Technical University of Denmark

DTU

Graph Decompositions

Merker, Martin; Thomassen, Carsten

Publication date:
2016

Document Version
Publisher's PDF, also known as Version of record

Link back to DTU Orbit

Citation (APA):
Merker, M., & Thomassen, C. (2016). Graph Decompositions. Kgs. Lyngby: Technical University of Denmark (DTU). (DTU Compute PHD-2016; No. 431).

DTU Library
Technical Information Center of Denmark

Graph Decompositions

Martin Merker

Summary (English)

The topic of this PhD thesis is graph decompositions. While there exist various kinds of decompositions, this thesis focuses on three problems concerning edge-decompositions. Given a family of graphs $\mathcal{H}$ we ask the following question: When can the edge-set of a graph be partitioned so that each part induces a subgraph isomorphic to a member of $\mathcal{H}$? Such a decomposition is called an $\mathcal{H}$-decomposition. Apart from the existence of an $\mathcal{H}$-decomposition, we are also interested in the number of parts needed in an $\mathcal{H}$-decomposition.

Firstly, we show that for every tree T there exists a constant $k(T)$ such that every $k(T)$-edge-connected graph whose size is divisible by the size of T admits a T-decomposition. This proves a conjecture by Barát and Thomassen from 2006.

Moreover, we introduce a new arboricity notion where we restrict the diameter of the trees in a decomposition into forests. We conjecture that for every natural number k there exists a natural number $d(k)$ such that the following holds: If G can be decomposed into k forests, then G can be decomposed into $k+1$ forests in which each tree has diameter at most $d(k)$. We verify this conjecture for $k \leq 3$. As an application we show that every 6-edge-connected planar graph

contains two edge-disjoint $\frac{18}{19}$-thin spanning trees.

Finally, we make progress on a conjecture by Baudon, Bensmail, Przybyło, and Woźniak stating that if a graph can be decomposed into locally irregular graphs, then there exists such a decomposition with at most 3 parts. We show that this conjecture is true if the number 3 is replaced by 328, establishing the first constant upper bound for this problem.

Summary (Danish)

Blandt de mange forskellige varianter der findes af graf-dekompositioner fokuseres her på tre specifikke problemer vedrørende kant-opspaltning. Givet en familie af grafer $\mathcal{H}$, betragter vi følgende spørgsmål: Hvornår kan kant-mængden af en graf opspaltes i dele så hver del inducerer en delgraf isomorf med et medlem af $\mathcal{H}$? En sådan kant-opspaltning kaldes en $\mathcal{H}$-dekomposition. Foruden eksistensen af en $\mathcal{H}$-dekomposition, interesserer vi os også for hvor mange dele en sådan dekomposition nødvendigvis må indeholde.

Først beviser vi at for ethvert træ T findes der en konstant $k(T)$ således, at hver $k(T)$-kant-sammenhængende graf G har en T-dekomposition forudsat kant-antallet af G er deleligt med kant-antallet af T. Dette beviser en formodning af Barát og Thomassen fra 2006.

Derudover introducerer vi et nyt arboricitet-begreb som begrænser diameteren af træerne i en dekomposition i skove. Vi formulerer den formodning, at for ethvert naturligt tal k findes der et naturligt tal $d(k)$, således at følgende holder: Hvis G kan opdeles i k skove, så kan G opdeles i $k+1$ skove, hvor hvert træ har diameter højst $d(k)$. Vi bekræfter denne formodning for $k \leq 3$. Som en anvendelse beviser vi at enhver 6-kant-sammenhængende planar graf har to kant-

disjunkte $\frac{18}{19}$-tynde udspændende træer.

Endelig gør vi fremskridt på en formodning af Baudon, Bensmail, Przybyło, og Woźniak om, at hvis en graf kan dekomponeres i lokalt irregulære grafer, så findes der en sådan dekomposition med højst 3 dele. Vi beviser at denne formodning er sand, hvis antallet 3 erstattes af 328 – det første bevis for en konstant øvre grænse for dette problem.

Preface

This thesis was prepared at the Department of Applied Mathematics and Computer Science of the Technical University of Denmark in fulfilment of the requirements for acquiring a PhD degree in mathematics. The research was financed by the EU-project GRACOL: "Graph Theory: Colourings, flows, and decompositions" under the supervision of Professor Carsten Thomassen. Part of the research was carried out during a 3 month research stay at the University of Waterloo, Canada.

The results presented in this thesis have been submitted for publication to various journals and can also be found on arXiv.org:

- M. Merker. Decomposing highly edge-connected graphs into homomorphic copies of a fixed tree. *J. Comb. Theory Ser. B*, to appear.

- J. Bensmail, M. Merker, and C. Thomassen. Decomposing graphs into a constant number of locally irregular subgraphs. arXiv:1604.00235

- J. Bensmail, A. Harutyunyan, T.-N. Le, M. Merker, and S. Thomassé. A Proof of the Barát-Thomassen Conjecture. arXiv:1603.00197

- M. Merker and L. Postle. Bounded Diameter Arboricity. arXiv:1608.05352

Lyngby, 31-August 2016   Martin Merker

Acknowledgements

First of all, I would like to thank my advisor Carsten Thomassen. His enthusiasm for graph theory is unrivalled and so is his kindness. I did not know Carsten when I applied to be his PhD student but after our first meeting I had no doubts that I would enjoy working with him. Despite all his commitments he usually found the time to meet me even on short notice to talk about anything I had on my mind. Not only did I appreciate his academic insights but also his stories and anecdotes he so often shared over a cup of coffee in the afternoon. Talking about coffee, one great benefactor of our graph theory group deserves special mention: Lis, Carsten's wife, who regularly impressed us with her incredible baking skills and marvellous birthday cakes.

I would also like to thank my academic brothers – Thomas Perrett, Seongmin Ok, and Kasper Szabo Lyngsie – whom I not only shared advisor and office with but also many great experiences. In particular, I am very grateful for Tom's friendship during these three years and the time we got to spend together. Both Kasper and Tom were a great help in the writing process of this thesis: Kasper assisted whenever I needed help with my Danish writing, while Tom proofread the entire manuscript and offered advice whenever I encountered difficulties with the typesetting.

Contents

Introduction

Decomposing a complex object into smaller pieces with a certain structure is a very common task in many areas of mathematics. Many problems in graph theory can in fact be phrased as a decomposition problem: the chromatic number of a graph G, for example, is the smallest number of independent sets needed to decompose the vertex set of G. This thesis, however, only considers edge-decompositions of graphs, i.e. partitions of the edge set of G.

1.1 $\mathcal{H}$-decompositions

All decompositions we consider in this thesis are types of $\mathcal{H}$-decompositions, where $\mathcal{H}$ is a family of graphs. An $\mathcal{H}$-*decomposition* of a graph G is a partition of the edge set of G, say $E(G) = E_1 \cup \cdots \cup E_k$, such that the subgraph of G induced by E_i is isomorphic to a member of $\mathcal{H}$ for every $i \in \{1, \ldots, k\}$. In

other words, we decompose the edge-set of G into parts that induce subgraphs of a certain structure. If a graph admits an $\mathcal{H}$-decomposition, then we call it *$\mathcal{H}$-decomposable.*

In general, edge-decomposition problems can also be considered edge-colouring problems and vice versa. Given a decomposition $E(G) = E_1 \cup \cdots \cup E_k$, we can define an edge-colouring $c : E(G) \to \{1, \ldots, k\}$ of G by setting $c(e) = i$ if and only if $e \in E_i$. Similarly, every edge-colouring of G gives rise to an edge-decomposition of G in a canonical way. Therefore we often use the terms interchangeably by thinking of the colour classes in an edge-colouring as the parts in an edge-decomposition.

If $\mathcal{H}$ consists of a single graph H, then we also speak of H-decompositions instead of $\mathcal{H}$-decompositions. Even in this very special case, it is usually very difficult to decide whether a graph has an H-decomposition: Dor and Tarsi [DT97] showed that it is NP-complete as soon as H has a connected component with at least 3 edges.

Typical questions concerning $\mathcal{H}$-decompositions are of the following type:

- When does a graph admit an $\mathcal{H}$-decomposition? What are sufficient conditions. What are necessary conditions?

- If a graph G has an $\mathcal{H}$-decomposition, what is the smallest number of parts in an $\mathcal{H}$-decomposition of G?

This thesis investigates three different problems which are all of the type described above. Each of the following three chapters is devoted to one of the problems and can be read independently of the other chapters.

1.2   Three problems

Firstly and most importantly, we consider the case where $\mathcal{H}$ consists of a single graph H.

Decomposing graphs into a given tree

One necessary condition for the existence of an H-decomposition is of course that $|E(H)|$ divides $|E(G)|$. Since this condition is obviously not sufficient in general, it is a natural question to ask what additional conditions guarantee the existence of an H-decomposition. In 2006, Barát and Thomassen [BT06] considered decompositions of graphs into trees and conjectured that sufficiently large edge-connectivity may be one such additional sufficient condition. More precisely, they conjectured the following, which became known as the Barát-Thomassen Conjecture or Tree Decomposition Conjecture.

CONJECTURE *For any tree T on m edges, there exists an integer k_T such that every k_T-edge-connected graph with size divisible by m has a T-decomposition.*

The motivation for this conjecture came from a perhaps surprising connection to Tutte's flow conjectures. Tutte's 3-flow conjecture states that every 4-edge-connected graph admits a nowhere-zero 3-flow. Now we know that every 6-edge-connected graph admits a nowhere-zero 3-flow by a result of Lovász et al. [LTWZ13] which improved on a previous result by Thomassen [Tho12]. In 2006 however it was not know whether any constant edge-connectivity guarantees the existence of a nowhere-zero 3-flow. Thus, it was remarkable when Barát and Thomassen [BT06] showed that if every 8-edge-connected graph with size divisible by 3 has a $K_{1,3}$-decomposition, then every 8-edge-connected graph has nowhere-zero 3-flow. Moreover, they also showed that if Tutte's 3-flow conjecture is true, then every 10-edge-connected graph with size divisible by 3 has a

$K_{1,3}$-decomposition.

Notice that if H contains a cycle then large edge-connectivity cannot guarantee the existence of H-decompositions since there exist graphs having both arbitrarily large girth and arbitrarily large edge-connectivity by a result of Mader [Mad72]. Therefore the Barát-Thomassen conjecture has no canonical extension to general H-decompositions.

When Barát and Thomassen made their conjecture, it was only known to hold for the paths of length 1 and 2, which are both trivial cases. Since then several papers have been published on this problem, many of them verifying the Barát-Thomassen Conjecture only for a specific tree T:

- Path of length 3 by Thomassen [Tho08a]

- Path of length 4 by Thomassen [Tho08b]

- Path of length 5 by [BMOW16b]

- Path of length 2^k by Thomassen [Tho13b]

- Path of any length independently by [BMOW16a] and [BHLT]

- Stars by Thomassen [Tho12]

- Bistar $S(2,3)$ by Barát and Gerbner [BG14]

- Bistars of the form $S(k, k+1)$ by Thomassen [Tho13a]

- Trees of diameter at most 4 by Merker [Mer16]

Finally, a proof of the full conjecture was obtained by the author of this thesis in joint work with Bensmail, Harutyunyan, Le, and Thomassé [BHL$^+$]. Roughly speaking, the proof consists of three parts.

In the first step, the problem is reduced to bipartite graphs. This was done by Thomassen [Tho13a] and independently by Barát and Gerbner [BG14]. We refer the interested reader to their papers for this part of the proof.

In the second step, we construct a special edge-colouring of G which we call T-equitable. This was done by Merker [Mer16] using modulo-k orientations. These T-equitable colourings give rise to some kind of approximate T-decomposition which we call a T-*pseudo-decomposition*. In certain cases, for example if the diameter of T is at most 3 or the girth of G is greater than the diameter of T, these T-pseudo-decompositions are already T-decompositions and the proof ends here.

In the third step, it is shown that any T-equitable colouring gives rise to a T-decomposition provided the minimum degree in each colour is large enough. Unlike the previous two parts, this part relies on probabilistic tools and the large edge-connectivity is only needed to guarantee large minimum degree.

Bounded diameter arboricity

The usual arboricity of a graph is defined as the number of forests needed to decompose the graph. An obvious necessary condition for a graph to have arboricity at most k is that $|E(H)| \leq k(|V(H)| - 1)$ for every subgraph H of G. In 1964, Nash-Williams [NW64] proved that this condition is also sufficient. Since then, several other concepts of arboricity have been studied in which the structure of the forest is further restricted. Perhaps the two arboricity variants which received most attention so far are *star arboricity* and *linear arboricity*.

In 1970, Harary [Har70] introduced the notion of *linear arboricity*, which is the smallest number of linear forests needed to decompose a graph where a *linear forest* is the disjoint union of paths. We denote the linear arboricity of a graph G by $\Upsilon_\ell(G)$. Clearly, the linear arboricity is intimately con-

nected to the maximum degree of a graph as $\Delta(G)/2 \leq \Upsilon_\ell(G) \leq \Delta(G) + 1$. The *linear arboricity conjecture* posed by Akiyama et al. [AEH80] states that $\Upsilon_\ell(G) \leq \frac{\Delta(G)+1}{2}$. Using probabilistic methods, Alon [Alo88] showed that $\Upsilon_\ell(G) = \Delta(G)/2 + O(\Delta(G) \log \log \Delta(G)/ \log \Delta(G))$.

The *star arboricity* of a graph G, denoted $\Upsilon_s(G)$, is the smallest number of star forests needed to decompose G, where a *star forest* is the disjoint union of stars. Clearly, the star arboricity is at most twice the arboricity since every forest can be decomposed into two star forests. However, this is best possible due to a construction of Alon et al. [AMR92]. Nevertheless for certain interesting graph classes the star arboricity can be lower. For example, Algor and Alon [AA89] showed that $\Upsilon_s(G) \leq d/2 + O(d^{2/3} \log^{1/3} d)$ for d-regular graphs. Hakimi et al. [HMS96] showed that $\Upsilon_s(G)$ is at most the acyclic chromatic number of G. In 1979, Borodin [Bor79] showed that planar graphs have acyclic chromatic number at most 5 and thus star arboricity at most 5, which is best possible as shown by Algor and Alon [AA89].

Following [MP], we introduce the following new arboricity variant. The *diameter-d arboricity* of a graph is the minimum number k such that the edges of the graph can be partitioned into k forests each of whose components has diameter at most d. This can be viewed as a generalisation of star arboricity which is identical to diameter-2 arboricity. We say a class of graphs has *bounded diameter arboricity k* if there exists a natural number d such that every graph in the class has diameter-d arboricity at most k.

CONJECTURE *For every natural number k, there exists a natural number $d(k)$ such that the following holds: If G is a graph of arboricity k, then G decomposes into $k+1$ forests in which each tree has diameter at most $d(k)$.*

In other words, this conjecture states that the class of graphs with arboricity at most k has bounded diameter arboricity at most $k + 1$.

In Chapter 3, we prove this conjecture for $k \in \{2, 3\}$ by proving the stronger assertion that the union of a forest and a star forest can be partitioned into two forests of diameter at most 18. We use these results to characterise the bounded diameter arboricity for the class of planar graphs of girth at least g for all $g \neq 5$.

Perhaps surprisingly, our result has implications for the existence of thin spanning trees in planar graphs. A spanning tree is called ε-thin if it contains at most an ε-proportion of the edges in every cut. We show that every 6-edge-connected planar (multi)graph contains two disjoint $\frac{18}{19}$-thin spanning trees. If the planarity condition is omitted, this turns into an important problem which is still wide open: Goddyn [God04] conjectured that for every ε there exists a number $f(\varepsilon)$ such that every $f(\varepsilon)$-edge-connected graph contains an ε-thin spanning tree.

Locally irregular subgraphs

The third topic of this thesis concerns locally irregular graphs. A *regular* graph is a graph in which all vertices have the same degree. It is well-known that in a simple graph on at least two vertices there always exist two vertices of the same degree. Thus, requiring that all vertices have distinct degrees is not a very interesting concept of irregularity. However, requiring only that adjacent vertices have distinct degrees leads to the concept of *local irregularity*. The famous 1,2,3-Conjecture by Karoński, Łuczak, and Thomason [KŁT04] states that every simple connected graph apart from K_2 can be made locally irregular by replacing some of its edges by two or three parallel edges.

The 1,2,3-Conjecture can also be phrased in terms of edge-colourings. A k-edge-colouring taking values in $\{1, \ldots, k\}$ is called *neighbour-sum-distinguishing* if for every two adjacent vertices the sums of the colours of the incident edges are distinct. The 1,2,3-Conjecture now states that every simple connected graph apart from K_2 has a neighbour-sum-distinguishing 3-edge-colouring. Currently

the best known result is due to Kalkowski, Karoński, and Pfender [KKP10] who showed the existence of a neighbour-sum-distinguishing 5-edge-colouring. We refer the reader to Seamone [Sea] for a survey on the 1,2,3-Conjecture, its variants and partial results.

In this thesis, we focus on a different conjecture about local irregularity. Baudon, Bensmail, Przybyło, and Woźniak [BBPW15] asked which graphs admit a decomposition into locally irregular subgraphs. This relates to the 1,2,3-Conjecture for regular graphs G since every neighbour-sum-distinguishing 2-edge-colouring of G corresponds to a decomposition into two locally irregular subgraphs. We write $\mathcal{L}$ for the class of locally irregular graphs. Let us call a graph *exceptional* if it is not $\mathcal{L}$-decomposable. Notice that not all graphs are $\mathcal{L}$-decomposable as can easily be seen by considering paths or cycles of odd length. Baudon, Bensmail, Przybyło, and Woźniak [BBPW15] completely characterised the exceptional graphs. They also asked the question how many locally irregular graphs are needed in a decomposition of an $\mathcal{L}$-decomposable graph. The *irregular chromatic index* of a graph G, denoted by $\chi'_{\mathrm{irr}}(G)$, is the smallest number of parts in an $\mathcal{L}$-decomposition of G. Baudon, Bensmail, and Sopena [BBS15] showed that determining the irregular chromatic index of a graph is NP-complete in general, and that, although infinitely many trees have irregular chromatic index 3, the same problem for trees can be solved in linear time.

Baudon et al. [BBPW15] made the following strong conjecture:

CONJECTURE *If G is $\mathcal{L}$-decomposable, then $\chi'_{\mathrm{irr}}(G) \leq 3$.*

They verified the conjecture for several classes of graphs such as trees, complete graphs, and regular graphs of minimum degree at least 10^7. Extending this result, Przybyło [Prz16] showed that $\chi'_{\mathrm{irr}}(G) \leq 3$ holds whenever G has minimum degree at least 10^{10}.

Despite these results it was not known until recently whether there exists a

constant c such that $\chi'_{\mathrm{irr}}(G) \leq c$ holds for every $\mathcal{L}$-decomposable graph G. This was even an open problem for $\mathcal{L}$-decomposable bipartite graphs, see [BBPW15, BBS15, BS16, Prz16]. The author answered this problem in the affirmative in joint work with Bensmail and Thomassen [BMT] by proving that $\chi'_{\mathrm{irr}}(G) \leq 10$ for $\mathcal{L}$-decomposable bipartite graphs. This constant upper bound for bipartite graphs together with Przybyło's result can be used to show that $\chi'_{\mathrm{irr}}(G) \leq 328$ for every $\mathcal{L}$-decomposable graph G. Chapter 4 is dedicated to the proofs of these two constant upper bounds.

1.3   Notation and Definitions

For basic graph theory terminology we refer the reader to the graph theory book by Diestel [Die12].

Unless stated otherwise, all graphs considered in this thesis are finite, simple and undirected. Given a graph G, we denote by $V(G)$ and $E(G)$ its vertex and edge sets, respectively. We sometimes write $e(G)$ for the number of edges of G, which we also call the size of G. For any subset S of vertices or edges of G, we denote by $G[S]$ the subgraph of G induced by S.

We denote the degree of a vertex v in G by $d(v, G)$, or by $d(v)$ if the graph is clear from the context. If the graph is directed, we denote the outdegree of a vertex v by $d^+(v)$ and the indegree by $d^-(v)$. The maximum degree of a graph G is denoted by $\Delta(G)$. The arboricity of a graph, which is defined as the smallest number of forests needed to edge-decompose G, is denoted by $\Upsilon(G)$.

We write P_k for the path on k vertices, thus P_k has length $k - 1$. Moreover, we denote the complete graph on n vertices by K_n, and the complete bipartite graphs with partition classes consisting of a and b vertices by $K_{a,b}$. A bistar is a tree with at most two vertices of degree greater than 1. If these two vertices have degrees k and ℓ, then we denote the bistar by $S(k, l)$.

Decomposing into copies of a given tree

In this chapter we give a proof of a conjecture by Barát and Thomassen from 2006, which became known as the Barát-Thomassen Conjecture or Tree Decomposition Conjecture. The material presented here essentially consists of two research articles [Mer16, BHL+].

2.1 Preliminaries

We begin by stating the main result we prove in this chapter.

THEOREM 2.1.1 *For any tree T on m edges, there exists an integer k_T such that every k_T-edge-connected graph G with size divisible by m has a T-decomposition.*

The proof presented here builds on several partial results on the Barát-Thomassen Conjecture. In particular, it is sufficient to only consider decompositions of bipartite graphs by the following theorem, which was proved by Barát and Gerbner [BG14] and independently by Thomassen [Tho13a].

THEOREM 2.1.2 *Let T be a tree on m edges. The following two statements are equivalent:*

(1) *There exists a natural number k_T such that every k_T-edge-connected graph with size divisible by m has a T-decomposition.*

(2) *There exists a natural number k'_T such that every k'_T-edge-connected bipartite graph with size divisible by m has a T-decomposition.*

In the remainder of this chapter we use m to denote the size of the tree T.

Another important reduction is the following decomposition result, which was shown by Thomassen in [Tho13a] and also applied by Botler et al. [BMOW16b] and Merker [Mer16].

THEOREM 2.1.3 *Let G be a bipartite graph with partition classes A_1 and A_2, and size divisible by m. If G is $(4\lambda + 6m)$-edge-connected, then G can be decomposed into two λ-edge-connected graphs G_1 and G_2 such that $d(v, G_i)$ is divisible by m for every v in A_i and $i \in \{1, 2\}$.*

By Theorems 2.1.2 and 2.1.3, it is sufficient to prove Theorem 2.1.1 for bipartite graphs G on partition classes A and B, where all vertices in A have degree divisible by m, the size of T.

A crucial part of the whole proof of the Barát-Thomassen Conjecture is a special kind of edge-colouring of G which was introduced by Merker [Mer16]. To define this, let T_A and T_B denote the partition classes of a bipartition of T. The T-decompositions we are going to construct will respect the bipartitions of G and

T in the sense that the vertices corresponding to T_A will lie in A for each copy of T.

DEFINITION 2.1.4 We say that vertices $v \in V(G)$ and $t \in V(T)$ are *compatible* if $v \in A$ and $t \in T_A$, or $v \in B$ and $t \in T_B$.

If the edges of G are coloured, then we denote the degree of vertex v in colour i by $d_i(v)$. For $t \in V(T)$, let $S(t)$ denote the set of edges incident with t.

DEFINITION 2.1.5 A *T-equitable edge-colouring* is a function $\phi : E(G) \to E(T)$ satisfying $d_j(v) = d_k(v)$ for any compatible vertices $v \in V(G), t \in V(T)$ and $j, k \in S(t)$.

We occasionally refer to the edges of T as colours since they appear as colours in T-equitable edge-colourings. Given a T-equitable colouring, we can group the edges as they appear in T to form coloured copies of T. Unfortunately, this does not necessarily result in a T-decomposition as the parts have the same size as T and a similar structure but might fail to be isomorphic to T. Therefore we introduce the following decomposition notion which is less restrictive than T-decompositions.

DEFINITION 2.1.6 A graph H is a *pseudo-copy* of T, if there exists a surjective graph homomorphism $h : V(T) \to V(H)$ that induces a bijection between $E(T)$ and $E(H)$.

