)$ such that the following statement is true:

$$Q(G, q) > 0 \text{ for } G \in \mathcal{G} \text{ and } q \in (1, q_0). \tag{2.2}$$

Suppose that G is a counterexample to our conjecture; in other words $G \in \mathcal{G}$ and there is $q \in (1, q_0)$ such that $Q(G, q) \le 0$. Furthermore, suppose that G is non-separable and has an edge e such that $G - e$ and G/e are both non-separable members of $\mathcal{G}$. By Proposition 2.4, we have

$$Q(G, q) = Q(G - e, q) + Q(G/e, q). \tag{2.3}$$

Since G is a smallest counterexample and $G - e, G/e \in \mathcal{G}$, we have that $Q(G - e, q)$ and $Q(G/e, q)$ are positive. But now by (2.3), $Q(G, q) > 0$ which is a contradiction. Thus, we deduce that a smallest counterexample to statement (2.2) has no such edge. Note that using the statement that $P(G, q)$ is non-zero in $(1, q_0)$ would not work here. Indeed, knowing that $Q(G - e, q) \ne 0$ and $Q(G/e, q) \ne 0$ is not enough to deduce that $Q(G, q) \ne 0$ from equation (2.3).

In a similar way, we will use Propositions 2.4 and 2.5 to deduce several structural properties of a smallest counterexample. This is possible if the class $\mathcal{G}$ is closed under certain operations.

DEFINITION 2.6 [DK08a] A class of graphs $\mathcal{G}$ is called **splitting-closed** if the following conditions are satisfied for each $G \in \mathcal{G}$.

(SC1) All components and blocks of G are elements of $\mathcal{G}$. Furthermore, if $\{x, y\}$ is a 2-cut of G and $xy \in E(G)$, then $\mathcal{G}$ includes each $\{x, y\}$-bridge of G.

(SC2) If G is non-separable and $\{x, y\}$ is a 2-cut of G with $xy \notin E(G)$, then $\mathcal{G}$ includes all $\{x, y\}$-bridges of $G + xy$ and all blocks of G/xy.

Many natural classes of graphs are splitting-closed. For example all minor-closed classes satisfy the definition, and in Chapter 3 we mention how classes defined by the existence of a certain spanning tree are also splitting-closed. Furthermore, for every positive integer k, Dong and Koh [DK08a] showed that the class of graphs with domination number at most k is splitting-closed.

LEMMA 2.7 [DK08a, Lemma 2.2] *Let $\mathcal{G}$ be a splitting-closed class of graphs. If $G \in \mathcal{G}$ is a smallest counterexample to statement* (2.2)*, then G satisfies properties* (GT1) *and* (GT3) *of Proposition* 2.1*.*

PROOF OF LEMMA 2.7. We first show that G is connected. Indeed, otherwise G is the disjoint union of graphs $G_1, \ldots, G_r$ and by Proposition 2.5, we have $Q(G, q) = \prod_{i=1}^{r} Q(G_i, q)$. Since $Q(G, q) \leq 0$, we have $Q(G_i, q) \leq 0$ for some $i \in \{1, \ldots, r\}$. But $G_i \in \mathcal{G}$ since G satisfies (SC1). This contradicts the fact that G is a smallest counterexample.

In a similar way, if G is separable, then there exist connected graphs $G_1, \ldots, G_r$ such that $G = G_1 \cup \cdots \cup G_r$ and $G_1 \cap \cdots \cap G_r$ is a single vertex. By Proposition 2.5, $Q(G, q) = q^{1-r} \prod_{i=1}^{r} Q(G_i, q)$. Again, since the left hand side of this equality is negative, there is some $i \in \{1, \ldots, r\}$ such that $Q(G_i, q) \leq 0$. However, by (SC1) we have $G_i \in \mathcal{G}$, so G_i is a smaller counterexample than G, a contradiction. This proves that G satisfies (GT1).

Finally, suppose that $\{x, y\}$ is a 2-cut of G with $\{x, y\}$-bridges $B_1, \ldots, B_r$. If $xy \in E(G)$, then by Proposition 2.5 we have

$$Q(G, q) = q^{1-r}(q - 1)^{1-r} \prod_{i=1}^{r} Q(B_i, q).$$

Now, as before, the left hand side of this equality is negative, so there is $i \in \{1, \ldots, r\}$ such that $Q(B_i, q) \leq 0$. However, by (SC1) we have $B_i \in \mathcal{G}$. This contradicts the fact that G is a smallest counterexample, which shows that G satisfies (GT3). $\qquad\square$

LEMMA 2.8 [DK08a, Lemma 2.3] *Let $\mathcal{G}$ be a splitting-closed class of graphs. If $G \in \mathcal{G}$ is a smallest counterexample to statement* (2.2), *then G has an odd number of $\{x, y\}$-bridges at each 2-cut $\{x, y\}$ of G.*

PROOF. By Lemma 2.7, we have that G satisfies (GT1) and (GT3) of Proposition 2.1. Thus, G is non-separable and for every 2-cut $\{x, y\}$, we have that $xy \notin E(G)$. Now suppose for a contradiction that G has $\{x, y\}$-bridges $B_1, \ldots, B_{2r}$ where $r \in \mathbb{N}$. Since G is non-separable, Proposition 2.4 gives

$$Q(G, q) = Q(G + xy, q) + Q(G/xy, q). \tag{2.4}$$

Now apply Proposition 2.5 to each of the terms in (2.4). Note that we have $Q(B_i + xy, q) > 0$ and $Q(B_i/xy, q) > 0$ for each $i \in \{1, \ldots, 2r\}$ since G is a smallest counterexample and $\mathcal{G}$ is splitting-closed. Thus, we conclude from (2.4) that $Q(G, q) > 0$, a contradiction. $\qquad\square$

We say that a class of graphs $\mathcal{G}$ is **connectivity-reducible** if for every 3-connected graph in $\mathcal{G}$, there exists an edge $e \in E(G)$ such that $G - e, G/e \in \mathcal{G}$.

LEMMA 2.9 *If $\mathcal{G}$ is connectivity-reducible, then a smallest counterexample to statement* (2.2) *satisfies* (GT2).

The proof of Lemma 2.9 follows simply by the deletion-contraction identity as in (2.3). Note that since G is 3-connected, each of the graphs $G - e$ and G/e will be non-separable. For some classes of graphs, the property in Lemma 2.9 is trivial to verify. Consider for example any minor-closed class of graphs. We will later show that it is also not difficult to verify when $\mathcal{G}$ is defined by the existence of certain spanning trees. However for some classes of graphs this is a much more difficult problem. Consider the following conjecture which has been open for some time.

CONJECTURE 2.10 [Tho96] *If G is a 3-connected Hamiltonian graph, then there is $e \in E(G)$ such that $G - e$ and G/e are both Hamiltonian.*
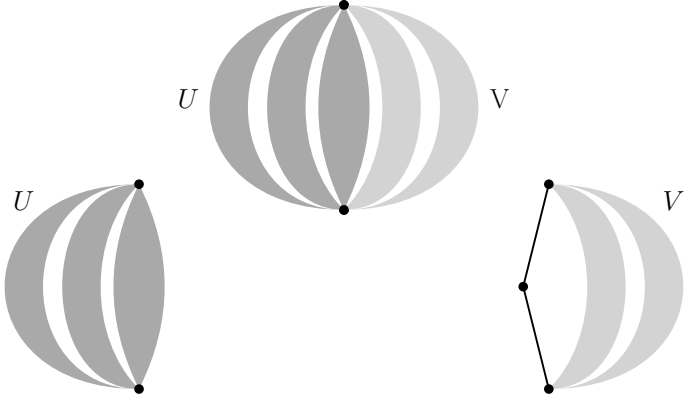
Figure 2.1: A bridge-partition (U, V').

Aside from being an interesting problem in itself, a positive resolution of Conjecture 2.10 would imply that $\omega(\mathcal{G}) = 2$ where $\mathcal{G}$ denotes the class of Hamiltonian graphs, see [Tho96].

Let $\mathcal{G}$ be a class of graphs and $G \in \mathcal{G}$ be non-separable. Let $\{x, y\}$ be a 2-cut of G such that $xy \notin E(G)$, and let U be a proper subgraph of G consisting of the union of an odd number of $\{x, y\}$-bridges. Possibly U is a single $\{x, y\}$-bridge. Let V denote the union of the other $\{x, y\}$-bridges, and let V' denote the graph formed from V by adding a new vertex z and the edges xz and yz, see Figure 2.1. In this case, we say that (U, V') is an $\{x, y\}$-**bridge-partition**. If the relevant 2-cut is obvious, then we will drop the qualifier $\{x, y\}$.

LEMMA 2.11 [DK08a, Lemma 2.5] *Let G be a non-separable graph and $\{x, y\}$ be a 2-cut of G with $\{x, y\}$-bridges $B_1, \ldots, B_r$ where r is an odd number such that $r > 1$. Let (U, V') be a bridge-partition of G and suppose that for fixed $q \in (1, 2)$, we have that $Q(U, q), Q(V', q), Q(B_i + xy, q)$, and $Q(B_i/xy, q)$ are positive for $i \in \{1, \ldots, r\}$. If U is non-separable, then $Q(G, q) > 0$.*

PROOF. First note that since $Q(B_i + xy, q)$ and $Q(B_i/xy, q)$ are positive for $i \in \{1, \ldots, r\}$, Proposition 2.5 implies that $Q(U + xy, q) > 0$, $Q(V + xy, q) > 0$ and $Q(V/xy, q) > 0$. Since G and U have an odd number of $\{x, y\}$-bridges, it

follows that V' has an odd number of $\{x, y\}$-bridges. Thus, Propositions 2.4 and 2.5 give that

$$
\begin{aligned}
Q(V', q) &= Q(V' + xy, q) - Q(V'/xy, q) \\
&= Q(V + xy, q)(2 - q) - Q(V/xy, q)(q - 1).
\end{aligned}
\tag{2.5}
$$

Since $Q(V', q) > 0$ and $1 < q < 2$, we have from equation (2.5) that

$$
\frac{Q(V + xy, q)}{q - 1} > \frac{Q(V/xy, q)}{2 - q} > Q(V/xy, q).
\tag{2.6}
$$

Recall that each of $Q(U, q)$, $Q(U + xy, q)$, $Q(V + xy, q)$ and $Q(V/xy, q)$ are positive. Using Proposition 2.4 on G and inequality (2.6) we have

$$
\begin{aligned}
Q(G, q) &= Q(G + xy, q) - Q(G/xy, q) \\
&= \frac{Q(U + xy, q)Q(V + xy)}{q(q - 1)} - \frac{Q(U/xy, q)Q(V/xy, q)}{q} \\
&> \frac{Q(V/xy, q)}{q} \left(Q(U + xy, q) - Q(U/xy, q) \right).
\end{aligned}
\tag{2.7}
$$

Note that since U is non-separable, and U/xy has an odd number of blocks, we have $Q(U, q) = Q(U + xy, q) - Q(U/xy, q)$ by Proposition 2.4. Finally, this together with (2.7) gives $Q(G, q) > q^{-1}Q(V/xy, q)Q(U, q) > 0$ as claimed. $\qquad\square$

If $\mathcal{G}$ is a splitting-closed class of graphs, and $G \in \mathcal{G}$ is a smallest counterexample to statement (2.2) under the additional assumption that $q_0 \in (1, 2]$, then the technical conditions of Lemma 2.11 are conveniently satisfied.

LEMMA 2.12 *Let $\mathcal{G}$ be a splitting-closed class of graphs, and let $G \in \mathcal{G}$ be a smallest counterexample to statement (2.2) where $q_0 \in (1, 2]$. If (U, V') is a bridge-partition of G, and $U, V' \in \mathcal{G}$, then U is separable.*

PROOF. Since $q_0 \in (1, 2]$ and G is a smallest counterexample to statement (2.2), there is $q \in (1, q_0)$ such that $Q(G, q) \leq 0$. In particular, $q \in (1, 2)$. Let $B_1, \ldots, B_r$ be the $\{x, y\}$-bridges of G. Since $\mathcal{G}$ is splitting-closed, we have by

Lemma 2.8 that r is odd. Furthermore, since G is a smallest counterexample and $\mathcal{G}$ is splitting-closed, we have that $Q(U, q), Q(V', q), Q(B_i + xy, q)$ and $Q(B_i/xy, q)$ are positive for each $i \in \{1, \ldots, r\}$. Thus, U must be separable. Otherwise, Lemma 2.11 implies that $Q(G, q) > 0$, which is a contradiction. $\qquad \square$

Let $\mathcal{G}$ be a class of graphs. If, for every non-separable graph $G \in \mathcal{G}$ and every bridge-partition (U, V') of G, the graphs $U, V' \in \mathcal{G}$, then we say that $\mathcal{G}$ is **partition-closed**.

LEMMA 2.13 *Let $\mathcal{G}$ be a splitting-closed and partition-closed class of graphs. If $G \in \mathcal{G}$ is a smallest counterexample to statement (2.2) where $q_0 \in (1, 2]$, then G satisfies properties (GT4) and (GT5) of Proposition 2.1.*

PROOF. Since $\mathcal{G}$ is splitting-closed, Lemma 2.7 implies that G is non-separable. Now let $\{x, y\}$ be a 2-cut and let the number of $\{x, y\}$-bridges of G be r. By Lemma 2.8, r is odd, so suppose for a contradiction that $r \geq 5$. Let (U, V') be a bridge-partition such that U is the union of three $\{x, y\}$-bridges. Since $\mathcal{G}$ is partition-closed, we have that $U, V' \in \mathcal{G}$. Furthermore, we clearly have that U is non-separable, which contradicts Lemma 2.12. Thus $r = 3$ and G satisfies (GT4). For the same reason, every $\{x, y\}$-bridge is separable. $\qquad \square$

THEOREM 2.14 *Let $\mathcal{G}$ be a splitting-closed, partition-closed and connectivity-reducible class of graphs. If $\omega(\mathcal{G} \cap \mathcal{K}) \in (1, 2]$, then $\omega(\mathcal{G}) = \omega(\mathcal{G} \cap \mathcal{K})$.*

PROOF. Since $\mathcal{G} \cap \mathcal{K} \subseteq \mathcal{G}$, we have $\omega(\mathcal{G}) \leq \omega(\mathcal{G} \cap \mathcal{K})$. Thus, it suffices to show that $Q(G, q) > 0$ for $G \in \mathcal{G}$ and $q \in (1, \omega(\mathcal{G} \cap \mathcal{K}))$. Let G be a smallest counterexample to this statement. So, there is $q \in (1, \omega(\mathcal{G} \cap \mathcal{K}))$ such that $Q(G, q) \leq 0$. Moreover, since $\omega(\mathcal{G} \cap \mathcal{K}) \in (1, 2]$, we have $q \in (1, 2)$. Now, since $\mathcal{G}$ is splitting-closed, partition-closed and connectivity-reducible, we deduce by Lemmas 2.7, 2.9, 2.13, and Proposition 2.1 that $G \in \mathcal{K}$, which is a contradiction. $\qquad \square$

It is easy to name a few classes of graphs which are splitting-closed, partition-

closed and connectivity-reducible. For example any minor-closed class of graphs $\mathcal{G}$ satisfies these properties. If $\mathcal{G}$ is a class of forests, then $R(\mathcal{G}) = \{0, 1\}$ and so $\omega(\mathcal{G}) = \infty$ and $\omega(\mathcal{G} \cap \mathcal{K}) = \omega(\emptyset) = \infty$ by definition. On the other hand, if $\mathcal{G}$ is not a class of forests, then $K_3 \in \mathcal{G}$. Since 2 is a chromatic root of K_3, we deduce that $\omega(\mathcal{G} \cap \mathcal{K}) \in (1, 2]$. These observations and Theorem 2.14 imply the following, which was first noticed by Dong and Koh [DK10].

THEOREM 2.15 [DK10] *If $\mathcal{G}$ is a minor-closed class of graphs, then $\omega(\mathcal{G}) = \omega(\mathcal{G} \cap \mathcal{K})$.*

In practice we will deal with classes of graphs $\mathcal{G}$ which are not so well behaved. However, it often turns out that by a variation of the techniques above, or by an ad-hoc argument, one can deduce that it suffices just to consider the generalised triangles in $\mathcal{G}$.

CHAPTER 3

Spanning Trees

3.1 Introduction

In this section we consider the chromatic roots of graphs which have a spanning tree with certain properties. This study was initiated by Thomassen [Tho00] who provided a new link between Hamiltonian paths and colourings. Thomassen showed that the zero-free interval of Jackson can be extended for graphs with a Hamiltonian path.

THEOREM 3.1 [Tho00] *If $\mathcal{H}$ denotes the class of graphs with a Hamiltonian path, then $\omega(\mathcal{H}) = t_2$, where $t_2 \approx 1.296$ is the unique real root of the polynomial $(q-2)^3 + 4(q-1)^2$.*

It is natural to ask if an analogous result holds for other spanning trees, and there are a number of possible generalisations of the class $\mathcal{H}$. Define a **k-leaf spanning tree** of a graph to be a spanning tree with at most k leaves. We denote the class of graphs which admit a k-leaf spanning tree by $\boldsymbol{\mathcal{G}_k}$. Thus,

Theorem 3.1 shows that the interval $(1, t_2)$ is zero-free for the class $\mathcal{G}_2$. The main result of this section will be to prove the following analogous result for the class $\mathcal{G}_3$.

THEOREM 3.2 $\omega(\mathcal{G}_3) = t_3$, where $t_3 \approx 1.290$ is the smallest real root of the polynomial $(q-2)^6 + 4(q-1)^2(q-2)^3 - (q-1)^4$.

A natural extension of this result would be to find $\varepsilon_k > 0$ so that $(1, 32/27 + \varepsilon_k)$ is zero-free for the class $\mathcal{G}_k$, $k \geq 4$. However, because of Theorem 1.9, it must be that $\varepsilon_k \to 0$ as $k \to \infty$. Nevertheless, we conjecture that $\omega(\mathcal{G}_k)$ is never equal to $32/27$.

CONJECTURE 3.3 For every $k \geq 2$, we have $\omega(\mathcal{G}_k) > 32/27$.

A more restricted generalisation to consider is the class of graphs having a spanning tree of maximum degree 3 and at most ℓ leaves. Here the possible implications are much more interesting since it is not clear if $\varepsilon_\ell \to 0$ as $\ell \to \infty$. Theorems 3.1 and 3.2 solve the cases $\ell = 2$ and $\ell = 3$ respectively, which leads us to conjecture the following.

CONJECTURE 3.4 If $\mathcal{G}$ denotes the class of graphs with a spanning tree of maximum degree 3, then $\omega(\mathcal{G}) > 32/27$.

An affirmative answer to Conjecture 3.4 would have interesting implications. Indeed, consider the following conjecture of Jackson.

CONJECTURE 3.5 [Jac03] The interval $(1, \alpha)$ is zero-free for the class of 3-connected graphs, where $\alpha \approx 1.781$ is a chromatic root of $K_{3,4}$.

Whilst there has been no progress on this conjecture, Dong and Jackson [DJ11] proved that there is a constant $t \approx 1.204$, such that the interval $(1, t)$ is zero-free for 3-connected *planar* graphs. For context, the Herschel graph is the 3-connected planar graph with the smallest known non-trivial chromatic root at approximately 1.840. Barnette [Bar66] proved that every 3-connected planar

graph has a spanning tree of maximum degree 3. Thus, an affirmative answer to Conjecture 3.4 would immediately imply a zero-free interval for the class of 3-connected planar graphs, and this interval could perhaps be larger than that found by Dong and Jackson.

One should obviously check that the graphs in Lemma 2.2 do not give a counterexample to Conjecture 3.4. Indeed they do not, as the following lemma shows.

PROPOSITION 3.6 *For every graph G, only finitely many of the graphs $\Diamond^k(G)$, $k \in \mathbb{N}$ have a spanning tree of maximum degree 3.*

PROOF. First note that for any graph with a spanning tree of maximum degree 3, deleting r vertices yields a graph with at most $2r + 1$ components.

Now suppose that G has n vertices and m edges. Clearly we may suppose that G is connected. A simple calculation shows that for $k \in \mathbb{N}_0$, $k \geq 2$, the graph $\Diamond^k(G)$ has $2m \cdot 4^{k-1}$ vertices of degree 2, and $r = n + \frac{2}{3}m(4^{k-1} - 1)$ vertices of degree greater than 2. Furthermore, for $k \geq 2$, the vertices of degree 2 form an independent set. Thus, deleting all r vertices of degree at least 3 yields precisely $2m \cdot 4^{k-1}$ components. Provided k is large enough, this exceeds $2r + 1 = 2n + \frac{4}{3}m(4^{k-1} - 1) + 1$, which contradicts the above assertion. $\qquad\square$

One can similarly check that the graphs in Lemma 2.2 do not constitute a counterexample to Conjecture 3.3. Indeed, in this case the number of leaves is bounded, so the number of vertices of degree more than 2 is also bounded. It follows that deleting r vertices can create at most $O(r)$ components, which can be used to deduce a contradiction in the same way.

Lemma 2.2 and the following proposition show that the zero-free interval for all graphs cannot be extended for graphs admitting spanning trees with an unbounded number of vertices of degree at least 4.
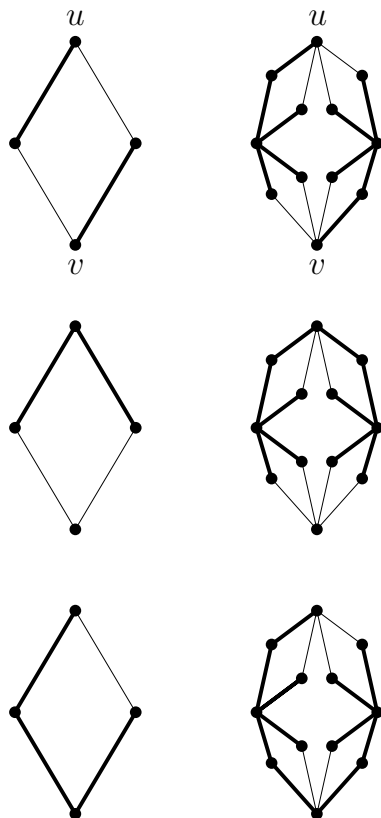
Figure 3.1: Construction of a spanning tree of maximum degree 4.

PROPOSITION 3.7 *If G is a graph such that $\Diamond(G)$ has a spanning tree of maximum degree 4, then for every $k \in \mathbb{N}$, the graph $\Diamond^k(G)$ also has a spanning tree of maximum degree 4.*

SKETCH OF PROOF. Let $k \in \mathbb{N}$ and let T_k be a spanning tree of $\Diamond^k(G)$ with maximum degree 4. Note that the edges of $\Diamond^k(G)$ decompose naturally into a collection of 4-cycles. Furthermore, in each 4-cycle, at most three of the edges are in T_k, and up to symmetry, there are only 3 possible configurations. To construct a spanning tree T_{k+1} of maximum degree 4 in $\Diamond^{k+1}(G)$, we use the constructions indicated by Figure 3.1 on each 4-cycle. Note that in T_{k+1}, the degrees of the vertices u and v are the same as in T_k. $\qquad\square$

3.2   Hamiltonian Paths

To prove Theorem 3.1, Thomassen effectively employed the generalised triangle method discussed in Section 2.2. He first showed that if $\mathcal{H}$ denotes the class of graphs with Hamiltonian paths, then $\mathcal{H}$ is splitting-closed and connectivity-reducible. Furthermore, he noted that if G is a graph with a Hamiltonian path, then G has at most three bridges at every 2-cut. These facts together with Lemmas 2.7 and 2.9 imply that a smallest counterexample to Theorem 3.1 satisfies properties (GT1), (GT2), (GT3) and (GT4) of Proposition 2.1. Finally, Thomassen employed an ad-hoc argument to show that in a smallest counterexample every bridge is separable. Consequently, we have the following.

LEMMA 3.8 [Tho00] $\omega(\mathcal{H}) = \omega(\mathcal{H} \cap \mathcal{K})$.

We now describe the structure of the graphs in $\mathcal{H} \cap \mathcal{K}$. To this end, for each natural number $\ell \geq 1$, let H_ℓ denote the graph obtained from a path $x_1 x_2 \dots x_{2\ell+3}$ by adding the edges $x_1 x_4$, $x_{2\ell} x_{2\ell+3}$, and all edges $x_i x_{i+4}$ for $i \in \{2, 4, \dots, 2\ell-2\}$. Figure 3.2 shows the graph H_3. Also let H_0 be a copy of K_3 with vertex set $\{x_1, x_2, x_3\}$, and let $\mathcal{H}' = \{H_\ell : \ell \in \mathbb{N}_0\}$.

For $\ell \geq 0$, define F_ℓ to be the graph $H_\ell - x_1 x_2$. If G is a graph, $\{x, y\}$ is a 2-cut of G, and B is an $\{x, y\}$-bridge, then we say that B is a **copy** of $F(x, y, \ell)$ to indicate that B is isomorphic to F_ℓ, where x is identified with x_1, and y is identified with x_2 in G.

In what follows, we often require the following simple proposition.

PROPOSITION 3.9 *Let G be a non-separable graph, and $\{x, y\}$ be a 2-cut of G. Let B be an $\{x, y\}$-bridge of G and v be a cut-vertex of B. Suppose that $B = B_x \cup B_y$ where B_x and B_y are connected graphs such that $x \in V(B_x)$, $y \in V(B_y)$, and $B_x \cap B_y = v$. If $\{x, v\}$ is not a 2-cut of G, then $B_x = xv$.*
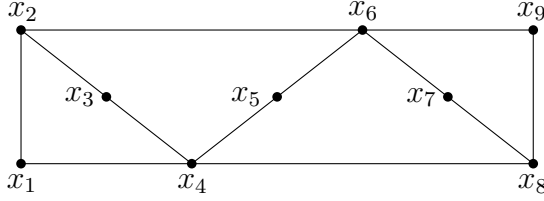
Figure 3.2: The graph H_3.

PROOF. Note that B_x clearly contains a path from x to v. However, if $|V(B_x)| \geq 3$, then $\{x, v\}$ is a 2-cut of G. Thus B_x is the edge xv as claimed. $\qquad \square$

The following lemma characterises the bridges of a generalised triangle under certain conditions. It is implicit in Thomassen [Tho00].

LEMMA 3.10 *Let G be a generalised triangle, $\{x, y\}$ be a 2-cut of G, and B be an $\{x, y\}$-bridge of G.*

(a) *If B contains a Hamiltonian path P starting at x and ending at y, then B is a copy of $F(x, y, 0)$, i.e. B is a path of length 2.*

(b) *If B contains a path P starting at y and covering all vertices of B except for x, then B is a copy of $F(x, y, \ell)$ for some $\ell \in \mathbb{N}_0$.*

PROOF.

(a) Since G is a generalised triangle, B is separable and has a cut-vertex v. The Hamiltonian path P of B shows that neither of $G - \{x, v\}$ and $G - \{y, v\}$ can have more than two components. Thus, since G is a generalised triangle, neither $\{x, v\}$ nor $\{y, v\}$ is a 2-cut of G. It follows by Proposition 3.9 that $|V(B)| = 3$ and $xv, yv \in E(G)$. Since G is a generalised triangle, $xy \notin E(G)$, and so B is a path of length 2 as required.

(b) We proceed by induction on $|V(B)|$. If $|V(B)| = 3$, then B is a copy of $F(x, y, 0)$, so we may assume that $|V(B)| \geq 4$ and the result is true for all bridges on fewer vertices. Since G is a generalised triangle, B has a cut-vertex v. Also, $|V(B)| \geq 4$ implies that at least one of $\{x, v\}$ or $\{y, v\}$ is

a 2-cut of G. If $\{x, v\}$ is a 2-cut, then G has three $\{x, v\}$-bridges, two of which lie inside B. But then P cannot cover all vertices of $B - x$. Thus, by Proposition 3.9, the vertex v is the unique neighbour of x in B, and $\{y, v\}$ is a 2-cut of G with precisely three $\{y, v\}$-bridges, two of which, say B_1 and B_2, are contained in B. Suppose without loss of generality that B_1 contains the subpath of P from y to v. Now, $P[V(B_1)]$ is a Hamiltonian path of B_1 and so by part (a), B_1 is a copy of $F(y, v, 0)$. On the other hand, $P[V(B_2)]$ is a path in B_2, starting at v and covering all vertices of B_2 except for y. By induction, B_2 is a copy of $F(y, v, \ell - 1)$ for some $\ell \in \mathbb{N}$. It can now be checked that B is a copy of $F(x, y, \ell)$. $\qquad \square$

If G is a generalised triangle with a Hamiltonian path, then at any 2-cut, one of the bridges must satisfy Lemma 3.10(a), and the other two must satisfy Lemma 3.10(b) up to changing the roles of the vertices x and y. Thus, we deduce the following easy corollary of Lemma 3.10.

COROLLARY 3.11 $\mathcal{H}' = \mathcal{H} \cap \mathcal{K}$.

