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An Analytical Solution Describing the Shape of a Yield Stress Material Subjected to an Overpressure

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Abstract. Many fluids and granular materials are able to withstand a limited shear stress without flowing. These materials are known as yields stress materials. Previously, an analytical solution was presented to quantify the yield stress for such materials. The yields stress is obtained based on the density as well as the spread length and height of the material when deformed in a box due to gravity. In the present work, the analytical solution is extended with the addition of an overpressure that acts over the entire body of the material. This extension enables finding the shape of a yield stress material with known density and yield stress when for instance deformed under water or subjected to a forced air pressure.

INTRODUCTION

Yield stress materials come in many varieties such as green sand [1], concrete [2], and ceramic slurries [?]. Recently, an analytical solution was derived to determine the yield stress of a given material (i.e. a very fluid concrete also known as a self-compacting concrete) [4]. The solution utilizes the density along with the height h_0 and spread length L measured in the LCPC-box test in order to back out the yield stress. The LCPC-box test is performed by pouring 6 liters of material into a box with a height of 0.12 m, a width of 0.2 m, and a length of 1 m, see fig. 1(a-b). This methodology is a cheap low-tech procedure to obtain the yield stress which often is a key parameter when evaluating the flowability of yield stress materials.
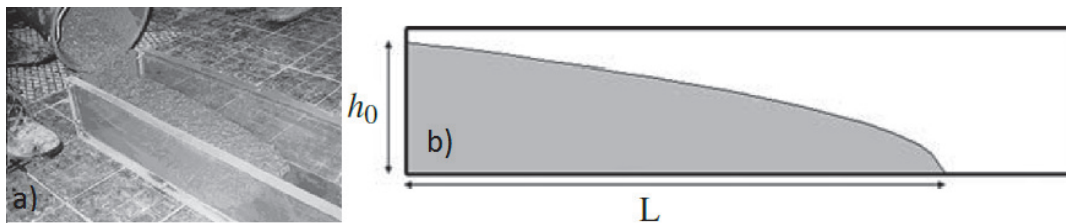
FIGURE 1. (a) LCPC BOX test and (b) height h_0 and length L in the LCPC BOX test are correlated to the yield stress, figures from [5].

In this paper, the analytical solution is presented with an overpressure. This new solution mimics a situation where the yield stress material deforms for instance subsurface or when exposed to an additional air pressure.The deformation is expressed as the length as a function of the height while the input parameters needed are the density and yield stress. In the following, the new solution is first derived in three- and two dimensions and afterwards plotted for different pressures.

3D ANALYTICAL SOLUTION WITH AN OVERPRESSURE

The forces acting on an infinitesimal slice of the yield stress material in the LCPC-box comes from the friction at the bottom and side walls, the hydrostatic pressure, and the overpressure, see fig. 2.

FIGURE 2. An illustration of the friction from the bottom and side walls, the hydrostatic pressure, and the overpressure that act on an infinitesimal slice of the yield stress material in the LCPC-box. The figure is edited from [4] and originates from [5].

The force equilibrium is given by,

$$(P + \frac{1}{2}\rho g h)h l_0 - (P + \frac{1}{2}\rho g(h+dh))(h+dh)l_0 - \tau_0 dx l_0 - 2\tau_0 h dx = 0 \tag{1}$$
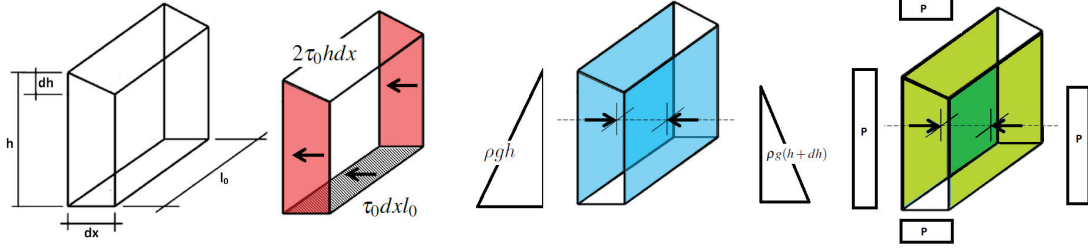
where P is the overpressure, ρ is the density, g is the gravitational acceleration, h is height, l_0 is the width of the box and τ_0 is the yield stress. Note that the increment dh is negative due to the decreasing height of the material in the box and that the vertical overpressure is eliminated by the reaction force from the bottom and thereby transferred into a horizontal overpressure.