A *T-pseudo-decomposition* of a graph is an edge-decomposition where each part is a pseudo-copy of T.

In other words, a graph H is a pseudo-copy of T, if it is isomorphic to a multigraph obtained from T by identifying vertices and keeping all edges. We also refer to pseudo-copies of T as *pseudo-trees*. Notice that pseudo-copies were also called homomorphic copies by Merker [Mer16].

In the following lemma we show that a T-pseudo-decomposition of a bipartite graph G can easily be constructed from a T-equitable colouring of G. Let us denote the set of edges coloured i incident with $v \in V(G)$ by $N_i(v)$. Furthermore, we set $N_{S(t)}(v) := \bigcup_{i \in S(t)} N_i(v)$ for every $t \in V(T)$ compatible with v.

LEMMA 2.1.7 *Let G be a bipartite graph on partition classes A and B. If G admits a T-equitable colouring then G has a T-pseudo-decomposition.*

PROOF. For each $v \in V(G)$ and compatible $t \in V(T)$, we partition $N_{S(t)}(v)$ into stars of size $|S(t)|$ that contain each of the colours in $S(t)$ exactly once. Let $\mathcal{S}$ be the collection of stars we get after having done this for every $v \in V(G)$ and compatible $t \in V(T)$. Consider an auxiliary graph $G_{\mathcal{S}}$ whose vertices are the stars in $\mathcal{S}$, and where two vertices are joined by an edge whenever the corresponding stars have an edge in common. By construction, each connected component of $G_{\mathcal{S}}$ is a tree isomorphic to T. For every connected component in $G_{\mathcal{S}}$, we take the union of all the stars corresponding to it in G. This yields a T-pseudo-decomposition of G. □

It was shown by Merker [Mer16], using modulo-k orientations, that highly edge-connected graphs admit T-equitable edge-colourings. Section 2.3 is devoted to a proof of the following theorem.

THEOREM 2.1.8 *For any natural numbers m and L there exists a natural number $k(m, L)$ such that the following holds: If G is a $k(m, L)$-edge-connected bipartite graph where all vertices in one side of the bipartition have degree divisible by m, then G admits a T-equitable colouring where the minimum degree in each colour is at least L.*

Using probabilistic methods similar to the ones used by Bensmail et al. [BHLT], we show that a T-equitable colouring can be turned into a T-decomposition provided the minimum degree in each colour is large enough.

THEOREM 2.1.9 *Let G be a bipartite graph admitting a T-equitable colouring. If the minimum degree in each colour is at least 10^{50m}, then G has a T-decomposition.*

The proof of Theorem 2.1.9 is given in Section 2.4. Combining the theorems above yields a complete proof of the Barát-Thomassen conjecture.

PROOF OF THEOREM 2.1.1. By Theorem 2.1.2, we may assume that G is bipartite. We show that every $(4k(m, 10^{50m}) + 6m)$-edge-connected bipartite graph has a T-decomposition, where $k(m, 10^{50m})$ is the number given by Theorem 2.1.8. By Theorem 2.1.3 we can decompose G into two spanning $k(m, 10^{50m})$-edge-connected graphs G_1 and G_2, such that in one side of the bipartition of each G_i all vertices have degree divisible by m. By Theorem 2.1.8, we can find a T-equitable colouring of G_i in which the minimum degree in each colour is at least 10^{50m}. This colouring can be turned into a T-decomposition by Theorem 2.1.9. $\qquad\qquad\square$

We collect the tools used in the proofs of Theorem 2.1.8 and Theorem 2.1.9 in Section 2.2. Section 2.5 shows how Theorem 2.1.8 can be used to construct a T-decomposition without using probabilistic methods if the diameter of T is at most 4. Finally, we make a short excursion to the world of infinite graphs in Section 2.6, where we show that highly edge-connected infinite graphs admit T-pseudo-decompositions.

2.2 Tools

The proofs of Theorem 2.1.8 and Theorem 2.1.9 require very different techniques. For Theorem 2.1.8 we use a variety of structural results that rely on the high

edge-connectivity of G. In the proof of Theorem 2.1.9 we use probabilistic methods to show the existence of a T-decomposition.

2.2.1   Edge-connectivity tools

Perhaps the most important tool in the study of T-decompositions is a recent result on modulo-k orientations. Thomassen [Tho12] showed that every $(2k^2 + k)$-edge-connected graph G has an orientation such that every vertex gets a prescribed outdegree modulo k, provided that the sum of all prescribed outdegrees is congruent to $e(G)$ modulo k. Lovász et al. [LTWZ13] improved the bound on the edge-connectivity to $3k-3$ for k odd, and to $3k-2$ for k even. An immediate consequence is that the Barát-Thomassen Conjecture holds if T is a star.

THEOREM 2.2.1 *Let G be a graph with m edges, k be a natural number, and $p : V(G) \to \mathbb{Z}$ be a function satisfying $\sum_{v \in V(G)} p(v) \equiv m \ (mod \ k)$. If G is $(3k-2)$-edge-connected, then there exists an orientation of the edges of G such that $d^+(v) \equiv p(v) \ (mod \ k)$ for every $v \in V(G)$.*

As an application of Theorem 2.2.1, it was shown by Thomassen [Tho13a] that a highly edge-connected bipartite graph G with size divisible by k can be decomposed into two k-edge-connected graphs G_1 and G_2 such that in G_1 all vertices of A have degree divisible by k, and in G_2 all vertices of B have degree divisible by k. This is essentially the statement Theorem 2.1.3, but for technical reasons we prove a slightly different version of it.

PROPOSITION 2.2.2 *Let m and ℓ be natural numbers, G be a bipartite graph on partition classes A_1 and A_2, and suppose the size of G is divisible by m. If G has $3m - 2 + 2\ell$ edge-disjoint spanning trees, then G can be decomposed into two spanning subgraphs G_1 and G_2 such that each G_i contains ℓ edge-disjoint spanning trees and all vertices of A_i have degree divisible by m in G_i,*

for $i \in \{1, 2\}$.

PROOF. Let H_1 and H_2 each be the union of ℓ of the spanning trees, and let G' be the graph on the remaining edges. For v in A_i define $p(v) = m - d(v, H_i)$. Observe that

$$\sum_{v \in V(G')} p(v) \equiv -|E(H_1)| - |E(H_2)| \equiv |E(G')| \pmod{m}.$$

Since G' is $(3m - 2)$-edge-connected, we can apply Theorem 2.2.1 to orient its edges so that each vertex v has outdegree congruent to $p(v)$ modulo m. For $i \in \{1, 2\}$, let G_i be the union of H_i and all edges oriented from A_i to A_{3-i}. □

Another important tool for working with high edge-connectivity is the following reduction method due to Mader [Mad78]. Let v be a vertex in a graph G, and let $e = vu_1$, $f = vu_2$ be two edges incident with v. A *lifting of the pair* $\{e, f\}$ is the operation of removing e and f from G and adding a new edge $u_1 u_2$. Notice that this operation might create multiple edges. Now let v be a vertex of even degree. A *lifting of* v is the operation of pairing up the edges incident with v, lifting each pair and deleting v. We say that the lifting is *connectivity-preserving*, if the edge-connectivity of the resulting graph is not smaller than the edge-connectivity of G. We shall use the following version of Mader's Theorem which was proved by Frank [Fra92].

THEOREM 2.2.3 *Let v be a vertex of even degree in a graph G. If v is not incident with a cut-edge, then there exists a connectivity-preserving lifting at v.*

Large edge-connectivity is also related to the existence of many edge-disjoint spanning trees. If a graph contains k edge-disjoint spanning trees, then it is clearly k-edge-connected. Conversely, Nash-Williams [NW61] and Tutte [Tut61] independently proved that large edge-connectivity implies the existence of many edge-disjoint spanning trees.

THEOREM 2.2.4 *Let k be a natural number. If G is a $2k$-edge-connected graph, then G contains k edge-disjoint spanning trees.*

Apart from many edge-disjoint spanning trees, large edge-connectivity also guarantees the existence of spanning trees with small vertex degrees. This has been investigated by several authors [CS97, ENV02, Has15, ZB98]. Small-degree spanning trees have already been used by Thomassen [Tho08a, Tho08b, Tho13b] and by Barát and Gerbner [BG14] to prove special cases of Theorem 2.1.1.

Our main interest is to prove the existence of k_T. Since the method presented here will not result in the best possible upper bound on k_T, we shall avoid the use of stronger but more technical statements for the sake of simplicity. The following theorem was proved by Ellingham, Nam, and Voss [ENV02] and is sufficient for our purposes.

THEOREM 2.2.5 *Let k be a natural number. If G is a $4k$-edge-connected graph, then G has a spanning tree T such that $d(v,T) < d(v,G)/k$ for every $v \in V(G)$.*

Repeated application of Theorem 2.2.5 also guarantees the existence of highly edge-connected subgraphs with small degrees.

LEMMA 2.2.6 *Let k and q be natural numbers. If G is a graph with $4kq$ edge-disjoint spanning trees, then G has a spanning q-edge-connected subgraph H such that $d(v,H) < d(v,G)/k$ for every $v \in V(G)$.*

PROOF. Let G_i consist of $4k$ of the spanning trees for $i \in \{1,\ldots,q\}$. By Theorem 2.2.5, for each G_i we can find a spanning tree T_i with $d(v,T_i) < d(v,G_i)/k$. Let H be the union of $T_1,\ldots,T_q$. Now H is q-edge-connected and we have

$$d(v,H) = \sum_{i=1}^{q} d(v,T_i) < \frac{1}{k} \sum_{i=1}^{q} d(v,G_i) \leq \frac{1}{k} d(v,G).$$

In Section 2.5.1 we take a closer at what value of k_T can be obtained from our proof if the tree has diameter at most 3. For this we use the following strengthening of Theorem 2.2.5, which was also proved by Ellingham et al. [ENV02].

LEMMA 2.2.7 *For every ε with $0 < \varepsilon < 1$, if G is $\lceil \frac{4}{\varepsilon} \rceil$-edge-connected, then G has a spanning tree T such that $d(v, T) < \varepsilon\, d(v, G)$ for every $v \in V(G)$.*

By combining Lemma 2.2.7 with the proof of Lemma 2.2.6 we get the following lemma.

LEMMA 2.2.8 *Let q be a natural number, and ε a real number with $0 < \varepsilon < 1$. If G is a graph with $\lceil \frac{4}{\epsilon} \rceil q$ edge-disjoint spanning trees, then G has a spanning q-edge-connected subgraph H such that $d(v, H) < \varepsilon d(v, G)$ for every $v \in V(G)$.*

Finally, we use the following quantitative version of Theorem 2.1.2, which was proved by Thomassen [Tho13a] for trees of diameter at most 3.

THEOREM 2.2.9 *Let T be a tree on m edges with diameter 3, and let k be a natural number. If G is a $(4k + 16m(m + 1))$-edge-connected graph, then G can be decomposed into a k-edge-connected bipartite graph G' and a graph H that admits a T-decomposition.*

2.2.2 Probabilistic tools

The probabilistic tools we list here can for example be found in the book by Molloy and Reed [MR02]. The first inequality we use is often called the Simple Concentration Bound.

PROPOSITION 2.2.10 *Let X be a random variable determined by n independent trials $T_1, ..., T_n$ such that changing the outcome of any one trial T_i can*

affect X by at most c. Then

$$\mathbb{P}[|X - \mathbb{E}[X]| > \lambda] \leq 2e^{-\lambda^2/(2c^2 n)}.$$

The following two lemmas are different versions of the so-called Lovász Local Lemma.

LEMMA 2.2.11 *Let $A_1, ..., A_n$ be a finite set of events in a probability space Ω, and suppose that for some $J_i \subset [n]$, A_i is mutually independent of $\{A_j : j \notin J_i \cup \{i\}\}$. If there exist real numbers $x_1, ..., x_n$ in $(0, 1)$ such that $\mathbb{P}[A_i] \leq x_i \prod_{j \in J_i}(1 - x_j)$ for every $i \in \{1, ..., n\}$, then $\mathbb{P}[\cap_{i=1}^n \overline{A_i}] > 0$.*

LEMMA 2.2.12 *Let $A_1, ..., A_n$ be events in a probability space Ω with $\mathbb{P}[A_i] \leq p$ for all $i \in \{1, ..., n\}$. Suppose that each A_i is mutually independent of all but at most d other events A_j. If $4pd < 1$, then $\mathbb{P}[\cap_{i=1}^n \overline{A_i}] > 0$.*

To show that each bad event occurs with low probability, we make use of an inequality due to McDiarmid [McD02] (see also Molloy and Reed [MR02]). In what follows, a *choice* is defined to be the position that a particular element gets mapped to in a permutation.

PROPOSITION 2.2.13 *Let X be a non-negative random variable, not identically 0, which is determined by m independent permutations $\Pi_1, ..., \Pi_m$. If there exist $d, r > 0$ such that*

- *interchanging two elements in any one permutation can affect X by at most d, and*

- *for any $s > 0$, if $X \geq s$ then there is a set of at most rs choices whose outcomes certify that $X \geq s$,*

then for any $0 \leq \lambda \leq \mathbb{E}[X]$,

$$\mathbb{P}\left[|X - \mathbb{E}[X]| > \lambda + 60d\sqrt{r\mathbb{E}[X]}\right] \leq 4e^{-\frac{\lambda^2}{8d^2 r\mathbb{E}[X]}}.$$

2.3 Construction of T-equitable colourings

In this section we show how to construct T-equitable colourings of highly edge-connected bipartite graphs as was done by Merker [Mer16]. The existence of such colourings is an easy consequence of the following technical theorem.

THEOREM 2.3.1 *For all natural numbers m and λ, there exists a natural number $f(m, \lambda)$ such that the following holds:*

If $m_1, \ldots, m_{b+1}$ are positive integers satisfying $m = m_1 + \ldots + m_{b+1}$, and if G is an $f(m, \lambda)$-edge-connected bipartite graph on partition classes A and B in which all vertices in A have degree divisible by m, then we can decompose G into $b + 1$ spanning λ-edge-connected subgraphs $G_1, \ldots, G_{b+1}$ such that

- *$d(v, G_i) = \frac{m_i}{m} d(v, G)$ for $v \in A$ and $i \in \{1, \ldots, b+1\}$, and*

- *$d(v, G_i)$ is divisible by m_i for $v \in B$ and $i \in \{1, \ldots, b\}$.*

Notice that it is not possible to achieve that also $d(v, G_{b+1})$ is divisible by m_{b+1} for $v \in B$, since for example all of $m_1, \ldots, m_{b+1}$ could be even, but B could have vertices of odd degree.

Before we begin the proof of Theorem 2.3.1, we show how it implies Theorem 2.1.8. Notice that the T-equitable colouring we construct satisfies the even stronger statement that each vertex in A has the same degree in each of the m colours.

PROOF OF THEOREM 2.1.8. We show that edge-connectivity $f(m, mL)$ suffices to construct a T-equitable edge-colouring where the minimum degree in each colour is at least L. Let A and B denote the partition classes of G and let us assume that the vertices in A have degrees divisible by m. Moreover, we choose the bipartition of T in such a way that T_B contains a leaf.

Let $\{v_1, \ldots, v_b\}$ denote the set of vertices of degree greater than 1 in T_B. More-over, we denote the set of edges incident with the vertex v_j by S_j and we write S_{b+1} for the set of edges incident with leaves in T_B. Thus, the edge-set of T is partitioned into $b + 1$ parts $S_1, \ldots, S_{b+1}$.

For $i \in \{1, \ldots, b + 1\}$, set $m_i = |S_i|$. Notice that m_{b+1} is the number of leaves in T_B and thus at least 1. Now we can apply Theorem 2.3.1 to get a decomposition of G into mL-edge-connected graphs $G_1, \ldots, G_{b+1}$ such that $d(v, G_i) = \frac{m_i}{m} d(v, G)$ for $v \in A$, $i \in \{1, \ldots, b + 1\}$, and $d(v, G_i)$ is divisible by m_i for $v \in B$, $i \in \{1, \ldots, b\}$.

For $i \in \{1, \ldots, b\}$, every vertex of G_i has degree divisible by m_i, so we can split each vertex of G_i and obtain an m_i-regular graph G'_i. We also split each vertex in G_{b+1} into vertices of degree m_{b+1} and possibly one vertex of degree less than m_{b+1}, resulting in a graph G'_{b+1}. A well-known result by König states that every k-regular bipartite graph has a proper edge-colouring with k colours. Thus, for every $i \in \{1, \ldots, b + 1\}$, there exists a proper edge-colouring of G'_i using the m_i edges in S_i as colours. This corresponds to an edge-colouring of G_i such that

$$d_j(v, G_i) = \frac{1}{m_i} d(v, G_i) = \frac{1}{m} d(v, G)$$

for $v \in A$, $i \in \{1, \ldots b + 1\}$ and $j \in S_i$. By construction, we also have $d_j(v) = d_k(v)$ for $v \in B$, $i \in \{1, \ldots, b\}$, $j, k \in S_i$, so the colouring is T-equitable. Since the minimum degree of G_i is at least mL, the minimum degree in each colour in G is at least L.   □

The following lemma is an easy application of Theorem 2.2.1 and Theorem 2.2.3. A similar argument was already used by Thomassen [Tho13a] to prove the Barát-Thomassen Conjecture for a class of bistars.

LEMMA 2.3.2 *Let G be a $(3k - 2)$-edge-connected bipartite graph on partition classes A and B, where each vertex in A has even degree. For every function*

$p : B \to \mathbb{Z}$ satisfying

$$\sum_{v \in B} p(v) \equiv \frac{e(G)}{2} \qquad (mod \ k)\,,$$

there exists a subgraph H of G with

$d(v, H) = \frac{1}{2} d(v, G)$ for $v \in A$, and

$d(v, H) \equiv p(v)$ (mod k) for $v \in B$.

PROOF. By Theorem 2.2.3, we can lift each vertex in A so that the resulting graph G' is still $(3k - 2)$-edge-connected. By Theorem 2.2.1, we can orient the edges of G' such that each vertex v has outdegree congruent to $p(v)$ modulo k. We can also orient the edges of G such that every directed edge of G' corresponds to a directed path of length 2 in G. This yields an orientation of G where each vertex v in B has outdegree congruent to $p(v)$ modulo k, and each vertex in A has the same out- and indegree. Now the subgraph consisting of the edges oriented from B to A is as required. $\qquad \square$

The case where we want the subgraph H to contain only $1/m$ of the edges at every vertex in A, for some $m \geq 3$, can easily be reduced to the case $m = 2$.

PROPOSITION 2.3.3 *Let m and k be natural numbers with $m \geq 2$, and let G be a bipartite graph on partition classes A and B with $12km$ edge-disjoint spanning trees, where each vertex in A has degree divisible by m. For every function $p : B \to \mathbb{Z}$ satisfying*

$$\sum_{v \in B} p(v) \equiv \frac{e(G)}{m} \qquad (mod \ k)\,,$$

there exists a subgraph H of G with

$d(v, H) = \frac{1}{m} d(v, G)$ *for $v \in A$ and*

$d(v, H) \equiv p(v)$ *(mod k) for $v \in B$.*

PROOF. By Lemma 2.2.6, we can find a spanning $3k$-edge-connected subgraph G' with $d(v, G') < \frac{1}{m} d(v, G)$. We add some edges of G to G' to get a graph

$G'' \subseteq G$ in which all vertices in A have degree exactly $\frac{2}{m}d(v, G)$. Now we can use Lemma 2.3.2 to find a subgraph H of G'' with $d(v, H) = \frac{1}{2}d(v, G'') = \frac{1}{m}d(v, G)$ for $v \in A$, and $d(v, H) \equiv p(v)$ modulo k for $v \in B$. $\qquad\square$

To get a decomposition into several graphs as in Theorem 2.3.1, we proceed by induction. To do so, we need that $G - E(H)$ still has large edge-connectivity. The following lemma shows that this can be achieved by increasing the edge-connectivity of G.

LEMMA 2.3.4 *Let k, m and λ be natural numbers with $m \geq 2$. Let G be a bipartite graph on partition classes A and B with $8\lambda m^2 + 12km$ edge-disjoint spanning trees, where each vertex in A has degree divisible by m. For every function $p : B \to \mathbb{Z}$ satisfying*

$$\sum_{v \in B} p(v) \equiv \frac{e(G)}{m} \qquad (\text{mod } k),$$

there exists a decomposition of G into λ-edge-connected subgraphs G_1 and G_2 with

$d(v, G_1) = \frac{1}{m}d(v, G)$ *for $v \in A$ and*

$d(v, G_1) \equiv p(v) (\text{mod } k)$ *for $v \in B$.*

PROOF. Let H_1 and H_2 each be the union of $4\lambda m^2$ of the spanning trees, and let H_3 be the union of the remaining spanning trees. By Lemma 2.2.6, we can find a spanning λ-edge-connected subgraph H_i' of H_i satisfying

$$d(v, H_i') < \frac{1}{m^2}d(v, H_i) < \frac{1}{m^2}d(v, G) \qquad (2.1)$$

for $i \in \{1, 2\}$, and a spanning $3k$-edge-connected subgraph H_3' of H_3 satisfying

$$d(v, H_3') < \frac{1}{m}d(v, H_3) < \frac{1}{m}d(v, G). \qquad (2.2)$$

We are going to colour the edges of G with colours 1 and 2 so that for $i \in \{1, 2\}$ the graph G_i induced by the edges coloured i will be as required. As before, we denote the degree of a vertex v in colour i by $d_i(v)$.

We start by colouring all edges in H_1' with colour 1, and all edges in H_2' with colour 2. This ensures that both G_1 and G_2 will be λ-edge-connected. We also want

$$(m-1)d_1(v) = d_2(v) \tag{2.3}$$

to hold for $v \in A$. For every vertex in A, we colour more of its edges with colours 1 or 2 so that (2.3) is satisfied. We do it in such a way that the number of edges we colour is minimal. By (2.1), we can give colour 1 to edges incident with v until $d_1(v) = \frac{1}{m^2}d(v,G)$, and colour 2 to other edges incident with v until $d_2(v) = \frac{m-1}{m^2}d(v,G)$. Thus, for every $v \in A$, we colour $\frac{1}{m}d(v,G)$ edges incident with v. Because of (2.2), we can assume that all these coloured edges are outside of H_3'.

Let G' be the graph consisting of all edges we have coloured so far, and let G'' be the graph induced by the remaining edges. In particular, G' satisfies (2.3) and G'' contains H_3'. In G' every vertex in A has degree divisible by m, so this must also be the case in G''. Since $d(v,G') = \frac{1}{m}d(v,G)$ for $v \in A$, we have $d(v,G'') = \frac{m-1}{m}d(v,G) \geq \frac{1}{2}d(v,G)$. Thus,

$$d(v,H_3') < \frac{1}{m}d(v,G) \leq \frac{2}{m}d(v,G'')$$

for every $v \in A$. Now we repeat the argument from the proof of Proposition 2.3.3: We find a subgraph G''' of G'' containing H_3' and satisfying $d(v,G''') = \frac{2}{m}d(v,G'')$ for $v \in A$. Let $p' : B \to \mathbb{Z}$ be the function defined by $p'(v) = p(v) - d_1(v,G')$ for $v \in B$. Note that

$$\begin{aligned}
\sum_{v \in B} p'(v) &\equiv \sum_{v \in B}\left(p(v) - d_1(v,G')\right)\\
&\equiv \frac{e(G)}{m} - \frac{e(G')}{m}\\
&\equiv \frac{e(G'')}{m}\\
&\equiv \frac{e(G''')}{2} \pmod{k}.
\end{aligned}$$

By Lemma 2.3.2, we can find a subgraph H of G''' satisfying

$$d(v, H) = \frac{1}{2}d(v, G''') = \frac{1}{m}d(v, G'')$$

for $v \in A$, and $d(v, H) \equiv p'(v)$ modulo k for $v \in B$. We colour the edges of H with colour 1 and the remaining edges of G'' with colour 2. Together with the edge-colouring of G', this completes the construction of G_1 and G_2. Observe that, for $v \in A$,

$$
\begin{aligned}
d(v, G_1) &= d_1(v, G') + d(v, H) \\
&= \frac{1}{m^2}d(v, G) + \frac{1}{m}d(v, G'') \\
&= \frac{1}{m^2}d(v, G) + \frac{m-1}{m^2}d(v, G) \\
&= \frac{1}{m}d(v, G) \,.
\end{aligned}
$$

For $v \in B$, we have $d(v, G_1) \equiv d_1(v, G') + p'(v) \equiv p(v) \pmod{k}$, so G_1 is as desired. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Repeated application of Lemma 2.3.4 results in the following proposition.