The final element in the proof of Theorem 3.1 is to show that t_2 is the infimum of $R(\mathcal{H}') \setminus \{0, 1\}$, where t_2 is the unique real root of the polynomial $(q - 2)^3 + 4(q - 1)^2$. To do this, Thomassen [Tho00] showed that for fixed $q \in (1, t_2]$, the value of the chromatic polynomial of H_ℓ at q can be expressed as

$$P(H_\ell, q) = A\alpha^\ell + B\beta^\ell. \tag{3.1}$$

where α and β are solutions, depending on q, of the auxiliary polynomial $x^2 = (q-2)^2 x + (q-1)^2(q-2)$, which is derived from a recurrence relation as we shall later show. The quantities A and B are constants depending on q. Alternatively A, B, α and β are defined by the following relations [Tho00].

$$\delta = \sqrt{(q - 2)^4 + 4(q - 1)^2(q - 2)}, \tag{3.2}$$

$$\alpha = \frac{1}{2}[(q - 2)^2 + \delta], \quad \beta = \frac{1}{2}[(q - 2)^2 - \delta], \tag{3.3}$$

$$A + B = q(q - 1)(q - 2), \tag{3.4}$$

$$A\alpha + B\beta = q(q - 1)[(q - 2)^3 + (q - 1)^2]. \tag{3.5}$$

Note that $\delta > 0$ and so $0 < \beta < \alpha < 1$ for $q \in (1, t_2)$. Multiplying (3.4) by β, subtracting it from (3.5), and noting that $\alpha - \beta = \delta$, we have more explicitly

$$A = \frac{1}{\delta} q(q - 1)[(q - 2)\alpha + (q - 1)^2], \text{ and} \tag{3.6}$$

$$B = q(q - 1)(q - 2) - A.$$

Since $\alpha > \frac{1}{2}(q - 2)^2$, it is easily checked that the contents of the square bracket in (3.6) is negative. Thus, it may be seen that $0 < B < -A$ for $q \in (1, t_2)$. Finally, using a computer algebra package such as MAPLE, it may be checked that $-A < 1$ for $q \in (1, t_3]$. These inequalities will be used in the proof of Theorem 3.2.

3.3   3-Leaf Spanning Trees

3.3.1   Generalised Triangles with 3-Leaf Spanning Trees

In this section we investigate the class of generalised triangles with 3-leaf spanning trees and analyse their chromatic roots. For $i, j, k \in \mathbb{N}_0$, define $G_{i,j,k}$ to be the graph composed of two vertices x and y, and three $\{x, y\}$-bridges which are copies of $F(x, y, i)$, $F(x, y, j)$ and $F(y, x, k)$ respectively. Figure 3.3 shows the graph $G_{4,2,3}$. We let $\mathcal{G}' = \{G_{i,j,k} : i, j, k \in \mathbb{N}_0\}$. Note that for $i, k \in \mathbb{N}_0$, the graph $G_{i,0,k}$ is isomorphic to H_{i+k+2}.

It is easy to see that each $G_{i,j,k}$ is a generalised triangle and contains a 3-leaf spanning tree. In fact, $\mathcal{G}'$ is a complete characterisation of $\mathcal{G}_3 \cap \mathcal{K}$ up to Whitney-equivalence.
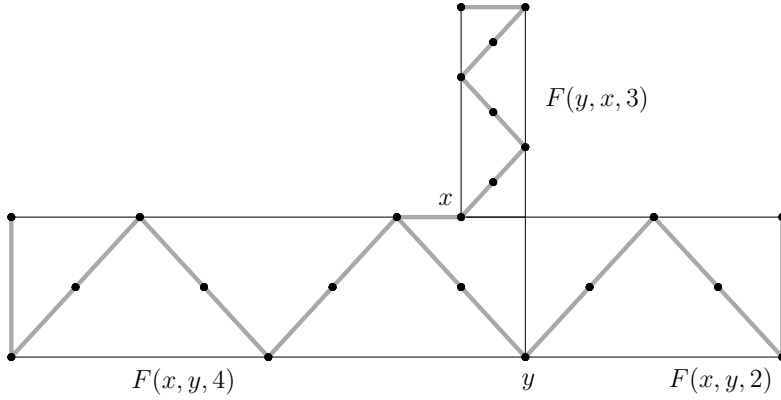
Figure 3.3: The graph $G_{4,2,3}$ and a 3-leaf spanning tree thereof.

LEMMA 3.12 *If G is a generalised triangle with a 3-leaf spanning tree, then G is Whitney-equivalent to $G_{i,j,k}$ for some $i, j, k \in \mathbb{N}_0$.*

PROOF. If G contains a Hamiltonian path, then by Corollary 3.11, we have that $G \in \mathcal{H}'$. Thus, G is isomorphic to $G_{i,0,k}$ for some $i, k \in \mathbb{N}_0$.

Now we may assume that G has a 3-leaf spanning tree T. Let v be the vertex of degree 3 in T. Since G is not 3-connected, there is a 2-cut of G. Consider a 2-cut $S = \{x, y\}$ so that the smallest S-bridge B_v containing v has as few vertices as possible. If $v \notin S$, then since B_v is separable it has a cut-vertex u. Also, since v has degree 3 in T, we have $|V(B_v)| \geq 4$. Thus, one of $\{x, u\}$ or $\{y, u\}$ contradicts the minimality of S, so we may assume that $v \in S$. We now find a 2-cut S such that $v \in S$, and the three neighbours of v in T, denoted v_1, v_2 and v_3, lie in three different S-bridges. If this is not already the case, then choose a 2-cut S such that $v \in S$, and the S-bridge containing two of v_1, v_2, v_3 is as small as possible. By a similar argument we find a 2-cut with the desired property.

Now fix the 2-cut $\{x, y\}$ so that y has degree 3 in T. For $i \in \{1, 2, 3\}$, let B_i be an $\{x, y\}$-bridge, and $y_i \in V(B_i)$ be the neighbours of y in T. Finally for $i \in \{1, 2, 3\}$ we let P_i be the unique path in T from y to a leaf of T, which
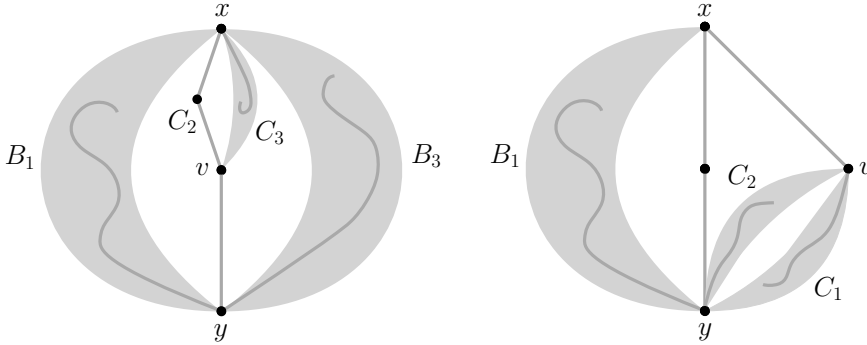
Figure 3.4: Cases 1 and 2 in the proof of Lemma 3.12.

contains the vertex y_i. Suppose without loss of generality that x lies on P_2. We distinguish two cases.

CASE 1. $V(P_2) = V(B_2)$.

In this case, P_1 and P_3 are paths which start at y, and cover all vertices of $B_1 - x$ and $B_3 - x$ respectively. By Lemma 3.10(b), B_1 is a copy of $F(x, y, i)$ and B_3 is a copy of $F(x, y, k)$ for some $i, k \in \mathbb{N}_0$. If P_2 ends at x then Lemma 3.10(a) implies B_2 is a copy of $F(x, y, 0)$ and by performing a Whitney 2-switch of B_3 about $\{x, y\}$ we are done. So suppose P_2 ends at some vertex z other than x, see Figure 3.4. Since B_2 is separable, it has a cut-vertex v, and since P contains a subpath connecting y and x, it follows that v lies between y and x on P. Now let Q_1, Q_2 and Q_3 be the subpaths of P_2 from y to v, v to x, and x to z respectively. Now $G - \{y, v\}$ contains at most two components, and so by Proposition 3.9, Q_1 is the edge yv. Since $|V(B)| \geq 4$, we have that $\{x, v\}$ is a 2-cut with precisely three $\{x, v\}$-bridges, two of which are contained in B_2. Let C_2 and C_3 be the $\{x, v\}$-bridges containing Q_2 and Q_3 respectively. Q_2 is a Hamiltonian path of C_2 from v to x. Thus, by Lemma 3.10(a), C_2 is a copy of $F(x, v, 0)$. Similarly, C_3 contains a path starting at x and covering all vertices of C_3 except for v. By Lemma 3.10(b) it follows that B_3 is a copy of $F(v, x, j - 1)$ for some $j \in \mathbb{N}_0$. Now performing a Whitney 2-switch of C_3 with respect to $\{x, v\}$ gives a new graph, where the $\{x, y\}$-bridge corresponding to B_2

is $F(x, y, j)$. Finally, performing a Whitney 2-switch of B_3 about $\{x, y\}$ gives a graph isomorphic to $G_{i,j,k}$ as required.

CASE 2. $V(P_2) \supset V(B_2)$.

Suppose without loss of generality that P_2 also contains vertices of $B_3 - x - y$. As before, Lemma 3.10(b) implies that B_1 is a copy of $F(x, y, i - 1)$ for some $i \in \mathbb{N}_0$, and Lemma 3.10(a) implies that B_2 is a copy of $F(x, y, 0)$. Now $T[V(B_3)]$ consists of two disjoint paths Q_1 and Q_2, starting at x and y respectively, see Figure 3.4. Since B_3 is separable, it has a cut-vertex v. We may assume that $v \in V(Q_1)$. If this is not the case, then we perform a Whitney 2-switch of B_3 with respect to $\{x, y\}$ and proceed similarly. Both Q_1 and Q_2 contain at least one edge, thus $|V(B_3)| \geq 4$ and at least one of $\{x, v\}$ and $\{y, v\}$ is a 2-cut of G. Suppose for a contradiction that $\{x, v\}$ is a 2-cut. Since G is a generalised triangle, G has precisely three $\{x, v\}$-bridges, two of which are contained in B_3. Since $v \in V(Q_1)$, all vertices of these two bridges must be covered by Q_1. Now, if $\{v, y\}$ is also a 2-cut, then similarly, there must be two $\{v, y\}$-bridges contained in B_3. But, since $v \in V(Q_1)$, it is not possible that Q_2 can cover all of these vertices. Thus, we deduce that $\{v, y\}$ is not a 2-cut, and hence by Proposition 3.9, v is the unique neighbour of y in B_3. Because $v \in V(Q_1)$, this means that Q_2 is a single vertex, which contradicts the fact that Q_2 contains at least one edge.

We may now assume that v is the unique neighbour of x in B_3, and that $\{y, v\}$ is a 2-cut of G. As before, G has precisely three $\{v, y\}$-bridges, two of which are contained in B_3. Denote the two $\{v, y\}$-bridges which are contained in B_3 by C_1 and C_2, where Q_2 is contained in C_2, see Figure 3.4. It is now easy to see that Q_2 is a path of C_2 starting at y and containing all vertices of C_2 except for v. Similarly, $Q_1[V(C_1)]$ is a path of C_1 starting at v and containing all vertices of C_1 except for y. Thus, by Lemma 3.10(b), C_1 is a copy of $F(y, v, k)$ and C_2 is a copy of $F(v, y, j)$ for some $j, k \in \mathbb{N}_0$.

Finally, consider the 2-cut $\{v, y\}$ of G. It has three bridges two of which are

C_1 and C_2 found in B_3. The third $\{v, y\}$-bridge, denoted C_3, is composed of the edge xv and the two $\{x, y\}$-bridges $F(x, y, 0)$ and $F(x, y, i-1)$ of G. By performing a Whitney 2-switch of $F(x, y, i-1)$ at $\{x, y\}$, we get a new graph G', where the $\{v, y\}$-bridge of G' corresponding to C_3 is precisely $F(v, y, i)$. Now G' is isomorphic to $G_{i,j,k}$ as required.   □

By the remark following Proposition 1.7, Lemma 3.12 implies that $\{P(G, q) : G \in \mathcal{G}_3 \cap \mathcal{K}\} = \{P(G, q) : G \in \mathcal{G}'\}$. This implies the following.

COROLLARY 3.13 $\omega(\mathcal{G}_3 \cap \mathcal{K}) = \omega(\mathcal{G}')$.

We now determine the behaviour of the chromatic roots of each $G \in \mathcal{G}'$.

LEMMA 3.14 $\omega(\mathcal{G}') = t_3$, where $t_3 \approx 1.290$ is the smallest real root of the polynomial $(q-2)^6 + 4(q-1)^2(q-2)^3 - (q-1)^4$.

PROOF. Recall that t_2 is the real number defined in Theorem 3.1 and $\mathcal{H}' = \{H_\ell : \ell \in \mathbb{N}_0\}$ is the class of graphs defined in Section 3.2. As mentioned, it is easily seen that $G_{i,0,k} = H_{i+k+2}$. If $j = 1$, then by Proposition 1.4 and 1.6 applied to y and the cut-vertex v of $F(x, y, j)$, we find that

$$P(G_{i,1,k}, q) = P(G_{i,1,k} + vy, q) + P(G_{i,1,k}/vy, q)$$
$$= (q-2)^2 P(H_{i+k+2}, q) + \frac{(q-1)}{q} P(H_{i+1}, q) P(H_k, q).$$

Also, if $j \geq 2$, then using Proposition 1.4 and 1.6 on the vertices x_{2j} and x_{2j+2} of $F(x, y, j)$ gives the recurrence

$$P(G_{i,j,k}, q) = P(G_{i,j,k} + x_{2j}x_{2j+2}, q) + P(G_{i,j,k}/x_{2j}x_{2j+2}, q)$$
$$= (q-2)^2 P(G_{i,j-1,k}, q) + (q-1)^2(q-2) P(G_{i,j-2,k}, q).$$

Solving this second order recurrence explicitly for fixed $q \in (1, t_2]$ gives a solution of the form $P(G_{i,j,k}, q) = C\alpha^j + D\beta^j$, where C and D are constants depending on i, k and q. Recall that α and β are defined in (3.3). The initial conditions

corresponding to $j = 0$ and $j = 1$ are

$$C + D = P(H_{i+k+2}, q), \text{ and} \tag{3.7}$$

$$C\alpha + D\beta = (q-2)^2 P(H_{i+k+2}, q) + \frac{q-1}{q} P(H_{i+1}, q) P(H_k, q). \tag{3.8}$$

Multiplying (3.7) by β, and subtracting the resulting equation from (3.8) gives

$$C(\alpha - \beta) = [(q-2)^2 - \beta] P(H_{i+k+2}, q) + \frac{q-1}{q} P(H_{i+1}, q) P(H_k, q)$$

$$= \alpha P(H_{i+k+2}, q) + \frac{q-1}{q} P(H_{i+1}, q) P(H_k, q). \tag{3.9}$$

For convenience let us write $\gamma = \gamma(q) = \alpha q/(q-1)$. Note that for $q \in (1, t_2]$, we have $\gamma(q) > 0$. Now let t_3 be the smallest real root of the polynomial

$$(q-2)^6 + 4(q-1)^2(q-2)^3 - (q-1)^4. \tag{3.10}$$

CLAIM 1 *For $q \in (1, t_2]$, we have $\gamma\beta < -A$. Furthermore, if $q \in (1, t_2]$, then $-A \leq \gamma\alpha$ if and only if $q \in (1, t_3]$.*

The first inequality follows since $\gamma\beta = -q(q-1)(q-2)$ and so by (3.4), we have $-A - \gamma\beta = B > 0$. The second assertion follows since by (3.6) and the subsequent inequalities, we have

$$
\begin{array}{rrcl}
& -A & \leq & \gamma\alpha \\
\iff & -\frac{1}{\delta}q(q-1)[(q-2)\alpha + (q-1)^2] & \leq & \frac{q}{q-1}(q-2)[(q-2)\alpha + (q-1)^2] \\
\iff & -(q-1)^2 & \geq & (q-2)\delta \\
\iff & (q-1)^4 & \leq & (q-2)^2\delta^2.
\end{array}
$$

Using (3.2), the final inequality is seen to be satisfied precisely when the polynomial in (3.10) is non-negative. For $q \in (1, t_2]$, this is the case if and only if $q \in (1, t_3]$, which completes the proof of Claim 1.

Since each $H \in \mathcal{H}'$ is non-separable and has an odd number of vertices, Theorems 2.3(v) and 3.1 imply that $P(H, q) < 0$ for $q \in (1, t_2]$. It now follows that

(3.9), and hence C, are negative if

$$\gamma |P(H_{i+k+2}, q)| > P(H_{i+1}, q)P(H_k, q).$$

Recall that $0 < \beta < \alpha < 1$ and $0 < B < -A < 1$ for $q \in (1, t_3]$. Together with (3.1) and Claim 1, we have that for $q \in (1, t_3]$,

$$\begin{aligned}
P(H_{i+1}, q)P(H_k, q) &= (A\alpha^{i+1} + B\beta^{i+1})(A\alpha^k + B\beta^k) \\
&= A^2\alpha^{i+k+1} + AB\alpha^{i+1}\beta^k + AB\alpha^k\beta^{i+1} + B^2\beta^{i+k+1} \\
&< -A\gamma\alpha^{i+k+2} - B^2\beta^{i+k+1} - B\gamma\beta^{i+k+2} + B^2\beta^{i+k+1} \\
&= \gamma(-A\alpha^{i+k+2} - B\beta^{i+k+2}) = \gamma|P(H_{i+k+2}, q)|.
\end{aligned}$$

Since $C < 0$, equality (3.7) implies that $D < -C$. Finally, since $\alpha > \beta$, we may conclude that $P(G_{i,j,k}, q) = C\alpha^j + D\beta^j < 0$ for $q \in (1, t_3]$.

Now suppose that $q \in (t_3, t_2)$ is fixed. By Claim 1 we have $-A > \gamma\alpha$. Setting $i + 1 = k$ for simplicity we see that

$$\frac{P(H_k, q)^2}{\gamma|P(H_{2k+1}, q)|} = \frac{A^2\alpha^{2k} + 2AB\alpha^k\beta^k + B^2\beta^{2k}}{\gamma(-A\alpha^{2k+1} - B\beta^{2k+1})} \longrightarrow \frac{A^2}{-\gamma A\alpha} = \frac{-A}{\gamma\alpha} > 1,$$

as $k \to \infty$. Thus, for large enough i and k, we have $\gamma|P(H_{i+k+2}, q)| < P(H_{i+1}, q)P(H_k, q)$. Hence C is positive. Though (3.7) implies that D is negative, since $\alpha > \beta$ it follows that for large enough j, we have $P(G_{i,j,k}, q) > 0$. Since we have proven that $P(G_{i,j,k}, q) < 0$ on $(1, t_3]$, we may conclude by continuity that $P(G_{i,j,k}, q)$ has a root in the interval (t_3, q). □

3.3.2 Reduction to Generalised Triangles

In this section we use the generalised triangle method presented in Section 2.2 to prove the following lemma. Together with Corollary 3.13 and Lemma 3.14, this completes the proof of Theorem 3.2.

LEMMA 3.15 $\omega(\mathcal{G}_3) = \omega(\mathcal{G}_3 \cap \mathcal{K})$.

We first note a number of properties of the class $\mathcal{G}_k$ and the graphs therein.

LEMMA 3.16 *For all $k \geq 2$, the class of graphs $\mathcal{G}_k$ is splitting-closed.*

PROOF. If G is a graph with a k-leaf spanning tree T, then clearly G is connected and for every block B of G, the graph $T[V(B)]$ is a k-leaf spanning tree of B. Now let $\{x, y\}$ be a 2-cut of G, let B be an $\{x, y\}$-bridge, and let $T' = T[V(B)]$. If T' is disconnected, then $T' + xy$ and T'/xy are k-leaf spanning trees of $B + xy$ and B/xy respectively. Similarly, if T' is connected, then T' is clearly a k-leaf spanning tree of B and $B + xy$, so it just remains to consider B/xy.

Note first that since T' is connected, there is a leaf vertex u of T in an $\{x, y\}$-bridge distinct from B. Thus, the graph T'/xy has at most $k - 1$ vertices of degree 1. There is also a unique cycle C of T'/xy containing the vertex v formed from the contraction of x and y. Now delete an edge e of C adjacent with v. If $d_{T'/xy}(v) \geq 3$, then this creates at most one new leaf and so $T'/xy - e$ is a k-leaf spanning tree of B/xy. On the other hand, if $d_{T'/xy}(v) \leq 2$, then both x and y are leaves of T'. This implies that T'/xy has at most $k - 2$ vertices of degree 1. Since deleting e creates at most 2 new leaves, the graph $T'/xy - e$ is again a k-leaf spanning tree of B/xy as required. $\qquad\square$

LEMMA 3.17 *For all $k \geq 2$, the class of graphs $\mathcal{G}_k$ is connectivity-reducible.*

PROOF. Let G be a 3-connected graph and let T be a k-leaf spanning tree of G. Let v be a leaf of T, and e be an edge of G which is incident to v but not in T. Such an edge exists since G is non-separable. It is easy to see that T is a k-leaf spanning tree of $G - e$ and $T - v$ is a k-leaf spanning tree of G/e. $\qquad\square$

We can now prove the main result of this section.

PROOF OF LEMMA 3.15. Clearly we have $\omega(\mathcal{G}_3) \leq \omega(\mathcal{G}_3 \cap \mathcal{K})$, thus it suffices to show that $Q(G, q) > 0$ for $G \in \mathcal{G}_3$ and $q \in (1, \omega(\mathcal{G}_3 \cap \mathcal{K}))$. Suppose G

is a smallest counterexample to this statement, so there is $q \in (1, \omega(\mathcal{G}_3 \cap \mathcal{K}))$ such that $Q(G, q) \leq 0$. By Lemmas 3.16 and 3.17, $\mathcal{G}_3$ is splitting-closed and connectivity-reducible. Thus, by Lemmas 2.7 and 2.9, we have that G is non-separable, not 3-connected, and for every 2-cut $\{x, y\}$, we have $xy \notin E(G)$. In other words, G satisfies properties (GT1), (GT2) and (GT3) of Proposition 2.1. Let $\{x, y\}$ be a 2-cut of G. Since $\mathcal{G}_3$ is splitting-closed, Lemma 2.8 implies that G has an odd number of $\{x, y\}$-bridges. Moreover, because G has a 3-leaf spanning tree, the graph $G - \{x, y\}$ has at most four components. It follows that G has precisely three $\{x, y\}$-bridges and thus satisfies property (GT4) of Proposition 2.1.

If, in addition, G satisfies property (GT5), then by Proposition 2.1, we deduce that $G \in \mathcal{K}$, which is a contradiction. Therefore it only remains to verify the following claim.

CLAIM 2 *If $\{x, y\}$ is a 2-cut of G, then every $\{x, y\}$-bridge is separable.*

Let T be a 3-leaf spanning tree of G. Let B be an arbitrary $\{x, y\}$-bridge and suppose for a contradiction that B is non-separable. Since $xy \notin E(G)$, we have $|V(B)| \geq 4$. We may assume that B contains at most two leaves of T, since if T has three leaves in B, then $G - \{x, y\}$ has at most two components. Relabelling x and y if necessary, there are four cases how $T[V(B)]$ may behave:

CASE 1: $T[V(B)]$ is connected.

CASE 2: $T[V(B)]$ consists of an isolated vertex x and a path starting at y and covering all vertices of $B - x$.

CASE 3: $T[V(B)]$ consists of an isolated vertex x and a tree with precisely three leaves, one of which is y, covering all vertices of $B - x$.

CASE 4: $T[V(B)]$ consists of two disjoint paths P_1 and P_2, starting at x and y respectively, which together cover all vertices of B.
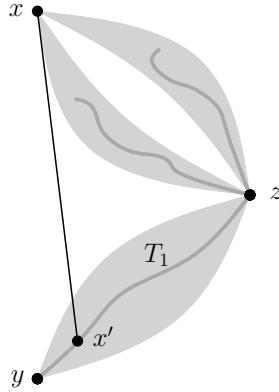
Figure 3.5: Subcase 3A of Claim 2.

In Case 1, the graph $T[V(B)]$ is a 3-leaf spanning tree of B. In Case 2, adding any edge of B incident with x to $T[V(B)]$ also shows that B contains such a spanning tree. Let (U, V') be a bridge-partition of G, where $U = B$ and V' denotes the graph defined before Lemma 2.11. In these two cases, we have $U \in \mathcal{G}_3$. Furthermore, it is easy to check that $V' \in \mathcal{G}_3$. Hence, by Lemma 2.12, B is separable. We now deal with the remaining cases.

CASE 3. $T[V(B)]$ consists of an isolated vertex x and a tree with precisely three leaves, one of which is y, covering all vertices of $B - x$.

Let v be the vertex of degree 3 in T, and let T_1 be the path in T from y to v.

SUBCASE 3A. The graph $B - x$ is separable.

Let z be a cut-vertex of $B - x$. So $\{x, z\}$ is a 2-cut of B and a 2-cut of G. Since G satisfies (GT4), G has precisely three $\{x, z\}$-bridges, two of which are contained in B. Due to the structure of T, this implies that $v = z$ and T_1 is a Hamiltonian path of the block of $B - x$ which contains y, see Figure 3.5. Since B is non-separable, $V(T_1) \geq 3$ and x has a neighbour in $V(T_1) \setminus \{y, z\}$. Choose such a neighbour, x', from which the distance to y on T_1 is minimal. Note that there are two internally vertex-disjoint paths from x to x' avoiding the edge xx'

itself, one through y and one through v. Thus, $G - xx'$ is non-separable and has a 3-leaf spanning tree. Since G has property (GT3), the set $\{x, x'\}$ is not a 2-cut of G. Thus, G/xx' is also non-separable. Now Proposition 2.4 gives

$$Q(G, q) = Q(G - xx', q) + Q(G/xx', q). \qquad (3.11)$$

Since G is a smallest counterexample, $Q(G - xx', q) > 0$. Thus, we have reached a contradiction if $Q(G/xx', q) > 0$. This follows immediately if G/xx' has a 3-leaf spanning tree. Otherwise we use Lemma 2.11 on G/xx' as follows.

Let (U, V') be an $\{x, y\}$-bridge-partition of G/xx' where $U = B/xx'$. Note that $U, U + xy, U/xy$ and V' all have a 3-leaf spanning tree. Furthermore, since $\mathcal{G}_3$ is splitting-closed, we have $B' + xy, B'/xy \in \mathcal{G}_3$ for all other $\{x, y\}$-bridges B' of G/xx'. We claim that U is non-separable. If this is not the case, then $\{x, x'\}$ is a 2-cut of B. By the choice of x', this implies that $\{x', y\}$ is a 2-cut of G. Now, by (GT4), G must have three $\{x', y\}$-bridges, two of which lie inside B. However, due to the structure of T_1, the vertices of these two bridges cannot be covered by T_1. We deduce that U is separable and hence all conditions of Lemma 2.11 are satisfied. Hence, we get $Q(G, q) > 0$ which is a contradiction. Therefore B is separable.

SUBCASE 3B. The graph $B - x$ is non-separable.

Since $B - x$ is non-separable, we have in particular that $G - e$ is non-separable for every edge $e \in E(B)$ incident to x. Choose a neighbour, x' of x, such that the distance on $T[V(B)]$ from v to x' is maximal. Note that if x' is a leaf of T, then B contains a 3-leaf spanning tree and so we are done by the same argument as in Cases 1 and 2. Since B is non-separable, x has at least two neighbours in B and so $x' \neq v$. By the same argument as in Case 3A, both $G - xx'$ and G/xx' are non-separable. Furthermore $G - xx'$ has a 3-leaf spanning tree. As before, by (3.11), we have reached a contradiction if $Q(G/xx', q) > 0$. This follows immediately if G/xx' has a 3-leaf spanning tree. Otherwise we apply

Lemma 2.11 to G/xx' as in Case 3A. The same argument shows that by the choice of x', the hypotheses of Lemma 2.11 hold.

CASE 4. The graph $T[V(B)]$ consists of two disjoint paths P_1 and P_2, starting at x and y respectively, which together cover all vertices of B.

Let $P_1 = x_1, \ldots, x_{n_1}$ and $P_2 = y_1, \ldots, y_{n_2}$ where $x = x_1$ and $y = y_1$. If x or x_{n_1} has a neighbour x' on P_2, then B contains a 3-leaf spanning tree. As in Cases 1 and 2, this is enough to reach a contradiction. Since B is non-separable, this means that $|V(P_1)| \geq 4$ and all neighbours of x_{n_1} lie on P_1. Also, apart from x_{n_1-1}, the vertex x_{n_1} has at least one other neighbour on P_1. If x_{n_1} has at least two other neighbours, say x_i, x_j with $i < j$, then $G - x_{n_1}x_j$ and $G/x_{n_1}x_j$ are non-separable and have a 3-leaf spanning tree. By Proposition 2.4 we reach a contradiction. So we may suppose that $d(x_{n_1}) = 2$ and $N(x_{n_1}) = \{x_{n_1-1}, x_i\}$ for some $i \in \{1, \ldots, n_1 - 2\}$. It follows that $\{x_i, x_{n_1-1}\}$ is a 2-cut of G. We now make the following claim about 2-cuts on P_1.