Assuming that dh^2 is negligible one obtains,

$$\left(\frac{P}{\rho g} + h\right)\frac{dh}{dx} = -\left(\tau_0 + 2\tau_0\frac{h}{l_0}\right)\frac{1}{\rho g} \tag{2}$$

Solving the differential equation yields the length as a function of the height,

$$x(h) = \frac{(h_0 - h)\rho g l_0}{2\tau_0} + \left(\frac{l_0^2 \rho g}{4\tau_0} - \frac{P l_0}{2\tau_0}\right) ln\left(\frac{2h + l_0}{2h_0 + l_0}\right) \tag{3}$$

Note that if $P = 0$, then the original solution from [5] is obtained $x(h) = \frac{(h_0 - h + \frac{l_0}{2}ln(\frac{2h+l_0}{2h_0+l_0}))\rho g l_0}{2\tau_0}$. The integral of the material is given by the following when assuming that h does not vary with l_0,

$$V = l_0 \int_0^{h_0} x(h)dh = l_0 \int_0^{h_0} \frac{(h_0 - h)\rho g l_0}{2\tau_0} + \left(\frac{l_0^2 \rho g}{4\tau_0} - \frac{P l_0}{2\tau_0}\right) ln\left(\frac{2h + l_0}{2h_0 + l_0}\right) dh \tag{4}$$

Based on the initial volume and width and the assumption that $A = \frac{V}{l_0}$, one obtains,

$$A = \int_0^{h_0} x(h)dh = \frac{\rho g l_0}{2\tau_0} \int_0^{h_0} h_0 - h + \left(\frac{l_0}{2} - \frac{P}{\rho g}\right) ln\left(\frac{2h + l_0}{2h_0 + l_0}\right) dh \tag{5}$$

which yields,

$$A = \frac{l_0}{8\tau_0}\left(2\rho g h_0^2 - 2l_0 h_0 \rho g + 4P h_0 + (2P l_0 - g l_0^2 \rho) ln\left(\frac{l_0}{l_0 + 2h_0}\right)\right) \tag{6}$$

Now h_0 can be found iteratively e.g. with the Newton Raphson method. Utilizing h_0 in eqn. (3) together with a known viscosity and density facilitates the solution for the deformed yield stress material.

2D ANALYTICAL SOLUTION WITH AN OVERPRESSURE

In the ideal 2D case, as opposed to the 3D case, there is no influence from the front and back lateral walls ($l_0 \to \infty$), see fig. 3.
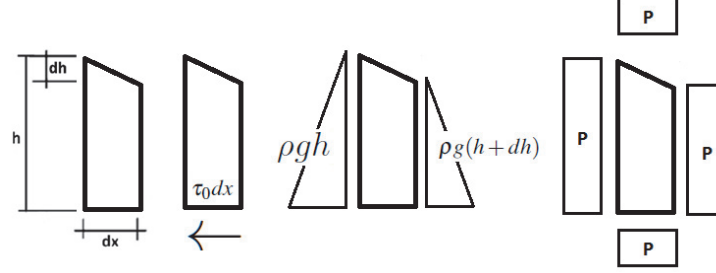

FIGURE 3. An illustration of the friction from the bottom, the hydrostatic pressure, and overpressure that act on an infinitesimal slice of the yield stress material in the ideal 2D case. The figure is edited from [4] and originates from [5].

The force equilibrium is now given by,

$$(P + \frac{1}{2}\rho gh)h - (P + \frac{1}{2}\rho g(h+dh))(h+dh) - \tau_0 dx = 0 \tag{7}$$

Assuming that dh^2 is negligible one obtains,

$$(P + \rho gh)\frac{dh}{dx} = -\tau_0 \tag{8}$$

Solving the differential equation yields the height as function of the length,

$$h(x) = \frac{-P + \sqrt{P^2 + 2\rho g(\tau_0 L - \tau_0 x)}}{\rho g} \tag{9}$$

or the length as function of the height,

$$x(h) = \frac{h_0^2 \rho g}{2\tau_0} - \frac{h^2 \rho g}{2\tau_0} + \frac{Ph_0}{\tau_0} - \frac{Ph}{\tau_0} \tag{10}$$