PROPOSITION 2.3.5 *For all natural numbers k, m, and λ, there exists a natural number $f_k(m, \lambda)$ such that the following holds:*

If G is an $f_k(m, \lambda)$-edge-connected bipartite graph on partition classes A and B, in which all vertices in A have degree divisible by m, and $p_1, \ldots, p_{m-1}$ are functions $p_i : B \to \mathbb{Z}$ satisfying

$$\sum_{v \in B} p_i(v) \equiv \frac{e(G)}{m} \pmod{k}$$

for $i \in \{1, \ldots, m-1\}$, then there exists a decomposition of G into m spanning λ-edge-connected subgraphs $G_1, \ldots, G_m$ such that
$d(v, G_i) = \frac{1}{m}d(v, G)$ for $v \in A$ and $i \in \{1, \ldots, m\}$, and
$d(v, G_i) \equiv p_i(v) \pmod{k}$ for $v \in B$ and $i \in \{1, \ldots, m-1\}$.

PROOF. We use induction on m. By Theorem 2.2.4 and Lemma 2.3.4, the statement is true for $m = 2$ and $f_k(2, \lambda) = 64\lambda + 48k$. Thus, we may assume $m \geq 3$ and $f_k(m-1, \lambda)$ exists. Set

$$f_k(m, \lambda) = 16 f_k(m-1, \lambda)m^2 + 24km.$$

If G is $f_k(m, \lambda)$-edge-connected, then we can use Lemma 2.3.4 to decompose G into $f_k(m-1, \lambda)$-edge-connected subgraphs G' and G_{m-1} such that $d(v, G_{m-1}) = d(v, G)/m$ for v in A and $d(v, G_{m-1}) \equiv p_{m-1}(v)$ modulo k for v in B. Now we can use the induction hypothesis for $m - 1$ with functions $p_1, \ldots, p_{m-2}$ to decompose G' into $m-1$ spanning λ-edge-connected subgraphs $G_1, \ldots, G_{m-2}, G_m$ satisfying the conditions above. These graphs together with G_{m-1} decompose G as desired. $\qquad\square$

Now Theorem 2.3.1 follows easily.

PROOF OF THEOREM 2.3.1. For a partition P of m into parts $m_1, \ldots, m_{b+1}$, we define $\pi(P)$ to be the product of $m_1, \ldots, m_{b+1}$. We are going to show that every $f_{\pi(P)}(m, \lambda)$-edge-connected graph has a decomposition satisfying the conditions in the conclusion of Theorem 2.3.1, where $f_{\pi(P)}$ is the function defined by Proposition 2.3.5. Since there are only finitely many partitions of m into positive integers, we can then choose $f(m, \lambda)$ as the maximum of all values $f_{\pi(P)}(m, \lambda)$ over all partitions P of m.

Let $m = m_1 + \ldots + m_{b+1}$ be a partition of m into positive integers, and let k be the product of $m_1, \ldots, m_{b+1}$. Let G be $f_k(m, \lambda)$-edge-connected. We pick some function $q : B \to \mathbb{Z}$ satisfying

$$\sum_{v \in B} q(v) \equiv \frac{e(G)}{m} \pmod{k},$$

and we apply Proposition 2.3.5 with $p_1 = \ldots = p_{m-1} = q$ to get λ-edge-connected graphs $H_1, \ldots, H_m$ satisfying $d(v, H_i) = \frac{1}{m}d(v, G)$ for $v \in A$, $i \in \{1, \ldots, m\}$, and $d(v, H_i) \equiv q(v) \pmod{k}$ for $v \in B$, $i \in \{1, \ldots, m-1\}$. We

construct graphs $G_1, \ldots, G_{b+1}$ such that G_i is the union of precisely m_i of the graphs H_j, every H_j is contained in precisely one of the G_i, and G_{b+1} contains H_m. Now we have

$$d(v, G_i) = \frac{m_i}{m} d(v, G)$$

for $v \in A$, $i \in \{1, \ldots, b+1\}$, and

$$d(v, G_i) \equiv m_i q(v) \pmod{k}$$

for $v \in B$, $i \in \{1, \ldots, b\}$. Since m_i divides k for $i \in \{1, \ldots, b\}$, we have that $d(v, G_i)$ is divisible by m_i for $v \in B$, $i \in \{1, \ldots, b\}$, so the graphs $G_1, \ldots, G_{b+1}$ are as desired. $\square$

2.4   From T-equitable colourings to T-decompositions

In this section we use probabilistic methods to transform a T-equitable edge-colouring with large minimum degree into a T-decomposition of G. The proof we present here was found in joint work with Bensmail, Harutyunyan, Le, and Thomassé [BHL+].

2.4.1   Definitions and sketch of proof

In our proof of Theorem 2.1.9, a T-decomposition of a graph G is obtained in two steps, which we describe more formally below. In the first step we use the T-equitable colouring of G to obtain a T-pseudo-decomposition. Instead of choosing any such decomposition, we use probabilistic methods to find one in which the vast majority of pseudo-copies at every vertex are isomorphic to T. In the second step, we use these isomorphic copies to repair the non-isomorphic copies of T by making subgraph switches. While the switching itself is a deter-

ministic operation, we again use probabilistic methods to find a suitable set of
isomorphic copies.

Step 1: Finding a good T-pseudo-decomposition

Notice that it might be the case that a graph H can be considered as a pseudo-
copy of T in different ways if there exists more than one homomorphism from
T to H with the required properties. However, we will only consider homo-
morphisms that induce the same edge-colouring of H as the given T-equitable
colouring. Furthermore, we only consider pseudo-copies of T in G that respect
the bipartition in the sense that vertices corresponding to T_A always lie in A.

Let $\mathcal{H}$ be a T-pseudo-decomposition of G. For every compatible $v \in V(G)$ and
$t \in V(T)$, we denote by $N_{\mathcal{H}}(v|t)$ the set of pseudo-trees in $\mathcal{H}$ in which v is the
image of t. Let $d_{\mathcal{H}}(v|t) = |N_{\mathcal{H}}(v|t)|$. Clearly, for any two different vertices u
and v of G, we have $N_{\mathcal{H}}(u|t) \cap N_{\mathcal{H}}(v|t) = \emptyset$. Notice also that

$$\bigcup_{v \in G} N_{\mathcal{H}}(v|t) = \mathcal{H}$$

for every $t \in V(T)$.

We often denote a T-pseudo-decomposition of G by $\mathcal{H} \cup \mathcal{I}$, where $\mathcal{I}$ denotes the
collection of pseudo-copies that are isomorphic to T and $\mathcal{H}$ denotes the collection
of the remaining pseudo-copies.

If the minimum degree in each colour is large in the T-equitable colouring, then
there are many possibilities at every vertex to decompose its incident edges into
stars. We show that there exists a T-pseudo-decomposition where $d_{\mathcal{H}}(v|t) \leq$
$\varepsilon d_{\mathcal{I}}(v|t)$ for some given $\varepsilon > 0$ and every compatible $v \in V(G)$, $t \in V(T)$. Now
for every non-isomorphic copy $H \in N_{\mathcal{H}}(v|t)$, there are many copies isomorphic
to T in $N_{\mathcal{I}}(v|t)$. We will use one of these isomorphic copies to improve the
T-pseudo-decomposition by repairing H. This is done by a subgraph switch

operation which is explained in more detail in Step 2. However, if the trees in $N_{\mathcal{I}}(v|t)$ overlap too much, then we might not be able to make any switch that improves the T-pseudo-decomposition. To avoid this, we need to find a large set of isomorphic copies in $N_{\mathcal{I}}(v|t)$ that pairwise intersect only in v. To measure how much the pseudo-trees in a T-pseudo-decomposition overlap, we use the following concept that was introduced by Bensmail et al. [BHLT].

DEFINITION 2.4.1 Let $\mathcal{H}$ be a collection of pseudo-copies of T in G, and $v \in V(G)$ and $t \in V(T)$ be compatible vertices. The *conflict ratio* of v with respect to t, denoted by $\mathrm{conf}_{\mathcal{H}}(v|t)$, is defined by

$$\mathrm{conf}_{\mathcal{H}}(v|t) := \frac{\max_{w \in V(G), w \neq v} \left| \{ H \in N_{\mathcal{H}}(v|t) : w \in V(H) \} \right|}{d_{\mathcal{H}}(v|t)} \, .$$

Intuitively, $\mathrm{conf}_{\mathcal{H}}(v|t)$ measures the maximum proportion of pseudo-copies in $N_{\mathcal{H}}(v|t)$ in which some fixed vertex w appears. Clearly, we always have $0 \leq \mathrm{conf}_{\mathcal{H}}(v|t) \leq 1$. If v and t are not compatible, then we set $\mathrm{conf}_{\mathcal{H}}(v|t) = 0$. Globally, we define

$$\mathrm{conf}(\mathcal{H}|t) := \max_{v \in V(G)} \mathrm{conf}(v|t)$$

and

$$\mathrm{conf}(\mathcal{H}) := \max_{t \in V(T)} \mathrm{conf}(\mathcal{H}|t) \, .$$

To ensure that the isomorphic copies in the T-pseudo-decomposition $\mathcal{H} \cup \mathcal{I}$ are sufficiently spread out, we also require $\mathrm{conf}(\mathcal{H}) \leq \delta$ for some given $\delta > 0$. In Section 2.4.2, we prove that such a T-pseudo-decomposition can always be obtained provided the minimum degree in each colour is large enough.

LEMMA 2.4.2 *Let T be a tree on m edges and ε, δ real numbers with $0 < \varepsilon, \delta < 1$. Let G be a T-equitably coloured bipartite graph where the minimum degree in each colour is at least $(10m)^{18}(\varepsilon\delta)^{-6}$. Then G admits a T-pseudo-decomposition $\mathcal{H} \cup \mathcal{I}$, where $\mathcal{I}$ denotes the collection of isomorphic copies of T, such that:*

(1) for every compatible $v \in V(G)$ and $t \in V(T)$, we have $d_{\mathcal{H}}(v|t) \leq \varepsilon d_{\mathcal{I}}(v|t)$;

(2) $\mathrm{conf}(\mathcal{I}) \leq \delta$.

Step 2: Repairing non-isomorphic copies

For this part of the proof we label the vertices $t_0, \ldots, t_m$ of T so that, for every $i \in \{1, \ldots m\}$, the subgraph induced by $t_0, \ldots, t_i$ is connected. Such an ordering can for example be obtained by applying a breadth-first search algorithm from some vertex t_0 of T. We also label the edges of T so that e_i denotes the edge joining t_i with $T[t_0, \ldots, t_{i-1}]$ for every $i \in \{1, \ldots m\}$. To indicate at which place a pseudo-copy H fails to be isomorphic to T, we introduce the following definition.

DEFINITION 2.4.3 Let H be a pseudo-copy of T, and let v_i denote the image of t_i in H for every $i \in \{0, \ldots, m\}$. For $i \in \{1, \ldots, m\}$, we say that H is *i-good* if the vertices $v_0, \ldots, v_i$ are pairwise distinct. If H is not i-good, then we say that H is *i-bad*.

Note that since G does not have multiple edges, every pseudo-copy of T in G is 2-good. Moreover, since G is bipartite, every pseudo-copy of T in G is even 3-good.

The idea is to use isomorphic copies to repair the pseudo-trees that are not isomorphic to T. We start by considering all pseudo-trees in $\mathcal{H}$ that are 4-bad. For each such H, we will find an isomorphic copy $f(H)$ in $\mathcal{I}$ such that $H \cup f(H)$ can be written as the union of two 5-good pseudo-copies of T, say $H_1 \cup H_2$. We then remove H from $\mathcal{H}$ and $f(H)$ from $\mathcal{I}$, and add $\{H_1, H_2\}$ to $\mathcal{H}$. The technical definition of this so-called *switch* is given below. We use this operation for all 4-bad pseudo-copies of T in $\mathcal{H}$. Let $\mathcal{H}' \cup \mathcal{I}'$ denote the resulting T-pseudo-decomposition, where $\mathcal{I}'$ again contains only isomorphic copies of T and all pseudo-copies in $\mathcal{H}'$ are 4-good. We repeat this step, this time repairing

all 5-bad pseudo-copies in $\mathcal{H}'$ by using isomorphic copies in $\mathcal{I}'$. We continue like this until we get a T-pseudo-decomposition in which all pseudo-copies are m-good and thus isomorphic to T.

To make sure that we can make a switch between H and $f(H)$, we need $f(H)$ to satisfy certain properties. Let v_j denote the image of t_j in H for $j \in \{0, \ldots, m\}$ and suppose i is chosen minimal such that H is i-bad. By the choice of our labelling, there exists $i' \in \{0, \ldots, i-1\}$ with $t_{i'}t_i \in E(T)$. To avoid that v_i collides with one of the previous vertices $v_0, \ldots, v_{i-1}$, we want to choose a different edge corresponding to e_i at $v_{i'}$. Since we take this edge from $f(H)$, we want $f(H)$ to also use the vertex $v_{i'}$ as the image of t_i'. However, this should be the only point of intersection with H to ensure that both copies will be i-good after the switch.

More precisely, for every edge $e_i \in E(T)$, let T^{i-} denote the connected component of $T - e$ containing t_0. Let T^{i+} be the subgraph of T induced by $E(T) \setminus E(T^{i-})$. If H is a pseudo-copy of T, then we denote the images of T^{i-} and T^{i+} under the homomorphism by H^{i-} and H^{i+}. Now we are ready to define the switching operation.

DEFINITION 2.4.4 Let $\mathcal{H}$ be a collection of pseudo-copies of T in G and $i \in \{1, \ldots, m\}$. Let $t_{i'}$ be the endpoint of the edge e_i that is different from t_i. Suppose $H_1, H_2 \in N_{\mathcal{H}}(v|t_{i'})$ for some $v \in G$. The i-switch of $\{H_1, H_2\}$ is defined by

$$\text{sw}_i(\{H_1, H_2\}) = \left\{ H_1^{i+} \cup H_2^{i-}, H_1^{i-} \cup H_2^{i+} \right\}.$$

By making an i-switch between two pseudo-copies H and $f(H)$, their vertices corresponding to $v_0, \ldots, v_{i-1}$ remain unchanged. In particular, if both H and $f(H)$ are $(i-1)$-good, then also both copies in $\text{sw}_i(\{H, f(H)\})$ will be $(i-1)$-good. Moreover, if $H \cap f(H) = \{v_{i'}\}$, then after the switch both pseudo-trees will be i-good. Notice that neither of the two new pseudo-trees is necessarily still isomorphic to T. In particular, the collection of isomorphic copies might

shrink with every step of the repairing process.

If the pseudo-trees in $\mathcal{I}$ overlap too much, we might not be able to find a single pseudo-tree $f(H)$ in $\mathcal{I}$ with $H \cap f(H) = \{v_{i'}\}$. A sufficiently low conflict ratio of $\mathcal{I}$ ensures that we can find such a function $f : \mathcal{H} \to \mathcal{I}$. However, to continue this process we also need that the remaining collection of isomorphic copies $\mathcal{I} \setminus f(\mathcal{H})$ has a low conflict ratio. To this end we use the Lóvasz local lemma to prove the following lemma in Section 2.4.3.

LEMMA 2.4.5 *Let T be a tree on m edges and ε, δ positive real numbers with $\varepsilon + \delta m < \frac{1}{2}$. Let $\mathcal{H}$ and $\mathcal{H}'$ be collections of pseudo-copies of T in G with $\mathrm{conf}(\mathcal{H}') \leq \delta$ and $d_{\mathcal{H}'}(v|t) > \max\{22/\varepsilon^7, d_{\mathcal{H}}(v|t)/\varepsilon\}$ for every compatible $v \in V(G)$, $t \in V(T)$.*
For every $t \in V(T)$, there exists an injective function $f_t : \mathcal{H} \to \mathcal{H}'$ such that

- *$f_t(N_{\mathcal{H}}(v|t)) \subset N_{\mathcal{H}'}(v|t)$ for every $v \in V(G)$ compatible with t,*

- *$H \cap f_t(H) = \{v\}$ for every $H \in N_{\mathcal{H}}(v|t)$, and*

- *$d_{f_t(\mathcal{H})}(v|t') \leq 3\varepsilon d_{\mathcal{H}'}(v|t')$ for every compatible $v \in V(G)$, $t' \in V(T)$.*

By using Lemma 2.4.5 with $\mathcal{H}' = \mathcal{I}$, we find a collection $f(\mathcal{H})$ in which the degrees are low compared to the degrees in $\mathcal{I}$. Thus, the conflict ratio of the collection of isomorphic copies only increases by a constant factor after each step of the repairing process. By choosing ε and δ sufficiently small, the proof of Theorem 2.1.9 follows from Lemma 2.4.2 and repeated application of Lemma 2.4.5. The details can be found at the end of Section 2.4.3.

2.4.2 Finding a good T-pseudo-decomposition

Given a graph with a T-equitable colouring and large minimum degree in each colour, we construct a T-pseudo-decomposition satisfying the conditions in Theorem 2.4.2. As described in Step 1 of Section 2.4.1, every T-equitable colouring

gives rise to several T-pseudo-decompositions. We form the pseudo-copies of T by grouping the edges at every vertex randomly into rainbow stars. If the degrees in each colour are large enough, we can ensure that most of the resulting pseudo-trees are isomorphic to T and also the conflict ratio of the resulting T-pseudo-decomposition is small. The proof of this is essentially an application of the Lóvasz Local Lemma.

A necessary condition to apply Lemma 2.2.12 is that each event is mutually independent of most other events. To make sure that this is the case, we start the proof by partitioning the edges at each vertex into so-called *fans* of roughly the same size. Recall that for $v \in V(G)$ and $t \in V(T)$, we denote by $N_i(v)$ the edges coloured i incident with v in G, and by $S(t)$ the set of edges incident with t in T.

PROOF OF LEMMA 2.4.2. Set $c = \lceil (10m)^9 (\varepsilon\delta)^{-3} \rceil$. For every $v \in V(G)$ and colour i, we choose $r_{v,i} \in \{0, \dots, c-2\}$ such that $d_i(v) \equiv r_{v,i} \pmod{c-1}$. Since the minimum degree in each colour in G is greater than $c(c-2)$, we can partition every set $N_i(v)$ into subsets of size c and $c-1$ so that precisely $r_{v,i}$ of them have size c. We call these subsets i-*blades*. Note that an edge uv of colour i in G appears both in an i-blade of $N_i(u)$ as well as in an i-blade of $N_i(v)$, but we do not require these two i-blades to have the same size.

For every compatible $t \in V(T)$, $v \in V(G)$, and $i, j \in S(t)$, we have $d_i(v) = d_j(v)$ since the colouring is T-equitable. Thus, the number of i-blades of size c (resp. of size $c-1$) in the partition of $N_i(v)$ is equal to the number of j-blades of size c (resp. of size $c-1$) in the partition of $N_j(v)$. We can therefore partition the edges of $N_{S(t)}(v)$ into *fans* which are unions of blades of the same size such that precisely one i-blade appears in the fan for every $i \in S(t)$. In other words, a fan φ at a vertex v (with relation to t) is a subset of $N_{S(t)}(v)$ of size $c|S(t)|$ or $(c-1)|S(t)|$ such that all colours in $S(t)$ appear c times or $c-1$ times in φ. We also call φ a t-*fan* to indicate the colours appearing in φ.

For every compatible $t \in V(T)$, $v \in V(G)$, and every t-fan φ at v, we uniformly at random select a rainbow matching between the blades of φ. More precisely, for every $i \in S(t)$ we choose a permutation $\Pi_{\varphi,i}$ independently and uniformly at random from all permutations on c elements (resp. on $c-1$ elements if the blades of φ have size $c-1$). By labeling the edges of each blade, each permutation $\Pi_{\varphi,i}$ corresponds to an ordering of the edges of the i-blade of φ. Now we partition the edges of φ into stars of size $|S(t)|$ by grouping the edges of different blades that were mapped to the same position. In other words, for every $s \in \{1,\ldots,c\}$ (resp. $s \in \{1,\ldots,c-1\}$) we form a star by choosing for every $i \in S(t)$ the edge labelled $\Pi_{\varphi,i}(s)$ in the i-blade of φ. These stars are centered at v and each colour in $S(t)$ appears precisely once. Note that every edge $uv \in E(G)$ belongs to exactly two stars, one centered at u and one centered at v. As described in Step 1 in Section 2.4.1, these stars correspond to a T-pseudo-decomposition of G in a canonical way. All that remains to show is that there exists an outcome of the random permutations such that the resulting T-pseudo-decomposition is as desired.

We denote the set of pseudo-trees using edges of a fan φ by $\mathcal{T}_\varphi$. Note that $|\mathcal{T}_\varphi|$ is either equal to $c-1$ or c. Now we formally define what the bad events at a t-fan φ at a vertex v are. Let A_φ be the event that more than $2m^2c^{2/3}$ of the pseudo-copies in $\mathcal{T}_\varphi$ are not isomorphic to T. Let B_φ be the event that there exists a vertex $u \in V(G)$ with $u \neq v$ such that more than $2mc^{2/3}$ pseudo-copies in $\mathcal{T}_\varphi$ contain u. Finally, let $C_\varphi = A_\varphi \cup B_\varphi$. We will prove the following two statements.

Claim 1: Each C_φ is mutually independent of all but at most $4(cm)^{2m}$ other events C_ψ.

Claim 2: $\mathbb{P}[C_\varphi] < 9(cm)^m m e^{-c^{2/3}/32}$.

Before we proceed to prove these claims, let us note that they allow us to use Lemma 2.2.12 to get our desired T-pseudo-decomposition $\mathcal{H} \cup \mathcal{I}$. Indeed, since

$e^{-x} < \frac{(9m)!}{x^{9m}}$ for $x > 0$, we have

$$
\begin{aligned}
4 \cdot 4(cm)^{2m} \cdot \mathbb{P}[C_\varphi] \quad &< \quad 2^8 \cdot (cm)^{3m} \cdot m \cdot e^{-c^{2/3}/32} \\
&< \quad 2^{45m+8} \cdot \left(\frac{m}{c}\right)^{3m} \cdot m \cdot (9m)! \\
&< \quad \left(2^{18} \cdot \frac{m}{c} \cdot (9m)^3\right)^{3m} \\
&< \quad \left(\frac{10^9 m^4}{c}\right)^{3m} \\
&< \quad 1,
\end{aligned}
$$

where the last inequality follows from $c \geq (10m)^9$. Thus, the Lemma 2.2.12 yields a T-pseudo-decomposition $\mathcal{H} \cup \mathcal{I}$ for which none of the events C_φ holds. Now $\mathcal{H} \cup \mathcal{I}$ has the desired properties:

- Since A_φ does not hold for any φ, at most $2m^2 c^{2/3}$ of the pseudo-copies in $\mathcal{T}_\varphi$ are not isomorphic to T. Since $c \geq (10m)^9 \varepsilon^{-3}$, we have $2m^2 c^{2/3} < \frac{\varepsilon}{1+\varepsilon} c$. Thus, less than $\frac{\varepsilon}{1+\varepsilon} c$ of the pseudo-copies in $\mathcal{T}_\varphi$ are in $\mathcal{H}$, while at least $\frac{1}{1+\varepsilon} c$ of them are in $\mathcal{I}$. This holds for every t-fan at v, so we have $d_\mathcal{H}(v|t) < \varepsilon d_\mathcal{I}(v|t)$.