CLAIM 3 *Let x_a and x_b be vertices of P_1 with $a < b < n_1$. If $\{x_a, x_b\}$ is a 2-cut of G, then $b = a + 2$ and $d(x_{a+1}) = 2$.*

First note that because G satisfies property (GT4), there are precisely three $\{x_a, x_b\}$-bridges, and both of x_a and x_b have a neighbour in each such bridge. Let B_y and B_{n_1} be the $\{x_a, x_b\}$-bridges of G containing y and x_{n_1} respectively, and let B' be the $\{x_a, x_b\}$-bridge containing the subpath of P_1 from x_a to x_b. Finally, let e_1 be an edge of B_{n_1} incident to a_i, and let e_2 be an edge of B_y incident to x_b. Since $T[V(B')]$ is a Hamiltonian path of B', it follows from Case 1 that B' is separable. Thus, it has a cut-vertex v. Because of the edges e_1 and e_2, we see that $G - \{x_a, v\}$ and $G - \{v, x_b\}$ both have at most two components. By property (GT4), neither of $\{x_a, v\}$ and $\{v, x_b\}$ are 2-cuts of G. Thus, by two applications of Proposition 3.9, we deduce that B' is a path of length 2, which implies that $b = a + 2$, and $d(x_{a+1}) = 2$. This completes the proof of Claim 3.
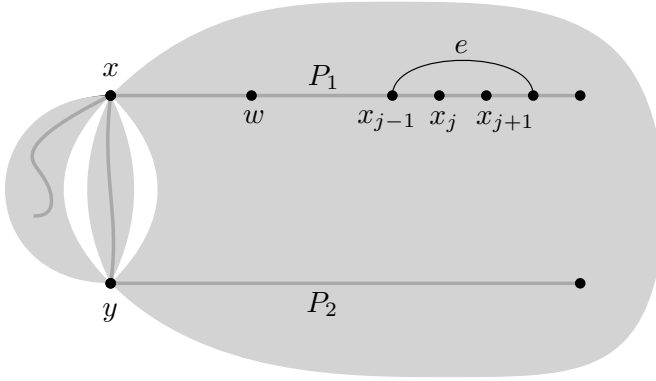
Figure 3.6: Case 4, when $w \in V(P_1)$.

By Claim 3 applied with $a = i$ and $b = n_1 - 1$, we conclude that $\{x_{n_1-3}, x_{n_1-1}\}$ is a 2-cut of G, and $d(x_{n_1-2}) = 2$. Thus, there is at least one vertex of degree 2 in the interior of P_1.

Now let $x_j \in V(P_1) \setminus \{x_1, x_{n_1}\}$ be a vertex of degree 2 with j as small as possible, see Figure 3.6. Then $\{x_{j-1}, x_{j+1}\}$ is a 2-cut and G contains precisely three $\{x_{j-1}, x_{j+1}\}$-bridges. Since each of x_{j-1} and x_{j+1} has a neighbour in each $\{x_{j-1}, x_{j+1}\}$-bridge, there is some edge e from x_{j-1} to one of $x_{j+2}, x_{j+3}, \ldots, x_{n_1}$. Now consider the $\{x_{j-1}, x_{j+1}\}$-bridge, B_y, containing y. This bridge contains a 3-leaf spanning tree covering all of its vertices apart from x_{j+1}. By Case 3, it is separable and has a cut-vertex w. It is easy to see that w must either lie on the subpath of P_1 from x to x_{j-2} or on P_2.

We claim that w lies on P_2, so suppose for a contradiction that w lies on the subpath of P_1 from x to x_{j-2}, see Figure 3.6. Since $|V(B_y)| \geq 4$, at least one of $\{x_{j-1}, w\}$ and $\{x_{j+1}, w\}$ is a 2-cut of G. However, because of the edge e and the presence of the other $\{x, y\}$-bridges, the graph $G - \{x_{j+1}, w\}$ has at most two components. Therefore $\{x_{j-1}, w\}$ is a 2-cut of G with three $\{x_{j-1}, w\}$-bridges, one of which contains the subpath of P_1 from w to x_{j-1}. This path has length at least 2 by property (GT3). By Claim 3, we deduce that $w = x_{j-3}$ and $d(x_{j-2}) = 2$, contradicting the minimality of j.
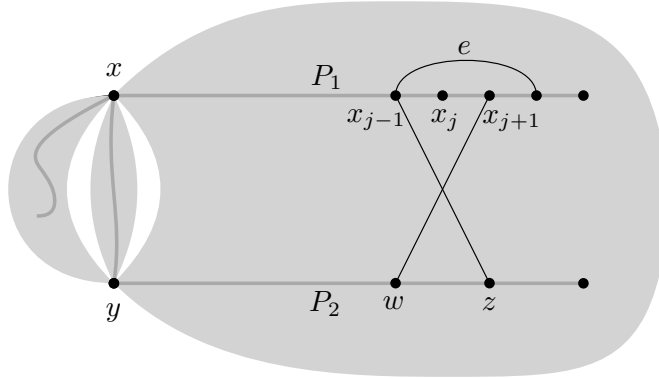
Figure 3.7: Case 4, when $w \in V(P_2)$.

Thus, the vertex w lies on the path P_2, see Figure 3.7. As before, at least one of $\{x_{j-1}, w\}$ or $\{x_{j+1}, w\}$ is a 2-cut of G. However, since $G - \{x_{j+1}, w\}$ has at most two components, property (GT4) implies that $\{x_{j+1}, w\}$ is not a 2-cut of G. By Proposition 3.9, it follows that $wx_{j+1} \in E(G)$ and $\{x_{j-1}, w\}$ is a 2-cut with precisely three bridges. Let B_5 be the $\{x_{j-1}, w\}$-bridge containing the vertex y_{n_2}. So the subpath of P_2 from w to y_{n_2} is a path of B_5 covering all vertices of B_5 except x_{j-1}. By Case 2, B_5 is separable and has a cut-vertex z on the subpath of P_2 from w to y_{n_2}. Because of the edge wx_{j+1}, the graph $G - \{x_{j-1}, z\}$ has at most two components. Therefore, by (GT4), $\{x_{j-1}, z\}$ is not a 2-cut of G and Proposition 3.9 implies that $zx_{j-1} \in E(G)$.

Finally note that $x_{j-1} \neq x$ and $w \neq y$, or else B would contain a 3-leaf spanning tree. Consider the $\{x_{j-1}, w\}$-bridge B_6 containing x and y. $T[V(B_6)]$ is connected, so by Case 1 the bridge B_6 has a cut-vertex z'. Since $|V(B_6)| \geq 4$, at least one of $\{x_{j-1}, z'\}$ and $\{w, z'\}$ is a 2-cut of G. However because of the edges zx_{j-1} and wx_{j+1}, both $G - \{x_{j-1}, z'\}$ and $G - \{w, z'\}$ have at most two components. This contradicts (GT4), so B is separable and Claim 2 holds. $\square$

Minor-Closed Classes of Graphs

4.1 Introduction

In Chapter 2 we noted a close connection between generalised triangles and minor-closed classes of graphs. In particular, if $\mathcal{G}$ is a minor-closed class of graphs, then Theorem 2.15 says that $\omega(\mathcal{G}) = \omega(\mathcal{G} \cap \mathcal{K})$. From this, it is easy to deduce a number of simple corollaries. For example, let $\mathcal{G}$ denote the class of $K_{2,3}$-minor-free graphs. Since every generalised triangle except K_3 contains $K_{2,3}$ as a minor, we have $\omega(\mathcal{G}) = \omega(\mathcal{G} \cap \mathcal{K}) = \omega(\{K_3\}) = 2$. On the other hand, if $\mathcal{G}$ denotes the series-parallel graphs, then $\omega(\mathcal{G}) = \omega(\mathcal{G} \cap \mathcal{K}) = \omega(\mathcal{K}) = 32/27$, since every generalised triangle is series-parallel. More generally, since $32/27$ is not a chromatic root of any graph, we have the following observation.

PROPOSITION 4.1 *If $\mathcal{G}$ is a minor-closed class of graphs such that $\mathcal{G} \cap \mathcal{K}$ is finite, then $\omega(\mathcal{G}) > 32/27$.*
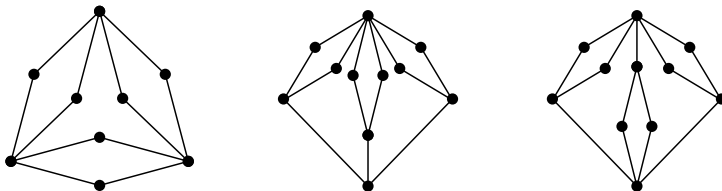
Figure 4.1: From left to right, the graphs H_0, H_1 and H_2.

Dong and Koh [DK10] noted that for any $r \geq 3$, the class of graphs not containing the complete bipartite graph $K_{2,r}$ as a minor is such a class.

For all previously investigated minor-closed classes $\mathcal{G}$, the set of generalised triangles in $\mathcal{G}$ is either finite or equal to $\mathcal{K}$. In such cases it is easy to determine $\omega(\mathcal{G})$. Indeed if $\mathcal{G} \cap \mathcal{K} = \mathcal{K}$, then $\omega(\mathcal{G}) = 32/27$, while if $\mathcal{G} \cap \mathcal{K}$ is finite, then $\omega(\mathcal{G})$ is the minimum of the non-trivial roots of $G \in \mathcal{G} \cap \mathcal{K}$, which is a finite problem. In this chapter we precisely determine $\omega(\mathcal{G})$ for three minor-closed classes of graphs, each of which has the property that $\mathcal{G} \cap \mathcal{K}$ is infinite, but not equal to $\mathcal{K}$.

THEOREM 4.2 *Let H_0, H_1 and H_2 be the graphs in Figure 4.1.*

(i) *If $\mathcal{G}$ is the class of H_0-minor-free graphs, then $\omega(\mathcal{G}) = 5/4$.*

(ii) *If $\mathcal{G}$ is the class of $\{H_1, H_2\}$-minor-free graphs, then $\omega(\mathcal{G}) = t_1$, where $t_1 \approx 1.225$ is the real root of $q^4 - 4q^3 + 4q^2 - 4q + 4$ in $(1, 2)$.*

(iii) *If $\mathcal{G}$ is the class of $\{H_0, H_1, H_2\}$-minor-free graphs, then $\omega(\mathcal{G}) = t_2$, where $t_2 \approx 1.296$ is the unique real root of $q^3 - 2q^2 + 4q - 4$.*

Since the graphs H_0, H_1 and H_2 are some of the smallest generalised triangles, Theorem 4.2 provides evidence for the following conjecture.

CONJECTURE 4.3 *If $\mathcal{G}$ is a minor-closed class of graphs, then $\omega(\mathcal{G}) > 32/27$ if and only if $\mathcal{G}$ does not contain all generalised triangles.*

One can easily show that Conjecture 4.3 is equivalent to the following.

Conjecture 4.4 *If $\mathcal{G}$ is a minor-closed class of graphs, then $\omega(\mathcal{G}) > 32/27$ if and only if $\mathcal{G}$ does not contain all series-parallel graphs.*

The intervals we find in Theorem 4.2 coincide with those obtained or conjectured for other, seemingly unrelated, classes of graphs. Notice for example that the interval in Theorem 4.2(iii) is the same as that of Theorem 3.1. This connection will be fully explained. Furthermore, the intervals in parts (i) and (ii) of Theorem 4.2 coincide precisely with those in important conjectures of Dong and Jackson. These conjectures would have implications for the chromatic roots of 3-connected graphs, about which very little is currently known. We describe how our results suggest that it might be fruitful to attack a relaxed version of these conjectures.

4.2 Generalised Triangles and Minors

In this section we collect a few preliminary results regarding generalised triangles and graph minors.

Proposition 4.5 *Let G be a generalised triangle and $\{x, y\}$ be a 2-cut of G. If B is an $\{x, y\}$-bridge, then B has precisely two blocks.*

Proof. Consider the block graph T of B, whose vertices are the cut-vertices and blocks of B, see [Die16, Lemma 3.1.4]. Since G is non-separable, T is a path. If T has more than three vertices, then there is a 2-cut of G which contradicts (GT4) of Proposition 2.1. Hence, B has precisely two blocks. $\square$

Proposition 4.6 *Suppose that G is a generalised triangle, $\{x, y\}$ is a 2-cut of G, and B is an $\{x, y\}$-bridge of G. The following hold.*

(i) *$B + xy$ is a generalised triangle.*

(ii) *If B is non-trivial, then there is a 2-cut $\{u, v\}$ of G such that $\{u, v\} \subseteq V(B)$, and the two $\{u, v\}$-bridges of G which are contained in B are trivial.*

Proof.

(i) It is easy to check that the conditions (GT1) - (GT4) in Proposition 2.1 hold for the graph $B + xy$. To see that (GT5) holds, let $\{u, v\}$ be a 2-cut of $B + xy$ such that some $\{u, v\}$-bridge B' of $B + xy$ is non-separable. Delete xy and reinstate the $\{x, y\}$-bridges of G distinct from B. It remains the case that the $\{u, v\}$-bridge of G corresponding to B' is non-separable, which contradicts the fact that G is a generalised triangle.

(ii) Let z be a vertex of G not in B. Choose a 2-cut $\{u, v\}$ of G such that $\{u, v\} \subseteq V(B)$, and the $\{u, v\}$-bridge of G containing z has as many vertices as possible. Suppose some $\{u, v\}$-bridge B' contained in B is not trivial. Proposition 2.1 implies that B' has a cut-vertex w. Since B' is not trivial, one of $\{u, w\}$ and $\{v, w\}$ is a 2-cut of G and the bridge of this 2-cut containing z is larger, a contradiction.   □

The double subdivision operation defines a partial order on the class of generalised triangles. More precisely, for $G, H \in \mathcal{K}$ we define $H \leq G$ if G can be obtained from H by a sequence of double subdivisions. The following lemma was presented in [Per16a].

LEMMA 4.7 *If $G, H \in \mathcal{K}$, then H is a minor of G if and only if G can be obtained from H by a sequence of double subdivisions.*

Unfortunately, Lemma 4.7 is false. Clearly, if G can be obtained from H by a sequence of double subdivision operations, then H is a minor of G, but the other direction does not hold. In [Per16a], the proofs of subsequent lemmas and Theorem 4.2 are shortened by a dependency on Lemma 4.7, however they hold nonetheless as we show in this chapter.

Let G and H be graphs such that H is a minor of G. We say that a graph J is a **fixed H-minor** of G if J is isomorphic to H, and is formed from G by deleting and contracting fixed sets of edges.
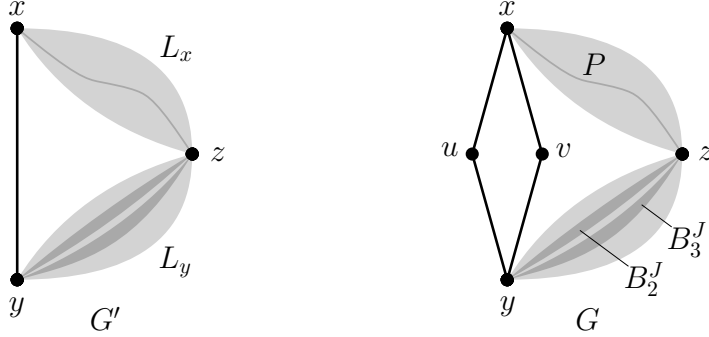
Figure 4.2: The structure of G' and G in the proof of Proposition 4.8.

PROPOSITION 4.8 *Let $G, G', H \in \mathcal{K}$ such that G is obtained from G' by applying a double subdivision operation to the edge $xy \in E(G')$, and let $u, v \in V(G)$ be the vertices created. If H is a minor of G but not a minor of G', then for any fixed H-minor J of G, we have $xu, xv, yu, yv \in E(J)$.*

PROOF. Let B be the $\{x, y\}$-bridge of G not containing u or v. Since G is a generalised triangle, Proposition 2.1 implies that B has a cut-vertex z which separates x from y. Furthermore, Proposition 4.5 shows that B has precisely two blocks, L_x and L_y, containing x and y respectively, see Figure 4.2.

Since H is a generalised triangle, it is clearly non-separable. Furthermore, since H is not a minor of the generalised triangle G', we deduce that $H \neq K_3$ and thus $|V(H)| > 3$. Because H is non-separable and a minor of G but not G', the vertices x and y are not identified to form J. Furthermore, at least one of u and v, say u, has neither of its adjacent edges deleted or contracted. Thus $xu, yu \in E(J)$ and u has degree 2 in J. This implies that $\{x, y\}$ is a 2-cut of J, and since H is a generalised triangle, (GT3) implies that the edge $xy \notin E(J)$ There are now two remaining possibilities: Either the edges xv and yv are deleted entirely, or $xv, yv \in E(J)$.

By (GT4), there are three $\{x, y\}$-bridges B_1^J, B_2^J and B_3^J, one of which, say B_1^J, is the path xuy. It remains to show that one of B_2^J or B_3^J is the path xvy,

so suppose for a contradiction that this is not the case, and that $B_2^J \cup B_3^J$ is a minor of B. Every path in B from x to y must go through the cut-vertex z of B. However, since B_2^J and B_3^J are distinct $\{x, y\}$-bridges in J, they each contain a path from x to y, and these paths are internally disjoint. It follows that to form J, the vertex z must be identified with either x or y, say x. In fact, since H is non-separable, J is a minor of the graph formed from G by contracting the whole of L_x to a single vertex. Thus $B_2^J \cup B_3^J$ is a minor of L_y, see Figure 4.2. Now let P be a path from x to z in L_x. Since P has at least one edge and B_1^J is a trivial bridge, it follows that B_1^J is a minor of the graph $P + xy$. But now J is a minor of the graph $L_y \cup P + xy$, which is a subgraph of G'. This contradicts the fact that H is not a minor of G'. $\qquad\qquad\Box$

4.3   Restricted Classes of Generalised Triangles

In this section we determine the value of $\omega(\mathcal{G})$ for several minor-closed classes of graphs. To do this, we investigate subsets $\mathcal{K}' \subseteq \mathcal{K}$ of generalised triangles defined by properties of their 2-cuts. We show that these subsets can be considered minor-closed within $\mathcal{K}$, in the sense that there is a minor-closed class of graphs $\mathcal{G}$ such that $\mathcal{G} \cap \mathcal{K} = \mathcal{K}'$. Using Theorem 2.15, we have $\omega(\mathcal{G}) = \omega(\mathcal{G} \cap \mathcal{K}) = \omega(\mathcal{K}')$, so determining $\omega(\mathcal{K}')$ gives the value $\omega(\mathcal{G})$ for the much larger class $\mathcal{G}$.

Definition 4.9 Let G be a graph.

(i) A 2-cut $\{x, y\}$ of G has **property P_0** if for every $\{x, y\}$-bridge B, at least one of x and y has degree 1 in B.

(ii) A 2-cut $\{x, y\}$ of G has **property P_1** if at least one $\{x, y\}$-bridge is trivial.

For $i \in \{0, 1\}$, define $\mathcal{K}_i$ to be the class of generalised triangles such that every 2-cut has property P_i.

We define a **generalised edge** to be either K_2 or any graph obtained from K_2 by a sequence of double subdivisions. When $V(K_2) = \{u, v\}$, we shall refer to a generalised edge obtained from this K_2 as a **generalised uv-edge**. Let G be a generalised uv-edge with $|V(G)| \geq 4$, and let B_1 and B_2 be the $\{u, v\}$-bridges of G. For $i \in \{0, 1\}$, we say G has property P_i if every 2-cut $\{x, y\}$ such that $\{x, y\} \subseteq V(B_j)$ for some $j \in \{1, 2\}$ has property P_i.

Let $\mathcal{A} \subseteq \mathcal{K}$. We say $\mathcal{A}$ is a **downward-closed** subset of $\mathcal{K}$ if for all $G \in \mathcal{A}$ and $H \in \mathcal{K}$, we have that $H \leq G$ implies $H \in \mathcal{A}$. In practice, if $\mathcal{K}'$ is a class of generalised triangles defined by some graph property, then $\mathcal{K}'$ is frequently downward-closed. It is often possible to exploit this to determine the value of $\omega(\mathcal{K}')$.

LEMMA 4.10 $\mathcal{K}_0$ and $\mathcal{K}_1$ are downward-closed subsets of $\mathcal{K}$.

SKETCH OF PROOF. Let $i \in \{0, 1\}$. It suffices to show that if $G \in \mathcal{K}_i$ and G is formed from G' by a single double subdivision, then $G' \in \mathcal{K}_i$. The contrapositive of this statement is much easier to see. Indeed if $G' \notin \mathcal{K}_i$, then there is some 2-cut $\{x, y\}$ of G' which does not satisfy property P_i. The same vertices form a 2-cut of G which does not satisfy property P_i. $\qquad\qquad\square$

4.3.1 The Class $\mathcal{K}_0$

The aim of this section is to prove Theorem 4.2(i). To do this we first show that $\omega(\mathcal{K}_0) = 5/4$.

LEMMA 4.11 If $G \in \mathcal{K}_0$, then $Q(G, q) > 0$ for $q \in (1, 5/4]$.

The proof of this lemma is simple but fairly lengthy and can be found in Section 4.4.1. The idea is to prove several inequalities simultaneously by induction, one of which is the statement above.

LEMMA 4.12 $\omega(\mathcal{K}_0) = 5/4$.

PROOF. Let $J_0 = K_3$ and $x \in V(J_0)$. For $i \in \mathbb{N}$, let J_i be obtained from J_{i-1} by applying the double subdivision operation to each edge of J_{i-1} incident with x. Though it is non-trivial to verify, Dong and Jackson [DJ11, pg. 1114] say that these graphs have chromatic roots converging to 5/4 from above. We shall show that $J_i \in \mathcal{K}_0$ for each $i \in \mathbb{N}_0$. It then follows that $\omega(\mathcal{K}_0) \leq 5/4$, which together with Lemma 4.11 implies that $\omega(\mathcal{K}_0) = 5/4$.

Let $i \in \mathbb{N}_0$ and note that, by construction, every 2-cut of J_i contains the vertex x. Consider a 2-cut $\{x, y\}$ and let B be an $\{x, y\}$-bridge of J_i. Since J_i is a generalised triangle, Proposition 2.1 gives that B has a cut-vertex z which separates x from y. If y has degree at least 2 in B, then $\{y, z\}$ is a 2-cut of G, contradicting the fact that each 2-cut contains x. Thus, for each 2-cut $\{x, y\}$, the vertex y has degree 1 in each $\{x, y\}$-bridge. We conclude that each 2-cut has property P_0, so $J_i \in \mathcal{K}_0$ as desired.   □

Recall that H_0 is the graph depicted in Figure 4.1. It is formed from K_3 by applying the double subdivision operation to each edge. We now show that H_0 is the forbidden minor which characterises $\mathcal{K}_0$ within $\mathcal{K}$.

LEMMA 4.13 *If $G \in \mathcal{K}$, then $G \in \mathcal{K}_0$ if and only if G is H_0-minor-free.*

PROOF. If $G \in \mathcal{K} \setminus \mathcal{K}_0$, then G has a 2-cut $\{x, y\}$ with three $\{x, y\}$-bridges B_1, B_2 and B_3, such that both x and y have degree at least 2 in B_1 say. Since G is a generalised triangle, B_1 has a cut-vertex z. Furthermore, since both x and y have degree at least 2 in B_1, both of $\{x, z\}$ and $\{y, z\}$ are 2-cuts of G, and have precisely three bridges, two of which are contained in B_1. It is now easy to see that G contains H_0 as a minor.

Now let $G \in \mathcal{K}$ such that G contains H_0 as a minor, and suppose for a contradiction that $G \in \mathcal{K}_0$. We may assume that G is minimal with respect to the double subdivision operation. Let $G' \in \mathcal{K}$ be such that G is obtained from G' by double subdividing the edge $xy \in E(G')$. Since $\mathcal{K}_0$ is downward-closed, we have $G' \in \mathcal{K}_0$.
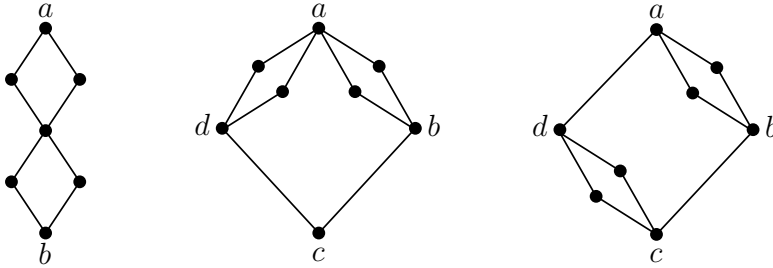
Figure 4.3: From left to right, the graphs H_0', H_1' and H_2'.

Let J be a fixed H_0-minor of G. By Proposition 4.8, we have that J contains both trivial $\{x, y\}$-bridges of G. Let B be the third $\{x, y\}$-bridge of G. Since $G \in \mathcal{K}_0$, one of x and y has degree 1 in B. Say $d_B(x) = 1$ and the neighbour of x in B is z. Since $|V(H_0)| = 9$ and H_0 is a minor of G, we have that $|V(G)| \geq 9$. Thus, $\{y, z\}$ is a 2-cut of G and gives rise to precisely three bridges, two of which, B_1 and B_2, are contained in B. Let H_0' be the graph consisting of two 4-cycles joined at a vertex, see Figure 4.3. It is easy to see that $B_1 \cup B_2$ must contain H_0' as a minor, where the vertices a and b of H_0' are identified with y and z respectively. Because of this, H_0' must in fact be a minor of either B_1 or B_2, say B_1. Finally, note that there are paths of length at least 2 from y to z in both B_2 and the $\{y, z\}$-bridge of G' containing the vertex x. Combining these paths with the H_0'-minor of B_1, we see that G' contains H_0 as a minor. This contradicts the minimality of G. $\qquad\square$

PROOF OF THEOREM 4.2(i). Let $\mathcal{G}$ be the class of H_0-minor-free graphs. By Theorem 2.15 and Lemma 4.13 we have $\omega(\mathcal{G}) = \omega(\mathcal{G} \cap \mathcal{K})$ and $\mathcal{G} \cap \mathcal{K} = \mathcal{K}_0$ respectively. Now, by Lemma 4.12, we have $\omega(\mathcal{K}_0) = 5/4$. The result follows. $\qquad\square$

Let $\mathcal{D}_0$ be the class of non-separable graphs for which there is a unique vertex which is contained in every 2-cut. Dong and Jackson [DJ11] conjecture that $\omega(\mathcal{D}_0) = 5/4$. This conjecture is significant because $\mathcal{D}_0$ contains the class of 3-connected graphs, and so a positive solution would give a lower bound on the non-trivial chromatic roots of 3-connected graphs, see Conjecture 3.5. Whilst

it can be shown that $\mathcal{D}_0 \cap \mathcal{K} \subset \mathcal{K}_0$, this does not prove the conjecture since it is not known if $\omega(\mathcal{D}_0) = \omega(\mathcal{D}_0 \cap \mathcal{K})$. In particular $\mathcal{D}_0$ is not minor-closed so Theorem 2.15 does not apply.

The fact that $\mathcal{D}_0 \cap \mathcal{K}$ is not the largest class of generalised triangles $\mathcal{K}'$ such that $\omega(\mathcal{K}') = 5/4$ suggests that a well chosen weaker property could be used to make progress on Dong and Jackson's conjecture.

PROBLEM 4.14 Find a class of graphs $\mathcal{D}$ such that $\mathcal{D}_0 \subseteq \mathcal{D}$, $\omega(\mathcal{D}) = \omega(\mathcal{D} \cap \mathcal{K})$ and $\mathcal{D} \cap \mathcal{K} = \mathcal{K}_0$.

4.3.2   The Class $\mathcal{K}_1$

In this section we prove Theorem 4.2(ii). To do this we first show that $\omega(\mathcal{K}_1) = t_1$, where $t_1 \approx 1.225$ is the unique real root of the polynomial $q^4 - 4q^3 + 4q^2 - 4q + 4$ in the interval $(1, 2)$.

LEMMA 4.15 *If $G \in \mathcal{K}_1$, then $Q(G, q) > 0$ for $q \in (1, t_1]$.*

The proof of this lemma is also simple but fairly lengthy and can be found in Section 4.4.2. The idea is the same as that of Lemma 4.11.

LEMMA 4.16 $\omega(\mathcal{K}_1) = t_1$.