Note that if $P = 0$, then the original solution from [5] is obtained $x(h) = \frac{h_0^2 \rho g}{2\tau_0} - \frac{h^2 \rho g}{2\tau_0}$.
The integral, giving the area, can be expressed as,

$$A = \int_0^{h_0} x(h)dh = \int_0^{h_0} \frac{h_0^2 \rho g}{2\tau_0} - \frac{h^2 \rho g}{2\tau_0} + \frac{Ph_0}{\tau_0} - \frac{Ph}{\tau_0} dh \tag{11}$$

Solving the integral yields,

$$A = \frac{\rho g h_0^3}{3\tau_0} + \frac{h_0^2 P}{2\tau_0} \tag{12}$$

h_0 is then given by,

$$h_0 = \frac{C^{\frac{1}{3}}}{2\rho g} + \frac{P^2}{2\rho g C^{\frac{1}{3}}} - \frac{P}{2\rho g} \tag{13}$$

Where $C = 12A\tau_0\rho^2 g^2 + 2\sqrt{6}\sqrt{A\tau_0(6A\rho^2 g^2 \tau_0 - P^3)}\rho g - P^3$. Utilizing h_0 in eqn. (10) together with a known viscosity and density facilitates the solution for the deformed yield stress material.

Results and Discussion

The final deformed shapes of a yield stress material obtained with the 3D and 2D analytical solutions are illustrated in Fig. 4. The material properties chosen are a density of 1000 kg/m^3 and a yield stress of 100 Pa. The applied over pressures are 0 Pa, 10 Pa, 100 Pa, 1000 Pa and 10000 Pa.
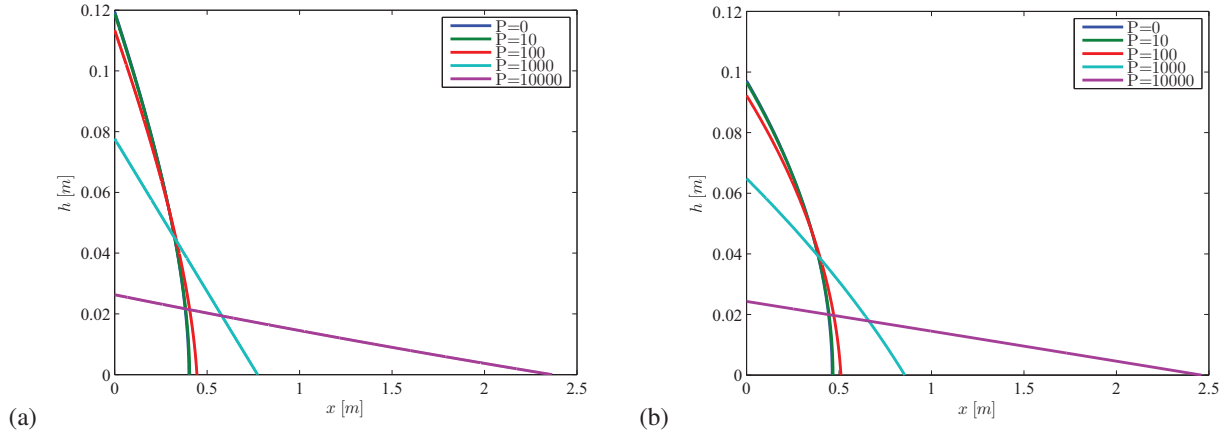

FIGURE 4. Final deformed shapes of a yield stress material obtained with the (a) the 3D analytical solution and an initial volume of 6 liters and a width of l_0=0.2 m, (b) the 2D analytical solution and an initial area of 0.03 m^2 and an infinte width $l_0 \to \infty$. The physical settings are the same for the two simulations: a density of 1000 kg/m^3, a yield stress of 100 Pa, and an overpressure that varies between 0 Pa and 10000 Pa. Note the different length scales on the axes.

Generally the horizontal pressure squeezes out the material, thereby increasing the spread length L and decreasing h_0 as seen in fig. 4. For small overpressures, the profile becomes more parabolic, while it becomes more and more linear when the overpressure starts to dominate. The figure also illustrates that the spread length for the 3D solution is somewhat less than for the 2D solution especially for low over pressures, since the 3D solution takes into account the friction at the front and back wall.

CONCLUSION

In the present work an extension of the original solution in [5] for the shape of a yield stress material under the influence of gravity has been derived with an over pressure. The basic force in both 2D and 3D equilibrium for an infinitesimal fluid element is established and the resulting differential equations are solved analytically. The final deformation shapes are plotted as a function of distance for increasing values of the overpressure and it is noticed that the shape tranfers from parabolic to a more linear shape for increased values of the overpressure.

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