- Since B_φ does not hold for any φ, there are at most $2mc^{2/3}$ trees in $\mathcal{T}_\varphi$ containing a given vertex u different from v. As argued above, at least $\frac{c}{1+\varepsilon}$ of the pseudo-copies in $\mathcal{T}_\varphi$ are in $\mathcal{I}$. Since $c \geq (10m)^9 (\varepsilon\delta)^{-3}$, we have $2mc^{2/3} < \delta\frac{c}{1+\varepsilon}$. Thus, the proportion of trees in $\mathcal{T}_\varphi \cap \mathcal{I}$ containing u is less than δ. This is true for every t-fan at v, so we have

$$
\frac{\left|\{H \in N_\mathcal{I}(v|t) : u \in V(H)\}\right|}{d_\mathcal{I}(v|t)} \leq \delta
$$

and thus $\mathrm{conf}(\mathcal{I}) \leq \delta$.

Now all that is left to verify is Claims 1 and 2.

Proof of Claim 1: The structure of $\mathcal{T}_\varphi$ depends on permutations in different fans. Let $J(\varphi)$ denote the set of fans ψ for which there exists an outcome such

that $\mathcal{T}_\varphi \cap \mathcal{T}_\psi$ is non-empty. Since each fan consists of at most cm edges, there are at most $cm + (cm)^2 + \ldots + (cm)^m$ fans we can reach from φ via a path of length at most m. Thus,

$$|J(\varphi)| \leq cm + (cm)^2 + \ldots + (cm)^m < 2(cm)^m \,.$$

This shows that there are at most $2(cm)^m$ fans where the outcome of the permutation affects the structure of $\mathcal{T}_\varphi$. The same calculation shows that each permutation affects the structure of at most $2(cm)^m$ sets $\mathcal{T}_\psi$. Hence, the event C_φ is mutually independent of all but at most $4(cm)^{2m}$ other events C_ψ. $\square$

Before we prove Claim 2, let us introduce more terminology. Let t_i and t_j be two distinct vertices of T. We say that a pseudo-copy H of T is (t_i, t_j)-*bad* if the images of t_i and t_j in H are identical. For a t-fan φ at a vertex v, let $A_\varphi(t_i, t_j)$ be the event that the number of (t_i, t_j)-bad pseudo-trees in $\mathcal{T}_\varphi$ is greater than $2c^{2/3}$. For a vertex $u \in V(G)$ with $u \neq v$, let $B_\varphi(u|t_i)$ be the event that the number of pseudo-trees in $\mathcal{T}_\varphi$ in which u is the image of t_i is greater than $2c^{2/3}$. The proof of Claim 2 consists of two parts:

Claim 2A: $\mathbb{P}[A_\varphi(t_i, t_j)] < 4e^{-c^{2/3}/32}$ for every $t_i, t_j \in V(T)$ with $t_i \neq t_j$.

Claim 2B: $\mathbb{P}[B_\varphi(u|t_i)] < 4e^{-c^{2/3}/8}$ for every $u \in V(G)$, $t_i \in V(T)$ and $u \neq v$.

The proofs of Claims 2A and 2B use Proposition 2.2.13 and have a very similar structure. We will therefore present all the details in the proof of Claim 2A, and only point out the differences in the proof of Claim 2B.

Proof of Claim 2A: Fix t_i and t_j as different vertices of T. Let P_i and P_j denote the paths in T from t to t_i and t_j. In the case that one is a subpath of the other, we may assume that P_i is contained in P_j. Let $t_{j'}$ denote the second last vertex of P_j and let j denote the edge joining $t_{j'}$ and t_j. Now $T - j$ consists of two components, one of which contains t_j while the other one contains t, t_i, and $t_{j'}$.

Let π be a fixed outcome of all permutations apart from those at the j-blades of $t_{j'}$-fans. In other words, given π, we only need to know the outcome of the permutations $\Pi_{\psi,j}$ for every $t_{j'}$-fan ψ to construct the T-pseudo-decomposition. For any such outcome π, we will show that the conditional probability $\mathbb{P}[A_\varphi(t_i, t_j)|\pi]$ is at most $4e^{-c^{2/3}/32}$. Clearly, since we condition on an arbitrary but fixed event, this uniform bound implies Claim 2A.

Let T' denote the component of $T - j$ containing t, t_i and $t_{j'}$, and let T'' denote the subgraph of T induced by $E(T) \setminus E(T')$. Let $\mathcal{T}'_\varphi$ denote the images of T' in the pseudo-trees of $\mathcal{T}_\varphi$. By fixing π, the set $\mathcal{T}'_\varphi$ is also fixed. The permutations of the j-blades at the $t_{j'}$-fans only decide how the images of T' and T'' get matched at the $t_{j'}$-fans.

Let Ψ denote the set of $t_{j'}$-fans which contain edges of pseudo-copies in $\mathcal{T}_\varphi$. Note that also the set Ψ is completely determined by π. Let X_φ denote the random variable counting the number of (t_i, t_j)-bad pseudo-trees in $\mathcal{T}_\varphi$ conditional on π. Notice that X_φ only depends on the random permutations $\Pi_{\psi,j}$ with $\psi \in \Psi$.

For each pseudo-tree $H \in \mathcal{T}'_\varphi$ at a $t_{j'}$-fan $\psi \in \Psi$, we already know what the image of t_i in H is. There are $c - 1$ or c different images of T'' that could get matched to H at ψ, each having a distinct vertex as the image of t_j. Thus, there are at least $c - 1$ different vertices that could be the image of t_j in H. Since the permutation $\Pi_{\psi,j}$ is chosen uniformly at random, the probability that H will be part of a (t_i, t_j)-bad pseudo-tree is at most $\frac{1}{c-1}$. Now, by linearity of expectation,

$$\mathbb{E}[X_\varphi] \leq |\mathcal{T}_\varphi| \cdot \frac{1}{c-1} \leq \frac{c}{c-1}.$$

We will apply Proposition 2.2.13 to the random variable Y_φ defined by $Y_\varphi := X_\varphi + c^{2/3}$. Clearly $\mathbb{E}[Y_\varphi] = \mathbb{E}[X_\varphi] + c^{2/3}$. Only the permutations $\Pi_{\psi,j}$ with $\psi \in \Psi$ affect X_φ and thus Y_φ. If two elements in one of these permutations are interchanged, then the structure of two pseudo-trees in $\mathcal{T}_\varphi$ changes. In particular, the number of (t_i, t_j)-bad trees in $\mathcal{T}_\varphi$ changes by at most 2. Thus, we can choose $d = 2$ in Proposition 2.2.13.

If $Y_\varphi \geq s$, then $X_\varphi \geq s - c^{2/3}$, and thus at least $s - c^{2/3}$ of the pseudo-trees in $\mathcal{T}_\varphi$ are (t_i, t_j)-bad. Let $H' \in \mathcal{T}'_\varphi$ be a part of a pseudo-tree H that is counted by X_φ. Let v_i and v_j denote the images of t_i and t_j in H. To verify that H is (t_i, t_j)-bad, we only need to know which edge in the j-blade of ψ gets mapped to the same position as the edges in H' in other blades of ψ. In other words, the vertex v_j is determined by the position of one element in the permutation $\Pi_{\psi,j}$, and thus $v_i = v_j$ can be certified by a single outcome. Thus, $X_\varphi \geq s - c^{2/3}$ can be certified by the outcomes of $s - c^{2/3} < s$ choices and we can choose $r = 1$ in Proposition 2.2.13.

By applying Proposition 2.2.13 to Y_φ with $\lambda = \mathbb{E}[Y_\varphi]$, $d = 2$, $r = 1$, we get

$$\mathbb{P}\left[|Y_\varphi - \mathbb{E}[Y_\varphi]| > \mathbb{E}[Y_\varphi] + 120\sqrt{\mathbb{E}[Y_\varphi]}\right] \leq 4e^{-\frac{\mathbb{E}[Y_\varphi]}{32}} \leq 4e^{-\frac{c^{2/3}}{32}}$$

and thus $\mathbb{P}\left[X_\varphi > 2c^{2/3}\right] \leq 4e^{-c^{2/3}/32}$. Now $\mathbb{P}[A_\varphi(t_i, t_j)|\pi] < 4e^{-c^{2/3}/32}$ and Claim 2A follows. $\square$

Proof of Claim 2B: Let $t_i \in V(T)$ be a fixed vertex different from t. Let P denote the path from t to t_i in T. Let t_j denote the second last vertex of P and let i denote the edge joining t_j and t_i. Now $T - i$ consists of two components, one of which contains t and t_j while the other one contains t_i. Let π be a fixed outcome of all permutations apart from those at the i-blades of t_j-fans. We show that the conditional probability $\mathbb{P}[B_\varphi(u|t_j)|\pi]$ is at most $4e^{-c^{2/3}/8}$. As in the proof of Claim 2A, this implies the general bound $\mathbb{P}[B_\varphi(u|t_j)] < 4e^{-c^{2/3}/8}$.

Let X_φ denote the random variable conditional on π which counts the number of pseudo-trees in $\mathcal{T}_\varphi$ where u is the image of t_j. The vertex u appears at most once in each t_j-fan, so by linearity of expectation we have

$$\mathbb{E}[X_\varphi] \leq |\mathcal{T}_\varphi| \cdot \frac{1}{c-1} \leq \frac{c}{c-1}\,.$$

We apply Proposition 2.2.13 to the random variable $X_\varphi + c^{2/3}$. Swapping two positions in a permutation $\Pi_{\psi,i}$ can affect X_φ by at most 1 since u is incident to

at most one edge of the i-blade of ψ. If $X_\varphi + c^{2/3} \geq s$, then this can be certified by revealing at most s positions in the random permutations. Thus, applying Proposition 2.2.13 to the random variable $X_\varphi + c^{2/3}$ with $\lambda = \mathbb{E}[X_\varphi] + c^{2/3}$, $r = 1$, $d = 1$ yields

$$\mathbb{P}\left[X_\varphi > 2c^{2/3}\right] \leq 4e^{-c^{2/3}/8}.$$

Now $\mathbb{P}[B_\varphi(u|t_i)|\pi] < 4e^{-c^{2/3}/8}$ and Claim 2B follows. $\square$

Now the proof of Claim 2 follows easily from Claims 2A and 2B.

Proof of Claim 2: By Claim 2A, we have

$$\mathbb{P}[A_\varphi] \leq \mathbb{P}\left[\bigcup_{\forall i<j} A_\varphi(t_i, t_j)\right] \leq \sum_{\forall i<j} \mathbb{P}\left[A_\varphi(t_i, t_j)\right] < 4m^2 e^{-c^{2/3}/32}.$$

Let $B_\varphi(u)$ be the event that the number of pseudo-trees in $\mathcal{T}_\varphi$ containing u is greater than $2mc^{2/3}$. Since u cannot be the image of t_k, we have, by Claim 2B,

$$B_\varphi(u) \leq \mathbb{P}\left[\bigcup_{\forall i, i \neq k} B_\varphi(u|t_i)\right] \leq \sum_{\forall i, i \neq k} \mathbb{P}[B_\varphi(u|t_i)] < 4me^{-c^{2/3}/8}.$$

Since each fan consists of at most cm edges, there are at most $cm + (cm)^2 + \ldots + (cm)^m$ vertices we can reach from φ via a path of length at most m. Thus, there are less than $2(cm)^m$ vertices u for which $B_\varphi(u)$ is positive. In particular, we have

$$\mathbb{P}[B_\varphi] = \mathbb{P}\left[\bigcup_{\forall u, u \neq v} B_\varphi(u)\right] \leq \sum_{\forall u, u \neq v} \mathbb{P}[B_\varphi(u)] < 8(cm)^m m e^{-c^{2/3}/8}$$

and Claim 2 follows from $\mathbb{P}[C_\varphi] \leq \mathbb{P}[A_\varphi] + \mathbb{P}[B_\varphi]$. $\square$

2.4.3   Repairing non-isomorphic copies

Let $\mathcal{H} \cup \mathcal{I}$ be the T-pseudo-decomposition given by Lemma 2.4.2. As described in Step 2 in Section 2, we use copies in $\mathcal{I}$ to repair the pseudo-trees in $\mathcal{H}$ that

are not isomorphic to T. We apply Lemma 2.4.5 to show the existence of a suitable subset of $\mathcal{I}$ to perform the switches.

PROOF OF LEMMA 2.4.5. Consider a pseudo-tree $H \in N_{\mathcal{H}}(v|t)$, and let $w \in V(H) \setminus \{v\}$. Since $\mathrm{conf}(\mathcal{H}'|t) \leq \delta$, there are no more than $\delta d_{\mathcal{H}'}(v|t)$ trees in $N_{\mathcal{H}'}(v|t)$ containing w. Thus, there are at least $(1 - \delta m)d_{\mathcal{H}'}(v|t)$ pseudo-copies of T in $N_{\mathcal{H}'}(v|t)$ that intersect H only in v. Since $d_{\mathcal{H}}(v|t) \leq \varepsilon d_{\mathcal{H}'}(v|t)$, we can associate a set $S(H)$ of $\lfloor \frac{1 - \delta m}{\varepsilon} \rfloor$ pseudo-copies in $N_{\mathcal{H}'}(v|t)$ with each $H \in N_{\mathcal{H}}(v|t)$ such that each element of $N_{\mathcal{H}'}(v|t)$ is contained in at most one of these sets. We define the function f_t by choosing $f_t(H)$ uniformly at random from one of the pseudo-trees in $S(H)$. Clearly, any such function will satisfy the first two conditions of the lemma. All that remains to show is that with positive probability $d_{f_t(\mathcal{H})}(v|t') \leq 3\varepsilon d_{\mathcal{H}'}(v|t')$ holds for every compatible $v \in V(G)$, $t' \in V(T)$.

The value of $d_{f_t(\mathcal{H})}(v|t')$ only depends on the set of pseudo-trees in $N_{\mathcal{H}'}(v|t')$ that are contained in some $S(H)$. Let $\mathcal{H}''$ be the collection of pseudo-copies of $\mathcal{H}'$ that are contained in some $S(H)$. Clearly, each tree in $N_{\mathcal{H}''}(v|t')$ can be matched with exactly one tree in $\mathcal{H}$ and this occurs with probability $\lfloor \frac{1 - \delta m}{\varepsilon} \rfloor^{-1}$. By linearity of expectation,

$$\mathbb{E}[d_{f_t(\mathcal{H})}(v|t')] = \left\lfloor \frac{1 - \delta m}{\varepsilon} \right\rfloor^{-1} d_{\mathcal{H}''}(v|t') < 2\varepsilon d_{\mathcal{H}'}(v|t').$$

Let $A_{v,t'}$ be the event that $d_{f_t(\mathcal{H})}(v|t') > 3\varepsilon d_{\mathcal{H}'}(v|t')$. Note that $d_{f_t(\mathcal{H})}(v|t')$ is completely determined by $d_{\mathcal{H}''}(v|t')$ independent trials. Since the outcome of each trial can affect $d_{f_t(\mathcal{H})}(v|t')$ by at most 1, Proposition 2.2.10 gives

$$\mathbb{P}[A_{v,t'}] < 2e^{-\varepsilon^2 d_{\mathcal{H}''}(v|t')/2}.$$

We claim that $A_{v,t'}$ is mutually independent of all but at most $m\lfloor \frac{1 - \delta m}{\varepsilon} \rfloor d_{\mathcal{H}''}(v|t')$ other events $A_{v',t''}$. Indeed, $A_{v,t'}$ depends on $d_{\mathcal{H}''}(v|t')$ random trials, and in each trial we have a choice of $\lfloor \frac{1 - \delta m}{\varepsilon} \rfloor$ trees to match. Each tree affects precisely m events other than $A_{v,t'}$.

Now we apply the Lemma 2.2.11 to show that with positive probability none of the events $A_{v,t'}$ occur. Set $x = \frac{\varepsilon^4}{8}$. It is sufficient to show that

$$x \left(1 - x\right)^{m \left\lfloor \frac{1 - \delta m}{\varepsilon} \right\rfloor d_{\mathcal{H}''}(v|t')} \geq \mathbb{P}[A_{v,t'}]$$

holds for all compatible $v \in V(G)$, $t' \in V(T)$. If $d_{\mathcal{H}''}(v|t') < \left(\frac{2}{\varepsilon}\right)^6$, then $d_{f_t(\mathcal{H})}(v|t') < \left(\frac{2}{\varepsilon}\right)^6 < 3\varepsilon d_{\mathcal{H}'}(v|t')$, so $\mathbb{P}[A_{v,t'}] = 0$. If $d_{\mathcal{H}''}(v|t') \geq \left(\frac{2}{\varepsilon}\right)^6$, then we have

$$
\begin{aligned}
x \left(1 - x\right)^{m \left\lfloor \frac{1 - \delta m}{\varepsilon} \right\rfloor d_{\mathcal{H}''}(v|t')} \quad &\geq \quad x \left(1 - x\right)^{d_{\mathcal{H}''}(v|t')/\varepsilon^2} \\
&\geq \quad x e^{-2x d_{\mathcal{H}''}(v|t')/\varepsilon^2} \\
&\geq \quad \mathbb{P}[A_{v,t'}] \cdot \frac{x}{2} \cdot e^{\frac{\varepsilon^2}{4} d_{\mathcal{H}''}(v|t')} \\
&\geq \quad \mathbb{P}[A_{v,t'}] \cdot \left(\frac{\varepsilon}{2}\right)^6 d_{\mathcal{H}''}(v|t') \\
&\geq \quad \mathbb{P}[A_{v,t'}].
\end{aligned}
$$

By Lemma 2.2.11, there is a positive probability that none of the bad events occur. Thus, there exists a function f_t with the desired properties. $\qquad\square$

We now have all ingredients for the proof of Theorem 2.1.9. Notice that by using Lemmas 2.4.2 and 2.4.5, the remaining part of the proof is completely deterministic.

PROOF OF THEOREM 2.1.9. As described in Step 2 of Section 2.4.1, let $t_0, \ldots, t_m$ be a labeling of the vertices of T such that $T[t_0, \ldots, t_i]$ is connected for every $i \in \{1, \ldots m\}$. We also label the edges of T so that e_i denotes the edge joining t_i with $T[t_0, \ldots, t_{i-1}]$ for every $i \in \{1, \ldots m\}$. Set $\varepsilon_i = 5^{i-m}/15m$ for $i \in \{1, \ldots, m\}$. We are going to construct a sequence $(\mathcal{H}_i \cup \mathcal{I}_i)_{i=1}^m$ of T-pseudo-decompositions of G such that the following holds:

- $\mathcal{I}_i$ is a collection of isomorphic copies of T for every $i \in \{1, \ldots m\}$;

- $\mathcal{H}_i$ is i-good for every $i \in \{1, \dots m\}$;

- $d_{\mathcal{I}_i}(v|t) \geq \max\{22/\varepsilon_i^7, d_{\mathcal{H}_i}(v|t)/\varepsilon_i\}$ for all compatible $v \in V(G)$, $t \in V(T)$;

- $\mathrm{conf}(\mathcal{I}_i) \leq \varepsilon_i$ for every $i \in \{1, \dots, m\}$.

Since the minimum degree in each colour in G is at least 10^{50m}, we can apply Lemma 2.4.2 with parameters $\varepsilon = \delta = 10^{-2m}$. Let $\mathcal{H} \cup \mathcal{I}$ denote the resulting T-pseudo-decomposition. Clearly $\mathcal{H} \cup \mathcal{I}$ satisfies the conditions for $\mathcal{H}_1 \cup \mathcal{I}_1$. Let $i \in \{2, \dots, m\}$ and suppose we have constructed $\mathcal{H}_{i-1} \cup \mathcal{I}_{i-1}$ such that the conditions above are satisfied. We need to repair the pseudo-trees in $\mathcal{H}_{i-1}$ that are not i-good. Since the pseudo-trees in $\mathcal{H}_{i-1}$ are all $(i-1)$-good, we can achieve this by making i-switches. Let t_j be the endpoint of e_i that is different from t_i. Let $f_j : \mathcal{H}_{i-1} \to \mathcal{I}_{i-1}$ be the function we get by applying Lemma 2.4.5 with $\mathcal{H} = \mathcal{H}_{i-1}$, $\mathcal{H}' = \mathcal{I}_{i-1}$, $\varepsilon = \delta = \varepsilon_{i-1}$, and $t = t_j$. Now $f_j(\mathcal{H}_{i-1})$ is the set of trees we use to repair the pseudo-trees in $\mathcal{H}_{i-1}$ that are not i-good. Set

$$
\begin{aligned}
\mathcal{H}_i &= \bigcup_{H \in \mathcal{H}_{i-1}} \mathrm{sw}_i(H, f_j(H)) \text{ and} \\
\mathcal{I}_i &= \mathcal{I}_{i-1} \setminus f_j(\mathcal{H}_{i-1}),
\end{aligned}
$$

where $\mathrm{sw}_i(H, f_j(H))$ denotes the i-switch of H and $f_j(H)$ as defined in Section 2. Since $H \cap f_j(H) = \{v\}$ for every $H \in N_{\mathcal{H}_{i-1}}(v|t_j)$, the two pseudo-copies in $\mathrm{sw}_i(H, f_j(H))$ are both i-good.

Notice that the degree $d_{\mathcal{H}_i}(v|t)$ of a vertex is invariant under i-switches between pseudo-trees in $\mathcal{H}_i$. Since $d_{f_j(\mathcal{H}_{i-1})}(v|t) \leq 3\varepsilon_{i-1} d_{\mathcal{I}_{i-1}}(v|t)$ holds for compatible $v \in V(G)$ and $t \in V(T)$, we have $d_{\mathcal{I}_i}(v|t) \geq (1 - 3\varepsilon_{i-1})d_{\mathcal{I}_{i-1}}(v|t)$ and $d_{\mathcal{H}_i}(v|t) \leq 4\varepsilon_{i-1} d_{\mathcal{I}_{i-1}}(v|t)$. Thus,

$$
d_{\mathcal{H}_i}(v|t) \leq \frac{4\varepsilon_{i-1}}{(1 - 3\varepsilon_{i-1})} d_{\mathcal{I}_i}(v|t) \leq 5\varepsilon_{i-1} d_{\mathcal{I}_i}(v|t) = \varepsilon_i d_{\mathcal{I}_i}(v|t),
$$

$$
d_{\mathcal{I}_i}(v|t) \geq (1 - 3\varepsilon_{i-1})d_{\mathcal{I}_{i-1}}(v|t) \geq 22\frac{1 - 3\varepsilon_{i-1}}{\varepsilon_{i-1}^7} \geq \frac{22}{\varepsilon_i^7}
$$

and

$$
\mathrm{conf}(\mathcal{I}_i) \leq \frac{\mathrm{conf}(\mathcal{I}_{i-1})}{1 - 3\varepsilon_{i-1}} \leq \frac{5}{4}\varepsilon_{i-1} < \varepsilon_i.
$$

Hence, the T-pseudo-decomposition $\mathcal{H}_i \cup \mathcal{I}_i$ has the desired properties. In particular, $\mathcal{H}_m$ is m-good and $\mathcal{H}_m \cup \mathcal{I}_m$ is a T-decomposition of G. $\hfill\square$

2.5   Trees of small diameter

The previous sections gave a complete proof of Theorem 2.1.1. In certain cases however, the proof simplifies substantially if we restrict the structure of T. For example, if T has diameter at most 3, then every T-pseudo-decomposition of a bipartite graph G is also a T-decomposition of G. Thus, it is in this case sufficient to construct a T-equitable colouring of G. Also if the diameter of T is 4, we can avoid using the probabilistic tools as in Section 2.4 by giving a much simpler argument instead. Moreover, we take a closer look at the value we obtain for k_T for these trees of small diameter.