PROOF. Define $J_0 = K_3$ and consider an embedding of J_0 in the plane. For $i \in \mathbb{N}$, let J_i be formed from J_{i-1} by applying the double subdivision operation to each edge of J_{i-1} on the outer face. In [DJ11, pg. 1114], Dong and Jackson say this sequence of graphs has chromatic roots converging to t_1 from above. The following claim implies that $J_i \in \mathcal{K}_1$ for $i \in \mathbb{N}_0$. It then follows that $\omega(\mathcal{K}_1) \leq t_1$, which together with Lemma 4.15 gives that $\omega(\mathcal{K}_1) = t_1$.

CLAIM 4 *For $i \in \mathbb{N}_0$, every 2-cut of J_i has a trivial bridge which lies in the interior of the outer cycle of J_i.*

We prove the claim by induction on i. The result holds vacuously for $i = 0$ and is easily checked for $i = 1$, so suppose the result is true for $k \in \mathbb{N}_0$. For the induction step, note that the 2-cuts of J_{k+1} consist of the 2-cuts of J_k, and $\{u, v\}$ for every edge uv of the outer cycle of J_k. If $\{x, y\}$ is a 2-cut of J_k, then by the induction hypothesis there is a trivial $\{x, y\}$-bridge which lies in the interior of the outer cycle of J_k. This bridge is left unchanged in J_{k+1} so $\{x, y\}$ still satisfies the hypothesis. Alternatively, if uv is an edge of the outer cycle of J_k, then in J_{k+1}, the edge uv is replaced with two trivial $\{u, v\}$-bridges. One of these bridges forms part of the outer cycle of J_{k+1}, whilst the other bridge lies in the interior of the new outer cycle as required. $\qquad\square$

Let H_1' and H_2' be the graphs formed from a 4-cycle with vertex set $\{a, b, c, d\}$ by applying the double subdivision operation to the edges ab, ad and ab, cd respectively, see Figure 4.3.

PROPOSITION 4.17 *If G is a generalised uv-edge with property P_1, then G does not contain a minor isomorphic to H_1' or H_2' where the vertices a and c are identified with the vertices u and v of G in some order.*

PROOF. Let G be a smallest counterexample, and let J be a fixed H_1'-minor or H_2'-minor of G such that the vertices a and c of J are identified with the vertices u and v of G in some order. Since G is a generalised uv-edge with property P_1, it follows that G has two $\{u, v\}$-bridges B_1 and B_2, one of which, say B_1, is trivial. Since a and c are at distance 3 in J, it follows that the edges of B_1 are deleted to form J from G. Thus, J is a minor of B_2. Now, since G is a generalised edge, Proposition 4.8 implies that B_2 has a cut-vertex u' which separates B_2 into two blocks, L_u and L_v, containing u and v respectively. Since J is non-separable, it follows that J must be a minor of one of these blocks, say L_v. Furthermore, J must be a minor of L_v in such a way that the vertices a and c of J correspond to the vertices u' and v of L_v in some order. Now notice that L_v is a generalised $\{u', v\}$-edge with property P_1. This contradicts the minimality of G. $\qquad\square$

Let $H = K_{2,3}$ with vertex partition $\{\{x, y\}, \{u, v, w\}\}$. The graph H_1 is formed from H by applying the double subdivision operation to each edge adjacent to x. The graph H_2 is formed from H by applying the double subdivision operation to the edges xu, xv and yw, see Figure 4.1. The next lemma shows that H_1 and H_2 are the forbidden minors which characterise $\mathcal{K}_1$ within $\mathcal{K}$.

LEMMA 4.18 *If $G \in \mathcal{K}$, then $G \in \mathcal{K}_1$ if and only if G is $\{H_1, H_2\}$-minor-free.*

PROOF. If $G \in \mathcal{K} \setminus \mathcal{K}_1$, then G has a 2-cut $\{x, y\}$ with three $\{x, y\}$-bridges, none of which are trivial. Thus, each $\{x, y\}$-bridge is obtained from a path of length 2 by at least one double subdivision operation. It is now easy to see that G contains either H_1 or H_2 as a minor.

Now let $G \in \mathcal{K}$ be such that G contains H_1 or H_2 as a minor, and suppose for a contradiction that $G \in \mathcal{K}_1$. We may assume that G is minimal with respect to the double subdivision operation. Let $G' \in \mathcal{K}$ be such that G is obtained from G' by double subdividing the edge $xy \in E(G')$. Since $\mathcal{K}_1$ is downward-closed, we have $G' \in \mathcal{K}_1$.

Let J be a fixed H_1-minor or H_2-minor of G. By Proposition 4.8, we have that J contains both trivial $\{x, y\}$-bridges of G. Let B be the third $\{x, y\}$-bridge of G. Since G is a generalised triangle, B has a cut-vertex z, and Proposition 4.5 implies that B has precisely two blocks. Let H_1' and H_2' be the graphs in Figure 4.3. Clearly, if J is H_1, then B contains H_1' as a minor. Similarly, if J is H_2, then B contains H_1' or H_2' as a minor. In either case, since H_1' and H_2' are non-separable, this minor must lie inside one of the two blocks of B, say the block L_x which contains x. Furthermore, it must lie in such a way that the vertices a and c are identified with the vertices x and z of G in some order. Now, since the block L_x is a generalised xz-edge with property P_1, we reach a contradiction by Proposition 4.17. □

PROOF OF THEOREM 4.2(ii). Let $\mathcal{G}$ be the class of $\{H_1, H_2\}$-minor-free graphs. By Theorem 2.15 and Lemma 4.18 we have $\omega(\mathcal{G}) = \omega(\mathcal{G} \cap \mathcal{K})$ and $\mathcal{G} \cap \mathcal{K} = \mathcal{K}_1$

respectively. Now, by Lemma 4.16, we have $\omega(\mathcal{K}_1) = t_1$. Hence the result follows. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Let $\mathcal{D}_1$ be the class of non-separable plane graphs such that every 2-cut is contained in the outer-cycle. In [DJ11], Dong and Jackson conjecture that $\omega(\mathcal{D}_1) = t_1$. Once again, this is an important conjecture since $\mathcal{D}_1$ contains the class of 3-connected planar graphs. Whilst it can be shown that $\mathcal{D}_1 \cap \mathcal{K} \subset \mathcal{K}_1$, this does not prove the conjecture since it is not known if $\omega(\mathcal{D}_1) = \omega(\mathcal{D}_1 \cap \mathcal{K})$. In particular $\mathcal{D}_1$ is not minor-closed so Theorem 2.15 does not apply.

Again, the fact that $\mathcal{D}_1 \cap \mathcal{K}$ is not the largest class of generalised triangles $\mathcal{K}'$ such that $\omega(\mathcal{K}') = t_1$ suggests that a well chosen weaker property could be used to make progress on Dong and Jackson's conjecture.

PROBLEM 4.19 Find a class of graphs $\mathcal{D}$ such that $\mathcal{D}_1 \subseteq \mathcal{D}$, $\omega(\mathcal{D}) = \omega(\mathcal{D} \cap \mathcal{K})$ and $\mathcal{D} \cap \mathcal{K} = \mathcal{K}_1$.

4.3.3   The Class $\mathcal{K}_0 \cap \mathcal{K}_1$

In this section we show that $\omega(\mathcal{K}_0 \cap \mathcal{K}_1) = t_2$, where $t_2 \approx 1.296$ is the unique real root of the polynomial $q^3 - 2q^2 + 4q - 4$. We use this to deduce Theorem 4.2(iii).

Recall the definition of a Whitney 2-switch from Section 1.3. We first note a useful property of the graphs in $\mathcal{K}_0 \cap \mathcal{K}_1$.

PROPOSITION 4.20 *Let $G \in \mathcal{K}_0$, let $\{x, y\}$ be a 2-cut of G, and let B be an $\{x, y\}$-bridge. Suppose that every 2-cut of G except possibly $\{x, y\}$ has property P_1. There is a graph G', which is Whitney-equivalent to G, such that the $\{x, y\}$-bridge B' of G' corresponding to B contains a path starting at y, and covering all vertices of B' except for x.*

PROOF. We proceed by induction on $|V(B)|$. If $|V(B)| = 3$, then B is trivial and the result is clear. Thus we may suppose that $|V(B)| > 3$. Since $G \in \mathcal{K}_0$, at
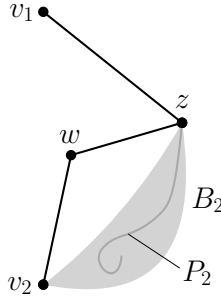
Figure 4.4: The bridge B in the proof of Proposition 4.20.

least one vertex in $\{x, y\}$ has degree 1 in B. Call this vertex v_1, and let v_2 be the other vertex of $\{x, y\}$. Now, let z be the neighbour of v_1 in B, see Figure 4.4. Since $|V(B)| > 3$, we have that $\{v_2, z\}$ is a 2-cut of G with precisely three $\{v_2, z\}$-bridges B_1, B_2 and B_3, two of which, say B_1 and B_2, are contained in B. Since $\{v_2, z\}$ has property P_1, and $|V(B_3)| > 3$, one of B_1 and B_2 is trivial. Without loss of generality, say that B_1 is the path $v_2 w z$, see Figure 4.4.

Now $|V(B_2)| < |V(B)|$, so by induction, there is a sequence of Whitney 2-switches of G such that in the resulting graph G', the bridge B_2' corresponding to B_2 contains a path P_2 starting at z, and covering all vertices of B_2 except for v_2. Now let $P' = v_2 w z \cup P_2$. Clearly, P' is a path which starts at v_2, and covers all vertices of the bridge B' of G' which corresponds to B in G. If $v_2 = y$, then the graph G' and path P' are as desired. Otherwise we perform a Whitney 2-switch of B' about $\{x, y\}$. The resulting graph and the path P' are as desired.

$\square$

As before, let $\mathcal{H}$ denote the class of graphs which have a Hamiltonian path. The next two lemmas show that $\{P(H, q) : H \in \mathcal{H} \cap \mathcal{K}\} = \{P(G, q) : G \in \mathcal{K}_0 \cap \mathcal{K}_1\}$.

LEMMA 4.21 $\mathcal{H} \cap \mathcal{K} \subseteq \mathcal{K}_0 \cap \mathcal{K}_1$.

PROOF. Suppose $G \in \mathcal{H} \cap \mathcal{K}$ and let P denote the Hamiltonian path of G. If $G = K_3$, then we are done, so we may assume that G contains a 2-cut. Let $\{x, y\}$ be an arbitrary 2-cut of G. Since G is a generalised triangle, by

Proposition 2.1 there are precisely three $\{x, y\}$-bridges of G which we call B_1, B_2 and B_3. Without loss of generality, assume that P begins in B_1, visits x before y, and ends in B_3.

Since the subpath of P from x to y is a Hamiltonian path of B_2, Lemma 3.10(a) implies that B_2 is a path of length 2. Thus, $\{x, y\}$ has property P_1. Since B_1 contains a path starting at y and covering all vertices of B_1 except for x, then by Lemma 3.10(b), we have that B_1 is a copy of $F(x, y, \ell)$ for some $\ell \in \mathbb{N}_0$. From the definition of the graph F, it follows that y has degree 1 in B_1. Similarly, it can be shown that x has degree 1 in B_3. Thus, $\{x, y\}$ has property P_0. $\qquad\square$

LEMMA 4.22 *If $G \in \mathcal{K}_0 \cap \mathcal{K}_1$, then there is $H \in \mathcal{H} \cap \mathcal{K}$ such that $P(G, q) = P(H, q)$.*

PROOF. Let $G \in \mathcal{K}_0 \cap \mathcal{K}_1$. By the characterisation of generalised triangles in Proposition 2.1, it is easy to see that $\mathcal{K}$ is invariant under Whitney 2-switches. Thus, we only need to show that G can be transformed into a graph with a Hamiltonian path by a sequence of Whitney 2-switches. The result clearly holds if $G = K_3$, so we may suppose that $|V(G)| > 3$.

Let $\{x, y\}$ be a 2-cut of G such that two of the $\{x, y\}$-bridges B_1 and B_2 are trivial with vertex-sets $\{x, y, u\}$ and $\{x, y, v\}$ respectively. Such a 2-cut can easily be found by considering the construction of G from a triangle by double subdivisions. Let B be the remaining $\{x, y\}$-bridge. By Proposition 4.20, there is a sequence of Whitney 2-switches in G such that the resulting graph has a path P starting at y and covering all vertices of B except for x. In the resulting graph, $uxvy \cup P$ is a Hamiltonian path. $\qquad\square$

PROOF OF THEOREM 4.2(iii). Let $\mathcal{G}$ be the class of $\{H_0, H_1, H_2\}$-minor-free graphs. By Theorem 2.15 we have $\omega(\mathcal{G}) = \omega(\mathcal{G} \cap \mathcal{K})$. Lemmas 4.13 and 4.18 imply that $\mathcal{G} \cap \mathcal{K} = \mathcal{K}_0 \cap \mathcal{K}_1$. Now, by Lemmas 4.21 and 4.22, we have $\omega(\mathcal{K}_0 \cap \mathcal{K}_1) = \omega(\mathcal{H} \cap \mathcal{K})$. Finally, Theorem 3.1 and Lemma 3.8 imply that $\omega(\mathcal{H} \cap \mathcal{K}) = t_2$. This completes the proof. $\qquad\square$
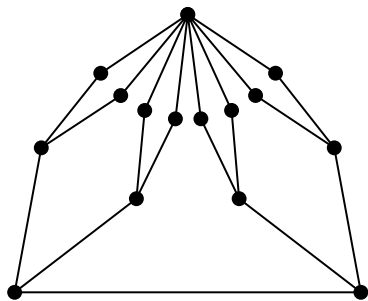
Figure 4.5: The graph H_3.

4.3.4　3-Leaf Spanning Trees

Recall Theorem 3.2 from the previous chapter, which says that $\omega(\mathcal{G}_3) = t_3$, where $\mathcal{G}_3$ denotes the class of graphs with 3-leaf spanning trees, and $t_3 \approx 1.290$. As in Section 4.3.3, we can deduce an excluded minor result for this class of graphs. To state it, let $\mathcal{W}$ be the class of graphs which are Whitney-equivalent to H_3, where H_3 is the generalised triangle depicted in Figure 4.5.

THEOREM 4.23 *If $\mathcal{G}$ denotes the class of $\{H_0\} \cup \mathcal{W}$-minor-free graphs, then $\omega(\mathcal{G}) = t_3$ where $t_3 \approx 1.290$ is the smallest real root of the polynomial $(q-2)^6 + 4(q-1)^2(q-2)^3 - (q-1)^4$.*

Let $\mathcal{K}_3$ denote the class of generalised triangles such that every 2-cut has property P_0, and all but at most one 2-cut have property P_1. Thus, we have $\mathcal{K}_0 \cap \mathcal{K}_1 \subseteq \mathcal{K}_3 \subseteq \mathcal{K}_0$. To prove Theorem 4.23 we first show that $\omega(\mathcal{K}_3) = t_3$. This follows from the following two lemmas.

LEMMA 4.24 $\mathcal{G}_3 \cap \mathcal{K} \subseteq \mathcal{K}_3$.

PROOF. Let $G \in \mathcal{G}_3 \cap \mathcal{K}$. By Lemma 3.12, G is Whitney-equivalent to the graph $G_{i,j,k}$ for some $i, j, k \in \mathbb{N}_0$. Furthermore, it is easy to see that $G_{i,j,k} \in \mathcal{K}_3$ for all $i, j, k \in \mathbb{N}_0$. Now the result follows because the properties P_0 and P_1 of a 2-cut are invariant under Whitney 2-switches. $\qquad\square$

LEMMA 4.25 *If $G \in \mathcal{K}_3$, then there exists $H \in \mathcal{G}_3 \cap \mathcal{K}$ such that $P(G,q) = P(H,q)$.*

PROOF. Let $G \in \mathcal{K}_3$. If $G \in \mathcal{K}_0 \cap \mathcal{K}_1$, then by Lemma 4.22 there is $H \in \mathcal{H} \cap \mathcal{K}$ such that $P(G,q) = P(H,q)$. Since $\mathcal{H} \subseteq \mathcal{G}_3$, the result follows. So suppose there is a 2-cut $\{x,y\}$ of G which does not have property P_1. By the definition of $\mathcal{K}_3$, all other 2-cuts satisfy P_1. Now, since $G \in \mathcal{K}$, there are three $\{x,y\}$-bridges B_1, B_2 and B_3. By three applications of Proposition 4.20, we find a graph G', which is Whitney-equivalent to G, and where the $\{x,y\}$-bridges B'_1, B'_2 and B'_3 of G' corresponding to B_1, B_2 and B_3 each contain a path starting at y and covering all vertices of that bridge except x. By Lemma 3.10(b), B'_1, B'_2 and B'_3 are each copies of $F(x,y,\ell)$ for some $\ell \in \mathbb{N}$. Performing a Whitney 2-switch of one of these bridges about $\{x,y\}$ yields a graph H which is isomorphic to $G_{i,j,k}$ for some $i,j,k \in \mathbb{N}_0$. Clearly $H \in \mathcal{G}_3 \cap \mathcal{K}$, and since H is Whitney-equivalent to G, we have $P(G,q) = P(H,q)$ as desired. $\qquad\square$

The next lemma implies that if $\mathcal{G}$ denotes the class of $\{H_0\} \cup \mathcal{W}$-minor-free graphs, then $\mathcal{G} \cap \mathcal{K} = \mathcal{K}_3$. Theorem 4.23 then follows from Theorem 2.15.

LEMMA 4.26 *If $G \in \mathcal{K}$, then $G \in \mathcal{K}_3$ if and only if G is $\{H_0\} \cup \mathcal{W}$-minor-free*

PROOF. If $G \in \mathcal{K} \setminus \mathcal{K}_3$, then either G has a 2-cut without property P_0, or two 2-cuts without property P_1. In the former case, Lemma 4.13 implies that G contains H_0 as a minor. Thus, we may suppose the latter case holds, and that in addition, every 2-cut has property P_0. Let $\{x,y\}$ and $\{u,v\}$ be 2-cuts of G without property P_1. Let B_1, B_2 and B_3 be the $\{x,y\}$-bridges of G, and suppose without loss of generality that $\{u,v\} \subseteq V(B_1)$. Since $\{x,y\}$ does not have property P_1, the bridges B_2 and B_3 are formed from a trivial bridge by at least one double subdivision. Thus, $B_2 \cup B_3$ contains H'_1 or H'_2 as a minor in such a way that the vertices a and c are identified with the vertices x and y of G.

Now, since $\{x, y\}$ has property P_0, one of x or y has degree 1 in B_1. Suppose without loss of generality that this is y, and let z be the unique neighbour of y in B_1. Thus, $\{x, z\}$ is a 2-cut of G. Let C_1, C_2 and C_3 be the $\{x, z\}$-bridges of G such that C_3 contains y. Clearly, we now have $\{u, v\} \subseteq V(C_1 \cup C_2)$. Since $\{u, v\}$ does not have property P_1, we deduce by the argument above that $C_1 \cup C_2$ contains H_1' or H_2' as a minor in such a way that the vertices a and c are identified with the vertices u and v of G in some order. Now consider the two minors of H_1' or H_2', together with two paths in B_3 joining distinct vertices from $\{u, v\}$ to the vertices x and y in some order. Because of the edge yz, one of these paths has at least one edge. Thus, we see that G contains a minor of some graph Whitney-equivalent to H_3 as required.

For the reverse implication, let $G \in \mathcal{K}_3$. Note that since $\mathcal{K}_3 \subseteq \mathcal{K}_0$, Lemma 4.13 implies that G is H_0-minor-free. Additionally, if $G \in \mathcal{K}_1$, then Lemma 4.18 implies that G is $\{H_1, H_2\}$-minor-free. Since every graph in $\mathcal{W}$ contains H_1 or H_2 as a minor, this shows that G is $\mathcal{W}$-minor-free. Thus we may assume that $G \in \mathcal{K}_3 \setminus \mathcal{K}_1$. Consequently, G has precisely one 2-cut $\{x, y\}$ which does not have property P_1.

Suppose for a contradiction that G contains some graph $W \in \mathcal{W}$ as a minor, and let J be a fixed W-minor of G. Replacing each $\{x, y\}$-bridge of G by a trivial $\{x, y\}$-bridge yields a graph in $\mathcal{K}_1$. Since such graphs are $\mathcal{W}$-minor-free, we deduce that $\{x, y\}$ must be a 2-cut of J with three non-trivial bridges. Thus, $\{x, y\}$ must correspond to one of the two 2-cuts of W which do not satisfy property P_1. Let B be the $\{x, y\}$-bridge of G such that the corresponding $\{x, y\}$-bridge in J contains 9 vertices. Since $G \in \mathcal{K}_0$, one of x and y has degree 1 in B. Suppose this is y and let z be the unique neighbour of y in B. Thus, $\{x, z\}$ is a 2-cut of G with precisely 3 bridges, two of which, say B_1 and B_2, are contained in B. It now follows that $B_1 \cup B_2$ must contain a minor isomorphic to H_1' or H_2' in such a way that the vertices a and c are identified with the vertices x and z of G in some order. However, $B_1 \cup B_2$ is a generalised xz-edge with property P_1. Thus, we deduce a contradiction from Proposition 4.17. $\square$

4.4   Proofs of the Lemmas

In this section we prove Lemma 4.11 and Lemma 4.15. The proofs are similar to Jackson's proof in [Jac93] of the result that $\omega(\mathcal{K}) \geq 32/27$, except that the additional structure of the classes $\mathcal{K}_0$ and $\mathcal{K}_1$ allows us to make some savings and get a larger value for $\omega(\mathcal{K}_0)$ and $\omega(\mathcal{K}_1)$. The proofs are fairly long, but nevertheless rely only on the basic identities introduced in Proposition 2.4 and Proposition 2.5.

4.4.1   Proof of Lemma 4.11

Lemma 4.11 is statement (e) in the following lemma.

LEMMA 4.27 *Let G be a graph and let $q \in (1, 5/4]$.*

(a) *If $G \in \mathcal{K}_0$, and v is a vertex of degree 2 in G with neighbours u and w, then $Q(G, q) \geq \frac{1}{2}Q(G/uv, q)$.*

(b) *If G is a generalised uw-edge with property P_0, and $|V(G)| \geq 4$, then $Q(G + uw, q) \geq \frac{1}{2}Q(G, q)$.*

(c) *If $G \in \mathcal{K}_0$, and v is a vertex of degree 2 in G with neighbours u and w, then $Q(G/uv, q) > 0$.*

(d) *If G is a generalised uw-edge with property P_0, then $Q(G, q) > 0$.*

(e) *If $G \in \mathcal{K}_0$, then $Q(G, q) > 0$.*

PROOF. We proceed simultaneously by induction on $|V(G)|$. If $|V(G)| \leq 4$, then either $G = K_3$ if $G \in \mathcal{K}_0$, or $G = C_4$ if G is a generalised edge with property P_0. Thus, parts (c), (d) and (e) are easily verified. Part (a) also holds since $Q(K_3, q) = (2 - q)Q(K_2, q) \geq \frac{3}{4}Q(K_2, q) > \frac{1}{2}Q(K_2, q) > 0$. Finally (b) holds when $G = C_4$ since $Q(C_4 + uw, q) - \frac{1}{2}Q(C_4, q) = \frac{1}{2}q(q-1)((q-2)^2 - (q-1)) > 0$.

Thus, we may suppose that $|V(G)| > 4$ and that (a) to (e) hold for all graphs with fewer vertices.

(a) Set $H = G - v$. Note that H is a generalised uw-edge with property P_0, and $G/uv = H + uw$. By Propositions 2.4 and 2.5, we have

$$Q(G, q) = -Q(G - uv, q) + Q(G/uv, q)$$
$$= (1 - q)Q(H, q) + Q(H + uw, q). \qquad (4.1)$$

By the induction hypothesis of (d) on H, we have $Q(H, q) > 0$. Furthermore, by the induction hypothesis of (b) on H, we have $Q(H + uw, q) \geq \frac{1}{2}Q(H, q)$. Now, using the fact that $q \in (1, 5/4]$, equation (4.1) becomes

$$Q(G, q) \geq 2(1 - q)Q(H + uw, q) + Q(H + uw, q)$$
$$= (3 - 2q)Q(H + uw, q)$$
$$\geq \tfrac{1}{2}Q(H + uw, q)$$
$$= \tfrac{1}{2}Q(G/uv, q).$$

(b) Let $s = Q(G + uw, q) - \frac{1}{2}Q(G, q)$. Also, let H_1 and H_2 be the $\{u, w\}$-bridges of the graph $G + uw$, and note that $H_1, H_2 \in \mathcal{K}_0$. By Propositions 2.4 and 2.5,

$$s = Q(G + uw, q) - \tfrac{1}{2}[Q(G + uw, q) + Q(G/uw, q)]$$
$$= \tfrac{1}{2}Q(G + uw, q) - \tfrac{1}{2}Q(G/uw, q)$$
$$= \tfrac{1}{2}q^{-1}(q - 1)^{-1}Q(H_1, q)Q(H_2, q) - \tfrac{1}{2}q^{-1}Q(H_1/uw, q)Q(H_2/uw, q).$$

Since the 2-cut $\{u, w\}$ of G has property P_0, in each of H_1 and H_2 at least one of the vertices u and w has degree 2. Therefore, by the induction hypotheses of (c) and (e), we have $Q(H_i/uw, q) > 0$ and $Q(H_i, q) > 0$ for $i \in \{1, 2\}$. Moreover, for $i \in \{1, 2\}$, the induction hypothesis of (a) on the edge uw of

H_i implies that $Q(H_i, q) \geq \frac{1}{2}Q(H_i/uw, q)$. Now since $q \in (1, 5/4]$,

$$2sq(q-1) = Q(H_1, q)Q(H_2, q) - (q-1)Q(H_1/uw, q)Q(H_2/uw, q)$$
$$\geq Q(H_1/uw, q)Q(H_2/uw, q)[(\tfrac{1}{2})^2 - \tfrac{1}{4}] = 0.$$

(c) Since v has degree 2, the set $\{u, w\}$ is a 2-cut of G/uv and $uw \in E(G/uv)$. Thus, the $\{u, w\}$-bridges H_1 and H_2 of G/uv are members of $\mathcal{K}_0$, and so $Q(H_i, q) > 0$ for $i \in \{1, 2\}$ by the induction hypothesis of (e). Finally, since H_1 and H_2 intersect in a complete subgraph, Proposition 2.5 gives

$$Q(G/uv, q) = q^{-1}(q-1)^{-1}Q(H_1, q)Q(H_2, q) > 0.$$

(d) Let H_1 and H_2 be the uw-bridges of $G + uw$, and note that $H_1, H_2 \in \mathcal{K}_0$. Propositions 2.4 and 2.5 give,

$$Q(G, q) = Q(G + uw, q) + Q(G/uw, q)$$
$$= \tfrac{1}{q(q-1)}Q(H_1, q)Q(H_2, q) + \tfrac{1}{q}Q(H_1/uw, q)Q(H_2/uw, q). \quad (4.2)$$

By the induction hypothesis of (e), we have $Q(H_i, q) > 0$ for $i \in \{1, 2\}$. Since G is a generalised edge with property P_0, the 2-cut $\{u, w\}$ of G has property P_0, and so in each of H_1 and H_2, at least one of u or w has degree 2. Thus, by the induction hypothesis of (c), we have $Q(H_i/uw, q) > 0$ for $i \in \{1, 2\}$. Now, by (4.2), we have $Q(G, q) > 0$ as claimed.

(e) Firstly, note that (a) and (c) have now been proved for a graph with $|V(G)|$ vertices. Let v be a vertex of degree 2 with neighbours u and w. By (a), we have $Q(G, q) \geq \frac{1}{2}Q(G/uv, q)$, and by (c), we have $Q(G/uv, q) > 0$. Therefore, $Q(G, q) > 0$ as claimed.

$\square$

4.4.2   Proof of Lemma 4.15

To prove Lemma 4.15 we require a few preliminary results. Recall that $t_1 \approx 1.225$ is the unique real root of the polynomial $q^4 - 4q^3 + 4q^2 - 4q + 4$ in $(1, 2)$ and define the constants

$$\gamma = \tfrac{1}{4}(t_1 - 2)(t_1^2 - 2t_1 - 2) \approx 0.571,$$
$$\alpha = (1 - \gamma)(2 - t_1)(2 - t_1 - \gamma)^{-1} \approx 1.632,$$
$$\beta = 1 - \alpha^{-1} = \gamma(t_1 - 1)(1 - \gamma)^{-1}(2 - t_1)^{-1} \approx 0.387.$$

LEMMA 4.28 *For all $q \in (1, t_1]$ we have*

(i) $q(q - 1)^{-1}\gamma^2 - 2\gamma + 1 \geq \alpha$

(ii) $(1 - q)\gamma^{-1} + 1 \geq \beta$

(iii) $(1 - \gamma)(2 - q)\beta - (q - 1)\gamma \geq 0.$

PROOF. For $q \in (1, t_1]$, the left hand sides of the three inequalities are decreasing functions of q. Thus, we only need to verify them for $q = t_1$. Now (iii) follows immediately from the definition of β. Inequality (i) holds for $q = t_1$ since substituting in the expressions for γ and α and simplifying gives

$$\tfrac{1}{16}(t_1^2 - 2t_1 - 2)(t_1^3 - 4t_1^2 + 4t_1 - 4)(t_1^4 - 4t_1^3 + 4t_1^2 - 4t_1 + 4) \geq 0,$$

which holds with equality by the definition of t_1. For inequality (ii) the same substitution gives

$$\tfrac{1}{16}(2 - t_1)(t_1^5 - 2t_1^4 - 8t_1^3 + 16t_1^2 - 12t_1 + 8) \geq 0.$$

This inequality holds loosely which can be checked for example by using a computer algebra package to approximate the roots of the quintic factor.   □

The following useful reduction lemma is due to Jackson.