2.5.1   Trees of diameter 3

Let T be a tree of diameter 3. As before, it is sufficient to consider the case where the graph G we want to decompose is bipartite. In particular, we can assume that the girth of G is at least 4. Thus, every T-pseudo-decomposition of G is also a T-decomposition of G, and so Theorem 2.1.8 immediately implies the Barát-Thomassen Conjecture in this case.

Let $S(k, \ell)$ denote the bistar with two adjacent vertices of degree k and l respectively, and all other vertices having degree 1. Every tree of diameter 3 is isomorphic to a bistar $S(k, \ell)$ for some natural numbers k and ℓ with $1 < k \leq \ell$. The following proposition is very similar to Proposition 2.3.3.

PROPOSITION 2.5.1 *Let k and ℓ be natural numbers with $1 < k \leq \ell$, and let*

$m = k + \ell - 1$. *Assume G is a bipartite graph on partition classes A and B where all vertices in A have degree divisible by m. If G has $3\ell \lceil \frac{2m}{k-1} \rceil$ edge-disjoint spanning trees, then G has a decomposition into two graphs G_1 and G_2 such that*

- $d(v, G_1) = \frac{k-1}{m} d(v, G)$ *for $v \in A$, and*

- $d(v, G_2)$ *is divisible by ℓ for $v \in B$.*

PROOF. By Lemma 2.2.8, we can find a spanning 3ℓ-edge-connected subgraph G' with $d(v, G') < \frac{2(k-1)}{m} d(v, G)$. Since $2(k-1) < m$, we can add some edges of G to G' to get a graph $G'' \subseteq G$ in which every vertex $v \in A$ has degree precisely $\frac{2(k-1)}{m} d(v, G)$. Let $p : B \to \mathbb{Z}$ be the function defined by $p(v) = d(v, G)$. Observe that

$$\sum_{v \in B} p(v) \equiv e(G)$$

$$\equiv e(G) - \frac{\ell}{m} e(G)$$

$$\equiv \frac{m - \ell}{m} e(G)$$

$$\equiv \frac{k - 1}{m} e(G)$$

$$\equiv \frac{e(G'')}{2} \pmod{\ell},$$

so we can apply Lemma 2.3.2 with the function p. The resulting subgraph G_1 of G'' satisfies $d(v, G_1) = \frac{1}{2} d(v, G'') = \frac{k-1}{m} d(v, G)$ for $v \in A$, and $d(v, G_1) \equiv p(v)$ modulo ℓ for $v \in B$. Let G_2 denote the graph $G - E(G_1)$, then $d(v, G_2) = d(v, G) - d(v, G_1) \equiv 0$ modulo ℓ for $v \in B$, so the graphs G_1 and G_2 are as desired. $\qquad\square$

Given a decomposition of a graph G into graphs G_1 and G_2 as above, we immediately get an $S(k, \ell)$-decomposition by the same arguments as in Section 2.3: We edge-colour G_2 with ℓ colours so that every vertex has the same degree in each colour, and we edge-colour G_1 with $k - 1$ different colours so that every

vertex in A has the same degree in all $k + \ell - 1$ colours. Now we get an $S(k, \ell)$-decomposition by Lemma 2.1.7, where the vertices of degree k lie in A and the vertices of degree ℓ lie in B.

Thomassen [Tho13a] proved that every $180k^4$-edge-connected bipartite graph with size divisible by $2k$ has an $S(k, k+1)$-decomposition. Combining Proposition 2.5.1 with Proposition 2.2.2, we get the following stronger result.

THEOREM 2.5.2 *Let k and ℓ be natural numbers with $1 < k \leq \ell$, and let $m = k + \ell - 1$. Every $(12\ell\lceil \frac{2m}{k-1} \rceil + 6m - 4)$-edge-connected bipartite graph with size divisible by m has an $S(k, \ell)$-decomposition.*
In particular, every $(72k+164)$-edge-connected bipartite graph with size divisible by $2k$ has an $S(k, k+1)$-decomposition.

PROOF. By Theorem 2.2.4, G contains $(6\ell\lceil \frac{2m}{k-1} \rceil + 3m - 2)$ edge-disjoint spanning trees. By Proposition 2.2.2, G can be decomposed into two graphs G_1 and G_2 satisfying the conditions of Proposition 2.5.1. This yields an $S(k, \ell)$-decomposition as described above.

To see that the second part of the statement holds, note that for $\ell = k + 1$ we have

$$12\ell\left\lceil \frac{2m}{k-1} \right\rceil + 6m - 4 = 12(k+1)\left(4 + \left\lceil \frac{4}{k-1} \right\rceil\right) + 12k - 4$$

$$= 60k + 44 + 12(k+1)\left\lceil \frac{4}{k-1} \right\rceil$$

$$= 72k + 56$$

for $k \geq 5$. It is easy to check that $(k+1)\left\lceil \frac{4}{k-1} \right\rceil \leq k + 10$ holds for $k \in \{2, 3, 4\}$, resulting in the general bound $12\ell\left\lceil \frac{2m}{k-1} \right\rceil + 6m - 4 \leq 72k + 164$.   □

As an application, Thomassen showed that every $784k^4$-edge-connected graph with size divisible by $2k$ has a $S(k, k+1)$-decomposition. Combining Theorem 2.5.2 with Theorem 2.2.9, we get the following more general result.

THEOREM 2.5.3 *Let k and ℓ be natural numbers with $1 < k \leq \ell$, and let $m = k + \ell - 1$. Every $112m^2$-edge-connected graph of size divisible by m has an $S(k, \ell)$-decomposition.*

For the proof, it suffices to see that $112m^2 \geq 4k' + 16(m^2 + m)$, where $k' = 12(m - k + 1)\left\lceil\frac{2m}{k-1}\right\rceil + 6m - 4$.

For $k = \ell = 2$, the bistar $S(k, \ell)$ is a path of length 3. This special case was investigated by Thomassen [Tho08a], who showed that every 171-edge-connected graph with size divisible by 3 admits a P_4-decomposition. In the proof it was shown that every 2-edge-connected bipartite graph where all vertices on one side have degree divisible by 3 admits a decomposition into paths of length 3. Note that for m odd $3m - 3 + 2\ell$ edge-disjoint spanning trees suffice in Proposition 2.2.2, so every bipartite graph with 10 edge-disjoint spanning trees has a P_4-decomposition. Replacing this part in Thomassen's proof, we get that every 63-edge-connected graph with size divisible by 3 can be decomposed into paths of length 3.

THEOREM 2.5.4 *If G is a 63-edge-connected graph with size divisible by 3, then G admits a P_4-decomposition.*

2.5.2 Trees of diameter 4

Let T be a tree of diameter 4. We may assume that the graph G we want to decompose is bipartite and thus has girth at least 4. Given a T-pseudo-decomposition, the only difference to a T-decomposition is that some pseudo-copies of T could contain 4-cycles. To take care of this, we start with a T-pseudo-decomposition and try to improve it by switching leaf edges between different pseudo-copies. It is essential that we have large degree in every colour of the T-equitable colouring, so that we have enough freedom to make switches. This method can be used whenever the girth of G is at least the diameter of T. Before

we see how this strategy works in a general setting, we investigate the path of length 4. Notice that the minimum degree condition in the next proposition cannot be omitted, since a cycle of length 4 satisfies all other conditions.

PROPOSITION 2.5.5 *Let G be a bipartite graph on partition classes A and B with size divisible by 4, where the vertices in A have even degree.*

If G is 2-edge-connected, then G has a P_5-pseudo-decomposition.

If G is 2-edge-connected and the vertices in A have minimum degree 4, then G has a P_5-decomposition.

PROOF. We lift the vertices in A in such a way that the resulting graph G' is still 2-edge-connected. Since G' is connected and has an even number of edges, it is possible to orient its edges so that every vertex has even outdegree. Indeed, it suffices to see that G' can be decomposed into paths of length 2, and one can orient each such path so that its central vertex has outdegree 2. Every directed edge in G' corresponds to a directed path of length 2 in G. We colour the first edge of each of these directed paths in G red and the second edge blue. Now every vertex in A has the same degree in red and blue, and the vertices in B have even degree in red.

We first pair up the red edges at every vertex in B arbitrarily, these will be the two middle edges of the paths of length 4. For each red path of length 2, we need to add a blue edge to each of its ends. Since the vertices in A have the same degree in red and blue, we can find a pairing up of the blue edges and the ends of the red paths resulting in a P_5-pseudo-decomposition. This proves the first part of the proposition, so we may now assume that the vertices in A have minimum degree $2d$ for some $d \geq 2$.

Let x be a vertex in B. We say that a pseudo-copy of P_5 has a *conflict* at x, if x is incident with both blue edges of that copy. We pair each red edge with a blue edge such that the number of conflicts, and thus the number of 4-cycles, is

minimal.

Suppose there is a conflict at some vertex x in B. Consider the directed graph $D(x)$ where the vertices are the pseudo-copies of P_5 in our decomposition. For two pseudo-copies T_1 and T_2, we add an edge oriented from T_1 to T_2 in $D(x)$ for every $a \in A$ such that ax is a blue edge of T_1, and there is a vertex $b \in B$ for which ab is a blue edge of T_2. The idea is that it is then possible to switch the blue edge ax of T_1 with the blue edge ab of T_2, obtaining T_1' and T_2', so that T_1' has no conflict at x. Notice that such a switch might create a new conflict at x, but not at any other vertex (possibly T_2' might have a conflict at x).

In $D(x)$, each vertex has either outdegree 0 (if x is not a leaf in the pseudo-copy), or it has at least outdegree $d-1$. Notice that every vertex with positive outdegree has indegree at most 1, since the corresponding pseudo-copy has at most one blue edge not incident with x, say ab with $a \in A$, $b \in B$, and there is at most one pseudo-copy in which ax is a blue edge.

Since we assumed there is a conflict at x, there is a vertex v in $D(x)$ with outdegree at least $2(d-1)$ and indegree 0. Let X be the set of vertices we can reach from v via a directed path, including v. Suppose every vertex in X has positive outdegree, then the subgraph induced by X contains at least $(|X|+1)(d-1)$ edges. However, it can contain at most $|X|-1$ edges, since every vertex has indegree at most 1, and v has indegree 0. Thus, there is a directed path in $D(x)$ from v to a vertex of outdegree 0, and making the switches corresponding to the edges on this path reduces the number of conflicts by 1, contradicting our assumption. $\square$

Thomassen [Tho08b] showed that every $10^{10^{10^{14}}}$-edge-connected graph of size divisible by 4 has a decomposition into paths of length 4. Using the proposition above, this bound on the edge-connectivity can be significantly improved. By Proposition 2.2.2, every bipartite graph with 14 edge-disjoint spanning trees and size divisible by 4 can be decomposed into two graphs satisfying the conditions

of Proposition 2.5.5. Combining this with the first part of Thomassen's proof we can conclude that every 107-edge-connected graph of size divisible by 4 has a P_5-decomposition.

THEOREM 2.5.6 *If G is a 107-edge-connected graph of size divisible by 4, then G admits a P_5-decomposition.*

We conclude this section by showing that we can avoid probabilistic tools also in the more general setting where the girth of G is at least the diameter of T.

THEOREM 2.5.7 *Let T be a tree of size m and diameter d, and let G be a bipartite graph on partition classes A and B in which all vertices in A have degree divisible by m. If G is $f(m, 2m)$-edge-connected and has girth at least d, then G has a T-decomposition, where f is the function defined by Theorem 2.3.1.*

PROOF. If d is odd, then G has girth at least $d + 1$, so the conclusion follows from Theorem 2.1.8 (even $f(m, 1)$-edge-connectivity suffices in this case). Thus, we may assume that d is even. Let T_A and T_B be the two partition classes defined by a proper 2-colouring of T. We may assume that T_B contains the ends of every longest path in T, since d is even. We colour the edges of T that are incident with leaves in T_B blue, and the remaining edges red.

Let λ be a natural number with $\lambda \geq 2m$, and assume G is $f(m, \lambda)$-edge-connected. We proceed as in the proof of Theorem 2.1.8. We use Theorem 2.3.1 and Lemma 2.1.7 to get a T-pseudo-decomposition, where all vertices in T_A correspond to vertices in A in the pseudo-copies. We colour the edges of G red and blue according to the colour of the edge they correspond to in T. Notice that by the proof of Theorem 2.1.8, the subgraph G_{b+1} in Theorem 2.3.1 corresponds precisely to the edges coloured blue in G, so every vertex in A is incident with at least λ blue edges.

Since G has girth d, the only way a pseudo-copy can fail to be an isomorphic copy of T is if it contains a cycle of length d or, equivalently, two blue edges

intersecting at a vertex in B. As in the previous proof, we shall repair this by switching one of the blue edges with a blue edge from another pseudo-copy. We are not going to make any changes to the red edges, every red part of a pseudo-copy in the T-pseudo-decomposition will be the red part of an isomorphic copy in the T-decomposition.

For $x \in B$, a *conflict* at x is a pair of blue edges contained in the same pseudo-copy of T such that both of them are incident with x. Notice that one pseudo-copy may have several conflicts at x. Out of all T-pseudo-decompositions we can get by switching blue edges between copies of our original T-pseudo-decomposition, we choose one for which the number of conflicts is minimal.

Suppose there is a conflict at some vertex $x \in B$. Consider the directed graph $D(x)$ where the vertices are the pseudo-copies of T in the T-pseudo-decomposition. For two pseudo-copies T_1 and T_2, we add an edge oriented from T_1 to T_2 in $D(x)$ for every $a \in A$, $b \in B - V(T_1)$ such that ax is a blue edge of T_1 and ab is a blue edge of T_2. Again, the idea is to switch the blue edge ax of T_1 with the blue edge ab of T_2 to decrease the number of occurences of x in T_1. Notice that such a switch might create a new conflict at x, but since b is not contained in T_1 it will not create a conflict at any other vertex. Since less than m of the blue edges at a are incident with another vertex of T_1, there are at least $\lambda - m$ blue edges we can choose for the switch. In particular, every vertex of positive outdegree in $D(x)$ has outdegree at least $\lambda - m$.

Let v be a vertex of $D(x)$ corresponding to a pseudo-copy containing a conflict at x, so v has outdegree at least $2(\lambda - m)$. Let X denote the set of vertices in $D(x)$ we can reach from v via a directed path, including v. If every vertex in X has positive outdegree, then the subgraph induced by X has more than $(\lambda - m)|X|$ edges. However, every vertex of $D(x)$ has indegree at most ℓ, where ℓ denotes the number of blue edges of T. Thus, the graph induced by $|X|$ has less than $\ell|X|$ edges, which is at most $(\lambda - m)|X|$ for $\lambda \geq 2m$. This shows that there must be a vertex u of outdegree 0 in X. Now making the switches corresponding to

the edges of the directed path from v to u results in a T-pseudo-decomposition with fewer conflicts, contradicting our assumption. $\square$

2.6   Extensions to infinite graphs

As before, let T be a tree of size m. In the following, k_T denotes the smallest natural number such that every k_T-edge-connected (finite) graph of size divisible by m has a T-decomposition. The existence of k_T is guaranteed by Theorem 2.1.1. We conjecture that an analogous version of the Barát-Thomassen Conjecture also holds for infinite graphs.

CONJECTURE 2.6.1 *For any tree T on m edges, there exists an integer k_T^* such that every k_T^*-edge-connected graph infinite graph has a T-decomposition.*

Here we prove that Conjecture 2.6.1 holds at least in a slightly weaker sense by proving the existence of T-pseudo-decompositions.

THEOREM 2.6.2 *Every $(k_T + m^2 - m)$-edge-connected infinite graph G has a T-pseudo-decomposition.*

We start by proving Theorem 2.6.2 for locally finite graphs. In this case we can even show the existence of T-decompositions by a standard argument using Königs infinity lemma.

LEMMA 2.6.3 *Every $(k_T + m - 1)$-edge-connected locally finite graph G has a T-decomposition.*

PROOF. Pick a vertex v and denote the set vertices of distance i from v by D_i. Let G_k denote the graph where all vertices of distance greater than k from v are contracted into a single vertex v_k. Let H_k denote the subgraph of G_k

we get by deleting v_k, i.e. the graph induced by v and its distance classes $D_1, \ldots, D_k$. Since contraction preserves edge-connectivity, we have that G_k is still $(k_T + m - 1)$-edge-connected. Note that G_k is a finite graph, so the number of edges might not be divisible by m. By deleting at most $m - 1$ of the edges incident with v_k, we get a graph G'_k with size divisible by m, and which is still k_T-edge-connected. We can decompose G'_k into copies of T, which results in a *near-decomposition* of H_k. By this we mean that H_k is decomposed into copies of T and some copies of proper subgraphs of T which intersect D_k. Let $\mathcal{N}(H_k)$ denote the set of all near-decompositions of H_k.

We consider an auxiliary graph H whose vertex set is the union of all $\mathcal{N}(H_k)$. In H, for every i, we have an edge between a member of $\mathcal{N}(H_{i-1})$ and a member of $\mathcal{N}(H_i)$ if and only if the deletion of D_i in the latter induces the former, and there are no other edges in H. Since G is locally finite, we have that each $\mathcal{N}(H_i)$ is finite. Clearly each vertex in $\mathcal{N}(H_i)$ has a neighbour in $\mathcal{N}(H_{i-1})$, so by König's infinity lemma there exists a ray $uu_1u_2\ldots$ in H with $u_i \in \mathcal{N}(H_i)$ for all i. This ray corresponds to a T-decomposition of G. □

To extend this result to all infinite graphs, we would like to split some vertices so that the resulting graph still has large edge-connectivity and is locally finite. A *splitting* of a graph G is a graph G' which can be obtained from G by replacing some vertices of G by independent sets of vertices. Each vertex v gets replaced by a set of vertices V_v, each edge $uv \in E(G)$ gets replaced by precisely one edge $u'v' \in E(G')$ with $u' \in V_u$, $v' \in V_v$, and there are no other edges in G'. In particular, if V_v is contracted into a single vertex for every $v \in V(G')$, then the resulting graph is G. If only one vertex v gets replaced by a set V_v then we also call this operation a *splitting of v*.

Note that if G' is a splitting of G, then every T-pseudo-decomposition corresponds to a T-pseudo-decomposition of G in the canonical way. The following theorem was proved by Thomassen (see Theorem 9 in [Tho]) and is a useful tool

in reducing the problem to locally finite graphs.

THEOREM 2.6.4 *Let k be a natural number and G be a countably infinite graph. If G is k-edge-connected, then G has a splitting such that the resulting graph is k-edge-connected, and each block of the resulting graph is locally finite.*

Theorem 2.6.4 reduces the problem of finding a T-pseudo-decomposition to graphs in which all vertices of infinite degree are cutvertices. We are going to split the graph further so that every connected component contains at most one cutvertex.

DEFINITION 2.6.5 We call a connected countably infinite graph G a *bad star*, if G has a cutvertex v of infinite degree such that all components of $G - v$ are finite.

We call these graphs bad stars since we do not know how to make them locally finite. However, as it turns out bad stars are not that bad at all, as we can find a T-pseudo-decomposition without too much effort.

LEMMA 2.6.6 *Every $(k_T + m^2 - m)$-edge-connected bad star G has a T-pseudo-decomposition.*

PROOF. We denote the components of $G - v$ by $H_1, H_2, \ldots$ and let G_i be the subgraph of G induced by H_i and v. We denote the size of G_i by m_i. For every natural number i, choose $k_i \in \{0, 1, \ldots, m-1\}$ to be congruent to $m_1 + \ldots + m_i$ modulo m. If $k_i \neq 0$, then we delete k_i edge-disjoint pseudo-copies of T in $G_i \cup G_{i+1}$, each having exactly one edge in G_i and the remaining $m-1$ edges in G_{i+1}. This is possible since the edge-connectivity of G_i is large enough. Notice that after having done this for every natural number i, we have deleted at most $m(m-1)$ of the edges in G_i. We denote the resulting graphs by G_i'. Since the edge-connectivity of every G_i is at least $k_T + m^2 - m$, the graphs G_i' are still

k_T-edge-connected. Furthermore, by the choice of k_i, every G_i' has size divisible by m and can thus be decomposed into pseudo-copies of T. $\qquad\square$

Notice that k_T-edge-connectivity might not suffice in Lemma 2.6.6, since dividing all but one edge of the infinite star results in a connected graph that cannot be decomposed into paths of length 2. To extend the decomposition result to countably infinite graphs, we show that every countable graph can be split into infinite locally finite graphs and bad stars – both of which we can decompose.

THEOREM 2.6.7 *Let k be a natural number and G be a countably infinite graph. If G is k-edge-connected, then G has a splitting such that every component of the resulting graph is infinite, k-edge-connected and either locally finite or a bad star.*

PROOF. By Theorem 2.6.4, there exists a splitting of G such that the resulting graph G' is still k-edge-connected and each block is locally finite. Notice that the only vertices of infinite degree in G' are cutvertices, so it suffices to split these cutvertices. We enumerate the cutvertices of infinite degree $v_1, v_2, \ldots$, and we split them in this order. Let $G_1 = G$, and for $i = 2, 3, \ldots$, let G_i be the connected component containing v_i after the splitting of vertex v_{i-1}. For $i = 1, 2, \ldots$, we have that $G_i - v_i$ has infinitely many components or at least one infinite component. We do the vertex splitting so that for every infinite component C of $G_i - v_i$ we have a new vertex $v_{i,C}$ joined to the neighbours of v_i in C. Since all blocks are locally finite, all vertices $v_{i,C}$ have finite degree. If $G_i - v_i$ has only finitely many finite components, then there exists an infinite component, so pick one of the vertices $v_{i,C}$ and join it to all the neighbours of v_i in the finite components. On the other hand, if $G_i - v_i$ has infinitely many finite components, then we add a new vertex $v_{i,*}$ and join it to all the neighbours of v_i in the finite components. That way, we split v_i into vertices of finite degree, and possibly one vertex of infinite degree which is then a cutvertex of a bad star.

Let G'' denote the resulting graph after all the vertex splittings. Since we never create finite components with our vertex splittings, all components of G'' are infinite. Since all blocks of G' are k-edge-connected and the splittings do not affect the blocks, the components of G'' are still k-edge-connected. Now suppose there exists a vertex v of infinite degree in G''. This vertex comes from the splitting of a cutvertex of infinite degree in G', but then it is the cutvertex of a bad star. Thus, all components of G'' are either locally finite or bad stars. □

By putting the pieces together we can extend our decomposition result to all countably infinite graphs.

COROLLARY 2.6.8 *Every* $(k_T + m^2 - m)$-*edge-connected countably infinite graph* G *has a* T-*pseudo-decomposition.*

PROOF. Let G' denote the graph we get by splitting the vertices as in Theorem 2.6.7. By Lemma 2.6.3 and Lemma 2.6.6, the graph G' admits a T-pseudo-decomposition. This corresponds to a T-pseudo-decomposition of the original graph, since identifications of vertices preserve the pseudo-copies of T. □

The general case now follows immediately by a decomposition result due to Laviolette [Lav05].

PROOF OF THEOREM 2.6.2. By Theorem 3 in [Lav05], every k-edge-connected uncountably infinite graph can be decomposed into k-edge-connected countable graphs. By Corollary 2.6.8, each of these admits a T-pseudo-decomposition and together they form a T-pseudo-decomposition of the whole graph. □

Decomposing into few forests with trees of small diameter

The results of this chapter were obtained by the author in joint work with Postle [MP].

3.1 Conjectures and results

Let $\mathcal{F}_d$ denote the class of all forests in which each tree has diameter at most d.

DEFINITION 3.1.1 The smallest number of parts in an $\mathcal{F}_d$-decomposition of a graph G is called the *diameter-d arboricity* of G and denoted by $\Upsilon_d(G)$.

Notice that $\Upsilon_1(G)$ is the chromatic index of G while $\Upsilon_2(G)$ is the star arboricity of G. If d is large enough, for example greater than the size of G, then the diameter-d arboricity of G is the same as the usual arboricity.

While we are interested in what diameters can be obtained, we are in general more interested in which graphs have any bound on the diameter. Such a notion though only makes sense when referring to graph classes, for example planar graphs or graphs of arboricity at most k.

DEFINITION 3.1.2 Let $\mathcal{G}$ be a family of graphs. The *bounded diameter arboricity* of $\mathcal{G}$, denoted by $\Upsilon_{bd}(\mathcal{G})$, is the smallest number k for which there exists a number d such that $\Upsilon_d(G) \leq k$ for every $G \in \mathcal{G}$.

Let $\mathcal{A}_k$ denote the class of graphs with arboricity at most k. Clearly $\Upsilon_{bd}(\mathcal{A}_k) \leq \Upsilon_2(\mathcal{A}_k) \leq 2k$ since every forest can be partitioned into two star forests. Similarly $\Upsilon_{bd}(\mathcal{A}_k)$ is strictly greater than k. To see this note that a graph which is the union of k spanning trees if decomposed into k forests must necessarily be decomposed into k spanning trees. Since there exists graphs of arbitrarily large diameter which are the union of k spanning trees it follows that every such decomposition has a forest with a component of large diameter. But is it possible that by allowing a few more forests we can in fact obtain components of bounded diameter? We make the following very strong conjecture.

CONJECTURE 3.1.3 *The class of graphs with arboricity at most k has bounded diameter arboricity $k + 1$, i.e. $\Upsilon_{bd}(\mathcal{A}_k) = k + 1$.*

To tackle this conjecture, we show that the union of a forest and a star forest can be decomposed into two forests with small diameter trees.

THEOREM 3.1.4 *If G is the union of a forest and a star forest, then $\Upsilon_{18}(G) \leq 2$.*

We postpone the proof of Theorem 3.1.4 to Section 3.2. The following is an easy application of it.

COROLLARY 3.1.5 *If* $\Upsilon(G) \leq 2$, *then* $\Upsilon_{18}(G) \leq 3$. *If* $\Upsilon(G) \leq 3$, *then* $\Upsilon_{18}(G) \leq 4$. *In particular,* $\Upsilon_{bd}(\mathcal{A}_2) = 3$ *and* $\Upsilon_{bd}(\mathcal{A}_3) = 4$.

PROOF. If G is the union of two forests F_1 and F_2, then we decompose the edges of F_2 into two star forests S_1 and S_2 and apply Theorem 3.1.4 to the union of F_1 and S_1.

If G is the union of three forests F_1, F_2 and F_3, then we decompose the edges of F_3 into two star forests S_1 and S_2. We apply Theorem 3.1.4 to the union of F_1 and S_1, and separately to the union of F_2 and S_2. □

Notice that this implies Conjecture 3.1.3 for $k = 2$ and $k = 3$. Moreover, Corollary 3.1.5 can be used to improve the general upper bound $\Upsilon_{bd}(\mathcal{A}_k) \leq 2k$.

COROLLARY 3.1.6 $\Upsilon_{bd}(\mathcal{A}_k) \leq \lceil \frac{4}{3}k \rceil$.

PROOF. By Corollary 3.1.5 there exists a natural number d such that every graph G with $\Upsilon(G) \leq 3$ satisfies $\Upsilon_d(G) \leq \Upsilon(G) + 1$. If $\Upsilon(G) = k$, then G can be written as the union of $\ell = \lceil \frac{k}{3} \rceil$ graphs $G_1, \ldots, G_\ell$ with $\Upsilon(G_i) = 3$ for $i \in \{1, \ldots, \ell - 1\}$ and $\Upsilon(G_\ell) = k - 3(\ell - 1)$. Now

$$\Upsilon_d(G) \leq \Upsilon_d(G_1) + \ldots + \Upsilon_d(G_\ell) \leq 4(\ell - 1) + k - 3(\ell - 1) + 1 = \left\lceil \frac{4k}{3} \right\rceil ,$$

and thus $\Upsilon_{bd}(\mathcal{A}_k) \leq \lceil \frac{4}{3}k \rceil$. □

For the general problem, Theorem 3.1.4 suggests a strategy for proving Conjecture 3.1.3. We conjecture the following generalisation of Theorem 3.1.4.

CONJECTURE 3.1.7 *For all natural numbers* $d \geq 1$, *there exists a natural number* $f(d)$ *such that the following holds: If* G *is the union of a forest and a*

*second forest whose components have diameter at most d, then G can be parti-
tioned into two forests each of whose components have diameter at most f(d).*

Theorem 3.1.4 confirms this conjecture when $d \leq 2$ with $f(2) \leq 18$.

Perhaps an even stronger variant of Conjecture 3.1.3 holds, in which we consider
the fractional arboricity instead. The *fractional arboricity* $\Upsilon_f(G)$ is defined as
$\max_{H \subseteq G} \frac{|E(H)|}{|V(H)|-1}$. Note that $\lceil \Upsilon_f(G) \rceil = \Upsilon(G)$ by Nash-Williams' result. A ma-
jor open question is whether the structure of the forests can be restricted when
the fractional arboricity is strictly smaller (asymptotically) than the arboricity.
In particular, Montassier et al. [MdMRZ12] formulated the Nine Dragon Tree
Conjecture as follows.

CONJECTURE 3.1.8 (NINE DRAGON TREE CONJECTURE) *Let G be a
graph and k, d be natural numbers with k, d ≥ 1. If $\Upsilon_f(G) \leq k + \frac{d}{k+d+1}$, then G
can be decomposed into k + 1 forests at least one of which has maximum degree
d.*

They proved Conjecture 3.1.8 for $k = 1$ and $d \leq 2$. Kim et al. [KKW$^+$13] proved
the conjecture for $k = 1$ and $d \leq 6$. The *Strong Nine Dragon Tree Conjecture*
states that for such graphs at least one of the forests in the decomposition has
components of size at most d (and hence diameter at most d as well). In light
of Conjecture 3.1.7 and the Strong Nine Dragon Tree Conjecture, we also make
the following strong conjecture.

CONJECTURE 3.1.9 *For every natural number k and real number ε > 0, there
exists d(k, ε) such that the following holds: If $\Upsilon_f(G) \leq k - \varepsilon$ for a graph G, then
$\Upsilon_{d(k,\varepsilon)}(G) \leq k$.*

Originally, our introduction of bounded diameter arboricity was motivated by
a question on the existence of thin subgraphs in highly edge-connected graphs.
Given a graph G and a set of vertices $A \subseteq V(G)$, we denote by $\sigma_G(A)$ the set

of edges of the form $\{ab \in E(G) : a \in A, b \notin A\}$. We call $\sigma_G(A)$ the *boundary* of A in G.

DEFINITION 3.1.10 Let ε be a real number with $0 < \varepsilon < 1$. We say a spanning subgraph H of a graph G is ε-*thin* if for every $A \subseteq V(G)$ we have $|\sigma_H(A)| \leq \varepsilon|\sigma_G(A)|$.

Of particular interest is the existence of ε-thin spanning trees. Goddyn [God04] made the following conjecture which is still wide open.

CONJECTURE 3.1.11 *For every ε with $0 < \varepsilon < 1$ there exists a number $f(\varepsilon)$ such that every $f(\varepsilon)$-edge-connected graph contains an ε-thin spanning tree.*

This conjecture would imply a qualitative version of the $(2 + \varepsilon)$-flow conjecture by Goddyn and Seymour, resulting in a proof different to the one found by Thomassen [Tho12].

In Section 3.3 we show how the bounded diameter arboricity for planar graphs of a certain girth has implications for the existence of ε-thin spanning trees in highly edge-connected planar graphs. In particular, we prove that every 6-edge-connected planar graph contains two edge-disjoint $\frac{18}{19}$-thin spanning trees.

3.2   Forest plus star forest

In this section we show that every simple graph G which is the union of a forest and a star forest can be decomposed into two forests in which every tree has diameter at most 18. Note that our proof also works if we allow G to be infinite.

An *out tree* is a rooted tree in which every edge is oriented away from the root. An *out star forest* is a directed forest in which every component is a star and the edges of every star are oriented from the center to the leaves. If a star has

size 1, then we arbitrarily choose one of the two vertices as the center and orient the edge away from it.

DEFINITION 3.2.1 An *outing* $G = (S, T)$ is the union of an out star forest S and an out tree T. We let $C(S)$ denote the set of centers of the star forest S and $L(S)$ denote the set of leaves of S.

Given an outing G, our goal is to construct a 2-edge-colouring of G such that there are no monochromatic cycles and no long monochromatic paths. Notice that in an outing every vertex has indegree at most 2. The first important property of the colouring we construct is that every vertex has indegree at most 1 in each colour. In such a colouring every monochromatic cycle is directed and every monochromatic path is the union of at most two directed paths which we call *dipaths* for brevity.

Ideally we would like to start with an edge-colouring of S in which every star is monochromatic and extend this colouring to all edges of G. Unfortunately, this additional constraint is too strong: If a monochromatic star has d leaves which form a path in T, then colouring the edges of this path with the alternate colour is necessary to avoid monochromatic triangles. Doing so would create a long monochromatic path in T. To avoid this problem, we allow some star edges to have a different colour. For technical reasons, we encode the colouring of the stars in a 2-colouring of the vertices of G. The colour of the center vertex is the colour assigned to the star, while the colour of a leaf shows how the edge is coloured. Note that vertex-colourings in this section are not necessarily proper.

DEFINITION 3.2.2 Let c be a vertex 2-colouring of an outing $G = (S, T)$. We say that an edge $\overrightarrow{uv} \in E(S)$ is *rebellious* if $c(u) \neq c(v)$. We also call $v \in V(G)$ *rebellious* if it is the head of a rebellious edge.

We are mainly concerned with colourings where the rebellious vertices behave nicely with respect to T in the following sense.

Definition 3.2.3 Let c be a vertex 2-colouring of an outing $G = (S,T)$. We say that c is *tame* if for every edge $\overrightarrow{uv} \in E(T)$ where v is rebellious, we have $c(u) \neq c(v)$ and u is not rebellious.

In particular, it follows that if the 2-colouring is tame then two rebellious vertices are never joined by an edge in T. Notice that in a tame 2-colouring it is possible that all edges of a star are rebellious.

Given a 2-vertex-colouring of an outing $G = (S,T)$, we now define a 2-edge-colouring of G as follows.

Definition 3.2.4 Let $c : V(G) \to \{1,2\}$ be a vertex 2-colouring of an outing $G = (S,T)$. The *extension* of c, denoted by $\text{Ext}(c)$, is the 2-edge-colouring $c' : E(G) \to \{1,2\}$ where:

1. For all edges $\overrightarrow{uv} \in E(S)$, we have $c'(\overrightarrow{uv}) = c(v)$.

2. For all edges $\overrightarrow{uv} \in E(T)$, we have

$$c'(\overrightarrow{uv}) = \begin{cases} c(v) & \text{if } v \in C(S), \, c(u) = c(v) \text{ and } u \text{ is not rebellious,} \\ 3 - c(v) & \text{otherwise.} \end{cases}$$

Notice that in the $\text{Ext}(c)$-colouring of G, every vertex $v \in V(G)$ has indegree at most 1 in each colour. This implies that each monochromatic cycle is directed and each monochromatic path is the union of two directed paths.

Definition 3.2.5 The *center graph* $\text{Center}(G)$ of an outing $G = (S,T)$ is a directed graph whose vertex set is $C(S)$ and for every $u, v \in C(S)$ with $u \neq v$, there is an edge $\overrightarrow{uv}$ if $\overrightarrow{uv} \in E(T)$ or if there exists a vertex $w \in L(S)$ such that $\overrightarrow{uw} \in E(S)$ and $\overrightarrow{wv} \in E(T)$.

Each vertex in $\text{Center}(G)$ has indegree at most 1. In particular, each cycle in $\text{Center}(G)$ is directed and each connected component contains at most one cycle.

Given a colouring of the vertices of G, this also corresponds to a colouring of Center(G) in a natural way.

DEFINITION 3.2.6 Let c be a vertex 2-colouring of an outing G. The *center restriction* of c, denoted by Res(c), is the vertex 2-colouring of Center(G) defined by colouring each vertex $v \in V(\text{Center}(G))$ with colour $c(v)$.

Our first lemma characterizes monochromatic paths in Ext(c) where the two endvertices of the path are in $C(S)$ and its interior vertices are in $L(S)$. Note that we phrase the lemma only for monochromatic paths in colour 1, but the analogous statement holds also for paths in colour 2.

LEMMA 3.2.7 *Let $c : V(G) \to \{1,2\}$ be a tame vertex 2-colouring of an outing $G = (S,T)$. Let $P = v_0 v_1 \ldots v_k$ be a dipath in G whose edges are coloured 1 in Ext(c). Suppose $v_0, v_k \in C(S)$ and $c(v_i) \in L(S)$ for $i \in \{1, \ldots, k-1\}$.*

If $c(v_0) = c(v_k)$, then $k \leq 2$ and $\overrightarrow{v_0 v_k} \in E(Center(G))$.

If $c(v_0) \neq c(v_k)$, then $k \leq 3$ and $c(v_0) = 1$, $c(v_k) = 2$.

PROOF. First note that $\overrightarrow{v_i v_{i+1}} \in E(T)$ for $i \in \{1, \ldots, k-1\}$ since $v_i \in L(S)$ for all such i. Now let us suppose $c(v_0) = c(v_k)$ and $k \geq 3$. Since $\overrightarrow{v_{k-2} v_{k-1}} \in E(T)$ and $v_{k-1} \in L(S)$, it follows from the definition of Ext(c) that the colour of $\overrightarrow{v_{k-2} v_{k-1}}$ in Ext(c) (which is 1) equals $3 - c(v_{k-1})$ and hence $c(v_{k-1}) = 2$. If $c(v_k) = 1$, then the edge $\overrightarrow{v_{k-1} v_k}$ would be coloured 2 by the definition of Ext(c), a contradiction. Thus $c(v_k) = 2$ and v_{k-1} is rebellious. Since c is tame, we have $c(v_{k-2}) = 1$ and v_{k-2} is not rebellious. By the definition of Ext(c), it follows that $\overrightarrow{v_{k-3} v_{k-2}} \in E(S)$. Thus, $v_{k-3} \in C(S)$ and $k = 3$. Since $c(v_0) = c(v_k) = 2$ and $c(v_1) = 1$, we have that v_1 is rebellious, a contradiction since v_{k-2} is not rebellious.

Notice that $c(v_0) = c(v_k)$ and $k \leq 2$ implies $\overrightarrow{v_0 v_k} \in E(\text{Center}(G))$ unless $k = 2$ and $\overrightarrow{v_0 v_1}, \overrightarrow{v_1 v_2} \in E(T)$. As before, this case implies $c(v_1) = 2$, $c(v_2) = 2$ and

v_1 is not rebellious. Since c is tame, it follows that $c(v_0) = 1$, contradicting $c(v_0) = c(v_2)$.

Next suppose $c(v_0) = 2$ and $c(v_k) = 1$. By the definition of $\text{Ext}(c)$, we have $c(v_{k-1}) = 1$ and v_{k-1} is not rebellious. It follows that $k \geq 2$. Once again, it follows that $\overrightarrow{v_{k-2}v_{k-1}} \in E(S)$. Thus, $v_{k-2} \in C(S)$ and $k = 2$. Now $c(v_0) = 2$ and $c(v_1) = 1$, so v_1 is rebellious, a contradiction.

Finally, suppose $c(v_0) = 1$, $c(v_k) = 2$ and $k > 3$. Since the edges $\overrightarrow{v_{k-3}v_{k-2}}$ and $\overrightarrow{v_{k-2}v_{k-1}}$ are in $E(T)$ and coloured 1, we have $c(v_{k-2}) = c(v_{k-1}) = 2$. Now v_{k-1} is not rebellious since c is tame, so the edge $\overrightarrow{v_{k-1}v_k}$ received colour 2 in $\text{Ext}(c)$, a contradiction. $\square$

Let c be a vertex-colouring (resp. edge-colouring) of a directed graph G. We say that c is *acyclic* if there exists no directed cycle in G in which all vertices (resp. edges) have the same colour. We want to find a vertex 2-colouring c of G such that $\text{Ext}(c)$ is acyclic. The next lemma shows that this goal is achieved whenever c is tame and the restriction of c is acyclic.

LEMMA 3.2.8 *Let $c : V(G) \to \{1, 2\}$ be a tame vertex 2-colouring of an outing $G = (S, T)$. If Res(c) is acyclic, then also Ext(c) is acyclic.*

PROOF. Suppose not. Let C be a monochromatic cycle in $\text{Ext}(c)$, say in colour 1. We set $C_C = V(C) \cap C(S)$ and $C_L = V(C) \cap L(S)$. Notice that both C_C and C_L are non-empty since C must contain an edge of S as T is a tree. Let $v_0 \in C_C$ and label the remaining vertices in C_C by $v_1, \ldots, v_n$ as they appear in C starting from v_0.

First let us suppose that not all vertices in C_C are coloured the same. Then there exists an $i \in \{0, \ldots n\}$ such that $c(v_i) = 2$ and $c(v_{i+1}) = 1$ (indices are considered modulo $n+1$). Now the directed path from v_i to v_{i+1} on C contradicts Lemma 3.2.7. We may thus assume that all vertices in C_C received the same

colour. By Lemma 3.2.7, the paths between v_i and v_{i+1} on C correspond to edges in Center(G). Thus, the vertices $v_0, \ldots, v_n$ correspond to a monochromatic cycle in Center(G), contradicting that Res(c) is acyclic. $\qquad\qquad\square$

Now we give an upper bound for the length of a monochromatic dipath in Ext(c).

LEMMA 3.2.9 *Let $c : V(G) \to \{1, 2\}$ be a tame vertex 2-colouring of an outing $G = (S, T)$ for which Res(c) is acyclic. Let d_T be the length of a longest vertex-monochromatic dipath in T whose vertices are all in $L(S)$. For $i \in \{1, 2\}$, let d_i be the length of a longest monochromatic dipath in Center(G) whose vertices are coloured i in Res(c). If P is a monochromatic dipath in the Ext(c)-colouring of G, then the length of P is at most $d_T + 2(d_1 + d_2) + 6$.*

PROOF. By Lemma 3.2.8, we know that Ext(c) is acyclic. We may assume that the edges of P are all coloured 1. Let $v_0, v_1 \ldots, v_n$ denote the vertices in $V(P) \cap C(S)$, labelled in the order they appear on P. By Lemma 3.2.7 there exists no $i \in \{0, \ldots, n-1\}$ with $c(v_i) = 2$ and $c(v_{i+1}) = 1$. Thus, there exists $k \in \{0, \ldots, n+1\}$ such that $c(v_i) = 1$ if and only if $i < k$. Notice that by Lemma 3.2.7, the vertices $v_0 v_1 \ldots v_{k-1}$ correspond to a monochromatic path of colour 1 and length $k - 1$ in Center(G), while the vertices $v_k v_{k+1} \ldots v_n$ correspond to a monochromatic path of colour 2 and length $n - k$. By definition of d_1 and d_2 we have $k - 1 \leq d_1$ and $n - k \leq d_2$. By Lemma 3.2.7, there are at most 3 edges on P between v_{k-1} and v_k, and at most 2 edges between v_{i-1} and v_i for every $i \in \{1, \ldots, n\} \setminus \{k\}$. Thus, the number of edges on P between v_0 and v_n is at most $2(k - 1) + 3 + 2(n - k) \leq 2(d_1 + d_2) + 3$.

Let $w_0, \ldots, w_{n'}$ denote the vertices encountered on P after v_n. Then $w_i \in L(S)$ for $i \in \{0, \ldots, n'\}$ and $\overrightarrow{w_i w_{i+1}} \in E(T)$ for $i \in \{0, \ldots, n'-1\}$. Since the edges of P are all coloured 1, we have $c(w_i) = 2$ for $i \in \{1, \ldots, n'\}$. Thus $n' - 1 \leq d_T$, and there are at most $d_T + 2$ edges on P after v_n.

Suppose there are at least 3 edges on P before v_0, say $\overrightarrow{u_0 u_1}$, $\overrightarrow{u_1 u_2}$, and $\overrightarrow{u_2 v_0}$.

Then all these three edges must be in T and $c(u_1) = c(u_2) = 2$. Thus, u_2 is not rebellious, and no matter what the the colour of v_0 is, the edge $\overrightarrow{u_2 v_0}$ is coloured 2 in Ext(c), a contradiction. Suppose there are two edges $\overrightarrow{u_1 u_2}$ and $\overrightarrow{u_2 v_0}$ before v_0. Then $c(u_2) = 2$ and since the edge $\overrightarrow{u_2 v_0}$ is coloured 1, it follows that $c(v_0) = 2$. In this case there are at most $2d_2$ edges between v_0 and v_n, so the length of P is at most $2 + 2d_2 + d_T + 2 < d_T + 2(d_1 + d_2) + 6$. Finally, suppose there is at most one edge preceding v_0 in P. Then the length of P is at most $1 + 2(d_1 + d_2) + 3 + d_T + 2 = d_T + 2(d_1 + d_2) + 6$. □

Finally, all that is left to show is that there exists a vertex 2-colouring of G satisfying the conditions of Lemma 3.2.9.

LEMMA 3.2.10 *Let $G = (S, T)$ be an outing. There exists a tame vertex 2-colouring c of G such that colour class 1 of Res(c) forms an independent set in Center(G), colour class 2 of Res(c) induces no directed path of length 2 in Center(G), and there is no vertex-monochromatic dipath of length 2 in T whose vertices are all in $L(S)$.*

PROOF. We start by colouring the vertices in $C(S)$. If a component of Center(G) is bipartite, then we choose a proper 2-colouring of its vertices. If a component is not bipartite, then it contains precisely one cycle and this cycle has odd length. In this case we delete an edge uv of that cycle and properly 2-colour the resulting tree so that $c(u) = 2$. Now the two colour classes of Res(c) are as desired.

We now extend this colouring to the vertices in $L(S)$. If the root of T is in $L(S)$, colour it arbitrarily. Let v be a vertex at distance i from the root in T and suppose all vertices at distance $i-1$ from the root are already coloured. Let u be the parent of v in T and let w be such that $\overrightarrow{wv} \in E(S)$. We set $c(v) = 3 - c(u)$ unless u is rebellious and $c(u) = c(w)$, in which case we set $c(v) = c(u)$. Notice that if $c(v) = c(u)$, then v is not rebellious. Thus if $c(v) \neq c(u)$ and v is rebellious, then $c(u) = c(w)$; in which case u is not rebellious given how we set

the colour of v. This implies that the resulting colouring c is tame. Furthermore, if $\overrightarrow{uv}$ is an edge with $u, v \in L(S)$ and $c(u) = c(v)$, then u is rebellious while v is not rebellious. It follows immediately that there are no vertex-monochromatic dipaths of length 2 in T whose vertices are in $L(S)$. $\qquad\qquad\square$

Now Theorem 3.1.4 follows easily.

PROOF OF THEOREM 3.1.4. Let G be the union of a forest and a star forest. Now let $G' = (S, T)$ be an outing such that the underlying undirected graph of G' contains G as a subgraph. Let c be a tame vertex 2-colouring of G' as given by Lemma 3.2.10. Let H' be a monochromatic connected subgraph of G' and let H be the underlying undirected graph of H'.