LEMMA 4.29 [Jac93] *Let G be a non-separable graph and $\{u, v\}$ be a 2-cut such that uv is not an edge of G. If G_1 and G_2 are non-separable subgraphs of G such that $G_1 \cup G_2 = G$, $V(G_1) \cap V(G_2) = \{u, v\}$, $|V(G_1)| \geq 3$ and $|V(G_2)| \geq 3$, then*

$$q(q-1)Q(G, q) = qQ(G_1 + uv, q)Q(G_2 + uv, q)$$
$$+ (q-1)[Q(G_1, q)Q(G_2, q) - Q(G_1 + uv, q)Q(G_2, q) - Q(G_1, q)Q(G_2 + uv, q)].$$

We also use the following proposition of Dong and Koh.

PROPOSITION 4.30 [DK10] *If G is a generalised triangle, then for every edge uv, there is a vertex z such that $G - uv = G_1 \cup G_2$, where G_1 is a generalised uz-edge, G_2 is a generalised vz-edge, and $G_1 \cap G_2 = \{z\}$.*

Now Lemma 4.15 is statement (e) in the following result.

LEMMA 4.31 *Let G be a simple graph and let $q \in (1, t_1]$.*

(a) *Suppose $G = G_1 \cup G_2$ where G_1 and G_2 are generalised uv-edges such that $G_1 \cap G_2 = \{u, v\}$, $|V(G_1)| \geq 4$ and $|V(G_2)| \geq 4$. If G_1 and G_2 have property P_1, then $Q(G, q) \geq \alpha q^{-1} Q(G_1, q)Q(G_2, q)$.*

(b) *Suppose $G = G_1 \cup G_2 + uv$ where G_1 is a generalised uw-edge, G_2 is a generalised vw-edge and $G_1 \cap G_2 = \{w\}$. If G_1 and G_2 have property P_1, then $Q(G, q) \geq \beta Q(G/uv, q)$.*

(c) *If G is a generalised uv-edge with property P_1, and $|V(G)| \geq 4$, then $Q(G + uv, q) \geq \gamma Q(G, q)$.*

(d) *Suppose $G = G_1 \cup G_2$ where G_1 and G_2 are generalised uw-edges such that $G_1 \cap G_2 = \{u, w\}$. If G_1 and G_2 have property P_1, then $Q(G, q) > 0$.*

(e) *If $G \in \mathcal{K}_1$, then $Q(G, q) > 0$.*

(f) *If G is a generalised uw-edge with property P_1, then $Q(G, q) > 0$.*

PROOF. We proceed simultaneously by induction on $|V(G)|$. For the base case, suppose that $|V(G)| \leq 4$. Thus, part (a) is vacuously true. Furthermore, if $G \in \mathcal{K}_1$ then $G = K_3$, and if G is a generalised edge with property P_1, then $G \in \{C_4, K_2\}$. Thus, parts (e) and (f) are easily verified. Part (b) also holds since $Q(K_3, q) = (2 - q)Q(K_2, q) > \frac{3}{4}Q(K_2, q) > \beta Q(K_2, q)$. Part (c) holds when $G = C_4$ since

$$Q(C_4 + uw, q) - \gamma Q(C_4, q) = q(q - 1)[(1 - \gamma)(q - 2)^2 - \gamma(q - 1)] > 0.$$

Since G is simple, the only graph satisfying part (d) is $C_4 + uw$, and $Q(C_4 + uw, q) = q(q - 1)(q - 2)^2 > 0$ as claimed. Thus we may suppose $|V(G)| > 4$ and that (a) to (f) hold for all graphs with fewer vertices.

(a) Applying Lemma 4.29 to G and rearranging, we have

$$\begin{aligned} qQ(G, q) =& Q(G_1 + uv, q) \left[\tfrac{1}{2}q(q - 1)^{-1}Q(G_2 + uv, q) - Q(G_2, q) \right] \\ &+ Q(G_2 + uv, q) \left[\tfrac{1}{2}q(q - 1)^{-1}Q(G_1 + uv, q) - Q(G_1, q) \right] \\ &+ Q(G_1, q)Q(G_2, q). \end{aligned} \tag{4.3}$$

By the induction hypothesis of (c), we have $Q(G_i + uv, q) \geq \gamma Q(G_i, q)$ for $i \in \{1, 2\}$. Together with Lemma 4.28(i), this gives

$$\begin{aligned} \tfrac{1}{2}q(q - 1)^{-1}Q(G_i + uv, q) - Q(G_i, q) &\geq \left[\tfrac{1}{2}q(q - 1)^{-1}\gamma - 1 \right] Q(G_i, q) \\ &\geq \tfrac{\alpha - 1}{2\gamma}Q(G_i, q). \end{aligned} \tag{4.4}$$

By the induction hypothesis of (f), we have $Q(G_i, q) > 0$ for $i \in \{1, 2\}$. Together with (4.4), this shows that the contents of the square brackets in (4.3) are positive. Substituting (4.4) into (4.3) and using the induction hypothesis of (c) finally gives $qQ(G, q) \geq \alpha Q(G_1, q)Q(G_2, q)$ as required.

(b) If one of G_1 and G_2 is a single edge, say $G_1 = uw$, then $G/uv = G_2 + vw$.

By Proposition 2.4 on uv, and Proposition 2.5,

$$Q(G, q) = (1 - q)Q(G_2, q) + Q(G_2 + vw, q). \tag{4.5}$$

The induction hypothesis of (f) gives that $Q(G_2, q) > 0$. Moreover the induction hypothesis of (c) on G_2 gives $Q(G_2 + vw, q) \geq \gamma Q(G_2, q)$. Substituting these inequalities into (4.5) and using Lemma 4.28(ii) we get

$$\begin{aligned} Q(G, q) &\geq [(1 - q)\gamma^{-1} + 1]Q(G_2 + vw, q) \\ &\geq \beta Q(G_2 + vw, q) \\ &= \beta Q(G/uv, q). \end{aligned} \tag{4.6}$$

So suppose that both G_1 and G_2 have at least 4 vertices. By Proposition 2.4 on uv, and Proposition 2.5, we have

$$\begin{aligned} Q(G, q) &= -Q(G - uv, q) + Q(G/uv, q) \\ &= -q^{-1}Q(G_1, q)Q(G_2, q) + Q(G/uv, q). \end{aligned} \tag{4.7}$$

The induction hypothesis of (f) gives $Q(G_i, q) > 0$ for $i \in \{1, 2\}$, and the induction hypothesis of (a) gives $q^{-1}Q(G_1, q)Q(G_2, q) \leq \alpha^{-1}Q(G/uv, q)$. Substituting into (4.7), we have

$$Q(G, q) \geq (1 - \alpha^{-1})Q(G/uv, q) = \beta Q(G/uv, q). \tag{4.8}$$

(c) Let $s = Q(G + uv, q) - \gamma Q(G, q)$. Since the 2-cut $\{u, v\}$ of G has property P_1, at least one $\{u, v\}$-bridge of G is trivial. Let H be the other $\{u, v\}$-bridge of G and notice that $H + uv \in \mathcal{K}_1$. By Proposition 2.4 on G, we get

$$\begin{aligned} s &= Q(G + uv, q) - \gamma[Q(G + uv, q) + Q(G/uv, q)] \\ &= (1 - \gamma)Q(G + uv, q) - \gamma Q(G/uv, q). \end{aligned} \tag{4.9}$$

Now using Proposition 2.5, equation (4.9) becomes

$$s = (1 - \gamma)(2 - q)Q(H + uv, q) - \gamma(q - 1)Q(H/uv, q). \qquad (4.10)$$

Note that $H = H_1 \cup H_2$ where H_1 is a generalised uw-edge with property P_1, H_2 is a generalised vw-edge with property P_1, and $H_1 \cap H_2 = \{w\}$. Thus, by the induction hypothesis of (d), we have $Q(H/uv, q) > 0$. Now by the induction hypothesis of (b), we have $Q(H + uv, q) \geq \beta Q(H/uv, q)$. Substituting these inequalities into (4.10) and using Lemma 4.28(iii) gives

$$s \geq [(1 - \gamma)(2 - q)\beta - \gamma(q - 1)]Q(H/uv, q) \geq 0.$$

(d) If one of G_1 and G_2 is a single edge, then G is either a single edge, or we can write $G = H_1 \cup H_2$, where $H_1, H_2 \in \mathcal{K}_1$, and $H_1 \cap H_2$ is the edge uw. By the induction hypothesis of (e) and Proposition 2.5, we conclude that $Q(G, q) > 0$. So suppose that both G_1 and G_2 have at least 4 vertices. By (a), which has now been proved for a graph on $|V(G)|$ vertices, we conclude that $Q(G, q) \geq \alpha q^{-1} Q(G_1, q)Q(G_2, q)$. By the induction hypothesis of (f), we have $Q(G_i, q) > 0$ for $i \in \{1, 2\}$. Therefore, $Q(G, q) > 0$ as claimed.

(e) Let $\{u, w\}$ be a 2-cut of G so that two of the $\{u, w\}$-bridges are trivial. Such a 2-cut is easily found by considering the construction of G from K_3 by the double subdivision operation. Let v be a vertex of degree 2 in G with neighbours u and w. By Proposition 4.30, we may write $G = G_1 \cup G_2 + uv$ where G_1 is a generalised vw-edge, G_2 is a generalised uw-edge, and $G_1 \cap G_2 = \{w\}$. By the choice of $\{u, w\}$, we have in particular that G_1 is the edge vw and G_2 is a generalised uw-edge with property P_1.

Now we may apply (b) to deduce that $Q(G, q) \geq \beta Q(G/uv, q)$. Note that $G/uv = H_1 \cup H_2$ where $H_1, H_2 \in \mathcal{K}_1$ and $H_1 \cap H_2$ is the edge uw. By the induction hypothesis of (e), we have that $Q(H_i, q) > 0$ for $i \in \{1, 2\}$. Finally, Proposition 2.5 gives $Q(G/uv, q) = q^{-1}(q - 1)^{-1}Q(H_1, q)Q(H_2, q) > 0$, whence $Q(G, q) > 0$.

(f) Let v be a vertex of degree 2 with neighbours u and w. Let $H = G - v$ and let z be a cut-vertex of H. Note that $H = H_1 \cup H_2$ where H_1 is a generalised uz-edge with property P_1, H_2 is a generalised wz-edge with property P_1, and $H_1 \cap H_2 = \{z\}$. Note also that this implies $H + uw \in \mathcal{K}_1$. By Propositions 2.4 and 2.5,

$$Q(G, q) = Q(G + uw, q) - Q(G/uw, q)$$
$$= (2 - q)Q(H + uw, q) + (q - 1)Q(H/uw, q). \qquad (4.11)$$

By the induction hypothesis of (e), we have $Q(H + uw, q) > 0$. If one of H_1 or H_2 is a single edge, then H/uw is either a single edge or an element of $\mathcal{K}_1$. In either case, we have $Q(H/uw, q) > 0$. Thus, we may suppose both H_1 and H_2 have at least 4 vertices. By the induction hypothesis of (f), we have $Q(H_i, q) > 0$ for $i \in \{1, 2\}$. Now we apply the induction hypothesis of (a) to get

$$Q(H/uw, q) \geq \alpha q^{-1} Q(H_1, q)Q(H_2, q) > 0. \qquad (4.12)$$

Finally, by (4.11) and (4.12), we have $Q(G, q) > 0$ as claimed. $\qquad\square$

CHAPTER 5

Roots of the Tutte Polynomial

5.1 Introduction

In this chapter we study the Tutte polynomial $T_G(x, y)$, a two-variable graph polynomial whose roots and evaluations encode many interesting graph properties. In particular, it contains the chromatic polynomial as a special case. Because of this more general setting, we allow graphs to have loops and multiple edges throughout this chapter. Accordingly, we invite the reader to review the remarks in Section 1.1.

Let G be a graph with vertex set V and edge set E. Consider the polynomial Z_G defined by

$$Z_G(q, v) = \sum_{A \subseteq E} q^{k(A)} v^{|A|}, \tag{5.1}$$

where q and v are commuting indeterminates, and $k(A)$ denotes the number of

components in the graph (V, A). The expression in (5.1) is sometimes called the random cluster formulation of the Tutte polynomial. However, since we deal only with this formulation, we will say that $Z_G(q, v)$ is the **Tutte polynomial** of G. We can retrieve the more classical formulation of the Tutte polynomial $T_G(x, y)$ from $Z_G(q, v)$ by a simple change of variables as follows.

$$T_G(x, y) = (x-1)^{-k(E(G))}(y-1)^{-|V(G)|}Z_G\big((x-1)(y-1), y-1\big).$$

Note that by Definition 1.5, the chromatic polynomial can be obtained from (5.1) by setting $v = -1$. In particular, the chromatic roots of a graph also yield roots of its Tutte polynomial along the line $v = -1$.

Recall that Theorems 1.9 and 1.10 show that the real chromatic roots of all graphs consist of 0, 1 and a dense subset of the interval $(32/27, \infty)$. Expanding on this, Jackson and Sokal [JS09] identified a **zero-free region** R_1 of the (q, v) plane where the Tutte polynomial never has a root. By using a multivariate version of the Tutte polynomial, their work also further elucidated the origin of the number $32/27$. They conjectured that R_1 is the first in an inclusion-wise increasing sequence of regions $R_1, R_2, \ldots$, such that for $i \geq 1$, the only non-separable graphs whose Tutte polynomials have a root inside R_i have fewer than i edges. Jackson and Sokal also conjectured that this sequence converges to a limiting region R^*, outside of which the roots of the Tutte polynomials of graphs are dense. The region R^* is depicted by the unshaded region in Figure 5.1.

We now state the conjecture of Jackson and Sokal precisely. Following [JS09], let $v_\Diamond^+(q)$ be the function describing the middle branch of the curve $v^3 - 2qv - q^2 = 0$ for $0 < q \leq 32/27$, see Figure 5.1 or [JS09, Figure 2]. Also, let $v_\Diamond^-(q)$ be defined by $v_\Diamond^-(q) = q/v_\Diamond^+(q)$ for $0 < q \leq 32/27$.

CONJECTURE 5.1 [JS09] *The roots of the Tutte polynomials of graphs are dense in the following regions:*

(a) $q < 0$ and $v < -2$,

(b) $q < 0$ and $0 < v < -q/2$,

(c) $0 < q \leq 32/27$ and $v < v_\diamond^-(q)$,

(d) $0 < q \leq 32/27$ and $v_\diamond^+(q) < v < 0$, and

(e) $q > 32/27$ and $v < 0$.

The union of the regions described in Conjecture 5.1 is illustrated by the shaded and hatched area in Figure 5.1.

The main result of this chapter shows that the roots of the Tutte polynomial form a dense subset of large regions of the (q, v) plane, and we prove most cases of Conjecture 5.1. Our main tool is a technique of Thomassen [Tho97], which, loosely speaking, describes how to construct graphs whose chromatic roots are dense in a given interval. Whilst Thomassen's technique was originally developed for the chromatic polynomials of graphs, we show that it extends naturally to the Tutte polynomial, and can be used to prove density results in regions of the (q, v) plane.

THEOREM 5.2 *The roots of the Tutte polynomials of graphs are dense in the following regions:*

(a) $q < 0$ and $v < -2$,

(b) $q < 0$ and $0 < v < -q/2$,

(c) $0 < q \leq 32/27$ and $v < v_\diamond^-(q)$,

(d) $0 < q \leq 32/27$ and $v_\diamond^+(q) < v < 0$,

(e) $32/27 < q < 4$ and $v < 0$, and

(f) $q > 4$ and $-q < v < 0$.

Figure 5.1: Regions in Conjecture 5.1.

Thus, the only region of Jackson and Sokal's conjecture which is not covered by Theorem 5.2 is the region defined by $q > 4$ and $v < -q$. This region is indicated by a hatched area and question mark in Figure 5.1. We later discuss the obstructions that arise in this region which are related to an open problem on the flow polynomials of graphs.

5.2   The Multivariate Tutte Polynomial

In this section we introduce the multivariate Tutte polynomial and briefly describe the advantages in using this more general version. We refer the reader to [Sok05] for a more comprehensive introduction. Let G be a graph with vertex set V and edge set E. The **multivariate Tutte polynomial** of G is the polynomial

$$Z_G(q, \boldsymbol{v}) = \sum_{A \subseteq E} q^{k(A)} \prod_{e \in A} v_e, \tag{5.2}$$

where q and $\boldsymbol{v} = \{v_e\}_{e \in E}$ are commuting indeterminates, and $k(A)$ denotes the number of components in the graph (V, A). We say that $(G, \boldsymbol{v})$ is a **weighted graph**, $\boldsymbol{v}$ is a **weight-function**, and v_e is the **weight** of the edge e. It can be seen from (5.1) that the Tutte polynomial is obtained by setting all edge weights equal to a single indeterminate v.

We define a **dipole** to be the loopless multigraph consisting of two vertices x and y connected with a number of parallel edges. Thus, a single edge xy is considered to be a dipole. If F is a dipole with s edges of weights $v_1, \ldots, v_s \in \mathbb{R}$, then from (5.2) it may be seen that

$$Z_F(q, \boldsymbol{v}) = q^2 - q + q \prod_{i=1}^{s} (1 + v_i). \tag{5.3}$$

Similarly, if P_s denotes the path with s edges of weights $v_1, \ldots, v_s \in \mathbb{R}$, then (5.2) gives

$$Z_F(q, \boldsymbol{v}) = \prod_{i=1}^{s} (q + v_i). \tag{5.4}$$

Despite being interested in the roots of the Tutte polynomial, we will find it useful to consider the multivariate version. This viewpoint has proven to be particularly useful in studying the computational complexity of the evaluations [GJ08, GJ12, GJ14, JVW90], and the roots of the Tutte polynomial [JS09, Sok04].

The major advantage of using the multivariate Tutte polynomial is that, in certain circumstances, one can replace a subgraph by a single edge with an appropriate weight. Indeed, suppose $(G, \boldsymbol{v})$ is a weighted graph, $\{x, y\}$ is a 2-cut of G and F is an $\{x, y\}$-bridge. We let $\boldsymbol{v}|_F$ denote the restriction of the weight-function $\boldsymbol{v}$ to the edges of F. Recall that F_{xy} denotes the graph formed by identifying the vertices x and y in F. This may introduce multiple edges, and any edges between x and y in F become loops in F_{xy}. Also, let $F + xy$ denote the graph formed by adding an edge xy to F, and let $Z_{F+xy}(q, \boldsymbol{v}|_F, w)$ denote the multivariate Tutte polynomial of $F + xy$ where the new edge xy has

weight w. Finally, let v_F be the **effective weight** of F in $(G, \boldsymbol{v})$, which is the function defined by

$$1 + v_F(q, \boldsymbol{v}) = \frac{(q-1) Z_{F_{xy}}(q, \boldsymbol{v}|_F)}{Z_{F+xy}(q, \boldsymbol{v}|_F, -1)} = \frac{(q-1) Z_{F_{xy}}(q, \boldsymbol{v}|_F)}{Z_F(q, \boldsymbol{v}|_F) - Z_{F_{xy}}(q, \boldsymbol{v}|_F)}. \qquad (5.5)$$

The second equality holds since if G is a graph and xy is an edge of G with weight $v \in \mathbb{R}$, then $Z_G(q, \boldsymbol{v}) = Z_{G-xy}(q, \boldsymbol{v}|_{G-xy}) + v \cdot Z_{G_{xy}}(q, \boldsymbol{v}|_{G-xy})$, see [Sok05]. If F is a graph with $x, y \in V(F)$, then we write $v_F(q, v)$ to indicate the effective weight of the weighted graph F where every edge has weight v. The following lemma shows that replacing the graph F with a single edge of weight v_F only changes the multivariate Tutte polynomial by a prefactor depending on F.

LEMMA 5.3 [DJ11] *If $(G, \boldsymbol{v})$ is a weighted graph and F and H are connected subgraphs of G such that $V(F) \cup V(H) = V(G)$ and $V(F) \cap V(H) = \{x, y\}$, then $Z_G(q, \boldsymbol{v}) = \frac{1}{q(q-1)} Z_{F+xy}(q, \boldsymbol{v}|_F, -1) Z_{H+xy}(q, \boldsymbol{v}|_H, v_F(q, \boldsymbol{v}))$.*

We often use Lemma 5.3 in the following way. Suppose that F is a graph with two non-adjacent vertices x and y, and that F has effective weight v_F. If G' denotes the graph obtained from G by replacing every edge xy with a copy of F, then by Lemma 5.3, $Z_{G'}(q, v)$ is equal to $Z_G(q, v_F(q, v))$ up to a prefactor.

$$Z_{G'}(q, v) = \left[q^{-1}(q-1)^{-1} Z_{F+xy}(q, v, -1) \right]^{|E(G)|} Z_G(q, v_F(q, v)). \qquad (5.6)$$

In particular, if $(q, v_F(q, v))$ is a root of Z_G, then (q, v) is a root of $Z_{G'}$. For the readers convenience, we remark that in [GJ14], this operation is referred to as *implementing* the point $(q, v_F(q, v))$ from the point (q, v), and it can easily be composed. Indeed, if we choose non-adjacent vertices $x', y' \in V(G)$, then the effective weight of G' satisfies

$$v_{G'}(q, v) = v_G(q, v_F(q, v)). \qquad (5.7)$$

To see this, note that for $G'_{x'y'}$ and $G' + x'y'$, a similar equality to (5.6) holds by repeated application of Lemma 5.3. These prefactors of these equalities cancel in (5.5), which gives (5.7).

We briefly note the effective weights of two common graphs which we will frequently use. See [Sok05] for a more detailed derivation. If F is a dipole with s edges of weights $v_1, \ldots, v_s \in \mathbb{R}$, then the effective weight v_F of F satisfies

$$1 + v_F = \prod_{i=1}^{s} (1 + v_i). \tag{5.8}$$

As before, let P_s denote the path with s edges. If the end vertices of P_s are labelled x and y, and the edges of P_s have weights $v_1, \ldots, v_s \in \mathbb{R}$, then the effective weight v_{P_s} of P_s satisfies

$$1 + \frac{q}{v_{P_s}} = \prod_{i=1}^{s} \left(1 + \frac{q}{v_i}\right). \tag{5.9}$$

We say that a connected loopless graph F with two vertices labelled x and y is a **two-terminal graph** if x and y are not adjacent in F. The vertices x and y are called **terminals**. The following lemma shows that these graphs satisfy a technical condition which will be required later.

LEMMA 5.4 *Let $v \in \mathbb{R}$ be fixed. If F is a two-terminal graph, then there exists $q_0 > 0$ such that $1 + v_F(q, v) > 0$ for all $q > q_0$. Furthermore, if $1 + v_F(q, v)$ is a constant function of q, then $1 + v_F(q, v) = 1$ for all $q \in \mathbb{R}$.*

PROOF. By (5.5), we have

$$1 + v_F(q, v) = \frac{(q-1)Z_{F_{xy}}(q, v)}{Z_{F+xy}(q, v, -1)} = \frac{qZ_{F_{xy}}(q, v) - Z_{F_{xy}}(q, v)}{Z_F(q, v) - Z_{F_{xy}}(q, v)}.$$

Since $xy \notin E(F)$, the graph F_{xy} is loopless. Hence, the terms with the highest powers of q in $Z_{F_{xy}}(q, v)$ and $Z_{F+xy}(q, v, -1)$ are $q^{|V(F)|-1}$ and $q^{|V(F)|}$ respectively. Thus, $1 + v_F(q, v) \to 1$ as $q \to \infty$, which implies that the desired q_0 exists. By the same reason, if $1 + v_F$ is a constant function of q then it is equal to 1 as claimed. $\qquad\square$

5.3   Strategy

In [Tho97], Thomassen showed that the set of chromatic roots contains a dense subset of the interval $(32/27, \infty)$. In this section we generalise his technique to the Tutte polynomial. At the heart of Thomassen's method lies the following lemma, which is proved implicitly in Proposition 2.3 of [Tho97].

LEMMA 5.5 [Tho97] *Let $I \subseteq \mathbb{R}$ be an interval of positive length, and let a, b and c be rational functions of q such that $0 < b(q) < 1 < a(q)$ and $c(q) > 0$ for all $q \in I$. If there is no $\alpha \in \mathbb{Q}$ such that $\log(a(q))/\log(b(q)) = \alpha$ for all $q \in I$, then there exist $s, t \in \mathbb{N}$ such that $a(q_0)^s b(q_0)^t = c(q_0)$ for some $q_0 \in I$. Moreover, s and t can be chosen to have prescribed parity.*

We remark that the proof of Lemma 5.5 proceeds by finding s and t such that the function $a(q)^s b(q)^t - c(q)$ has different signs at two points in I. Between two such points there is a root, and since there are finitely many roots, at least one of them has odd multiplicity. We will use this extra fact later.

Let F be a two-terminal graph or dipole. For real numbers q and v, we say that F has one of four types at (q, v) defined by the following conditions.

Type A^+: $1 + v_F(q, v) > 1$,

Type A^-: $1 + v_F(q, v) < -1$,

Type B^+: $0 < 1 + v_F(q, v) < 1$,

Type B^-: $-1 < 1 + v_F(q, v) < 0$.

Let A and B be two-terminal graphs or dipoles. Note that if A is a graph of type A^+ or A^- at (q, v), then the rational function $1 + v_A(q, v)$ or $-1 - v_A(q, v)$ respectively can play the role of $a(q)$ in Lemma 5.5. The corresponding property holds for a graph of type B^+ or B^-.

DEFINITION 5.6 Let A and B be graphs. We say that (A, B) is a **complementary pair** at (q, v) if A and B are two-terminal graphs or dipoles, at most one of A and B is a dipole, and

- A has type A^+ and B has type B^- at (q, v), or

- A has type A^- and B has type B^+ at (q, v).

Let $G_1, \ldots, G_n$ be two-terminal graphs or dipoles. We say that the graph G formed by identifying all vertices labelled x into a single vertex, and all vertices labelled y into another is the **parallel composition** of $G_1, \ldots, G_n$. The definition of complementary pairs is motivated by the following key lemma, which is based on an argument implicit in [Tho97].

LEMMA 5.7 *Let $q_0, v \in \mathbb{R}$ be fixed such that $q_0 \neq 1$, and let A and B be two-terminal graphs or dipoles. If (A, B) is complementary at (q_0, v), then for all $\varepsilon > 0$, there is a graph G such that $Z_G(q_1, v) = 0$ for some $q_1 \in (q_0 - \varepsilon, q_0 + \varepsilon)$. Furthermore, G is a parallel composition of copies of A and B.*

PROOF. Suppose that A has type A^-, and B has type B^+ at (q_0, v). The other case is analogous. Since the functions $v_A(q, v)$ and $v_B(q, v)$ are continuous in q on their domain, there exists an interval $I \subseteq \mathbb{R}$ of positive length such that $q_0 \in I \subseteq (q_0 - \varepsilon, q_0 + \varepsilon)$ and the graphs A and B have types A^- and B^+ respectively for all $q \in I$. For $s, t \in \mathbb{N}$, let G be a parallel composition of s copies of A, and t copies of B. Using Lemma 5.3 and (5.3), one can see that the Tutte polynomial of G is

$$Z_G(q, v) = q \left(\frac{Z_{A+xy}(q, v, -1)}{q(q-1)} \right)^s \left(\frac{Z_{B+xy}(q, v, -1)}{q(q-1)} \right)^t f(q, v),$$

where $f(q, v) = q - 1 + (1 + v_A(q, v))^s (1 + v_B(q, v))^t$. Define $a(q) = -1 - v_A(q, v)$ and $b(q) = 1 + v_B(q, v)$. Moreover, define $c(q) = q - 1$ or $c(q) = 1 - q$ such that $c(q) > 0$ for all $q \in I$. In doing this it may be necessary to replace I with a subinterval of I containing q_0. This is possible since $q_0 \neq 1$ by assumption. If

the functions a, b and c satisfy the conditions in Lemma 5.5, then there exist $s, t \in \mathbb{N}$, of any prescribed parity, such that $a(q_1)^s b(q_1)^t = c(q_1)$ for some $q_1 \in I$. Since $f(q, v)$ is $c(q) + (-a(q))^s b(q)^t$ or $-c(q) + (-a(q))^s b(q)^t$, we may choose the parity of s and t such that this factor becomes zero for some $q_1 \in I$.