Suppose H contains a cycle C. Since the indegree of every vertex in H' is at most one, the cycle C is directed. By Lemma 3.2.8, there are no monochromatic directed cycles in $\text{Ext}(c)$, a contradiction. So we may assume that H is a tree. By Lemma 3.2.9, the length of a monochromatic dipath in $\text{Ext}(c)$ is at most $1 + 2 \cdot (0 + 1) + 6 = 9$. Thus, every dipath in H' has length at most 9. Since the indegree of every vertex in H' is at most one, every path in H is the union of at most two dipaths in H'. Thus, the diameter of H is at most 18. Hence, c induces a 2-edge-colouring of G in which every connected monochromatic subgraph is a tree with diameter at most 18. $\qquad\qquad\square$

3.3   Planar graphs and ε-thin spanning trees

Thomassen observed that there exists no real number ε with $0 < \varepsilon < 1$ such that every 4-edge-connected planar graph contains an ε-thin spanning tree (personal communication). Here we give a short proof inspired by his argument.

THEOREM 3.3.1 *For every real number ε with $0 < \varepsilon < 1$ there exists a planar*

4-edge-connected graph with no ε-thin spanning tree.

PROOF. We fix ε and set $k > \max\{\lceil \frac{3}{1-\varepsilon} \rceil, 1000\}$. Let G be the cartesian product of a path of length $4k$ and a cycle of length $4k$. The graph G is planar but not 4-edge-connected since there exist $8k$ vertices of degree 3 which lie on two faces each containing $4k$ vertices of degree 3. We add new vertices inside these faces and join each new vertex to 4 vertices of degree 3 so that the resulting graph is planar, 4-regular and 4-edge-connected. Moreover, it is easy to see that the resulting graph G' has the property that every sufficiently large set of vertices has a large neighbourhood. We leave the verification of the following statement to the reader: For every $A \subseteq V(G')$ with $k^2 \leq |A| \leq |V(G')| - k^2$, we have $|\sigma_{G'}(A)| \geq k$.

Suppose for a contradiction that G' has an ε-thin spanning tree T. Since T is ε-thin, the graph $G' - E(T)$ is connected. Let T' be a spanning tree of $G' - E(T)$. Since G' is 4-regular, we have $|E(G') \setminus E(T \cup T')| = 2n - 2(n - 1) = 2$. Let e be an edge of T' such that $T' - e$ has two connected components A and B each having size at least k^2 (such an edge exists since the maximum degree of T' is 4). Thus, $|\sigma_{G'}(A)| \geq k$, but only one of the edges in $\sigma_{G'}(A)$ is contained in T'. Since there exist only two edges in G' outside of T and T', the proportion of $\sigma_{G'}(A)$ contained in T is at least

$$\frac{|\sigma_{G'}(A)| - 3}{|\sigma_{G'}(A)|} \geq \frac{k - 3}{k} = 1 - \frac{3}{k} > \varepsilon,$$

contradicting T being ε-thin. $\qquad\square$

The following lemma shows that bounded diameter arboricity of planar graphs is related to the existence of ε-thin spanning trees. In the following, we denote the dual of a plane graph G by G^*.

LEMMA 3.3.2 *If G is a plane graph with $\Upsilon_d(G) = 2$, then G^* contains two edge-disjoint $\frac{d}{d+1}$-thin spanning trees.*

PROOF. Since $\Upsilon_d(G) = 2$, we can edge-colour G, say in colours 1 and 2, so that there are no monochromatic cycles and every monochromatic path has length at most d. By the usual bijection $E(G) \to E(G^*)$, this gives a 2-edge-colouring of G^*. Consider a set $A \subseteq V(G^*)$. The edges in $\sigma_{G^*}(A)$ correspond to an edge-disjoint union of cycles in G. Consider one such cycle C in the union. Since there are no monochromatic cycles in G, both colours appear in C. Moreover, since every path of length at least $d+1$ contains an edge in colour 1, at least $\frac{1}{d+1}|E(C)|$ edges of C are coloured 1. Thus, at most $\frac{d}{d+1}|\sigma_{G^*}(A)|$ edges of $\sigma_{G^*}(A)$ are coloured 2. Since $\sigma_{G^*}(A)$ also contains at least $\frac{1}{d+1}|\sigma_{G^*}(A)|$ edges in colour 2, the subgraph coloured 2 is both spanning and $\frac{d}{d+1}$-thin. The same holds for the subgraph in colour 1. Since subgraphs of ε-thin graphs are again ε-thin, we can choose one spanning tree of G^* in each colour to finish the proof. $\square$

We should note that planar graphs of various girths have received much attention for star arboricity (their arboricity is at most 3 for all planar graphs, and at most 2 for triangle-free planar graphs by Euler's formula). Thus we wondered what the bounded diameter arboricity of planar graphs of various girths was. Upon studying the problem, we began to conjecture that planar graphs have bounded diameter arboricity at most 4; similarly, we conjectured that planar triangle-free graphs have bounded diameter arboricity at most 3. Indeed, this is what led us to Conjecture 3.1.3. Theorem 3.1.4 has allowed us to prove these conjectures. To see that the bounded diameter arboricity of these classes is greater than the usual arboricity, we use the following lemma.

LEMMA 3.3.3 *Let $\mathcal{G} \subseteq \mathcal{A}_k$ be a family of graphs and c a natural number. If there exists a sequence of graphs $G_1, G_2, \ldots$ in $\mathcal{G}$ such that the diameter of G_i is at least i and $|E(G_i)| \geq k|V(G_i)| - c$ for all i, then $\Upsilon_{bd}(\mathcal{G}) \geq k + 1$.*

PROOF. Suppose $\Upsilon_{bd}(\mathcal{G}) \leq k$, then there exists a natural number d such that $\Upsilon_d(G) \leq k$ for all $G \in \mathcal{G}$. Consider the graph $H = G_{cd+1}$. Let $\mathcal{F} = \{F_1, \ldots, F_k\}$

be a decomposition of H into k forests in which each tree has diameter at most d. For $i \in \{1, \ldots, k\}$, let $\mathcal{T}_i$ denote the connected components of F_i (if a vertex of H is not contained in F_i then we include it in $\mathcal{T}_i$ as an isolated vertex). Now $\mathcal{T} = \bigcup_{i=1}^{k} \mathcal{T}_i$ is a collection of trees decomposing H, each having diameter at most d. Notice that

$$k|V(H)| - c \le |E(H)| = \sum_{i=1}^{k} |E(F_i)| = \sum_{i=1}^{k} |V(H)| - |\mathcal{T}_i| \le k|V(H)| - |\mathcal{T}|,$$

so $|\mathcal{T}| \le c$. Since the diameter of H is at least $cd + 1$, there exists a path P of length at least $cd+1$ in H such that P is a shortest path between its endpoints. Since P contains $cd + 1$ edges and every edge is contained in a tree of $\mathcal{T}$, there exists a tree T in $\mathcal{T}$ containing at least $d + 1$ edges of P. However, since P is a shortest path, this implies that the diameter of T is greater than d, contradicting our choice of $\mathcal{F}$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

For planar graphs of higher girth, we were led to conjecture that planar graphs of girth at least 5 have bounded diameter arboricity at most 2. We were only able to prove this for girth at least 6 and only then by using the result of Kim et al. [KKW$^+$13] that a planar graph of girth at least 6 can be decomposed into a forest and a matching.

THEOREM 3.3.4 *If we let $\mathcal{P}_g$ denote the class of planar graphs of girth at least g, then*

- $\Upsilon_{bd}(\mathcal{P}_3) = 4$,

- $\Upsilon_{bd}(\mathcal{P}_4) = 3$,

- $\Upsilon_{bd}(\mathcal{P}_g) = 2$ *for all $g \ge 6$.*

PROOF. By Euler's formula $\Upsilon(\mathcal{P}_3) = 3$ and hence by Corollary 3.1.5, $\Upsilon_{bd}(\mathcal{P}_3) \le 4$. Since there exist planar triangulations of arbitrary diameter (and hence

$|E(G)| = 3|V(G)| - 6$), it follows from Lemma 3.3.3 that $\Upsilon_{bd}(\mathcal{P}_3) = 4$. Similarly by Euler's formula $\Upsilon(\mathcal{P}_4) = 2$. By Corollary 3.1.5, $\Upsilon(\mathcal{P}_4) \leq 3$. Since there exist triangle-free planar graphs of arbitrary diameter with $|E(G)| = 2|V(G)| - 4$, it follows from Lemma 3.3.3 that $\Upsilon(\mathcal{P}_4) = 3$.

For $g \geq 5$, clearly $\Upsilon_{bd}(\mathcal{P}_g) \geq 2$. By Kim et al. [KKW+13], every planar graph of girth at least six can be decomposed into a forest and a matching. Thus by Theorem 3.1.4, every planar graph of girth at least six can be decomposed into two forests whose components have diameter at most 18. Hence $\Upsilon_{bd}(\mathcal{P}_6) = 2$ and $\Upsilon_{bd}(\mathcal{P}_g) = 2$ for all $g \geq 6$.   □

Notice that Lemma 3.3.2 still holds when G^* has multiple edges. Thus we have the following corollary.

COROLLARY 3.3.5 *Every 6-edge-connected planar (multi)graph contains two edge-disjoint $\frac{18}{19}$-thin spanning trees.*

PROOF. Let G be a 6-edge-connected planar (multi)graph. As G is 6-edge-connected, it follows that the dual G^* of G is a simple planar graph of girth at least 6. As in Theorem 3.3.4, we find that $\Upsilon_{18}(G^*) = 2$. By Lemma 3.3.2, $(G^*)^* = G$ contains two edge-disjoint $\frac{18}{19}$-thin spanning trees.   □

As we have seen, bounded diameter arboricity differs from star arboricity for the class of planar graphs (5 instead of 4). The only missing case in Theorem 3.3.4 is $g = 5$. Clearly, $2 \leq \Upsilon_{bd}(\mathcal{P}_5) \leq \Upsilon_{bd}(\mathcal{P}_4) = 3$. We conjecture that the following holds.

CONJECTURE 3.3.6 $\Upsilon(\mathcal{P}_5) = 2$.

This conjecture would be implied by Theorem 3.1.4 if the answer to the following question is affirmative.

QUESTION 3.3.7 *Is every planar graph of girth 5 the union of a forest and a star forest?*

As before, a positive answer to this question would also imply that every 5-edge-connected planar graph contains two disjoint $\frac{18}{19}$-thin spanning trees. It is not even known whether there exists an ε such that every 5-edge-connected planar graph contains an ε-thin spanning tree.

CHAPTER 4

Decomposing into locally irregular subgraphs

The results of this chapter were obtained by the author in joint work with Bensmail and Thomassen [BMT].

4.1 Definitions and basic observations

We start this section by recalling some of the definitions from Chapter 1.

DEFINITION 4.1.1 A graph G is *locally irregular* if any two adjacent vertices have distinct degrees. We call an edge-colouring *locally irregular* if each colour class induces a locally irregular subgraph.

We write $\mathcal{L}$ for the class of locally irregular graphs. Let us call a graph *exceptional* if it is not $\mathcal{L}$-decomposable.

DEFINITION 4.1.2 For every $\mathcal{L}$-decomposable graph G, the *irregular chromatic index* of G, denoted by $\chi'_{\mathrm{irr}}(G)$, is defined as the smallest number of colours in a locally irregular edge-colouring of G.

Baudon, Bensmail, Przybyło, and Woźniak [BBPW15] characterised the graphs admitting an $\mathcal{L}$-decomposition. To state this characterisation, we define a family $\mathcal{E}$ of graphs. A connected graph G belongs to $\mathcal{E}$ if and only if G has a nonempty collection of triangles, G has no other cycles, G has maximum degree at most 3, all vertices not in a triangle have degree at most 2, and if P is path in G whose intermediate vertices all have degree 2 in G and P is maximal with this property, then P has odd length if and only if both its ends are in triangles.

THEOREM 4.1.3 *A connected graph is exceptional if and only if it is a path of odd length, a cycle of odd length, or a member of $\mathcal{E}$.*

While it is known which graphs admit an $\mathcal{L}$-decomposition, it is still an open problem how many parts are needed in an $\mathcal{L}$-decomposition. The following strong conjecture was made by Baudon et al. [BBPW15].

CONJECTURE 4.1.4 *For every $\mathcal{L}$-decomposable graph G, we have $\chi'_{\mathrm{irr}}(G) \leq 3$.*

The number 3 in Conjecture 4.1.4 cannot be decreased to 2, as shown for example by cycles with lengths congruent to 2 modulo 4 and complete graphs.

The strongest evidence for Conjecture 4.1.4 so far is due to Przybyło [Prz16] who verified it for graphs of large minimum degree.

THEOREM 4.1.5 *For every graph G with minimum degree at least 10^{10}, we have $\chi'_{\mathrm{irr}}(G) \leq 3$.*

Here we show that $\chi'_{\mathrm{irr}}(G) \leq 328$ holds for every $\mathcal{L}$-decomposable graph G. This proof provides the first general constant upper bound on the irregular chromatic index.

Notice that every connected graph of even size can be decomposed into paths of length 2 and is thus $\mathcal{L}$-decomposable. For this reason we start our proof by showing that we can restrict our attention to connected graphs of even size. We show that every $\mathcal{L}$-decomposable graph G of odd size contains a locally irregular subgraph H such that all connected components of $G - E(H)$ have even size.

In Section 4.3 we show that for bipartite graphs of even size the irregular chromatic index is at most 9. In Section 4.4, we decompose a connected graph G of even size into a graph H of minimum degree 10^{10} and a $(2 \cdot 10^{10})$-degenerate graph D in which every component has even size. We use Theorem 4.1.5 to decompose H, and we further decompose D into 36 bipartite graphs of even size. By using our result for bipartite graphs, this results in a decomposition of G into $3 + 9 \cdot 36 = 327$ locally irregular subgraphs.

4.2 Reduction to graphs of even size

In this section we show that we can always remove a locally irregular subgraph H from an $\mathcal{L}$-decomposable graph G of odd size, so that all connected components of $G - E(H)$ have even size. This implies that if every graph of even size has irregular chromatic index at most c, then every $\mathcal{L}$-decomposable graph has irregular chromatic index at most $c + 1$.

LEMMA 4.2.1 *Let G be a connected graph of odd size. For every vertex $v \in V(G)$ there exists an edge e incident with v such that every connected component of $G - e$ has even size.*

PROOF. Let $E(v)$ denote the set of edges incident with v. If $e \in E(v)$ is not a cut-edge, then $G - e$ is connected and of even size. We may thus assume that all edges in $E(v)$ are cut-edges. For every $e \in E(v)$, let H_e denote the connected component of $G - e$ not containing v. Now

$$E(G) = \bigcup_{e \in E(v)} E(H_e) \cup \{e\}.$$

Since $|E(G)|$ is odd, there exists $e \in E(v)$ for which $|E(H_e) \cup \{e\}|$ is odd. Thus, H_e is of even size, and so is the other connected component of $G - e$.   □

LEMMA 4.2.2 *Let G be a connected graph of even size. For every vertex $v \in V(G)$ there exists a path P of length 2 containing v such that every connected component of $G - E(P)$ has even size.*

PROOF. Let e be an edge incident with v. Then $G - e$ has precisely one connected component of odd size, and e is incident with a vertex u of that component, possibly $u = v$. By Lemma 4.2.1 we can delete an edge f incident with u so that every component of $G - \{e, f\}$ has even size. Since e and f are incident, they form a path P of length 2.   □

THEOREM 4.2.3 *Let G be a connected graph of odd size. If G is $\mathcal{L}$-decomposable, then G contains a locally irregular subgraph H such that every connected component of $G - E(H)$ has even size.*

PROOF. We show that we can choose H to be isomorphic to $K_{1,3}$ or to $K_{1,3}$ where two edges are subdivided once. Assume that G is a graph for which we cannot delete one of these two graphs such that every connected component in the resulting graph is of even size. If G has maximum degree at most 2 and odd size, then G is exceptional. We can thus assume that G has maximum degree at least 3. Notice that every vertex v of degree at least 3 in G must be a

cutvertex, since otherwise we can delete a claw (i.e. a subgraph isomorphic to $K_{1,3}$) centred at v.

First, suppose that G contains a cycle C. Let V_C denote the vertices of C with degree at least 3. For every $v \in V_C$, let $E_C(v)$ denote the two edges of C that are incident with v. If $G - E_C(v)$ is connected, then we can use Lemma 4.2.1 to delete one more edge at v so that every connected component in the resulting graph has even size. We may thus assume that $G - E_C(v)$ is disconnected. Let $G_C(v)$ denote the connected component of $G - E_C(v)$ containing v. If $|E(G_C(v))|$ is odd, then we can again use Lemma 4.2.1 to delete one more edge at v to reach the desired conclusion. Thus we may assume that $|E(G_C(v))|$ is even for all $v \in V_C$. By Lemma 4.2.2, there exists a path P_v of length 2 in $G_C(v)$ incident with v such that every connected component of $G_C(v) - E(P_v)$ has even size. If v is the middle vertex of P_v, then P_v together with one of the two edges in $E_C(v)$ forms a claw whose removal leaves a graph where every connected component has even size. Thus, we may assume that v is an endvertex of P_v. If C has length at least 4, then let P_C be a path of length 3 in C in which v has degree 2. The graph $P_v \cup P_C$ is locally irregular and it is easy to see that every connected component of $G - E(P_v) - E(P_C)$ has even size.

Thus we may assume that all cycles of G have length 3. Suppose two cycles C_1, C_2 have a vertex v in common. Choose an edge e_i incident with v in C_i for $i \in \{1, 2\}$. Now $G - \{e_1, e_2\}$ is connected, so we can apply Lemma 4.2.1 to delete one more edge at v so that every connected component has even size.

So far, we have shown that triangles are the only cycles in G and that any two triangles are disjoint. Now we show that there exists no induced claw in G. Suppose for a contradiction that v is a vertex of degree at least 3 which is a center of a claw. If v is contained in a triangle, then we assume that the degree of v is at least 4. Since any two triangles are disjoint, there exists at most one edge between the neighbours of v. By Lemma 4.2.1, we can delete an edge uv so that every component of $G' = G - uv$ has even size. By our choice of v, there exists

two neighbours u_1 and u_2 of v such that $\{u, u_1, u_2\}$ is an independent set in G. Let G_1 denote the connected component of odd size in $G' - u_1v$. If G_1 contains v, then we can delete a third edge e at v by Lemma 4.2.1 such that all components of $G' - u_1v - e$ have even size. Thus, we can assume that G_1 contains u_1 but not v. Similarly, we may assume that the odd component G_2 of $G' - u_2v$ contains u_2 but not v. Now we can apply Lemma 4.2.1 to delete an edge e_i incident with u_i in G_i such that every connected component of $G_i - e_i$ has even size for $i \in \{1, 2\}$. Thus, every connected component of $G' - e_1 - e_2 - u_1v - u_2v$ has even size. Since G_1, G_2 are distinct components of $G - v$, the graph we removed is isomorphic to $K_{1,3}$ where two edges are subdivided once. This contradicts our choice of G, implying that G has no induced claw.

Thus we may assume that the maximum degree in G is 3 and that every vertex of degree 3 is contained in a triangle. Since there are no other cycles, this implies that the contraction of all triangles results in a tree of maximum degree 3. All that remains to show is that the parities of the path lengths are the same as for the exceptional graphs. Let P be a path joining a leaf in G with a triangle C. Let v be the common vertex of P and C. Now $P = G_C(v)$ and since $|E(G_C(v))|$ is even, the length of P is even. Finally, let P be a path joining two different triangles C_1 and C_2. If v_1 and v_2 denote the endvertices, then

$$|E(G)| = |E(G_{C_1}(v_1))| + |E(G_{C_2}(v_2))| - |E(P)|.$$

Since $|E(G_{C_1}(v_1))|$ and $|E(G_{C_2}(v_2))|$ are even and $|E(G)|$ is odd, we get that $|E(P)|$ must also be odd. This shows that G is exceptional. $\square$

4.3 Locally irregular decompositions of bipartite graphs

We now focus on the irregular chromatic index of bipartite graphs. Recall that the only bipartite exceptional graphs are odd length paths. In Corollary 4.3.10 we show that $\chi'_{\mathrm{irr}}(G) \leq 10$ for every $\mathcal{L}$-decomposable bipartite graph G, which is the first constant upper bound on χ'_{irr} for bipartite graphs.

If all vertices in one partition class of the bipartite graph G have even degree, while the vertices in the other partition class have odd degree, then G is locally irregular. The idea of the proof is to remove some well-behaved subgraphs from G to obtain a graph which is very close to this structure. These well-behaved subgraphs include a particular kind of forest, which is defined as follows.

DEFINITION 4.3.1 We say a forest is *balanced* if it has a bipartition such that all vertices in one of the partition classes have even degree.

Since a balanced forest cannot contain an odd length path as a connected component, it follows from [BBPW15] that $\chi'_{\mathrm{irr}}(F) \leq 3$ for every balanced forest F. The characterisation of trees T with $\chi'_{\mathrm{irr}}(T) \leq 2$ in [BBS15] implies that even $\chi'_{\mathrm{irr}}(F) \leq 2$ holds for balanced forests F. For the sake of completeness, we present a short proof of this special case.

LEMMA 4.3.2 *If F is a balanced forest, then F admits a 2-edge-colouring such that each colour induces a locally irregular graph and, for each vertex v in the partition class with no odd degree vertex, all edges incident with v have the same colour. In particular, $\chi'_{\mathrm{irr}}(F) \leq 2$.*

PROOF. The proof is by induction on the number of edges of F. Clearly, we may assume that F is connected. Let A and B be the partition classes of F, where all vertices in B have even degree. We may assume that some vertex in

A has even degree since otherwise we can give all edges of F the same colour. Let v be a vertex in A of even degree q. We delete v but keep the edges incident with v and let them go to q new vertices $v_1, v_2, \ldots, v_q$ each of degree 1. In other words, we split F into q new trees $T_1, T_2, \ldots, T_q$ such that the union of their edges is the edge set of F. Each of the trees $T_1, T_2, \ldots, T_q$ is balanced and has therefore a colouring of its edges in colours red and blue satisfying the conclusion of Lemma 4.3.2. This also gives a colouring of the edges of F in colours red and blue. By switching colours in some of the T_i, if necessary, we can ensure that the red degree of v is 1. This shows that also F satisfies the conclusion of Lemma 4.3.2. $\square$

Apart from balanced forests we also delete a subgraph which is the union of a path and an induced cycle. The following lemma gives an upper bound on the irregular chromatic index in this case.

LEMMA 4.3.3 *Let G be a bipartite graph and let v be a vertex in G. If G is the edge-disjoint union of an induced cycle C through v and a path P starting at v, then $\chi'_{\mathrm{irr}}(G) \leq 4$.*

PROOF. If the length of P is 0, then $\chi'_{\mathrm{irr}}(G) \leq 3$, so we may assume P has positive length. First suppose that P has odd length. Let e denote the edge of P incident with v. It is easy to see that $\chi'_{\mathrm{irr}}(C + e) \leq 2$. Thus,

$$\chi'_{\mathrm{irr}}(G) \leq \chi'_{\mathrm{irr}}(C + e) + \chi'_{\mathrm{irr}}(P - e) \leq 2 + 2 = 4 \,.$$

Now suppose the length of P is even. If the length of C is divisible by 4, then

$$\chi'_{\mathrm{irr}}(G) \leq \chi'_{\mathrm{irr}}(C) + \chi'_{\mathrm{irr}}(P) \leq 2 + 2 = 4 \,.$$

We may therefore assume the length of C is congruent to 2 modulo 4. Let e denote the edge of P incident with v, and let f denote the edge incident with e on P. It is easy to check that if e, f and all edges of C incident to e or f are coloured 1, then this colouring can be extended to a locally irregular

$\{1, 2\}$-edge-colouring of $C + e + f$. Thus, we have

$$\chi'_{\text{irr}}(G) \leq \chi'_{\text{irr}}(C + e + f) + \chi'_{\text{irr}}(P - e - f) \leq 2 + 2 = 4 \,.$$

$\square$

The following lemma is well-known.

LEMMA 4.3.4 *Let G be a connected graph and let S be a set of vertices. If S is even, then there exists a collection of $\frac{|S|}{2}$ edge-disjoint paths in G such that each vertex in S is an endvertex of precisely one of them.*

PROOF. Take a collection of paths having the vertices in S as endvertices for which the total length is minimal. $\square$

COROLLARY 4.3.5 *If G is a connected bipartite graph of even size with partition classes A and B, then there exists a balanced forest F with leaves in A such that in $G - E(F)$ all vertices in A have even degree.*

PROOF. Notice that since G has even size, the number of vertices in A with odd degree is even. The statement follows by choosing S to be the set of odd-degree vertices in A, and F as the union of the paths given by Lemma 4.3.4. $\square$

COROLLARY 4.3.6 *Let G be a connected bipartite graph with partition classes A and B, and let v be a vertex in B. If all vertices in A have even degree, then there exists a balanced forest F with leaves in B such that in $G - E(F)$ all vertices in $B \setminus \{v\}$ have odd degree.*

PROOF. Choose S as the set of even-degree vertices in B. If $|S|$ is odd, then we apply Lemma 4.3.4 to the set $S \cup \{v\}$ or $S \setminus \{v\}$, and if $|S|$ is even we

apply Lemma 4.3.4 to the set S. Now the union of the paths forms the desired balanced forest.   □

LEMMA 4.3.7 *Let G be a bipartite graph with partition classes A and B, and let v be a vertex in B. If all vertices in A have even degree and all vertices in $B \setminus \{v\}$ have odd degree, then there exists a path P starting in v such that $G - E(P)$ is locally irregular.*

PROOF. If v has odd degree, then we can choose P as a path of length 0. If v has even degree and G is not locally irregular, then v is adjacent to a vertex u_1 of the same degree. We choose the edge vu_1 as the first edge of P and define $G_1 = G - vu_1$. If G_1 is not locally irregular, then u_1 is adjacent to a neighbour u_2 of the same degree. In this case we extend P by the edge u_1u_2 and define $G_2 = G_1 - u_1u_2$. We continue like this, defining G_{i+1} if G_i is not locally irregular by deleting a conflict edge u_iu_{i+1}. We claim that this process stops with a locally irregular graph G_k and that the deleted edges form a path. Notice that if G_i is not locally irregular, then u_i is incident to a vertex u_{i+1} of the same degree. Moreover, the degree of u_i in G_i is $d(v) - i$, so the degrees $d(u_i)$ form a decreasing sequence. In particular, $u_i \neq u_j$ for $i \neq j$ and $u_i \neq v$ for all i. Thus, eventually the process stops with a locally irregular graph G_k and $G - E(G_k)$ is a path of length k.   □

LEMMA 4.3.8 *Let G be a bipartite graph with partition classes A and B. If all vertices in A have even degree, then $\chi'_{\text{irr}}(G) \leq 7$.*

PROOF. We may assume that G is connected. By Lemma 4.3.6, we can delete a balanced forest F with leaves in B such that in the resulting graph G' there is at most one vertex of even degree in B, say v. If v does not exist or if v is an isolated vertex in G', then G' is locally irregular and $\chi'_{\text{irr}}(G) \leq \chi'_{\text{irr}}(F) + \chi'_{\text{irr}}(G') \leq 3$. Thus, we may assume that v exists. Notice that G' might consist of

several connected components, but every component not containing v is locally irregular. Let H denote the connected component of G' containing v.

If there exists no cycle through v in H, then all edges incident with v are cut-edges. Let e be an edge incident with v, and let H_1 and H_2 denote the two connected components of $H - e$. We may assume that H_1 contains v. Notice that the degree of v in H_1 and in $H_2 + e$ is odd, while the degrees of its neighbours are even. It follows that both H_1 and $H_2 + e$ are locally irregular and hence

$$\chi'_{\text{irr}}(G) \le \chi'_{\text{irr}}(F) + \chi'_{\text{irr}}(H_1) + \chi'_{\text{irr}}(H_2 + e) \le 4\,.$$

Thus, we may assume that there exists a cycle going through v. Let C be a cycle through v of shortest length and set $H' = H - E(C)$. Since the parities of the degrees remain unchanged, the vertex v is still the only vertex in B that could have positive even degree in H', while all vertices in A have even degree. By Lemma 4.3.7, there exists a path P in H' starting in v such that $H' - E(P)$ is locally irregular. Now $\chi'_{\text{irr}}(C \cup P) \le 4$ by Lemma 4.3.3 and we have

$$\chi'_{\text{irr}}(G) \le \chi'_{\text{irr}}(F) + \chi'_{\text{irr}}(H' - E(P)) + \chi'_{\text{irr}}(C \cup P) \le 2 + 1 + 4 = 7\,.$$

$\square$

We are now ready for the main result of this section.

THEOREM 4.3.9 *If G is a connected bipartite graph of even size, then $\chi'_{\text{irr}}(G) \le 9$.*

PROOF. By Lemma 4.3.5, we can delete a balanced forest F of G so that the degrees in A in the resulting graph G' are even. By Lemma 4.3.2 we have $\chi'_{\text{irr}}(F) \le 2$, and $\chi'_{\text{irr}}(G') \le 7$ follows from Lemma 4.3.8. Thus $\chi'_{\text{irr}}(G) \le \chi'_{\text{irr}}(F) + \chi'_{\text{irr}}(G') \le 2 + 7 = 9$. $\square$

COROLLARY 4.3.10 *If G is a connected bipartite graph and not an odd length path, then $\chi'_{\mathrm{irr}}(G) \leq 10$.*

PROOF. Since paths of odd lengths are the only exceptional bipartite graphs, this follows immediately from Theorems 4.2.3 and 4.3.9. $\square$

4.4   Locally irregular decompositions of degenerate graphs

Here we apply the result from the previous section by decomposing degenerate graphs into bipartite graphs of even size. We show that every connected d-degenerate graph of even size can be decomposed into at most $\lceil \log_2(d+1) \rceil + 1$ bipartite graphs whose components all have even size. The proof makes repeated use of the following easy lemma.

LEMMA 4.4.1 *If G is a graph with a vertex v such that $G - v$ is bipartite, then there exists a set E of at most $\lfloor \frac{d(v)}{2} \rfloor$ edges incident with v such that $G - E$ is bipartite.*

PROOF. Since $G - v$ is bipartite, there exists a partition class containing at most $\lfloor \frac{d(v)}{2} \rfloor$ neighbours of v. Deleting all edges in G from v to these vertices results in a bipartite graph. $\square$

LEMMA 4.4.2 *Let d be an even natural number, $\ell \geq \lceil \log_2 d \rceil + 1$, and v a vertex of degree d in a graph G. If $G - v$ is the edge-disjoint union of ℓ bipartite graphs in which every component has even size, then so is G.*

PROOF. Notice that it suffices to prove the statement for $\ell = \lceil \log_2 d \rceil + 1$. We use induction on d. In the case $d = 2$ we colour $G - v$ with colours 1 and 2 so that the monochromatic connected components are bipartite subgraphs of even size. Let u_1, u_2 be the neighbours of v in G. If u_1 and u_2 are not connected by an odd length path in colour 1, then colouring both vu_1 and vu_2 with colour 1 will keep all monochromatic components bipartite and of even size.

Thus, we may assume that u_1 and u_2 are connected by a monochromatic path of odd length in each colour. Let $P = v_0 v_1 \ldots v_k$ be a monochromatic path from u_1 to u_2 in colour 2, so $v_0 = u_1$ and $v_k = u_2$. Suppose that for every $i \in \{0, \ldots, k-1\}$ there exists an even length path in colour 1 from v_i to v_{i+1}. By concatenating them, we get a walk of even length from v_0 to v_k. Since there is also a path of odd length joining v_0 and v_k in colour 1, this contradicts the assumption that the subgraph in colour 1 is bipartite. Thus, there exists $i \in \{0, \ldots, k-1\}$ for which there is no even length path in colour 1 from v_i to v_{i+1}. Choose i minimal with this property. We change the colour of $v_i v_{i+1}$ to colour 1. By the choice of i, all monochromatic components in colour 1 are still bipartite. Now there exists precisely one monochromatic component of odd size in each colour. Notice that the monochromatic component of odd size in colour 1 is incident with both u_1 and u_2, while the one in colour 2 is incident with at least one of u_1 and u_2. Thus, we can colour one of the edges at v with colour 2 so that all monochromatic components in colour 2 are bipartite and of even size. Colouring the other edge at v with colour 1 yields the desired decomposition.

Now suppose $d \geq 4$ and that the statement is true for all smaller even numbers. Set $d' = \frac{d}{2}$ if d is divisible by 4, and $d' = \frac{d}{2} + 1$ otherwise. Notice that d' is even and $\lceil \log_2 d \rceil = \lceil \log_2 d' \rceil + 1$. Let $\mathcal{H}$ be the collection of $\lceil \log_2 d \rceil + 1$ bipartite graphs in $G - v$ with even component sizes. Choose $H \in \mathcal{H}$ and denote by G_H the graph we get by adding v and all its incident edges to H. By Lemma 4.4.1, there exists a set E of d' edges incident with v such that $G_H - E$ is bipartite. Since $d - d'$ is even, all connected components of $G_H - E$ have even size. We add the edges in E to the union of the graphs in $\mathcal{H} \setminus \{H\}$ to obtain a graph G'.

By the induction hypothesis, we can decompose G' into $\lceil \log_2 d' \rceil + 1$ bipartite graphs where every component has even size. Together with $G_H - E$, this is a collection of $\lceil \log_2 d' \rceil + 2 = \lceil \log_2 d \rceil + 1$ such graphs. $\qquad\square$

Notice that the bound $\lceil \log_2 d \rceil + 1$ can in general not be decreased by more than 1. The complete graph K_{d+1} is d-degenerate and at least $\lceil \log_2(d + 1) \rceil$ bipartite graphs are needed to decompose it. Moreover, we might need more bipartite graphs to achieve that all components have even size. For example, the complete graph K_4 can be decomposed into two bipartite graphs, but three bipartite graphs are necessary to achieve even component sizes.

THEOREM 4.4.3 *Let $d \geq 1$ be a natural number. If G is a d-degenerate graph in which every connected component has even size, then G can be decomposed into $\lceil \log_2(d + 1) \rceil + 1$ bipartite graphs in which all connected components have even size.*

PROOF. Suppose not, and let G be a smallest counterexample. Clearly G is connected.

CLAIM 1 *If v is a cutvertex of G, then v is adjacent to precisely one vertex u of degree 1 and $G - u - v$ is connected.*

To prove the claim, suppose there exists a 1-separation $\{V_1, V_2\}$ of G with $V_1 \cap V_2 = \{v\}$ and $|V_1|, |V_2| \geq 3$. If $G[V_1]$ and $G[V_2]$ have even size, then we can decompose $G[V_1]$ and $G[V_2]$ by induction. If $G[V_1]$ and $G[V_2]$ have odd size, then we construct two new graphs H_1 and H_2 by adding a new vertex v_i to $G[V_i]$ together with the single edge vv_i. Since $|V_1|, |V_2| \geq 3$, both H_1 and H_2 are smaller than G so we can decompose them by induction. We think of the decomposition as an edge-colouring, and we permute colours so that the edges vv_i receive the same colour in both subgraphs. This corresponds to a colouring of G in which every monochromatic component is bipartite and of even size.

This proves the claim.

In particular, every vertex is adjacent to at most one vertex of degree 1. Among all vertices of degree greater than 1, let v be one of minimal degree. Since G is d-degenerate, we have $d(v) \leq d+1$. Suppose first that $d(v)$ is even. Since G is a smallest counterexample, we can decompose $G-v$ into $\lceil \log_2(d+1) \rceil + 1$ bipartite graphs in which all connected components have even size. By Lemma 4.4.2, this gives rise to the desired decomposition of G.

We may thus assume that $d(v)$ is odd. Set $d' = \frac{1}{2}(d(v) - 1)$ if $d(v)$ is congruent to 1 modulo 4, and $d' = \frac{1}{2}(d(v) + 1)$ otherwise. Notice that d' is even and $\lceil \log_2(d+1) \rceil \geq \lceil \log_2 d' \rceil + 1$. Let u be a neighbour of v of degree greater than 1. If $G - v$ has an isolated vertex, then we let w denote that vertex. Otherwise we add an isolated vertex w. The graph $G - v + uw$ has even size, so we can decompose it as in the previous case. This gives us a decomposition of $G - v$ into $\lceil \log_2(d+1) \rceil + 1$ bipartite graphs in which all connected components are of even size, apart from one component of odd size which is incident with u. Let H be the bipartite subgraph of odd size, and let H_o be the connected component of odd size. Let G_H be the graph we get by adding v and all its incident edges to H.

By Lemma 4.4.1, there exists a set E of precisely d' edges incident with v such that $G_H - E$ is bipartite. We may assume that E does not contain all edges that are incident with H_o. Since $d(v) - d'$ is odd, all connected components of $G_H - E$ have even size. We add the edges in E to $G - v - E(H)$ to obtain a graph G'. Notice that $G - v - E(H)$ is the union of $\lceil \log_2(d+1) \rceil$ bipartite graphs with components of even size. By Lemma 4.4.2, we can decompose G' into $\lceil \log_2(d+1) \rceil$ bipartite graphs where every component has even size. Together with $G_H - E$, this is a collection of $\lceil \log_2(d+1) \rceil + 1$ such graphs. $\qquad\square$

Now we can use our result on bipartite graphs to get an upper bound on the irregular chromatic index of d-degenerate graphs.

Corollary 4.4.4 *If G is a connected d-degenerate graph of even size, then*

$$\chi'_{\mathrm{irr}}(G) \leq 9(\lceil \log_2(d+1) \rceil + 1).$$

Proof. This follows immediately from Theorems 4.3.9 and 4.4.3. □

To get a constant upper bound for $\mathcal{L}$-decomposable graphs in general, we combine Corollary 4.4.4 with Przybyło's result on graphs with large minimum degree. For this purpose, we need the following lemma.

Lemma 4.4.5 *Let d be a natural number. If G is a connected graph of even size, then G can be decomposed into two graphs D and H such that D is $2d$-degenerate, every connected component of D has even size, and the minimum degree of H is at least d.*

Proof. Starting from $D = \emptyset$ and $H = G$, we remove vertices of degree at most $2d$ from H and add them to D. Once this process stops, the graph D is $2d$-degenerate and H has minimum degree at least $2d + 1$. Every connected component C of D with odd size intersects H; let $v(C)$ be a vertex in the intersection. Notice that $v(C) \neq v(C')$ for different connected components C and C' of D. We choose an almost-balanced orientation of H, i.e. an orientation where the out-degree and in-degree at every vertex differ by at most 1. For each connected component C of odd size, we choose an out-edge $e(C)$ at $v(C)$ in H. We remove $e(C)$ from H and add it to D. Since every vertex in H might lose all of its in-edges but at most one out-edge, the minimum degree in H remains at least d. The edges we add to D in this step induce a 2-degenerate subgraph, so D will still be $2d$-degenerate. Moreover, every connected component of odd size gains an edge and possibly gets joined to other connected components of even size. In any case, all connected components of D now have even size. □

Now we are ready for the proof of the main result.

THEOREM 4.4.6 *If G is an $\mathcal{L}$-decomposable graph, then $\chi'_{\mathrm{irr}}(G) \le 328$.*

PROOF. By Theorem 4.2.3 it suffices to show that $\chi'_{\mathrm{irr}}(G) \le 327$ holds for connected graphs G of even size. By Lemma 4.4.5, we can decompose G into two graphs D and H such that D is $(2 \cdot 10^{10})$-degenerate, every connected component of D has even size, and the minimum degree of H is at least 10^{10}. By Theorem 4.1.5, we have $\chi'_{\mathrm{irr}}(H) \le 3$ and by Corollary 4.4.4 we have

$$\chi'_{\mathrm{irr}}(D) \le 9(\lceil \log_2(2 \cdot 10^{10} + 1) \rceil + 1) = 324.$$

Hence, $\chi'_{\mathrm{irr}}(G) \le \chi'_{\mathrm{irr}}(H) + \chi'_{\mathrm{irr}}(D) \le 3 + 324 = 327$. □

Bibliography

[AA89]   I. Algor and N. Alon. The star arboricity of graphs. *Discrete Math.*, 75:11–22, 1989.

[AEH80]   L. Akiyama, G. Exoo, and F. Harary. Covering and packing in graphs III. cyclic and acyclic invariants. *Math. Slovaca*, 30:405–417, 1980.

[Alo88]   N. Alon. The linear arboricity of graphs. *Israel J. Math.*, 62:311–325, 1988.

[AMR92]   N. Alon, C. McDiarmid, and B. Reed. Star arboricity. *Combinatorica*, 12:375–380, 1992.

[BBPW15]   O. Baudon, J. Bensmail, J. Przybyło, and M. Woźniak. On decomposing regular graphs into locally irregular subgraphs. *European J. Combin.*, 49:90–104, 2015.

[BBS15]   O. Baudon, J. Bensmail, and É. Sopena. On the complexity of determining the irregular chromatic index of a graph. *J. Discrete Algorithms*, 30:113–127, 2015.

[BG14]   J. Barát and D. Gerbner. Edge-decomposition of graphs into
             copies of a tree with four edges. *Electr. J. Comb.*, 21:P1.55, 2014.

[BHL⁺]   J. Bensmail, A. Harutyunyan, T.-N. Le, M. Merker, and
             S. Thomassé. A Proof of the Barát–Thomassen Conjecture.
             `arXiv:1603.00197`.

[BHLT]   J. Bensmail, A. Harutyunyan, T.-N. Le, and S. Thomassé. Edge-
             partitioning a graph into paths: beyond the Barát-Thomassen
             conjecture. `arXiv:1507.08208`.

[BMOW16a]   F. Botler, G.O. Mota, M.T.I. Oshiro, and Y. Wakabayashi. De-
             composing highly edge-connected graphs into paths of any given
             length. *J. Combin. Theory Ser. B*, 2016. To appear.

[BMOW16b]   F. Botler, G.O. Mota, M.T.I. Oshiro, and Y. Wakabayashi. De-
             compositions of highly connected graphs into paths of length five.
             Discrete Applied Mathematics, 2016. To appear.

[BMT]   J. Bensmail, M. Merker, and C. Thomassen. Decomposing
             graphs into a constant number of locally irregular subgraphs.
             `arXiv:1604.00235`.

[Bor79]   O.V. Borodin. On acyclic colorings of planar graphs. *Discrete
             Math.*, 25:211–236, 1979.

[BS16]   J. Bensmail and B. Stevens. Edge-partitioning graphs into regular
             and locally irregular components. *Discrete Math. Theor. Comput.
             Sci.*, 17, 2016.

[BT06]   J. Barát and C. Thomassen. Claw-decompositions and Tutte-
             orientations. *J. Graph Theory*, 52:135–146, 2006.

[CS97]   A. Czumaj and W.-B. Strothmann. Bounded degree spanning
             trees. In *Algorithms - ESA '97*, volume 1284 of *Lecture Notes in
             Computer Science*, pages 104–117. Springer, 1997.

[Die12]   R. Diestel. *Graph Theory*. Springer, 4th edition, 2012.

[DT97]   D. Dor and M. Tarsi. Graph decomposition is np-complete: A complete proof of Holyer's conjecture. *SIAM J. Comput.*, 26:1166–1187, 1997.

[ENV02]   M.N. Ellingham, Y. Nam, and H.-J. Voss. Connected (g, f)-factors. *J. Graph Theory*, 39:62–75, 2002.

[Fra92]   A. Frank. On a theorem of Mader. *Discrete Math.*, 101:49–57, 1992.

[God04]   L. Goddyn. Some open problems I like. March 2004. http://people.math.sfu.ca/∼goddyn/Problems/problems.html.

[Har70]   F. Harary. Covering and packing in graphs I. *Ann. N.Y. Acad. Sci.*, 175:198–205, 1970.

[Has15]   M. Hasanvand. Spanning trees and spanning eulerian subgraphs with small degrees. *Discrete Math.*, 338:1317–1321, 2015.

[HMS96]   S.L. Hakimi, J. Mitchem, and E. Schmeichel. Star arboricity of graphs. *Discrete Math.*, 149:93–98, 1996.

[KKP10]   M. Kalkowski, M. Karoński, and F. Pfender. Vertex-coloring edge-weightings: Towards the 1-2-3-conjecture. *J. Combin. Theory Ser. B*, 100:347–349, 2010.

[KKW⁺13]   S. Kim, A.V. Kostochka, D.B. West, H. Wu, and X. Zhu. Decomposition of sparse graphs into forests and a graph with bounded degree. *J. Graph Theory*, 74:369–391, 2013.

[KŁT04]   M. Karoński, T. Łuczak, and A. Thomason. Edge weights and vertex colours. *J. Combin. Theory Ser. B*, 91:151–157, 2004.

[Lav05]   F. Laviolette. Decompositions of infinite graphs: I-bond-faithful decompositions. *J. Combin. Theory Ser. B*, 94:259–277, 2005.

[LTWZ13]   L.M. Lovász, C. Thomassen, Y. Wu, and C.-Q. Zhang. Nowhere-zero 3-flows and modulo k-orientations. *J. Combin. Theory Ser. B*, 103:587–598, 2013.

[Mad72]   W. Mader. Existenz n-fach zusammenhängender Teilgraphen in Graphen genügend grosser Kantendichte. *Abh. Math. Seminar Hamburg*, 37:86–97, 1972.

[Mad78]   W. Mader. A reduction method for edge-connectivity in graphs. *Ann. Discrete Math.*, 3:145–164, 1978.

[McD02]   C. McDiarmid. Concentration for independent permutations. *Combin. Probab. Comput.*, 11:163–178, 2002.

[MdMRZ12] M. Montassier, P. Ossona de Mendez, A. Raspaud, and X. Zhu. Decomposing a graph into forests. *J. Combin. Theory Ser. B*, 102:38–52, 2012.

[Mer16]   M. Merker. Decomposing highly edge-connected graphs into homomorphic copies of a fixed tree. *J. Combin. Theory Ser. B*, 2016. To appear.

[MP]   M. Merker and L. Postle. Bounded Diameter Arboricity. `arXiv:1608.05352`.

[MR02]   M. Molloy and B. Reed. *Graph colouring and the probabilistic method.* Springer, 2002.

[NW61]   C.St.J.A. Nash-Williams. Edge-disjoint spanning trees of finite graphs. *J. Lond. Math. Soc.*, 36:445–450, 1961.

[NW64]   C.St.J.A. Nash-Williams. Decompositions of finite graphs into forests. *J. London Math. Soc.*, 39:12, 1964.

[Prz16]   J. Przybyło. On decomposing graphs of large minimum degree into locally irregular subgraphs. *Electron. J. Combin.*, 23, 2016.

[Sea]   B. Seamone. The 1-2-3 Conjecture and related problems: a survey. `arXiv:1211.5122`.

[Tho]   C. Thomassen. Orientations of infinite graphs with prescribed edge-connectivity. *Combinatorica*. To appear.

[Tho08a]   C. Thomassen. Decompositions of highly connected graphs into paths of length 3. *J. Graph Theory*, 58:286–292, 2008.

[Tho08b]   C. Thomassen. Edge-decompositions of highly connected graphs into paths. *Abh. Math. Seminar Hamburg*, 78:17–26, 2008.

[Tho12]   C. Thomassen. The weak 3-flow conjecture and the weak circular flow conjecture. *J. Combin. Theory Ser. B*, 102:521–529, 2012.

[Tho13a]   C. Thomassen. Decomposing a graph into bistars. *J. Combin. Theory Ser. B*, 103:504–508, 2013.

[Tho13b]   C. Thomassen. Decomposing graphs into paths of fixed length. *Combinatorica*, 33:97–123, 2013.

[Tut61]   W.T. Tutte. On the problem of decomposing a graph into n connected factors. *J. Lond. Math. Soc.*, 36:221–230, 1961.

[ZB98]   L. Zhenhong and X. Baoguang. On low bound of degree sequences of spanning trees in k-edge-connected graphs. *J. Graph Theory*, 28:87–95, 1998.