It remains to check that a, b and c satisfy the conditions of Lemma 5.5. Indeed, by assumption, we have that $0 < b(q) < 1 < a(q)$ and $c(q) > 0$ for all $q \in I$. Suppose for a contradiction that there is $\alpha \in \mathbb{Q}$ such that $\log(a(q))/\log(b(q)) = \alpha$ for all $q \in I$. Equivalently $a(q) = b(q)^\alpha$ for $q \in I$. Since a and b are rational functions and $\alpha \in \mathbb{Q}$, it follows that this equality is satisfied for all $q \in \mathbb{R}$ except for any singularities. Also, since $a(q) > 0$ for $q \in I$, we take the principal branch of any fractional power. Since (A, B) is complementary, at least one of A and B is a two-terminal graph. If A is a two-terminal graph, then since A has type A^-, we have that $1 + v_A(q_0, v) \neq 1$. By Lemma 5.4, this implies that $a(q)$ is not constant. Similarly, if B is a two-terminal graph, then $b(q)$ is not constant. Now, if precisely one of a and b is a constant function, then we immediately deduce a contradiction. Thus, we may assume that both of A and B are two-terminal graphs. By Lemma 5.4, we see that $a(q) < 0$ and $b(q) > 0$ for large enough q. This contradicts the assertion that $a(q) = b(q)^\alpha$ for all $q \in \mathbb{R}$.   □

COROLLARY 5.8 *If R is an open subset of the (q, v) plane such that for every $(q, v) \in R$, there is a complementary pair of graphs, then the roots of the Tutte polynomials of graphs are dense in R.*

To obtain density in some regions we will use planar duality. Let G be a plane graph and let G^* be its planar dual. The following relation is easily derived from (5.2) and Eulers formula, see [Sok05].

$$Z_{G^*}(q, v) = q^{1 - |V(G)|} v^{|E(G)|} Z_G(q, \tfrac{q}{v}). \tag{5.10}$$

Notice that the graphs constructed in Lemma 5.7 are planar if $A + xy$ and $B + xy$ are both planar. Thus, we have a second corollary of Lemma 5.7.

COROLLARY 5.9 *Let R be an open subset of the (q, v) plane. If for every $(q, v) \in \mathbb{R}$ there is a complementary pair (A, B) such that $A + xy$ and $B + xy$ are planar, then the roots of the Tutte polynomials of graphs are dense in R^* where*

$$R^* = \{(q, v) : (q, q/v) \in R\}. \tag{5.11}$$

5.4   Complementary Pairs

In this section, we find complementary pairs of graphs for points in several regions of the (q, v) plane. Combining this with Corollaries 5.8 and 5.9, we deduce Theorem 5.2. In what follows it will be useful to partition the (q, v) plane into a number of regions, which are illustrated in Figures 5.2 and 5.3. Note that taken together, the closure of the regions below is equal to the union of the regions in Theorem 5.2. Thus, if the roots of the Tutte polynomials of graphs are dense in these regions, then Theorem 5.2 follows. The regions are defined as follows.

- Region I: $q < 0$ and $v < -2$.

- Region II: $0 < q < 1$ and $v < -2$.

- Region III: $1 < q < 2$ and $v < -2$.

- Region IV: $2 < q < 4$, $q \neq 3$ and $v < -q$.

- Region V: $q > 2$ and $-q < v < -2$.

- Region VI: $2 < q < 4$ and $-2 < v < -q/2$.

- Region VII: $q > 2$ and $-1 < v < 0$.

- Region VIII: $0 < q < 32/27$ and $-2 < v < v_{\diamond}^{-}$.

- Region IX: $32/27 < q < 2$ and $-2 < v < -1$.
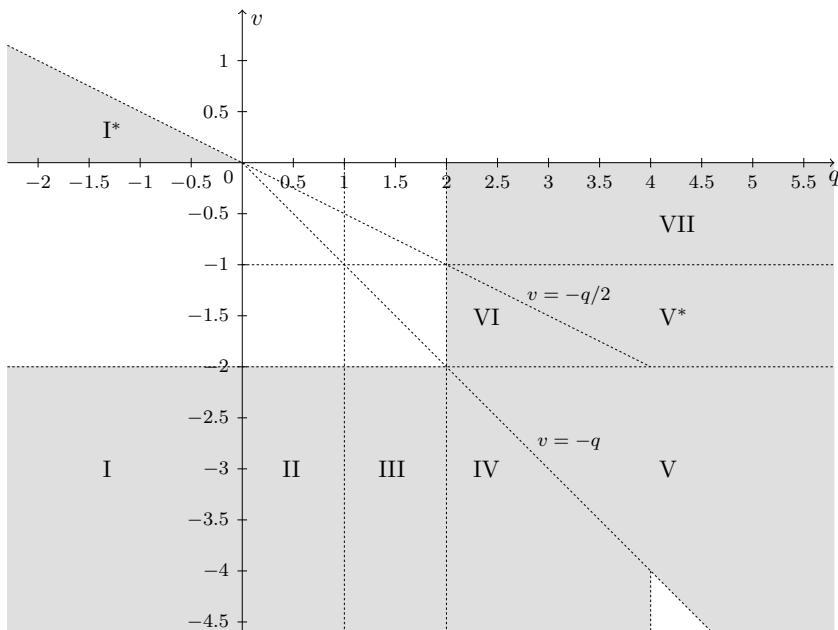
Figure 5.2: Regions I - VII, I* and V*.

We also define the following dual regions in the sense of (5.11). It is easy to check that the dual of each region below is contained in the corresponding region above.

- Region I*: $q < 0$ and $0 < v < -q/2$.

- Region II*: $0 < q < 1$ and $-q/2 < v < 0$.

- Region III*: $1 < q < 2$ and $-q/2 < v < 0$.

- Region V*: $-2 < v < -1$ and $q > -2v$.

- Region VIII*: $0 < q < 32/27$ and $v_\diamond^+ < v < -q/2$.

- Region IX*: $32/27 < q < 2$ and $-1 < v < -q/2$.

In the following lemma, we show that the path of length $s \in \mathbb{N}$ gives graphs of varying types depending on the point (q, v). We impose the condition $s > 1$
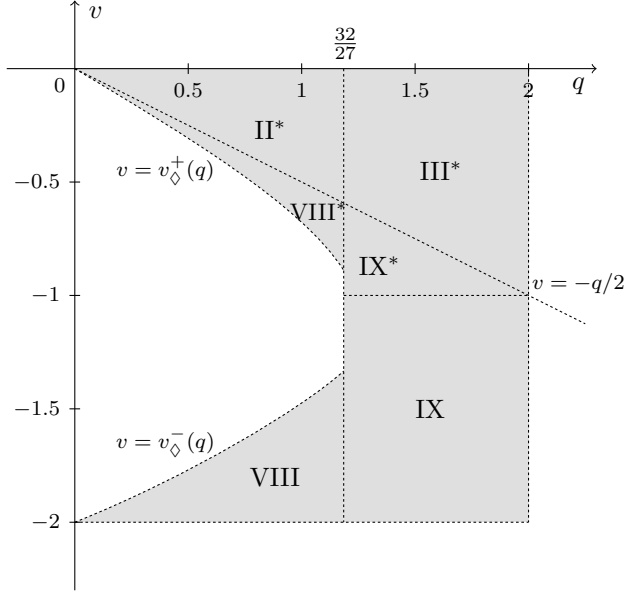
Figure 5.3: Regions VIII, IX, II*, III*, VIII* and IX*.

so that the resulting graphs are two-terminal graphs. We note that parts (ii) and (iii) are equivalent to Lemmas 21 and 22 in [GJ14] respectively.

LEMMA 5.10 *Let q and v be real numbers, and let P_s denote the path of length s where every edge has weight v.*

 (i) *If $v < 0$ and $q < 0$, then there is $s > 1$ such that P_s has type B^+ at (q, v).*

 (ii) *If $v < -2$ and $0 < q < 1$, then there is $s > 1$ such that P_s has type B^+ at (q, v).*

 (iii) *If $v < -2$ and $1 < q < 2$, then there is $s > 1$ such that P_s has type B^- at (q, v).*

 (iv) *If $v < 0$ and $q > -2v$, then there is $s > 1$ such that P_s has type A^+ at (q, v).*

 (v) *If $v < 0$ and $2 < q < -2v$, then there is $s > 1$ such that P_s has type A^- at (q, v).*

PROOF. By (5.9), the effective weight of P_s is given by

$$v_{P_s} = \frac{q}{(1 + \frac{q}{v})^s - 1}. \tag{5.12}$$

(i) Since $q/v > 0$, the denominator of (5.12) tends to infinity with s. Because $q < 0$, we deduce that there exists $s > 1$ such that $-1 < v_{P_s} < 0$. Thus, P_s has type B^+ as claimed.

(ii) The conditions of the lemma imply that $0 < 1 + \frac{q}{v} < 1$. Thus, for all $\varepsilon > 0$, there is $s > 1$ such that $-q - \varepsilon < v_{P_s} < -q$. Since $0 < q < 1$, there exists $s > 1$ such that P_s has type B^+.

(iii) By the same argument as in (ii), for every $\varepsilon > 0$, there exists $s > 1$ such that $-q - \varepsilon < v_{P_s} < -q$. Since $1 < q < 2$, there exists $s > 1$ such that P_s has type B^-.

(iv) The conditions imply $q/v < -2$. Thus, for any even s we have $v_{P_s} > 0$. It follows that there is $s > 1$ such that P_s has type A^+.

(v) Since $2 < q < -2v$, we have $-1 < 1 + \frac{q}{v} < 1$. Thus, for $s = 2$, we have $v_{P_s} < -q$. In particular, $v_{P_s} < -2$. Thus, there exists $s > 1$ such that P_s has type A^-.   □

We will also require some less simple two-terminal graphs. Many of these we take from [GJ14] and [JS09] where a similar technique is used. The following lemma is an intermediate step in the proof of Lemma 11 from [GJ14].

LEMMA 5.11 *If $q > 2$ and $-q < v < -2$, then there is a two-terminal graph of type B^+ at (q, v).*

PROOF. Let F be the dipole having two edges of weight v. Note that by (5.8) we have $v_F = v(v + 2) > 0$. Now let G be the two-terminal graph consisting of s copies of F and one edge of weight v in series. By (5.9) we have that

$$1 + \frac{q}{v_G} = \left(1 + \frac{q}{v_F}\right)^s \left(1 + \frac{q}{v}\right). \tag{5.13}$$

By the conditions of the lemma, we have that $1 + q/v < 0$. Since $1 + q/v_F > 1$, the right hand side of (5.13) tends to minus infinity as s tends to infinity. It follows that there is s such that $-1 < v_G < 0$, and for this s, the graph G is a two-terminal graph of type B^+ at (q, v). $\qquad \square$

LEMMA 5.12 *Let F be a two-terminal graph with effective weight v_F and let q be a real variable. Also let $F^{(2)}$ be the parallel composition of two copies of F. If $v_F(q) < -2$, then $F^{(2)}$ has effective weight satisfying $v_{F^{(2)}}(q) > 0$. If $-2 < v_F(q) < -1$, then $F^{(2)}$ has effective weight satisfying $-1 < v_{F^{(2)}}(q) < 0$.*

PROOF. Let D denote the dipole with two edges of weight v_F. By (5.7), the effective weight of $F^{(2)}$ is equal to the effective weight of D. Thus, by (5.8), we have $v_{F^{(2)}}(q) = v_F(q)(v_F(q) + 2)$. If $v_F(q) < -2$, this is positive. If $-2 < v_F(q) < -1$, then $-1 < v_{F^{(2)}}(q) < 0$. $\qquad \square$

In the following lemma we invoke Lemma 23 of [GJ14], which uses the two-terminal graph obtained from the Petersen graph by deleting an edge xy.

LEMMA 5.13 *If $2 < q < 4$ is non-integer and $v < -q$, then there is a two-terminal graph of type B^+ at (q, v).*

PROOF. By the argument in Lemma 23 of [GJ14], there is a two-terminal graph F satisfying $-q < v_F < 0$. If $-1 < v_F < 0$, then the result follows immediately. If $-q < v_F < -2$, then the result follows by Lemma 5.11. If $-2 < v_F < -1$, then by Lemma 5.12, the two-terminal graph $F^{(2)}$ formed by taking two copies of F in parallel has effective weight $v_{F^{(2)}}$ satisfying $-1 < v_{F^{(2)}} < 0$ as required. Thus, it just remains to consider the cases when $v_F \in \{-1, -2\}$.

Let J_s denote the graph consisting of s copies of F in series. The effective weight of J_s is equal to the effective weight of P_s, where P_s denotes the path with s edges of weight v_F. Suppose $v_F = -1$. Thus, by (5.9), we have $v_{J_3} = \frac{-1}{q^2 - 3q + 3}$. It may be checked that for $q > 2$, we have $-1 < v_{J_3} < 0$. So J_3 has type B^+ as required. Now suppose that $v_F = -2$. By (5.9), we have $1 + \frac{q}{v_{J_s}} = (1 - \frac{q}{2})^s$.

Since $2 < q < 4$, we have that $-1 < 1 - \frac{q}{2} < 0$, and so for any $\varepsilon > 0$, there exists a large and odd s such that $-q < v_{J_s} < -q + \varepsilon$. Thus, we can ensure that $-q < v_{J_s} < -2$. The result now follows by an application of Lemma 5.11. $\square$

The following lemma uses a gadget based on large complete graphs and consequently, the resulting two-terminal graph is non-planar.

LEMMA 5.14 [GJ14, Lemma 18] *If $q > 2$ and $-1 < v < 0$ then there is a two-terminal graph of type A^- or B^- at (q, v).*

Recall that $v_\diamond^+(q)$ is the function describing the middle branch of the curve $v^3 - 2qv - q^2 = 0$ for $0 < q < 32/27$, and that $v_\diamond^-(q)$ is defined by $v_\diamond^-(q) = q/v_\diamond^+(q)$ for $0 < q < 32/27$.

LEMMA 5.15 *If $0 < q < 32/27$ and $-2 < v < v_\diamond^-(q)$, then there is a two-terminal graph F of type A^+ at (q, v). Furthermore, we can choose F such that $F + xy$ is planar.*

PROOF. Let H be the graph consisting of two edges of weight v in parallel. We claim that $v_H > v_\diamond^+(q)$. By (5.8), the effective weight v_H of H satisfies $v_H = v(v + 2)$, which is a decreasing function of v for $-2 < v < v_\diamond^-$. Note that $\frac{q}{v}(\frac{q}{v} + 2) = v$ is precisely the equation satisfied by $v_\diamond^+(q)$. Thus, $v_\diamond^-(q)(v_\diamond^-(q) + 2) = v_\diamond^+(q)$ for $0 < q < 32/27$. Since $v < v_\diamond^-$, it follows that $v_H = v(v + 2) > v_\diamond^-(q)(v_\diamond^-(q) + 2) = v_\diamond^+(q)$ as claimed. Now by Lemmas 8.5(a) and 8.5(b) in [JS09], there is a two-terminal graph obtained from H which has type A^+. $\square$

LEMMA 5.16 [GJ14, Lemma 12] *If $q > 32/27$ and $-2 < v < -q/2$, then there is a two-terminal graph F of type A^+ at (q, v). Moreover, $F + xy$ is planar.*

We now combine the results of this section and Section 5.3 to prove Theorem 5.2.

PROOF OF THEOREM 5.2. We first show that the roots of the Tutte polynomial are dense in regions I - IX. By Corollary 5.8, it suffices to show that for each point (q, v) in regions I - IX, there exists a complementary pair of graphs. In

regions I - V, a single edge of weight v has type A^-. By Lemma 5.12, the graph consisting of two such edges in parallel has type A^+. Thus, in regions I - V, it only remains to find a two-terminal graph of type B^+ or B^-.

Region I: By Lemma 5.10(i), there is a two-terminal graph of type B^+.

Region II: By Lemma 5.10(ii), there is a two-terminal graph of type B^+.

Region III: By Lemma 5.10(iii), there is a two-terminal graph of type B^-.

Region IV: By Lemma 5.13, there is a two-terminal graph of type B^+.

Region V: By Lemma 5.11, there is a two-terminal graph of type B^+.

We deal with the remaining regions individually.

Region VI: For (q, v) in region VI, a single edge of weight v has type B^-. By Lemma 5.10(v), there exists a two-terminal graph of type A^-. By Lemma 5.12, taking two copies of this graph in parallel gives a two-terminal graph of type A^+ as required.

Region VII: A single edge of weight v has type B^+. By Lemma 5.14 there is a two-terminal graph F of type A^- or B^- at (q, v). If F has type A^- then we are done. If F has type B^-, then the effective weight v_F of F satisfies $-2 < v_F < -1$. Thus, the point (q, v_F) lies in region VI or V*. If $(q, v_F) \in$ VI, then we use the argument for region VI to obtain a two-terminal graph of type A^+. If $(q, v_F) \in$ V*, then we use Lemma 5.10(iv) to obtain a two-terminal graph of type A^+.

Region VIII: A single edge of weight v has type B^-. By Lemma 5.15, there is a two-terminal graph of type A^+ at (q, v).

Region IX: A single edge of weight v has type B^-. By Lemma 5.16, there is a two-terminal graph of type A^+ at (q, v).

We note that in regions I, II, III, V, VIII and IX, each graph F that we use has the property that $F + xy$ is planar. Thus, by Corollary 5.9, the roots of the Tutte polynomials of planar graphs are also dense in the regions I*, II*, III*, V*, VIII* and IX*. $\square$

We briefly remark on the region in which we have been unable to prove density, namely the points satisfying $q > 4$ and $v < -q$. For (q, v) in this region, the sequence of paths P_s, $s \in \mathbb{N}$ have effective weights converging to the point $-q$ as $s \to \infty$. Along the line $v = -q$, the multivariate Tutte polynomial is nothing other than the flow polynomial $F(G, q)$ multiplied by a prefactor. More precisely, we have

$$F(G, q) = q^{-|V(G)|}(-1)^{|E(G)|} Z_G(q, -q). \tag{5.14}$$

Goldberg and Jerrum [GJ14] have shown that if G is a graph and $xy \in E(G)$ such that $F_G(q)$ and $F_{G-xy}(q)$ have opposite signs, then it is possible to implement a weight v' satisfying $-q < v' < 0$. Using an argument similar to that of Lemma 5.13, it would then be possible to find a two-terminal graph which has type B^+ at (q, v). It is conjectured [JS13b] that there exists $q_0 \in \mathbb{R}$ such that $F_G(q) > 0$ for all 2-edge connected graphs G and all $q > q_0$. Thus, it seems unlikely that this technique can be used to prove density for all $q > 4$.

The dual of the unsolved region lies inside region VII. Unfortunately, the graphs we used to prove density in region VII are non-planar, and so we cannot use duality as we have done above. However, if we allow ourselves to use all matroids instead of all graphs, then we can apply the duality argument, since every matroid has a dual. It is easy to define the Tutte polynomial for matroids by replacing the term $q^{k(A)}$ with $q^{r(G)-r(A)+1}$ where r is the rank function, see [Sok05].

Density of Chromatic Roots

6.1 Introduction

For many natural classes of graphs $\mathcal{G}$, we do not know the topological closure of $R(\mathcal{G})$. The class of planar graphs is a prominent example of this, despite the fact that the chromatic polynomial was initially introduced and studied for planar graphs. In one of the first monographs on the subject of chromatic polynomials, Birkhoff and Lewis [BL46] proved that all chromatic roots of planar graphs are less than 5.

THEOREM 6.1 [BL46] *The interval $[5, \infty)$ is zero-free for the class of planar graphs.*

In the same monograph, the authors made the following conjecture which is still an open problem.

CONJECTURE 6.2 [BL46] *The interval $[4, 5)$ is zero-free for the class of planar graphs.*

In fact, the only point at which we know that Conjecture 6.2 holds is $q = 4$, which is equivalent to the four colour theorem.

At the other end of the number line, it is clear from Theorem 1.9 that there are no non-trivial chromatic roots of planar graphs below 32/27. In addition, Thomassen [Tho97] used the method described in Chapter 5 to prove the following density result.

THEOREM 6.3 [Tho97] *If $\mathcal{G}$ denotes the class of planar graphs, then we have* $[32/27, 3] \subseteq \overline{R(\mathcal{G})}$.

Thomassen also conjectured a natural extension of Theorem 6.3 which, if true, implies that the only remaining unknown interval is that in Conjecture 6.2.

CONJECTURE 6.4 [Tho97] *If $\mathcal{G}$ denotes the class of planar graphs, then we have* $[3, 4] \subseteq \overline{R(\mathcal{G})}$.

Royle [Roy08] proved Conjecture 6.4 for the point 4.

THEOREM 6.5 [Roy08] *For all $\varepsilon > 0$, there is a planar triangulation with a chromatic root in the interval $(4 - \varepsilon, 4)$.*

In this chapter, we build on Royle's result by proving Conjecture 6.4 except in a small interval around $\tau + 2$, where τ is the golden ratio.

THEOREM 6.6 *If $\mathcal{G}$ denotes the class of planar graphs, then $[3, t_\ell] \cup [t_r, 4] \subseteq \overline{R(\mathcal{G})}$ where t_ℓ and t_r are real numbers with $t_\ell \approx 3.618032$ and $t_r \approx 3.618356$.*

We first show that for certain classes of graphs $\mathcal{G}$, the set $R(\mathcal{G})$ of chromatic roots is closely related to the set of real numbers q_0 such that $P(G, q_0) < 0$ for some $G \in \mathcal{G}$. This new observation makes it much easier to apply the method of Thomassen discussed in Chapter 5, and we use this to deduce Theorem 6.6.

An intriguing conjecture of Beraha [Ber75] suggests a connection between chromatic roots and the sequence of numbers defined by $B_n = 2 + 2\cos(\frac{2\pi}{n})$, now known as the **Beraha numbers**. Indeed, in light of the results mentioned above, it is plain to see that the numbers $B_1 = 4$, $B_2 = 0$, $B_3 = 1$, $B_4 = 2$, $B_5 = \tau^2$, $B_6 = 3$ and $B_{10} = \tau + 2$ are of significance. While Beraha's conjecture has been interpreted in several ways, see [JS13a], the version presented by Jackson [Jac03], and by Jensen and Toft [JT95] is as follows.

CONJECTURE 6.7 *There exists a planar triangulation with a real chromatic root in $(B_n - \varepsilon, B_n + \varepsilon)$ for all $n \geq 1$ and all $\varepsilon > 0$.*

Conjecture 6.7 trivially holds for B_2, B_3, B_4 and B_6 by considering the chromatic polynomial of K_4, say. Theorem 6.5 settles the case B_1, and Beraha, Kahane and Weiss [BKW80] solved the cases B_5 and B_7. They also proved that B_{10} is an accumulation point of *complex* chromatic roots of planar triangulations. Nonetheless, the conjecture remains open for $n \geq 8$. Interestingly, Salas and Sokal [SS01] proved that no non-integer Beraha number is a chromatic root except possibly B_{10}, which is not the chromatic root of any planar graph by a result of Tutte.

It seems that the condition of being a triangulation in Beraha's conjecture arises only because of the applications to physics and the study of lattice graphs therein. Thus, we find the question of Beraha to also be of interest for general planar graphs. Since $B_n \in (3, t_\ell) \cup (t_r, 4)$ for $n \geq 8$, $n \neq 10$, a consequence of our density result is that there are planar graphs with real chromatic roots arbitrarily close to each B_n, $n \geq 8$, except possibly for $B_{10} = \tau + 2$. Moreover, our methods can be used to show that if there are planar graphs with real chromatic roots arbitrarily close to B_{10}, then there are planar triangulations with the same property.

After considering the planar graphs, we also investigate the classes of K_5-minor-free graphs and $K_{3,3}$-minor-free graphs. It is well known that there is a nice

interplay between these three graph classes, starting with Wagner's equivalence theorem [Wag37], which says that the four colour theorem is equivalent to Hadwiger's conjecture for $k = 5$. The chromatic root distribution of these three classes will further strengthen this interplay. We also briefly consider the density of chromatic roots of graphs on surfaces, bipartite graphs and the roots of the flow polynomial.

6.2   A Method Revisited

In this section, we reinterpret the technique of Chapter 5 for the chromatic polynomial. For this reason, we fix $v = -1$ for the rest of this chapter and define $Z_G(q) = Z_G(q, -1)$. Recall the definition of effective weights and complementary pairs from Section 5.3. Since $v = -1$, the effective weight v_F of a two-terminal graph F with terminals x and y is a function only of q. Thus, we write $v_F(q)$ to mean $v_F(q, -1)$. Note that accordingly, we have

$$1 + v_F(q) = \frac{(q-1)Z_{F_{xy}}(q)}{Z_F(q) - Z_{F_{xy}}(q)} = \frac{(q-1)P(F_{xy}, q)}{P(F, q) - P(F_{xy}, q)}. \tag{6.1}$$

Since we now deal exclusively with the chromatic polynomial, we only consider loopless graphs. Also, multiple edges have no effect on the chromatic polynomial, which is reflected in the properties of (5.1) when $v = -1$. For this reason, the polynomial $Z_{F_{xy}}(q)$ in (6.1) is nothing other than $Z_{F/xy}(q)$. Thus, by Proposition 1.3, the effective weight of F at q is given by

$$1 + v_F(q) = \frac{(q-1)P(F/xy, q)}{P(F + xy, q)}. \tag{6.2}$$

Let $\mathcal{G}$ be a class of graphs. We let $\boldsymbol{R_o(\mathcal{G})}$ be the set of all chromatic roots of odd multiplicity of graphs in $\mathcal{G}$. We also denote by $\boldsymbol{N(\mathcal{G})}$ the set of all real numbers q_0 such that the chromatic polynomial of some graph in $\mathcal{G}$ is negative at q_0. Theorem 2.3 implies the following fact.

PROPOSITION 6.8 *Let $\mathcal{G}$ be a class of graphs. If $\mathcal{G}$ contains connected graphs G and H such that $|V(G)|$ and $|V(H)|$ are at least 2 and have different parity, then $N(\mathcal{G}) \cap (-\infty, 1] = (-\infty, 0) \cup (0, 1)$.*

We now demonstrate a relationship between the sets $R_o(\mathcal{G})$ and $N(\mathcal{G})$.

LEMMA 6.9 *If $\mathcal{G}$ is a class of graphs, then $R_o(\mathcal{G}) \subseteq \overline{N(\mathcal{G})}$.*

PROOF. Let q_0 be a real number such that $q_0 \in R_o(\mathcal{G})$. For some graph G in $\mathcal{G}$, the chromatic polynomial $P(G, q)$ is of the form $(q - q_0)^r S(q)$ where r is an odd natural number and S is a polynomial such that $S(q_0) \neq 0$. By continuity, $S(q)$ is non-zero in an interval $(q_0 - \varepsilon, q_0 + \varepsilon)$ for some $\varepsilon > 0$. Now, for any ε' such that $0 < \varepsilon' < \varepsilon$, it follows that one of $P(G, q_0 - \varepsilon')$ and $P(G, q_0 + \varepsilon')$ is negative. Thus, q_0 is in the closure of $N(\mathcal{G})$. $\qquad\square$

For two graphs G and H, and an integer $k \geq 2$, a **k-clique sum** of G and H is any graph formed by identifying a clique of size k in G with a clique of size k in H. We say that $\mathcal{G}$ is closed under taking k-clique sums if all k-clique sums of graphs $G, H \in \mathcal{G}$ are members of $\mathcal{G}$.

In fact, if $\mathcal{G}$ satisfies certain conditions, then the sets $R_o(\mathcal{G})$ and $N(\mathcal{G})$ have essentially the same closure.

THEOREM 6.10 *Let $\mathcal{G}$ be a class of graphs such that $C_4 \in \mathcal{G}$. If $\mathcal{G}$ is closed under edge deletion and taking 2-clique sums, then $\overline{R_o(\mathcal{G})} \cap [2, \infty) = \overline{N(\mathcal{G})} \cap [2, \infty)$.*

PROOF. By Lemma 6.9, it suffices to show that $N(\mathcal{G}) \cap [2, \infty) \subseteq \overline{R_o(\mathcal{G})}$, so suppose that $q_0 \in N(\mathcal{G}) \cap [2, \infty)$. Thus, there is a graph $G \in \mathcal{G}$, such that $P(G, q_0) < 0$. This implies that $q_0 \neq 2$, because the chromatic polynomial of any graph evaluated at a positive integer is non-negative. Assume that G is edge-minimal with this property, so for each edge e we have $P(G - e, q_0) \geq 0$. Note that G is not edgeless, since the chromatic polynomial of a graph with no

edges is positive at any $q_0 > 0$. By the deletion-contraction formula, we have

$$P(G - e, q_0) = P(G, q_0) + P(G/e, q_0).$$

Hence $P(G/e, q_0) > 0$, and $P(G/e, q_0) \geq -P(G, q_0)$. Since $q_0 > 2$, we have by (6.2) that $F = G - e$ satisfies $1 + v_F(q_0) < -1$. In other words, F has type A^-. By (6.2) or (5.9) it is easy to deduce that P_3, the path with three edges, is of type B^+ for $q > 2$. Thus, (F, P_3) is a complementary pair. For every $\varepsilon > 0$, Lemma 5.7 implies that there is a graph H such that $P(H, q_1) = 0$ for some $q_1 \in (q_0 - \varepsilon, q_0 + \varepsilon)$. By the remark following Lemma 5.5, these roots have odd multiplicity. Furthermore, each graph H is a parallel composition of copies of F and P_3. Hence, such graphs can also be formed by taking the 2-clique sum of copies of G and C_4, and then subsequently deleting the edge xy. The conditions on $\mathcal{G}$ ensure that all graphs obtained by this construction stay in the class $\mathcal{G}$. $\square$

For a minor-closed class of graphs $\mathcal{G}$, we denote by **Forb($\mathcal{G}$)** the minor-minimal graphs not in $\mathcal{G}$. By the Robertson-Seymour theory [RS04], the set Forb($\mathcal{G}$) is finite. If $\mathcal{G}$ is minor-closed, and all graphs in Forb($\mathcal{G}$) are 3-connected, then $\mathcal{G}$ satisfies the conditions of Theorem 6.10.

COROLLARY 6.11 *If $\mathcal{G}$ is a minor-closed class of graphs, and all graphs in Forb($\mathcal{G}$) are 3-connected, then $\overline{R_o(\mathcal{G})} \cap [2, \infty) = \overline{N(\mathcal{G})} \cap [2, \infty)$.*

Suppose that $\mathcal{G}$ is minor-closed, and all graphs in Forb($\mathcal{G}$) are 3-connected. Let $\mathcal{G}_{\mathbf{max}}$ denote the set of edge-maximal graphs in $\mathcal{G}$. Equivalently, a graph $G \in \mathcal{G}$ is in $\mathcal{G}_{\max}$ if the addition of any edge produces a graph which is not in $\mathcal{G}$. If $G \in \mathcal{G}_{\max}$, then it is easy to see that G has no cut-vertex. Indeed, if x is a cut-vertex and we add an edge e between two neighbours of x in distinct components of $G - x$, then the resulting graph $G + e$ does not have a minor in Forb($\mathcal{G}$). This is because, as all graphs in Forb($\mathcal{G}$) are 3-connected, such a minor would also exist in G. Thus, $G + e \in \mathcal{G}$, which contradicts the maximality of G. Moreover, if x and y are vertices such that $G - x - y$ is disconnected, then x and y are joined by

an edge in G. Otherwise, we could add the edge xy and obtain a contradiction as above. Therefore, G is a 2-clique sum of the 3-connected graphs in $\mathcal{G}_{\max}$ and triangles. We let $\boldsymbol{\mathcal{G}_{3\max}}$ denote the set of graphs in $\mathcal{G}_{\max}$ which are 3-connected or K_3.

Theorem 6.12 *Let $\mathcal{G}$ be a minor-closed class of graphs. If all graphs in $Forb(\mathcal{G})$ are 3-connected, then*

$$N(\mathcal{G}) \cap (1, \infty) = N(\mathcal{G}_{max}) \cap (1, \infty) = N(\mathcal{G}_{3max}) \cap (1, \infty).$$

Proof. As the three sets are decreasing, it suffices to prove that any real number q_0 in $N(\mathcal{G}) \cap (1, \infty)$ is also in $N(\mathcal{G}_{3\max})$. To this end, let G be a graph in $\mathcal{G}$ such that $P(G, q_0) < 0$ and, subject to this, G has as few vertices as possible. Subject to these conditions, assume further that G has as many edges as possible. We claim that G is in $\mathcal{G}_{\max}$, so suppose for a contradiction that this is not the case. Thus, there are vertices $x, y \in V(G)$, such that $G + xy$ is in $\mathcal{G}$, and since $\mathcal{G}$ is minor-closed, this implies that G/xy is also in $\mathcal{G}$. Now, by the deletion-contraction formula, we have

$$P(G, q_0) = P(G + xy, q_0) + P(G/xy, q_0).$$

Since the left hand side is negative, at least one of $P(G+xy, q_0)$ and $P(G/xy, q_0)$ is also negative. Therefore, either $G + xy$ contradicts the edge maximality of G, or G/xy contradicts the vertex minimality of G.

We next claim that G is in $\mathcal{G}_{3\max}$. Otherwise, the remarks above imply that G is the union of two graphs $G_1, G_2 \in \mathcal{G}$, such that $|V(G_1)| \geq 3$, $|V(G_2)| \geq 3$, and the intersection $G_1 \cap G_2$ is a single edge. Now Proposition 1.6 gives that

$$P(G, q_0) = P(G_1, q_0)P(G_2, q_0)/q_0(q_0 - 1).$$

Since the left hand side is negative and $q_0 > 1$, it follows that one of $P(G_1, q_0)$ and $P(G_2, q_0)$ is negative. Therefore, one of G_1 and G_2 contradicts the mini-

mality property of G. This completes the proof.   □

In the following sections, we apply Corollary 6.11 and Theorem 6.12 to three classes of graphs, namely the planar graphs, the graphs having no K_5-minor, and the graphs having no $K_{3,3}$-minor. We find that for each of these classes, the chromatic roots are essentially the same in the sense that the closures are essentially the same.

We conclude this section with a remark about multiplicities. It is easy to see that a chromatic root of odd multiplicity is also a chromatic root of even multiplicity. For if G is a graph in $\mathcal{G}$ and $2G$ denotes the union of two copies of G, then $P(2G, q) = P(G, q)^2$, and hence all chromatic roots of $2G$ have even multiplicity. Consider the chromatic roots of graphs in $\mathcal{G}$ which are not chromatic roots of a graph in $\mathcal{G}$ of odd multiplicity. We do not know if such roots exist, but the next observation implies that we can say something about them if $\mathcal{G}_{3\mathrm{max}}$ is well understood. As we later show, this is indeed the case for the three classes of graphs that we focus on.

THEOREM 6.13 *Let $\mathcal{G}$ be a minor-closed class of graphs. If all graphs in Forb($\mathcal{G}$) are 3-connected, then $R(\mathcal{G}) \subseteq R(\mathcal{G}_{3max}) \cup N(\mathcal{G})$.*

PROOF. The proof is a repetition of the proof of Theorem 6.12. Let $q_0 \in R(\mathcal{G})$. By Theorem 1.9, we may clearly assume that $q_0 > 1$. Now let G be a graph in $\mathcal{G}$ such that q_0 is a chromatic root of G. Subject to this, assume that G has as few vertices as possible, and further, that G has as many edges as possible. If $G \in \mathcal{G}_{\mathrm{max}}$, then q_0 is a chromatic root of a graph in $\mathcal{G}_{\mathrm{max}}$ and hence also of a graph in $\mathcal{G}_{3\mathrm{max}}$ by the same argument as above using Proposition 1.6 and the fact that $q_0 > 1$. Otherwise, there are vertices $x, y \in V(G)$, such that $G + xy$ is in $\mathcal{G}$. By the deletion-contraction formula, we have

$$P(G, q_0) = P(G + xy, q_0) + P(G/xy, q_0).$$

Now, if $P(G+xy, q_0) = P(G/xy, q_0) = 0$, then $G+xy$ contradicts the maximality property of G. Therefore, one of $P(G+xy, q_0)$ and $P(G/xy, q_0)$ is negative, and thus $q_0 \in N(\mathcal{G})$. $\qquad \square$

An application of Corollary 6.11 yields the following.

COROLLARY 6.14 *If $\mathcal{G}$ is a minor-closed class of graphs such that all graphs in Forb$(\mathcal{G})$ are 3-connected, then $R(\mathcal{G}) \cap [2, \infty) \subseteq R(\mathcal{G}_{3max}) \cup \overline{R_o(\mathcal{G})}$.*

6.3   Planar Graphs

Before the solution of the 4-colour problem, Tutte [Tut70] proved a fascinating result regarding the chromatic polynomials of planar triangulations at the number $\tau + 2 = (5 + \sqrt{5})/2 \approx 3.618033$, where τ is the golden ratio. A consequence of Tutte's result is the following theorem.

THEOREM 6.15 [Tut70] *If G is a planar triangulation, then $P(G, \tau + 2) > 0$.*

Tutte's result gave new hope of an analytic solution to the 4-colour problem. However, regarding the interval $(\tau + 2, 4)$, Read and Tutte wrote in [RT88]:

> "It is tempting to conjecture that the chromatic polynomial of a triangulation must be positive throughout this interval, but counterexamples are known."

Theorem 6.5 shows that there is no interval of the form $(4 - \varepsilon, 4)$ which is zero-free for planar graphs. Theorem 6.6 shows that there are no such intervals in most of the interval $(3, 4)$. However, since $\tau + 2$ is in the interval (t_ℓ, t_r), it is possible that Tutte's result can be extended to a small neighbourhood around $\tau + 2$. There is strong evidence that this is not the case. Indeed using the results of this chapter, Royle (Private communication, 2016) has shown that

	p_1	p_2
q^0	567438039	-152643002
q^1	-2138431325	543148340
q^2	3863934525	-923047942
q^3	-4454394269	996884682
q^4	3673360947	-766835653
q^5	-2299985643	445549685
q^6	1131241573	-202024440
q^7	-445623415	72739711
q^8	141942413	-20939539
q^9	-36631145	4811140
q^{10}	7622430	-873498
q^{11}	-1263582	122848
q^{12}	163358	-12934
q^{13}	-15902	961
q^{14}	1098	-45
q^{15}	-48	1
q^{16}	1	

Figure 6.1: Coefficients of the polynomials p_1 and p_2.

the chromatic roots of planar graphs are dense in the interval $(t_\ell, \tau + 2)$.

Let $\mathcal{G}$ be a class of graphs. We say that a real number q_0 is **$\mathcal{G}$-positive** if $P(G, q_0) > 0$ for every graph G in $\mathcal{G}$. We let $\boldsymbol{A(\mathcal{G})}$ denote the real numbers which are $\mathcal{G}$-positive. If $\mathcal{G}$ is minor-closed, then a repetition of the proof of Theorem 6.12 shows that $A(\mathcal{G}) = A(\mathcal{G}_{\max})$. From this, one may deduce that Tutte's theorem [Tut70], which was originally stated for planar triangulations, also holds for all planar graphs.

THEOREM 6.16 *If G is a planar graph, then $P(G, \tau + 2) > 0$.*

Now Conjecture 6.2 is equivalent to the statement that all real numbers in $[4, \infty)$ are $\mathcal{G}$-positive when $\mathcal{G}$ denotes the class of planar graphs.

CONJECTURE 6.17 *If $\mathcal{G}$ denotes the class of planar graphs, then $\tau + 2$ is the only $\mathcal{G}$-positive real number less than 4.*
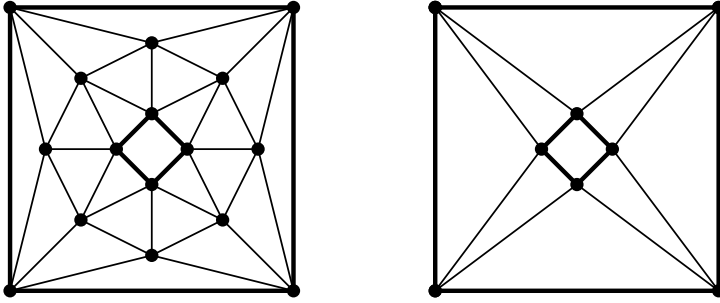
Figure 6.2: The graphs K and L with distinguished 4-cycles in bold.

By Wagner's theorem, the planar graphs are precisely the $\{K_5, K_{3,3}\}$-minor-free graphs. Since K_5 and $K_{3,3}$ are 3-connected, Corollary 6.11 shows that Conjecture 6.17 implies Conjecture 6.4. In this chapter we prove a statement only slightly weaker than Conjecture 6.17. Let p_1 and p_2 be the polynomials whose coefficients are listed in Figure 6.1. Let $t_\ell \approx 3.618032$ be the largest real root of p_1, and let $t_r \approx 3.618356$ be the second largest real root of p_2. Note that $\tau + 2 \in (t_\ell, t_r)$.

THEOREM 6.18 *For every $q \in (3, t_\ell) \cup (t_r, 4)$, there is a planar graph whose chromatic polynomial is negative at q.*

Theorem 6.18 and Corollary 6.11 imply Theorem 6.6. To prove part of Theorem 6.18, we use an infinite class of planar triangulations $X(n)$, $n \in \mathbb{N}$. These graphs are formed from the graphs K, L and W depicted in Figure 6.2 and Figure 6.3. The graph K is part of a graph found by Woodall, and the graph L is a layer of the triangular lattice, see [Roy08]. In Section 6.9, we use a transfer matrix approach to analyse the chromatic polynomials of $X(n)$, $n \in \mathbb{N}$ and obtain the following result.

LEMMA 6.19 *For every $q \in (3.7, 4)$, there exists a natural number n such that $P(X(n), q) < 0$.*

Let F be the graph in Figure 6.3. This graph was found by a computer search in order to prove Theorem 6.18.

PROOF OF THEOREM 6.18. Let $q \in (3, t_\ell) \cup (t_r, 4)$ be fixed. If $q \in (3.7, 4)$, then Lemma 6.19 implies that there exists $n \in \mathbb{N}$ such that $P(X(n), q) < 0$ as claimed.

Let G_1 be the unique graph obtained from the union of F and W by identifying the two distinguished 4-cycles. Using a computer algebra package such as MAPLE, it may be calculated that G_1 has chromatic polynomial

$$q(q^7 - 18q^6 + 141q^5 - 619q^4 + 1627q^3 - 2525q^2 + 2107q - 714),$$

which is negative in the interval $(3, 3.6)$.

Now let K be the graph in Figure 6.2. Let G_2 denote the planar triangulation formed from K and two copies of F, say F_1 and F_2, by identifying the two distinguished 4-cycles in K with the distinguished 4-cycles in F_1 and F_2 respectively. We do this in such a way that the vertices of degree 3 in F_1 and F_2 are at distance 2 in G_2. Again, using MAPLE, it may be computed that this graph has chromatic polynomial $q(q - 1)(q - 2)(q - 3)^3 p_1(q)$, where p_1 is the polynomial in Figure 6.1. Among other places, the polynomial $P(G_2, q)$ is negative in the interval $(3.5, t_\ell)$.

Finally, let G_3 be the unique planar triangulation formed from the disjoint union of F, K and W by identifying the two distinguished 4-cycles in K with those in F and W respectively. It may be computed that the chromatic polynomial of this graph is $q(q-1)(q-2)(q-3)^2 p_2(q)$, where p_2 is the polynomial in Figure 6.1. The polynomial $P(G_3, q)$ is negative in the interval $(t_r, 3.8)$. This completes the proof. $\square$
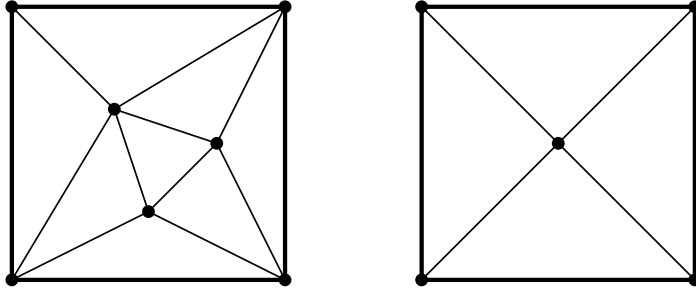
Figure 6.3: The graphs F and W with distinguished 4-cycles in bold.

	$(-\infty, 0)$	$(0, 1)$	$(1, 32/27]$	$(32/27, 3)$
$\overline{N(\mathcal{G})}$	$(-\infty, 0)$ Consider K_1	$(0, 1)$ Consider K_2	$(1, 32/27]$ Consider K_3	$(32/27, 3)$ Consider K_3, K_4
$\overline{A(\mathcal{G})}$	$\emptyset$ Consider K_1	$\emptyset$ Consider K_2	$\emptyset$ Consider K_3	$\emptyset$ Consider K_3, K_4
$\overline{R(\mathcal{G})}$	$\emptyset$ Theorem 2.3	$\emptyset$ Theorem 2.3	$\emptyset$ Theorem 2.3	$(32/27, 3)$ Theorem 6.3

	$(3, 4)$	$(4, 5)$	$(5, \infty)$
$\overline{N(\mathcal{G})}$	Contains $(3, t_\ell) \cup (t_r, 4)$ Theorem 6.18	Unknown	$\emptyset$ Theorem 6.1
$\overline{A(\mathcal{G})}$	Contains $\tau + 2$ Theorem 6.16	Unknown	$(5, \infty)$ Theorem 6.1
$\overline{R(\mathcal{G})}$	Contains $(3, t_\ell) \cup (t_r, 4)$ Theorem 6.6	Unknown	$\emptyset$ Theorem 6.1

Figure 6.4: A summary of known results regarding $\overline{N(\mathcal{G})}$, $\overline{A(\mathcal{G})}$ and $\overline{R(\mathcal{G})}$, where $\mathcal{G}$ denotes the class of planar graphs. In each case, the intersection of this set with an interval of $\mathbb{R}$ is given, along with a reason or theorem number in this thesis.

6.4 K_5-Minor-Free Graphs

The proof of Wagner's equivalence theorem [Wag37] is based on a complete description of the set $\mathcal{W}$ of 3-connected graphs with no K_5-minor. That characterisation says that the elements of $\mathcal{W}$ are 3-clique sums of 3-connected planar graphs and one other graph, J, which is a cycle of length 8 with the four diagonals added. The graph J is sometimes called the Wagner graph.

Let $\mathcal{F}$ denote the class of graphs with no K_5-minor. Since J has no triangle, it follows that $\mathcal{F}_{3\mathrm{max}}$ consists of K_3, J, and 3-clique sums of maximal planar graphs.

PROPOSITION 6.20 *If $\mathcal{G}$ denotes the class of planar graphs, and $\mathcal{F}$ denotes the graphs with no K_5-minor, then $N(\mathcal{F}_{3max}) = N(\mathcal{G}_{3max})$ and $R(\mathcal{F}_{3max}) \cap [3, \infty) = R(\mathcal{G}_{3max}) \cap [3, \infty)$.*

PROOF. Since both $\mathcal{F}_{3\mathrm{max}}$ and $\mathcal{G}_{3\mathrm{max}}$ contain K_3 and K_4, we have

$$N(\mathcal{F}_{3\mathrm{max}}) \cap (-\infty, 3) = N(\mathcal{G}_{3\mathrm{max}}) \cap (-\infty, 3) = (-\infty, 3) \setminus \{0, 1, 2\}.$$

Since $\mathcal{G}_{3\mathrm{max}} \subseteq \mathcal{F}_{3\mathrm{max}}$, it only remains to check that $N(\mathcal{F}_{3\mathrm{max}}) \cap [3, \infty) \subseteq N(\mathcal{G}_{3\mathrm{max}})$ and $R(\mathcal{F}_{3\mathrm{max}}) \cap [3, \infty) \subseteq R(\mathcal{G}_{3\mathrm{max}})$.

First let $q_0 \in N(\mathcal{F}_{3\mathrm{max}}) \cap [3, \infty)$. Note that J has chromatic polynomial

$$q(q-1)(q-2)(q^3 - 5q^2 + 12q - 14)(q^2 - 4q + 5),$$

which is negative only in the open intervals from 0 to 1, and from 2 to approximately 2.43. Thus, since $q_0 \geq 3$, we may assume that $P(F, q_0) < 0$ for some graph F which is a 3-clique sum of maximal planar graphs. If F is planar then we are done. Otherwise, by Proposition 1.6 we have $P(F, q) = [q(q-1)(q-2)]^{1-s} \prod_{i=1}^{s} P(F_i, q)$, where $F_1, \ldots, F_s$ are maximal 3-connected planar graphs. Since $q_0 > 2$ and $P(F, q_0) < 0$, it follows that $P(F_i, q_0) < 0$ for

some $i \in \{1, \ldots, s\}$. Thus $q_0 \in N(\mathcal{G}_{3\text{max}})$ as required.

By an analogous argument, if $q_0 \in R(\mathcal{F}_{3\text{max}}) \cap [3, \infty)$, then there is a 3-connected maximal planar graph G such that $P(G, q_0) = 0$. Thus, $q_0 \in R(\mathcal{G}_{3\text{max}})$ as required. □

As we now show, Proposition 6.20 together with Theorems 6.10, 6.12, 6.13 and Corollary 6.14 demonstrates a close relationship between the class of planar graphs and those with no K_5-minor. We use several facts about the chromatic roots of planar graphs, and accordingly we invite the reader to view the summary of these results provided in Figure 6.4.

THEOREM 6.21 *If $\mathcal{G}$ denotes the planar graphs, and $\mathcal{F}$ denotes the graphs with no K_5-minor, then*

(a) $N(\mathcal{F}) = N(\mathcal{G})$.

(b) $A(\mathcal{F}) = A(\mathcal{G})$.

(c) $\overline{R(\mathcal{F})} = \overline{R(\mathcal{G})}$.

PROOF.

(a) Using Proposition 6.8 it is easy to see that $N(\mathcal{F}) \cap (-\infty, 1] = N(\mathcal{G}) \cap (-\infty, 1]$. By Theorem 6.12, we have $N(\mathcal{F}) \cap (1, \infty) = N(\mathcal{F}_{3\text{max}}) \cap (1, \infty)$ and $N(\mathcal{G}) \cap (1, \infty) = N(\mathcal{G}_{3\text{max}}) \cap (1, \infty)$. Thus, Proposition 6.20 implies the result.

(b) By considering the graphs K_3 and K_4, it is easy to see that $A(\mathcal{F}) \subseteq A(\mathcal{G}) \subseteq (3, \infty)$. Now let $q_0 \in A(\mathcal{G})$, and suppose for a contradiction that $q_0 \notin A(\mathcal{F})$. Thus, $q_0 \in R(\mathcal{F}) \cup N(\mathcal{F})$. Part (a) gives a contradiction if $q_0 \in N(\mathcal{F})$, so we may assume that $q_0 \in R(\mathcal{F})$. Now, a similar argument and Theorem 6.13 imply that $q_0 \in R(\mathcal{F}_{3\text{max}})$. Finally, by Proposition 6.20, and because $q_0 \geq 3$, we deduce that $q_0 \in R(\mathcal{G}_{3\text{max}}) \subseteq R(\mathcal{G})$, a contradiction.

(c) Note that since $\mathcal{G} \subseteq \mathcal{F}$, we have $R(\mathcal{G}) \subseteq R(\mathcal{F})$. Theorems 2.3 and 6.3 now imply that

$$\overline{R(\mathcal{F})} \cap (-\infty, 3] = \overline{R(\mathcal{G})} \cap (-\infty, 3] = \{0, 1\} \cup [32/27, 3].$$

Let $q_0 \in R(\mathcal{F})$ with $q_0 \geq 3$. It suffices to show that $q_0 \in \overline{R(\mathcal{G})}$. First note that by Corollary 6.14, we have that $q_0 \in R(\mathcal{F}_{3\mathrm{max}}) \cup \overline{R_o(\mathcal{F})}$. If $q_0 \in R(\mathcal{F}_{3\mathrm{max}})$, then by Proposition 6.20 we have $q_0 \in R(\mathcal{G}_{3\mathrm{max}}) \subseteq R(\mathcal{G})$. If, on the other hand, $q_0 \in \overline{R_o(\mathcal{F})}$, then by continuity, $q_0 \in \overline{N(\mathcal{F})}$, and so $q_0 \in \overline{N(\mathcal{G})}$ by part (a). Theorem 6.10 now implies that $q_0 \in \overline{R_o(\mathcal{G})}$, which proves part (c). $\qquad\square$

By Theorem 6.21(b) and Theorem 6.16, we deduce as a consequence that $\tau + 2$ is $\mathcal{F}$-positive when $\mathcal{F}$ denotes the class of graphs with no K_5-minor.

6.5 $K_{3,3}$-Minor-Free Graphs

It is an easy exercise to prove that, if a graph has a K_5-minor, then it also has either a K_5-subdivision or a $K_{3,3}$-subdivision. It is also easy to prove that if a 3-connected graph G contains a K_5-subdivision, then G contains a $K_{3,3}$-subdivision unless $G = K_5$. From these observations it follows that, if $\mathcal{F}$ denotes the class of graphs having no $K_{3,3}$-minor, then $\mathcal{F}_{3\mathrm{max}}$ consists of K_3, the 3-connected maximal planar graphs and just one more graph, namely K_5. From these remarks we obtain the following fact.

PROPOSITION 6.22 *If $\mathcal{G}$ denotes the planar graphs and $\mathcal{F}$ denotes the graphs with no $K_{3,3}$-minor, then $N(\mathcal{F}_{3max}) = N(\mathcal{G}_{3max}) \cup (3, 4)$ and $R(\mathcal{F}_{3max}) = R(\mathcal{G}_{3max}) \cup \{4\}$.*

We now show that Proposition 6.22 and the results of the previous sections imply the following close relationship between the class of planar graphs, and

those with no $K_{3,3}$-minor. Again, we invite the reader to view the summary of these results provided in Figure 6.4.

THEOREM 6.23 *If $\mathcal{G}$ denotes the planar graphs, and $\mathcal{F}$ denotes the graphs with no $K_{3,3}$-minor, then*

(a) $N(\mathcal{F}) = N(\mathcal{G}) \cup (t_\ell, t_r)$.

(b) $A(\mathcal{F}) = A(\mathcal{G}) \cap (4, \infty)$.

(c) $\overline{R(\mathcal{F})} \cap (-\infty, 4] = \{0, 1\} \cup [32/27, 4]$ *and* $\overline{R(\mathcal{F})} \cap [4, \infty) = \overline{R(\mathcal{G})} \cap [4, \infty)$.

PROOF.

(a) By Proposition 6.8, it is easy to see that $N(\mathcal{F}) \cap (-\infty, 1] = N(\mathcal{G}) \cap (-\infty, 1]$. Now by Theorem 6.12, we have $N(\mathcal{F}) \cap (1, \infty) = N(\mathcal{F}_{3\max}) \cap (1, \infty)$ and $N(\mathcal{G}) \cap (1, \infty) = N(\mathcal{G}_{3\max}) \cap (1, \infty)$. Proposition 6.22 and Theorem 6.18 now imply the result.

(b) By considering the graphs K_4 and K_5, it is easy to see that $A(\mathcal{F}) \subseteq (4, \infty)$. Furthermore, since $\mathcal{G} \subseteq \mathcal{F}$, we have $A(\mathcal{F}) \subseteq A(\mathcal{G})$. Now, it only remains to show that $A(\mathcal{G}) \cap (4, \infty) \subseteq A(\mathcal{F})$. So let $q_0 \in A(\mathcal{G}) \cap (4, \infty)$ and suppose for a contradiction that $q_0 \notin A(\mathcal{F})$. Thus, $q_0 \in R(\mathcal{F}) \cup N(\mathcal{F})$. Part (a) yields a contradiction if $q_0 \in N(\mathcal{F})$, so we may assume that $q_0 \in R(\mathcal{F})$. By the same argument, Theorem 6.13 implies that $q_0 \in R(\mathcal{F}_{3\max})$. Finally, by Proposition 6.22, and because $q_0 > 4$, we deduce that $q_0 \in R(\mathcal{G}_{3\max}) \subseteq R(\mathcal{G})$, a contradiction.

(c) Theorems 2.3 and 6.3 imply that $\overline{R(\mathcal{F})} \cap (-\infty, 3] = \{0, 1\} \cup [32/27, 3]$. Furthermore, the graph K_5 shows that $(3, 4) \subseteq N(\mathcal{F})$, whence Theorem 6.10 gives that $[3, 4] \subseteq \overline{R(\mathcal{F})}$.

Since $R(\mathcal{G}) \subseteq R(\mathcal{F})$, it suffices to show that $R(\mathcal{F}) \cap [4, \infty) \subseteq \overline{R(\mathcal{G})}$. To this end, let $q_0 \in R(\mathcal{F})$ with $q_0 \geq 4$. By Corollary 6.14, we have that $q_0 \in R(\mathcal{F}_{3\max}) \cup \overline{R_o(\mathcal{F})}$. If $q_0 \in R(\mathcal{F}_{3\max})$, then by Proposition 6.22, we

have $q_0 \in R(\mathcal{G}_{3\max}) \cup \{4\} \subseteq \overline{R(\mathcal{G})}$. If, on the other hand, $q_0 \in \overline{R_o(\mathcal{F})}$, then by continuity $q_0 \in \overline{N(\mathcal{F})}$, and so $q_0 \in \overline{N(\mathcal{G})}$ by part (a). Theorem 6.10 now implies that $q_0 \in \overline{R_o(\mathcal{G})}$, which proves part (c). $\square$

6.6   Graphs on a Fixed Surface

For three important minor-closed classes of graphs, we have obtained insight into the root distribution of chromatic polynomials using real numbers at which some chromatic polynomials are negative. In this section, we comment on another important minor-closed class, namely the graphs that can be embedded in a fixed surface S. We call this class of graphs $\mathcal{G}(S)$. By **surface** we mean the sphere with g handles added, denoted S_g, or the sphere with k crosscaps added, denoted N_k. If S is distinct from the sphere and the Klein bottle N_2, then we can characterise completely $N(\mathcal{G}(S))$ and the $\mathcal{G}(S)$-positive numbers. Ironically, we have no density results for chromatic roots, except those for planar graphs.

The Heawood number $H(k)$ is defined as $H(k) = \lfloor (7 + \sqrt{1 + 24k})/2 \rfloor$. We shall need the following facts:

(i) Every graph which can be embedded on S_g, $g > 1$, contains a vertex of degree at most $H(2g) - 1$.

(ii) Every graph which can be embedded on N_k, $k \geq 1$, contains a vertex of degree at most $H(k) - 1$.

(iii) The complete graph $K_{H(2g)}$ can be embedded in S_g.

(iv) The complete graph $K_{H(k)}$ can be embedded in N_k, except that K_7 cannot be embedded in the Klein bottle N_2.

Facts (i) and (ii) are easy consequences of Euler's formula. Facts (iii) and (iv) constitute the solution by Ringel and Youngs of the Heawood Map Colour Problem, see [MT01].

Theorem 6.24 *Let* $g, k \in \mathbb{N}$.

(a) *If* $g > 0$, *then* $A(\mathcal{G}(S_g)) = (H(2g) - 1, \infty)$ *and*

$$N(\mathcal{G}(S_g)) = (-\infty, H(2g) - 1) \setminus \{0, 1, \ldots, H(2g) - 2\}.$$

(b) *If* $k \neq 2$, *then* $A(\mathcal{G}(N_k)) = (H(k) - 1, \infty)$ *and*

$$N(\mathcal{G}(N_k)) = (-\infty, H(k) - 1) \setminus \{0, 1, \ldots, H(k) - 2\}.$$

(c) $(6, \infty) \subseteq A(\mathcal{G}(N_2))$ *and* $N(\mathcal{G}(N_2)) \cap (-\infty, 5) = (-\infty, 5) \setminus \{0, 1, \ldots, 4\}.$

The statements about the $\mathcal{G}(S_g)$-positive numbers and the $\mathcal{G}(N_k)$-positive numbers follow from a recursion formula discovered for matroids by Oxley [Oxl78] and rediscovered for graphs in [Woo97] and [Tho97]. That formula implies that, if $\mathcal{G}$ is a minor-closed class of graphs in which each graph has a vertex of degree at most d, then every real number greater than d is $\mathcal{G}$-positive. The statements regarding $N(\mathcal{G}(S_g))$ and $N(\mathcal{G}(N_k))$ follow by considering the complete graphs which can be embedded on S_g and N_k.

We cannot use Theorems 6.10, 6.12 and Corollary 6.14 to get insight into the chromatic root distribution for $\mathcal{G}(S)$ when S is distinct from the sphere because $\mathrm{Forb}(\mathcal{G}(S))$ contains graphs that are not 3-connected. This is well-known and also easy to see as follows: Let d be the largest natural number such that dK_5 (the union of d pairwise disjoint copies of K_5) can be embedded in S. If S is distinct from the sphere, then $d > 0$. Now $(d + 1)K_5$ is a disconnected graph which belongs to $\mathrm{Forb}(\mathcal{G}(S))$. Thus, density results for the chromatic roots of graphs in $\mathcal{G}(S)$ would require a new construction.

Question 6.25 Is there a surface S, and an interval in $[4, \infty)$ where $R(\mathcal{G}(S))$ is dense?

6.7   Bipartite Graphs

Since the chromatic polynomial enumerates the proper colourings of a graph, it is natural to ask how chromatic roots behave under assumptions about these colourings. If G is a graph of chromatic number k, then it is clearly true that $P(G, q) > 0$ for all $q \in \mathbb{N}$ with $q \geq k$. However the same is not necessarily true for $q \in \mathbb{R}$. Indeed Woodall [Woo77] showed the following, which can also be found in [DKT05].

THEOREM 6.26 *For every $q \in \mathbb{R} \setminus \mathbb{N}$ such that $q > 2$, there exist $m, n \in \mathbb{N}$ such that $P(K_{m,n}, q) < 0$.*

Using Theorem 6.26 and the above remarks, Woodall deduced that all integers $q \geq 2$ are limit points of the chromatic roots of bipartite graphs. In particular, there is no upper bound on the chromatic roots of graphs with bounded chromatic number. In terms of lower bounds, Dong and Koh [DK08b] proved that $\omega(\mathcal{G}) = 32/27$ for the class of bipartite graphs $\mathcal{G}$. This can also be derived from a result of Jackson and Sokal as we shall see.

Using Woodall's result and the techniques of this section, we are able to deduce a stronger version of these results.

THEOREM 6.27 *If $\mathcal{G}$ denotes the class of bipartite graphs, then $\overline{R(\mathcal{G})} = \{0, 1\} \cup [32/27, \infty)$.*

We shall require the following result of Jackson and Sokal regarding the double subdivision operation defined in Section 2.1.

LEMMA 6.28 [JS09] *Let G be a copy of K_2, and for $k \in \mathbb{N}$, let G_k denote the weighted two-terminal graph $\lozenge^k(G)$, where every edge has weight -1. For all $q > 32/27$, there exists $k \in \mathbb{N}$ such that $v_{G_k}(q) > 0$.*

PROOF OF THEOREM 6.27. By Theorem 6.26, we have that $[2, \infty) \subseteq \overline{N(\mathcal{G})}$. Furthermore, $\mathcal{G}$ is closed under edge deletion and taking 2-clique sums. Thus, by Theorem 6.10, we have that $[2, \infty) \subseteq \overline{R(\mathcal{G})}$.

Let $q \in (1.5, 2)$. If P denotes the path of length 2, then by (5.12) we have $v_P(q) = 1/(q-2)$. Thus, P has type A^- at q. Moreover, by Lemma 5.12, the two-terminal graph $P^{(2)}$, obtained by placing two copies of P in parallel, has type A^+. If P_s denotes the path of length s, then by (5.12) we have $v_{P_s}(q) = \frac{q}{(1-q)^s - 1}$, so for large and even s, the two-terminal graph P_s has type B^-. It follows that for every $q \in (1.5, 2)$, there exists an even $s \in \mathbb{N}$ such that the pair $(P^{(2)}, P_s)$ is complementary. Furthermore, the parallel composition of these two graphs results in a bipartite graph. Thus, by Lemma 5.7, we have $[1.5, 2] \subseteq \overline{R(\mathcal{G})}$.

Similarly, we have that P has type B^- for $q \in (32/27, 1.5)$. Also, for each $q \in (32/27, 1.5)$, Lemma 6.28 implies that there exists $k \in \mathbb{N}$ such that G_k has type A^+. It follows that the pair (G_k, P) is complementary. Since any parallel composition of these two graphs is bipartite, we have by Lemma 5.7 that $[32/27, 1.5] \subseteq \overline{R(\mathcal{G})}$. $\qquad\square$

6.8 Flow Roots

The **flow polynomial** of a graph G is a polynomial $F(G, q)$ such that for each $q \in \mathbb{N}$, the number of nowhere-zero $\mathbb{Z}_q$-flows of G is precisely $F(G, q)$. It can be verified that this function is indeed a polynomial as was done in Section 1.3 for the chromatic polynomial. We say that a real number q is a **flow root** of a graph G if $F(G, q) = 0$.

For planar graphs, the flow polynomial and chromatic polynomial are dual, see [Sok05]. Indeed, if G^* denotes the planar dual of G, then

$$P(G, q) = qF(G^*, q). \tag{6.3}$$

From Theorem 2.3 and (6.3), it follows that that no bridgeless planar graph has a flow root in the set $(-\infty, 0) \cup (0, 1) \cup (1, 32/27]$. Wakelin [Wak94] removed the planarity assumption.

THEOREM 6.29 [Wak94] *If G is a bridgeless graph with n vertices, m edges, b blocks, and no isolated vertices, then*

   (i) $(-1)^{m-n+1} F(G, q) > 0$ *for $q \in (-\infty, 1)$.*

   (ii) $F(G, q)$ *has a zero of multiplicity b at $q = 1$.*

   (iii) $(-1)^{m-n+b+1} F(G, q) > 0$ *for $q \in (1, 32/27]$.*

By (6.3) and Theorem 6.3, the flow roots of bridgeless planar graphs are dense in the interval $(32/27, 3)$. In the same way, we deduce the following corollary from Theorem 6.6.

COROLLARY 6.30 *The flow roots of planar graphs contains a dense subset of the set $(3, t_\ell) \cup (t_r, 4)$.*

Jacobsen and Salas [JS13b] showed that $t_f \approx 5.235260$ is an accumulation point of real roots of the flow polynomial. Currently these are the largest known such roots, and it would not be surprising if the roots of the flow polynomial are dense in $(32/27, t_f)$. However, unless Conjecture 6.2 is false, it will require the use of non-planar graphs to prove this.

6.9   Proof of Lemma 6.19

Let L and K be the graphs in Figure 6.2. We denote by L^n the graph obtained from n copies of L, say $L_1, \ldots, L_n$, by identifying the inner 4-cycle of L_i with the outer 4-cycle of L_{i+1} for each $i \in \{1, \ldots, n-1\}$. The resulting graph is planar and has two distinguished 4-cycles. Also let W denote the wheel on 5

	p_3	p_4	p_5	p_6
q^{11}		1	1	1
q^{10}		-34	-34	-34
q^9	1	538	538	538
q^8	-28	-5244	-5244	-5246
q^7	362	35078	35078	35148
q^6	-2846	-169490	-169490	-170548
q^5	15036	604806	604806	613920
q^4	-55448	-1595807	-1595807	-1645010
q^3	142716	3051803	3051803	3222645
q^2	-246724	-4024676	-4024676	-4397342
q^1	258889	3286881	3286881	3753346
q^0	-124884	-1255163	-1255163	-1511254

Figure 6.5: Coefficients of the polynomials $p_3, \ldots, p_6$, see [Roy08, Table 1].

vertices, see Figure 6.3, and let K' denote the graph formed from K by adding a single edge, triangulating one of the distinguished 4-cycles. In this section, we analyse the class of planar triangulations $X(n)$, $n \in \mathbb{N}$, formed by identifying the distinguished 4-cycles of L^n with the distinguished 4-cycles of W and K' respectively.

The graphs $X(n)$, $n \in \mathbb{N}$ were first studied by Royle [Roy08], who took inspiration from a graph found by Woodall and from graphs studied in the field of statistical mechanics. Indeed, the class of graphs L^n, $n \in \mathbb{N}$ can be viewed as $4 \times n$ strips of the infinite triangular lattice, see [Roy08, Figure 1], where one side of the strip is wrapped around and identified with the other. A standard technique to compute the chromatic polynomial of such graphs is the so-called transfer matrix approach, see [SS01]. We employ this technique in the form used by Royle [Roy08], and give here a fairly condensed analysis. A more detailed presentation can be found in [Roy08].

Let A be a graph with a distinguished 4-cycle $a_1 a_2 a_3 a_4$. We may partition the colourings ϕ of A into four types.

Type 1: $\phi(a_1) = \phi(a_3)$ and $\phi(a_2) = \phi(a_4)$,

Type 2: $\phi(a_1) = \phi(a_3)$ and $\phi(a_2) \neq \phi(a_4)$,

Type 3: $\phi(a_1) \neq \phi(a_3)$ and $\phi(a_2) = \phi(a_4)$,

Type 4: $\phi(a_1) \neq \phi(a_3)$ and $\phi(a_2) \neq \phi(a_4)$.

Let $P_i(A, q)$ denote the number of q-colourings of A of type i. Note that iden-
tifying the vertices a_1 and a_3, say, gives a graph whose colourings correspond
bijectively to the colourings ϕ of A such that $\phi(a_1) = \phi(a_3)$. Alternatively,
adding the edge a_1a_3 produces a graph whose colourings correspond bijectively
to the colourings ϕ of A such that $\phi(a_1) \neq \phi(a_3)$. Thus, for computational pur-
poses, we have for example that $P_2(A, q) = P(A/a_1a_3 + a_2a_4, q)$. We collect this
information in a vector $S(A, q)$ called the **partitioned chromatic polynomial**

$$S(A, q) = \begin{pmatrix} P_1(A, q) \\ P_2(A, q) \\ P_3(A, q) \\ P_4(A, q) \end{pmatrix}.$$

Let $\langle q \rangle_k$ denote the k'th **falling factorial** $q(q-1)\cdots(q-k+1)$ and let $p_3, \ldots, p_6$
be the polynomials whose coefficients are listed in Figure 6.5. The partitioned
chromatic polynomials of W and K' are given below. The formula for $S(W, q)$
is trivial, whereas the formula for $S(K', q)$ appears in [Roy08] and was verified
by the author.

$$S(W, q) = \begin{pmatrix} \langle q \rangle_3 \\ \langle q \rangle_4 \\ \langle q \rangle_4 \\ \langle q \rangle_5 \end{pmatrix} \qquad S(K', q) = \begin{pmatrix} \langle q \rangle_5 \cdot p_3(q) \\ \langle q \rangle_4 \cdot p_4(q) \\ \langle q \rangle_4 \cdot p_5(q) \\ \langle q \rangle_4 \cdot (q - 3) \cdot p_6(q) \end{pmatrix}$$

In a similar way, for a graph with two distinguished 4-cycles, we may define
a square matrix whose entries capture the types of colourings on those two
4-cycles. More precisely, we let the element in the ith row and jth column
be the number of q-colourings of the graph which are of type i on the outer
distinguished 4-cycle and type j on the inner distinguished 4-cycle. The matrix

M corresponding to the graph L in Figure 6.2 is as follows. The expression appears in [Roy08] and was verified by the author.

$$
\begin{pmatrix}
\langle q \rangle_4 & \langle q \rangle_5 & \langle q \rangle_5 & \langle q \rangle_6 \\
\langle q \rangle_5 & \langle q \rangle_4 + 2\langle q \rangle_5 + \langle q \rangle_6 & \langle q \rangle_4 + 2\langle q \rangle_5 + \langle q \rangle_6 & 4\langle q \rangle_5 + 4\langle q \rangle_6 + \langle q \rangle_7 \\
\langle q \rangle_5 & \langle q \rangle_4 + 2\langle q \rangle_5 + \langle q \rangle_6 & \langle q \rangle_4 + 2\langle q \rangle_5 + \langle q \rangle_6 & 4\langle q \rangle_5 + 4\langle q \rangle_6 + \langle q \rangle_7 \\
\langle q \rangle_6 & 4\langle q \rangle_5 + 4\langle q \rangle_6 + \langle q \rangle_7 & 4\langle q \rangle_5 + 4\langle q \rangle_6 + \langle q \rangle_7 & M_{44}
\end{pmatrix},
$$

where $M_{44} = 2\langle q \rangle_4 + 16\langle q \rangle_5 + 20\langle q \rangle_6 + 8\langle q \rangle_7 + \langle q \rangle_8$.

LEMMA 6.31 [Roy08] *Let A and B be graphs with distinguished 4-cycles. If $X_{A,B}(n)$ denotes the graph obtained from L^n by identifying its two distinguished 4-cycles with those of A and B respectively, then the chromatic polynomial of $X_{A,B}(n)$ is the sole entry of the 1×1 matrix $S(A)^T D(MD)^n S(B)$, where*

$$
D =
\begin{pmatrix}
1/\langle q \rangle_2 & 0 & 0 & 0 \\
0 & 1/\langle q \rangle_3 & 0 & 0 \\
0 & 0 & 1/\langle q \rangle_3 & 0 \\
0 & 0 & 0 & 1/\langle q \rangle_4
\end{pmatrix}.
$$

The matrix MD is called the **transfer matrix** of the graph L, as it describes how colourings transfer from one 4-cycle to the other. As we let n tend to infinity, the matrix $(MD)^n$ is determined by the spectral properties of MD. Indeed let $\lambda_1, \ldots, \lambda_4$ and $v_1, \ldots, v_4$ denote the eigenvalues and eigenvectors of MD respectively. Furthermore, let $\|v\|$ denote the norm of the vector v with respect to the inner product $\langle u, v \rangle = u^T D v$. Royle proves the following.

LEMMA 6.32 [Roy08] *If A and B are graphs with distinguished 4-cycles, and q_0 is a fixed real number in $(\tau + 2, 4)$, then*

$$
P(X_{A,B}(n), q_0) = \sum_{i=1}^{4} \alpha_i \beta_i \lambda_i^n \|v_i\|^2,
$$

where $\alpha_i \|v_i\|^2 = S(A, q_0)^T D v_i$ and $\beta_i \|v_i\|^2 = S(B, q_0)^T D v_i$ for $i \in \{1, 2, 3, 4\}$.

In essence, Lemma 6.32 says that if $\lambda_i > 0$ is the eigenvalue of largest magnitude, then in the limit as n tends to infinity, the chromatic polynomial of $X_{A,B}(n)$ has the sign of $\alpha_i \beta_i$. Moreover, this sign can be determined simply by looking at the partitioned chromatic polynomials of A and B and the eigenvector v_i. With this in mind we first find the eigenvalues and eigenvectors of MD. Indeed, if a is the polynomial $q^4 - 10q^3 + 43q^2 - 106q + 129$, then we have that $\lambda_1 = 2$,

$$\lambda_2 = \frac{1}{2}(q-3)(q^3 - 9q^2 + 33q - 48 - (q-4)a(q)^{1/2}),$$
$$\lambda_3 = \frac{1}{2}(q-3)(q^3 - 9q^2 + 33q - 48 + (q-4)a(q)^{1/2}),$$

and $\lambda_4 = 0$. The corresponding eigenvectors are $v_1 = (1, -1, -1, 1)^T$,

$$v_2 = \begin{pmatrix} (q^2 - 7q + 15 + a(q)^{1/2})/(2(q-2)(q-3)^2) \\ (q^2 - 9q + 21 + a(q)^{1/2})/(4(q-3)^2) \\ (q^2 - 9q + 21 + a(q)^{1/2})/(4(q-3)^2) \\ -1 \end{pmatrix},$$

$$v_3 = \begin{pmatrix} (q^2 - 7q + 15 - a(q)^{1/2})/(2(q-2)(q-3)^2) \\ (q^2 - 9q + 21 - a(q)^{1/2})/(4(q-3)^2) \\ (q^2 - 9q + 21 - a(q)^{1/2})/(4(q-3)^2) \\ -1 \end{pmatrix},$$

and $v_4 = (0, 1, -1, 0)^T$.

We remark that in the notation of this section, the graphs $X(n)$ in Lemma 6.19 can be denoted $X_{K',W}(n)$. We can now prove Lemma 6.19.

PROOF OF LEMMA 6.19. Let q_0 be a fixed real number in $(3.7, 4)$. At q_0, we have $\lambda_1 > \lambda_2 > \lambda_3 > \lambda_4 = 0$. For $i \in \{1, 2, 3, 4\}$, let α_i be such that $\alpha_i \|v_i\|^2 = S(K', q_0)^T D v_i$, and let β_i be the corresponding value for the graph W. It may be calculated that $\alpha_1 \|v_1\|^2 = 0$ and $\beta_1 \|v_1\|^2 = 0$. Thus, by Lemma 6.32, the dominant term in the expression for $P(X(n), q_0)$ is $\alpha_2 \beta_2 \lambda_2^n \|v_2\|^2$, and for large enough n, the sign of $P(X(n), q_0)$ depends on the sign of $\alpha_2 \beta_2$. Let b be

	c	d
q^{13}	1	
q^{12}	-38	
q^{11}	676	1
q^{10}	-7473	-33
q^9	57452	506
q^8	-325572	-4770
q^7	1405168	30784
q^6	-4698525	-143070
q^5	12224002	489214
q^4	-24510275	-1231299
q^3	36830333	2234215
q^2	-39168919	-2778891
q^1	26277660	2126441
q^0	-8337189	-755627

Figure 6.6: Coefficients of the polynomials c and d.

the polynomial $q^3 - 9q^2 + 25q - 24$, and let c and d be the polynomials defined in Figure 6.6. It may be calculated that

$$\alpha_2\|v_2\|^2 = \frac{d(q_0)a(q_0)^{1/2} - c(q_0)}{2(q_0 - 3)},$$

and

$$\beta_2\|v_2\|^2 = \frac{(q_0 - 2)a(q_0)^{1/2} - b(q_0)}{2(q_0 - 3)^2}.$$

From these expressions, a short calculation gives that $\alpha_2\|v_2\|^2 < 0$ for $q_0 \in (3.7, 4)$ and $\beta_2\|v_2\|^2 > 0$ for $q_0 \in (3, 4)$. This implies that $\alpha_2\beta_2$ is negative in $(3.7, 4)$. Therefore, there is $n \in \mathbb{N}$ such that $P(X(n), q_0) < 0$. □

We remark that the calculation of chromatic polynomials in this chapter was done with MAPLE. The calculation of eigenvalues and eigenvectors of the matrix MD was performed with MATLAB and cross-checked with the series expansion expressions presented in [Roy08].

Bibliography

[Bar66]   D. Barnette. Trees in polyhedral graphs. *Canad. J. Math.*, 18:731–736, 1966.

[Ber75]   S. Beraha. *Infinite non-trivial families of maps and chromials*. PhD thesis, Johns Hopkins University, 1975.

[Bir12]   G.D. Birkhoff. A determinant formula for the number of ways of coloring a map. *Ann. of Math.*, 12:42–46, 1912.

[BKW80] S. Beraha, J. Kahane, and N.J. Weiss. Limits of chromatic zeros of some families of maps. *J. Combin. Theory Ser. B*, 28:52–65, 1980.

[BL46]   G.D. Birkhoff and D.C. Lewis. Chromatic polynomials. *Trans. Amer. Math. Soc.*, 60:355–451, 1946.

[Die16]   R. Diestel. *Graph Theory, 5th Edition*, volume 173 of *Graduate texts in mathematics*. Springer, 2016.

[DJ11]   F.M. Dong and B. Jackson. A zero-free interval for chromatic polynomials of nearly 3-connected plane graphs. *SIAM J. Discrete Math.*, 25(3):1103–1118, 2011.

[DK08a] F.M. Dong and K.M. Koh. Domination numbers and zeros of chromatic polynomials. *Discrete Math.*, 308(10):1930–1940, 2008.

[DK08b] F.M. Dong and K.M. Koh. A maximal zero-free interval for chromatic polynomials of bipartite planar graphs. *Discrete Math.*, 308(11):2285–2287, 2008.

[DK10] F.M. Dong and K.M. Koh. On zero-free intervals in (1,2) of chromatic polynomials of some families of graphs. *SIAM J. Discrete Math.*, 24(2):370–378, 2010.

[DKT05] F.M. Dong, K.M. Koh, and K.L. Teo. *Chromatic Polynomials and Chromaticity of Graphs*. World Scientific, 2005.

[GJ08] L.A. Goldberg and M. Jerrum. Inapproximability of the Tutte polynomial. *Inform. and Comput.*, 206(7):908–929, 2008.

[GJ12] L.A. Goldberg and M. Jerrum. Inapproximability of the Tutte polynomial of a planar graph. *Comput. Complexity*, 21(4):605–642, 2012.

[GJ14] L.A. Goldberg and M. Jerrum. The complexity of computing the sign of the Tutte polynomial. *SIAM J. Comput.*, 43(6):1921–1952, 2014.

[Jac93] B. Jackson. A zero-free interval for chromatic polynomials of graphs. *Combin. Probab. Comput.*, 2(3):325–336, 1993.

[Jac03] B. Jackson. Zeros of chromatic and flow polynomials of graphs. *J. Geom.*, 73:95–109, 2003.

[JS09] B. Jackson and A.D. Sokal. Zero-free regions for multivariate Tutte polynomials (alias Potts-model partition functions) of graphs and matroids. *J. Combin. Theory, Ser. B*, 99(6):869–903, 2009.

[JS13a] J.L. Jacobsen and J. Salas. A generalized Beraha conjecture for non-planar graphs. *Nucl. Phys. B*, 875:678–718, 2013.

[JS13b] J.L. Jacobsen and J. Salas. Is the five-flow conjecture almost false? *J. Combin. Theory, Ser. B*, 103(4):532–565, 2013.

[JT95]   T. Jensen and B. Toft. *Graph Coloring Problems*. John Wiley, New York, 1995.

[JVW90] F. Jaeger, D.L. Vertigan, and D.J.A. Welsh. On the computational complexity of the Jones and Tutte polynomials. *Math. Proc. Cambridge Philos. Soc.*, 108(1):35–53, 1990.

[MT01]   B. Mohar and C. Thomassen. *Graphs on Surfaces*. Johns Hopkins University Press, 2001.

[OP]   S. Ok and T.J. Perrett. Density of real zeros of the Tutte polynomial. Submitted, `arXiv:1608.08747`.

[Oxl78]   J. Oxley. Colouring, packing and the critical problem. *Quart. J. Math.*, 29:11–22, 1978.

[Per16a] T.J. Perrett. Chromatic roots and minor-closed families of graphs. *SIAM J. Discrete Math.*, 30(3):1883–1897, 2016.

[Per16b] T.J. Perrett. A zero-free interval for chromatic polynomials of graphs with 3-leaf spanning trees. *Discrete Math.*, 339(11):2706–2714, 2016.

[PT]   T.J. Perrett and C. Thomassen. Density of chromatic roots in minor-closed graph families. Submitted.

[Roy08]   G. Royle. Planar triangulations with real chromatic roots arbitrarily close to four. *Annals Combin.*, 12:195–210, 2008.

[RS04]   N. Robertson and P.D. Seymour. Graph minors XX. Wagner's conjecture. *J. Combin. Theory Ser. B*, 92:325–357, 2004.

[RT88]   R.C. Read and W.T. Tutte. Chromatic polynomials. In L.W. Beineke and R.J. Wilson, editors, *Selected Topics in Graph Theory 3*, pages 15–42. Academic Press, London, 1988.

[Sok04]   A.D. Sokal. Chromatic roots are dense in the whole complex plane. *Combin. Probab. Comput.*, 13:221–261, 2004.

[Sok05]   A.D. Sokal. The multivariate Tutte polynomial (alias Potts model) for
          graphs and matroids. In B.S. Webb, editor, *Surveys in Combinatorics*,
          pages 173–226. Cambridge University Press, 2005.

[SS01]   J. Salas and A.D. Sokal. Transfer matrices and partition-function ze-
          ros for antiferromagnetic Potts models. I. General theory and square-
          lattice chromatic polynomial. *J. Stat. Phys.*, 104:609–699, 2001.

[Tho96]   C. Thomassen. On the number of Hamiltonian cycles in bipartite
          graphs. *Combin. Probab. Comput.*, 5(4):437–442, 1996.

[Tho97]   C. Thomassen. The zero-free intervals for chromatic polynomials of
          graphs. *Combin. Probab. Comput.*, 6(4):497–506, 1997.

[Tho00]   C. Thomassen. Chromatic roots and Hamiltonian paths. *J. Combin.
          Theory Ser. B*, 80(2):218–224, 2000.

[Tut70]   W.T. Tutte. The golden ratio in the theory of chromatic polynomials.
          Ann. New York Acad. Sci., 175:391–402, 1970.

[Tut74]   W.T. Tutte. Chromials. In C. Berge and D. Ray-Chaudhuri, editors,
          Hypergraph Seminar, volume 411 of *Lecture Notes in Math.*, pages
          243–266. Springer Berlin Heidelberg, 1974.

[Wag37]   K. Wagner. Über eine Eigenschaft der ebenen Komplexe. *Math. Ann.*,
          114:570–590, 1937.

[Wak94]   C.D. Wakelin. *Chromatic Polynomials*. PhD thesis, University of
          Nottingham, 1994.

[Whi32a] H. Whitney. The coloring of graphs. *Ann. of Math.*, 33(4):688–718,
          1932.

[Whi32b] H. Whitney. A logical expansion in mathematics. *Bull. Amer. Math.
          Soc.*, 38(8):572–579, 1932.

[Whi33]   H. Whitney. 2-isomorphic graphs. *Amer. J. Math.*, 55(1):245–254,
          1933.

[Woo77]   D.R. Woodall. Zeros of chromatic polynomials. In P. Cameron, editor, *Combinatorial Surveys, Proceedings of the Sixth British Combinatorial Conference*, pages 199–223. Academic Press, London, 1977.

[Woo97]   D.R. Woodall. The largest zero of a chromatic polynomial. *Discrete Math.*, 172:141–153, 1997.